Vector Balancing in Lebesgue Spaces

Victor Reis * Thomas Rothvoss [†]

The *Komlós* conjecture suggests that for any vectors $a_1, \ldots, a_n \in B_2^m$ there exist $x_1, \ldots, x_n \in \{-1, 1\}$ so that $\|\sum_{i=1}^n x_i a_i\|_{\infty} \leq O(1)$. It is a natural extension to ask what ℓ_q -norm bound to expect for $a_1, \ldots, a_n \in B_p^m$. We prove a tight partial coloring result for such vectors, implying a nearly tight full coloring bound. As a corollary, this implies a special case of Beck-Fiala's conjecture. We achieve this by showing that, for any $\delta > 0$, a symmetric convex body $K \subseteq \mathbb{R}^n$ with Gaussian measure at least $e^{-\delta n}$ admits a partial coloring. Previously this was known only for a *small* enough δ . Additionally, we show that a hereditary volume bound suffices to provide such Gaussian measure lower bounds.

Abstract

^{*}University of Washington, Seattle. Email: voreis@uw.edu.

[†]University of Washington, Seattle. Email: rothvoss@uw.edu. Supported by NSF CAREER grant 1651861 and a David & Lucile Packard Foundation Fellowship.

1 Introduction

The celebrated *Spencer's Theorem* in discrepancy theory [Spe85] shows that "six standard deviations suffice" for balancing vectors in the ℓ_{∞} -norm: for any $a_1, \ldots, a_n \in [-1,1]^n$, there exist signs $\mathbf{x} \in \{-1,1\}^n$ such that $\|\sum_{i=1}^n x_i \mathbf{a}_i\|_{\infty} \le 6\sqrt{n}$. More generally, Spencer showed that for vectors in $[-1,1]^m$ with $n \le m$ one can achieve a bound of $O(\sqrt{n\log(2m/n)})$. While his proof used a nonconstructive form of the *partial coloring lemma* based on the pigeonhole principle, in the past decade several approaches starting with the breakthrough work of Bansal [Ban10] did succeed in computing such signs in polynomial time [LM12, Rot14, LRR16, ES18].

As for balancing vectors of bounded ℓ_2 -norm, the situation has been more delicate. In the same paper, Spencer [Spe85] showed a nonconstructive bound of $O(\log n)$ for the ℓ_{∞} discrepancy of vectors $a_1, \ldots, a_n \in B_2^m$ and also stated a discrete version of a conjecture of Komlós that this may be improved to O(1). This was improved to $O(\sqrt{\log n})$ by Banaszczyk [Ban98] who showed that in fact for any set of *n* vectors of ℓ_2 -norm at most 1 and any convex body $K \subseteq \mathbb{R}^m$ of Gaussian measure at least 1/2, some ±1 combination of such vectors lies in 5 \cdot K. For the general setting of ℓ_q discrepancy, Matoušek [Mat98] gave an upper bound of $O(q) \cdot m^{1/q}$ for balancing vectors from ℓ_2 to ℓ_q . More recently, the work of Barthe, Guédon, Mendelson and Naor [BGMN05] (see Prop. 25) shows that, for $q \ge 2$, *n*-dimensional slices of the ℓ_q ball in \mathbb{R}^m scaled by a factor of $O(\sqrt{q}) \cdot n^{1/q}$ do have Gaussian measure at least 1/2 (we include an alternate proof in the appendix), thus improving the bound to $O(\sqrt{q}) \cdot n^{1/q}$. For $q = \log n$, this matches the ℓ_2 to ℓ_{∞} bound of $O(\sqrt{\log n})$. Banaszczyk's proof was nonconstructive and the first polynomial time algorithm in the general convex body setting was found only recently by Bansal, Dadush, Garg and Lovett [BDGL18], while the Komlós conjecture remains an open problem. The work of [BDGL18] actually shows that for any vectors $a_1, \ldots, a_n \in B_2^m$ there exists an efficiently computable distribution over signs $x \in \{-1, 1\}^n$ so that the sum $X := \sum_{i=1}^n x_i a_i$ is O(1)subgaussian, meaning that $\mathbb{E}[e^{\langle \boldsymbol{\theta}, \boldsymbol{X} \rangle}] \leq e^{O(1) \|\boldsymbol{\theta}\|_2^2}$ for every $\boldsymbol{\theta} \in \mathbb{R}^m$, and will be in $O(1) \cdot K$ with good probability. Interestingly, this means their algorithm is *oblivious* to the body K, which is a striking difference to the regime of $\gamma_n(K) = e^{-\Theta(n)}$ where any algorithm needs to be dependent on K. The connection between Banaszczyk's theorem and subgaussianity is due to Dadush et al. [DGLN16].

For the general setting of balancing vectors from ℓ_p to ℓ_q , where we are given vectors $a_1, \ldots, a_n \in B_p^m$ and wish to find signs x_1, \ldots, x_n that minimize the ℓ_q norm of $\sum_{i=1}^n x_i a_i$ (also called ℓ_q discrepancy), not much was known beyond Spencer's theorem $(p = \infty)$ or what can be deduced from Banaszczyk's theorem as above: any vector in B_p^m also belongs to $m^{\max(0,1/2-1/p)} \cdot B_2^m$, thus implying a discrepancy bound of $O(\sqrt{q}) \cdot m^{\max(0,1/2-1/p)} \cdot n^{1/q}$. Even in the square case m = n, in spite of tight partial coloring bounds [Spe85], it has been an open problem to remove the dependency on \sqrt{q} [DNTT18]. The goal of this paper is to provide a unified approach for balancing from ℓ_p to ℓ_q via optimal constructive fractional partial colorings, which yield optimal bounds for most of the range $1 \le p \le q \le \infty$. We obtain such fractional partial colorings by proving a new measure lower bound on the relevant linear preimages of ℓ_q balls (Section 3) and an improved algorithm for sets of Gaus-

sian measure $e^{-\delta n}$ for any $\delta > 0$ (Section 4), as opposed to previous work ([Rot14, ES18]) which required measure $e^{-\delta n}$ for *sufficiently small* $\delta > 0$. Finally, we show that a *hereditary* volume lower bound is sufficient to imply such Gaussian measure bound (Section 5).

As an application, we show a slight improvement to the bounds for the well-known Beck-Fiala conjecture [BF81], a discrete version of Komlós. It asks for a $O(\sqrt{t})$ bound on the ℓ_{∞} discrepancy of any $a_1, \ldots, a_n \in \{0, 1\}^m$, each with at most *t* ones. We establish the conjecture for $t \ge n$ and show slightly improved bounds when *t* is close to *n* (Corollary 4).

Notation. Let $B_p^m := \{ \mathbf{x} \in \mathbb{R}^m : \|\mathbf{x}\|_p \le 1 \}$ denote the unit ball in the ℓ_p -norm. The *Gaussian measure* of a measurable set $K \subseteq \mathbb{R}^n$ is given by $\gamma_n(K) := \Pr_{\mathbf{x} \sim N(\mathbf{0}, I_n)}[\mathbf{x} \in K]$. We denote the *mean width* of a convex set as $w(K) := \mathbb{E}_{\boldsymbol{\theta} \in S^{n-1}}[\sup_{\mathbf{x} \in K} \langle \boldsymbol{\theta}, \mathbf{x} \rangle]$. The Euclidean distance to a set $S \subseteq \mathbb{R}^n$ is denoted by $d(\mathbf{x}, S) := \min\{\|\mathbf{x} - \mathbf{y}\|_2 : \mathbf{y} \in S\}$. A function $f : \mathbb{R}^m \to \mathbb{R}$ is α -Lipschitz if $|f(\mathbf{x}) - f(\mathbf{y})| \le \alpha \cdot \|\mathbf{x} - \mathbf{y}\|_2$ for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m$. If $\mathbf{A} \in \mathbb{R}^{m \times n}$ is a matrix, we denote its rows by $\mathbf{A}_1, \ldots, \mathbf{A}_m \in \mathbb{R}^n$ and its columns by $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathbb{R}^m$. Naturally, a matrix can also be interpreted as a (not necessarily invertible) linear map. Then for any set $K \subseteq \mathbb{R}^m$, we use the notation $\mathbf{A}^{-1}(K) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} \in K\}$. The *C*-scaling of a symmetric convex body *K* is the body $C \cdot K = \{c\mathbf{x} : \mathbf{x} \in K\}$.

1.1 Our contribution

Our main contribution is a tight bound on partial colorings for balancing from ℓ_p to ℓ_q :

Theorem 1. Let $n \le m$ and $2 \le p \le q \le \infty$.¹ Then for any $a_1, ..., a_n \in B_p^m$, there exists a polynomial-time computable partial coloring $x \in [-1,1]^n$ with $|\{i : x_i^2 = 1\}| \ge n/2$ so that

$$\left\|\sum_{i=1}^{n} x_i \boldsymbol{a}_i\right\|_q \le C \sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)} \cdot n^{1/2 - 1/p + 1/q}$$

for some universal constant C > 0.

By a linear algebraic argument due to Bárány and Grinberg [BG81], the condition $n \le m$ does not weaken the theorem: in fact for n > m the upper bound can only be larger than that of n = m by a factor of two. On the other hand, the condition $p \le q$ is natural, for otherwise if p > q we would need a polynomial dependence on the dimension m, even for n = 1. By iteratively applying Theorem 1 we can obtain a full coloring at the expense of another factor of $\frac{1}{1/2-1/p+1/q}$, with the caveat that p > 2 whenever $q = \infty$:

Theorem 2. Let $n \le m$ and $2 \le p \le q \le \infty$ with $\{p, q\} \ne \{2, \infty\}$. Then for any $a_1, \ldots, a_n \in B_p^m$, there exist polynomial-time computable signs $\mathbf{x} \in \{-1, 1\}^n$ so that

$$\left\|\sum_{i=1}^{n} x_{i} \boldsymbol{a}_{i}\right\|_{q} \leq \frac{C\sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)}}{1/2 - 1/p + 1/q} \cdot n^{1/2 - 1/p + 1/q}$$

for some universal constant C > 0.

¹When $p \le 2$, uniformly random signs achieve a tight bound of $\Theta(n^{1/q})$ (see Theorem 5), so we focus on the more interesting case $p \ge 2$.

This significantly improves upon the general $\sqrt{q} \cdot m^{1/2-1/p} \cdot n^{1/q}$ bound from Banaszczyk's theorem in [DNTT18] when $p = 2 + \varepsilon$ for (not too small) $\varepsilon > 0$ and $q \gg 1$. It is also worth noting that we may always assume $q \le \log(m)$ as larger norms are equivalent up to a constant by Lemma 8. When p = q and m = n, we get the following corollary which matches, up to a constant, the lower bound $\Omega(\sqrt{n})$ of [Ban93] known to hold for any norm:

Corollary 3 (ℓ_p version of Spencer's theorem). Let $2 \le p \le \infty$ and $n \in \mathbb{N}$. Then for any $a_1, \ldots, a_n \in B_p^n$, there exist polynomial-time computable signs $x \in \{-1, 1\}^n$ so that

$$\left\|\sum_{i=1}^n x_i \boldsymbol{a}_i\right\|_p \le C\sqrt{n},$$

for some universal constant C > 0.

The following corollary shows the Beck-Fiala conjecture holds for $t \ge n$ and slightly improves upon the best known bound of $O(\sqrt{t \log n})$ [Ban98] when *t* is close to *n*:

Corollary 4 (Bound for Beck-Fiala). Let $n \le m$ and $a_1, ..., a_n \in \{0, 1\}^m$, each with at most $t \in [m]$ ones. Then there exist polynomial-time computable signs $\mathbf{x} \in \{-1, 1\}^n$ so that

$$\left\|\sum_{i=1}^n x_i \boldsymbol{a}_i\right\|_{\infty} \leq C\sqrt{t} \log\left(\frac{2\max(n,t)}{t}\right),$$

for some universal constant C > 0.

We show the partial coloring bound in Theorem 1 is tight at least when m = n:

Theorem 5. Let $1 \le p \le q \le \infty$. There exist infinitely many positive integers *n* for which we can find $a_1, \ldots, a_n \in B_p^n$ such that for any $x \in [-1, 1]^n$ with $|\{i : x_i^2 = 1\}| \ge n/2$ one has

$$\left\|\sum_{i=1}^{n} x_{i} \boldsymbol{a}_{i}\right\|_{q} \geq C \cdot n^{\max(0,1/2-1/p)+1/q},$$

for some universal constant C > 0.

A result of Giannopoulos [Gia97] shows that for a *small enough* constant, a symmetric convex body *K* with $\gamma_n(K) \ge e^{-\alpha n}$ contains a partial coloring $\mathbf{x} \in \{-1, 0, 1\}^n$ with a linear number of entries in ±1. We can prove that for fractional colorings *any* constant $\alpha > 0$ suffices. Our argument even works for intersections with a large enough subspace.

Theorem 6. For all $\alpha, \beta, \gamma > 0$, there is a constant $C := C(\alpha, \beta, \gamma) > 0$ so that the following holds: There is a randomized polynomial time algorithm which for a symmetric convex set $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \ge e^{-\alpha n}$, a shift $\mathbf{y} \in [-1,1]^n$ and a subspace $H \subseteq \mathbb{R}^n$ with dim $(H) \ge \beta n$, finds an $\mathbf{x} \in (C \cdot K \cap H)$ with $\mathbf{x} + \mathbf{y} \in [-1,1]^n$ and $|\{i \in [n] : (\mathbf{x} + \mathbf{y})_i \in \{\pm 1\}\}| \ge (\beta - \gamma)n$.

Finally, we show that a weaker *hereditary* volume lower bound suffices to provide Gaussian measure lower bounds for arbitrary convex bodies. Previously such an implication was known only for the Gaussian measure of intersections with subspaces [DNTT18]:

Theorem 7. Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body. Given $S \subseteq [n]$, denote by K_S the intersection with the coordinate subspace: $K_S := K \cap \{x : x_i = 0 \forall i \notin S\} \subseteq \mathbb{R}^S$. Then we have

$$\gamma_n(K) \ge \min_{S \subseteq [n]} \operatorname{vol}_{|S|}(K_S) \cdot 2^{-O(n)},$$

with the convention that $vol_0(\{0\}) = 1$. More generally, for any $\delta \in (0, 1]$,

$$\gamma_n(K) \ge \min_{S \subseteq [n], |S| \le \delta n} \operatorname{vol}_{|S|}(K_S)^{1/\delta} \cdot 2^{-O(n/\delta)}$$

2 Preliminaries

We will use two elementary inequalities dealing with ℓ_p -norms. The first one estimates the ratio between different norms:

Lemma 8. For any $z \in \mathbb{R}^m$ and $1 \le p \le q \le \infty$, we have $||z||_q \le ||z||_p \le m^{1/p-1/q} ||z||_q$.

It is instructive to note that this bound implies $\|\boldsymbol{z}\|_{\infty} \leq \|\boldsymbol{z}\|_{\log_2(m)} \leq 2\|\boldsymbol{z}\|_{\infty}$. If one has an upper bound on the largest entry in a vector — say $\|\boldsymbol{z}\|_{\infty} \leq 1$ — then one can strengthen the first inequality to $\|\boldsymbol{z}\|_q^q \leq \|\boldsymbol{z}\|_p^p$. More generally:

Lemma 9. For any $z \in \mathbb{R}^m$ and $1 \le p \le q \le \infty$, we have $\|z\|_q^q \le \|z\|_p^p \cdot \|z\|_{\infty}^{q-p}$.

We will also need the following version of *Khintchine's inequality*, see e.g. the excellent textbook of Artstein-Avidan, Giannopoulos and Milman [AAGM15].

Lemma 10 (Khintchine's inequality). Given p > 0, $a_1, \ldots, a_n \in \mathbb{R}$ and $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_n)$, we have

$$\mathbb{E}\left[\left|\sum_{i=1}^{n} x_{i} a_{i}\right|^{p}\right] \leq C\sqrt{p} \cdot \left(\sum_{i=1}^{n} a_{i}^{2}\right)^{p/2}$$

where C > 0 is a universal constant.

This fact can be derived from a standard Chernov bound which guarantees that for a vector with $\|\boldsymbol{a}\|_2 = 1$ one has $\Pr[|\langle \boldsymbol{a}, \boldsymbol{x} \rangle| > \lambda] \le 2e^{-\lambda^2/2}$; then one can analyze that the regime of $\lambda = \Theta(\sqrt{p})$ dominates the contribution to $\mathbb{E}[|\langle \boldsymbol{a}, \boldsymbol{x} \rangle|^p]$. We use it to show the following standard estimate on the type constants of ℓ_p spaces (see Appendix A):

Lemma 11. Given $p \ge 1$ and $a_1, \ldots, a_n \in B_p^m$ and $x \sim N(0, I_n)$, we have

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} x_{i} \boldsymbol{a}_{i}\right\|_{p}\right] \leq O(\sqrt{p} \cdot n^{\max(1/2, 1/p)}).$$

A well-known correlation inequality for Gaussian measure is the following:

Lemma 12 (Šidak [Šid67] and Kathri [Kha67]). For any symmetric convex set $K \subseteq \mathbb{R}^n$ and strip $S = \{x \in \mathbb{R}^n : |\langle a, x \rangle| \le 1\}$, one has $\gamma_n(K \cap S) \ge \gamma_n(K) \cdot \gamma_n(S)$.

It is worth noting that a recent result of Royen [Roy14] extends this to any two arbitrary symmetric sets, though its full power will not be needed. We refer to the exposition of Latała and Matlak [LM17]. We also need a one-dimensional estimate:

Lemma 13. For a strip $S = \{x \in \mathbb{R}^n : |\langle a, x \rangle| \le 1\}$, one has

$$\gamma_n(S) = \gamma_1(\{x \in \mathbb{R} : |x| \le \|\boldsymbol{a}\|_2^{-1}\}) \ge 1 - \exp(-\|\boldsymbol{a}\|_2^{-2}/2).$$

We use the following scaling lemma to deal with constant factors, see [Tko15]:

Lemma 14. Let $K \subset \mathbb{R}^n$ be a measurable set and *B* be a closed Euclidean ball such that $\gamma_n(K) = \gamma_n(B)$. Then $\gamma_n(tK) \ge \gamma_n(tB)$ for all $t \in [0,1]$. In particular, if $\gamma_n(C \cdot K) \ge 2^{-O(n)}$ for some constant C > 1 then also $\gamma_n(K) \ge 2^{-O(n)}$.

For Section 4 we also need three helpful results. For the first one, see [vH14].

Theorem 15. If $F : \mathbb{R}^m \to \mathbb{R}$ is 1-Lipschitz, then for $t \ge 0$ one has

$$\Pr_{\boldsymbol{y} \sim N(\boldsymbol{0}, \boldsymbol{I}_m)} \left[F(\boldsymbol{y}) > \mathbb{E}[F(\boldsymbol{y})] + t \right] \le e^{-t^2/2}.$$

The classical *Urysohn Inequality* states that among all convex bodies of identical volume, the Euclidean ball minimizes the width. We will need a variant that is phrased in terms of the Gaussian measure rather than volume. For a proof, see Eldan and Singh [ES18].

Theorem 16 (Gaussian Variant of Urysohn's Inequality). Let $K \subseteq \mathbb{R}^n$ be a convex body and let r > 0 be so that $\gamma_n(K) = \gamma_n(rB_2^n)$. Then $w(K) \ge w(rB_2^n) = r$.

For a symmetric convex body *K* and a subspace *H*, the Gaussian measure of the section $K \cap (\mathbf{x} + H)$ is maximized when $\mathbf{x} = \mathbf{0}$ by log-concavity. Thus we have the following:

Lemma 17 (Gaussian measure of sections). Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body and $H \subseteq \mathbb{R}^n$ a subspace. Then $\gamma_H(K \cap H) \ge \gamma_n(K)$.

3 Main technical result

In this section we show our measure lower bound for balancing vectors from ℓ_p to ℓ_q :

Theorem 18. Let $n \le m$ and $1 \le p \le q \le \infty$. Then for any $a_1, \ldots, a_n \in B_p^m$,

$$\gamma_n\left(\left\{\boldsymbol{x} \in \mathbb{R}^n : \left\|\sum_{i=1}^n x_i \boldsymbol{a}_i\right\|_q \le \sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)} \cdot n^{\max(0, 1/2 - 1/p) + 1/q}\right\}\right) \ge 2^{-O(n)}$$

In order to show Theorem 18, roughly speaking it will suffice to show the corresponding bounds for the two special cases of $q \in \{p, \infty\}$, which can be bootstrapped into a general bound. First we address the simpler case p = q which at heart is based on Khintchine's inequality: **Lemma 19.** Let $n \le m$ and $p \ge 1$. Then for any $a_1, \ldots, a_n \in B_p^m$,

$$\gamma_n\left(\left\{\boldsymbol{x}\in\mathbb{R}^n: \left\|\sum_{i=1}^n x_i \boldsymbol{a}_i\right\|_p \le \sqrt{p} \cdot n^{\max(1/2,1/p)}\right\}\right) \ge 2^{-O(n)}.$$

Proof. By Lemma 11 we know that, for some constant C > 0,

$$\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} \left[\left\| \sum_{i=1}^n x_i \boldsymbol{a}_i \right\|_p \right] \leq C \sqrt{p} \cdot n^{\max(1/2, 1/p)}.$$

By Markov's inequality it follows that

$$\gamma_n\left(\left\{\boldsymbol{x}\in\mathbb{R}^n: \left\|\sum_{i=1}^n x_i \boldsymbol{a}_i\right\|_p \le 2C\sqrt{p}\cdot n^{\max(1/2,1/p)}\right\}\right) \ge 1/2,$$

so that the result follows by Lemma 14.

Next, we deal with the crucial case $q = \infty$:

Lemma 20. Let $n \le m$ and $p \ge 1$. Then for any $A \in \mathbb{R}^{m \times n}$ with columns $a_1, \ldots, a_n \in B_p^m$ and rows $A_1, \ldots, A_m \in \mathbb{R}^n$, the body $K := \{ \mathbf{x} \in \mathbb{R}^n : \|\sum_{i=1}^n x_i \mathbf{a}_i\|_{\infty} \le \sqrt{p} \cdot n^{\max(0, 1/2 - 1/p)} \}$ satisfies

$$\gamma_n(K) \ge \prod_{j \in [m]} \gamma_n(\{ \boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{A}_j \rangle| \le \sqrt{p} n^{\max(0, 1/2 - 1/p)} \}) \ge 2^{-O(n)}$$

Proof. The main idea in the proof is that we can convert the bound on the ℓ_p -norm of the columns a_i into information about the ℓ_2 -norm of the rows A_j . Namely,

$$\left(\frac{1}{n}\sum_{j\in[m]}\|A_j\|_2^p\right)^{1/p} \stackrel{\text{Lem 8}}{\leq} n^{\max(0,1/2-1/p)} \cdot \left(\frac{1}{n}\sum_{\substack{j\in[m]\\\leq n}}\|A_j\|_p^p\right)^{1/p} \le n^{\max(0,1/2-1/p)}.$$
 (1)

We rescale the row vectors to $V_j := (\sqrt{p}n^{\max(0,1/2-1/p)})^{-1}A_j$ and abbreviate $y_j := ||V_j||_2^2$, so that Eq. (1) simplifies to $\sum_{j=1}^m y_j^{p/2} \le n \cdot p^{-p/2}$. We may then apply Šidak's Lemma 12 and bound the one-dimensional measure:

$$\begin{split} \gamma_n(K) &= \gamma_n\left(\left\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{V}_j \rangle| \le 1 \ \forall j \in [m]\right\}\right) \\ \stackrel{\text{Lem 12}}{\ge} &\prod_{j \in [m]} \gamma_n\left(\left\{\boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{V}_j \rangle| \le 1\right\}\right) \\ \stackrel{\text{Lem 13}}{\ge} &\prod_{j \in [m]} \left(1 - \exp(-y_j^{-1}/2)\right) \\ \stackrel{\text{Claim I}}{\ge} &\prod_{j \in [m]} \exp\left(-C'p^{p/2}y_j^{p/2}\right) = \exp\left(-C'p^{p/2}\sum_{j \in [m]} y_j^{p/2}\right) \ge \exp(-C'n) \end{split}$$

Here we have used an estimate that remains to be proven: **Claim I.** For any $p \ge 1$ and y > 0 one has $1 - \exp(-\frac{1}{2y}) \ge \exp(-C'p^{p/2}y^{p/2})$ where C' > 0 is a

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universal constant.

Proof of Claim I. It will suffice to show for any y > 0:

$$-\log(1 - \exp(-y^{-1}/2)) \le O(p^{p/2}y^{p/2}).$$

To see this, let $z = \sqrt{2y}$ and note that it suffices to show

$$-\log(1 - \exp(-z^{-2})) \cdot z^{-p} \le O((p/2)^{p/2}).$$

First, by convexity of $x \mapsto -\log(1-x)$, we have $-\log(1-x) \le O(x)$ for $x \in [0, 1/e]$. It follows that for $z \le 1$, we have

$$-\log(1 - \exp(-z^{-2})) \le O(\exp(-z^{-2})) \le O(\lceil p/2 \rceil! / z^{-2\lceil p/2 \rceil}),$$

and therefore $-\log(1 - \exp(-z^{-2})) \cdot z^{-p} \leq O(\lceil p/2 \rceil!) \leq O((p/2)^{p/2}).$

Next, we claim that $-\log(1 - \exp(-z^{-2})) \le 4z$ for all z > 0. Indeed, both sides tend to 0 as $z \to 0$ and the derivative of the left side is

$$\frac{2}{z^3 \left(\exp\left(\frac{1}{2z^2}\right) - 1 \right)} < \frac{2}{z^3 \left(\frac{1}{2z^2} + \frac{1}{8z^4}\right)} = \frac{16z}{4z^2 + 1} \le 4,$$

where we used $e^x > 1 + x + x^2/2$ for $x = \frac{1}{2z^2}$ and $(2z - 1)^2 \ge 0$. It follows that when $z \ge 1$, $-\log(1 - \exp(-z^{-2})) \cdot z^{-p} \le 4z^{1-p} \le 4 \le O((p/2)^{p/2})$.

Remark 1. This argument is largely motivated by the result of Ball and Pajor [BP90] which bounds volume instead of Gaussian measure. More specifically, [BP90] prove that for $1 \le p \le \infty$ and any matrix $A \in \mathbb{R}^{m \times n}$, the set

$$K = \left\{ \boldsymbol{x} \in \mathbb{R}^{n} : |\langle \boldsymbol{A}_{j}, \boldsymbol{x} \rangle| \leq \sqrt{p} \cdot \left(\frac{1}{n} \sum_{j=1}^{m} \|\boldsymbol{A}_{j}\|_{2}^{p}\right)^{1/p} \forall j \in [m] \right\}$$

satisfies $\operatorname{vol}_n(K) \ge 1$. In contrast, our Lemma 20 provides a simpler proof of a stronger result (up to a constant scaling), since the volume of a convex body is always at least its Gaussian measure. On the other hand, it is also possible to recover Lemma 20 directly from this result together with Theorem 7.

We are now ready to show Theorem 18:

Proof of Theorem 18. Let $1 \le p \le q \le \infty$ and let $A \in \mathbb{R}^{m \times n}$ denote the matrix with columns $a_1, ..., a_n \in B_p^m$. By Lemma 9 we know that for any $z \in \mathbb{R}^m$ with $||z||_p \le n^{1/p}$ and $||z||_{\infty} \le 1$ one has $||z||_q \le (||z||_p^{q-p})^{1/q} \le n^{1/q}$. Phrased in geometric terms this means $n^{1/q}B_q^m \ge n^{1/p}B_p^m \cap B_{\infty}^m$. We would like to point out that this is a crucial point to obtain a dependence solely on *n* rather than the larger parameter *m*. Next, note the fact that $A^{-1}(S \cap T) =$

 $A^{-1}(S) \cap A^{-1}(T)$ for any sets *S* and *T* which we use together with the inequality of Šidak and Kathri (Lemma 12) to obtain the estimate

$$\begin{split} &\gamma_n \Big(A^{-1} \big(\sqrt{p} \cdot n^{\max(0,1/2-1/p)+1/q} B_q^m \big) \Big) \\ &\geq &\gamma_n \Big(A^{-1} \big(\sqrt{p} \cdot n^{\max(0,1/2-1/p)} (n^{1/p} B_p^m \cap B_\infty^m) \big) \Big) \\ &\geq &\gamma_n \Big(A^{-1} \big(\sqrt{p} \cdot n^{\max(1/2,1/p)} B_p^m \big) \Big) \cdot \prod_{j \in [m]} \gamma_n \big(\big\{ \boldsymbol{x} \in \mathbb{R}^n : |\langle \boldsymbol{x}, \boldsymbol{A}_j \rangle| \leq \sqrt{p} n^{\max(0,1/2-1/p)} \big\} \big) \\ &\geq & 2^{-O(n)} \cdot 2^{-O(n)} = 2^{-O(n)}, \end{split}$$

where we have used the measure lower bounds from Lemmas 19 and 20. This shows the claimed bound whenever $p \le O(\log(\frac{2m}{n}))$, where the hidden constant can be removed by scaling the corresponding convex body, see Lemma 14.

It remains to prove that we can bootstrap the existing bound for the regime of large p. So let us assume that $p \ge 2 \cdot \max\{1, \log(m/n)\}$. Let $p_0 \in [2, p]$ be a parameter to be determined and remark that Lemma 8 gives $\|\boldsymbol{a}_i\|_{p_0} \le m^{1/p_0-1/p} \cdot \|\boldsymbol{a}_i\|_p \le m^{1/p_0-1/p}$. Applying the above measure lower bound for p_0 implies

$$\gamma_n \Big(\Big\{ \boldsymbol{x} \in \mathbb{R}^n : \Big\| \sum_{i=1}^n x_i \boldsymbol{a}_i \Big\|_q \le \sqrt{p_0} \cdot n^{1/2 - 1/p_0 + 1/q} \cdot m^{1/p_0 - 1/p} \Big\} \Big) \ge 2^{-O(n)}.$$

We can rewrite the above upper bound on ℓ_q -norm as

$$\sqrt{p_0} \cdot n^{1/2 - 1/p_0 + 1/q} \cdot m^{1/p_0 - 1/p} = n^{1/2 - 1/p + 1/q} \cdot \underbrace{\left(\frac{m}{n}\right)^{-1/p}}_{\leq 1} \cdot \sqrt{p_0} \cdot \left(\frac{m}{n}\right)^{1/p_0}$$

Taking $p_0 := 2 \cdot \max\{1, \log(m/n)\}$ gives the desired result as then $(m/n)^{1/p_0} \le \sqrt{e}$ and Lemma 14 can again deal with such constant scaling.

Now our main result on existence of partial colorings easily follows:

Proof of Theorem 1. Apply Theorem 6 to the set

$$K := \left\{ \boldsymbol{x} \in \mathbb{R}^n : \left\| \sum_{i=1}^n x_i \boldsymbol{a}_i \right\|_q \le \sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)} \cdot n^{1/2 - 1/p + 1/q} \right\},$$

which by Theorem 18 indeed has a Gaussian measure of $\gamma_n(K) \ge 2^{-O(n)}$.

Next, we show how to obtain a full coloring by iteratively finding partial colorings.

Proof of Theorem 2. Let again $2 \le p \le q \le \infty$ and let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n \in B_p^m$. We begin with $\boldsymbol{x}^{(0)} := \boldsymbol{0}$ and given $\boldsymbol{x}^{(0)}, \ldots, \boldsymbol{x}^{(t)}$ we set $S^{(t)} := \{i \in [n] : -1 < x_i^{(t)} < 1\}$ as the *active variables*. Then combining Theorem 6 and Theorem 18 we can find a partial coloring $\boldsymbol{x}^{(t+1)} \in [-1,1]^n$ in polynomial time so that $|S^{(t+1)}| \le |S^{(t)}|/2$ and $\|\sum_{i=1}^n (x_i^{(t+1)} - x_i^{(t)})\boldsymbol{a}_i\|_q \le C_1 \sqrt{\min(p,\log(\frac{2m}{|S^{(t)}|}))}$.

 $|S^{(t)}|^{1/2-1/p+1/q}$. Let $\mathbf{x}^{(T)}$ be the first iterate with $\mathbf{x}^{(T)} \in \{-1, 1\}^n$. Clearly $|S^{(t)}| \le n2^{-t}$ and $T \le \log_2(n)$. Using the triangle inequality we get

$$\begin{split} \left\|\sum_{i=1}^{n} x_{i}^{(T)} \boldsymbol{a}_{i}\right\|_{q} &\leq \sum_{t=0}^{T-1} \left\|\sum_{i=1}^{n} (x_{i}^{(t+1)} - x_{i}^{(t)}) \boldsymbol{a}_{i}\right\|_{q} \\ &\leq C_{1} \sum_{t=0}^{T-1} \sqrt{\min\left(p, \log\left(\frac{2m}{2^{-t} \cdot n}\right)\right)} \cdot (2^{-t} \cdot n)^{1/2 - 1/p + 1/q} \\ &\leq \frac{C_{1} C_{2} \sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)}}{1/2 - 1/p + 1/q} \cdot n^{1/2 - 1/p + 1/q}. \quad \Box \end{split}$$

The intuition behind the extra factor for obtaining a full coloring is as follows: abbreviate the exponent as $\beta := 1/2 - 1/p + 1/q$. Then it takes $\frac{1}{\beta}$ iterations until the term $|S^{(t)}|^{\beta}$ decreases by a factor of 1/2 which dominates the miniscule growth of the logarithmic term. Then indeed the overall discrepancy is dominated by the discrepancy from the first $\frac{1}{\beta}$ iterations.

We can now demonstrate how a nontrivial choice of ℓ_p -norms can be beneficial in classical discrepancy settings:

Proof of Corollary 4. Consider column vectors $a_1, ..., a_n \in \{0, 1\}^m$ with at most t nonzero entries per a_i . First let us study the case $t \ge n/10$. Since for each column $||a_i||_4 \le t^{1/4}$, Theorem 2 provides a coloring $x \in \{-1, 1\}^n$ with $||\sum_{i=1}^n x_i a_i||_{\infty} \le O(n^{1/4} \cdot t^{1/4}) = O(\sqrt{t})$.²

Now if t < n/10, we take $p \in [2, 16)$ with $1/2 - 1/p = 1/\log(n/t)$. Then $||a_i||_p \le t^{1/p}$ and Theorem 2 gives $x \in \{-1, 1\}^n$ with

$$\left\|\sum_{i=1}^{n} x_{i} a_{i}\right\|_{\infty} \leq \frac{C \cdot n^{1/2 - 1/p} \cdot t^{1/p}}{1/2 - 1/p} = C\sqrt{t} \log(n/t) \cdot \underbrace{(n/t)^{1/\log(n/t)}}_{=e}.$$

We conclude this section by showing that the term $n^{\max(0,1/2-1/p)+1/q}$ in our bounds is necessary:

Proof of Theorem 5. Consider the case $p \ge 2$. Consider an $n \times n$ *Hadamard matrix*, which is a matrix $\mathbf{H} \in \{-1, 1\}^{n \times n}$ so that all rows and columns are orthogonal. Such matrices are known to exist at least whenever n is a power of 2. The columns satisfy $\|\mathbf{h}_i\|_p = n^{1/p}$ and for any $\mathbf{x} \in [-1, 1]^n$ with $|\{i : x_i^2 = 1\}| \ge n/2$ we know that $\|\mathbf{x}\|_2 \ge \Omega(\sqrt{n})$ and $\|\mathbf{H}\mathbf{x}\|_2 \ge \Omega(n)$, so that by Lemma 8 we have

$$\|H\mathbf{x}\|_q \ge \|H\mathbf{x}\|_2 \cdot n^{1/q - 1/2} = \Omega(n^{1/2 + 1/q}).$$

For $p \in [1,2]$, take an identity matrix I_n . For every $\mathbf{x} \in [-1,1]^n$ with $|\{i : x_i^2 = 1\}| \ge n/2$ we have $||I_n \mathbf{x}||_q = ||\mathbf{x}||_q \ge \Omega(n^{1/q})$, and the columns of I_n are certainly in B_p^m .

²In fact for $t \ge n$ a more careful choice of $p = \log(2t/n)$ gives a better ℓ_{∞} discrepancy bound of $O(\sqrt{n\log(2t/n)})$, even though the Beck-Fiala conjecture asks only for $O(\sqrt{t})$.

Partial coloring via measure lower bound 4

In this section, we want to show the existence of partial fractional colorings for bodies K with $\gamma_n(K) \ge e^{-\alpha n}$ as promised in Theorem 6. The main innovation of this work compared to e.g. [Rot14] is to handle an arbitrarily small constant $\alpha > 0$. We will show how to find a partial coloring that colors a small constant fraction of coordinates; then iterating the argument will color the promised $\beta - \gamma$ fraction. Also, instead of working with a shift y and a scaling of K, it will be notationally easier to work with a shifted and scaled box. Hence, for vectors $L, R \in \mathbb{R}^n_{>0}$, we write $[-L, R] := [-L_1, R_1] \times ... \times [-L_n, R_n]$ as the box defined by constraints $-L_i \le x_i \le R_i$ for i = 1, ..., n. We use $N(\mathbf{0}, H)$ to denote the Gaussian distribution restricted to a subspace $H \subseteq \mathbb{R}^n$. Then the main technical result for this section will be:

Theorem 21. For all constants $\alpha, \beta > 0$ there are $\varepsilon := \varepsilon(\alpha, \beta) > 0$ and $\delta := \delta(\alpha, \beta) > 0$ so that the following holds: Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body with $K \subseteq H$ for a subspace $H \subseteq \mathbb{R}^n$ with dim $(H) \ge \beta n$ and $\gamma_H(K) \ge e^{-\alpha n}$; also let $L, R \in [0, \varepsilon]^n$. Assuming a weak separation oracle for K, there is a randomized polynomial time algorithm which finds an $x \in K \cap [-L, R]$ so that $|\{i \in [n] : x_i \in \{-L_i, R_i\}\}| \ge \delta n$ with probability at least $1 - e^{-\Theta_{\varepsilon,\delta}(n)}$.

Note that the considered box satisfies $[-L, R] \subseteq [-\varepsilon, \varepsilon]^n$. We would like to point out that applying the standard nonconstructive proof by Gluskin [Glu89] and Giannopoulos [Gia97] to a find a partial coloring $\mathbf{x} \in \{-\varepsilon, 0, \varepsilon\}^n$ with support $\Omega(n)$ will require either a small enough constant $\alpha > 0$, or ε needs to be exponentially small in n. In fact, it is not hard to construct a thin strip K with $\gamma_n(K) \ge e^{-\Omega(n)}$ so that K does not intersect $\{-1, 0, 1\}^n \setminus \{0\}$ (even after a subexponential scaling). We show the construction in Appendix B.

For our proof we make use of the mean width $w(Q) := \mathbb{E}_{\boldsymbol{\theta} \in S^{n-1}}[\sup_{\boldsymbol{x} \in Q} \langle \boldsymbol{\theta}, \boldsymbol{x} \rangle]$ of a body. We should point out that the connection between partial coloring arguments and mean width is due to Eldan and Singh [ES18]. Several of the claims require that n is chosen large enough.

Lemma 22. Let $Q \subseteq \mathbb{R}^n$ be a symmetric convex body with $\gamma_n(Q) \ge e^{-\alpha n}$ for $\alpha > 0$. Then $w(Q) \geq \frac{1}{2}e^{-\alpha}\sqrt{n}.$

Proof. Let r > 0 be the radius so that $\gamma_n(rB_2^n) = \gamma_n(Q)$. By Urysohn's Inequality (Theorem 16) one has $w(Q) \ge w(rB_2^n) = r$ so it suffices to give a lower bound on the radius r. A simple but useful estimate is that $2^n \leq \text{Vol}_n(\sqrt{n}B_2^n) \leq 5^n$ for any $n \geq 1$. Moreover, the Gaussian density is maximized at $\gamma_n(\mathbf{0}) = \frac{1}{(\sqrt{2\pi})^n}$. Then for $\beta := 2e^{\alpha} \geq 2$ we have

$$\gamma_n \left(\frac{\sqrt{n}}{\beta} B_2^n\right) \le \operatorname{Vol}_n \left(\frac{\sqrt{n}}{\beta} B_2^n\right) \cdot \gamma_n(\mathbf{0}) \le \left(\frac{5}{\beta}\right)^n \cdot \frac{1}{(\sqrt{2\pi})^n} \le \left(\frac{2}{\beta}\right)^n \stackrel{\beta = 2e^\alpha}{\le} e^{-\alpha n}$$

$$P r \ge \frac{\sqrt{n}}{\beta} = \frac{\sqrt{n}}{2e^\alpha}.$$

and so

The key modification of our work in contrast to [Rot14] is a finer upper bound on the distance of a Gaussian to K:

Lemma 23. Let $K \subseteq \mathbb{R}^n$ be a symmetric convex set with $\gamma_n(K) \ge e^{-\alpha n}$ where $\alpha \ge 1$ and n is large enough. Then

$$\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} [d(\boldsymbol{x}, K)] \leq \sqrt{n} \cdot \left(1 - \frac{1}{512\alpha e^{4\alpha}}\right)$$

Proof. Note that by Theorem 15 we have $\Pr_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)}[\|\boldsymbol{x}\|_2 \ge 4\sqrt{\alpha n}] \le e^{-2\alpha n}$, hence the restriction $Q := K \cap 4\sqrt{\alpha n}B_2^n$ still has $\gamma_n(Q) \ge \gamma_n(K) - e^{-2\alpha n} \ge e^{-2\alpha n}$ for *n* large enough. Then by the previous Lemma we know that $w(Q) \ge \frac{\sqrt{n}}{2e^{2\alpha}}$. For a vector \boldsymbol{x} , let $\boldsymbol{z}(\boldsymbol{x}) := \operatorname{argmax}\{\langle \boldsymbol{z}, \boldsymbol{x} \rangle : \boldsymbol{z} \in Q\}$. As we just showed, $\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)}[\langle \boldsymbol{z}(\boldsymbol{x}), \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|_2} \rangle] \ge \frac{\sqrt{n}}{2e^{2\alpha}}$. Let $\lambda \in [0, 1]$ be a parameter that we determine later. Note that the point $\lambda \cdot \boldsymbol{z}(\boldsymbol{x})$ lies in Q.



This point can be used to bound

$$\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} [\|\boldsymbol{x} - \lambda \boldsymbol{z}(\boldsymbol{x})\|_2^2] = \mathbb{E}[\|\boldsymbol{x}\|_2^2] - 2\lambda \mathbb{E}[\langle \boldsymbol{x}, \boldsymbol{z} \rangle] + \mathbb{E}[\lambda^2 \|\boldsymbol{z}\|_2^2]$$

$$= \underbrace{\mathbb{E}[\|\boldsymbol{x}\|_2^2]}_{=n} - 2\lambda \underbrace{\mathbb{E}[\|\boldsymbol{x}\|_2]}_{\geq \frac{1}{2}\sqrt{n}} \cdot \underbrace{\underbrace{\mathbb{E}[\langle \boldsymbol{\theta}, \boldsymbol{z}(\boldsymbol{\theta}) \rangle]}_{\geq \sqrt{n}/(2e^{2\alpha})} + \mathbb{E}[\lambda^2 \|\boldsymbol{z}\|_2^2]}_{\leq 16\alpha n}$$

$$\leq n - \frac{1}{2}e^{-2\alpha}\lambda n + \lambda^2 \cdot 16\alpha n^{\lambda := \frac{1}{\frac{64}{2}\alpha e^{2\alpha}}} n \cdot \left(1 - \frac{1}{256\alpha e^{4\alpha}}\right)$$

Then

$$\mathbb{E}[d(\boldsymbol{x}, Q)] \stackrel{\lambda \boldsymbol{z} \in Q}{\leq} \mathbb{E}[\|\boldsymbol{x} - \lambda \boldsymbol{z}\|_{2}] \stackrel{\text{Jensen}}{\leq} \mathbb{E}[\|\boldsymbol{x} - \lambda \boldsymbol{z}\|_{2}^{2}]^{1/2} \leq \sqrt{n} \cdot \sqrt{1 - \frac{1}{256\alpha e^{4\alpha}}} \leq \sqrt{n} \cdot \left(1 - \frac{1}{512\alpha e^{4\alpha}}\right)$$

using $\sqrt{1 - y} \leq 1 - \frac{y}{2}$ for $0 \leq y \leq 1$.

Lemma 23 can be extended to the case that *K* is included in a not too small subspace *H*.

Lemma 24. Let $\alpha \ge 1$, $0 < \beta \le 1$ be constants. Let $H \subseteq \mathbb{R}^n$ be a subspace with dim $(H) \ge \beta n$ and let $K \subseteq H$ be a symmetric convex body with $\gamma_H(K) \ge e^{-\alpha n}$. For *n* large enough, one has

$$\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} [d(\boldsymbol{x}, K)] \leq \sqrt{n} \cdot \left(1 - \frac{\beta}{512\alpha e^{4\alpha}}\right)$$

Proof. Note that one can generate a Gaussian $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_n)$ as $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1 \sim N(\mathbf{0}, H^{\perp})$ and $\mathbf{x}_2 \sim N(\mathbf{0}, H)$ independently. Then $d(\mathbf{x}, K)^2 = d(\mathbf{x}_1, H)^2 + d(\mathbf{x}_2, K)^2$ by Pythagoras. Hence

$$\mathbb{E}_{\boldsymbol{x} \sim N(\boldsymbol{0}, \boldsymbol{I}_n)} \left[d(\boldsymbol{x}, \boldsymbol{K})^2 \right] \leq \mathbb{E}_{\boldsymbol{x}_1 \sim N(\boldsymbol{0}, H^{\perp})} \left[d(\boldsymbol{x}_1, H)^2 \right] + \mathbb{E}_{\boldsymbol{x}_2 \sim N(\boldsymbol{0}, H)} \left[d(\boldsymbol{x}_2, \boldsymbol{K})^2 \right] \\
\stackrel{\text{Lem 23}}{\leq} \dim(H^{\perp}) + \dim(H) \cdot \left(1 - \frac{1}{256\alpha e^{4\alpha}} \right) \\
\stackrel{\dim(H) \geq \beta n}{\leq} n \cdot \left(1 - \frac{\beta}{256\alpha e^{4\alpha}} \right)$$

As in the proof of Lemma 23, the claim follows after applying Jensen inequality with the fact that $\sqrt{1-y} \le 1 - \frac{y}{2}$ for $0 \le y \le 1$.

Next, we show the average distance of a Gaussian to the cube $[-\varepsilon, \varepsilon]^n$ is $\sqrt{n} \cdot (1 - \Theta(\varepsilon))$.

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Lemma 25. Let $\varepsilon > 0$. Then for *n* large enough one has

$$\Pr_{\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_n)} \left[d(\mathbf{x}, \left[-\varepsilon, \varepsilon\right]^n) \ge (1 - 5\varepsilon)\sqrt{n} \right] \ge 1 - \exp\left(-\frac{\varepsilon^2}{2}n\right)$$

Proof. Let $\mathbf{y} := \mathbf{y}(\mathbf{x}) := \operatorname{argmin}\{\|\mathbf{x} - \mathbf{y}\|_2 : \mathbf{y} \in [-\varepsilon, \varepsilon]^n\}$ be the closest point in the cube to \mathbf{x} . For an individual coordinate $i \in [n]$ the expected contribution to the distance is

$$\mathbb{E}\left[d(x_i, [-\varepsilon, \varepsilon])^2\right] = \mathbb{E}\left[|x_i - y_i|^2\right] = \underbrace{\mathbb{E}[x_i^2]}_{=1} - 2\underbrace{\mathbb{E}[x_i y_i]}_{\leq \varepsilon \mathbb{E}[|x_i|]} + \underbrace{\mathbb{E}[y_i^2]}_{\geq 0} \ge 1 - 2\sqrt{\frac{2}{\pi}} \cdot \varepsilon \ge 1 - 2\varepsilon.$$

Then by linearity $\mathbb{E}[d(\mathbf{x}, [-\varepsilon, \varepsilon]^n)^2]^{1/2} \ge \sqrt{n \cdot (1 - 2\varepsilon)} \ge \sqrt{n} \cdot (1 - 2\varepsilon)$. Recall that the distance function $F(\mathbf{x}) := d(\mathbf{x}, [-\varepsilon, \varepsilon]^n)$ is 1-Lipschitz and for such functions the difference $|\mathbb{E}[F(\mathbf{x})] - \mathbb{E}[F(\mathbf{x})^2]^{1/2}|$ is bounded by an absolute constant. Then $\mathbb{E}[F(\mathbf{x})] \ge \sqrt{n} \cdot (1 - 4\varepsilon)$ for *n* large enough. Finally by Theorem 15 one has $\Pr[F(\mathbf{x}) < \mathbb{E}[F(\mathbf{x})] - \varepsilon \sqrt{n}] \le e^{-\varepsilon^2 n/2}$ for $\mathbf{x} \sim N(\mathbf{0}, \mathbf{I}_n)$ which then gives the claim as $\mathbb{E}[F(\mathbf{x})] - \varepsilon \sqrt{n} \ge (1 - 5\varepsilon)\sqrt{n}$.

We will now prove Theorem 21. Let $H \subseteq \mathbb{R}^n$ be a subspace with dim $(H) \ge \beta n$ and let $K \subseteq H \subseteq \mathbb{R}^n$ be a symmetric convex body with $\gamma_H(K) \ge e^{-\alpha n}$. Moreover, let $L_i, R_i \in [0, \varepsilon]$ be given parameters where the choice of $\varepsilon := \varepsilon(\alpha, \beta) > 0$ will be made in the upcoming proof of Lemma 26. We will use the following algorithm:

- (1) Pick $\mathbf{x}^* \sim N(\mathbf{0}, \mathbf{I}_n)$ at random.
- (2) Compute $y^* := \operatorname{argmin} \{ \| x^* y \|_2 : y \in K \cap [-L, R]^n \}.$



Note that the step (2) is a convex program which can be solved in polynomial time, see [GLS88]. Now we can finish the proof of Theorem 21.

Lemma 26. If $\varepsilon, \delta > 0$ are chosen small enough (depending on α), then with probability $1 - e^{-\Omega_{\varepsilon,\delta}(n)}$ one has $|\{i \in [n] : y_i^* \in \{-L_i, R_i\}\}| \ge \delta n$.

Proof. For a set of indices $I \subseteq [n]$ we abbreviate the subspace $H(I) := \{x \in H \mid x_i = 0 \forall i \in I\}$. Moreover we abbreviate $K(I) := \{x \in K \mid -L_i \le x_i \le R_i \forall i \in I\}$ as the intersection of K with the slabs corresponding to coordinates in I. Consider the two events

$$\mathcal{E}_1 := ``d(\mathbf{x}^*, K \cap [-L, \mathbf{R}]) \ge (1 - 5\varepsilon) \cdot \sqrt{n}"$$

$$\mathcal{E}_2 := ``for all I \subseteq [n] with |I| \le \delta n \text{ one has } d(\mathbf{x}^*, K \cap H(I)) \le (1 - 10\varepsilon)\sqrt{n}"$$

We will see that both events \mathcal{E}_1 and \mathcal{E}_2 happen with overwhelming probability. **Claim I.** One has $\Pr[\mathcal{E}_1] \ge 1 - \exp(-\frac{\varepsilon^2}{2}n)$.

Proof of Claim I. Follows from Lemma 25 as $d(\mathbf{x}^*, K \cap [-L, \mathbf{R}]) \ge d(\mathbf{x}^*, K \cap [-\varepsilon, \varepsilon]^n) \ge d(\mathbf{x}^*, [-\varepsilon, \varepsilon]^n)$.

Claim II. If $\varepsilon, \delta > 0$ are small enough, then $\Pr[\mathcal{E}_2] \ge 1 - e^{-\Theta_{\varepsilon}(n)}$.

Proof of Claim II. For any index set *I* one can lower bound the measure as $\gamma_{H(I)}(K \cap H(I)) \ge \gamma_H(K) \ge e^{-\alpha n}$ by Lemma 17. Let us abbreviate $\mathcal{I} := \{I \subseteq [n] : |I| \le \delta n\}$ as the family of small index sets. For $I \in \mathcal{I}$ we have dim $(H(I)) \ge \dim(H) - |I| \ge \frac{\beta}{2}n$, if we choose $\delta \le \frac{\beta}{2}$. Then by Lemma 24 we know that a fixed $I \in \mathcal{I}$ has $\mathbb{E}_{\boldsymbol{x} \sim N(\mathbf{0}, I_n)}[d(\boldsymbol{x}, K \cap H(I))] \le \sqrt{n} \cdot \left(1 - \frac{\beta/2}{512 \cdot \alpha e^{4\alpha}}\right) \le (1 - 20\varepsilon)\sqrt{n}$, if we choose $\varepsilon \le \frac{\beta/2}{20 \cdot 512 \alpha e^{4\alpha}}$. Then by concentration one has $\Pr_{\boldsymbol{x} \sim N(\mathbf{0}, I_n)}[d(\boldsymbol{x}, K \cap H(I)) > (1 - 10\varepsilon)\sqrt{n}] \le \exp(-50\varepsilon^2 n)$, see Theorem 15. A useful bound is $|\mathcal{I}| \le e^{2\delta \log_2(\frac{1}{\delta})n} \le e^{\varepsilon^2 n}$ if we choose δ small enough compared to ε . Then

$$\Pr[\mathcal{E}_2] \stackrel{\text{union bound}}{\leq} \sum_{I \in \mathcal{I}} \Pr\left[d(\boldsymbol{x}^*, K \cap H(I)) > (1 - 10\varepsilon)\sqrt{n}\right]$$
$$\leq e^{\varepsilon^2 n} \cdot \exp(-50\varepsilon^2 n) \leq \exp\left(-40\varepsilon^2 n\right). \square$$

Now we have everything to finish the proof. Fix an outcome of the vector \mathbf{x}^* so that the events \mathcal{E}_1 and \mathcal{E}_2 are both true, and abbreviate $I^* := \{i \in [n] : y_i^* \in \{-L_i, R_i\}\}$. Suppose for the sake of contradiction that $|I^*| < \delta n$. Then

$$(1-10\varepsilon)\sqrt{n} \quad \stackrel{\mathcal{E}_2 \text{ true } \& I^* \in \mathcal{I}}{\geq} \quad d(\mathbf{x}^*, K \cap H(I^*))$$
$$\stackrel{K \cap H(I^*) \subseteq K(I^*)}{\geq} \quad d(\mathbf{x}^*, K(I^*))$$
$$\stackrel{(*)}{=} \quad d(\mathbf{x}^*, K \cap [-L, R])$$
$$\stackrel{\mathcal{E}_1 \text{ true}}{\geq} \quad (1-5\varepsilon)\sqrt{n}$$

which is a contradiction. Here the crucial argument for (*) is that $d(\mathbf{x}^*, K \cap [-L, R]) = \min\{\|\mathbf{x}^* - \mathbf{y}\|_2 : \mathbf{y} \in K \text{ and } -L_i \leq y_i \leq R_i \ \forall i \in [n]\}$ is a *convex minimization* problem and the optimum value will not change if linear constraints are discarded that are not tight for the optimum \mathbf{y}^* , and the box constraints for coordinates $I^* \setminus [n]$ are indeed not tight. \Box

We stated such a result earlier in Theorem 6. Now we are ready to prove it:

Proof of Theorem 6. The basic idea is to simply apply Theorem 21 a constant number of times until the desired number of elements is colored. We assume $\beta > \gamma$ since otherwise there is nothing to prove. Let $\varepsilon := \varepsilon(\alpha, \gamma), \delta := \delta(\varepsilon, \gamma) > 0$ be the constants from Theorem 21 that work for the given α and $\beta' := \gamma > 0$.

We set $\mathbf{y}^{(0)} := \mathbf{y}$ and for $t \ge 0$ we set $F^{(t)} := \{i \in [n] : y_i^{(t)} \in \{-1, 1\}\}$ as the variables that are *frozen*. Suppose for some *t* we have constructed a sequence $\mathbf{y}^{(0)}, \dots, \mathbf{y}^{(t)}$ and still $|F^{(t)}| < (\beta - \gamma)n$. Set $H^{(t)} := \{\mathbf{x} \in H \mid x_i = 0 \forall i \in F^{(t)}\}$ be the subspace of *H* where we fix frozen coordinates to be 0. Note that $\dim(H^{(t)}) \ge \dim(H) - |F^{(t)}| \ge \gamma n$. Moreover $\gamma_{H^{(t)}}(K \cap H^{(t)}) \ge \gamma_H(K) \ge e^{-\alpha n}$ by Lemma 17. We set $R_i := \frac{\varepsilon}{2} \cdot (1 - y_i^{(t)})$ and $L_i := \frac{\varepsilon}{2} \cdot (y_i^{(t)} - (-1))$ for $i \in [n] \setminus F^{(t)}$ and $R_i := L_i := \varepsilon$ for $i \in F^{(t)}$ and apply Theorem 21. With high probability, the algorithm succeeds and provides a vector $\mathbf{x}^{(t)}$. We update $\mathbf{y}^{(t+1)} := \mathbf{y}^{(t)} + \frac{2}{\varepsilon} \mathbf{x}^{(t)} \in [-1, 1]^n$ where $\|\mathbf{y}^{(t+1)}\|_K \le \|\mathbf{y}^{(t)}\|_K + \frac{2}{\varepsilon}$ by the triangle inequality. Moreover, the number of frozen coordinates increases³ to $|F^{(t+1)}| \ge |F^{(t)}| + \delta n$. We will terminate after at most $\frac{1}{\delta}$ iterations and if *T* is the final iteration, then $\mathbf{y}^{(T)} \in [-1, 1]^n \cap \frac{2}{\varepsilon\delta}K$ as desired.

We would like to mention that Theorem 6 may also be deduced, after some work, from the Gaussian measure amplification techniques derived in [DNTT18] with the use of α regular M-ellipsoids. We believe the analysis presented here is simpler, since the existence of such regular M-ellipsoids is a deep result in convex geometry.

5 From hereditary volume bounds to Gaussian measure

This section is devoted to the proof of Theorem 7, which provides a connection between hereditary volume and Gaussian measure. For a brief motivation, note that for any convex body $K \subseteq \mathbb{R}^n$ and any $S \subseteq [n]$ one has $\operatorname{vol}_{|S|}(K_S) \ge \gamma_{|S|}(K_S) \ge \gamma_n(K)$. It is therefore a natural question whether a converse holds, and Theorem 7 shows that this is indeed the case. As a corollary, we settle up to an exponential factor a conjecture of [BGMN05] that coordinate sections minimize the Gaussian measure among all sections of scaled ℓ_p balls.

We would also like to mention that we cannot hope for a refinement of the right side to only sections of dimension δn . For example when $K = \varepsilon \cdot B_2^{\delta n-1} \times \mathbb{R}^{n-\delta n+1}$, all δn -dimensional sections have infinite volume yet $\gamma(K) \to 0$ as $\varepsilon \to 0$.

While relatively short, our proof does use several auxilliary results. The key ingredient is the following formula which expresses the volume of the Minkowski sum of a convex body and an Euclidean ball as a weighted sum of *quermassintegrals* $W_i(K)$ which are average volumes of projections. Recall that given $A, B \subseteq \mathbb{R}^n$, $A + B := \{a + b : a \in A, b \in B\}$.

Lemma 27 (Kubota's Integral Formula [Pis89]). For any convex body $K \subset \mathbb{R}^n$, we have

$$\operatorname{vol}_n(K + \lambda B_2^n) = \sum_{i=0}^n \lambda^i \binom{n}{i} W_i(K)$$

³For frozen coordinates *i* we did set $L_i = R_i = \varepsilon$ so that $\mathbf{x}^{(t)}$ will indeed contain δn "fresh" coordinates that become tight, rather than rediscovering the coordinates in $F^{(t)}$.

with

$$W_{n-i}(K) := \frac{\operatorname{vol}_n(B_2^n)}{\operatorname{vol}_i(B_2^i)} \int_{G(n,i)} \operatorname{vol}_i(\pi_L(K)) dL,$$

where the integral is over the uniform measure over G(n, i), which is the set of *i*-dimensional linear subspaces $L \subseteq \mathbb{R}^n$ and $\pi_L(K)$ denotes the orthogonal projection of *K* onto *L*.

In order to relate projections to slices, we use polarity. Given a symmetric convex set $K \subseteq \mathbb{R}^n$, its *polar* is $K^\circ := \{y \in \text{span}(K) \mid \langle x, y \rangle \le 1 \ \forall x \in K\}$. The following lemma elucidates the reason polars are helpful to transform projections into slices:

Lemma 28. Given a symmetric convex body $K \subseteq \mathbb{R}^n$ and any subspace $H \subseteq \mathbb{R}^n$, we have $(K \cap H)^\circ = \pi_H(K^\circ)$.



It is also well-known that polarity transforms intersections into convex hulls:

Lemma 29. Given symmetric convex bodies $K, L \subseteq \mathbb{R}^n$, we have $(K \cap L)^\circ = conv(K^\circ, L^\circ)$.

For a detailed introduction to polarity we refer to Rockefellar [Roc70]. Finally, we need the Blaschke-Santaló Inequality and its deep converse due to Bourgain-Milman [AAGM15]:

Lemma 30. Given a symmetric convex body $K \subseteq \mathbb{R}^n$, we have $2^{O(n)} \ge \frac{\operatorname{vol}_n(K) \cdot \operatorname{vol}_n(K^\circ)}{\operatorname{vol}_n(B_2^n)^2} \ge 2^{-O(n)}$.

The starting point of the proof, which connects the Gaussian measure to the Minkowski sum with an Euclidean ball, is given by the following bound:

Lemma 31. Given a symmetric convex body $K \subseteq \mathbb{R}^n$, $\gamma_n(K) \ge \operatorname{vol}_n \left(K^\circ + \frac{1}{\sqrt{n}} B_2^n \right)^{-1} \cdot n^{-n} \cdot 2^{O(n)}$.

Proof. We start by noting that we can lower bound the Gaussian measure upon restriction to a \sqrt{n} -radius ball:

$$\gamma_n(K) = \frac{1}{(2\pi)^{n/2}} \int_K e^{-\|\boldsymbol{x}\|_2^2/2} \, \mathrm{d}\boldsymbol{x} \ge \frac{1}{(2\pi e)^{n/2}} \mathrm{vol}_n(K \cap \sqrt{n}B_2^n),$$

and since $(K \cap \sqrt{n}B_2^n)^\circ = \operatorname{conv}(K^\circ, \frac{1}{\sqrt{n}}B_2^n)$ by Lemma 29, we conclude

$$\begin{split} \gamma_n(K) &\geq & \operatorname{vol}_n(K \cap \sqrt{n}B_2^n) \cdot 2^{-O(n)} \\ & \stackrel{\operatorname{Lem 30}}{\geq} & \operatorname{vol}_n\left(\operatorname{conv}\left(K^\circ, \frac{1}{\sqrt{n}}B_2^n\right)\right)^{-1} \cdot n^{-n} \cdot 2^{-O(n)} \\ & \geq & \operatorname{vol}_n\left(K^\circ + \frac{1}{\sqrt{n}}B_2^n\right)^{-1} \cdot n^{-n} \cdot 2^{O(n)}, \end{split}$$

since $\operatorname{conv}(K^{\circ}, \frac{1}{\sqrt{n}}B_2^n) \subseteq K^{\circ} + \frac{1}{\sqrt{n}}B_2^n$.

In order to connect slices to *coordinate* slices, we apply a result of [DNTT18] for ellipsoids. Thus we will need to use the existence of M-ellipsoids [AAGM15]:

Lemma 32. For any symmetric convex body $K \subseteq \mathbb{R}^n$ there exists an ellipsoid $E \subseteq \mathbb{R}^n$ for which there exist collections of centers $S_E, S_K \subseteq \mathbb{R}^n$ with $|S_E|, |S_K| \leq 2^{O(n)}$ so that $K \subseteq \bigcup_{c \in S_E} (c + E)$ and $E \subseteq \bigcup_{c' \in S_K} (c' + K)$.

Proof of the first inequality in Theorem 7. Kubota's integral formula (Lemma 27) applied to K° yields

$$W_{n-i}(K^{\circ}) = \frac{\operatorname{vol}_n(B_2^n)}{\operatorname{vol}_i(B_2^i)} \int_{G(n,i)} \operatorname{vol}_i(\pi_L(K^{\circ})) dL.$$

By Lemma 28 and Santaló's inequality (Lemma 30) we know that for any subspace L,

$$\operatorname{vol}_i(\pi_L(K^\circ)) \le \operatorname{vol}_i(B_2^i)^2 \cdot \operatorname{vol}_i(K \cap L)^{-1} \le M^{-1} \cdot i^{-i} \cdot 2^{O(i)}$$

where we choose to denote $M := \min_{\dim L = i \le n} \operatorname{vol}_i(K \cap L)$. We conclude

$$W_{n-i}(K^{\circ}) \le M^{-1} \cdot n^{-n/2} \cdot i^{-i/2} \cdot 2^{O(n)},$$

so that

$$n^{-(n-i)/2} \cdot W_{n-i}(K^{\circ}) \le M^{-1} \cdot n^{-n} \cdot 2^{O(n)},$$

by using $(n/i)^i \le 2^{O(n)}$ for $i \in [n]$. Taking $\lambda := 1/\sqrt{n}$ and summing over $i \in [n]$ in Lemma 27 gives

$$\operatorname{vol}_n\left(K^\circ + \frac{1}{\sqrt{n}}B_2^n\right) \le M^{-1} \cdot n^{-n} \cdot 2^{O(n)},$$

so that by Lemma 31 we obtain $\gamma_n(K) \ge M \cdot 2^{-O(n)}$. It remains to show that the minimal *coordinate* sections are not much larger than the minimal sections. With this purpose in mind, let *E* be an M-ellipsoid of *K*. By Lemma 32, there exist collections S_E, S_K with $|S_E|, |S_K| \le 2^{O(n)}$ so that $K \subseteq \bigcup_{c \in S_E} (c + E)$ and $E \subseteq \bigcup_{c' \in S_K} (c' + K)$. Note that for any *i*-dimensional subspace *L* we have

$$\operatorname{vol}_i(K \cap L) \le \sum_{c \in S_E} \operatorname{vol}_i((c+E) \cap L) \le 2^{O(n)} \cdot \operatorname{vol}_i(E \cap L)$$

and similarly

$$\operatorname{vol}_i(E \cap L) \leq \sum_{c' \in S_K} \operatorname{vol}_i((c' + K) \cap L) \leq 2^{O(n)} \cdot \operatorname{vol}_i(K \cap L),$$

where by Brunn's concavity principle the sections with largest volume are those through the origin. Thus it suffices to show that

$$\min_{\dim L=i} \operatorname{vol}_i(E \cap L) \geq \min_{S \subseteq [n], |S|=i} \operatorname{vol}_i(E_S) \cdot 2^{-O(n)}.$$

Indeed this follows a form of restricted invertibility in the work of Dadush, Nikolov, Talwar and Tomczak-Jaegermann, who showed in [DNTT18] (see p. 8) an improved bound of

$$\min_{\dim L=i} \operatorname{vol}_i(E \cap L) \ge \min_{S \subseteq [n], |S|=i} \operatorname{vol}_i(E_S) \cdot {\binom{n}{i}}^{-1}.$$

We now prove the second part of Theorem 7 which restricts our attention to sections of dimension $\leq \delta n$. For this we need the following inequality for quermassintegrals which can be seen as a strenghtening of the isoperimetric inequality:

Theorem 33 (Alexandrov Inequality [Pis89]). *Given* $i \ge j$ *we have*

$$\left(\frac{W_{n-i}(K)}{\operatorname{vol}_{i}(B_{2}^{i})}\right)^{1/i} \leq \left(\frac{W_{n-j}(K)}{\operatorname{vol}_{j}(B_{2}^{j})}\right)^{1/j}.$$

Proof of the second inequality in Theorem 7. We proceed as in the proof of the first inequality. Setting $\lambda := 1/\sqrt{n}$ we still have, for $j \le \delta n$,

$$\begin{split} \lambda^{n-j} W_{n-j}(K^{\circ}) &\leq \max_{\dim L = i \leq \delta n} \operatorname{vol}_{i}^{-1}(K \cap L) \cdot n^{-n} \cdot 2^{O(n)} \\ &\leq \max_{\dim L = i \leq \delta n} \operatorname{vol}_{i}^{-1/\delta}(K \cap L) \cdot n^{-n} \cdot 2^{O(n)}, \end{split}$$

as the maximum is at least one (for i = 0). For $j > \delta n$ we use Theorem 33 to see that

$$\lambda^{n-j} W_{n-j}(K^{\circ}) \leq \lambda^{n-j} (W_{n-\delta n}(K^{\circ}))^{j/(\delta n)} \cdot \mathrm{vol}_{j}(B_{2}^{j}) \cdot \mathrm{vol}_{\delta n}(B_{2}^{\delta n})^{-j/\delta n}$$

and proceed as in the first half of the proof:

$$\begin{split} \lambda^{n-j} W_{n-j}(K^{\circ}) &\leq \lambda^{n-j} \cdot (W_{n-\delta n}(K^{\circ}))^{j/(\delta n)} \cdot \operatorname{vol}_{j}(B_{2}^{j}) \cdot \operatorname{vol}_{\delta n}(B_{2}^{\delta n})^{-j/\delta n} \\ &\leq \lambda^{n-j} \cdot \left(\max_{\dim L=i \leq \delta n} \operatorname{vol}_{i}^{-1}(K \cap L) \cdot n^{-n/2} \cdot (\delta n)^{-\delta n/2} \right)^{j/(\delta n)} \cdot (\delta n/j)^{j/2} \cdot 2^{O(n/\delta)} \\ &\leq \lambda^{n-j} \cdot n^{-j/(2\delta)} \cdot j^{-j/2} \cdot \max_{\dim L=i \leq \delta n} \operatorname{vol}_{i}^{-1/\delta}(K \cap L) \cdot 2^{O(n/\delta)} \\ &= n^{-n/2} \cdot n^{-j/(2\delta)} \cdot \underbrace{(n/j)^{j/2}}_{\leq 2^{O(n)}} \max_{\dim L=i \leq \delta n} \operatorname{vol}_{i}^{-1/\delta}(K \cap L) \cdot 2^{O(n/\delta)} \\ &\leq n^{-n} \cdot \max_{\dim L=i \leq \delta n} \operatorname{vol}_{i}^{-1/\delta}(K \cap L) \cdot 2^{O(n/\delta)}. \end{split}$$

The statement follows as before: by summing over $j \in [n]$ in Lemma 27 we obtain

$$\gamma_n(K) \ge \min_{\dim L = i \le \delta n} \operatorname{vol}_i^{1/\delta}(K \cap L) \cdot 2^{-O(n/\delta)}$$

and we can pass to coordinate sections via M-ellipsoids.

Remark 2. Barthe, Guédon, Mendelson, and Naor conjectured that coordinate slices maximize the Gaussian volume among all slices of a (scaled) ℓ_p ball [BGMN05] (see the remark in p. 28). We can use the above result to give an affirmative answer up to $2^{-O(n)}$:

Corollary 34. Let $p \ge 2$, r > 0 and $H \subseteq \mathbb{R}^m$ an *n*-dimensional subspace. Then

$$\gamma_H(rB_p^m \cap H) \ge \gamma_n(rB_p^n) \cdot 2^{-O(n)}$$

Proof. If $r > n^{1/p}$, the right side is already $2^{-O(n)}$ so we may assume that $r \le n^{1/p}$. A well-known result of Meyer-Pajor asserts that coordinate sections minimize the *volume* among all sections of the ℓ_p ball [MP88]. Applying Theorem 7 and using Meyer-Pajor we get

$$\gamma_H(rB_p^m \cap H) \ge \min_{L \subseteq H, \dim L=i} \operatorname{vol}_i(rB_p^m \cap L) \ge \min_{i \le n} \operatorname{vol}_i(rB_p^i) \ge \gamma_n(rB_p^n) \cdot 2^{-O(n)}.$$

Remark 3. We mention another application of Theorem 7. For a symmetric convex $K \subseteq \mathbb{R}^n$, denote the *hereditary discrepancy* hd(K) as the minimum $t \ge 0$ so that tK_S intersects $\{-1,1\}^S \times \{0\}^{[n]\setminus S}$ for all $S \subseteq [n]$. In [DNTT18] it is shown that we have a lower bound hd(K) $\ge \max_{S \subseteq [n]} \inf\{t : \operatorname{vol}_{|S|}(tK_S) \ge 1\}$, where the left side is known as the *volume lower bound* volLB(K). In fact an analogous argument also shows the lower bound hd(K) $\ge \max_{S \subseteq [n]} \inf\{t : \gamma_{|S|}(tK_S) \ge 2^{-C|S|}\}$ for a universal constant C > 0. Since the volume of a convex body is always lower bounded by its Gaussian measure, this lower bound is at least volLB(K) up to a factor of 2^C . Theorem 7 immediately implies that it is also at most volLB(K) up to a constant.

6 Open problems

We conjecture that Theorem 2 can be improved to match Theorem 1:

Conjecture 1 ($\ell_p \rightarrow \ell_q$ version of Komlós conjecture). *Given* $n \le m$, $2 \le p \le q \le \infty$ and $a_1, ..., a_n \in B_p^m$, do there always exist signs $x \in \{-1, 1\}^n$ so that

$$\left\|\sum_{i=1}^n x_i \boldsymbol{a}_i\right\|_q \leq C \sqrt{\min\left(p, \log\left(\frac{2m}{n}\right)\right)} \cdot n^{1/2 - 1/p + 1/q},$$

for some universal constant C > 0?

Since Conjecture 1 is at least as hard as the Komlós conjecture, a more realistic goal would be to improve the full coloring of Theorem 2 by a factor of $(1/2 - 1/p + 1/q)^{-1/2}$ so as to match the best known bound of $O(\sqrt{\log n})$ for Komlós.

Recall that for a matrix $A \in \mathbb{R}^{n \times n}$ and $1 \le p \le \infty$, the *Schatten-p norm* is defined as $\|A\|_{S(p)} := (\sum_{i=1}^{n} \sigma_i(A)^p)^{1/p}$ where $\sigma_i(A) \ge 0$ is the *i*th *singular value* of the matrix. In particular $\|A\|_{S(\infty)}$ is the maximum singular value and $\|A\|_{S(1)}$ is known as *Trace norm* or *Nuclear norm*. One might wonder whether Theorem 1 could be extended for *matrices* instead of vectors in the corresponding Schatten norms. In fact this is not possible: even for p = 2 and $q = \infty$, there exist *n* rank-one matrices $A_i := v_i v_i^\top \in \mathbb{R}^{n \times n}$ with unit v_i for which any fractional coloring has discrepancy $\Omega(\sqrt{n})$ in the operator norm ([Wea02], Section 3). It is still possible nevertheless that Corollary 3 extends in the following way:

Conjecture 2 (ℓ_p version of Matrix Spencer). *Given* $2 \le p \le \infty$ *and symmetric* $A_1, \ldots, A_n \in \mathbb{R}^{n \times n}$ with Schatten-p norm at most 1, can we always find signs $\mathbf{x} \in \{-1, 1\}^n$ so that

$$\left\|\sum_{i=1}^n x_i A_i\right\|_{S(p)} \le C\sqrt{n}$$

for some universal constant C > 0?

This is a more general form of the Matrix Spencer conjecture [Zou12], and one can show a weaker bound of $O(\sqrt{pn})$ with random signs similar to Lemma 11. In fact, it is an open problem to show even a partial coloring for Conjecture 2. This would be implied by the following, which at least holds for diagonal matrices by the proof of Lemma 20:

Conjecture 3. Given $1 \le p \le \infty$ and symmetric $A_1, \ldots, A_n \in \mathbb{R}^{n \times n}$, can we show that

$$K := \left\{ \boldsymbol{x} \in \mathbb{R}^n : \left\| \sum_{i=1}^n x_i \boldsymbol{A}_i \right\|_{S(p)} \le \left\| \left(\sum_{i=1}^n \boldsymbol{A}_i^2 \right)^{1/2} \right\|_{S(p)} \right\}$$

satisfies $\gamma_n(K) \ge 2^{-O(n)}$?

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A Proof of Lemma 11

Proof of Lemma 11. By convexity of $z \mapsto |z|^p$, Jensen's inequality in (*) and Khintchine's inequality in (**) (Lemma 10) we have

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} x_{i} \boldsymbol{a}_{i}\right\|_{p}\right] \stackrel{(*)}{\leq} \mathbb{E}\left[\left\|\sum_{i=1}^{n} x_{i} \boldsymbol{a}_{i}\right\|_{p}^{p}\right]^{1/p}$$
$$= \left(\sum_{j \in [m]} \mathbb{E}\left[\left|\sum_{i \in [n]} x_{i} a_{ij}\right|^{p}\right]\right)^{1/p}$$
$$\stackrel{(**)}{\leq} C\sqrt{p} \cdot \left(\sum_{j \in [m]} \left(\sum_{i \in [n]} a_{ij}^{2}\right)^{p/2}\right)^{1/p}$$

If $p \in [1, 2]$, write $A_j \in \mathbb{R}^n$ as $(A_j)_i := a_{ij}$. Then by Lemma 8,

$$\left(\sum_{j\in[m]} \left(\sum_{i\in[n]} a_{ij}^2\right)^{p/2}\right)^{1/p} = \left(\sum_{j\in[m]} \|A_j\|_2^p\right)^{1/p} \le \left(\sum_{j\in[m]} \|A_j\|_p^p\right)^{1/p} = \left(\sum_{i\in[n]} \|a_i\|_p^p\right)^{1/p} \le n^{1/p}.$$

Now suppose that $p \ge 2$. Define $(a_i)^2 \in \mathbb{R}^m$ to be the vector with *j*th coordinate a_{ij}^2 . Since $\|\cdot\|_{p/2}$ is a norm, we can use the triangle inequality to get

$$\left(\sum_{j\in[m]} \left(\sum_{i\in[n]} a_{ij}^2\right)^{p/2}\right)^{1/p} = \left\|\sum_{i\in[n]} (\boldsymbol{a}_i)^2\right\|_{p/2}^{1/2} \le \left(\sum_{i\in[n]} \|(\boldsymbol{a}_i)^2\|_{p/2}\right)^{1/2} = \left(\sum_{i\in[n]} \|\boldsymbol{a}_i\|_p^2\right)^{1/2} \le n^{1/2}.$$

Either way, we conclude that $\mathbb{E}[\|\sum_{i=1}^{n} x_i \boldsymbol{a}_i\|_p] \le O(\sqrt{p} \cdot n^{\max(1/2, 1/p)})$, as desired. \Box

Remark 4. A similar approach gives an alternate proof of Prop. 25 in [BGMN05], which states that a $r := O(\sqrt{p} \cdot n^{1/p})$ scaling of an *n*-dimensional section *H* of B_p^m has Gaussian measure $\gamma_H(H \cap rB_p^m) \ge 1/2$ for $p \ge 2$. Indeed, by Markov's inequality, it suffices to note that given an orthonormal basis a_1, \ldots, a_n of *H* we have

$$\mathbb{E}\left[\left\|\sum_{i=1}^{n} x_{i} \boldsymbol{a}_{i}\right\|_{p}\right] \leq C\sqrt{p} \cdot \left(\sum_{j \in [m]} \left(\sum_{i \in [n]} a_{ij}^{2}\right)^{p/2}\right)^{1/p} \leq C\sqrt{p} \cdot n^{1/p},$$

where the last inequality follows from convexity of $z \mapsto z^{p/2}$ and from the fact that the *m* terms $\sum_{i \in [n]} a_{ii}^2$ sum to *n* and are at most 1 by orthonormality.

B Large convex sets without partial colorings

We have mentioned earlier that a symmetric convex set *K* with measure $\gamma_n(K) \ge e^{-\delta n}$ contains a partial coloring $\mathbf{x} \in \{-1, 0, 1\}^n$ with a linear number of nonzero coordinates if the constant δ is small enough — but we claimed that this is false for constants beyond a certain threshold, even if one is allowed to rescale the body by some parameter dependent on δ . The construction for such a set is a thin strip that avoids any point in $\{-1, 0, 1\}^n \setminus \{\mathbf{0}\}$.

Lemma 35. For any $C \ge 1$, there exists a $\delta > 0$ so that the following holds: for any $n \in \mathbb{N}$ large enough there is a symmetric convex body $K \subseteq \mathbb{R}^n$ so that (i) $(C^n K) \cap (\{-1, 0, 1\}^n \setminus \{0\}) = \emptyset$ and (ii) $\gamma_n(K) \ge e^{-\delta n}$.

Proof. The construction is probabilistic. We sample a Gaussian $\mathbf{g} \sim N(\mathbf{0}, \mathbf{I}_n)$ and for a tiny parameter s > 0 that we determine later, we consider the strip $K := \{\mathbf{x} \in \mathbb{R}^n : |\langle \mathbf{g}, \mathbf{x} \rangle| \le s\}$. Consider the set of nontrivial partial colorings $X := \{-1, 0, 1\}^n \setminus \{\mathbf{0}\}$ and recall that $|X| \le 3^n$. For any $\mathbf{x} \in X$, the distribution of $\langle \mathbf{g}, \mathbf{x} \rangle$ is Gaussian with variance $\|\mathbf{x}\|_2^2 \ge 1$ and hence the density of this 1-dimensional Gaussian is at most $\frac{1}{\sqrt{2\pi}}e^0 \le \frac{1}{2}$ everywhere. In particular for a fixed $\mathbf{x} \in X$, one can obtain the simple estimate of $\Pr[|\langle \mathbf{g}, \mathbf{x} \rangle| \le t] \le 4t$ for any t > 0. Then choosing $s := \frac{1}{16} \cdot C^{-n}3^{-n}$ we obtain

$$\Pr_{\boldsymbol{g}}\left[(C^{n}K) \cap X \neq \emptyset\right] \le \sum_{\boldsymbol{x} \in X} \Pr_{\boldsymbol{g}}\left[|\langle \boldsymbol{g}, \boldsymbol{x} \rangle| > C^{n}s\right] \le \frac{1}{4} \cdot |X| \cdot 3^{-n} \le \frac{1}{4} \qquad (*)$$

Moreover using Markov's Inequality we obtain the (rather weak) estimate

$$\Pr\left[\|\boldsymbol{g}\|_{2}^{2} > 4n\right] \le \frac{1}{4} \qquad (**)$$

Then with probability at least 1/2 none of the events (*) and (**) happen. We fix such an outcome of g and estimate that the measure of our strip is

$$\gamma_n(K) = \int_{-s/\|\mathbf{g}\|_2}^{s/\|\mathbf{g}\|_2} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx \ge \frac{1}{\sqrt{2\pi}} e^{-1/2} \frac{2s}{\sqrt{n}} \ge e^{-\delta n}$$

for a suitable choice of δ using $\frac{s}{\|\mathbf{g}\|_2} \leq 1$.