

ON THE THRESHOLD OF CHAOS IN RANDOM BOOLEAN CELLULAR AUTOMATA

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ABSTRACT. A random boolean cellular automaton is a network of boolean gates where the inputs, the boolean function, and the initial state of each gate are chosen randomly. In this article, each gate has two inputs. Let a (respectively c) be the probability the gate is assigned a constant function (respectively a non-canalyzing function, i.e., EQUIVALENCE or EXCLUSIVE OR). Previous work has shown that when $a > c$, with probability asymptotic to 1, the random automaton exhibits very stable behavior: almost all of the gates stabilize, almost all of them are weak, i.e., they can be perturbed without affecting the state cycle that is entered, and the state cycle is bounded in size. This article gives evidence that the condition $a = c$ is a threshold of chaotic behavior: with probability asymptotic to 1, almost all of the gates are still stable and weak, but the state cycle size is unbounded. In fact, the average state cycle size is superpolynomial in the number of gates.

1. INTRODUCTION

A topic of current interest in the theory of complex systems is the existence of sharp boundaries between highly ordered and chaotic behavior. Evidence for this phenomenon has been provided by computer simulations, where some parameter is varied. As the parameter passes through a certain critical region, the behavior of the system rapidly changes between the two extremes of stability and chaos [5]. In this article, we examine one of the simplest, yet most intensively studied, models of complex systems — the random boolean cellular automaton. We present analytic results proving that there is such a threshold for these systems.

Boolean cellular automata were introduced by Kauffman in [3]. He was interested in determining the conditions when complex systems exhibit stable behavior. Three ways of measuring the stability are:

1. The proportion of gates that stabilize, i.e. eventually stop changing.
2. The proportion of weak gates, i.e., gates that can be perturbed without affecting the state cycle that is entered.
3. The size of the state cycle that the system eventually enters.

The second and third of these measures are finite discrete analogues of criteria that are used to characterize chaos in dynamical systems. A small proportion of weak gates is similar to sensitivity to initial conditions, and a large state cycle is similar to nonperiodicity.

Computer simulations, beginning with those described in [3], have suggested that certain classes of randomly constructed boolean cellular automata possess all three forms of stability with high probability. The basic random model is where each gate

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has two inputs, and the inputs, the boolean functions assigned to the gates, and the initial state are all chosen with uniform probability distributions. In particular, for each gate, each of the 16 boolean functions of two arguments has probability $1/16$ of being assigned to the gate.

In spite of extensive experimental work on these automata, comparatively little has actually been proven about them. The first article containing formal proofs of stability in the basic model is by Luczak and Cohen [6]. They show that as $n \rightarrow \infty$, for almost all random boolean cellular automata with n gates, the number of stable gates and the number of weak gates is asymptotic to n . They also give a nontrivial upper bound on the state cycle size. In Lynch [8], it was shown that by giving a slight bias to the probability of certain of the boolean functions assigned to the gates (on the order of $\log \log n / \log n$), for almost all random boolean cellular automata with n gates, the state cycle size can be bounded above by n^γ , for some γ . However, the proof failed when the bias was reduced to 0, i.e. for the basic random model. This suggested two lines of research. First, a more extensive analysis of random boolean cellular automata with nonuniform probabilities of the boolean functions might be possible. This could be a step toward understanding more realistic models of complex systems. Also, the breakdown of the proof at the uniform distribution hinted at a threshold phenomenon.

Treating all 16 of the two argument boolean functions individually seems to be a complex undertaking. A classification of the boolean functions due to Kauffman [4] has proven useful. He referred to certain boolean functions as canalyzing. We will define this precisely in the next section, but for now it suffices to note that among the canalyzing functions are the constant functions; i.e. the function that outputs 0 regardless of its inputs and its negation that always outputs 1. Further, among the two-argument boolean functions, there are only two non-canalyzing functions: the EQUIVALENCE function that outputs 1 if and only if both of its inputs have the same value, and its negation the EXCLUSIVE OR.

Let a (respectively c) be the probability that the boolean function assigned to a gate is constant (respectively noncanalyzing). In Lynch [9] it was shown that when $a > c$, with probability asymptotic to 1, the random boolean cellular automaton is very stable in all three senses: almost all of the gates are stable and weak, and the state cycle size is bounded.

In this article, we investigate the case $a = c \neq 0$. This includes the basic model as the special case $a = c = 1/8$. We prove that the first two kinds of stability still hold (although the bounds here are not as tight), but the state cycle size is unbounded for almost all automata. In fact, the average state cycle size is greater than any polynomial in n . Thus, the automaton still appears to be stable when viewed locally, i.e. at the level of a typical gate, but large state cycles are a global symptom of the beginning of instability. In a future article, we will describe the behavior when $a < c$. At present, it is known that the proportion of weak gates is less than n by a nontrivial factor.

2. DEFINITIONS

Let n be a natural number. A *boolean cellular automaton* B with n gates is a triple $\langle D, F, x \rangle$ where D is a directed graph with vertices $1, \dots, n$ (referred to as *gates*), $F = (f_1, \dots, f_n)$ is a sequence of boolean functions, and $x = (x_1, \dots, x_n) \in \{0, 1\}^n$ (the set of 0-1 sequences of length n). In this article, each gate will have

indegree two, and each boolean function will have two arguments. We say that gate j is an *input* to gate i if (j, i) is an edge of D . B is a finite state automaton with state set $\{0, 1\}^n$ and initial state x . The pair $\langle D, F \rangle$ defines the transition function of B in the following way. For each $i = 1, \dots, n$ let $j_i < k_i$ be the inputs of i . Given $y = (y_1, \dots, y_n) \in \{0, 1\}^n$, $B(y) = (f_1(y_{j_1}, y_{k_1}), \dots, f_n(y_{j_n}, y_{k_n}))$. That is, the state of B at time 0 is x , and if its state at time t is $y \in \{0, 1\}^n$, then its state at time $t + 1$ is $B(y)$.

Our first set of definitions pertains to the aspects of stability that will be studied.

Definitions 2.1. Let $B = \langle D, F, x \rangle$ be a boolean cellular automaton.

1. We put $B^t(x)$ for the state of B at time t , and $f_i^t(x)$ for the value of its i th component, or gate, at time t .
2. Since the number of states is finite, i.e. 2^n , there exist times $t_0 < t_1$ such that $B^{t_0}(x) = B^{t_1}(x)$. Let t_1 be the first time at which this occurs. Then $B^{t+t_1-t_0}(x) = B^t$ for all $t \geq t_0$. We refer to the set of states $\{B^t(x) : t \geq t_0\}$ as the *state cycle* of B , to distinguish it from a cycle of D in the graph-theoretic sense.
3. Gate i *stabilizes* in t steps if for all $t' \geq t$, $f_i^{t'}(x) = f_i^t(x)$.
4. Gate i is *weak* if, letting \bar{x}^i be identical to x except that its i th component is $1 - x_i$,

$$\exists t_0 \exists d \forall t (t \geq t_0 \Rightarrow B^t(x) = B^{t+d}(\bar{x}^i)).$$

That is, changing the state of i does not affect the state cycle that is entered.

The next definitions describe a property of boolean functions that plays a key role in the characterization of the threshold between order and chaos.

Definitions 2.2. Let $f(x_1, x_2)$ be a boolean function of two arguments.

1. We say that f *depends* on argument x_1 if for some $v \in \{0, 1\}$, $f(0, v) \neq f(1, v)$. A symmetric definition applies when f depends on x_2 . Similarly, if $\langle D, F, x \rangle$ is a boolean cellular automaton, $f_i = f$, and the inputs of gate i are j_{i1} and j_{i2} , then for $m = 1, 2$, i depends on j_{im} if f depends on x_m .
2. The function f is said to be *canalyzing* if there is some $m = 1$ or 2 and some values $u, v \in \{0, 1\}$ such that for all $x_1, x_2 \in \{0, 1\}$, if $x_m = u$ then $f(x_1, x_2) = v$. Argument x_m of f is said to be a *forcing argument* with *forcing value* u and *forced value* v . Likewise, if $\langle D, F, x \rangle$ is a boolean cellular automaton and f_i is a canalyzing function with forcing argument x_m , forcing value u and forced value v , then input j_{im} is a *forcing input* of gate i . That is, if the value of j_{im} is u at time t , then the value of i is guaranteed to be v at time $t + 1$.

All of these definitions generalize immediately to boolean functions of arbitrarily many arguments. In the case of two argument boolean functions, the only non-canalyzing functions are EQUIVALENCE and EXCLUSIVE OR. The two constant functions $f(x, y) = 0$ and $f(x, y) = 1$ are trivially canalyzing, as are the four functions that depend on only one argument:

$$\begin{aligned} f(x, y) &= x, \\ f(x, y) &= \neg x, \\ f(x, y) &= y, \text{ and} \\ f(x, y) &= \neg y. \end{aligned}$$

The remaining eight boolean functions of two arguments are canalizing, and they are all similar in the sense that both arguments are forcing with a single value, and there is one forced value. A typical example is the OR function. Both arguments are forcing with 1, and the forced value is 1.

The notion of forcing, defined next, is a combinatorial condition that is useful in characterizing stability. It depends on D and F , but not on x .

Definition 2.3. Again, $\langle D, F, x \rangle$ is a boolean cellular automaton. Using induction on t , we define what it means for gate i to be *forced to a value v in t steps*.

If f_i is the constant function $f(x_1, x_2) = v$, then i is forced to v in t steps for all $t \geq 0$.

If the inputs j_{i1} and j_{i2} of i are forced to u_1 and u_2 respectively in t steps, then i is forced to $f_i(u_1, u_2)$ in $t + 1$ steps.

If f_i is a canalizing function with forcing argument x_m , forcing value u , and forced value v , and j_{im} is forced to u in t steps, then i is forced to v in $t + 1$ steps.

By induction on t it can be seen that if i is forced in t steps, then it stabilizes for all initial states x in t steps.

The following combinatorial notions will be used in characterizing forcing structures. We assume the reader is familiar with the basic concepts of graph theory (see e.g. Harary [2]). Unless otherwise stated, *path* and *cycle* shall mean directed path and cycle in the digraph D .

Definitions 2.4. 1. For any gate i in D with inputs j_{i1} and j_{i2} , let

$$S_0^-(i) = \{i\} \text{ and} \\ S_{d+1}^-(i) = S_d^-(j_{i1}) \cup S_d^-(j_{i2}).$$

2. Then

$$N_d^-(i) = \bigcup_{c \leq d} S_c^-(i).$$

That is, $N_d^-(i)$ is the set of all gates that are connected to i by a path of length at most d .

3. If I is a set of gates, then $N_d^-(I) = \cup_{i \in I} N_d^-(i)$.

4. In a similar way we define $S_d^+(i)$ and $N_d^+(i)$, the set of all gates reachable from i by a path of length at most d .

Note that whether i is forced in d steps is completely determined by the restriction of D and F to $N_d^-(i)$.

We will examine the asymptotic behavior of *random* boolean cellular automata. For each boolean function f of two arguments, we associate a probability $a_f \in [0, 1]$, where $\sum_f a_f = 1$. The random boolean cellular automaton with n gates is the result of three random processes. First, a random digraph where every gate has indegree two is generated. Independently for each gate, its two inputs are selected from the $\binom{n}{2}$ equally likely possibilities. Next, each gate is independently assigned a boolean function of two arguments, using the probability distribution $\langle a_f : f : \{0, 1\}^2 \rightarrow \{0, 1\} \rangle$. Lastly, the initial state x is chosen using the uniform distribution on $\{0, 1\}^n$. We will use $\tilde{B} = \langle \tilde{D}, \tilde{F}, \tilde{x} \rangle$ to denote a random boolean cellular automaton generated as above. For any properties \mathcal{P} and \mathcal{Q} pertaining to boolean cellular automata, we put $\text{pr}(\mathcal{P}, n)$ for the probability that the random boolean cellular automaton on n gates has property \mathcal{P} and $\text{pr}(\mathcal{P}|\mathcal{Q}, n)$ for the conditional probability that \mathcal{P} holds,

given that \mathcal{Q} holds. Usually, we will omit the n in these expressions since it will be understood. Some of the properties we will investigate depend only on D and F . In that case, the expression describing \mathcal{P} will involve $\langle \tilde{D}, \tilde{F} \rangle$ instead of \tilde{B} , and pr can be regarded as the probability measure on random $\langle \tilde{D}, \tilde{F} \rangle$. Similar notation will be used for properties that depend only on D . Random variables will be denoted by boldface capital letters, and $\mathbf{E}(\mathbf{X})$ will be the expectation of \mathbf{X} .

We classify the two argument boolean functions as follows:

1. \mathcal{A} contains the two constant functions.
2. \mathcal{B}_1 contains the four canalyzing functions that depend on one argument.
3. \mathcal{B}_2 contains the eight canalyzing functions that depend on both arguments.
4. \mathcal{C} contains the two non-canalyzing functions.

Then the probabilities that a gate is assigned a function in each of the categories are:

$$\begin{aligned} a &= \sum_{f \in \mathcal{A}} a_f \\ b_1 &= \sum_{f \in \mathcal{B}_1} a_f \\ b_2 &= \sum_{f \in \mathcal{B}_2} a_f \\ c &= \sum_{f \in \mathcal{C}} a_f \end{aligned}$$

Lastly, we put $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $b = b_1 + b_2$, the probability that a gate is assigned a nonconstant canalyzing function. Throughout the rest of the article, we assume the following symmetry conditions on our distributions:

$$\begin{aligned} a_{f(x,y)} &= a_{f(y,x)} \text{ for all } f \in \mathcal{B}_1 \\ a_{f(x,y)} &= a_{f(\neg x, \neg y)} \text{ for all } f \in \mathcal{B}_2 \\ a_{f(x,y)} &= a_{\neg f(x,y)} \text{ for all } f \in \mathcal{C}. \end{aligned}$$

Also, \log shall always mean \log_2 , and \ln is the natural logarithm.

3. LOCAL STABILITY

A key idea, first stated in [6], is that almost all of the gates have sufficiently large neighborhoods that are trees. We will use the following version of this fact.

Lemma 3.1. *For any positive α and unbounded increasing function $\omega(n)$,*

$$\lim_{n \rightarrow \infty} \text{pr}(\tilde{D} \text{ has at most } \omega(n)(\log n)^3 n^{2\alpha} \text{ gates } i \text{ such that } N_{\alpha \log n}^-(i) \text{ is not a tree}) = 1.$$

The same is true for $N_{\alpha \log n}^+$.

Proof. For each gate i , let \mathbf{X}_i be the indicator random variable that is 1 if and only if $N_{\alpha \log n}^-(i)$ is not a tree, and let $\mathbf{X} = \sum_{i=1}^n \mathbf{X}_i$. If $\mathbf{X}_i = 1$, then there exists a path P of length $p \leq \alpha \log n$ beginning at some gate k and ending at i and another path Q of length q , $1 \leq q \leq \alpha \log n$, beginning at k , disjoint from P except at k and its other endpoint, which must be in P . There are no more than n^p ways of

choosing P and no more than $n^{q-1} \times p$ ways of choosing Q . The probability of any such choice is bounded above by $(2/n)^{p+q}$. Therefore

$$\begin{aligned} \mathbf{E}(\mathbf{X}_i) &\leq \sum_{p=0}^{\alpha \log n} \sum_{q=1}^{\alpha \log n} 2^{p+q} p n^{-1} \\ &\leq (\alpha \log n)^3 n^{2\alpha-1}. \end{aligned}$$

Then $\mathbf{E}(\mathbf{X}) \leq (\alpha \log n)^3 n^{2\alpha}$, and the Lemma follows by Markov's inequality. A similar argument applies to $N_{\alpha \log n}^+$. \square

Another result we will need, from [9], is a recurrence relation for the probability that a gate is forced, given that its in-neighborhood is treelike.

Lemma 3.2. *For $d \geq 0$ and $v \in \{0, 1\}$ let*

$$\begin{aligned} p_d(v) &= \text{pr}(\text{gate } i \text{ is forced to } v \text{ in } d \text{ steps} \mid N_d^-(i) \text{ is a tree}) \text{ and} \\ p_d &= p_d(0) + p_d(1). \end{aligned}$$

Then

$$p_d(0) = p_d(1)$$

and p_d satisfies the following recurrence.

$$\begin{aligned} p_0 &= a \text{ and} \\ p_{d+1} &= a + b p_d + c p_d^2. \end{aligned} \tag{3.3}$$

The fixed points of the recursion (3.3) are a/c and 1. Consequently, when $a \geq c$, p_d converges to 1. We will prove this for $a = c$, but Figure 1 gives a graphical explanation of this fact. Part (a) illustrates a typical case when $a > c$. In this case, as proven in [9], the convergence is geometric. The convergence when $a = c$, shown in Part (b), is not as rapid, but is still sufficiently fast.

Lemma 3.4. *Let d be a natural number. Then*

$$p_d \geq 1 - \frac{1}{ad}.$$

Proof. Let $q_d = 1 - p_d$. Then from (3.3), the recurrence for q_d is

$$q_{d+1} = q_d - a q_d^2 \tag{3.5}$$

Letting $r_d = 1/q_d$ and using induction on d , we will finish the proof by showing that $r_d \geq ad$. When $d = 0$, this is evident. By (3.5),

$$\frac{1}{r_{d+1}} = \frac{1 - a/r_d}{r_d}$$

and so

$$\begin{aligned} r_{d+1} &= \frac{r_d}{1 - a/r_d} \\ &\geq r_d + a, \end{aligned}$$

which establishes the induction step. \square

Our two main results on local stability are essentially generalizations of similar results in [6]. Theorem 3.8 also improves the lower bound on the number of weak gates that was given in [6].

FIGURE 1. Examples of the convergence of p_d . The dotted line \cdots indicates the successive iterations of (3.3) from $p_0 = a$ towards 1.

(a) $a = 1/2, c = 1/4$.

(b) $a = c = 1/4$.

Theorem 3.6. *Let $\alpha < 1/2$ and $\omega(n)$ be any unbounded increasing function. Then*

$$\lim_{n \rightarrow \infty} \text{pr}(\langle \tilde{D}, \tilde{F} \rangle \text{ has at least } n(1 - \omega(n)/\log n) \text{ gates that are forced in } \alpha \log n \text{ steps}) = 1.$$

Proof. Let \mathbf{Y} be the random variable that counts the number of gates i in $\langle \tilde{D}, \tilde{F} \rangle$ such that $N_{\alpha \log n}^-(i)$ is a tree and i is not forced in $\alpha \log n$ steps. By Lemma 3.4,

$$\mathbf{E}(\mathbf{Y}) \leq \frac{\mathbf{n}}{\mathbf{a}\alpha \log \mathbf{n}}.$$

By Markov's inequality,

$$\text{pr}\left(\mathbf{Y} \geq \frac{\mathbf{n}\omega(\mathbf{n})}{\mathbf{a}\alpha \log \mathbf{n}}\right) \leq \frac{1}{\omega(n)} \rightarrow 0.$$

Therefore, together with Lemma 3.1, with probability asymptotic to 1, there are at most

$$\omega(n) \left[\frac{n}{\mathbf{a}\alpha \log n} + (\log n)^3 n^{2\alpha} \right] = O\left(\frac{n\omega(n)}{\log n}\right)$$

gates not forced in $\alpha \log n$ steps. \square

Recalling that the notion of forcing is stronger than stability, we have

Corollary 3.7. *Let $\alpha < 1/2$ and $\omega(n)$ be any unbounded increasing function. Then*

$$\lim_{n \rightarrow \infty} \text{pr}(\langle \tilde{D}, \tilde{F} \rangle \text{ has at least } n(1 - \omega(n)/\log n) \text{ gates that stabilize in } \alpha \log n \text{ steps}) = 1.$$

Theorem 3.8. *Let $\omega(n)$ be any unbounded increasing function. Then*

$$\lim_{n \rightarrow \infty} \text{pr}(\tilde{B} \text{ has at least } n(1 - \omega(n)/\log n) \text{ weak gates}) = 1.$$

Proof. We will use the following fact from [6].

Fact . For any gate i and natural number r ,

$$\text{pr}(|S_1^+(i)| = r) = \frac{2^r}{r!} e^{-2} \left(1 + O\left(\frac{r}{n}\right)\right).$$

Thus, for $r > \log n$,

$$\begin{aligned} \text{pr}(|S_1^+(i)| = r) &= O\left(\frac{(2e)^r}{r^r}\right) \\ &= O(2^{-r \log r/2}) \\ &= o(n^{-2}) \end{aligned}$$

and the probability that there exists some gate with $|S_1^+(i)| > \log n$ is asymptotic to 0. For $r \leq \log n$,

$$\text{pr}(|S_1^+(i)| = r) = \frac{2^r}{r!} e^{-2} + o(n^{-1/2}).$$

By Lemma 3.1, this remains true even when the probability is conditioned on $N_{\alpha \log n}^+(i)$ being a tree, $\alpha < 1/4$.

For any gate i and natural number $d \leq \alpha \log n$, assuming $N_{\alpha \log n}^+(i)$ is a tree, let ϕ_d be the probability that there is some gate $j \in N_d^+(i)$ whose value is affected at

step d , if the value of i is changed at step 0. That is, taking \bar{x}^i as in Definitions 2.1 (4), $f_j^d(\bar{x}^i) \neq f_j^d(x)$. We will show by induction on $d = 1, \dots, \alpha \log n$ that $\phi_d \leq 4/d$. Clearly $\phi_1 \leq 1$. Assuming $N_{\alpha \log n}^+(i)$ is a tree, let $j \in S_1^+(i)$ and ρ be the probability that a change to i affects j in step 1. Since $N_{d+1}^+(i)$ is a tree, for any $k \in N_d^+(j)$, a change to i affects k in step $d+1$ if and only if a change to i affects j in step 1 and a change to j affects k in step d . Therefore, assuming $|S_1^+(i)| \leq \log n$,

$$\phi_{d+1} = 1 - \sum_{r=0}^{\lfloor \log n \rfloor} \text{pr}(|S_1^+(i)| = r) \times [(1 - \rho) + \rho(1 - \phi_d)]^r. \quad (3.9)$$

We show that $\rho = 1/2$. The three possibilities to consider are that $f_j \in \mathcal{B}_1$, $f_j \in \mathcal{B}_2$, and $f_j \in \mathcal{C}$. Let k be the other input of j . Assuming $f_j \in \mathcal{B}_1$, two out of the four functions in \mathcal{B}_1 result in i affecting j in step 1. That is, if $i < k$ they are $f(x, y) = x$ and $f(x, y) = \neg x$, and similarly for $k < i$. Altogether, the probability of the first case is $b_1/2$ by the symmetry property $a_{f(x,y)} = a_{f(y,x)}$. Now suppose $f_j \in \mathcal{B}_2$, and say $i < k$ and $x_k = 0$. (The cases when $k < i$ or $x_k = 1$ are similar.) Then $f_j(0, 0) \neq f_j(1, 0)$. But f_j is canalyzing on both inputs, so $f_j(0, 1) = f_j(1, 1)$. Four out of the eight functions in \mathcal{B}_2 satisfy these conditions, and the sum of their probabilities is $b_2/2$ by the symmetry property $a_{f(x,y)} = a_{f(\neg x, \neg y)}$. The probability of the third case is c , so altogether $\rho = b/2 + c = 1/2$. Therefore by the Fact and Equation (3.9),

$$\begin{aligned} \phi_{d+1} &= 1 - e^{-2} \sum_{r=0}^{\lfloor \log n \rfloor} \frac{(2 - \phi_d)^r}{r!} + o(n^{-1/2} \log n) \\ &= 1 - e^{-\phi_d} + o(n^{-1/2} \log n) \\ &\leq \phi_d - \frac{\phi_d^2}{2} + \frac{\phi_d^3}{6} + o(n^{-1/2} \log n). \end{aligned}$$

If $\phi_d \leq 1/(\log n)^2$, then $\phi_{d+1} \leq 4/(\alpha \log n) \leq 4/(d+1)$. If $\phi_d > 1/(\log n)^2$, then $\phi_{d+1} \leq \phi_d - \phi_d^2/4$, and using the same argument that was applied to Equation (3.5), $\phi_{d+1} \leq 4/(d+1)$.

Now let \mathbf{Y} be the random variable that counts the number of gates i in \tilde{B} such that $N_{\alpha \log n}^+(i)$ is a tree and i is not weak. Then by what we have just shown,

$$\mathbf{E}(\mathbf{Y}) \leq \frac{4\mathbf{n}}{\alpha \log \mathbf{n}}.$$

The rest of the proof proceeds as in Theorem 3.6. \square

4. LOWER BOUNDS ON AVERAGE STATE CYCLE SIZE

4.1. Main Results. Let the random variable \mathbf{C} denote the size of the state cycle of \tilde{B} .

Theorem 4.1. *For any constant γ and sufficiently large n ,*

$$\mathbf{E}(\mathbf{C}) > \mathbf{n}^\gamma.$$

In the next theorem, $\mathbf{E}(\mathbf{C} | \langle \tilde{\mathbf{D}}, \tilde{\mathbf{F}} \rangle)$ is the expected state cycle size of a random $\langle \tilde{D}, \tilde{F} \rangle$ averaged over all $x \in \{0, 1\}^n$.

Theorem 4.2. *There is a constant $\gamma > 0$ such that*

$$\lim_{n \rightarrow \infty} \text{pr}(\mathbf{E}(\mathbf{C} | \langle \tilde{\mathbf{D}}, \tilde{\mathbf{F}} \rangle) \geq \mathbf{n}^\gamma) = \mathbf{1}.$$

These theorems will follow from a key result (Lemma 4.15) on the probability of existence of certain kinds of structures in $\langle \tilde{\mathbf{D}}, \tilde{\mathbf{F}} \rangle$. We first define these structures and prove some basic facts about them. Let α be a fixed real number such that $0 < \alpha < 1/2$. In the following we will put m for $\lceil \alpha \log n \rceil$.

4.2. Vortices.

Definition 4.3. Let $B = \langle D, F, x \rangle$ be a boolean cellular automaton on n gates. A *vortex* of circumference d consists of two disjoint subsets of gates $R = \{r_0, \dots, r_{d-1}\}$ and $S = \{s_0, \dots, s_{d-1}\}$ satisfying the following conditions for $0 \leq i < d$.

1. $(r_i, r_{i+1 \pmod{d}}) \in D$.
2. $(s_i, r_i) \in D$.
3. s_i is forced in m steps.
4. The value that s_i is forced to is not a forcing value for f_{r_i} .

We refer to it as a vortex on R, S or simply $R \cup S$ if we do not need to distinguish R and S .

An example is given in Figure 2.

The essential characteristics of such a vortex are captured by the directed labeled graph

$$V = \langle R \cup S, D \upharpoonright (R \cup S), F \upharpoonright R, v_0, \dots, v_{d-1} \rangle$$

where v_i is the value that s_i is forced to, for $i = 0, \dots, d-1$. That is, V is simply the restriction of $\langle \tilde{\mathbf{D}}, \tilde{\mathbf{F}} \rangle$ to $R \cup S$, with the functions labeling the gates in S replaced by their forced values. The isomorphism class of V is called a *vortex type*.

For any such vortex type τ , and any $V \in \tau$ as above, we put $\lambda(\tau)$ for the size of the automorphism group on V and $\pi(\tau)$ for $\prod_{i=0}^{d-1} a_{f_{r_i}}$. Clearly $\lambda(\tau)$ and $\pi(\tau)$ do not depend on the choice of $V \in \tau$. The significance of these two quantities is that $(2d)!/\lambda(\tau)$ is the number of distinct labelings of the gates in any $V \in \tau$, and $\pi(\tau)$ is the conditional probability that two disjoint subsets R and S , each of size d , form a vortex of type τ , given that conditions (1)–(3) in Definition 4.3 hold. The following two facts will be used later in the combinatorial analysis of vortices. Let T be the set of all vortex types of circumference d .

FIGURE 2. A schematic diagram of a vortex of circumference 8. Shaded circles are members of S , and unshaded circles are in R . The enlargement shows a typical (s_i, r_i) pair. In this example, s_i is forced to 0 while $f_{r_i} = \vee$ (the OR function).

Lemma 4.4. *There exists $\rho \in (0, 1)$ such that*

$$\sum_{\substack{\tau \in T \\ \lambda(\tau) > 1}} \pi(\tau) \leq d\rho^{d/2}.$$

Proof. The only nontrivial automorphisms of $V \in \tau$ are those that take each r_i to $r_{i+p \pmod{d}}$, where $1 \leq p < d$. But this implies

$$f_{r_i} = f_{r_{i+p \pmod{d}}} \text{ for } i = 0, \dots, d-1. \quad (4.5)$$

We may assume p is the minimal number satisfying (4.5), and therefore $p|d$, so $p \leq d/2$. Let $\rho = \max\{a_f : f \notin \mathcal{A}\}$, $q = d/p$, and T_p be the set of vortex types satisfying (4.5). Then

$$\begin{aligned} \sum_{\tau \in T_p} \pi(\tau) &\leq (\rho^p)^{q-1} \\ &= \rho^{d-p} \\ &\leq \rho^{d/2}. \end{aligned}$$

The factor d in the Lemma is a crude upper bound on the number of divisors of d . \square

Lemma 4.6. *We have*

$$1 - d2^{-d/2} \leq \sum_{\tau \in T} \pi(\tau) \leq 1.$$

Proof. For any sequence $v = (v_0, \dots, v_{d-1}) \in \{0, 1\}^d$, let T_v be the set of all vortex types in T such that the labeling of S is isomorphic to v . Let U consist of all sequences $v \in \{0, 1\}^d$ that do not have any nontrivial cyclic permutations, and let $T' = T - \cup_{v \in U} T_v$. Then, using the same methods as in Lemma 4.4, $|U| \geq 2^d - d2^{d/2}$. Since

$$\sum_{\tau \in T} \pi(\tau) = \sum_{v \in U} \sum_{\tau \in T_v} \pi(\tau) + \sum_{\tau \in T'} \pi(\tau),$$

we will be done by showing that for all $v \in \{0, 1\}^d$

$$\sum_{\tau \in T_v} \pi(\tau) = 2^{-d}. \quad (4.7)$$

For every $i = 0, \dots, d-1$, v_i does not force f_{r_i} . Therefore one of the following possibilities must hold.

1. $f_{r_i} \in \mathcal{B}_1$, and the input on which r_i depends is $r_{i-1 \pmod{d}}$.
2. $f_{r_i} \in \mathcal{B}_2$, and v_i is not a forcing value for f_{r_i} .
3. $f_{r_i} \in \mathcal{C}$.

Case (1) has probability $b_1/2$ by the symmetry property $a_{f(x,y)} = a_{f(y,x)}$, Case (2) has probability $b_2/2$ by the symmetry property $a_{f(x,y)} = a_{f(-x,-y)}$, and Case (3) has probability c . Therefore, given that s_i is labeled with v_i , the probability that one of the three cases above holds is $1/2$, and (4.7) follows. \square

The existence of vortices of sufficiently large prime circumference will be used to prove the lower bounds on average state cycle size. This is the relevance of the next two basic facts. When we refer to the state of a vortex, we simply mean the state of B restricted to $R \cup S$.

Lemma 4.8. *A vortex enters its state cycle in at most m steps. Its state cycle is completely determined by the initial state of $R \cup N_m^-(S)$.*

Proof. After m steps, for each $i = 0, \dots, d-1$, s_i is forced to some value v . Since v is not a forcing value for f_{r_i} , assuming $s_i < r_{i-1 \pmod{d}}$ (the case when $s_i > r_{i-1 \pmod{d}}$ is symmetric), $f_{r_i}(v, y) = y$ or $\neg y$. Let us use the notation $f_{r_i}(v, y) = g_i(y)$ where $g_i(y) = y$ or $\neg y$, depending on which case holds.

In other words, after m steps, the vortex is equivalent to a cycle of 1-input gates, none of which are constants. Let $u = (u_0, \dots, u_{d-1})$ be the state of these gates after m steps. We need only show that u reoccurs.

Suppose there is an even number of gates r_i such that $g_i(y) = \neg y$. Then after $m + d$ steps, the state of each r_i will be u_i . If there is an odd number, then the state of each r_i after $m + 2d$ steps will again be u_i . In either case, the state cycle has been reentered in not more than $2d$ steps. \square

Lemma 4.9. *If the circumference of the vortex is prime, then the size of its state cycle is 1, 2, d , or $2d$.*

Proof. From the proof of Lemma 4.8, we know that the state repeats every $2d$ steps, and thus the state cycle size is a factor of $2d$. \square

To simplify calculations in the remainder of the proofs, we condition all events on the following two properties. Let $\beta > \alpha$ be fixed.

1. There are no distinct vortices on R, S and R', S' respectively of circumference less than or equal to $2\beta \log n$ such that

$$(R \cup N_m^-(S)) \cap (R' \cup N_m^-(S')) \neq \emptyset.$$

2. For every vortex of circumference less than or equal to $2\beta \log n$ on any R, S , for all $s, s' \in S$,

$$\begin{aligned} N_m^-(s) \text{ is a tree,} \\ N_m^-(s) \cap N_m^-(s') = \emptyset, \text{ and} \\ N_m^-(s) \cap R = \emptyset. \end{aligned}$$

A boolean cellular automaton satisfying these conditions is said to be *simple*. By the next lemma, this will not affect the asymptotic probabilities that will be computed.

Lemma 4.10. *We have*

$$\text{pr}(\langle \tilde{D}, \tilde{F} \rangle \text{ is simple}) = 1 - n^{-\Omega(1)}.$$

Proof. One way that a boolean cellular automaton can fail Condition (1) above is if there exist distinct vortices on R, S and R', S' such that $R \cap R' \neq \emptyset$. Then there are gates $r_i, r_j \in R$ (possibly the same) and a path of gates in R' beginning at r_i and ending at r_j , disjoint from R except at the endpoints. If the circumference of R is d and the length of the path is l , then $p = d + l - 2$ is the number of gates in R and the path. Letting κ range over all choices of (d, l, i, j, C) such that $d, l \leq 2\beta \log n$, $0 \leq i, j < d$, and C is a subset of $\{1, \dots, n\}$ of size p , we put \mathbf{X}_κ for the indicator random variable that is 1 if and only if the gates in C form a cycle R and a path as above. Then $\mathbf{X} = \sum_\kappa \mathbf{X}_\kappa$ is an upper bound on the expected number of pairs of vortices such that $R \cap R' \neq \emptyset$.

Now

$$\begin{aligned} \mathbf{E}(\mathbf{X}_\kappa) &\leq p! \times \frac{1}{\binom{n}{2}} \times \left(\frac{n-1}{\binom{n}{2}} \right)^{p-1} \times \left(\frac{1}{2} \right)^p \\ &= \frac{p!}{(n-1)n^p} \end{aligned}$$

because $p!$ is an upper bound on the number labelings of C , $1/\binom{n}{2}$ is the probability that Condition (1) of Definition 4.3 holds for r_j , $(n-1)/\binom{n}{2}$ is the probability that Condition (1) holds for all other gates in C , and $1/2$ is the probability that Condition (4) holds for a gate, given that Condition (3) holds. There are $O((\log n)^4)$ choices for d, l, i , and j , and for each of these choices, there are $\binom{n}{p}$ choices for C . Therefore

$$\begin{aligned} \mathbf{E}(\mathbf{X}) &= O((\log n)^4 n^{-1}) \\ &= n^{-\Omega(1)}. \end{aligned}$$

On the other hand, if $R \cap R' = \emptyset$, but Condition (1) of simplicity is still violated, then there exists a gate g and two paths P and P' of lengths $p, p' \leq m+1$ beginning at g and disjoint everywhere else, one path ending in R and the other in R' . There are at most n ways of choosing g , $(m+2)^2$ ways of choosing p and p' , and $n^{p+p'-2} \times (2\beta \log n)^2$ ways of choosing the remaining gates in P and P' . The probability of such a choice is bounded above by $(2/n)^{p+p'}$. Therefore, by Markov's inequality, the probability that P and P' exist is bounded above by

$$\begin{aligned} n^{p+p'-1} \times (m+2)^2 \times (2\beta \log n)^2 \times \left(\frac{2}{n} \right)^{p+p'} &= O((\log n)^4 2^{2\alpha \log n} n^{-1}) \\ &= n^{-\Omega(1)}. \end{aligned}$$

A similar proof enables us to show that Condition (2) holds with probability $1 - n^{-\Omega(1)}$. \square

One final condition on vortices that will be needed is that they should enter a large (relative to their circumference) state cycle from many initial states. This is formalized by the next definition.

Definition 4.11. A vortex of circumference d is *strong* if for at least $1/2$ of the initial states of B , the state cycle of the vortex is greater than or equal to d .

Lemma 4.12. *If \tilde{B} is simple, then for any vortex V of circumference $d \geq m+2$ where d is prime, the probability that V is strong is greater than or equal to $1/2 - o(1)$.*

Proof. If at least $1/2$ of the initial states take V to a state cycle of size $\geq d$, then we are done. Otherwise, by Lemma 4.9, at least $1/2$ of the inputs take V to a fixed point or a 2-cycle.

Since $d \geq m+2$, with probability $\geq 1 - o(1)$, V has at least one i such that $f_{r_i} \in \mathcal{C}$. Without loss of generality, let us assume $f_{r_0} \in \mathcal{C}$, and let x be any input that takes V to a fixed point or 2-cycle. Let V' be the vortex obtained from V by changing f_{r_0} to $\neg f_{r_0}$. Using the notation of Lemma 4.8, this has the effect of changing g_0 to $\neg g_0$.

Let w_0 and w_1 be the values of r_{m+1} in V at times m and $m+1$ respectively. Then, by Condition (2) in the definition of simplicity, w_0 and w_1 are also the values of r_{m+1} in V' at those times. Since V enters a fixed point or 2-cycle from x , the sequence of values of r_{m+1} beginning at time m must be $w_0, w_1, w_0, w_1, \dots$ (possibly $w_0 = w_1$). If V' also enters a fixed point or 2-cycle, then the sequence of values of r_{m+1} beginning at m is also $w_0, w_1, w_0, w_1, \dots$. In particular, assuming m is even, its state at time $2m+2$ is w_0 . If m is odd, a similar argument applies.

For $j = 0, \dots, m+1$, let u_j (respectively u'_j) be the value of r_j in V (respectively V') at time $m+j+1$. Then by induction on j , $u'_j = \neg u_j$. But then $w_0 = u'_{m+1} = \neg u_{m+1} = \neg w_0$, contradiction. Therefore V' must enter a state cycle of size $\geq d$ when started in state x .

To summarize, we have shown that with probability $1 - o(1)$, there is some gate in R , say r_0 , such that $f_{r_0} \in \mathcal{C}$, and V is strong when r_0 is assigned one of the functions in \mathcal{C} . By symmetry, the two choices are equally likely, and the Lemma follows. \square

4.3. Combinatorial Lemmas. We now derive lower bounds on the probability of existence of sets of vortices of various circumference. Let $D_n \subseteq [\beta \log n, 2\beta \log n]$ and $|D_n| = k(n)$ for each positive integer n . Our goal is to find an asymptotic estimate for the probability that \tilde{B} has strong vortices of circumference d , for all $d \in D_n$. The approach is based on sieve methods that are extensions of Ch. Jordan's formula and Bonferroni's inequality. The monograph of Bollobás [1] contains an exposition of these formulas. The extensions that we will use are described in full generality in Lynch [7].

Fixing n , put $k = k(n)$ and index the elements of D_n by d_1, \dots, d_k . For each $i = 1, \dots, k$ let \mathcal{B}_i be an indexed set of all subsets of $\{1, \dots, n\}$ of size $2d_i$, say $\mathcal{B}_i = \{C_{ij} : 1 \leq j \leq \binom{n}{2d_i}\}$. For each C_{ij} let \mathcal{P}_{ij} be the property " B has a strong vortex of circumference d_i on C_{ij} ."

Take any family of sets

$$\vec{S} = \{S_i : 1 \leq i \leq k\}$$

such that $S_i \subseteq \mathcal{B}_i$. Let

$$E^{\geq}(\vec{S}) = \bigcap_{i=1}^k \left(\bigcap_{C_{ij} \in S_i} \mathcal{P}_{ij} \right).$$

That is, $E^{\geq}(\vec{S})$ is the set of boolean cellular automata on n gates that have strong vortices on C_{ij} for each $C_{ij} \in S_i$, $i = 1, \dots, k$. Let $\vec{s} = \langle s_i : 1 \leq i \leq k \rangle$ be a sequence of positive integers and

$$L(\vec{s}) = \sum_{|\vec{S}_i|=s_i} \text{pr}(E^{\geq}(\vec{S}) | \tilde{B} \text{ is simple}).$$

We put $\sum(\vec{s})$ for $\sum_{i=1}^k s_i$ and $\langle r \rangle^k$ for $\langle r, \dots, r \rangle$, the sequence of k r 's, for any real number r . We use $\vec{s} \geq \langle r \rangle^k$ to mean $s_i \geq r$ for $i = 1, \dots, k$. The next two lemmas are applications of the extensions of Ch. Jordan's formula and Bonferroni's inequality mentioned earlier.

Lemma 4.13. *We have*

$$\begin{aligned} \text{pr} \left(\bigwedge_{i=1}^k \tilde{B} \text{ has a strong vortex of circumference } d_i \mid \tilde{B} \text{ is simple} \right) \\ = \sum_{\vec{s} \geq \langle 1 \rangle^k} (-1)^{\Sigma(\vec{s})-k} L(\vec{s}). \end{aligned}$$

Lemma 4.14. *For any $K \geq k$*

$$\sum_{\substack{\vec{s} \geq \langle 1 \rangle^k \\ \Sigma(\vec{s}) \geq K}} (-1)^{\Sigma(\vec{s})-K} L(\vec{s}) \geq 0.$$

The main result of this subsection is the next lemma.

Lemma 4.15. *Let p_m be as given in Lemma 3.2, $k(n) = O(\log n / \log \log n)$, and σ_i be the probability that a vortex of circumference d_i is strong, for $i = 1, \dots, k(n)$. Then*

$$\begin{aligned} \text{pr} \left(\bigwedge_{i=1}^k \tilde{B} \text{ has a strong vortex of circumference } d_i \wedge \tilde{B} \text{ is simple} \right) \\ = (1 - n^{-\Omega(1)}) \prod_{i=1}^k (1 - e^{-p_m^{d_i} \sigma_i}) + n^{-\Omega(\log \log n)}. \end{aligned}$$

Proof. We will show that

$$\begin{aligned} \text{pr} \left(\bigwedge_{i=1}^k \tilde{B} \text{ has a strong vortex of circumference } d_i \mid \tilde{B} \text{ is simple} \right) \\ = (1 - n^{-\Omega(1)}) \prod_{i=1}^k (1 - e^{-p_m^{d_i} \sigma_i}) + n^{-\Omega(\log \log n)}. \end{aligned}$$

The Lemma will follow by Lemma 4.10. For $i = 1, \dots, k$ let T_i be the set of all vortex types of circumference d_i . Take any $\vec{S} = \{S_i : 1 \leq i \leq k\}$ such that each $S_i \subseteq \mathcal{B}_i$, $|S_i| = s_i$, and $C_{gh} \cap C_{ij} = \emptyset$ for all $(g, h) \neq (i, j)$, $C_{gh} \in S_g$, $C_{ij} \in S_i$. By Lemma 3.2,

$$\text{pr}(E^{\geq}(\vec{S}) \mid \tilde{B} \text{ is simple}) = \prod_{i=1}^k \left[\sum_{\tau \in T_i} \frac{(2d_i)!}{\lambda(\tau)} \times \left(\frac{1}{\binom{n}{2}} \right)^{d_i} \times \left(\frac{p_m}{2} \right)^{d_i} \times \pi(\tau) \times \sigma_i \right]^{s_i}.$$

By Lemmas 4.4 and 4.6, this is

$$\prod_{i=1}^k \left[(1 - n^{-\Omega(1)}) \left(\frac{p_m}{n(n-1)} \right)^{d_i} \times (2d_i)! \times \sigma_i \right]^{s_i}.$$

Then, using the falling factorial power notation $n^{\underline{k}} = \prod_{i=0}^{k-1} (n-i)$,

$$L(\vec{s}) = \frac{n^{\Sigma 2d_i s_i}}{\prod_{i=1}^k ((2d_i)!)^{s_i} s_i!} \times \prod_{i=1}^k \left[(1 - n^{-\Omega(1)}) \left(\frac{p_m}{n(n-1)} \right)^{d_i} \times (2d_i)! \times \sigma_i \right]^{s_i}.$$

Let us approximate $L(\vec{s})$ when $\sum(\vec{s}) \leq (\log n)^2$. Since $1 - x = e^{-x - O(x^2)}$ for $x \rightarrow 0$, $(1 - n^{-\Omega(1)})^{\sum(\vec{s})} = 1 - n^{-\Omega(1)}$. Then, using Stirling's formula,

$$L(\vec{s}) = (1 - n^{-\Omega(1)}) \prod_{i=1}^k \frac{(p_m^{d_i} \sigma_i)^{s_i}}{s_i!}.$$

The number of sequences \vec{s} such that $\vec{s} \geq \langle 1 \rangle^k$ and $\sum(\vec{s}) = (\log n)^2$ is bounded above by

$$\binom{(\log n)^2}{k-1} = \log n^{O(\log n / \log \log n)} = n^{O(1)}.$$

For any such \vec{s} , there is some i such that $s_i \geq \log n$. Therefore

$$\begin{aligned} \sum_{\substack{\vec{s} \geq \langle 1 \rangle^k \\ \sum(\vec{s}) = (\log n)^2}} L(\vec{s}) &= \frac{n^{O(1)}}{(\log n)!} \\ &= n^{-\Omega(\log \log n)}. \end{aligned}$$

By Lemmas 4.13 and 4.14 (taking $K = (\log n)^2$),

$$\begin{aligned} &\text{pr} \left(\bigwedge_{i=1}^k \tilde{B} \text{ has a strong vortex of circumference } d_i \mid \tilde{B} \text{ is simple} \right) \\ &= (1 - n^{-\Omega(1)}) \left[\sum_{\substack{\vec{s} \geq \langle 1 \rangle^k \\ \sum(\vec{s}) \leq (\log n)^2}} (-1)^{\sum(\vec{s}) - k} \prod_{i=1}^k \frac{(p_m^{d_i} \sigma_i)^{s_i}}{s_i!} \right] + n^{-\Omega(\log \log n)} \\ &= (1 - n^{-\Omega(1)}) \left[\sum_{\langle 1 \rangle^k \leq \vec{s} \leq \langle (\log n)^2 \rangle^k} (-1)^{\sum(\vec{s}) - k} \prod_{i=1}^k \frac{(p_m^{d_i} \sigma_i)^{s_i}}{s_i!} \right] + n^{-\Omega(\log \log n)} \\ &= (1 - n^{-\Omega(1)}) \prod_{i=1}^k \left(\sum_{1 \leq s \leq (\log n)^2} (-1)^{s-1} \frac{(p_m^{d_i} \sigma_i)^s}{s!} \right) + n^{-\Omega(\log \log n)} \\ &= (1 - n^{-\Omega(1)}) \prod_{i=1}^k \left(1 - e^{-p_m^{d_i} \sigma_i} \right) + n^{-\Omega(\log \log n)}. \quad \square \end{aligned}$$

Corollary 4.16. *If $k(n) = O(\log n / \log \log n)$, then*

$$\text{pr} \left(\bigwedge_{i=1}^k \tilde{B} \text{ has a vortex of circumference } d_i \wedge \tilde{B} \text{ is simple} \right) = n^{-o(1)}.$$

Proof. By Lemma 3.4

$$p_m^{d_i} \geq \left(1 - \frac{1}{a\alpha \log n} \right)^{2\beta \log n} \sim e^{-2\beta / (a\alpha)},$$

and by Lemma 4.12, $\sigma_i \geq 1/2 - o(1)$. Therefore

$$\begin{aligned} \prod_{i=1}^k \left(1 - e^{-p_m^{d_i} \sigma_i}\right) &\geq \prod_{i=1}^k \frac{p_m^{d_i} \sigma_i}{2} \\ &\geq (e^{-2\beta/(a\alpha)}/5)^{O(\log n / \log \log n)} \text{ (any constant } > 4 \text{ will do)} \\ &= n^{-o(1)}. \quad \square \end{aligned}$$

4.4. Completion of Proofs.

Proof of Theorem 4.1. For each n let D_n be the set of primes in $[\beta \log n, 2\beta \log n]$. By the Prime Number Theorem [10],

$$k(n) \sim \frac{\beta \log n}{\ln \log n}.$$

Therefore by Corollary 4.16,

$$\text{pr} \left(\bigwedge_{i=1}^k \tilde{B} \text{ has a strong vortex of circumference } d_i \wedge \tilde{B} \text{ is simple} \right) = n^{-o(1)}.$$

Take any \tilde{B} satisfying the above condition. Since \tilde{B} is simple, with probability $\geq 2^{-k(n)} = n^{-o(1)}$, a random starting state takes each strong vortex of circumference d_i , $i = 1, \dots, k(n)$, to a state cycle of size d_i or $2d_i$. That is, for such a starting state, \tilde{B} enters a state cycle of size greater than or equal to

$$\begin{aligned} (\beta \log n)^{k(n)} &= e^{(1-o(1))\beta \log n} \\ &= n^{\beta \log e - o(1)}. \end{aligned}$$

Thus, with probability $\geq n^{-o(1)}$, \tilde{B} enters a state cycle larger than $n^{\beta \log e - o(1)}$. By Markov's inequality,

$$\mathbf{E}(\mathbf{C}) \geq \mathbf{n}^\beta.$$

Since β was arbitrarily large, the Theorem follows. \square

Proof of Theorem 4.2. Take D_n as in the previous proof. Fixing n , for $i = 1, \dots, k(n)$ let \mathbf{X}_i be the indicator random variable that is 1 if and only if \tilde{B} has a strong vortex of circumference d_i , and $\mathbf{X} = \sum_{i=1}^{k(n)} \mathbf{X}_i$. Then, still assuming simplicity, by Lemma 4.15,

$$\mathbf{E}(\mathbf{X}_i) = (\mathbf{1} - \mathbf{n}^{-\Omega(1)}) \left(\mathbf{1} - e^{-p_m^{d_i} \sigma_i} \right) + \mathbf{n}^{-\Omega(\log \log \mathbf{n})}.$$

Since $k(n) = \Theta(\log n / \log \log n)$ and $1 - e^{-p_m^{d_i} \sigma_i} \geq e^{-2\beta/(a\alpha)}/5$,

$$\begin{aligned} \mathbf{E}(\mathbf{X}) &\sim \sum_{i=1}^{k(n)} 1 - e^{-p_m^{d_i} \sigma_i} \\ &\rightarrow \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{E}(\mathbf{X}^2) &= \sum_{i=1}^{k(n)} \mathbf{E}(\mathbf{X}_i) + 2 \sum_{1 \leq i < j \leq k(n)} \mathbf{E}(\mathbf{X}_i \mathbf{X}_j) \\ &\sim \sum_{i=1}^{k(n)} 1 - e^{-p_m^{d_i} \sigma_i} + 2 \sum_{1 \leq i < j \leq k(n)} \left(1 - e^{-p_m^{d_i} \sigma_i}\right) \left(1 - e^{-p_m^{d_j} \sigma_j}\right) \\ &\sim (\mathbf{E}(\mathbf{X}))^2. \end{aligned}$$

Therefore by Chebyshev's inequality, for any $\delta < 1$,

$$\begin{aligned} \text{pr}(\mathbf{X} \leq \delta \mathbf{E}(\mathbf{X}) | \tilde{\mathbf{B}} \text{ is simple}) &\leq \frac{\mathbf{E}(\mathbf{X}^2) - (\mathbf{E}(\mathbf{X}))^2}{(1 - \delta)^2 (\mathbf{E}(\mathbf{X}))^2} \\ &\rightarrow 0. \end{aligned}$$

That is, almost all \tilde{B} have at least $\delta k(n) e^{-2\beta/(a\alpha)}/5$ strong vortices of distinct prime circumferences in $[\beta \log n, 2\beta \log n]$. For all such automata, with probability $\geq 2^{-\delta k(n) e^{-2\beta/(a\alpha)}/5} = n^{-o(1)}$, the starting state leads to a state cycle larger than or equal to

$$\begin{aligned} (\beta \log n)^{\delta k(n) e^{-2\beta/(a\alpha)}/5} &\geq e^{\beta \delta e^{-2\beta/(a\alpha)} \log n / 5} \\ &= n^{\beta \delta e^{-2\beta/(a\alpha)} \log e / 5}. \end{aligned}$$

By Markov's inequality,

$$\mathbf{E}(\mathbf{C} | \langle \tilde{\mathbf{D}}, \tilde{\mathbf{F}} \rangle) \geq \mathbf{n}^{\beta \delta e^{-2\beta/(a\alpha)} \log e / 5 - o(1)},$$

and we can take any $\gamma < \beta \delta e^{-2\beta/(a\alpha)} \log e / 5$. In fact, as noted in Corollary 4.16, the 5 can be replaced by 4. \square

Note that $\beta e^{-2\beta/(a\alpha)}$ has a unique maximum when $\beta = a\alpha/2$. Therefore, since the only restrictions on α , β , and δ are that $\alpha < 1/2$, $\alpha < \beta$, and $\delta < 1$, the γ in Theorem 4.2 can be arbitrarily close to $e^{-2/a} \log e / 8$.

5. DISCUSSION

As mentioned in the Introduction, there have been many computer simulations of random boolean cellular automata, specifically the uniform distribution model where $a = c = 1/8$. The results indicate a rather slow, even sublinear, growth rate of the average state cycle size as a function of the number of gates. At first glance, the superpolynomial average size of state cycles given by Theorem 4.1 seems to contradict the experimental evidence. There are two possible resolutions to this. First, $a = c$ is the border where large state cycles are just beginning to appear. This may not be noticeable until the number of gates is quite large. Perhaps the simulated automata were not large enough.

Second, our proof shows that the large average state cycle size is due to a small fraction of the automata that have very large state cycles. It may be that most of the automata have relatively small state cycles. Our other main result (Theorem 4.2) is consistent with this. It gives a n^γ lower bound on state cycle size averaged over all inputs, for almost all networks $\langle D, F \rangle$. The exponent γ is quite small. For $a = c = 1/8$, it is less than 2×10^{-8} . Two relevant open problems are to improve the lower bound in Theorem 4.2 and the upper bound for state cycle size in [6].

Other computer experiments indicate that systems on the edge of chaos show complex computational capability. To formalize this notion in terms of the model in this article, we should consider random boolean cellular automata with inputs and outputs. Then, instead of looking at stability measures, we should try to determine the conditions that result in substructures that compute complex functions. If the experimental evidence is correct, then the $a = c$ threshold is the region where these substructures arise. The techniques used here to prove the existence of large vortices may be applicable.

The model studied in this article is essentially a metaphor for complex biological systems. Future work in this area will inevitably lead to models with more biological detail and accuracy. Whether such models will be mathematically tractable cannot be answered now, but there are some simple generalizations of our model that may be pertinent to this question. One example is random boolean cellular automata where the probabilities of the functions assigned to gates do not necessarily satisfy any symmetry conditions. An immediate question is whether the results of [9] and this article extend to non-symmetric probabilities. Another generalization is to random boolean cellular automata whose gates need not have exactly two inputs. One-input gates are just a special type of two-input gates, but the population of three-input gates seems quite different because of the large proportion of non-analyzing functions.

Lastly, two technical problems are to analyze the stability of random boolean cellular automata without constant gates, i.e., $a = 0$ and those where $a < c$. Results on the proportion of weak gates indicate that $a < c$ is the chaotic region, but the proportion of stable gates and nontrivial bounds on state cycle size are not known. We make the following conjectures:

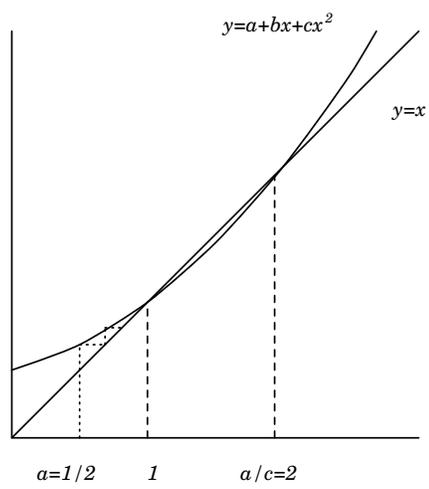
1. If $a < c$ then asymptotically a/c of the gates are stable. Recall that in this case, a/c is the smaller of the fixed points of the recurrence (3.3).
2. As $a - c$ increases, stability of the system increases. That is, the proportions of stable and weak gates increase, and the size of the state cycle decreases.

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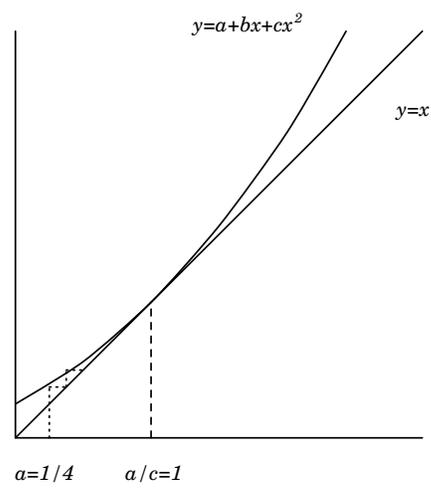
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(a)



(b)

FIGURE 1

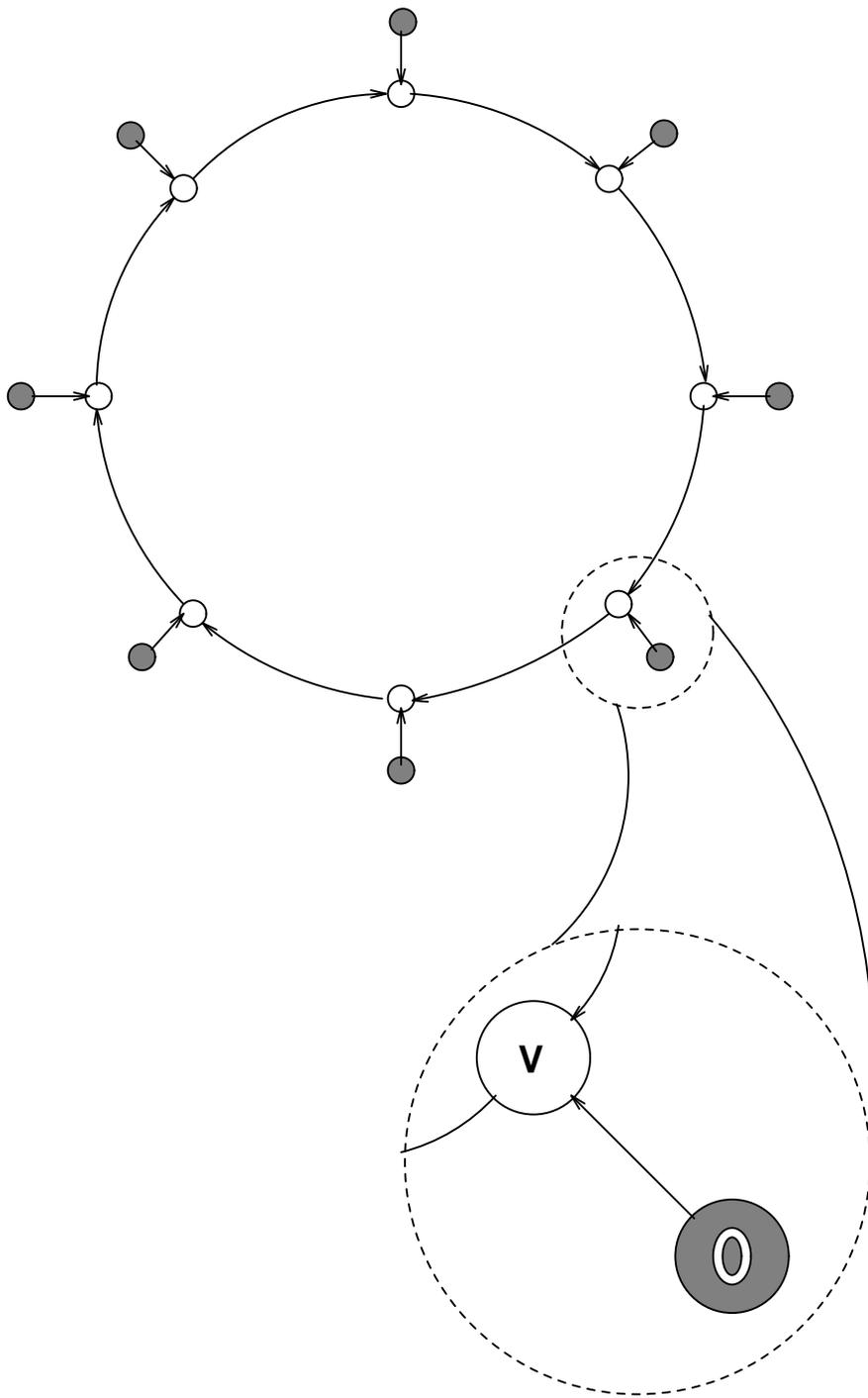


FIGURE 2