# Self-avoiding walk on the hypercube 

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#### Abstract

We study the number $c_{n}^{(N)}$ of $n$-step self-avoiding walks on the $N$-dimensional hypercube, and identify an $N$-dependent connective constant $\mu_{N}$ and amplitude $A_{N}$ such that $c_{n}^{(N)}$ is $O\left(\mu_{N}^{n}\right)$ for all $n$ and $N$, and is asymptotically $A_{N} \mu_{N}^{n}$ as long as $n \leq 2^{p N}$ for any fixed $p<\frac{1}{2}$. We refer to the regime $n \ll 2^{N / 2}$ as the dilute phase. We discuss conjectures concerning different behaviours of $c_{n}^{(N)}$ when $n$ reaches and exceeds $2^{N / 2}$, corresponding to a critical window and a dense phase. In addition, we prove that the connective constant has an asymptotic expansion to all orders in $N^{-1}$, with integer coefficients, and we compute the first five coefficients $\mu_{N}=N-1-N^{-1}-4 N^{-2}-26 N^{-3}+O\left(N^{-4}\right)$. The proofs are based on generating function and Tauberian methods implemented via the lace expansion, for which an introductory account is provided.


## 1 Introduction

The self-avoiding walk in a much studied model in combinatorics, probability theory, statistical physics, and polymer chemistry [32,37]. Typically it has been studied on an infinite graph such as the hypercubic lattice $\mathbb{Z}^{d}$. More recently its critical behaviour has been analysed on finite graphs including the complete graph $[14,47]$ and (for weakly self-avoiding walk) a discrete torus in dimensions $d>4[40,41,46]$. Our goal here is to investigate the critical behaviour of the self-avoiding walk on the hypercube. We analyse its dilute phase in detail using the lace expansion, and identify the connective constant, whose reciprocal is the critical value. We also raise open questions about its critical window and dense phase.

### 1.1 Self-avoiding walk on the hypercube

Let $\mathbb{Q}^{N}=\mathbb{Z}_{2}^{N}$ denote the $N$-dimensional hypercube. Thus an element $x \in \mathbb{Q}^{N}$ is a binary string of length $N$. Addition on $\mathbb{Q}^{N}$ is defined coordinate-wise modulo 2. The volume of $\mathbb{Q}^{N}$ is $V=V(N)=2^{N}$. The Hamming norm $|x|$ of $x \in \mathbb{Q}^{N}$ is the number of coordinates of $x$ which are equal to 1 . In particular $|x|$ is an integer between 0 and $N$.

An $n$-step walk on $\mathbb{Q}^{N}$ is a function $\omega:\{0,1, \ldots, n\} \rightarrow \mathbb{Q}^{N}$ with $|\omega(i)-\omega(i-1)|=1$ for $1 \leq i \leq n$. An $n$-step self-avoiding walk on $\mathbb{Q}^{N}$ is an $n$-step walk for which $\omega(i) \neq \omega(j)$ for all $i \neq j$. Typically we take $\omega(0)=0$. Let $c_{n}^{(N)}$ be the number of $n$-step self-avoiding walks on $\mathbb{Q}^{N}$ with $\omega(0)=0$. For $n=0$ we set $c_{0}^{(N)}=1$.

For example, the 3 -step walk 00000 , $00100,01100,01000$ is counted in $c_{3}^{(5)}$, and for any $N \geq 1$ we have $c_{0}^{(N)}=1, c_{1}^{(N)}=N, c_{2}^{(N)}=N(N-1), c_{3}^{(N)}=N(N-1)^{2}, c_{4}^{(N)}=N^{2}(N-1)(N-2)$. Since an $n$-step self-avoiding walk visits $n+1$ distinct vertices, $c_{n}^{(N)}=0$ if $n \geq V$. Also, $c_{V-1}^{(N)}$ is the number of Hamilton paths on $\mathbb{Q}^{N}$ which start at 0 . Our aim is to study the asymptotic behaviour of $c_{n}^{(N)}$ for large $n$ and $N$.

[^0]The susceptibility is the generating function for the sequence $c_{n}^{(N)}$ (for fixed $N$ ), and thus is the polynomial in $z \in \mathbb{C}$ defined by

$$
\begin{equation*}
\chi_{N}(z)=\sum_{n=0}^{\infty} c_{n}^{(N)} z^{N}=\sum_{n=0}^{V-1} c_{n}^{(N)} z^{N} \tag{1.1}
\end{equation*}
$$

Motivated by the definition of the critical value for self-avoiding walk on a finite graph proposed in [47], which itself was motivated by finite-graph percolation [6], given any $\lambda>0$ we define the critical value $z_{N}=z_{N}(\lambda)>0$ by

$$
\begin{equation*}
\chi_{N}\left(z_{N}\right)=\lambda V^{1 / 2}=\lambda 2^{N / 2} . \tag{1.2}
\end{equation*}
$$

To ensure that $z_{N}$ is well-defined, we always assume that $\lambda V^{1 / 2} \geq \chi_{N}(0)=1$. Then we define the connective constant $\mu_{N}=\mu_{N}(\lambda)$ to be the reciprocal of the critical value:

$$
\begin{equation*}
\mu_{N}=\mu_{N}(\lambda)=\frac{1}{z_{N}(\lambda)} \tag{1.3}
\end{equation*}
$$

The term "constant" is used despite the dependence of $\mu_{N}$ on $N$ and $\lambda$. By definition, $z_{N}$ is an increasing function of $\lambda$, and $\mu_{N}$ is decreasing.

### 1.2 Main results

Our main results are the following five theorems, Theorems 1.1-1.5. We expect that with minor additional effort it would be possible to extend our results to more general graphs including the Hamming graph, as in [7]. However we prefer to restrict attention to the hypercube to develop methods in a concrete setting.

### 1.2.1 Connective constant and number of self-avoiding walks

As a first indication that the connective constant is useful, the following theorem shows that it provides an exponential upper bound on the number of $n$-step self-avoiding walks, valid for all $n \in \mathbb{N}$.

Theorem 1.1. There exist $\lambda_{0}>0$ and $K>0$ (depending on $\lambda_{0}$ ) such that for all $n, N \in \mathbb{N}$ (with $\lambda_{0} V^{1 / 2} \geq 1$ ),

$$
\begin{equation*}
c_{n}^{(N)} \leq K \mu_{N}\left(\lambda_{0}\right)^{n} \tag{1.4}
\end{equation*}
$$

The next theorem establishes that the connective constant truly is the exponential growth rate of the number of $n$-step self-avoiding walks as long as $n \leq V^{p}$ for any fixed $p \in\left(0, \frac{1}{2}\right)$. We regard this range of $n$ as the regime in which the self-avoiding walk does not yet "feel" the finite volume of the hypercube. A more detailed error estimate is given in Theorem 2.3.

Theorem 1.2. There exists $\lambda_{0}>0$ such that with $\mu_{N}=\mu_{N}(\lambda)$ defined by (1.3) for any $\lambda \in\left(0, \lambda_{0}\right]$, and for any choice of $p \in\left(0, \frac{1}{2}\right)$, there exists $\epsilon_{p}>0$ such that

$$
\begin{equation*}
c_{n}^{(N)}=A_{N} \mu_{N}^{n}\left[1+O\left(n^{-\epsilon_{p}}\right)\right] \tag{1.5}
\end{equation*}
$$

for all $n, N$ such that $n \leq V^{p}$ (and $\lambda V^{1 / 2} \geq 1$ ). The sequence $A_{N}$ is independent of $n$ (but depends on $\lambda$ and $p$ ). The constant in the error term depends on $\lambda$ and $p$ but not on $n$ or $N$ as long as $n \leq V^{p}$.

A possibly surprising feature of (1.5) is that its left-hand side does not depend on the choice of $\lambda$ but both $A_{N}$ and the exponential term $\mu_{N}^{n}$ on the right-hand side do depend on $\lambda$. This is not contradictory, as we will prove in Section 6.2 (see (6.26)) that, for $0<\lambda^{\prime} \leq \lambda_{1}<\lambda_{2} \leq \lambda_{0}$,

$$
\begin{equation*}
\frac{\mu_{N}\left(\lambda_{1}\right)}{\mu_{N}\left(\lambda_{2}\right)}=1+O\left(V^{-1 / 2}\right) \tag{1.6}
\end{equation*}
$$

where the constant in the error term depends on $\lambda^{\prime}, \lambda_{0}$. Thus the replacement of one fixed choice of $\lambda$ by another in $\mu_{N}^{n}$ produces a factor $\left[1+O\left(V^{-1 / 2}\right)\right]^{n}$, and for $n \leq V^{p}$ with $p<\frac{1}{2}$ this is $1+O\left(n V^{-1 / 2}\right)$ and hence can be absorbed by the error term $n^{-\epsilon_{p}}$ since when $n \leq V^{p}$ we have

$$
\begin{equation*}
\frac{n}{V^{1 / 2}} \leq \frac{1}{n^{(1-2 p) /(2 p)}} . \tag{1.7}
\end{equation*}
$$

The next theorem gives another sense in which the connective constant $\mu_{N}$ depends only weakly on $\lambda$ and the amplitude $A_{N}$ depends only weakly on $\lambda$ and $p$.

Theorem 1.3. Let $\lambda_{0}>0$ be sufficiently small. Let $m \in \mathbb{N}$, fix $c>0$ (independent of $N$ but possibly depending on $m$ ), and suppose that $z$ obeys $\chi_{N}(z) \in\left[c N^{m}, \lambda_{0} V^{1 / 2}\right]$. Then there are integers $a_{n}$ for $n \in \mathbb{N}$, which are universal constants that do not depend on the particular choice of $z$, such that

$$
\begin{equation*}
z=\sum_{n=1}^{m} a_{n} N^{-n}+O\left(N^{-m-1}\right) \tag{1.8}
\end{equation*}
$$

The constant in the error term depends on $m, \lambda_{0}, c$, but does not depend otherwise on $z$. The first five terms are given by

$$
\begin{equation*}
z=\frac{1}{N}+\frac{1}{N^{2}}+\frac{2}{N^{3}}+\frac{7}{N^{4}}+\frac{39}{N^{5}}+O\left(\frac{1}{N^{6}}\right) \tag{1.9}
\end{equation*}
$$

For any $\lambda \in\left(0, \lambda_{0}\right], p \in\left(0, \frac{1}{2}\right)$, and $m \in \mathbb{N}$, the amplitude $A_{N}$ in (1.5) has an asymptotic expansion

$$
\begin{equation*}
A_{N}=\sum_{n=1}^{m} a_{n}^{\prime} N^{-n}+O\left(N^{-m-1}\right) \tag{1.10}
\end{equation*}
$$

with universal integer coefficients $a_{n}^{\prime}$ (which in particular do not depend on $p, \lambda$ ) and with an error depending on $m, \lambda, p$. The first five terms are given by

$$
\begin{equation*}
A_{N}=1+\frac{1}{N}+\frac{4}{N^{2}}+\frac{26}{N^{3}}+\frac{231}{N^{4}}+O\left(\frac{1}{N^{5}}\right) . \tag{1.11}
\end{equation*}
$$

By Theorem 1.3, any choice of $z$ for which $\chi_{N}(z) \in\left[c N^{m}, \lambda_{0} V^{1 / 2}\right]$ has the same expansion up to an error $O\left(N^{-m-1}\right)$, with the error independent of the particular choice made for $z$. The expansion (1.8) is valid simultaneously to all orders $m$ if we choose an $N$-dependent sequence $z$ for which $\chi_{N}(z)$ lies eventually in all intervals $\left[c N^{m}, \lambda_{0} V^{1 / 2}\right]$. In particular, (1.8) holds simultaneously for all $m$ when $z=z_{N}(\lambda)$ with $\lambda \in\left(0, \lambda_{0}\right]$, with the coefficients $a_{n}$ independent of $\lambda$. It also holds if $z$ is chosen, e.g., to satisfy $\chi_{N}(z)=2^{\sqrt{N}}$. The connective constant therefore also has an asymptotic expansion in $N^{-1}$ to all orders and with integer coefficients, and in particular by taking the reciprocal of (1.9) we find that, for any $\lambda \in\left(0, \lambda_{0}\right]$,

$$
\begin{equation*}
\mu_{N}=N-1-\frac{1}{N}-\frac{4}{N^{2}}-\frac{26}{N^{3}}+O\left(\frac{1}{N^{4}}\right) . \tag{1.12}
\end{equation*}
$$

The existence proof for the expansions for $z_{N}$ and $A_{N}$ presents an algorithm for the computation of any number of coefficients, and more terms could be computed with computer assistance as has been done for $\mathbb{Z}^{d}$ (see Section 1.5, in fact the hypercube computations appear to be substantially easier than for $\mathbb{Z}^{d}$ ).

### 1.2.2 Susceptibility and expected length

The following theorem provides upper and lower bounds on the susceptibility. As the proof will show, the lower bound in (1.13) is a general consequence of submultiplicativity and holds on any finite or infinite transitive graph, while the upper bound relies on the proof of a "bubble condition."

Theorem 1.4. Fix $\lambda \in\left(0, \lambda_{0}\right]$, assume that $\lambda V^{1 / 2} \geq 1$, and let $z_{N}=z_{N}(\lambda)$. Let $\beta=N^{-1}+\lambda^{2}$. For all $z \in\left[0, z_{N}\right]$,

$$
\begin{equation*}
\frac{1}{\lambda^{-1} V^{-1 / 2}+1-z / z_{N}} \leq \chi_{N}(z) \leq \frac{2-z / z_{N}}{\lambda^{-1} V^{-1 / 2}+(1-O(\beta))\left(1-z / z_{N}\right)} \tag{1.13}
\end{equation*}
$$

The expected length of a self-avoiding walk is defined as follows. The length $L$ is the discrete random variable with $z$-dependent probability mass function

$$
\begin{equation*}
\mathbb{P}_{z}^{(N)}(L=n+1)=\frac{1}{\chi_{N}(z)} c_{n}^{(N)} z^{n} \tag{1.14}
\end{equation*}
$$

with fixed $N$ and fixed $z \geq 0$, and for all nonnegative integers $n$. With this definition using $n+1$ on the left-hand side of (1.14), $L$ reflects the number of vertices in the walk rather than the number of steps. The expected length is

$$
\begin{equation*}
\mathbb{E}_{z}^{(N)} L=\sum_{n=0}^{\infty}(n+1) \mathbb{P}_{z}^{(N)}(L=n+1)=\frac{1}{\chi_{N}(z)} \partial_{z}\left[z \chi_{N}(z)\right] \tag{1.15}
\end{equation*}
$$

The next theorem concerns the asymptotic behaviour of the expected length. The upper bound is a consequence of submultiplicativity and holds on any finite or infinite transitive graph, while the lower bound is a consequence of the bubble condition.

Theorem 1.5. Fix $\lambda \in\left(0, \lambda_{0}\right]$, assume that $\lambda V^{1 / 2} \geq 1$, and let $z_{N}=z_{N}(\lambda)$. Let $\beta=N^{-1}+\lambda^{2}$. For $z \in\left[0, z_{N}\right]$,

$$
\begin{equation*}
[1-O(\beta)] \chi_{N}(z) \leq \mathbb{E}_{z}^{(N)} L \leq \chi_{N}(z) \tag{1.16}
\end{equation*}
$$

In particular, at the critical value,

$$
\begin{equation*}
\mathbb{E}_{z_{N}}^{(N)} L=\lambda V^{1 / 2}[1+O(\beta)] \tag{1.17}
\end{equation*}
$$

### 1.3 Notation

We write $f \sim g$ to mean $\lim f / g=1, f \prec g$ to mean $f \leq c_{1} g$ with $c_{1}>0$ and $f \succ g$ to mean $g \prec f$. We also write $f \asymp g$ when $g \prec f \prec g$. Constants in these relations are not permitted to depend on $N$ but may depend on the choice of $\lambda$ used to define $z_{N}$, and also on $p \in\left(0, \frac{1}{2}\right)$ when it is part of the discussion.

### 1.4 Conjectured phase transition

In the hypotheses of Theorem 1.2 it is assumed that $p \in\left(0, \frac{1}{2}\right)$. At the upper limit $p=\frac{1}{2}$, which Theorem 1.2 does not address, the error estimate is no longer small. We believe that this is not an artifact of our proof but that the asymptotic behaviour does change once $n$ reaches $V^{1 / 2}$. The nature of this conjectured change can be anticipated by comparison with self-avoiding walk on the complete graph, which is exactly solvable - its susceptibility is essentially an incomplete Gamma function - and which has been analysed recently in [47] (see also [14]). In [47], it is conjectured that the susceptibility $\chi_{N}(z)$ for the hypercube remains of order $V^{1 / 2}$ throughout the critical window consisting of ( $N$-dependent) $z$ values such that $\left|1-z / z_{N}\right|$ is of order $V^{-1 / 2}$. A related conjecture for self-avoiding walk on a discrete torus of dimension $d>4$ is discussed in [41].

On the complete graph on $V$ vertices, the number $k_{n}$ of $n$-step self-avoiding walks starting from a fixed vertex is simply

$$
\begin{equation*}
k_{n}=\frac{v!}{(v-n)!}, \tag{1.18}
\end{equation*}
$$

where $v=V-1$. In the limit in which $v \rightarrow \infty$, and assuming for simplicity that $n=o\left(v^{2 / 3}\right)$ (so in particular $v-n \rightarrow \infty$ ), it follows from Stirling's formula that

$$
\begin{equation*}
k_{n}=v^{n} e^{-n^{2} / 2 v}[1+o(1)] . \tag{1.19}
\end{equation*}
$$

We expect similar asymptotics to apply to the hypercube in and around the critical window, with dominant behaviour $\mu_{N}^{n} e^{-\alpha n^{2} / V}$ for $c_{n}^{(N)}$, for some $\alpha>0$. This is consistent with the susceptibility remaining of order $V^{1 / 2}$ in the critical window.

By analogy with the theory of self-avoiding walk on the complete graph developed in detail in [47] (see also [14]), we are led to the conjecture for the hypercube that the interval $z \in(0, \infty)$ is divided into three regimes. With $z$ written as $z=z_{N}(1+\epsilon)$ with $\epsilon \in(-1, \infty)$, these regimes are:

- the dilute phase $\epsilon \ll-V^{-1 / 2}$ :

$$
\chi_{N} \asymp \epsilon^{-1}, \quad c_{n}^{(N)} \sim A_{N} \mu_{N}^{n} \text { for } n \ll V^{1 / 2}, \quad \mathbb{E}_{z}^{(N)} L \asymp \epsilon^{-1} ;
$$

- the critical window $|\epsilon| \asymp V^{-1 / 2}$ :

$$
\chi_{N} \asymp V^{1 / 2}, \quad c_{n}^{(N)} \asymp \mu_{N}^{n} \text { for } n \asymp V^{1 / 2}, \quad \mathbb{E}_{z}^{(N)} L \asymp V^{1 / 2} ;
$$

- the dense phase $\epsilon \gg V^{-1 / 2}$ :

$$
\chi_{N} \text { exponential in } V, \quad c_{n}^{(N)} \ll \mu_{N}^{n} \text { for } n \gg V^{1 / 2}, \quad \mathbb{E}_{z}^{(N)} L \asymp V \frac{\epsilon}{1+\epsilon} .
$$

In particular, if $\epsilon=V^{-p}$ with $p \in\left(0, \frac{1}{2}\right)$ then the above states that $\mathbb{E}_{z}^{(N)} L \asymp V^{1-p}$, whereas if $\epsilon \geq c>0$ then it states that $\mathbb{E}_{z}^{(N)} L \asymp V$. For the case $\epsilon=-V^{-p}$ with $p \in\left(0, \frac{1}{2}\right)$, the above states that $\chi_{N} \asymp V^{p} \asymp$ $\mathbb{E}_{z}^{(N)} L$.

Theorem 1.4 proves the above behaviour for the susceptibility in the dilute phase and in the critical window up to and including $z=z_{N}$. Theorem 1.2 proves the dilute behaviour of $c_{n}^{(N)}$ as long as $n \leq V^{p}$ for some $p<\frac{1}{2}$. Theorem 1.5 proves the above behaviour for the expected length in the dilute phase and in the critical window up to and including $z=z_{N}$. It is an open problem to prove (or disprove) any of the remaining statements.

For general graphs, the mathematical analysis of the dense phase of self-avoiding walk is not yet very well developed. Various aspects of the dense phase are studied in $[9,17,23,50]$.

For percolation on the hypercube, a related and much-studied parallel to the above picture is developed in $[2,5,8,28-31,33]$. Our analysis takes inspiration in particular from the general study of the percolation phase transition on finite graphs including the hypercube from [6], though we also rely on complex analytic methods that were not used for percolation.

### 1.5 The connective constant on infinite graphs

It is something of a misnomer to refer to $\mu_{N}$ as the connective "constant" since it depends on $N$ and also on the choice of $\lambda$. However the terminology is natural in the sense that on an infinite lattice the term "connective constant" is used for the exponential growth rate for the number $c_{n}$ of $n$-step self-avoiding walks started from a given vertex. On any transitive graph, finite or infinite, $c_{n}$ obeys $c_{n+m} \leq c_{n} c_{m}$ and by Fekete's lemma this implies existence of the limit

$$
\begin{equation*}
\mu=\lim _{n \rightarrow \infty} c_{n}^{1 / n}=\inf _{n \geq 0} c_{n}^{1 / n} \tag{1.20}
\end{equation*}
$$

where $\mu$ of course depends on the graph. However on a finite graph, such as the hypercube, $c_{n}$ is eventually zero so $\mu$ takes the uninformative value $\mu=0$. On an infinite lattice such as $\mathbb{Z}^{d}$ or the hexagonal lattice, $\mu$ is not zero and it gives the exponential growth rate of $c_{n}$ in the sense of (1.20). There are numerical estimates and rigorous bounds for the value of $\mu\left(\mathbb{Z}^{d}\right)$ but its exact value is not known for any $d \geq 2$. Exceptionally, for the hexagonal lattice it was predicted in [43] and proved in [18] that $\mu(\operatorname{Hex})=\sqrt{2+\sqrt{2}}$. Connective constants for more general graphs are studied in $[3,25,36,38,44]$. Expansions for the connective constant have been considered in other settings, e.g., two terms were computed in [44] for hyperbolic graphs. The lace expansion (when applicable) provides a systematic method for computation of many terms.

Indeed, for $\mathbb{Z}^{d}$ it is proved in [27] that the connective constant has an asymptotic expansion to all orders in $(2 d)^{-1}$, with integer coefficients, and in [12] thirteen of these coefficients are computed with the result that

$$
\begin{align*}
\mu\left(\mathbb{Z}^{d}\right)=2 d & -1-\frac{1}{2 d}-\frac{3}{(2 d)^{2}}-\frac{16}{(2 d)^{3}}-\frac{102}{(2 d)^{4}}-\frac{729}{(2 d)^{5}}-\frac{5533}{(2 d)^{6}}-\frac{42229}{(2 d)^{7}} \\
& -\frac{288761}{(2 d)^{8}}-\frac{1026328}{(2 d)^{9}}+\frac{21070667}{(2 d)^{10}}+\frac{780280468}{(2 d)^{11}}+O\left(\frac{1}{(2 d)^{12}}\right) \tag{1.21}
\end{align*}
$$

Equivalently, the critical value $z_{c}\left(\mathbb{Z}^{d}\right)=1 / \mu\left(\mathbb{Z}^{d}\right)$ satisfies

$$
\begin{align*}
z_{c}\left(\mathbb{Z}^{d}\right)=\frac{1}{2 d} & +\frac{1}{(2 d)^{2}}+\frac{2}{(2 d)^{3}}+\frac{6}{(2 d)^{4}}+\frac{27}{(2 d)^{5}}+\frac{157}{(2 d)^{6}}+\frac{1065}{(2 d)^{7}}+\frac{7865}{(2 d)^{8}}+\frac{59665}{(2 d)^{9}} \\
& +\frac{422421}{(2 d)^{10}}+\frac{1991163}{(2 d)^{11}}-\frac{16122550}{(2 d)^{12}}-\frac{805887918}{(2 d)^{13}}+O\left(\frac{1}{(2 d)^{14}}\right) . \tag{1.22}
\end{align*}
$$

Also, in the asymptotic formula $c_{n}=A \mu^{n}\left[1+O\left(n^{-\epsilon}\right)\right]$ for $\mathbb{Z}^{d}$ with $d \geq 5$ proved in [26], the amplitude $A$ is proved in [12] to have an asymptotic expansion to all orders, with integer coefficients, and in particular

$$
\begin{align*}
A\left(\mathbb{Z}^{d}\right)= & 1+\frac{1}{2 d}+\frac{4}{(2 d)^{2}}+\frac{23}{(2 d)^{3}}+\frac{178}{(2 d)^{4}}+\frac{1591}{(2 d)^{5}}+\frac{15647}{(2 d)^{6}}+\frac{164766}{(2 d)^{7}}+\frac{1825071}{(2 d)^{8}} \\
& +\frac{20875838}{(2 d)^{9}}+\frac{240634600}{(2 d)^{10}}+\frac{2684759873}{(2 d)^{11}}+\frac{26450261391}{(2 d)^{12}}+O\left(\frac{1}{(2 d)^{13}}\right) . \tag{1.23}
\end{align*}
$$

The possibility that the above series are Borel summable is investigated but not resolved in [24]. See [49] for a sufficient condition for Borel summability. We believe that these series and also the series for the hypercube in Theorem 1.3 have radius of convergence zero but are Borel summable; to prove any of these statements is an open problem. Numerical results of Padé-Borel resummation [35] of the above series for $\mu\left(\mathbb{Z}^{d}\right)$ and $A\left(\mathbb{Z}^{d}\right)$ are reported in $[12$, Table 15$]$. For the related question of the $1 / d$ expansion for the critical point for the Berlin-Kac spherical model, it is resolved affirmatively in [22] that the radius of convergence of the expansion is zero. There is a substantial literature concerning such $1 / d$ expansions going back as early as 1964 where the first six coefficients of (1.21) were determined [19], and decades later confirmed with rigorous error estimate $[27]$. Earlier expansions for the amplitude $A\left(\mathbb{Z}^{d}\right)$ including terms up to and including order $(2 d)^{-2}$ (with rigorous error estimate) and to $(2 d)^{-5}$ (without rigorous error estimate) were given respectively in [27] and in [21, 42].

Such expansions have also been studied for other models including lattice animals [39] and percolation $[27,30,31]$. In particular, a theorem analogous to Theorem 1.3 is proved for the critical value of percolation on the hypercube and on $\mathbb{Z}^{d}$, this time with rational rather than integer coefficients, in $[30,31]$.

### 1.6 Organisation

Sections $2-7$ provide the proofs of Theorems 1.1-1.5, which are organised as follows.
In Section 2.1 we state Proposition 2.1 which gives a lower bound on the reciprocal of the susceptibility as a function of complex $z$ in the disk $|z| \leq z_{N}$, where $z_{N}=z_{N}(\lambda)$ for a sufficiently small choice of $\lambda>0$. In conjunction with the elementary Tauberian theorem stated in Lemma 2.2, this leads to a short proof
of the general upper bound on $c_{n}^{(N)}$ stated in Theorem 1.1. In Section 2.2, a version of Theorem 1.2 with a more accurate error estimate is stated as Theorem 2.3, and the proof of Theorem 2.3 is given subject to Propositions 2.4-2.5. These two propositions give more refined information on the reciprocal of the susceptibility than Proposition 2.1 but in a smaller disk $|z| \leq z_{N}\left(1-V^{-p}\right)$ for arbitrary but fixed $p \in\left(0, \frac{1}{2}\right)$. This detailed information allows for the extraction of a leading term from the susceptibility, and thereby from its coefficients $c_{n}^{(N)}$, with an error that can be estimated using the Tauberian theorem. This proves Theorems 1.1-1.2 subject to the control of the reciprocal of the susceptiblity stated in Propositions 2.1, 2.4 , and 2.5 , which are all proved using the lace expansion.

The lace expansion was introduced by Brydges and Spencer in 1985 to study weakly self-avoiding walk on $\mathbb{Z}^{d}$ in dimensions $d>4$ [11]. Since then, it has been developed into a flexible method for the analysis of critical behaviour in many high-dimensional settings, including self-avoiding walk, lattice trees, lattice animals, percolation on finite and infinite graphs, oriented percolation, the contact process, and spin systems (Ising and $\varphi^{4}$ models). In Section 3, we review the lace expansion in our present context of self-avoiding walk on the hypercube.

The convergence of the lace expansion employs some elementary estimates for simple random walk on the hypercube which are proved in Section 4. The convergence of the lace expansion is established in Section 5 for complex $z$ in the disk $|z| \leq z_{N}$, via the Fourier approach used previously for percolation in [7] and adapted to self-avoiding walk in [45]. The zero mode of the Fourier transform plays a special and key role, and is what forces the choice of a small $\lambda$ for the definition of the critical value $z_{N}=z_{N}(\lambda)$. The fact that we work on the hypercube results in a convergence proof that is strikingly simple. The centrepiece for high-dimensional percolation is the triangle condition [1, 28]; its role is played here by the bubble condition which is established in Section 5.2. The importance of the bubble condition for self-avoiding walk goes back at least as far as [10]. The bulk of our analysis would apply generally to other transitive graphs for which the bubble condition holds.

Once the convergence of the lace expansion has been proved, it is short work in Section 6.3 to prove Propositions 2.1 and 2.4 , as well as the estimates for the susceptibility and expected length in Theorems 1.4 and 1.5. The proof of Proposition 2.5 makes use of the fractional derivative methodology developed in [26], which is briefly reviewed in Section 6.4, before proving Proposition 2.5 in Section 6.5.

Finally, in Section 7 we prove the existence of the $1 / N$ expansions for $z_{N}$ and $A_{N}$ stated in Theorem 1.3 and compute the first five coefficients. The general approach to the existence proof is related to the approach used for $\mathbb{Z}^{d}$ in [27], but improvements to that approach which were introduced in [12] are adapted here to the hypercube to obtain a relatively simple existence proof. The computation of the expansion coefficients follows a straightforward iterative procedure and could be extended to more terms with further effort to enumerate lace graphs on the hypercube. For small lace graphs, enumeration on the hypercube is not difficult to adapt from the enumerations on $\mathbb{Z}^{d}$ provided in [13], and in this way we avoid any difficult counting in the computation of the five coefficients given in Theorem 1.3.

## 2 Analysis of the susceptibility

In this section, we prove Theorems 1.1 and 1.2 subject to Proposition 2.1 (for Theorem 1.1) and Propositions 2.4-2.5 (for Theorem 1.2). These propositions give estimates on the susceptibility which can be converted into estimates for $c_{n}^{(N)}$ via the Tauberian theorem in Lemma 2.2.

### 2.1 Upper bound: proof of Theorem 1.1

### 2.1.1 Use of the Tauberian theorem

The susceptibility is a polynomial, so its reciprocal

$$
\begin{equation*}
F_{N}(z)=\frac{1}{\chi_{N}(z)} \tag{2.1}
\end{equation*}
$$

is a meromorphic function of $z \in \mathbb{C}$. Since $\chi_{N}$ is a polynomial with positive coefficients, $F_{N}$ has no poles on the nonnegative real axis. We will prove the following proposition in Section 6 using the lace expansion.

Proposition 2.1. There is a $\lambda_{0}>0$ such that, with $z_{N}=z_{N}(\lambda)$ for any $\lambda \in\left(0, \lambda_{0}\right]$, and with $N$ sufficiently large depending on $\lambda$, the function $F_{N}$ obeys the bounds $\left|F_{N}^{\prime}(z)\right| \leq 2 N$ and $\left.\left|F_{N}(z)\right| \geq \frac{1}{2} \right\rvert\, 1-$ $z / z_{N} \mid$ uniformly in $z \in \mathbb{C}$ with $|z| \leq z_{N}$. In addition, $z_{N} \leq 2 N^{-1}$.

To prove Theorem 1.1, we use Proposition 2.1 in combination with the Tauberian theorem from [20, Theorem 4] stated in the next lemma.

Lemma 2.2. Let $b>1$. Suppose that the power series $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ obeys $|f(z)| \leq K_{1}|1-z / \rho|^{-b}$ for all $|z|<\rho$. Then $\left|a_{n}\right| \leq K_{2} K_{1} n^{b-1} \rho^{-n}$ with $K_{2}$ depending only on $b$.

Proof of Theorem 1.1. By Proposition 2.1,

$$
\begin{equation*}
\left|\chi_{N}^{\prime}(z)\right|=\left|\frac{F_{N}^{\prime}(z)}{F_{N}(z)^{2}}\right| \leq \frac{8 N}{\left|1-z / z_{N}\right|^{2}} \tag{2.2}
\end{equation*}
$$

holds uniformly in $|z| \leq z_{N}$. Since the coefficient of $z^{n}$ in $\chi_{N}^{\prime}(z)$ is $(n+1) c_{n+1}^{(N)}$, it follows from Lemma 2.2 (with $f=\chi_{N}^{\prime}, \rho=z_{N}$ and $b=2$ ) that there is a constant $K$ such that

$$
\begin{equation*}
(n+1) c_{n+1}^{(N)} \leq K N n^{2-1} z_{N}^{-n} \tag{2.3}
\end{equation*}
$$

for all $n$. Since $N z_{N} \leq 2$ by Proposition 2.1,

$$
\begin{equation*}
c_{n}^{(N)} \leq K N z_{N}^{-(n-1)}=K N z_{N} \mu_{N}^{n} \leq 2 K \mu_{N}^{n} \tag{2.4}
\end{equation*}
$$

which is the desired upper bound.
In the above we have assumed that $N$ is sufficiently large, say $N \geq N_{0}\left(\lambda_{0}\right)$. However, for $N<N_{0}$ there are only finitely many choices of $(n, N)$ and we can therefore obtain (2.4) for all ( $n, N$ ) (with $\lambda V^{1 / 2} \geq 1$ ) by increasing $K$.

### 2.1.2 Remarks on Tauberian theorems

1. Extensions of Lemma 2.2 in [15, Lemma 3.2] include the case $b=1$ which instead has upper bound $\rho^{-n} \log n$. This is the reason why $\chi_{N}^{\prime}$ appears rather than $\chi_{N}$ in the above application of Lemma 2.2 to obtain Theorem 1.1: applied directly to $\chi_{N}$, the extension to Lemma 2.2 would produce an unwanted logarithm in the upper bound. Lemma 2.2 is false for $b<1$, a counterexample is given in the Remark following [37, Lemma 6.3.3].
2. We have chosen to prove Theorem 1.1 using Lemma 2.2 because Lemma 2.2 is also required for the proof of Theorem 1.2. However, for Theorem 1.1 we could instead have applied Hutchcroft's

Tauberian theorem [34, Lemma 3.4] for submultiplicative sequences (since we do have $c_{n+m}^{(N)} \leq$ $c_{n}^{(N)} c_{m}^{(N)}$ ), which implies that for all $n \geq 1$ and all $z \geq w>0$ it is the case that

$$
\begin{equation*}
c_{n}^{(N)} \leq \frac{z^{n}}{w^{2 n}}\left(\frac{\chi_{N}(w)}{n+1}\right)^{2} \tag{2.5}
\end{equation*}
$$

With the choices $z=z_{N}$ and $w=\frac{n}{n+1} z_{N}$, and with the upper bound $\chi_{N}(w) \leq 2\left|1-w / z_{N}\right|^{-1}$ of Proposition 2.1, the upper bound of Theorem 1.1 follows from (2.5) and without the need to consider the derivative $\chi_{N}^{\prime}$ nor to consider complex $z$. However the application of Lemma 2.2 cannot be replaced by [34, Lemma 3.4] in Section 2.2 because the generating function used in that application is not for a submultiplicative sequence, and also (2.5) fails to provide sharp powers of $n$ for generating functions that diverge faster than linearly.

### 2.2 Asymptotic formula: proof of Theorem 1.2

### 2.2.1 Extended version of Theorem 1.2

The following theorem, whose statement is not limited to $n \leq V^{p}$ as in Theorem 1.2, implies Theorem 1.2.
Theorem 2.3. There exists $\lambda_{0}>0$ such that with $\mu_{N}=\mu_{N}(\lambda)$ defined by (1.3) for any $\lambda \in\left(0, \lambda_{0}\right]$, with any choice of $p \in\left(0, \frac{1}{2}\right)$ and $a \in(0,1)$, and for all $n, N \in \mathbb{N}$ (with $\lambda V^{1 / 2} \geq 1$ ),

$$
\begin{equation*}
c_{n}^{(N)}=A_{N} \mu_{N}^{n}\left[1+O\left(n^{-a}\left(N^{-1}+V^{(2+a) p-1}\right)\right)\right]\left[1+O\left(V^{-p}\right)\right]^{n} . \tag{2.6}
\end{equation*}
$$

The sequence $A_{N}$ is independent of $n$ (but depends on $\lambda$ and $p$ ) and obeys $A_{N}=1+O\left(N^{-1}\right)$. The constants in error terms depend on $p, a$ and $\lambda$.

Theorem 2.3 has most significance for the largest values of $n$ which give a small error, so $n \leq V^{p}$ for $p$ close to $\frac{1}{2}$. To understand this, consider first the factor $\left[1+O\left(V^{-p}\right)\right]^{n}$, which is bounded for $n \leq V^{p}$ but is not close to 1 when $n=V^{p}$. However when $n \leq V^{p}$ we can also apply Theorem 2.3 for any choice of $p^{\prime} \in\left(p, \frac{1}{2}\right)$ and in this case $V^{-p^{\prime}}=\left(V^{-p}\right)^{p^{\prime} / p} \leq n^{-p^{\prime} / p}$ and hence

$$
\begin{equation*}
\left[1+O\left(V^{-p^{\prime}}\right)\right]^{n}=1+O\left(n V^{-p^{\prime}}\right) \leq 1+O\left(n n^{-p^{\prime} / p}\right)=1+O\left(n^{-\left(p^{\prime}-p\right) / p}\right) \tag{2.7}
\end{equation*}
$$

Also, given any $p^{\prime} \in\left(\frac{1}{3}, \frac{1}{2}\right)$, we can choose $a=\frac{1-2 p^{\prime}}{p^{\prime}} \in(0,1)$ in which case $V^{(2+a) p^{\prime}-1}=1$. Thus, (2.6) can be simplified in this case of $n \leq V^{p}$ as (with $p^{\prime}$ and $a$ as above)

$$
\begin{equation*}
c_{n}^{(N)}=A_{N} \mu_{N}^{n}\left[1+O\left(n^{-a}\right)+O\left(n^{-\left(p^{\prime}-p\right) / p}\right)\right] \quad\left(n \leq V^{p}\right) \tag{2.8}
\end{equation*}
$$

Therefore, as long as $n \leq V^{p}$ for some $p<\frac{1}{2}$, the leading asymptotic behaviour of $c_{n}^{(N)}$ is $A_{N} \mu_{N}^{n}$ and hence $\mu_{N}$ is the exponential growth rate in this regime. In this way, Theorem 2.3 implies Theorem 1.2. We will therefore prove Theorem 2.3. It suffices to consider $N$ large in the proof, since (2.6) holds for any finite set of $(n, N)$ by adjusting the constants.

### 2.2.2 Proof of Theorem 2.3

The proof of Theorem 2.3 also uses Lemma 2.2, but for this it is necessary to extract leading behaviour and then apply the Tauberian theorem to bound the remainder term. This requires an extension of Proposition 2.1 in which the linear part of $F_{N}$ is extracted with a higher-order remainder. In this section,
we reduce the proof of Theorem 2.3 to Propositions 2.4-2.5, which are proved in Section 6 using the lace expansion. We always assume that $N$ is large enough that $\lambda V^{1 / 2} \geq 1$ so that $z_{N}(\lambda)$ is well defined.

To extract the linear term, our method gives useful results only if we restrict $z$ to a smaller disk than the disk $|z| \leq z_{N}$ of Proposition 2.1. Thus, for $p>0$, we define $\zeta_{p}=\zeta_{p}(N, \lambda)>0$ by

$$
\begin{equation*}
\zeta_{p}=z_{N}(\lambda)\left(1-V^{-p}\right), \tag{2.9}
\end{equation*}
$$

and we will work in the disk $|z| \leq \zeta_{p}$. It will be necessary to restrict to $p \in\left(0, \frac{1}{2}\right)$. The linear approximation to $F_{N}(z)$ near $\zeta_{p}$ is the linear function

$$
\begin{equation*}
\Phi_{N}(z)=F_{N}\left(\zeta_{p}\right)+F_{N}^{\prime}\left(\zeta_{p}\right)\left(z-\zeta_{p}\right), \tag{2.10}
\end{equation*}
$$

with remainder

$$
\begin{equation*}
R_{N}(z)=F_{N}(z)-\Phi_{N}(z) . \tag{2.11}
\end{equation*}
$$

Thus we have

$$
\begin{equation*}
\chi_{N}(z)=\frac{1}{F_{N}(z)}=\frac{1}{\Phi_{N}(z)}+H_{N}(z), \quad H_{N}(z)=-\frac{R_{N}(z)}{\Phi_{N}(z) F_{N}(z)} \tag{2.12}
\end{equation*}
$$

We write the coefficients of the power series representations of $1 / \Phi_{N}(z)$ and $H(z)$ as

$$
\begin{equation*}
\frac{1}{\Phi_{N}(z)}=\sum_{n=0}^{\infty} \varphi_{n} z^{n}, \quad H_{N}(z)=\sum_{n=0}^{\infty} h_{n} z^{n} \tag{2.13}
\end{equation*}
$$

Both $\varphi_{n}$ and $h_{n}$ depend on $N$. By definition,

$$
\begin{equation*}
c_{n}^{(N)}=\varphi_{n}+h_{n} . \tag{2.14}
\end{equation*}
$$

The next proposition provides what is needed for good estimates on the linear approximation $\Phi_{N}$ to $F_{N}$.
Proposition 2.4. There is a $\lambda_{0}>0$ such that for any $\lambda \in\left(0, \lambda_{0}\right]$, for any $p \in\left(0, \frac{1}{2}\right)$, with $\zeta_{p}=$ $z_{N}(\lambda)\left(1-V^{-p}\right)$, and with $\lambda$-dependent error bounds,

$$
\begin{equation*}
\zeta_{p}=N^{-1}\left[1+O\left(N^{-1}\right)\right], \quad F_{N}\left(\zeta_{p}\right) \asymp V^{-p}, \quad F_{N}^{\prime}\left(\zeta_{p}\right)=-N+O(1) \tag{2.15}
\end{equation*}
$$

The next proposition gives a bound on the remainder term $R_{N}(z)$ and its derivative in the disk $|z| \leq \zeta_{p}$ in the complex plane.

Proposition 2.5. There is a $\lambda_{0}>0$ such that for any $\lambda \in\left(0, \lambda_{0}\right]$, any $p \in\left(0, \frac{1}{2}\right)$, any $a \in(0,1)$, any $z \in \mathbb{C}$ with $|z| \leq \zeta_{p}=z_{N}(\lambda)\left(1-V^{-p}\right)$, and with $\lambda$-dependent error bounds,

$$
\begin{align*}
& \left|R_{N}(z)\right| \prec N^{-1}\left(1+N V^{(2+a) p-1}\right)\left|1-z / \zeta_{p}\right|^{1+a},  \tag{2.16}\\
& \left|R_{N}^{\prime}(z)\right| \prec\left(1+N V^{(2+a) p-1}\right)\left|1-z / \zeta_{p}\right|^{a} . \tag{2.17}
\end{align*}
$$

An indication of deterioration for $p \geq \frac{1}{2}$ can be seen from the term $V^{(2+a) p-1}$ in (2.16). We desire a remainder $R_{N}$ of higher order than linear, so $a>0$, and when $p \geq \frac{1}{2}$ the term $V^{(2+a) p-1}$ grows exponentially in $N$ and spoils control unless $a \leq 0$ which we do not permit.

We prove Theorem 2.3 by using the Tauberian theorem Lemma 2.2 in conjunction with Propositions $2.4-2.5$. To prepare for this we have the following two corollaries of the above propositions. The first corollary is for the leading behaviour of $c_{n}^{(N)}$.

Corollary 2.6. With $z_{N}=z_{N}(\lambda)$ for $\lambda \in\left(0, \lambda_{0}\right]$, for all $p \in\left(0, \frac{1}{2}\right)$, all $N$ sufficiently large, and all $n \in \mathbb{N}$, the coefficient $\varphi_{n}$ of $z^{n}$ in $1 / \Phi_{N}(z)$ obeys

$$
\begin{equation*}
\varphi_{n}=A_{N} \mu_{N}^{n}\left(1+O\left(V^{-p}\right)\right)^{n} \tag{2.18}
\end{equation*}
$$

with $A_{N}=1+O\left(N^{-1}\right)$ independent of $n$ (but dependent on $p$ and $\lambda$ ), and with $\lambda$-dependent error bounds. Proof. We define

$$
\begin{equation*}
\alpha_{N}=F_{N}\left(\zeta_{p}\right)-\zeta_{p} F_{N}^{\prime}\left(\zeta_{p}\right), \quad \beta_{N}=-F_{N}^{\prime}\left(\zeta_{p}\right) \tag{2.19}
\end{equation*}
$$

which are both positive for large $N$ since then $F_{N}^{\prime}\left(\zeta_{p}\right)$ is negative by Proposition 2.4. By definition, $\Phi(z)=\alpha_{N}-\beta_{N} z$, so expansion of the geometric series gives

$$
\begin{equation*}
\frac{1}{\Phi_{N}(z)}=\frac{1}{\alpha_{N}} \sum_{n=0}^{\infty}\left(\frac{\beta_{N} z}{\alpha_{N}}\right)^{n} \quad\left(|z|<\alpha_{N} / \beta_{N}\right) . \tag{2.20}
\end{equation*}
$$

Let $A_{N}=1 / \alpha_{N}$, so $A_{N}$ depends on $\lambda$ and $p$. Then $A_{N}=1+O\left(N^{-1}\right)$ by Proposition 2.4 and we have

$$
\begin{equation*}
\varphi_{n}=A_{N}\left(\frac{\beta_{N}}{\alpha_{N}}\right)^{n} \tag{2.21}
\end{equation*}
$$

By definition and by Proposition 2.4,

$$
\begin{align*}
\frac{\beta_{N}}{\alpha_{N}} & =\mu_{N} \frac{z_{N} \beta_{N}}{\alpha_{N}}=\mu_{N} \frac{1}{1-V^{-p}} \frac{\zeta_{p} \beta_{N}}{\alpha_{N}} \\
& =\mu_{N} \frac{1}{1-V^{-p}} \frac{1}{1+\frac{F_{N}\left(\zeta_{p}\right)}{-\zeta_{p} F_{N}^{\prime}\left(\zeta_{p}\right)}}=\mu_{N}\left(1+O\left(V^{-p}\right)\right) . \tag{2.22}
\end{align*}
$$

This gives the desired result

$$
\begin{equation*}
\varphi_{n}=A_{N}\left(\frac{\beta_{N}}{\alpha_{N}}\right)^{n}=A_{N} \mu_{N}^{n}\left(1+O\left(V^{-p}\right)\right)^{n} \tag{2.23}
\end{equation*}
$$

and the proof is complete.
To prove Theorem 2.3, it now suffices to prove that

$$
\begin{equation*}
h_{n} \mu_{N}^{-n}=O\left(n^{-a}\left(N^{-1}+V^{(2+a) p-1}\right)\right)\left(1+O\left(V^{-p}\right)\right)^{n} . \tag{2.24}
\end{equation*}
$$

To do so, we will use the following corollary of Propositions 2.1 and 2.4-2.5.
Corollary 2.7. With $z_{N}=z_{N}(\lambda)$ for $\lambda \in\left(0, \lambda_{0}\right]$, and for all $p \in\left(0, \frac{1}{2}\right)$, all $a \in(0,1)$, and all $z \in \mathbb{C}$ with $|z| \leq \zeta_{p}$,

$$
\begin{equation*}
\left|H_{N}^{\prime}(z)\right| \prec \frac{1+N V^{(2+a) p-1}}{\left|1-z / \zeta_{p}\right|^{2-a}} \tag{2.25}
\end{equation*}
$$

Proof. Let $|z| \leq \zeta_{p}$. By definition,

$$
\begin{equation*}
H_{N}^{\prime}(z)=-\frac{R_{N}^{\prime}(z)}{F_{N}(z) \Phi_{N}(z)}+\frac{R_{N}(z) F_{N}^{\prime}(z)}{F_{N}(z)^{2} \Phi_{N}(z)}-\frac{\beta_{N} R_{N}(z)}{F_{N}(z) \Phi_{N}(z)^{2}} \tag{2.26}
\end{equation*}
$$

To bound the denominators of (2.26) we proceed as follows. With the notation from the proof of Corollary 2.6, it follows from the facts that $\beta_{N} \zeta_{p} \sim 1$ and

$$
\begin{equation*}
\frac{\alpha_{N}}{\zeta_{p} \beta_{N}}=1+\frac{F_{N}\left(\zeta_{p}\right)}{-\zeta_{p} F_{N}^{\prime}\left(\zeta_{p}\right)} \geq 1 \tag{2.27}
\end{equation*}
$$

that for $|z| \leq \zeta_{p}$ and for large $N$ we have

$$
\begin{equation*}
\left|\Phi_{N}(z)\right|=\left|\alpha_{N}-\beta_{N} z\right|=\zeta_{p} \beta_{N}\left|\frac{\alpha_{N}}{\zeta_{p} \beta_{N}}-\frac{z}{\zeta_{p}}\right| \geq \frac{1}{2}\left|1-z / \zeta_{p}\right|, \tag{2.28}
\end{equation*}
$$

where in the last inequality we used the geometric fact that if $a>1$ and $|w| \leq 1$ then $|a-w| \geq|1-w|$. Similarly, it follows from the linear lower bound on $F_{N}$ from Proposition 2.1 that on the disk $|z| \leq \zeta_{p}$ we have

$$
\begin{equation*}
\left|F_{N}(z)\right| \succ z_{N}^{-1}\left|z_{N}-z\right| \geq z_{N}^{-1}\left|\zeta_{p}-z\right| \succ\left|1-z / \zeta_{p}\right| \tag{2.29}
\end{equation*}
$$

where we used $z_{N}>\zeta_{p}$ for the second inequality and $z_{N} \sim \zeta_{p}$ for the third. Proposition 2.1 also gives $\left|F_{N}^{\prime}(z)\right| \prec N$. Therefore, by (2.26), (2.28), (2.29), $\beta_{N} \leq 2 N$, and Proposition 2.5,

$$
\begin{align*}
\left|H_{N}^{\prime}(z)\right| & \prec \frac{\left|R_{N}^{\prime}(z)\right|}{\left|1-z / \zeta_{p}\right|^{2}}+\frac{N\left|R_{N}(z)\right|}{\left|1-z / \zeta_{p}\right|^{3}} \\
& \prec \frac{1+N V^{(2+a) p-1}}{\left|1-z / \zeta_{p}\right|^{2-a}}+\frac{1+N V^{(2+a) p-1}}{\left|1-z / \zeta_{p}\right|^{2-a}}, \tag{2.30}
\end{align*}
$$

and the proof is complete.
We now apply Lemma 2.2 to prove Theorem 2.3. Corollary 2.7 is formulated for $H^{\prime}$, rather than for $H$, for the reason mentioned in the first remark of Section 2.1.2.

Proof of Theorem 2.3. The combination of Corollary 2.7 with Lemma 2.2 (with $f=H^{\prime}, \rho=\zeta_{p}, b=2-a$ ) immediately gives

$$
\begin{equation*}
(n+1)\left|h_{n+1}\right| \prec n^{2-a-1} \zeta_{p}^{-n}\left(1+N V^{(2+a) p-1}\right) . \tag{2.31}
\end{equation*}
$$

With $\zeta_{p} \sim N^{-1}$, this implies that

$$
\begin{equation*}
\left|h_{n}\right| \prec n^{-a} \zeta_{p}^{-(n-1)}\left(1+N V^{(2+a) p-1}\right) \prec n^{-a} \zeta_{p}^{-n}\left(N^{-1}+V^{(2+a) p-1}\right) . \tag{2.32}
\end{equation*}
$$

It suffices now to observe that

$$
\begin{equation*}
\left(\mu_{N} \zeta_{p}\right)^{-n}=\left(1-V^{-p}\right)^{-n} . \tag{2.33}
\end{equation*}
$$

This proves (2.24) and therefore completes the proof.
Thus to prove Theorem 2.3 (and thereby prove Theorem 1.2) it suffices to prove Propositions 2.4-2.5. The proofs of Propositions 2.1, 2.4 and 2.5, as well as of Theorems 1.3-1.5, are based on the lace expansion for self-avoiding walk, which we discuss next.

## 3 The lace expansion

In this section, we summarise the derivation of the lace expansion as well as its diagrammatic estimates. More extensive treatments can be found in the original paper by Brydges and Spencer [11] or in the books [37,45]. The setting in those references is $\mathbb{Z}^{d}$ rather than the hypercube but the differences for the derivation of the expansion and for its diagrammatic estimates in these two settings are merely superficial. Although we do not adopt this perspective here, the lace expansion can alternatively be understood as arising from repeated application of the inclusion-exclusion relation (see [37, Section 5.1]).

### 3.1 Fourier transform on the hypercube

The proofs of Propositions 2.1, 2.4 and 2.5 rely heavily on Fourier transformation on the hypercube. Given a function $f: \mathbb{Q}^{N} \rightarrow \mathbb{C}$, its Fourier transform is

$$
\begin{equation*}
\hat{f}(k)=\sum_{x \in \mathbb{Q}^{N}} f(x)(-1)^{k \cdot x} \quad\left(k \in \mathbb{Q}^{N}\right), \tag{3.1}
\end{equation*}
$$

where the dot product is defined by $k \cdot x=\sum_{i=1}^{N} k_{i} x_{i}$ with $k_{i}$ and $x_{i}$ respectively the $i^{\text {th }}$ components of $k$ and $x$. The inverse Fourier transform is

$$
\begin{equation*}
f(x)=\frac{1}{V} \sum_{k \in \mathbb{Q}^{N}} \hat{f}(k)(-1)^{k \cdot x} \quad\left(x \in \mathbb{Q}^{N}\right) . \tag{3.2}
\end{equation*}
$$

The convolution $(f * g)(x)=\sum_{y \in \mathbb{Q}^{N}} f(x-y) g(y)$ obeys $\widehat{f * g}=\hat{f} \hat{g}$.
An important example is when $f$ is the transition probability $D$ for simple random walk, defined by

$$
D(x)=\left\{\begin{array}{ll}
N^{-1} & |x|=1  \tag{3.3}\\
0 & |x| \neq 1
\end{array} \quad\left(x \in \mathbb{Q}^{N}\right) .\right.
$$

Its Fourier transform is

$$
\begin{equation*}
\hat{D}(k)=1-\frac{2|k|}{N} \quad\left(k \in \mathbb{Q}^{N}\right) . \tag{3.4}
\end{equation*}
$$

### 3.2 The recursion relation

Let $c_{0}(x)=\delta_{0, x}$, and, for $n \geq 1$, let $c_{n}(x)$ denote the number of $n$-step self-avoiding walks that begin at the origin and end at $x \in \mathbb{Q}^{N}$. The two-point function is the generating function for the sequence $c_{n}^{(N)}(x)$, defined by

$$
\begin{equation*}
G_{z}(x)=\sum_{n=0}^{\infty} c_{n}^{(N)}(x) z^{n} \quad\left(x \in \mathbb{Q}^{N}, z \in \mathbb{C}\right) . \tag{3.5}
\end{equation*}
$$

Since $c_{n}(x)=0$ for all $n \geq V$, the two-point function is a polynomial in $z$.
For $m \geq 2$, the lace expansion produces a function $\pi_{m}: \mathbb{Q}^{N} \rightarrow \mathbb{Z}$, which we will define below. We write its generating function, which is not a polynomial, as $\Pi_{z}(x)=\sum_{m=2}^{\infty} \pi_{m}(x) z^{m}$. The following proposition is a statement of the lace expansion. The detailed derivation of the formulas in Proposition 3.1 can be found in $[45,(3.14),(3.27)]$, following the original proof in [11]. Although these references are for $\mathbb{Z}^{d}$, the discussion in [45] applies verbatim to the hypercube, or indeed to any finite or infinite transitive graph after suitable adaptation of the transition function $D$. Each of $D, G_{z}, \pi_{m}, \Pi_{z}$ depends on $N$ but to lighten the notation we do not make this explicit.

Proposition 3.1. For $n \geq 1$ and for $x \in \mathbb{Q}^{N}$,

$$
\begin{equation*}
c_{n}^{(N)}(x)=N\left(D * c_{n-1}^{(N)}\right)(x)+\sum_{m=2}^{n}\left(\pi_{m} * c_{n-m}^{(N)}\right)(x), \tag{3.6}
\end{equation*}
$$

and hence, for $z \in \mathbb{C}$ such that $\Pi_{z}(x)$ converges for all $x$,

$$
\begin{equation*}
G_{z}(x)=\delta_{0, x}+z N\left(D * G_{z}\right)(x)+\left(\Pi_{z} * G_{z}\right)(x) \tag{3.7}
\end{equation*}
$$



Figure 1: Laces in $\mathcal{L}^{(M)}[0, m]$ for $M=1,2,3,4$, with $s_{1}=0$ and $t_{M}=m$.

Consequently,

$$
\begin{equation*}
\hat{G}_{z}(k)=1+z N \hat{D}(k) \hat{G}_{z}(k)+\hat{\Pi}_{z}(k) \hat{G}_{z}(k) . \tag{3.8}
\end{equation*}
$$

This can be rewritten as

$$
\begin{equation*}
\hat{G}_{z}(k)=\frac{1}{1-z N \hat{D}(k)-\hat{\Pi}_{z}(k)} . \tag{3.9}
\end{equation*}
$$

Since the susceptibility is equal to $\chi_{N}(z)=\hat{G}_{z}(0)$, we obtain the identity

$$
\begin{equation*}
\chi_{N}(z)=\frac{1}{F_{N}(z)}=\frac{1}{1-z N-\hat{\Pi}_{z}(0)} \tag{3.10}
\end{equation*}
$$

which is central to the proof of our main results Theorems 1.1-1.5. In order to make use of (3.9) and (3.10), it will be necessary to obtain good estimates on $\hat{\Pi}_{z}(k)$. These will be achieved via diagrammatic estimates in Section 3.5, where the convergence of $\Pi_{z}(x)$ will be studied.

### 3.3 Graphs and laces

The derivation of (3.6) uses the following definitions. More detailed discussion and interpretation of these definitions can be found in [45, Section 3.3].

Definition 3.2. (i) Given an interval $I=[a, b]$ of positive integers, an edge is a pair $\{s, t\}$ of elements of $I$, often written st (with $s<t$ ). A set of edges (possibly the empty set) is called a graph. Let $\mathcal{B}[a, b]$ denote the set of all graphs.
(ii) A graph $\Gamma$ is connected if both $a$ and $b$ are endpoints of edges in $\Gamma$, and if in addition, for any $c \in(a, b)$, there is an edge $s t \in \Gamma$ such that $s<c<t$. Let $\mathcal{G}[a, b]$ denote the set of all connected graphs on $[a, b]$.
(iii) A lace is a minimally connected graph: a connected graph for which the removal of any edge would result in a disconnected graph. The set of laces on $[a, b]$ is denoted by $\mathcal{L}[a, b]$, and the set of laces on $[a, b]$ which consist of exactly $M$ edges is denoted $\mathcal{L}^{(M)}[a, b]$. Figure 1 shows laces in $\mathcal{L}^{(M)}[0, m]$ for $M=1,2,3,4$.

The above definition of connectivity is not the usual notion of path-connectivity in graph theory. Instead, connected graphs are those $\Gamma$ for which the union of open real intervals $\cup_{s t \in \Gamma}(s, t)$ is equal to the connected interval $(a, b)$. This is the useful concept of connectivity for the lace expansion.

A lace $L \in \mathcal{L}^{(M)}[a, b]$ can be written by listing its edges as $L=\left\{s_{1} t_{1}, \ldots, s_{M} t_{M}\right\}$, with $s_{l}<t_{l}$ for each $l$. For $M=1$, there is a unique lace and $a=s_{1}<t_{1}=b$. For $M \geq 2$, a graph is a lace $L \in \mathcal{L}^{(M)}[a, b]$ if and only if its edge endpoints can be ordered as

$$
\begin{equation*}
a=s_{1}<s_{2}, \quad s_{l+1}<t_{l} \leq s_{l+2} \quad(l=1, \ldots, M-2), \quad s_{M}<t_{M-1}<t_{M}=b \tag{3.11}
\end{equation*}
$$

(for $M=2$ the middle inequalities are absent). Thus $L$ divides $[a, b]$ into $2 M-1$ subintervals:

$$
\begin{equation*}
\left[s_{1}, s_{2}\right],\left[s_{2}, t_{1}\right],\left[t_{1}, s_{3}\right],\left[s_{3}, t_{2}\right], \ldots,\left[s_{M}, t_{M-1}\right],\left[t_{M-1}, t_{M}\right] \tag{3.12}
\end{equation*}
$$

Of these, intervals number $3,5, \ldots,(2 M-3)$ can have zero length for $M \geq 3$, whereas all others have length at least 1. This last fact will be important, as intervals which cannot have zero length yield good factors for convergence of the lace expansion.

Definition 3.3. Given a connected graph $\Gamma$ on $[a, b]$, the following prescription associates to $\Gamma$ a lace $\mathrm{L}_{\Gamma} \subset \Gamma$ : The lace $\mathrm{L}_{\Gamma}$ consists of edges $s_{1} t_{1}, s_{2} t_{2}, \ldots$, with $t_{1}, s_{1}, t_{2}, s_{2}, \ldots$ determined, in that order, by

$$
\begin{gathered}
t_{1}=\max \{t: a t \in \Gamma\}, \quad s_{1}=a \\
t_{i+1}=\max \left\{t: \exists s<t_{i} \text { such that st } \in \Gamma\right\}, \quad s_{i+1}=\min \left\{s: s t_{i+1} \in \Gamma\right\}
\end{gathered}
$$

The procedure terminates when $t_{i+1}=b$. Given a lace $L$, the set of all edges st $\notin L$ such that $L_{L \cup\{s t\}}=L$ is denoted $\mathcal{C}(L)$. Edges in $\mathcal{C}(L)$ are said to be compatible with $L$.

Given a lace $L$ and the closed intervals (3.12) it determines, any edge st with each of $s, t$ lying in the same one of those closed intervals is a compatible bond in $\mathcal{C}(L)$.

### 3.4 Definition of $\pi_{m}(x)$

For $m \geq 1$ and $x \in \mathbb{Q}^{N}$, let $\mathcal{W}_{m}(x)$ denote the set of all $m$-step walks $\omega=(\omega(0), \omega(1), \ldots, \omega(m))$ on $\mathbb{Q}^{N}$ (possibly self-intersecting), with $|\omega(i)-\omega(i-1)|=1$ for $i=1, \ldots, m$, and with $\omega(0)=0$ and $\omega(m)=x$. Given $\omega \in \mathcal{W}_{m}(x)$, let

$$
U_{s t}(\omega)=\left\{\begin{align*}
-1 & \text { if } \omega(s)=\omega(t)  \tag{3.13}\\
0 & \text { if } \omega(s) \neq \omega(t)
\end{align*}\right.
$$

Then

$$
\begin{equation*}
c_{n}(x)=\sum_{\omega \in \mathcal{W}_{n}(x)} \prod_{0 \leq s<t \leq n}\left(1+U_{s t}(\omega)\right) \tag{3.14}
\end{equation*}
$$

since the product is equal to 1 if $\omega$ is a self-avoiding walk and is equal to 0 otherwise. By expanding the product in (3.14) we obtain

$$
\begin{equation*}
c_{n}(x)=\sum_{\omega \in \mathcal{W}_{n}(x)} \sum_{\Gamma \in \mathcal{B}[0, n]} \prod_{s t \in \Gamma} U_{s t}(\omega) \tag{3.15}
\end{equation*}
$$

In the sum over all graphs $\Gamma$ in (3.15), we partition according to whether:

$$
\text { (a) } 0 \text { does not occur in an edge in } \Gamma, \quad \text { or } \quad \text { (b) } 0 \text { does occur in an edge in } \Gamma \text {. }
$$

This gives the identity (3.6) (see [45, p. 22] for details), namely

$$
\begin{equation*}
c_{n}(x)=N\left(D * c_{n-1}\right)(x)+\sum_{m=2}^{n}\left(\pi_{m} * c_{n-m}\right)(x) \tag{3.16}
\end{equation*}
$$

with, for $m \geq 2$,

$$
\begin{equation*}
\pi_{m}(x)=\sum_{\omega \in \mathcal{W}_{m}(x)} \sum_{\Gamma \in \mathcal{G}[0, m]} \prod_{s t \in \Gamma} U_{s t}(\omega) \tag{3.17}
\end{equation*}
$$

Indeed, Case (a) gives rise to the first term on the right-hand side of (3.16), and Case (b) gives rise to the second term with $[0, m]$ the support of the connected component of $\Gamma$ containing 0 .

## 

Figure 2: Self-intersections required for a walk $\omega$ with $\prod_{s t \in L} U_{s t}(\omega) \neq 0$, for the laces with $M=1,2,3,4$ edges depicted in Figure 1. The configuration for $M=11$ is also shown. A slashed subwalk may have length zero, whereas subwalks which are not slashed must take at least one step.

The sum over connected graphs can be reorganised by summing over laces $L$ and over connected graphs for which the prescription of Definition 3.3 produces $L$. Then a resummation of the sum over those connected graphs leads to the formula

$$
\begin{equation*}
\pi_{m}(x)=\sum_{\omega \in \mathcal{W}_{m}(x)} \sum_{L \in \mathcal{L}[0, m]} \prod_{s t \in L} U_{s t}(\omega) \prod_{s^{\prime} t^{\prime} \in \mathcal{C}(L)}\left(1+U_{s^{\prime} t^{\prime}}(\omega)\right) \tag{3.18}
\end{equation*}
$$

More details of this resummation can be found in [45, Section 3.3] or in either of [11,37]. The formula (3.18) is the useful formula for application of Proposition 3.1.

A refinement of (3.18) is obtained by restricting the sum in (3.18) to laces with $M$ edges, and we define

$$
\begin{equation*}
\pi_{m}^{(M)}(x)=\sum_{\omega \in \mathcal{W}_{m}(x)} \sum_{L \in \mathcal{L}^{(M)}[0, m]} \prod_{s t \in L}\left(-U_{s t}(\omega)\right) \prod_{s^{\prime} t^{\prime} \in \mathcal{C}(L)}\left(1+U_{s^{\prime} t^{\prime}}(\omega)\right) . \tag{3.19}
\end{equation*}
$$

The minus sign has been introduced in the first product of (3.19) in order to make $\pi_{m}^{(M)}(x)$ a nonnegative integer. The right-hand side of (3.19) is zero unless $M<m$ (since $U_{s t}(\omega)=0$ if $t=s+1$ and the subset of $\mathcal{L}^{(M)}[0, m]$ consisting of laces with all edges of length at least two is empty if $M \geq m$ ), and hence

$$
\begin{equation*}
\pi_{m}(x)=\sum_{M=1}^{m-1}(-1)^{M} \pi_{m}^{(M)}(x) \tag{3.20}
\end{equation*}
$$

Each term in the double sum (3.19) is either 0 or 1 , with the first product in (3.19) equal to 1 if and only if $\omega(s)=\omega(t)$ for each edge $s t \in L$, while the second product is equal to 1 if and only if $\omega\left(s^{\prime}\right) \neq \omega\left(t^{\prime}\right)$ for each $s^{\prime} t^{\prime} \in \mathcal{C}(L)$. Thus $\pi_{m}^{(M)}(x)$ counts the $m$-step "lace graphs" starting at the origin and ending at $x$, with the specific self-intersections that are enforced by the lace and with the specific self-avoidance conditions enforced by the compatible edges. The required self-intersections are illustrated in Figure 2. For $M \geq 1$, the generating function of $\pi_{m}^{(M)}(x)$ is written as

$$
\begin{equation*}
\Pi_{z}^{(M)}(x)=\sum_{m=2}^{\infty} \pi_{m}^{(M)}(x) z^{m} \tag{3.21}
\end{equation*}
$$

The simplest term is $\pi_{m}^{(1)}(x)$, which is zero if $x \neq 0$. Since every edge except $0 m$ is compatible with the unique 1-edge lace $L=0 m, \pi_{m}^{(1)}(x)$ is the number of $m$-step self-avoiding returns to the origin when $x=0$. Thus $\pi_{m}^{(1)}(0)$ is simply equal to $\sum_{i=1}^{N} c_{m-1}\left(e_{i}\right)$, where here the unit vector $e_{i}$ represents the penultimate vertex visited by the self-avoiding return before it takes its final step to the origin. Its generating function is therefore

$$
\begin{equation*}
\Pi_{z}^{(1)}(x)=\delta_{0, x} \sum_{m=2}^{\infty} \pi_{m}^{(1)}(x) z^{m}=\delta_{0, x} z N\left(D * G_{z}\right)(0) . \tag{3.22}
\end{equation*}
$$

For $M \geq 2, \pi_{m}^{(M)}(x)$ counts $m$-step $M$-loop walk configurations as indicated in Figure 2. The number of loops in a diagram is equal to the number of edges in the corresponding lace. Each of the $2 M-1$
subwalks in a diagram is self-avoiding due to the compatible edges. The compatible edges also enforce specific mutual avoidances between subwalks, which can be neglected in upper bounds but which must be taken into account to compute the coefficients $a_{n}$ in the asymptotic expansion of Theorem 1.3 (we return to this point in Section 7.4). The slashed lines in Figure 2 correspond to subwalks which may consist of zero steps, but the others correspond to subwalks consisting of at least one step (recall the discussion below (3.12)).

As an example of how to estimate $\Pi_{z}^{(M)}(x)$, we consider the case $M=2$ in further detail. A walk giving a contribution to $\pi_{m}^{(2)}(x)$ must travel from 0 to $x$, then back to 0 , and then finally return to $x$, as in the two-loop diagram in Figure 2. Due to the product over compatible bonds in (3.19), each of these three subwalks must itself be self-avoiding, and $x$ cannot equal 0 . By relaxing the avoidance between the three subwalks we obtain an upper bound

$$
\begin{equation*}
\pi_{m}^{(2)}(x) \leq \sum_{\substack{m_{1}+m_{2}+m_{3}=m \\ m_{1}, m_{2}, m_{3} \geq 1}} c_{m_{1}}(x) c_{m_{2}}(x) c_{m_{3}}(x) . \tag{3.23}
\end{equation*}
$$

We define the generating function

$$
\begin{equation*}
H_{z}(x)=G_{z}(x)-\delta_{0, x}=\sum_{n=1}^{\infty} c_{n}^{(N)}(x) z^{n} \tag{3.24}
\end{equation*}
$$

for the sequence $c_{n}^{(N)}(x)$ with its $n=0$ term omitted. The generating function for $\pi_{m}^{(2)}(x)$ converts the convolution in (3.23) into a product, so that (since $\pi_{m}^{(2)}(x)=0$ when $x=0$ )

$$
\begin{equation*}
\Pi_{z}^{(2)}(x) \leq H_{z}(x)^{3} . \tag{3.25}
\end{equation*}
$$

We can then estimate the sum over $x$ of $\Pi_{z}^{(2)}(x)$ using

$$
\begin{equation*}
\sum_{x \in \mathbb{Q}^{N}} \Pi_{z}^{(2)}(x) \leq \sum_{x \in \mathbb{Q}^{N}} H_{z}(x)^{3}=\left(H_{z} * H_{z}^{2}\right)(0) \leq\left(G_{z} * H_{z}^{2}\right)(0) . \tag{3.26}
\end{equation*}
$$

This is the $M=2$ version of the inequality (3.31) that will appear below. The right-hand side of (3.26) can be further estimated as

$$
\begin{equation*}
\sum_{x \in \mathbb{Q}^{N}} \Pi_{z}^{(2)}(x) \leq\left\|H_{z}\right\|_{\infty}\left(G_{z} * H_{z}\right)(0) \leq\left\|H_{z}\right\|_{\infty}\left\|G_{z} * H_{z}\right\|_{\infty} \tag{3.27}
\end{equation*}
$$

which is the $M=2$ version of the inequality (3.37) that will appear below.

### 3.5 Diagrammatic estimates

We define the multiplication operator $\mathcal{M}_{z}$ and the convolution operator $\mathcal{G}_{z}$ by

$$
\begin{align*}
\left(\mathcal{M}_{z} f\right)(x) & =H_{z}(x) f(x)  \tag{3.28}\\
\left(\mathcal{G}_{z} f\right)(x) & =\left(G_{z} * f\right)(x), \tag{3.29}
\end{align*}
$$

for $f: \mathbb{Q}^{N} \rightarrow \mathbb{C}$ and $x \in \mathbb{Q}^{N}$. For such functions $f$, we use the norms $\|f\|_{\infty}=\max _{x \in \mathbb{Q}^{N}}|f(x)|$ and $\|f\|_{p}=\left[\sum_{x \in \mathbb{Q}^{N}}|f(x)|^{p}\right]^{1 / p}$ for $p \in[1, \infty)$.

A proof of the following diagrammatic estimate can be found at [45, (4.40)]. For $\Pi_{z}^{(1)}$ it is (3.22), since $G_{z}$ in (3.22) can be replaced by $H_{z}$ since $D(0)=0$. Although presented in [45] for $\mathbb{Z}^{d}$, the proof applies to the hypercube mutatis mutandis. Each of the $2 M+1$ factors on the right-hand side of (3.31) arises from one of the $2 M+1$ lines in the $M$-loop diagrams depicted in Figure 2.

Proposition 3.4. For $z \geq 0$,

$$
\begin{equation*}
\Pi_{z}^{(1)}(x)=\delta_{0, x} z N\left(D * H_{z}\right)(0) \tag{3.30}
\end{equation*}
$$

and for $M \geq 2$,

$$
\begin{equation*}
\left\|\Pi_{z}^{(M)}\right\|_{1} \leq\left[\left(\mathcal{G}_{z} \mathcal{M}_{z}\right)^{M-1} H_{z}\right](0) . \tag{3.31}
\end{equation*}
$$

Note that estimates for $z \geq 0$ as in Proposition 3.4 also imply bounds for complex $z \in \mathbb{C}$ via

$$
\begin{equation*}
\left|\Pi_{z}^{(M)}(x)\right| \leq \Pi_{|z|}^{(M)}(x) \tag{3.32}
\end{equation*}
$$

since $\pi_{m}^{(M)}(x) \geq 0$. The following lemma provides a way to bound the right-hand side of (3.31). For its elementary proof see, e.g., [45, Lemma 4.6]; the assumption there that the $f_{i}$ be even functions is vacuous for $\mathbb{Q}^{N}$ since $x=-x$ for all $x \in \mathbb{Q}^{N}$.

Lemma 3.5. Given nonnegative functions $f_{0}, f_{1}, \ldots, f_{2 q}$ on $\mathbb{Q}^{N}$, for $j=1, \ldots, q$ let $\mathcal{C}_{j}$ and $\mathcal{M}_{j}$ be the operators $\left(\mathcal{C}_{j} f\right)(x)=\left(f_{2 j} * f\right)(x)$ and $\left(\mathcal{M}_{j} f\right)(x)=f_{2 j-1}(x) f(x)$. Then for any $k \in\{0, \ldots, 2 q\}$,

$$
\begin{equation*}
\left\|\mathcal{C}_{q} \mathcal{M}_{q} \cdots \mathcal{C}_{1} \mathcal{M}_{1} f_{0}\right\|_{\infty} \leq\left\|f_{k}\right\|_{\infty} \prod\left\|f_{i} * f_{i^{\prime}}\right\|_{\infty} \tag{3.33}
\end{equation*}
$$

where the product is over disjoint consecutive pairs ii' taken from the set $\{0, \ldots, 2 q\} \backslash\{k\}$ (e.g., for $k=3$ and $q=3$, the product has factors with $i i^{\prime}$ equal to $01,24,56$ ).

Given a function $f: \mathbb{Q}^{N} \rightarrow \mathbb{C}$ and $k \in \mathbb{Q}^{N}$, we define $f_{k}: \mathbb{Q}^{N} \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
f_{k}(x)=\left(1-(-1)^{k \cdot x}\right) f(x) . \tag{3.34}
\end{equation*}
$$

Also, given a power series $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ and a real number $\epsilon>0$, we define the "fractional derivative"

$$
\begin{equation*}
\delta_{z}^{\epsilon} f(z)=\sum_{n=1}^{\infty} n^{\epsilon} a_{n} z^{n} \tag{3.35}
\end{equation*}
$$

For $\epsilon$ equal to a positive integer, $\delta_{z}^{\epsilon}$ does not give the usual derivative but gives instead $\left(z \partial_{z}\right)^{\epsilon}$.
The following proposition gives norm estimates for $\Pi_{z}$, for $\Pi_{z, k}$ (defined by taking $f=\Pi_{z}$ in (3.34)), and for fractional $z$-derivatives of $\Pi_{z}$. Its proof is a very minor modification of the proof of [45, Theorem 4.1] (which is inspired by [11]) to which we refer the interested reader for the somewhat lengthy details. Rather than repeating those details here, we instead illustrate the ideas in the proof of (3.38)-(3.39) by focussing on the cases $M=1,2$.

Proposition 3.6. Let $z \geq 0, k \in \mathbb{Q}^{N}$, and $\epsilon \geq 1$. For $M=1, \Pi_{z, k}^{(1)}(x)=0$ and

$$
\begin{equation*}
\left\|\Pi_{z}^{(1)}\right\|_{1} \leq z N\left\|H_{z}\right\|_{\infty}, \quad\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(1)}\right\|_{1} \leq N\left\|\delta_{z}^{\epsilon}\left(z H_{z}\right)\right\|_{\infty} \tag{3.36}
\end{equation*}
$$

For $M \geq 2$,

$$
\begin{align*}
\left\|\Pi_{z}^{(M)}\right\|_{1} & \leq\left\|H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1},  \tag{3.37}\\
\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(M)}\right\|_{1} & \leq(2 M-1)^{\epsilon}\left\|\delta_{z}^{\epsilon} H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1},  \tag{3.38}\\
\left\|\Pi_{z, k}^{(M)}\right\|_{1} & \leq\lfloor M / 2\rfloor\left\|H_{z, k}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1} . \tag{3.39}
\end{align*}
$$

Proof. Since $\Pi_{z}^{(1)}(x)=\delta_{0, x} z N\left(D * H_{z}\right)(0)$ by (3.30), it follows that $\Pi_{z, k}^{(1)}(x)=0$ as claimed, and also the second bound of (3.36) follows from the identity

$$
\begin{equation*}
\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(1)}\right\|_{1}=\delta_{z}^{\epsilon}\left[z N\left(D * H_{z}\right)(0)\right]=N \sum_{i=1}^{N} N^{-1} \delta_{z}^{\epsilon}\left[z H_{z}\left(e_{i}\right)\right], \tag{3.40}
\end{equation*}
$$

where the $e_{i}$ are the unit vectors in $\mathbb{Q}^{N}$. The first estimate of (3.36) follows similarly, with $\delta_{z}^{\epsilon}$ omitted.
We restrict attention now to $M \geq 2$. The bound (3.37) is a consequence of (3.31) and Lemma 3.5 (with $k=0$ ), since the right-hand side of (3.31) is bounded by the left-hand side of (3.33) with $f_{0}=H_{z}$ and with $f_{1}, \ldots, f_{2(M-1)}$ alternating between $H_{z}$ and $G_{z}$.

For (3.38), by definition,

$$
\begin{equation*}
\delta_{z}^{\epsilon} \Pi_{z}^{(M)}(x)=\sum_{m=2}^{\infty} m^{\epsilon} \pi_{m}^{(M)}(x) z^{m} \tag{3.41}
\end{equation*}
$$

In the diagrammatic representation, $\pi_{m}^{(M)}(x)$ is represented by a diagram with $2 M-1$ subwalks of total length $m$. Let $m_{i}$ be the length of the $i^{\text {th }}$ subwalk. By Hölder's inequality with exponents $\epsilon$ and $\frac{\epsilon}{\epsilon-1}$ (here is where the restriction $\epsilon \geq 1$ is convenient),

$$
\begin{equation*}
m^{\epsilon}=\left(\sum_{i=1}^{2 M-1} m_{i} \cdot 1\right)^{\epsilon} \leq(2 M-1)^{\epsilon-1} \sum_{i=1}^{2 M-1} m_{i}^{\epsilon} \tag{3.42}
\end{equation*}
$$

To see how this can be used in the simplest example, consider the case $M=2$. In this case, (3.23) and (3.42) give

$$
\begin{equation*}
m^{\epsilon} \pi_{m}^{(2)}(x) \leq 3^{\epsilon-1} \sum_{i=1}^{3} \sum_{\substack{m_{1}+m_{2}+m_{3}=m \\ m_{1}, m_{2}, m_{3} \geq 1}} m_{i}^{\epsilon} c_{m_{1}}(x) c_{m_{2}}(x) c_{m_{3}}(x) . \tag{3.43}
\end{equation*}
$$

We can bound the sum over $x$ of the generating function of the left-hand side, term-by-term in the sum over $i$ as in (3.26), by

$$
\begin{equation*}
\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(2)}\right\|_{1} \leq 3^{\epsilon} \sum_{x \in \mathbb{Q}^{N}}\left(\delta_{z}^{\epsilon} H_{z}(x)\right) H_{z}(x)^{2} \tag{3.44}
\end{equation*}
$$

Observe that, along with the factor $3^{\epsilon}$, one of the $H_{z}$ factors in (3.26) has now been replaced by $\delta_{z}^{\epsilon} H_{z}$. As in (3.26)-(3.27), we can continue the estimate with

$$
\begin{equation*}
\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(2)}\right\|_{1} \leq 3^{\epsilon}\left\|\delta_{z}^{\epsilon} H_{z}\right\|_{\infty} \sum_{x \in \mathbb{Q}^{N}} H_{z}(x)^{2} \leq 3^{\epsilon}\left\|\delta_{z}^{\epsilon} H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}, \tag{3.45}
\end{equation*}
$$

in agreement with (3.38) for $M=2$. For general $M \geq 3$, use of the inequality (3.42) leads to an upper bound for $\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(M)}\right\|_{1}$ equal to $(2 M-1)^{\epsilon-1}$ times a sum of $2 M-1$ terms with the $i^{\text {th }}$ term being the modification of (3.31) in which the $i^{\text {th }}$ of the factors $\mathcal{G}_{z}, \mathcal{M}_{z}, H_{z}$ has its function ( $G_{z}$ or $H_{z}$ ) replaced by $\delta_{z}^{\epsilon} H_{z}$ (note that $\delta_{z}^{\epsilon} G_{z}=\delta_{z}^{\epsilon} H_{z}$ by definition). Consequently, with (3.33) and choosing the modified factor as the distinguished one in (3.33), we see that

$$
\begin{equation*}
\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(M)}\right\|_{1} \leq(2 M-1)^{\epsilon}\left\|\delta_{z}^{\epsilon} H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1} \tag{3.46}
\end{equation*}
$$

as claimed.

Finally, for (3.39), we again illustrate this for the case $M=2$, as follows. By definition, and by (3.25),

$$
\begin{align*}
\left\|\Pi_{z, k}^{(2)}\right\|_{1} & =\sum_{x \in \mathbb{Q}^{N}}\left[1-(-1)^{k \cdot x}\right] \Pi_{z}^{(2)}(x) \\
& \leq \sum_{x \in \mathbb{Q}^{N}}\left(\left[1-(-1)^{k \cdot x}\right] H_{z}(x)\right) H_{z}(x)^{2}=\sum_{x \in \mathbb{Q}^{N}} H_{z, k}(x) H_{z}(x)^{2} . \tag{3.47}
\end{align*}
$$

This is reminiscent of (3.44), with the difference that one factor on the right-hand side is $H_{z, k}(x)$ rather than $\delta_{z}^{\epsilon} H_{z}(x)$. By using the supremum norm on that factor, we can similarly obtain an upper bound

$$
\begin{equation*}
\left\|\Pi_{z, k}^{(2)}\right\|_{1} \leq\left\|H_{z, k}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty} \tag{3.48}
\end{equation*}
$$

which is the $M=2$ case of (3.39). For general $M \geq 3$, the proof is a very small adaptation of the proof of $[45,(4.10)]$. We divide the displacement $x$ in $\pi_{m}(x)$ as a sum $x=\sum_{i=1}^{\lfloor M / 2\rfloor} x_{i}$ over displacements $x_{i}$ along the subwalks along the bottom of the $M$-loop diagram depicted in Figure 2. We use the inequality

$$
\begin{equation*}
1-(-1)^{k \cdot x} \leq \sum_{i=1}^{\lfloor M / 2\rfloor}\left[1-(-1)^{k \cdot x_{i}}\right] \tag{3.49}
\end{equation*}
$$

which holds if $k \cdot x$ is even since the left-hand side is then zero, and holds if $k \cdot x$ is odd since then at least one of the $k \cdot x_{i}$ must also be odd. Use of this inequality leads to an upper bound for $\left\|\Pi_{z, k}^{(M)}\right\|_{1}$ consisting of a sum of $\lfloor M / 2\rfloor$ terms, each of which is the modification of (3.31) in which one of the factors $\mathcal{G}_{z}, \mathcal{M}_{z}, H_{z}$, has its function ( $G_{z}$ or $H_{z}$ ) replaced by $H_{z, k}$ (note that $H_{z, k}=G_{z, k}$ by definition). With (3.33) and with the modified factor chosen as the distinguished one in (3.33), this leads to (3.39).

## 4 Random walk on the hypercube

The convergence proof for the lace expansion makes use of a comparison with simple random walk on the hypercube. In this section, we prove the two estimates needed for that task, in Lemma 4.1.

Recall from (3.3) that $D(x)=N^{-1} \delta_{|x|, 1}$ is the transition probability for simple random walk on the hypercube. Its Fourier transform is given in (3.4) as $\hat{D}(k)=1-\frac{2|k|}{N}$. The following lemma, which is similar to but simpler than what appears in [7, Section 2] due to our restriction to the hypercube, provides essential estimates for the convergence proof for the lace expansion in Section 5.
Lemma 4.1. For $i \geq 0$ there is a constant $c_{i}$ such that

$$
\begin{equation*}
\max _{x \in \mathbb{Q}^{N}} \frac{1}{V} \sum_{k \in \mathbb{Q}^{N}} \hat{D}(k)^{i}(-1)^{k \cdot x} \leq c_{i} N^{-\lceil i / 2\rceil} \tag{4.1}
\end{equation*}
$$

For $i, j \geq 0$ there is a constant $c_{i, j}$ such that for all $t \in[0,1]$

$$
\begin{equation*}
\frac{1}{V} \sum_{k \in \mathbb{Q}^{N}: k \neq 0} \frac{\left|\hat{D}(k)^{i}\right|}{[1-t \hat{D}(k)]^{j}} \leq c_{i, j} N^{-i / 2} \tag{4.2}
\end{equation*}
$$

Proof. By inverse Fourier transformation, the normalised sum in (4.1) is the transition probability for simple random walk to travel from 0 to $x$ in $i$ steps:

$$
\begin{equation*}
D^{* i}(x)=\frac{1}{V} \sum_{k \in \mathbb{Q}^{N}} \hat{D}(k)^{i}(-1)^{k \cdot x} \tag{4.3}
\end{equation*}
$$

and hence it is nonnegative and equals zero if $i$ and $|x|$ have different parity. It is equal to $\delta_{0, x}$ for $i=0$ so we may assume that $i \geq 1$. Closely related explicit transition probabilities are written in terms of Krawtchouk polynomials in [16] but we can instead proceed crudely here with an elementary counting argument. Without loss of generality we may assume by symmetry that $x$ consists of a string of $|x|$ 's followed by $(N-|x|) 0$ 's. There are $N^{i}$ possible $i$-step walks starting from 0 . The number of those that end at $x$ can be bounded as follows. First we observe that the first $|x|$ coordinates of $x$ must flip an odd number of times (each at least once), whereas the remaining coordinates must flip an even number of times (possibly zero). Let $\delta$ be the total number of coordinates that do flip. Since the first $|x|$ coordinates flip at least once and the remaining $\delta-|x|$ flip at least twice, it must be the case that $|x|+2(\delta-|x|) \leq i$, which implies that $\delta-|x| \leq \frac{1}{2}(i-|x|)$. The number of ways to choose which of the $N-|x|$ coordinates are the $\delta-|x|$ coordinates that flip a positive even number of times is at most $(N-|x|)^{\delta-|x|} \leq N^{(i-|x|) / 2}$. Since the number of $i$-step walks that flip $\delta$ specific coordinates is $\delta^{i}$, we find that the transition probability obeys the inequality

$$
\begin{equation*}
D^{* i}(x) \leq \frac{1}{N^{i}} \sum_{\delta=|x|}^{i} N^{(i-|x|) / 2} \delta^{i} \leq \frac{1}{N^{(i+|x|) / 2}}(i+1) i^{i} \tag{4.4}
\end{equation*}
$$

When $i$ is even the factor $N^{-(i+|x|) / 2}$ on the right-hand side is at most $N^{-i / 2}$ (since $|x| \geq 0$ ), whereas for $i$ odd it is at most $N^{-(i+1) / 2}$ (since $|x|$ must also be odd so at least 1 ). This completes the proof of (4.1).

Next we consider (4.2). By the Cauchy-Schwarz inequality,

$$
\begin{equation*}
\frac{1}{V} \sum_{k \in \mathbb{Q}^{N}: k \neq 0} \frac{\left|\hat{D}(k)^{i}\right|}{[1-t \hat{D}(k)]^{j}} \leq\left[\frac{1}{V} \sum_{k \in \mathbb{Q}^{N}: k \neq 0} \hat{D}(k)^{2 i}\right]^{1 / 2}\left[\frac{1}{V} \sum_{k \in \mathbb{Q}^{N}: k \neq 0} \frac{1}{[1-t \hat{D}(k)]^{2 j}}\right]^{1 / 2} . \tag{4.5}
\end{equation*}
$$

The first factor on the right-hand side is at most a multiple of $N^{-i / 2}$ by (4.1) (applied just for $x=0$ ), so it suffices to prove that for any $j \geq 1$ (the case $j=0$ is clear)

$$
\begin{equation*}
\frac{1}{V} \sum_{k \neq 0} \frac{1}{[1-t \hat{D}(k)]^{j}} \leq c_{j} \tag{4.6}
\end{equation*}
$$

for some positive $c_{j}$. Since $\hat{D}(k) \in[-1,1]$ the left-hand side is bounded above by $2^{j}$ if $t \in\left[0, \frac{1}{2}\right]$. For the more substantial case of $t \in\left[\frac{1}{2}, 1\right]$ we have

$$
\begin{equation*}
1-t \hat{D}(k)=1-t+\frac{2 t|k|}{N} \geq \frac{|k|}{N} \tag{4.7}
\end{equation*}
$$

and therefore in this case

$$
\begin{equation*}
\frac{1}{V} \sum_{k \neq 0} \frac{1}{[1-t \hat{D}(k)]^{j}} \leq N^{j} \frac{1}{V} \sum_{k \neq 0} \frac{1}{|k|^{j}}=N^{j} \frac{1}{V} \sum_{m=1}^{N}\binom{N}{m} \frac{1}{m^{j}} . \tag{4.8}
\end{equation*}
$$

We divide the sum over $m$ according to whether $m \leq \frac{1}{4} N$ or $m>\frac{1}{4} N$. For the second case we use

$$
\begin{equation*}
N^{j} \frac{1}{V} \sum_{m=N / 4}^{N}\binom{N}{m} \frac{1}{m^{j}} \leq N^{j} \frac{4^{j}}{N^{j}}=4^{j} . \tag{4.9}
\end{equation*}
$$

For the first case, with $X_{N}$ a random variable with $\operatorname{Bin}\left(N, \frac{1}{2}\right)$ distribution, we have

$$
\begin{equation*}
N^{j} \frac{1}{V} \sum_{m=1}^{N / 4}\binom{N}{m} \frac{1}{m^{j}} \leq N^{j} \frac{1}{V} \sum_{m=0}^{N / 4}\binom{N}{m}=N^{j} \mathbb{P}\left(X_{N} \leq N / 4\right) \leq N^{j} e^{-N / 8}, \tag{4.10}
\end{equation*}
$$

where we used the Chernoff bound $\mathbb{P}\left(X_{N} \leq a\right) \leq \exp \left[-(N-2 a)^{2} /(2 N)\right]$ for $2 a<N$ in the last step (see, e.g, [4]). Since the right-hand side is bounded by a $j$-dependent constant, the proof is complete.

## 5 Convergence of the lace expansion

In this section we prove the convergence of the lace expansion, using the strategy of [45, Section 5.2] which itself is based on the strategy used in [7]. Like most lace expansion convergence proofs, we use a bootstrap argument. The bootstrap argument is presented in Section 5.2. The fact that we are working on the hypercube makes for considerable simplification and this convergence proof is simpler than that in $[7,45]$ (for a different kind of simplification see [48] for weakly self-avoiding walk on $\mathbb{Z}^{d}$ for $d>4$ ).

### 5.1 Preparation

We recall from (3.34) the definition

$$
\begin{equation*}
f_{k}(x)=\left(1-(-1)^{k \cdot x}\right) f(x) \tag{5.1}
\end{equation*}
$$

In particular, $f_{0}(x)=0$ for all $f$. The Fourier transform of $f_{k}$ is

$$
\begin{equation*}
\hat{f}_{k}(\ell)=\sum_{x \in \mathbb{Q}^{N}}\left(1-(-1)^{k \cdot x}\right) f(x)(-1)^{\ell \cdot x}=\hat{f}(\ell)-\hat{f}(k+\ell) \quad\left(k, \ell \in \mathbb{Q}^{N}\right) . \tag{5.2}
\end{equation*}
$$

In particular, $\hat{f}_{k}(0)=\hat{f}(0)-\hat{f}(k)$, and we will use bounds on $\hat{f}_{k}(\ell)$ to control differences of this type (see (5.32)-(5.33) and (5.38)-(5.39)).

For $x \in \mathbb{Q}^{N}$, let $w_{n}(x)$ denote the number of $n$-step walks (not necessarily self-avoiding) from 0 to $x$. Then $w_{n}(x)=N^{n} D^{* n}(x)$ and $\hat{w}_{n}(k)=[N \hat{D}(k)]^{n}$. For $p \in[0,1 / N)$ we define the generating function

$$
\begin{equation*}
C_{p}(x)=\sum_{n=0}^{\infty} w_{n}(x) p^{n} . \tag{5.3}
\end{equation*}
$$

The Fourier transform of $C_{p}$ is (recall (3.4))

$$
\begin{equation*}
\hat{C}_{p}(k)=\frac{1}{1-p N \hat{D}(k)}=\frac{1}{1-p N+2 p|k|} . \tag{5.4}
\end{equation*}
$$

If we evaluate the above right-hand side at $p=1 / N$ then it becomes $N /(2|k|)$, and the zero mode (namely the case $k=0$ ) is divergent. This is a symptom of the recurrence of simple random walk on the hypercube. The zero mode was excluded in (4.2) where the denominator is zero for $k=0$ if $t=1$. The zero mode will play an important role in the bootstrap argument and also subsequently in Section 6.

For later use, we observe that for $p \in[0,1 / N)$ it follows from (5.2) with $f=C_{p}$ and from (5.4) that

$$
\begin{align*}
\hat{C}_{p, k}(\ell) & =p N[\hat{D}(\ell)-\hat{D}(k+\ell)] \hat{C}_{p}(\ell) \hat{C}_{p}(k+\ell) \\
& \leq[1-\hat{D}(k)] \hat{C}_{p}(\ell) \hat{C}_{p}(k+\ell) \\
& =\bar{C}_{p}(k, \ell), \tag{5.5}
\end{align*}
$$

where the last equality defines $\bar{C}_{p}(k, \ell)$.


Figure 3: The definition of $p_{z}$, illustrated for $z>1 / N$.

### 5.2 The bootstrap argument

The following lemma is the basis for the bootstrap argument.
Lemma 5.1. Let $a<b$, let $f$ be a continuous function on the interval $\left[z_{1}, z_{2}\right]$, and assume that $f\left(z_{1}\right) \leq a$. Suppose for each $z \in\left(z_{1}, z_{2}\right]$ that if $f(z) \leq b$ then in fact $f(z) \leq a$. Then $f(z) \leq a$ for all $z \in\left[z_{1}, z_{2}\right]$.

Proof. By hypothesis, $f(z)$ cannot lie in the interval ( $a, b]$ for any $z \in\left(z_{1}, z_{2}\right]$. Since $f\left(z_{1}\right) \leq a$, it follows by continuity that $f(z) \leq a$ for all $z \in\left[z_{1}, z_{2}\right]$.

For $z \in[0, \infty)$, we define $p_{z} \in[0,1 / N)$ as in Figure 3 by

$$
\begin{equation*}
\hat{G}_{z}(0)=\chi_{N}(z)=\frac{1}{1-p_{z} N}=\hat{C}_{p_{z}}(0) . \tag{5.6}
\end{equation*}
$$

Equivalently, from (3.10) we see that

$$
\begin{equation*}
p_{z} N=1-\frac{1}{\chi_{N}(z)}=1-F_{N}(z)=z N+\hat{\Pi}_{z}(0) . \tag{5.7}
\end{equation*}
$$

Our choice of $f$ in Lemma 5.1 is motivated by the intuition that $\hat{G}_{z}(k)$ and $\hat{C}_{p_{z}}(k)$ are comparable in size, not just for $k=0$ where they are equal by definition but also for all $k \in \mathbb{Q}^{N}$. We also anticipate that $\hat{G}_{z, k}(\ell)$ and $\hat{C}_{p_{z}, k}(\ell)$ should be comparable, but it is convenient and also sufficient to compare instead $\hat{G}_{z, k}(\ell)$ and the upper bound $\bar{C}_{p_{z}}(k, \ell)$ for $\hat{C}_{p_{z}, k}(\ell)$ from (5.5).

We will apply Lemma 5.1 with $z_{1}=0, z_{2}=z_{N}, a=2, b=4$, and

$$
\begin{equation*}
f(z)=\max \left\{f_{1}(z), f_{2}(z), f_{3}(z)\right\}, \tag{5.8}
\end{equation*}
$$

where

$$
\begin{equation*}
f_{1}(z)=z N, \quad f_{2}(z)=\max _{k \in \mathbb{Q}^{N}} \frac{\left|\hat{G}_{z}(k)\right|}{\hat{C}_{p_{z}}(k)}, \quad f_{3}(z)=\max _{k \in \mathbb{Q}^{N} \backslash\{0\}} \max _{l \in \mathbb{Q}^{N}} \frac{\left|\hat{G}_{z, k}(\ell)\right|}{\bar{C}_{p_{z}}(k, \ell)} . \tag{5.9}
\end{equation*}
$$

The omission of $k=0$ in the definition of $f_{3}$ avoids the ratio $\frac{0}{0}$. By definition $p_{0}=0$, and since $\hat{G}_{0}(k)=\hat{C}_{0}(k)=1$, it follows that $f_{1}(0)=0, f_{2}(0)=1, f_{3}(0)=0$ and hence $f(0)=1 \leq 2$. The continuity of $f$ on $\left[0, z_{N}\right]$ also follows easily from the continuity of $z \mapsto p_{z}$ and the continuity (for fixed $k$ )
of $\hat{G}_{z}(k)$ and $\hat{C}_{p}(k)$ in $z$ and $p$. We will verify that the remaining and substantial bootstrap hypotheses of Lemma 5.1 holds when $z_{N}=z_{N}(\lambda)$ is defined with a $\lambda$ that is sufficiently small. From this, we can conclude that $f(z) \leq a=2$ uniformly in $z \in\left[0, z_{N}\right]$.

Once we conclude that $f(z) \leq 2$ we of course also know that $f(z) \leq 4$ and hence all the conclusions of Lemmas $5.2,5.4,5.5$ and Remark 5.3, which are initially conditional on $f(z) \leq 4$, in fact hold unconditionally. One of those conclusions, in the proof of Lemma 5.5, is that the ratio $\hat{G}_{z}(k) / \hat{C}_{p_{z}}(k)$ is close to 1 without the absolute value taken in the definition of $f_{2}$. In particular $\hat{G}_{z}(k) \geq 0$; absolute values are included in $f_{2}$ since this is not obvious a priori. The upper bound $\hat{G}_{z}(k) \leq 2 \hat{C}_{p_{z}}(k)$ is an example of what is known as an infrared bound, so named to place emphasis on its significance for $k$ near zero (low frequency).

We define

$$
\begin{equation*}
\beta_{z}=\frac{1}{N}+\frac{\chi_{N}(z)^{2}}{V} \tag{5.10}
\end{equation*}
$$

If we assume that $z \leq z_{N}$ with $z_{N}=z_{N}(\lambda)$ defined by some $\lambda>0$ in (1.2), then for all $z \in\left[0, z_{N}\right]$ we have $\beta_{z} \leq N^{-1}+\lambda^{2}$. Thus we can use $\beta_{z}$ as a small parameter, assuming (as we will) that $N^{-1}+\lambda^{2}$ is indeed small by demanding that $\lambda \in\left(0, \lambda_{0}\right]$ for sufficiently small $\lambda_{0}$.

For $p \in[1, \infty)$, we use the norms $\|\hat{f}\|_{\hat{p}}=\left[V^{-1} \sum_{k \in \mathbb{Q}^{N}}|\hat{f}(k)|^{p}\right]^{1 / p}$ for the Fourier transform as well as $\|f\|_{p}=\left[\sum_{x \in \mathbb{Q}^{N}}|f(x)|^{p}\right]^{1 / p}$ for untransformed functions, so it is necessary to notice hats with norms to distinguish between the presence or not of the volume factor. We also use $\|f\|_{\infty}=\max _{x \in \mathbb{Q}^{N}}|f(x)|$. The bound $\|f\|_{\infty} \leq\|\hat{f}\|_{\hat{1}}$ follows from (3.2). The Parseval relation asserts that $\|f\|_{2}=\|\hat{f}\|_{\hat{2}}$. The convolution $(f * g)(x)=\sum_{y \in \mathbb{Q}^{N}} f(x-y) g(y)$ obeys $\|f * g\|_{\infty} \leq\|f\|_{2}\|g\|_{2}$ by the Cauchy-Schwarz inequality, and $\widehat{f * g}=\hat{f} \hat{g}$.
Lemma 5.2. Fix $z \in\left(0, z_{N}\right]$ and assume that $f$ of (5.8) obeys $f(z) \leq K$. Then there is a constant $c_{K}$, independent of $z$, such that

$$
\begin{equation*}
\left\|H_{z, k}\right\|_{\infty} \leq c_{K}\left(1+\lambda^{2}\right)[1-\hat{D}(k)], \quad\left\|H_{z}\right\|_{2}^{2} \leq c_{K} \beta_{z}, \quad\left\|H_{z}\right\|_{\infty} \leq c_{K} \beta_{z} \tag{5.11}
\end{equation*}
$$

Proof. Special attention is required for the zero mode, which is the origin of the term $V^{-1} \chi_{N}(z)^{2}$ in $\beta_{z}$.
Since $f_{3}(z) \leq K$, it follows from the definition of $\bar{C}_{p}(k, \ell)$ in (5.5) and the Cauchy-Schwarz inequality that

$$
\begin{equation*}
\left\|H_{z, k}\right\|_{\infty}=\left\|G_{z, k}\right\|_{\infty} \leq\left\|\hat{G}_{z, k}\right\|_{\hat{1}} \leq K\left\|\bar{C}_{p_{z}}(k, \cdot)\right\|_{\hat{1}} \leq K[1-\hat{D}(k)]\left\|\hat{C}_{p_{z}}\right\|_{\hat{2}}^{2} . \tag{5.12}
\end{equation*}
$$

Now we apply the definition of $p_{z}$, Lemma 4.1, and the fact that $z \leq z_{N}$ (so $\chi_{N}(z) \leq \lambda V^{1 / 2}$ ) to see that

$$
\begin{align*}
\left\|\hat{C}_{p_{z}}\right\|_{\hat{2}}^{2} & =\frac{1}{V} \sum_{k \in \mathbb{Q}^{N}} \hat{C}_{p_{z}}(k)^{2}=\frac{1}{V} \hat{C}_{p_{z}}(0)^{2}+\frac{1}{V} \sum_{k \neq 0} \frac{1}{\left(1-p_{z} N \hat{D}(k)\right)^{2}} \\
& =\frac{\chi_{N}(z)^{2}}{V}+O(1) \leq \lambda^{2}+O(1) . \tag{5.13}
\end{align*}
$$

This proves the first bound of (5.11).
To estimate $\left\|H_{z}\right\|_{2}^{2}$, we first use submultiplicativity in the form of the inequality $c_{n}(x) \leq\left(c_{1} * c_{n-1}\right)(x)=$ $N\left(D * c_{n-1}\right)(x)$, along with $f_{1}(z) \leq K$, to obtain

$$
\begin{equation*}
H_{z}(x) \leq z N\left(D * G_{z}\right)(x) \leq K\left(D * G_{z}\right)(x) . \tag{5.14}
\end{equation*}
$$

With the Parseval relation and $f_{2}(z) \leq K$, this implies that

$$
\begin{equation*}
\left\|H_{z}\right\|_{2}^{2} \leq K^{2}\left\|D * G_{z}\right\|_{2}^{2}=K^{2}\left\|\hat{D} \hat{G}_{z}\right\|_{\hat{2}}^{2} \leq K^{4}\left\|\hat{D} \hat{C}_{p_{z}}\right\|_{\hat{2}}^{2} . \tag{5.15}
\end{equation*}
$$

Then we estimate the right-hand side by extracting the zero mode and using Lemma 4.1 for the nonzero $k$, as we did above. This gives

$$
\begin{equation*}
\left\|\hat{D} \hat{C}_{p_{z}}\right\|_{\hat{2}}^{2}=\frac{1}{V} \hat{C}_{p_{z}}(0)^{2}+\frac{1}{V} \sum_{k \neq 0} \frac{\hat{D}(k)^{2}}{\left(1-p_{z} N \hat{D}(k)\right)^{2}}=\frac{\chi_{N}(z)^{2}}{V}+O\left(N^{-1}\right) \leq O\left(\beta_{z}\right) \tag{5.16}
\end{equation*}
$$

which proves the second bound of (5.11), for a suitable constant $c_{K}$.
Iteration of (5.14) using $G_{z}(x)=\delta_{0, x}+H_{z}(x)$ gives $H_{z}(x) \leq K D(x)+K^{2}\left(D * D * G_{z}\right)(x)$. Therefore,

$$
\begin{equation*}
\left\|H_{z}\right\|_{\infty} \leq K\|D\|_{\infty}+K^{2}\left\|\hat{D}^{2} \hat{G}_{z}\right\|_{\hat{1}} \leq K N^{-1}+K^{3}\left\|\hat{D}^{2} \hat{C}_{p_{z}}\right\|_{\hat{1}} \tag{5.17}
\end{equation*}
$$

where we used $D(x) \leq N^{-1}$ and our assumption $f_{2}(z) \leq K$ to bound $\left\|\hat{D}^{2} \hat{G}_{z}\right\|_{\hat{1}}$. The norm on the right-hand side is equal to

$$
\begin{equation*}
\frac{\chi_{N}(z)}{V}+\frac{1}{V} \sum_{k \neq 0} \frac{\hat{D}(k)^{2}}{1-p_{z} N \hat{D}(k)} \leq \frac{\chi_{N}(z)^{2}}{V}+O\left(N^{-1}\right) \leq O\left(\beta_{z}\right) \tag{5.18}
\end{equation*}
$$

where we used the inequality (4.2) of Lemma 4.1 (with $i=2$ and $j=1$ ) to bound the sum in the last line. This proves the third bound of (5.11).

Remark 5.3. As mentioned below (5.9), once the bootstrap proof is complete, statements that follow from the assumption $f(z) \leq K$ with $K=4$ in fact will then be known to hold unconditionally with $K=2$. In particular, we can conclude from (5.11) (together with the fact that $H_{z}(0)=0$ by definition) that for any $z \in\left[0, z_{N}\right]$ the bubble diagram

$$
\begin{equation*}
\left\|G_{z}\right\|_{2}^{2}=1+\left\|H_{z}\right\|_{2}^{2} \tag{5.19}
\end{equation*}
$$

is bounded above by $1+O\left(\beta_{z}\right)$. When $z=z_{N}$, this is a statement of a bubble condition analogous to the triangle condition for percolation on a finite graph studied in [6].

Lemma 5.4. Fix $z \in\left(0, z_{N}\right]$, and suppose that $f$ of (5.8) obeys $f(z) \leq K$. Then there is a constant $\bar{c}_{K}$ (independent of $z$ ) such that if $\lambda \in\left(0, \lambda_{0}\right]$ with $\lambda_{0}$ sufficiently small (independent of $z$ ) then

$$
\begin{equation*}
\left\|\Pi_{z}\right\|_{1} \leq \bar{c}_{K} \beta_{z}, \quad\left\|\Pi_{z, k}\right\|_{1} \leq \bar{c}_{K} \beta_{z}[1-\hat{D}(k)] \tag{5.20}
\end{equation*}
$$

Proof. From Proposition 3.6 we have that $\Pi_{z, k}^{(1)}(x)=0$ and

$$
\begin{equation*}
\left\|\Pi_{z}^{(1)}\right\|_{1} \leq z N\left\|H_{z}\right\|_{\infty} \tag{5.21}
\end{equation*}
$$

and, for $M \geq 2$, also that

$$
\begin{align*}
\left\|\Pi_{z}^{(M)}\right\|_{1} & \leq\left\|H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1}  \tag{5.22}\\
\left\|\Pi_{z, k}^{(M)}\right\|_{1} & \leq\lfloor M / 2\rfloor\left\|H_{z, k}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1} \tag{5.23}
\end{align*}
$$

Since $H_{z} * G_{z}=H_{z}+\left(H_{z} * H_{z}\right)$ by definition, the Cauchy-Schwarz inequality and Lemma 5.2 give

$$
\begin{equation*}
\left\|H_{z} * G_{z}\right\|_{\infty} \leq\left\|H_{z}\right\|_{\infty}+\left\|H_{z} * H_{z}\right\|_{\infty} \leq\left\|H_{z}\right\|_{\infty}+\left\|H_{z}\right\|_{2}^{2} \leq 2 c_{K} \beta_{z} \tag{5.24}
\end{equation*}
$$

Since $z N \leq K$ by assumption,

$$
\begin{equation*}
\left\|\Pi_{z}\right\|_{1} \leq \sum_{M=1}^{\infty}\left\|\Pi_{z}^{(M)}\right\|_{1} \leq \bar{c}_{K} \beta_{z} \tag{5.25}
\end{equation*}
$$

For the second bound of (5.20), we similarly use

$$
\begin{align*}
\left\|\Pi_{z, k}\right\|_{1} & \leq \sum_{M=2}^{\infty}\lfloor M / 2\rfloor\left\|H_{z, k}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1} \\
& \leq \sum_{M=2}^{\infty}\lfloor M / 2\rfloor c_{K}\left(1+\lambda^{2}\right)[1-\hat{D}(k)]\left(2 c_{K} \beta_{z}\right)^{M-1} \leq \bar{c}_{K} \beta_{z}[1-\hat{D}(k)] \tag{5.26}
\end{align*}
$$

Here we have taken $\lambda_{0}$ sufficiently small to control the geometric sum over $M$ (and we always consider large $N$ ).

The next lemma completes the bootstrap argument by establishing the substantial hypothesis of Lemma 5.1 for small $\lambda_{0}$.
Lemma 5.5. Fix $z \in\left(0, z_{N}\right]$ and suppose that $f(z) \leq 4$. For $\lambda \in\left(0, \lambda_{0}\right]$ with $\lambda_{0}$ sufficiently small (independent of $z$ ), it is in fact the case that $f(z) \leq 1+c \beta_{z}$ for some $c>0$ independent of $z$.
Proof. We consider $f_{1}, f_{2}, f_{3}$ in that order.
Bound on $f_{1}(z)$. For $f_{1}(z)$, we simply note that $\chi_{N}(z)>0$ and hence also

$$
\begin{equation*}
\chi_{N}(z)^{-1}=1-z N-\hat{\Pi}_{z}(0)>0 . \tag{5.27}
\end{equation*}
$$

Therefore, by Lemma 5.4 and the fact that any function $h$ obeys $|\hat{h}(k)| \leq\|h\|_{1}$ for all $k$ (including the present case of $k=0$ ),

$$
\begin{equation*}
f_{1}(z)=z N<1-\hat{\Pi}_{z}(0) \leq 1+\bar{c}_{4} \beta_{z}, \tag{5.28}
\end{equation*}
$$

assuming $\lambda_{0}$ is sufficiently small.
Bound on $f_{2}(z)$. For $f_{2}$, we first recall (3.9) and write

$$
\begin{equation*}
\hat{F}_{z}(k)=\frac{1}{\hat{G}_{z}(k)}=1-z N \hat{D}(k)-\hat{\Pi}_{z}(k) \tag{5.29}
\end{equation*}
$$

(this gives an alternate notation $\hat{F}_{z}(0)$ for the reciprocal $F_{N}(z)$ of the susceptibility), so that

$$
\begin{equation*}
\frac{\hat{G}_{z}(k)}{\hat{C}_{p_{z}}(k)}=\frac{1-p_{z} N \hat{D}(k)}{\hat{F}_{z}(k)}=1+\hat{E}_{z}(k), \quad \hat{E}_{z}(k)=\frac{1-p_{z} N \hat{D}(k)-\hat{F}_{z}(k)}{\hat{F}_{z}(k)} . \tag{5.30}
\end{equation*}
$$

We will show that $\hat{E}_{z}(k)=O\left(\beta_{z}\right)$, which implies that $f_{2}(z)=1+O\left(\beta_{z}\right)$. By (5.7), $p_{z} N=1-\hat{F}_{z}(0)=$ $z N+\hat{\Pi}_{z}(0)$, and thus by (3.9) the numerator of $\hat{E}_{z}(k)$ is

$$
\begin{equation*}
1-p_{z} N \hat{D}(k)-\hat{F}_{z}(k)=-\hat{\Pi}_{z}(0) \hat{D}(k)+\hat{\Pi}_{z}(k)=\hat{\Pi}_{z}(0)[1-\hat{D}(k)]-\hat{\Pi}_{z, k}(0) \tag{5.31}
\end{equation*}
$$

We can now use our bound on $\hat{\Pi}_{z, k}(0)=\hat{\Pi}_{z}(0)-\hat{\Pi}_{z}(k)$. Indeed, by (5.20) (again with $|\hat{h}(k)| \leq\|h\|_{1}$ )

$$
\begin{equation*}
\left|\hat{E}_{z}(k)\right| \leq 2 \bar{c}_{4} \beta_{z} \frac{1-\hat{D}(k)}{\left|\hat{F}_{z}(k)\right|} . \tag{5.32}
\end{equation*}
$$

For $z \leq \frac{1}{2 N}$, we can use the crude bound $c_{n}^{(N)} \leq N^{n}$ to see that $\left|\hat{G}_{z}(k)\right| \leq \chi_{N}(z) \leq \hat{C}_{z}(0) \leq 2$ and hence that $\left|\hat{E}_{z}(k)\right| \leq 8 \bar{c}_{4} \beta_{z}$. For $\frac{1}{2 N} \leq z \leq z_{N}$, we use

$$
\begin{align*}
\left|\hat{F}_{z}(k)\right| & =\left|\hat{F}_{z}(0)+\left[\hat{F}_{z}(k)-\hat{F}_{z}(0)\right]\right| \\
& =\left|\hat{F}_{z}(0)+z N[1-\hat{D}(k)]+\hat{\Pi}_{z, k}(0)\right| \\
& \geq \hat{F}_{z}(0)+\frac{1}{2}[1-\hat{D}(k)]-\bar{c}_{4} \beta_{z}[1-\hat{D}(k)] \\
& \geq \frac{1}{4}[1-\hat{D}(k)] . \tag{5.33}
\end{align*}
$$

Therefore $\left|\hat{E}_{z}(k)\right| \leq 16 \bar{c}_{4} \beta_{z}$ (for small $\lambda_{0}$ ), and we have proved that $f_{2}(z)=1+O\left(\beta_{z}\right)$.
Bound on $f_{3}(z)$. Although we only need a bound for $k \neq 0$, the following applies in fact for all $k \in \mathbb{Q}^{N}$. We write

$$
\begin{equation*}
\hat{g}_{z}(k)=z N \hat{D}(k)+\hat{\Pi}_{z}(k), \tag{5.34}
\end{equation*}
$$

so that

$$
\begin{equation*}
\hat{G}_{z}(k)=\frac{1}{1-\hat{g}_{z}(k)} . \tag{5.35}
\end{equation*}
$$

By (5.2), for all $k, \ell \in \mathbb{Q}^{N}$,

$$
\begin{align*}
\left|\hat{G}_{z, k}(\ell)\right| & =\left|\hat{G}_{z}(\ell)\right|\left|\hat{G}_{z}(k+\ell)\right|\left|\hat{g}_{z}(\ell)-\hat{g}_{z}(k+\ell)\right| \\
& \leq\left|\hat{G}_{z}(\ell)\right|\left|\hat{G}_{z}(k+\ell)\right| \sum_{x \in \mathbb{Q}^{N}}\left[1-(-1)^{k \cdot x}\right]\left|g_{z}(x)\right|, \tag{5.36}
\end{align*}
$$

where

$$
\begin{equation*}
g_{z}(x)=z N D(x)+\Pi_{z}(x) \tag{5.37}
\end{equation*}
$$

is the inverse Fourier transform of $\hat{g}_{z}(k)$ (recall (3.2)). Since $f_{2}(z) \leq 1+O\left(\beta_{z}\right)$, we can bound each factor of $\left|\hat{G}_{z}\right|$ by $\left[1+O\left(\beta_{z}\right)\right] \hat{C}_{p_{z}}$. With $f_{1}(z)=z N \leq 1+O\left(\beta_{z}\right)$, and with the definition of the Fourier transform in (3.1), we obtain

$$
\begin{align*}
\sum_{x \in \mathbb{Q}^{N}}\left[1-(-1)^{k \cdot x}\right]\left|g_{z}(x)\right| & \leq \sum_{x \in \mathbb{Q}^{N}}\left[1-(-1)^{k \cdot x}\right]\left[z N D(x)+\left|\Pi_{z}(x)\right|\right] \\
& \leq\left(1+O\left(\beta_{z}\right)\right)[1-\hat{D}(k)]+\sum_{x \in \mathbb{Q}^{N}}\left[1-(-1)^{k \cdot x}\right]\left|\Pi_{z}(x)\right| \tag{5.38}
\end{align*}
$$

In the last term the absolute values inside the sum prevent a direct application of (5.20), but the proof of (5.20) bounds the sum over $M$ absolutely, so (5.20) also holds for the above sum and we see that

$$
\begin{equation*}
\sum_{x \in \mathbb{Q}^{N}}\left[1-(-1)^{k \cdot x}\right]\left|g_{z}(x)\right| \leq\left[1+O\left(\beta_{z}\right)\right][1-\hat{D}(k)] \tag{5.39}
\end{equation*}
$$

Together, for all $k, \ell \in \mathbb{Q}^{N}$ these bounds give

$$
\begin{equation*}
\left|\hat{G}_{z, k}(\ell)\right| \leq\left[1+O\left(\beta_{z}\right)\right][1-\hat{D}(k)] \hat{C}_{p_{z}}(\ell) \hat{C}_{p_{z}}(k+\ell)=\left[1+O\left(\beta_{z}\right)\right] \bar{C}_{p_{z}}(k, \ell) \tag{5.40}
\end{equation*}
$$

which implies that $f_{3}(z) \leq 1+O\left(\beta_{z}\right)$.
This completes the proof that $f(z) \leq 1+O\left(\beta_{z}\right)$.

## 6 Consequences of lace expansion convergence

In this section, we first prove Theorems 1.4 and 1.5 , which follow from a well-known differential inequality together with the bubble condition discussed in Remark 5.3. We also complete the proofs of Theorems 1.1 and 1.2 by proving Propositions 2.1, 2.4 and 2.5 which we have seen in Section 2 imply Theorems 1.1 and 1.2. The proofs of Propositions 2.4 and 2.5 make use of fractional derivatives. Once this has been accomplished only Theorem 1.3 remains; its proof is given in Section 7.

Note that $F_{N}(z)$ which was natural notation in Section 2 is identical to $\hat{F}_{z}(0)$ which is natural in the context of the lace expansion where we also used the Fourier transform $\hat{F}_{z}(k)$ (recall (5.29)). In this
section, we favour the notation $F_{N}(z)$ since the Fourier transform reappears only within the proof of Lemma 6.5.

Our concern is with small positive values of the parameter $\lambda$ used to define $z_{N}=z_{N}(\lambda)$ by $\chi_{N}\left(z_{N}\right)=$ $\lambda V^{1 / 2}$. By definition of $\beta_{z}$ in (5.10), we see that for $z \in\left[0, z_{N}\right]$ we have

$$
\begin{equation*}
\beta_{z} \leq N^{-1}+\lambda^{2} \tag{6.1}
\end{equation*}
$$

We take $\lambda \in\left(0, \lambda_{0}\right.$ ] where $\lambda_{0}$ will be chosen to be sufficiently small.

### 6.1 Proof of Theorems 1.4 and 1.5

To prove Theorems 1.4 and 1.5, we recall a differential inequality from [10] which we state here as in [45, Theorem 2.3] (or [37, Lemma 1.5.2]), namely

$$
\begin{equation*}
\frac{1}{\mathrm{~B}(z)} \chi_{N}(z)^{2} \leq \partial_{z}\left[z \chi_{N}(z)\right] \leq \chi_{N}(z)^{2} \tag{6.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathrm{B}(z)=\left\|G_{z}\right\|_{2}^{2} \tag{6.3}
\end{equation*}
$$

is the bubble diagram introduced in Remark 5.3. The upper bound is a very elementary consequence of submultiplicativity, as we discuss below (6.11). Both inequalities in (6.2) hold on any finite or infinite transitive graph, but for the lower bound to be useful it is necessary to have control of the bubble diagram.

Proof of Theorem 1.5. Let $z \in\left[0, z_{N}\right]$. By (6.2), the definition (1.15) of the expected length, and the monotonicity of the bubble diagram,

$$
\begin{equation*}
\frac{1}{\mathrm{~B}\left(z_{N}\right)} \chi_{N}(z) \leq \mathbb{E}_{z}^{(N)} L \leq \chi_{N}(z) \tag{6.4}
\end{equation*}
$$

This gives the upper bound claimed in (1.16), and the lower bound follows from the fact that $\left\|G_{z_{N}}\right\|_{2}^{2}=$ $1+\left\|H_{z_{N}}\right\|_{2}^{2}=1+O\left(\beta_{z_{N}}\right)$ (see Remark 5.3).

Proof of Theorem 1.4. The inequality (6.2) can equivalently be written in terms of the reciprocal $F_{N}$ of $\chi_{N}$ as

$$
\begin{equation*}
\frac{1}{\mathrm{~B}(z)}-F_{N}(z) \leq-z \partial_{z} F_{N}(z) \leq 1-F_{N}(z) \tag{6.5}
\end{equation*}
$$

Integration of the upper bound over the interval $[z, w]$ with $w>z>0$ leads to

$$
\begin{equation*}
\log \left(\frac{1-F_{N}(w)}{1-F_{N}(z)}\right) \leq \log \left(\frac{w}{z}\right) \tag{6.6}
\end{equation*}
$$

which rearranges to the general lower bound

$$
\begin{equation*}
\chi_{N}(z) \geq \frac{1}{\chi_{N}(w)^{-1} z / w+1-z / w} \tag{6.7}
\end{equation*}
$$

The choice $w=z_{N}$, together with replacement of $z / w$ by 1 for the first ratio in the denominator to give a further lower bound, proves the lower bound of (1.13).

For the lower bound of (6.5), for $z \in\left[z^{\prime}, z_{N}\right]$ we observe that

$$
\begin{equation*}
-z_{N} \partial_{z} F_{N}(z) \geq \frac{1}{\mathrm{~B}\left(z_{N}\right)}-F_{N}\left(z^{\prime}\right) \tag{6.8}
\end{equation*}
$$

and integrate to obtain

$$
\begin{equation*}
z_{N}\left(F_{N}\left(z^{\prime}\right)-F_{N}\left(z_{N}\right)\right) \geq\left(\frac{1}{\mathrm{~B}\left(z_{N}\right)}-F_{N}\left(z^{\prime}\right)\right)\left(z_{N}-z^{\prime}\right) \tag{6.9}
\end{equation*}
$$

After replacement of $z^{\prime}$ by $z$, this rearranges to

$$
\begin{equation*}
\left(2 z_{N}-z\right) F_{N}(z) \geq z_{N} F_{N}\left(z_{N}\right)+\frac{1}{\mathrm{~B}\left(z_{N}\right)}\left(z_{N}-z\right) \tag{6.10}
\end{equation*}
$$

which is equivalent to the upper bound on $\chi_{N}$ of (1.13) since $\mathrm{B}\left(z_{N}\right)=1+O\left(\beta_{z_{N}}\right)$.

### 6.2 First derivative of $\Pi$

Next, we prove a bound on the $z$-derivative $\partial_{z} \hat{\Pi}_{z}(k)$. For the proof of Proposition 2.5 we will also consider fractional derivatives in Section 6.5.

We begin with the observation that for any $z \geq 0$ and $j \in \mathbb{N}$,

$$
\begin{equation*}
z^{j} \partial_{z}^{j} H_{z}(x) \leq j!\left(H_{z}^{* j} * G_{z}\right)(x) . \tag{6.11}
\end{equation*}
$$

A proof can be found in [37, Lemma 6.2.8] and we illustrate the idea for $j=1$ as follows (we will only use $j=1,2)$. For $j=1$,

$$
\begin{equation*}
z \partial_{z} H_{z}(x)=\sum_{n=1}^{\infty} n c_{n}^{(N)}(x) z^{n}=\sum_{n=1}^{\infty} \sum_{i=1}^{n} c_{n}^{(N)}(x) z^{n} \leq \sum_{n=1}^{\infty} \sum_{i=1}^{n}\left(c_{i}^{(N)} * c_{n-i}^{(N)}\right)(x) z^{i} z^{n-i} \tag{6.12}
\end{equation*}
$$

since $c_{n}^{(N)}(x) \leq\left(c_{i}^{(N)} * c_{n-i}^{(N)}\right)(x)$ follows by relaxing the self-avoidance constraint between the first $i$ and last $n-i$ steps. After interchange of sums, the right-hand side rearranges to $\left(H_{z} * G_{z}\right)(x)$. This is in fact the essential step in the proof of the upper bound of (6.2).

Lemma 6.1. There is a $\lambda_{0}>0$ such that for any $\lambda \in\left(0, \lambda_{0}\right]$, and for all $z \in \mathbb{C}$ with $|z| \leq z_{N}$ and for all $k \in \mathbb{Q}^{N}$,

$$
\begin{equation*}
\left|\partial_{z} \hat{\Pi}_{z}(k)\right| \prec N \beta_{|z|} . \tag{6.13}
\end{equation*}
$$

Proof. As in (3.32), it suffices to prove that there is a constant $c$ such that, for all real $z \in\left[0, z_{N}\right]$,

$$
\begin{equation*}
\sum_{M=1}^{\infty}\left\|\partial_{z} \Pi_{z}^{(M)}\right\|_{1} \leq c N \beta_{z} \tag{6.14}
\end{equation*}
$$

Since $\delta_{z}^{1}=z \partial_{z}$, it follows from Proposition 3.6 with $\epsilon=1$ that for $z \geq 0$ and $M=1$,

$$
\begin{equation*}
\left\|\partial_{z} \Pi_{z}^{(1)}\right\|_{1} \leq N\left\|\partial_{z}\left(z H_{z}\right)\right\|_{\infty} \tag{6.15}
\end{equation*}
$$

and for $z \geq 0$ and $M \geq 2$,

$$
\begin{align*}
\left\|\partial_{z} \Pi_{z}^{(M)}\right\|_{1} & \leq(2 M-1)\left\|\partial_{z} H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1} \\
& \leq(2 M-1)\left\|\partial_{z} H_{z}\right\|_{\infty}\left(2 c_{2} \beta_{z}\right)^{M-1} . \tag{6.16}
\end{align*}
$$

where we used the bound $\left\|H_{z} * G_{z}\right\|_{\infty} \leq 2 c_{2} \beta_{z}$ from (5.24) (with $K=2$ ) for the last inequality.

By (6.11) and Lemma 5.2,

$$
\begin{equation*}
\partial_{z}\left(z H_{z}(x)\right)=H_{z}(x)+z \partial_{z} H_{z}(x) \leq H_{z}(x)+\left(H_{z} * G_{z}\right)(x) \prec \beta_{z}, \tag{6.17}
\end{equation*}
$$

and with (6.16) this gives an $O\left(N \beta_{z}\right)$ bound for the $M=1$ term in (6.14).
For $M \geq 2$, a slight manoeuvre is needed to deal with small $z$ due to the fact that we desire a bound on $\partial_{z} H_{z}$ whereas (6.11) provides a bound on $z \partial_{z} H_{z}$. Suppose that $z \in\left[0, \frac{1}{2 N}\right]$. In this case, as in (5.14) we use $H_{z}(x) \leq z N\left(D * C_{z}\right)(x)$. This inequality in fact holds term-by-term as power series, so it is preserved upon differentiation and

$$
\begin{equation*}
\partial_{z} H_{z}(x) \leq N\left(D * C_{z}\right)(x)+z N \partial_{z}\left(D * C_{z}\right)(x) . \tag{6.18}
\end{equation*}
$$

Since $C_{z}(x)=\delta_{0, x}+z N\left(D * C_{z}\right)(x)$, and since $\hat{C}_{z}(k) \leq \hat{C}_{z}(0) \leq 2$ for $z \leq \frac{1}{2 N}$, the first term on the above right-hand side obeys

$$
\begin{align*}
N\left(D * C_{z}\right)(x) & \leq N\left(D(x)+z N\left(D * D * C_{z}\right)(x)\right) \\
& \leq 1+\frac{1}{2} N\left\|\hat{D}(k)^{2} \hat{C}_{z}(k)\right\|_{\hat{1}} \prec 1+N\left\|\hat{D}(k)^{2}\right\|_{\hat{1}} \prec N \beta_{z}, \tag{6.19}
\end{align*}
$$

since the norm on the right-hand side is of order $N^{-1}$ by Lemma 4.1. Similarly, with (5.4) we see that

$$
\begin{equation*}
z N \partial_{z}\left(D * C_{z}\right)(x)=z N \frac{1}{V} \sum_{k \in \mathbb{Q}^{N}} \frac{N \hat{D}(k)^{2}(-1)^{k \cdot x}}{[1-z N \hat{D}(k)]^{2}} \prec \frac{N}{V}+N \cdot N^{-1} \prec N \beta_{z}, \tag{6.20}
\end{equation*}
$$

where we separated the $k=0$ term and used Lemma 4.1 to bound the sum over nonzero $k$. On the other hand, for $z \in\left[\frac{1}{2 N}, z_{N}\right]$ it follows from (6.17) that

$$
\begin{equation*}
\partial_{z} H_{z}(x) \leq 2 N z \partial_{z} H_{z}(x) \prec N \beta_{z} . \tag{6.21}
\end{equation*}
$$

Thus, using the above bound $N \beta_{z}$ for the $M=1$ term, together with the above considerations to bound $\left\|\partial_{z} H_{z}\right\|_{\infty}$ by $N \beta_{z}$ when $M \geq 2$, altogether we find from (6.16) that

$$
\begin{equation*}
\sum_{M=1}^{\infty}\left\|\partial_{z} \Pi_{z}^{(M)}\right\|_{1} \prec N \beta_{z}+N \beta_{z} \sum_{M=2}^{\infty} M\left(2 c_{2} \beta_{z}\right)^{M-1} \prec N \beta_{z}, \tag{6.22}
\end{equation*}
$$

where we used small $\lambda_{0}$ to bound the last sum.
We can now easily prove (1.6), as follows. As a first and elementary observation, since the generating function for self-avoiding walks is smaller than the generating function for all walks, we have $\chi_{N}(z) \leq$ $\hat{C}_{z}(0)=(1-z N)^{-1}$ for $z<\frac{1}{N}$, and hence $z_{N}$ is larger than the value $p_{z_{N}}=N^{-1}\left(1-\lambda^{-1} V^{-1 / 2}\right)$ for which $\left(1-p_{z_{N}} N\right)^{-1}=\lambda V^{1 / 2}$ (recall Figure 3). This shows that $z_{N} \geq \frac{3}{4} N^{-1}$ (we take $V$ large depending on $\lambda$ ) and therefore

$$
\begin{equation*}
\frac{3}{4} N^{-1} \leq z_{N} \leq 2 N^{-1} \tag{6.23}
\end{equation*}
$$

since we have seen in Lemma 5.5 that $z_{N} N \leq 2$.
Since $F_{N}\left(z_{N}(\lambda)\right)=\lambda^{-1} V^{-1 / 2}$, the chain rule and the formula for $F_{N}(z)$ in (3.10) give

$$
\begin{equation*}
z_{N}^{\prime}(\lambda)=\frac{1}{-F_{N}^{\prime}\left(z_{N}(\lambda)\right)} V^{-1 / 2} \frac{1}{\lambda^{2}}=\frac{1}{N+\left.\partial_{z} \hat{\Pi}_{z}(0)\right|_{z_{N}(\lambda)}} V^{-1 / 2} \frac{1}{\lambda^{2}} \tag{6.24}
\end{equation*}
$$

The first fraction on the right-hand side is at most $2 N^{-1}$ (for small $\lambda_{0}$ ) by Lemma 6.1, so for $\lambda^{\prime} \leq \lambda_{1}<$ $\lambda_{2} \leq \lambda_{0}$,

$$
\begin{equation*}
z_{N}\left(\lambda_{2}\right)-z_{N}\left(\lambda_{1}\right)=\int_{\lambda_{1}}^{\lambda_{2}} z_{N}^{\prime}(\lambda) \mathrm{d} \lambda \prec \frac{V^{-1 / 2}}{N} \int_{\lambda_{1}}^{\lambda_{2}} \frac{1}{\lambda^{2}} \mathrm{~d} \lambda=\frac{V^{-1 / 2}}{N} \frac{\lambda_{2}-\lambda_{1}}{\lambda_{1} \lambda_{2}} \tag{6.25}
\end{equation*}
$$

and hence, as claimed in (1.6),

$$
\begin{equation*}
\frac{z_{N}\left(\lambda_{2}\right)}{z_{N}\left(\lambda_{1}\right)}-1 \prec \frac{V^{-1 / 2}}{N z_{N}\left(\lambda_{1}\right)} \frac{\lambda_{2}-\lambda_{1}}{\lambda_{1} \lambda_{2}} \prec \frac{\lambda_{2}-\lambda_{1}}{\lambda_{1} \lambda_{2}} V^{-1 / 2} . \tag{6.26}
\end{equation*}
$$

In the last step, we used the lower bound of (6.23). As usual, the constant in (6.26) deteriorates as $\lambda_{1}, \lambda_{2}$ decrease, so depends on $\lambda^{\prime}$.

### 6.3 Proofs of Propositions 2.1 and 2.4

We are now in a position to prove Propositions 2.1 and 2.4. The following proposition concerns the susceptibility in the complex plane.

Proposition 6.2. There is a $\lambda_{0}>0$ such that for any $\lambda \in\left(0, \lambda_{0}\right]$, and for any $w_{N} \in \mathbb{C}$ with $\left|w_{N}\right| \leq z_{N}$ and $\lim _{N \rightarrow \infty} V^{1 / 2}\left|1-w_{N} / z_{N}\right|=\infty$,

$$
\begin{equation*}
\left|\chi_{N}\left(w_{N}\right)\right| \asymp \frac{1}{\left|1-w_{N} / z_{N}\right|} . \tag{6.27}
\end{equation*}
$$

In addition, for $z \in \mathbb{C}$ with $|z| \leq z_{N}$, if $N$ is large then

$$
\begin{equation*}
\left|\chi_{N}(z)\right| \leq \frac{2}{\left|1-z / z_{N}\right|} \tag{6.28}
\end{equation*}
$$

Proof. Let $|z| \leq z_{N}$. By the Fundamental Theorem of Calculus and (3.10),

$$
\begin{equation*}
F_{N}(z)=F_{N}\left(z_{N}\right)+z_{N} N\left(1-z / z_{N}\right)+\int_{z}^{z_{N}} \partial_{w} \hat{\Pi}_{w}(0) \mathrm{d} w \tag{6.29}
\end{equation*}
$$

with the integral along the line segment joining $z$ to $z_{N}$. By Lemma 6.1, the integral on the right-hand side is $O\left[N \beta_{z_{N}}\left|z_{N}-z\right|\right]=O\left[\left(N^{-1}+\lambda^{2}\right)\left|1-z / z_{N}\right|\right]$, since $z_{N} \leq 2 N^{-1}$. This gives

$$
\begin{equation*}
F_{N}(z)=\lambda^{-1} V^{-1 / 2}+z_{N} N\left(1-z / z_{N}\right)+O\left[\left(N^{-1}+\lambda^{2}\right)\left|1-z / z_{N}\right|\right] . \tag{6.30}
\end{equation*}
$$

Since $z_{N} N \geq \frac{3}{4}$, the last term is comparable to the middle term but with much smaller prefactor. When we set $z=w_{N}$ with the assumption that $\lim _{N \rightarrow \infty} V^{1 / 2}\left|1-w_{N} / z_{N}\right|=\infty$, the term $\lambda^{-1} V^{-1 / 2}$ becomes negligible and (6.27) follows.

Also, it follows from (6.30) that, with $\epsilon=\left(z_{N} N \lambda V^{1 / 2}\right)^{-1}$,

$$
\begin{align*}
\left|F_{N}(z)\right| & \geq z_{N} N\left|\epsilon+1-z / z_{N}\right|-O\left[\left(N^{-1}+\lambda^{2}\right)\left|1-z / z_{N}\right|\right] \\
& \geq z_{N} N\left|1-z / z_{N}\right|-O\left[\left(N^{-1}+\lambda^{2}\right)\left|1-z / z_{N}\right|\right], \tag{6.31}
\end{align*}
$$

with the first inequality a consequence of the triangle inequality and the second due to the geometric fact that any point $z / z_{N}$ in the unit disk is closer to 1 than it is to $1+\epsilon$. Since $z_{N} N \geq \frac{3}{4}$, this gives (6.28) and completes the proof.

The proofs of Propositions 2.1 and 2.4 now follow easily.

Proof of Proposition 2.1. This is an immediate consequence of (6.23) (for the bound $N z_{N} \leq 2$ ), of (6.28) (for the lower bound on $\left|F_{N}(z)\right|$ in the disk $|z| \leq z_{N}$ ), and of the fact that $F_{N}^{\prime}(z)=-N-\partial_{z} \hat{\Pi}_{z}(0)$ together with Lemma 6.1 which implies that $\left|\partial_{z} \hat{\Pi}_{z}(0)\right| \prec N \beta_{z_{N}} \prec 1+N \lambda^{2}$ uniformly in $|z| \leq z_{N}$ (for the bound $\left.\left|F_{N}^{\prime}(z)\right| \leq 2 N\right)$.

Proof of Proposition 2.4. Let $p \in\left(0, \frac{1}{2}\right)$ and recall that $\zeta_{p}=z_{N}\left(1-V^{-p}\right)$. Proposition 2.4 asserts that

$$
\begin{equation*}
\zeta_{p}=N^{-1}\left[1+O\left(N^{-1}\right)\right], \quad F_{N}\left(\zeta_{p}\right) \asymp V^{-p}, \quad F_{N}^{\prime}\left(\zeta_{p}\right)=-N+O(1) . \tag{6.32}
\end{equation*}
$$

For the second of these three statements, we first observe that by definition $V^{1 / 2}\left(1-\zeta_{p} / z_{N}\right)=V^{1 / 2-p} \rightarrow$ $\infty$. It then follows from (6.27) that

$$
\begin{equation*}
\chi_{N}\left(\zeta_{p}\right) \asymp V^{p}, \tag{6.33}
\end{equation*}
$$

which is equivalent to the desired relation $F_{N}\left(\zeta_{p}\right) \asymp V^{-p}$. By the definition of $\beta_{z}$ in (5.10), this also implies that

$$
\begin{gather*}
\beta_{\zeta_{p}} \prec V^{2 p-1}+N^{-1} \prec N^{-1} .  \tag{6.34}\\
\chi_{N}\left(\zeta_{p}\right)^{-1}=1-N \zeta_{p}-\hat{\Pi}_{\zeta_{p}}(0), \tag{6.35}
\end{gather*}
$$

so by Lemma 5.4, (6.33) and (6.34),

$$
\begin{equation*}
\zeta_{p}=N^{-1}\left[1-\hat{\Pi}_{\zeta_{p}}(0)-\chi_{N}\left(\zeta_{p}\right)^{-1}\right]=N^{-1}\left[1+O\left(N^{-1}\right)\right], \tag{6.36}
\end{equation*}
$$

which proves the first statement of (6.32). Finally, since

$$
\begin{equation*}
F_{N}^{\prime}\left(\zeta_{p}\right)=-N-\left.\partial_{z} \hat{\Pi}_{z}(0)\right|_{z=\zeta_{p}}, \tag{6.37}
\end{equation*}
$$

the third statement of (6.32) follows from Lemma 6.1 and (6.34).

### 6.4 Fractional derivatives

The proof of Proposition 2.5 uses the fractional derivative methods introduced in [26] (see also [37, Section 6.3]), and we begin with a summary of what is needed from that theory.

The fractional derivative $\delta_{z}^{\epsilon} f(z)=\sum_{n=1}^{\infty} n^{\epsilon} a_{n} z^{n}$ defined in (3.35) converges absolutely at least strictly within the circle of convergence of $f(z)$. For $\rho>0$ and for $j=1,2$ we define

$$
\begin{equation*}
A_{j}^{(\epsilon)}(\rho)=\sum_{n=j}^{\infty} n^{\epsilon}\left|a_{n}\right| \rho^{n} . \tag{6.38}
\end{equation*}
$$

The following lemma provides an error estimate analogous to the error estimate in Taylor's theorem. It is a restatement of [37, Lemma 6.3.2], apart from the replacement of $A_{1}^{(\epsilon)}(\rho)$ in [37, Lemma 6.3.2] by $A_{2}^{(\epsilon)}(\rho)$ in (6.40). The replacement is justified since the terms $a_{0}+a_{1} z$ in $f(z)$ cancel on the left-hand side of (6.40) and hence we can assume $a_{0}=a_{1}=0$ there.

Lemma 6.3. Let $\epsilon \in(0,1), f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$, and $\rho>0$. If $A_{1}^{(\epsilon)}(\rho)<\infty$ (so in particular $f(z)$ converges for $|z| \leq \rho)$, then for any $z \in \mathbb{C}$ with $|z| \leq \rho$,

$$
\begin{equation*}
|f(z)-f(\rho)| \leq 2^{1-\epsilon} A_{1}^{(\epsilon)}(\rho)|1-z / \rho|^{\epsilon} . \tag{6.39}
\end{equation*}
$$

If $A_{2}^{(1+\epsilon)}(\rho)<\infty$ (so in particular $f^{\prime}(z)=\sum_{n=1}^{\infty} n a_{n} z^{n-1}$ converges for $|z| \leq \rho$ ), then for any $z \in \mathbb{C}$ with $|z| \leq \rho$,

$$
\begin{equation*}
\left|f(z)-f(\rho)-f^{\prime}(\rho)(z-\rho)\right| \leq \frac{2^{1-\epsilon}}{1+\epsilon} A_{2}^{(1+\epsilon)}(\rho)|1-z / \rho|^{1+\epsilon} . \tag{6.40}
\end{equation*}
$$

For $\epsilon \in(0,1)$, the following elementary lemma provides an integral formula for $\delta_{z}^{\epsilon} f$ in terms of $f^{\prime}$. The statement and proof of the lemma involve the Gamma function $\Gamma(x)=\int_{0}^{\infty} t^{x-1} e^{-t} \mathrm{~d} t$.
Lemma 6.4. Let $f(z)=\sum_{n=0}^{\infty} a_{n} z^{n}$ have radius of convergence at least $\rho$. Let $\epsilon \in(0,1)$ and $\gamma_{\epsilon}=$ $1 / \Gamma(1-\epsilon)$. For any $z \in[0, \rho)$ (and also for $z=\rho$ if $a_{n} \geq 0$ for all $n$ ),

$$
\begin{equation*}
\delta_{z}^{\epsilon} f(z)=\gamma_{\epsilon} z \int_{0}^{\infty} f^{\prime}\left(z e^{-t}\right) e^{-t} t^{-\epsilon} \mathrm{d} t \tag{6.41}
\end{equation*}
$$

Proof. It is proved in [37, Lemma 6.3.1] that

$$
\begin{equation*}
\delta_{z}^{\epsilon} f(z)=C_{\epsilon} z \int_{0}^{\infty} f^{\prime}\left(z e^{-\lambda^{1 /(1-\epsilon)}}\right) e^{-\lambda^{1 /(1-\epsilon)}} \mathrm{d} \lambda \tag{6.42}
\end{equation*}
$$

with $C_{\epsilon}=[(1-\epsilon) \Gamma(1-\epsilon)]^{-1}$. We make the change of variables $t=\lambda^{1 /(1-\epsilon)}$ to obtain (6.41).

### 6.5 Fractional derivatives of $\Pi$ : proof of Proposition 2.5

We now prove Proposition 2.5. The proof is based on an estimate for the $(1+a)^{\text {th }}$ derivative of $\hat{\Pi}_{z}$ at $z=\zeta_{p}$. This in turn requires a fractional-derivative bound on $H_{z}$ and we begin with the crucial lemma that provides this bound. The proof of Lemma 6.5 requires delicate attention to the zero mode. Constants in estimates are permitted to depend on $p$ and $a$ as well as on $\lambda$.

Lemma 6.5. There is a $\lambda_{0}>0$ such that for any $\lambda \in\left(0, \lambda_{0}\right]$, and for any $p \in\left(0, \frac{1}{2}\right)$ and $a \in(0,1)$,

$$
\begin{align*}
N\left\|\left.\delta_{z}^{1+a}\right|_{z=\zeta_{p}}\left(z H_{z}\right)\right\|_{\infty} & \prec N^{-1}+V^{(2+a) p-1}  \tag{6.43}\\
\left\|\left.\delta_{z}^{1+a}\right|_{z=\zeta_{p}} H_{z}\right\|_{\infty} & \prec N^{-1}+V^{(2+a) p-1} \tag{6.44}
\end{align*}
$$

Proof. The values of $p \in\left(0, \frac{1}{2}\right)$ and $a \in(0,1)$ are fixed throughout the proof. To simplify the notation, we will write simply $H_{z}$ in place of $H_{z}(x)$; all upper bounds are uniform in $x \in \mathbb{Q}^{N}$. Also, all derivatives $\partial_{z}$ and $\delta_{z}$ are with respect to $z$. Since we need to evaluate these derivatives at both $\zeta_{p}$ and at $\zeta_{p} e^{-t}$, to lighten the notation we write $\partial_{\zeta_{p}}$ and $\delta_{\zeta_{p}}$ for derivatives evaluated at $\zeta_{p}$, and we introduce $\eta_{t}=\zeta_{p} e^{-t}$ and similarly write $\partial_{\eta_{t}}$ and $\delta_{\eta_{t}}$.

For (6.43) we use the general fact that $\delta_{z}^{1+a} f(z)=\delta_{z}^{a}\left[z \partial_{z} f(z)\right]$ and then (6.41) to see that

$$
\begin{equation*}
\delta_{\zeta_{p}}^{1+a}\left(z H_{z}\right)=\delta_{\zeta_{p}}^{a}\left[z \partial_{z}\left(z H_{z}\right)\right]=\gamma_{a} \zeta_{p} \int_{0}^{\infty} \partial_{\eta_{t}}\left[z \partial_{z}\left(z H_{z}\right)\right] e^{-t} t^{-a} \mathrm{~d} t \tag{6.45}
\end{equation*}
$$

The derivative on the right-hand side satisfies

$$
\begin{align*}
\partial_{z}\left[z \partial_{z}\left(z H_{z}\right)\right] & =H_{z}+3 z \partial_{z} H_{z}+z^{2} \partial_{z}^{2} H_{z} \\
& =c_{1}^{(N)}(x) z+3 c_{1}^{(N)}(x) z+z^{2} \sum_{n=2}^{\infty}(1+3 n+n(n-1)) c_{n}^{(N)}(x) z^{n-2} \tag{6.46}
\end{align*}
$$

and hence since $c_{1}^{(N)}(x) \leq 1$, and since $\eta_{t} \leq \zeta_{p} \sim N^{-1}$ by (6.32),

$$
\begin{equation*}
\partial_{\eta_{t}}\left[z \partial_{z}\left(z H_{z}\right)\right] \prec N^{-1}+N^{-2} \partial_{\eta_{t}}^{2} H_{z} \tag{6.47}
\end{equation*}
$$

Therefore, by (6.45) and again using $\zeta_{p} \prec N^{-1}$, we have the preliminary estimate

$$
\begin{equation*}
N \delta_{\zeta_{p}}^{1+a}\left(z H_{z}\right) \prec N^{-1}+N^{-2} \int_{0}^{\infty} \partial_{\eta_{t}}^{2} H_{z} e^{-t} t^{-a} \mathrm{~d} t \tag{6.48}
\end{equation*}
$$

For (6.44), and for $z \leq \zeta_{p} \prec N^{-1}$, as above and with (6.21) we have

$$
\begin{equation*}
\partial_{z}\left(z \partial_{z} H_{z}\right)=\partial_{z} H_{z}+z \partial_{z}^{2} H_{z} \prec 1+z \partial_{z}^{2} H_{z} \prec 1+N^{-1} \partial_{z}^{2} H_{z}, \tag{6.49}
\end{equation*}
$$

so again (with $\zeta_{p} \prec N^{-1}$ ) we find that

$$
\begin{align*}
\left.\delta_{\zeta_{p}}^{1+a} H_{z}\right|_{z=\zeta_{p}} & =\delta_{\zeta_{p}}^{a}\left(z \partial_{z} H_{z}\right)=\gamma_{a} \zeta_{p} \int_{0}^{\infty} \partial_{\eta_{t}}\left(z \partial_{z} H_{z}\right) e^{-t} t^{-a} \mathrm{~d} t \\
& \prec N^{-1}+N^{-2} \int_{0}^{\infty} \partial_{\eta_{t}}^{2} H_{z} e^{-t} t^{-a} \mathrm{~d} t . \tag{6.50}
\end{align*}
$$

To complete the proof of (6.43)-(6.44), it therefore suffices to prove that

$$
\begin{equation*}
\int_{0}^{\infty} \partial_{\eta_{t}}^{2} H_{z} e^{-t} t^{-a} \mathrm{~d} t \prec N+N^{2} V^{(2+a) p-1} . \tag{6.51}
\end{equation*}
$$

We consider small $t$ and large $t$ separately. The delicate part is small $t$.
For $z \leq \frac{1}{2 N}$, we first observe that since the number of $n$-step self-avoiding walks is bounded above by the number of $n$-step simple random walks, we have $\partial_{z}^{2} H_{z}(x) \leq \partial_{z}^{2} C_{z}(x)$. We rewrite $C_{z}(x)$ using the inverse Fourier transform (3.2) in conjunction with the formula for $\hat{C}_{z}(k)$ in (5.4), and thereby obtain

$$
\begin{align*}
\partial_{z}^{2} H_{z}(x) & \leq \partial_{z}^{2} C_{z}(x)=\partial_{z}^{2} \frac{1}{V} \sum_{k \in \mathbb{Q}^{N}} \frac{(-1)^{k \cdot x}}{1-z N \hat{D}(k)} \\
& =\frac{2 N^{2}}{V} \sum_{k \in \mathbb{Q}^{N}} \frac{\hat{D}(k)^{2}(-1)^{k \cdot x}}{(1-z N \hat{D}(k))^{3}} \leq \frac{2 N^{2}}{V} \sum_{k \in \mathbb{Q}^{N}} \frac{\hat{D}(k)^{2}}{\left(1-\frac{1}{2}\right)^{3}} \prec N, \tag{6.52}
\end{align*}
$$

where we used $z N|\hat{D}(k)| \leq z N \leq \frac{1}{2}$ in the penultimate step and used (4.1) (with $x=0$ ) in the last step. Thus in (6.51) the integral over $t$ such that $\eta_{t} \leq \frac{1}{2 N}$ is bounded by $N$ as required. Such $t$ include those for which $e^{-t} \leq \frac{1}{4}$ since $N \zeta_{p} \leq 2$. It therefore suffices to prove that

$$
\begin{equation*}
\int_{0}^{\log 4} \partial_{\eta_{t}}^{2} H_{z} t^{-a} \mathrm{~d} t \prec N+N^{2} V^{(2+a) p-1} \tag{6.53}
\end{equation*}
$$

where we have neglected a now unimportant exponential factor in the integrand using $e^{-t} \leq 1$.
The derivative in the integrand of (6.53) can be bounded using (6.11), with the result that

$$
\begin{align*}
\int_{0}^{\log 4} \partial_{\eta_{t}}^{2} H_{z}(x) t^{-a} \mathrm{~d} t & \prec \int_{0}^{\log 4}\left(\zeta_{p} e^{-t}\right)^{-2}\left(H_{\eta_{t}} * H_{\eta_{t}} * G_{\eta_{t}}\right)(x) t^{-a} \mathrm{~d} t \\
& \prec N^{2} \int_{0}^{\log 4}\left(H_{\eta_{t}} * H_{\eta_{t}} * G_{\eta_{t}}\right)(x) t^{-a} \mathrm{~d} t . \tag{6.54}
\end{align*}
$$

Now we use $\|f\|_{\infty} \leq\|\hat{f}\|_{\hat{1}}$ to see that

$$
\begin{equation*}
\left(H_{\eta_{t}} * H_{\eta_{t}} * G_{\eta_{t}}\right)(x) \leq \frac{1}{V} \sum_{k \in \mathbb{Q}^{N}} \hat{H}_{\eta_{t}}(k)^{2} \hat{G}_{\eta_{t}}(k) \tag{6.55}
\end{equation*}
$$

Recall from (5.29) that we write $\hat{F}_{z}(k)=1 / \hat{G}_{z}(k)$. As in (5.34), we define $\hat{g}_{z}(k)$ by

$$
\begin{equation*}
\hat{H}_{z}(k)=\hat{G}_{z}(k)-1=\frac{z N \hat{D}(k)+\hat{\Pi}_{z}(k)}{\hat{F}_{z}(k)}=\frac{\hat{g}_{z}(k)}{\hat{F}_{z}(k)}=\hat{g}_{z}(k) \hat{G}_{z}(k) \tag{6.56}
\end{equation*}
$$

For $z \leq \zeta_{p}$, since $z N \prec 1$ and $\left|\hat{\Pi}_{z}(k)\right| \prec \beta_{\zeta_{p}} \prec N^{-1}($ recall (6.34)),

$$
\begin{equation*}
\hat{g}_{z}(k)^{2} \prec \hat{D}(k)^{2}+\hat{\Pi}_{z}(k)^{2} \prec \hat{D}(k)^{2}+N^{-2} . \tag{6.57}
\end{equation*}
$$

We know from the bootstrap proof in Section 5.2 that $f_{2}(z) \leq 2$ for all $z \leq z_{N}$, so

$$
\begin{equation*}
\hat{G}_{z}(k) \leq 2 \hat{C}_{p_{z}}(k)=\frac{2}{1-p_{z} N \hat{D}(k)} . \tag{6.58}
\end{equation*}
$$

The sum over nonzero $k$ in (6.55) is therefore at most

$$
\begin{align*}
\frac{1}{V} \sum_{k \neq 0} \hat{g}_{\eta_{t}}(k)^{2} \hat{G}_{\eta_{t}}(k)^{3} & \prec \frac{1}{V} \sum_{k \neq 0} \frac{\hat{D}(k)^{2}}{\left[1-p_{\eta_{t}} N \hat{D}(k)\right]^{3}}+\frac{1}{V} \sum_{k \neq 0} \frac{N^{-2}}{\left[1-p_{\eta_{t}} N \hat{D}(k)\right]^{3}} \\
& \prec N^{-1}+N^{-2} \prec N^{-1}, \tag{6.59}
\end{align*}
$$

where we used Lemma 4.1 for the second line. When we insert this into the right-hand side of (6.54) the result is $N^{+1}$ which is consistent with (6.53).

It remains to consider the zero mode. The $k=0$ term in the sum in (6.55) is

$$
\begin{equation*}
\frac{1}{V} \hat{g}_{\eta_{t}}(0)^{2} \chi_{N}\left(\eta_{t}\right)^{3} \prec \frac{1}{V}\left(1+N^{-2}\right) \chi_{N}\left(\eta_{t}\right)^{3}, \tag{6.60}
\end{equation*}
$$

so it now suffices to prove that

$$
\begin{equation*}
\int_{0}^{\log 4} \chi_{N}\left(\eta_{t}\right)^{3} t^{-a} \mathrm{~d} t \prec V^{(2+a) p} . \tag{6.61}
\end{equation*}
$$

By (6.28) and the fact that $\eta_{t}=z_{N}\left(1-V^{-p}\right) e^{-t}$ by definition, for $t \leq \log 4$ we have

$$
\begin{equation*}
\chi_{N}\left(\eta_{t}\right) \leq \frac{2}{1-\eta_{t} / z_{N}}=\frac{2}{V^{-p} e^{-t}+1-e^{-t}} \prec \frac{1}{V^{-p}+t} . \tag{6.62}
\end{equation*}
$$

It follows that (with the change of variables $t=s V^{-p}$ )

$$
\begin{equation*}
\int_{0}^{\log 4} \chi_{N}\left(\eta_{t}\right)^{3} t^{-a} \mathrm{~d} t \prec \int_{0}^{\log 4} \frac{1}{\left(V^{-p}+t\right)^{3}} t^{-a} \mathrm{~d} t \prec \frac{1}{\left(V^{-p}\right)^{2+a}} \int_{0}^{\infty} \frac{1}{(1+s)^{3}} s^{-a} \mathrm{~d} s \prec V^{(2+a) p} . \tag{6.63}
\end{equation*}
$$

This completes the proof.
Corollary 6.6. There is $a \lambda_{0}>0$ such that for any $\lambda \in\left(0, \lambda_{0}\right]$, and for any $p \in\left(0, \frac{1}{2}\right)$ and $a \in(0,1)$,

$$
\begin{equation*}
\sum_{M=1}^{\infty}\left\|\delta_{\zeta_{p}}^{1+a} \Pi_{z}^{(M)}\right\|_{1} \prec N^{-1}+V^{(2+a) p-1} \tag{6.64}
\end{equation*}
$$

Proof. It was shown in Proposition 3.6 that for $z \geq 0$ and $\epsilon \geq 1$ we have

$$
\begin{equation*}
\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(1)}\right\|_{1} \leq N\left\|\delta_{z}^{\epsilon}\left(z H_{z}\right)\right\|_{\infty}, \tag{6.65}
\end{equation*}
$$

and for $M \geq 2$,

$$
\begin{equation*}
\left\|\delta_{z}^{\epsilon} \Pi_{z}^{(M)}\right\|_{1} \leq(2 M-1)^{\epsilon}\left\|\delta_{z}^{\epsilon} H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1} . \tag{6.66}
\end{equation*}
$$

Also, by (5.24),

$$
\begin{equation*}
\left\|H_{z} * G_{z}\right\|_{\infty} \leq 2 c_{K} \beta_{z} . \tag{6.67}
\end{equation*}
$$

Then the desired result follows by using Lemma 6.5 to estimate the fractional derivatives of $H_{z}$, with small $\lambda_{0}$ used to control the sum over $M$.

We are now ready to prove Proposition 2.5 which we restate here for convenience as Proposition 6.7. Recall that by definition

$$
\begin{equation*}
F_{N}(z)=\Phi_{N}(z)+R_{N}(z) \tag{6.68}
\end{equation*}
$$

where $\Phi_{N}(z)$ is the linear function

$$
\begin{equation*}
\Phi_{N}(z)=F_{N}\left(\zeta_{p}\right)+F_{N}^{\prime}\left(\zeta_{p}\right)\left(z-\zeta_{p}\right) \tag{6.69}
\end{equation*}
$$

Proposition 6.7. There is a $\lambda_{0}>0$ such that for any $\lambda \in\left(0, \lambda_{0}\right]$, and for any $p \in\left(0, \frac{1}{2}\right)$, any $a \in(0,1)$, and any $z \in \mathbb{C}$ with $|z| \leq \zeta_{p}$,

$$
\begin{align*}
& \left|R_{N}(z)\right| \prec\left(N^{-1}+V^{(2+a) p-1}\right)\left|1-z / \zeta_{p}\right|^{1+a},  \tag{6.70}\\
& \left|R_{N}^{\prime}(z)\right| \prec\left(1+N V^{(2+a) p-1}\right)\left|1-z / \zeta_{p}\right|^{a} . \tag{6.71}
\end{align*}
$$

Proof. For (6.70) we apply the fractional Taylor error estimate (6.40) to

$$
\begin{equation*}
f(z)=F_{N}(z)=1-z N-\hat{\Pi}_{z}(0) \tag{6.72}
\end{equation*}
$$

with $\epsilon=a$ and $\rho=\zeta_{p}$. The linear part $1-z N$ on the right-hand side does not contribute to $A_{2}^{(1+a)}\left(\zeta_{p}\right)$ defined by (6.38), which therefore only involves $\hat{\Pi}_{\zeta_{p}}(0)$ and according to Corollary 6.6 is bounded by $N^{-1}+V^{(2+a) p-1}$. Thus the claimed bound on $R_{N}(z)$ follows from (6.40).

For (6.71), we first use the definition of $R_{N}(z)$ to see that

$$
\begin{equation*}
R_{N}^{\prime}(z)=F_{N}^{\prime}(z)-F_{N}^{\prime}\left(\zeta_{p}\right)=\partial_{\zeta_{p}} \hat{\Pi}_{z}(0)-\partial_{z} \hat{\Pi}_{z}(0) \tag{6.73}
\end{equation*}
$$

Since

$$
\begin{equation*}
\partial_{z} \hat{\Pi}_{z}(0)=\sum_{m=1}^{\infty}(m+1) \sum_{x \in \mathbb{Q}^{N}} \pi_{m+1}(x) z^{m} \tag{6.74}
\end{equation*}
$$

by (6.39) it suffices to have the upper bound

$$
\begin{equation*}
\sum_{m=1}^{\infty} m^{a}(m+1)\left\|\pi_{m+1}\right\|_{1} \zeta_{p}^{m} \prec N \sum_{m=1}^{\infty}(m+1)^{1+a}\left\|\pi_{m+1}\right\|_{1} \zeta_{p}^{m+1} \prec 1+N V^{(2+a) p-1} \tag{6.75}
\end{equation*}
$$

which follows from Corollary 6.6. This completes the proof.

## 7 1/ $N$ expansion: proof of Theorem 1.3

In this section we prove Theorem 1.3, which in particular states that for $m \geq 1$ and $c>0$ (independent of $N$ but possibly depending on $m$ ), and for any $z$ such that $\chi_{N}(z) \in\left[c N^{m}, \lambda_{0} V^{1 / 2}\right]$, there are integers $a_{n}$, which are universal constants that are independent of the particular choice of $z$, such that

$$
\begin{equation*}
z=\sum_{n=1}^{m} a_{n} N^{-n}+O\left(N^{-m-1}\right) \quad \text { as } N \rightarrow \infty \tag{7.1}
\end{equation*}
$$

As usual, $\lambda_{0}$ is assumed to be sufficiently small. The constant in the error term depends on $m, \lambda_{0}, c$, but does not depend otherwise on $z$. Theorem 1.3 also includes a similar statement concerning an asymptotic expansion for the amplitude $A_{N}$, and identifies the first five coefficients in each expansion.

### 7.1 Independence of $z$

We first establish the independence of $z$ for the coefficients $a_{n}$, assuming they exist.
Proposition 7.1. Let $c>0$, and let $\lambda_{0}>0$ be sufficiently small. Let $m \geq 1$. If $z$ obeys (7.1) for some $z$ such that $\chi_{N}(z) \in\left[c N^{m}, \lambda_{0} V^{1 / 2}\right]$, then (7.1) is valid for every such $z$.

Proof. Let $z^{\prime}$ and $z_{N}$ be the solutions to $\chi_{N}\left(z^{\prime}\right)=c N^{m}$ and $\chi_{N}\left(z_{N}\right)=\lambda_{0} V^{1 / 2}$. For large $N, z^{\prime} \leq z_{N}$. It suffices to prove that

$$
\begin{equation*}
z_{N}-z^{\prime} \leq O\left(N^{-m-1}\right) \tag{7.2}
\end{equation*}
$$

By (6.28),

$$
\begin{equation*}
1-z^{\prime} / z_{N} \leq \frac{2}{\chi_{N}\left(z^{\prime}\right)}=\frac{2}{c N^{m}}, \tag{7.3}
\end{equation*}
$$

and therefore, since $z_{N} \leq 2 N^{-1}$ by (6.23),

$$
\begin{equation*}
z_{N}-z^{\prime} \leq z_{N} \frac{2}{c N^{m}} \leq \frac{4}{c N^{m+1}} \tag{7.4}
\end{equation*}
$$

This completes the proof.

### 7.2 Lace expansion estimates

To simplify the notation, for $k \geq 2$ and $M \geq 1$ we define

$$
\begin{equation*}
\pi_{k}^{(M)}=\sum_{x \in \mathbb{Q}^{N}} \pi_{k}^{(M)}(x) \tag{7.5}
\end{equation*}
$$

Thus $\pi_{k}^{(M)}$ counts the number of $k$-step $M$-loop lace graphs depicted in Figure 2, with no restriction to end at a specific vertex $x$ and with the mutual avoidance of subwalks dictated by the compatible edges in (3.19).

The following proposition prepares two estimates for the proof of Theorem 1.3. Its first inequality will be used to see that we can neglect large $M$ in the computation of the asymptotic expansions of Theorem 1.3 to a certain order, and the second says that for small $M$ we can neglect large $k$.

Proposition 7.2. Let $\lambda_{0}>0$ be sufficiently small and let $p \in\left(0, \frac{1}{2}\right)$ and $\zeta_{p}=z_{N}\left(\lambda_{0}\right)\left(1-V^{-p}\right)$. For each $j \geq 1$ there is a constant $C_{j}$, independent of $N$, such that

$$
\begin{equation*}
\sum_{k=2}^{\infty} \sum_{M=j}^{\infty} k \pi_{k}^{(M)} \zeta_{p}^{k} \leq C_{j} N^{-j} \tag{7.6}
\end{equation*}
$$

For each $j \geq 2$ and $M \geq 1$, there is a constant $C_{M, j}$, independent of $N$, such that

$$
\begin{equation*}
\sum_{k=j}^{\infty} k \pi_{k}^{(M)} \zeta_{p}^{k} \leq C_{M, j} N^{-\lceil j / 2\rceil} . \tag{7.7}
\end{equation*}
$$

Proof. The inequality (7.6) follows from the proof of Lemma 6.1, together with the facts that here we have an extra factor $\zeta_{p} \prec N^{-1}$ compared to $\partial_{z} \Pi_{z}^{(M)}$ and that $\beta_{\zeta_{p}} \prec N^{-1}$ by (6.34). Indeed, the inequality (6.22) is a term-by-term bound which gives

$$
\begin{equation*}
\left\|\partial_{z} \Pi_{z}^{(M)}\right\|_{1} \prec N \beta_{z} M\left(2 c_{2} \beta_{z}\right)^{M-1}, \tag{7.8}
\end{equation*}
$$

and hence (with $j$-dependent constants)

$$
\begin{equation*}
\sum_{k=2}^{\infty} \sum_{M=j}^{\infty} k \pi_{k}^{(M)} \zeta_{p}^{k}=\sum_{M=j}^{\infty} \zeta_{p}\left\|\partial_{z} \Pi_{z}^{(M)}\right\|_{1} \prec \zeta_{p} \beta_{\zeta_{p}}^{j-1} \prec N^{-j} \tag{7.9}
\end{equation*}
$$

It remains to prove (7.7). Constants are permitted to depend on $M$ and $j$ in the rest of the proof.
For (7.7), we first obtain preliminary estimates using Lemma 4.1. The bound on the random walk transition probability from (4.1) asserts that for any integer $i \geq 0$ we have

$$
\begin{equation*}
\left\|D^{* i}\right\|_{\infty} \leq c_{i} N^{-\lceil i / 2\rceil} . \tag{7.10}
\end{equation*}
$$

To avoid a notational clash, we temporarily write the value $p_{z}$ defined by (5.7) as $q_{z}$. Then, by using the bound $\hat{G}_{\zeta_{p}}(k) \leq 2 \hat{C}_{q_{\zeta_{p}}}(k)$ obtained from the bootstrap estimate $f_{2}(z) \leq 2$ (recall (5.9)), we see that

$$
\begin{equation*}
\left\|\hat{D}^{i} \hat{G}_{\zeta_{p}}\right\|_{\hat{2}}^{2} \leq 4\left\|\hat{D}^{i} \hat{C}_{q_{\zeta_{p}}}\right\|_{\hat{2}}^{2} . \tag{7.11}
\end{equation*}
$$

By extracting the zero mode, we can conclude from the fact that $\hat{C}_{q_{\zeta_{p}}}(0)=\chi_{N}\left(\zeta_{p}\right)$, together with (6.33) and the second bound of Lemma 4.1 that, for any integer $i \geq 0$,

$$
\begin{equation*}
\left\|\hat{D}^{i} \hat{G}_{\zeta_{p}}\right\|_{2}^{2} \leq \frac{4}{V} \chi_{N}\left(\zeta_{p}\right)^{2}+\frac{4}{V} \sum_{k \neq 0} \frac{\hat{D}(k)^{2 i}}{\left[1-q_{\zeta_{p}} N \hat{D}(k)\right]^{2}} \prec \frac{1}{V^{1-2 p}}+N^{-i} \prec N^{-i} \tag{7.12}
\end{equation*}
$$

Consider first the case $M=1$ of (7.7). Recall that $\Pi_{z}^{(1)}(x)$ is nonzero only for $x=0$, in which case it is the generating function for self-avoiding returns to the origin. We are interested in returns that take $k \geq j$ steps, and the factor $k$ in (7.7) counts the number of ways to label the distinct $k$ vertices of the walk forming the return. By relaxing the avoidance constraint for the first $j$ steps, and also the avoidance between the remaining two parts of the walk separated by the labelled vertex (in the worst case that it is not among the first $j$ steps) then we see that

$$
\begin{equation*}
\sum_{k=j}^{\infty} k \pi_{k}^{(M)} \zeta_{p}^{k} \leq\left(\left(\zeta_{p} N D\right)^{* j} * G_{\zeta_{p}} * G_{\zeta_{p}}\right)(0) \tag{7.13}
\end{equation*}
$$

One contribution to the right-hand side arises from the zero-step walks in each factor of $G_{\zeta_{p}}$, and in other contributions there is at least one step from those factors, so (as in (5.14))

$$
\begin{equation*}
\sum_{k=j}^{\infty} k \pi_{k}^{(M)} \zeta_{p}^{k} \leq\left(\zeta_{p} N D\right)^{* j}(0)+\left(\left(\zeta_{p} N D\right)^{*(j+1)} * G_{\zeta_{p}} * G_{\zeta_{p}}\right)(0) \tag{7.14}
\end{equation*}
$$

By (7.10) and the fact that $\zeta_{p} N \prec 1$, the first term is of order $N^{-\lceil j / 2\rceil}$. Via the inverse Fourier transform and the Cauchy-Schwarz inequality, by (7.12) the second term is at most

$$
\begin{equation*}
\left(\zeta_{p} N\right)^{j+1}\left\|D^{*(j+1)} * G_{\zeta_{p}} * G_{\zeta_{p}}\right\|_{\infty} \prec\left\|\hat{D}^{j+1} \hat{G}_{\zeta_{p}}\right\|_{\hat{2}}\left\|\hat{G}_{\zeta_{p}}\right\|_{\hat{2}} \prec N^{-(j+1) / 2} \prec N^{-\lceil j / 2\rceil}, \tag{7.15}
\end{equation*}
$$

and this proves (7.7) for the case $M=1$.
For $M \geq 2$, by (6.16) and (6.11) we have

$$
\begin{align*}
\sum_{k=2}^{\infty} k \pi_{k}^{(M)} z^{k}=\left\|z \partial_{z} \Pi_{z}^{(M)}\right\|_{1} & \leq(2 M-1)\left\|z \partial_{z} H_{z}\right\|_{\infty}\left\|H_{z} * G_{z}\right\|_{\infty}^{M-1} \\
& \leq(2 M-1)\left\|H_{z} * G_{z}\right\|_{\infty}^{M} \tag{7.16}
\end{align*}
$$

The effect of the restriction $k \geq j$ on this bound is to require that the factors $\left\|H_{z} * G_{z}\right\|_{\infty}$ become modified to ensure that at least $j$ steps are taken in total, so there is a restriction that the $i^{\text {th }}$ factor must take at least $j_{i}$ steps with $j_{1}+\cdots j_{M} \geq j$. As in the case $M=1$, with this restriction the $i^{\text {th }}$ factor of $\left\|H_{z} * G_{z}\right\|_{\infty}$ can be replaced via an upper bound by $\left\|D^{* j_{i}} * G_{z} * G_{z}\right\|_{\infty}$. Then, as in (7.14)-(7.15), we find that the unrestricted upper bound is replaced by

$$
\begin{equation*}
\sum_{k=j}^{\infty} k \pi_{k}^{(M)} \zeta_{p}^{k} \leq C_{M, j} N^{-\sum_{i=1}^{M}\left\lceil j_{i} / 2\right\rceil} \leq C_{M, j} N^{-\lceil j / 2\rceil} \tag{7.17}
\end{equation*}
$$

This completes the proof.

### 7.3 Asymptotic expansion to all orders

We now prove the part of Theorem 1.3 concerning existence of the asymptotic expansions for $\mu_{N}$ and $A_{N}$ to all orders with integer coefficients. The numerical computation of coefficients is deferred to Section 7.4.

Theorem 7.3. Let $\lambda_{0}>0$ be sufficiently small. Let $m \in \mathbb{N}$, fix $c>0$ (independent of $N$ but possibly depending on $m$ ), and suppose that $z$ obeys $\chi_{N}(z) \in\left[c N^{m}, \lambda_{0} V^{1 / 2}\right]$. Then there are integers $a_{n}$ for $n \in \mathbb{N}$, which are universal constants that do not depend on the particular choice of $z$, such that

$$
\begin{equation*}
z=\sum_{n=1}^{m} a_{n} N^{-n}+O\left(N^{-m-1}\right) \tag{7.18}
\end{equation*}
$$

The constant in the error term depends on $m, \lambda_{0}, c$, but does not depend otherwise on $z$. The same is true for the amplitude $A_{N}$.

Proof. We first consider the expansion for $z$. Fix $p \in\left(0, \frac{1}{2}\right)$. By Proposition 7.1, since $\chi_{N}\left(\zeta_{p}\right) \asymp V^{p}$ by Proposition 2.4, it suffices to prove that $\zeta_{p}$ has an asymptotic expansion to all orders with integer coefficients. By (3.10) we have

$$
\begin{equation*}
N \zeta_{p}=1-\hat{\Pi}_{\zeta_{p}}(0)+O\left(V^{-p}\right) \tag{7.19}
\end{equation*}
$$

The last term on the right-hand side is negligible compared to any fixed inverse power of $N$ so it plays no role.

We follow the basic approach of [27] but incorporate the significant simplifications used in [12]. To lighten the notation we write $s=N^{-1}$. We will prove by induction on $m \geq 1$ that there are integers $a_{n}$ such that

$$
\begin{equation*}
\zeta_{p}=\sum_{n=1}^{m} a_{n} s^{n}+O\left(s^{m+1}\right) \tag{7.20}
\end{equation*}
$$

The integers $a_{n}$ will be shown to be universal constants. The starting point is

$$
\begin{equation*}
\zeta_{p}=s\left[1-\hat{\Pi}_{\zeta_{p}}(0)\right]+O\left(s V^{-p}\right)=s\left[1-\sum_{M=1}^{\infty}(-1)^{M} \sum_{i=2}^{\infty} \pi_{i}^{(M)} \zeta_{p}^{i}\right]+O\left(s V^{-p}\right) \tag{7.21}
\end{equation*}
$$

Since $\hat{\Pi}_{\zeta_{p}}(0)=O(s)$ (e.g., by (7.6) with $j=1$ ), (7.21) gives the base case $m=1$ for the induction, with $a_{1}=1$.

To advance the induction, we assume now that (7.20) holds for some $m \geq 1$ and we will prove that it holds for $m+1$. It follows from Proposition 7.2 that for $j \geq 1$,

$$
\begin{equation*}
\sum_{k=2}^{\infty} \sum_{M=j}^{\infty} \pi_{k}^{(M)} \zeta_{p}^{k} \leq C_{j} s^{j}, \tag{7.22}
\end{equation*}
$$

and that for $j \geq 2$ and $M \geq 1$,

$$
\begin{equation*}
\sum_{k=j}^{\infty} \pi_{k}^{(M)} \zeta_{p}^{k} \leq C_{M, j} s^{\lceil j / 2\rceil} \tag{7.23}
\end{equation*}
$$

(We do not yet need the factor $k$ included in the sums of Proposition 7.2 to advance this induction, that factor is needed only later for $A_{N}$. Of course the bounds remain valid without that factor since the left-hand sides are smaller without it.) With (7.22)-(7.23), we see from (7.21) that

$$
\begin{equation*}
\zeta_{p}=s\left[1-\sum_{M=1}^{m}(-1)^{M} \sum_{k=2}^{2 m} \pi_{k}^{(M)} \zeta_{p}^{k}\right]+O\left(s^{m+2}\right)=s\left[1-\sum_{k=2}^{2 m} b_{N, k} \zeta_{p}^{k}\right]+O\left(s^{m+2}\right), \tag{7.24}
\end{equation*}
$$

with

$$
\begin{equation*}
b_{N, k}=\sum_{M=1}^{m}(-1)^{M} \pi_{k}^{(M)} . \tag{7.25}
\end{equation*}
$$

We classify contributions to $\pi_{k}^{(M)}$ according to the total number $\delta$ of dimensions explored by a $k$-step walk $\omega$. Let $\pi_{k, \delta}^{(M)}$ denote the contribution to $\pi_{k}^{(M)}$ due to walks starting at 0 which explore exactly $\delta$ dimensions, with the first step taken with the first coordinate, the first subsequent step involving a new coordinate taken with the second coordinate, the first subsequent step with a new coordinate taken with the third coordinate, and so on. Then, because the number of dimensions cannot exceed $k-1$ (because the last step must close a loop), it follows by symmetry that

$$
\begin{equation*}
\pi_{k}^{(M)}=\sum_{\delta=1}^{k-1} \pi_{k, \delta}^{(M)} \prod_{j=0}^{\delta-1}(N-j) . \tag{7.26}
\end{equation*}
$$

By definition, $\pi_{k, \delta}^{(M)}$ is a nonnegative integer which is independent of $N$, a universal number which counts certain lace diagrams. With (7.25), we find that

$$
\begin{equation*}
b_{N, k}=\sum_{q=1}^{k-1} \beta_{q, k} N^{q} \tag{7.27}
\end{equation*}
$$

with integer coefficients $\beta_{q, k}$ which are independent of $N$.
By the induction hypothesis (7.20),

$$
\begin{equation*}
\zeta_{p}^{k}=\left[\sum_{n=1}^{m} a_{n} s^{n}+O\left(s^{m+1}\right)\right]^{k}=s^{k}\left[\sum_{n=0}^{m-1} \gamma_{n, k} s^{n}+O\left(s^{m}\right)\right] \tag{7.28}
\end{equation*}
$$

with $N$-independent integer coefficients $\gamma_{n, k}$. With (7.24) and (7.27), this gives

$$
\begin{equation*}
\zeta_{p}=s\left(1-\sum_{k=2}^{2 m} \sum_{q=1}^{k-1} \beta_{q, k} s^{-q} s^{k}\left[\sum_{n=0}^{m-1} \gamma_{n, k} s^{n}+O\left(s^{m}\right)\right]\right)+O\left(s^{m+2}\right) \tag{7.29}
\end{equation*}
$$

The only term in the above product which can give rise to an non-integer power or coefficient is the $O\left(s^{m}\right)$ term. However this term is multiplied by $s^{1-q+k} \leq s^{2}$, and hence gives rise to a contribution which is $O\left(s^{m+2}\right)$. Therefore

$$
\begin{equation*}
\zeta_{p}=\sum_{n=1}^{m+1} d_{n} s^{n}+O\left(s^{m+2}\right), \tag{7.30}
\end{equation*}
$$

with integer coefficients $d_{n}$, which must agree with $a_{n}$ for $n \leq m$. This gives (7.20) with $m$ replaced by $m+1$, so the induction is advanced and the proof of (7.18) is complete.

Recall that the amplitude

$$
\begin{equation*}
A_{N}=\frac{1}{F_{N}\left(\zeta_{p}\right)-\zeta_{p} F_{N}^{\prime}\left(\zeta_{p}\right)}=\frac{1}{F_{N}\left(\zeta_{p}\right)+\zeta_{p} N+\zeta_{p} \partial_{z} \hat{\Pi}_{\zeta_{p}}(0)} \tag{7.31}
\end{equation*}
$$

was defined in the proof of Corollary 2.6. The existence of the expansion for $A_{N}$ then follows similarly from (7.6)-(7.7) (now we do need their factor $k$ ), by substitution of the expansion for $\zeta_{p}$ into

$$
\begin{equation*}
\frac{1}{A_{N}}=\zeta_{p} N+\sum_{k=2}^{2 m} \sum_{M=1}^{m}(-1)^{M} k \pi_{k}^{(M)} \zeta_{p}^{k}+O\left(N^{-m-1}\right) \tag{7.32}
\end{equation*}
$$

Again the coefficients in the expansion are universal integers.

### 7.4 Coefficient computation

The expansion coefficients for $z$ in (7.18) (equivalently, for $\zeta_{p}$ ) can be computed from (7.24) and (7.26) once suitably many of the lace graph counts $\pi_{k, \delta}^{(M)}$ are known. To compute to within an error of order $s^{6}$, only $M \leq 4$ and $k \leq 8$ are needed, by the first equality of (7.24). Also, we can restrict to $k-\delta \leq 4$ because in a term $\pi_{k, \delta}^{(M)} \zeta_{p}^{k}$ the largest possible contribution is of order $N^{\delta} s^{k}=s^{k-\delta}$ and if $k-\delta \geq 5$ then this contributes to $\zeta_{p}$ only at order $s^{6}$, due to the prefactor $s$ in (7.24).

To discuss the enumeration of lace graphs, we elaborate on the definition of an $M$-loop diagram as follows. Recall the definition of compatible edges from Definition 3.3. If the $2 M-1$ subwalks in the $M$-loop diagram of Figure 2 are labelled in the order they occur in a walk as $1,2, \ldots, 2 M-1$, then the subwalks are mutually avoiding (apart from the required intersections) due to the compatible edges, with the following patterns: [123] for $M=2$; [1234], [345] for $M=3 ;[1234],[3456],[567]$ for $M=4$. This means, e.g., for $M=4$, that subwalks $1,2,3,4$ are mutually avoiding apart from the enforced intersections explicitly depicted in Figure 2, as are subwalks $3,4,5,6$ and subwalks $5,6,7$. However, subwalks not grouped together are permitted to intersect, e.g., for $M=4$, subwalks 1,2 are permitted to intersect subwalks $5,6,7$, and subwalks 3 and 4 can intersect subwalk 7 .

Extensive computer assisted enumerations of lace graphs for $\mathbb{Z}^{d}$ are given in [13]. Lace graphs on the hypercube are a subset of those on $\mathbb{Z}^{d}$. The relevant nonzero counts are found to be:

$$
\begin{align*}
& \pi_{2,1}^{(1)}=1, \quad \pi_{4,2}^{(1)}=1, \quad \pi_{6,3}^{(1)}=4, \quad \pi_{8,4}^{(1)}=27,  \tag{7.33}\\
& \pi_{3,1}^{(2)}=1, \quad \pi_{5,2}^{(2)}=3, \quad \pi_{7,3}^{(2)}=15, \quad \pi_{4,1}^{(3)}=1, \quad \pi_{6,2}^{(3)}=5, \quad \pi_{5,1}^{(4)}=1 . \tag{7.34}
\end{align*}
$$

These numbers arise by inspection from Tables 18, 20, 22, 24 of [13] by discounting configurations which are possible on $\mathbb{Z}^{d}$ but not on $\mathbb{Q}^{N}$, as well as from the trivial counts $\pi_{k, 1}^{(M)}=\delta_{k, M+1}$. E.g., $\pi_{6,2}^{(1)}=3$ on $\mathbb{Z}^{d}$ but it is zero on $\mathbb{Q}^{N}$ where the 2 -dimensional subspace contains only 4 vertices. With computer assistance, it would be possible to enumerate more lace graphs on $\mathbb{Q}^{N}$ and thereby compute more terms in the asymptotic expansion for $z_{N}$.

To compute the expansion coefficients for $z_{N}=z_{N}\left(\lambda_{0}\right)$ (recall Proposition 7.1), as in (7.24) we use

$$
\begin{equation*}
z_{N}=s\left[1-\sum_{M=1}^{m}(-1)^{M} \sum_{k=2}^{2 m} \pi_{k}^{(M)} z_{N}^{k}\right]+O\left(s^{m+2}\right) \tag{7.35}
\end{equation*}
$$

iteratively with

$$
\begin{equation*}
\pi_{k}^{(M)}=\sum_{\delta=1}^{k-1} \pi_{k, \delta}^{(M)} \prod_{j=0}^{\delta-1}(N-j) \tag{7.36}
\end{equation*}
$$

as follows. We insert $z_{N}=s+O\left(s^{2}\right)$ into (7.35) with $m=1$ and obtain

$$
\begin{align*}
z_{N} & =s\left[1-(-1)^{1} \pi_{2}^{(1)} z_{N}^{2}\right]+O\left(s^{3}\right) \\
& =s\left[1+N \cdot 1 \cdot s^{2}\right]+O\left(s^{3}\right)=s+s^{2}+O\left(s^{3}\right) \tag{7.37}
\end{align*}
$$

Another iteration gives

$$
\begin{align*}
z_{N} & =s\left[1-(-1)^{1}\left(\pi_{2}^{(1)} z_{N}^{2}+\pi_{4}^{(1)} z_{N}^{4}\right)-(-1)^{2} \pi_{3}^{(2)} z_{N}^{3}\right]+O\left(s^{4}\right) \\
& =s\left[1+\left(N \cdot 1 \cdot\left(s+s^{2}\right)^{2}+N(N-1) \cdot 1 \cdot s^{4}\right)-N \cdot 1 \cdot s^{3}\right]+O\left(s^{4}\right) \\
& =s+s^{2}+2 s^{3}+O\left(s^{4}\right) . \tag{7.38}
\end{align*}
$$

Continuing in this way, we find that

$$
\begin{equation*}
z_{N}=s+s^{2}+2 s^{3}+7 s^{4}+39 s^{5}+O\left(s^{6}\right) . \tag{7.39}
\end{equation*}
$$

Now we can simply substitute the expansion for $z_{N}$ into (7.32) to get the expansion for $A_{N}^{-1}$ (and hence for $A_{N}$ ) up to and including terms of order $s^{4}$. The result is

$$
\begin{equation*}
A_{N}=1+s+4 s^{2}+26 s^{3}+231 s^{4}+O\left(s^{5}\right) . \tag{7.40}
\end{equation*}
$$

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