Computing Surfaces via pq-Permutations *

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Abstract. Q-permutations are very easy mathematical structures (essentially consisting in a permutation with attached a natural number) able to encode the basic information concerning any compact and connected orientable surface. At the first place, we introduce the notion of pq-permutation: an enrichment of q-permutations able to characterize surfaces in general: not only orientable but also non-orientable. In the second place, we provide a rewriting system on sets of pq-permutations in which the handling of topological surfaces throughout their polygonal presentations, is reduced to a pure and very intuitive combinatorial calculus. The system at issue is shown to enjoy both the properties of strong normalization and strict strong confluence. Finally, we recover the classification theorem by using the specific combinatorial tools here provided.

1 Introduction

The notion of *q-permutation* has been gradually introduced in a few contributions concerning theoretical computer science and, in particular, the specific ambit of linear proof theory [6]. A recent work on the geometry of linear logic proofs has established an interesting correspondence between logical demonstrations and orientable surfaces with boundary [10]. The need of reflecting this kind of geometrical structures at level of logical sequents, has induced to isolate a "syntactical counterpart" of orientable surfaces, called q-permutation [9, 1]. The idea leading to q-permutations consists in remarking that the basic information concerning any compact and connected orientable surface (possibly with boundary) can be encoded, modulo isomorphisms, by a very easy mathematical structure simply consisting in a permutation σ paired with a natural number q. Roughly speaking, whereas σ denotes, cycle by cycle, each boundary-component, q is a counter for the number of handles on the surface. The notion of q-permutation is clearly rooted in the classification theorem which states that any orientable surface turns out to be homeomorphic to either a connected sum of tori or a sphere (no tori in the connected sum) [7,8].

A more general structure able to characterize surfaces in general, not only orientable but also non-orientable, is once again suggested by the classification theorem which ensures that any non-orientable surface is always homeomorphic

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to a connected sum of projective planes. Thus, whereas the part of the structure encoding the boundary is kept unchanged (orientability does not affect boundaries), we replace our single counter with a couple of natural numbers: the first one for counting, as usual, tori (handles) and the second one for indicating projective planes. This kind of enriched structures are here called *pq-permutations*.

A standard result in algebraic topology establishes that any surface \mathscr{S} can be univocally determined by a finite set of polygons $W_{\mathscr{S}} = \{w_1, w_2, \ldots, w_n\}$, each one having edges labeled and oriented (triangularization theorem). The idea is that a surface \mathscr{S} is characterized by a finite set of polygons $W_{\mathscr{S}}$ if, modulo identification of paired edges, the quotient surface turning out by $W_{\mathscr{S}}$ is exactly \mathscr{S} . Such a set of polygons $W_{\mathscr{S}}$ is said to be a polygonal presentation of \mathscr{S} and it is usually presented as a set of words [7,8]. An effective procedure for computing surfaces from their polygonal presentations can be found, for instance, in [8]. In this classical reference, Massey reports an algorithm for transforming any given polygon into an equivalent one having the perimeter in canonical form: a standard shape in which all the fundamental information concerning the presented surface is explicitly displayed.

In this paper we provide a rewriting system \mathcal{P} on sets of pq-permutations which allows to reduce the handling of topological surfaces to a pure and very intuitive combinatorial context. The system \mathcal{P} should be seen as a computer-scientific improvement of Massey's algorithm and, more generally, of all the treatment of surfaces based on traditional word rewriting. This improvement is essentially due to the fact that pq-permutations together with their transformations are able to clearly "mimic", step by step, the process of forming a surface starting from any its polygonal presentation. The system \mathcal{P} is shown to enjoy both the properties of strong normalization and strict strong confluence [2].

In the last lines of this paper, we recover the classification theorem by using the specific combinatorial tools previously provided.

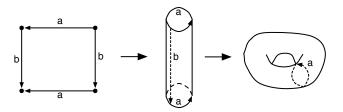
2 Polygonal Presentations of Topological Surfaces

2-dimensional connected and compact manifolds are currently called *surfaces*. It is a well-known achievement in algebraic topology that any surface $\mathscr S$ can be completely characterized by a finite set of polygons forming an its *polygonal presentation* [8, 7]. In particular, a presentation $W_{\mathscr S}$ of a surface $\mathscr S$ consists in a finite set of polygons $\{w_1, \ldots, w_n\}$ whose perimeters are constituted by labelled and oriented edges, such that:

- no more than two edges can have the same label;
- the quotient of $W_{\mathscr{S}}$, modulo identification of paired edges, is the surface \mathscr{S} .

Since fixed a clockwise or an anticlockwise orientation, any polygon w turns out to be completely determined by its perimeter, namely by a cycle of oriented edges. Edges having orientation opposite to the fixed one, are indicated by raising them at the minus one power. Thus, polygonal presentations are usually written as sets of words on an alphabet $A \cup A^{-1}$, where $A = \{a, b, c, ...\}$ and $A^{-1} = \{a, b, c, ...\}$

Fig. 1. The polygon $ab\bar{a}\bar{b}$ becomes a torus.



 $\{a^{-1}, b^{-1}, c^{-1}, \ldots\}$, considered up to circular permutations. In the sequel of this paper we will adopt the simplified notation x and \bar{x} ($x \in \mathcal{A} \cup \mathcal{A}^{-1}$), for meaning that the pair of edges labeled with x have opposite orientations¹.

We recall some basic polygonal presentations: $sphere: a\bar{a}; torus: ab\bar{a}\bar{b}$ (see Figure 1); $projective\ plane: abab;$ Klein bottle: $ab\bar{a}b$.

Theorem 1 (classification theorem). Any compact connected surface (possibly with boundary) is homeomorphic to exactly one of the following surfaces: a sphere, a finite connected sum² of tori, or a finite connected sum of projective planes (possibly with boundary). The sphere and connected sums of tori are orientable surfaces, whereas connected sums of projective planes are non-orientable.

Notation. W, U, V, \ldots denote sets of words, whereas we adopt small letters w, u, v, \ldots for indicating single words. Moreover, if $w = a_1 a_2 \ldots a_n$, then $\bar{w} = \bar{a}_n \bar{a}_{n-1} \ldots \bar{a}_1$. Words of the shape $a_1 b_1 \bar{a}_1 \bar{b}_1 \ldots a_n b_n \bar{a}_n \bar{b}_n$ and $a_1 a_1 \ldots a_n a_n$ are respectively abbreviated with tor_n and pjp_n .

A detailed proof for Theorem 1 can be found in [8], where Massey provides an algorithm for rewriting any given one-polygon presentation into an equivalent one (i.e. denoting the same surface) having perimeter in so-called canonical form. The advantage of dealing with presentations in canonical form consists in the fact that they make easily understood the fundamental information concerning the presented surface. In particular, the following three canonical shapes

$$a\bar{a}x_1u_1\bar{x}_1\dots x_qu_q\bar{x}_q$$
 $tor_nx_1u_1\bar{x}_1\dots x_qu_q\bar{x}_q$ $pjp_nx_1u_1\bar{x}_1\dots x_qu_q\bar{x}_q$

respectively denote a sphere, a connected sum of n tori and a connected sum of n projective planes, in all cases with the boundary decomposed into q components: u_1, u_2, \ldots, u_q .

¹ The bar-operation (⁻) is clearly an involution without fix point, namely, for any $x \in \mathcal{A} \cup \mathcal{A}^{-1}$, $\bar{x} = x$ and $x \neq \bar{x}$.

² Roughly speaking, the connected sum operation consists in connecting two surfaces with a tube after cutting out holes in the surfaces where the tubes are attached.

3 A Rewriting System on Words

Definition 1 (rewriting system). A rewriting system \mathcal{R} consists of a set of terms $\{t_1, t_2, \ldots\}$ closed with respect to a set of transformations $\{r_1, r_2, \ldots, r_n\}$.

Notation. In the specific jargon of term rewriting, an application of a single rule is called *step of reduction*. Consider a generic rewriting system \mathcal{R} : we write $t \to_{r_i} t'$ and $t \to_{\mathcal{R}} t'$ for meaning that t' is obtained from t respectively by applying the (single) specific transformation r_i and a (single) generic transformation of \mathcal{R} . $t \leadsto_{\mathcal{R}} t'$ indicates that t' is obtained from t throughout a sequence of reduction steps [2].

Definition 2. The rewriting system W is defined by taking polygonal presentations as terms together with the following six rules:

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- glue: W, wa, \bar{a}v \to W, wv

- split: W, wv \to W, wa, \bar{a}v

- cutting-out: W, wa\bar{a} \to W, w

- pump: W, w \to W, wa\bar{a}

- invert: W, w \to W, \bar{w}

- shift: W, wxu\bar{x}v \to W, wvx\sigma(u)\bar{x}, where \sigma is a cyclic permutation.
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The set of rules listed in the previous definition is a slight variant of that one already proposed in [5]: in particular, the primary list has been here closed under inversion of rules (e.g., pump is nothing else but the leftward reading of the cutting-out rule). This kind of closure allows to state that, if $W \leadsto_{\mathcal{W}} W'$, then $W' \leadsto_{\mathcal{W}} W$, which is a very natural property for the specific topological context we consider in these pages.

Lemma 1. The following rule is admissible in $W: W, wava \rightarrow_{Mobius} W, w\bar{v}aa$.

Proof. The working of this rule is intuitively explained in Figure 2. Nevertheless, for being more precise, we show that W, $wava \leadsto_{\mathcal{W}} W$, $w\bar{v}aa$:

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W, wava \rightarrow_{\text{split}} W, waz, \bar{z}va \rightarrow_{\text{inv.}} W, waz, \bar{a}\bar{v}z \rightarrow_{\text{glue}} W, w\bar{v}zz =_{\text{rename}} W, w\bar{v}aa.
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Lemma 2. Segments indicating a torus or a projective plane behave as central elements, namely they can be freely moved inside words.

Proof. The proof consists in detailing the following two chains:

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W, waav \leadsto_{\mathcal{W}} W, wvaa and W, wab\bar{a}\bar{b}v \leadsto_{\mathcal{W}} W, wvab\bar{a}\bar{b}.
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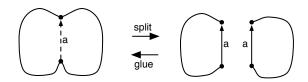
By the leftward reading of the chain used to prove the previous lemma, we have the admissibility of $w\bar{v}aa \rightarrow_{\text{Mob.}^{-1}} wava$; thus, we can write:

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W, waav \rightarrow_{\text{Mob.}^{-1}} W, a\bar{w}av \rightarrow_{\text{inv.}} W, \bar{v}\bar{a}w\bar{a} \rightarrow_{\text{Mob.}} W, \bar{v}\bar{a}\bar{a}\bar{w} \rightarrow_{\text{inv.}} W, waav. For what concerns the other chain, we have:
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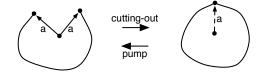
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W, wab\bar{a}\bar{b}v \rightarrow_{\text{shift}} W, wavb\bar{a}\bar{b} \rightarrow_{\text{shift}} W, w\bar{b}avb\bar{a} \rightarrow_{\text{shift}}
\rightarrow_{\text{shift}} W, w\bar{a}\bar{b}avb \rightarrow_{\text{shift}} W, wvb\bar{a}\bar{b}a =_{\text{rename}} W, wvab\bar{a}\bar{b}.
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Table 1. Geometrical visualization of the rules in \mathcal{W} .

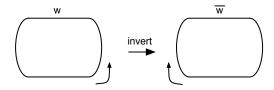
Split and glue:



Cutting-out and pump:



Invert:



Shift:

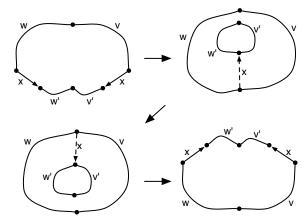
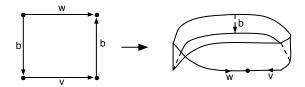


Fig. 2. Intuitive explanation of the Möbius rule.



Definition 3. Two polygonal presentations W and V are said to be equivalent, $W \sim V$, if they present the same surface.

Theorem 2. If W and W' are two presentations such that $W \to_{\mathcal{W}} W'$, then $W \sim W'$.

Proof. We sketch an intuitive version of the proof (the reader can find more details in [8]). All the rules listed in Definition 2 are geometrically explained in Table 1. In algebraic topology surfaces can be considered modulo "reversible" cuts: roughly speaking, we can arbitrarily cut a surface provided that we leave the traces for recomposing it without ambiguities (this is the reason for considering edges oriented and labeled). The two rules of split and pump (together with their relative inverses glue and cutting-out) exactly express this idea. The rule of invert just says that the perimeter of a polygon can be read following both the possible orientations (clockwise or anticlockwise) without changing the presented surface. The rule of *shift* is the most meaningful one. The idea is that a segment of perimeter u included between a pair of opposite letters, $xu\bar{x}$, can be always "carried inside" the polygon by identifying the edges labeled with x (see the last figure in Table 1). Since u is an "hole" inside the polygon, it can be once again "extracted" on the perimeter by performing a new cut on the surface. The shift rule is induced by the fact this new cut can be performed from an arbitrary vertex on the perimeter to an arbitrary vertex on the hole u.

Lemma 3 ([8]). The connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes.

Proof. In terms of words, connected sum is nothing else but concatenation, so the connected sum of a torus with a projective plane with boundary can be presented by a polygon of perimeter $tor_1pjp_1w = ab\bar{a}\bar{b}ccw$, where w indicates a generic word encoding boundary. Then, we rewrite our word as follows:

 $ab\bar{a}\bar{b}ccw \rightarrow_{\text{shift}} acb\bar{a}\bar{b}cw \rightarrow_{\text{Lemma1}} aba\bar{b}ccw \rightarrow_{\text{Lemma1}} \bar{b}aa\bar{b}ccw \rightarrow_{\text{Lemma2}} \bar{b}\bar{b}aaccw,$ namely pjp_3w .

4 Pq-Permutations

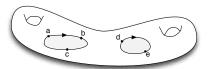
If we consider a surface $\mathscr S$ as the final result of identifying paired edges in a set of polygons forming an its topological presentation, we have that each boundary-component will be formed by at least one edge. Let $\partial\mathscr S$ be the set of labels occurring on the boundary of $\mathscr S$; since fixed an orientation, we can notice that $\mathscr S$ induces a cyclic order on each one of the subsets of $\partial\mathscr S$ corresponding to boundary-components; in other words, we obtain a permutation on $\partial\mathscr S$. The idea leading to the notion of pq-permutation is that the basic information concerning any surface $\mathscr S$ can always be encoded by a very easy mathematical structure consisting in a permutation σ (denoting cycle by cycle the boundary $\partial\mathscr S$) together with a couple of natural numbers $\langle p,q\rangle$ respectively counting tori and projective planes in the connected sum forming $\mathscr S$.

Notation. pq-permutations are denoted with small Greek letters α, β, \ldots , whereas sets of pq-permutations with big Greek letters $\Sigma, \Xi, \Psi, \ldots$ When letters W, V, U, \ldots and w, v, u, \ldots appear in pq-permutations they respectively stand for sets of cyles and series of elements (i.e. $w = a_1, a_2, \ldots, a_n$). The support of a pq-permutation α is denoted with $|\alpha|$. By abuse of notation, the permutation having empty support is simply indicated with \varnothing .

Definition 4 (pq-permutation). A pq-permutation α consists in a permutation σ written by cycles $\{(w_1), (w_2), \dots, (w_n)\}$, possibly the empty one \emptyset , indexed with an ordered couple of natural numbers $\langle p, q \rangle$:

$$\alpha = \{(w_1), (w_2), \dots, (w_n)\}_{\langle p, q \rangle}.$$

Example 1. The oriented surface illustrated below induces the pq-permutation $\{(a,b,c),(d,e)\}_{(2,0)}$.



Example 2. For taking an example of a non-orientable surface, a Klein bottle without boundary will induce the pq-permutation $\emptyset_{\langle 0,2\rangle}$ (it is in fact homeomorphic to the connected sum of two projective planes [8]).

Remark 1. Pq-permutations can be also seen as a way of making the structure of canonical words more perspicuous, by avoiding useless bureaucracy. In particular segments of the shapes \mathtt{tor}_p and \mathtt{pjp}_q , respectively used for storing tori and projective planes, are discarded throughout the two indices $\langle p,q\rangle$ and the part concerning the boundary is considered modulo shift rule: $x_1u_1\bar{x}_1x_2u_2\bar{x}_2\dots x_ru_r\bar{x}_r$ becomes the set of cycles $\{(u_1),(u_2),\dots,(u_r)\}$.

It is now clear that the structure of pq-permutations provides an invariant for considering surfaces modulo isomorphisms, namely modulo homeomorphisms preserving orientation. In the remaining part of this section, we will show that the specific combinatorial framework provided by pq-permutations allows to handle surfaces in a much more perspicuous and efficient way with respect to the approach based on words. We underline again, that this efficiency is due to the fact that pq-permutations admit transformations able to "mimic", step by step, the whole process of forming a surface $\mathscr S$ starting from any its polygonal presentation $W_{\mathscr S}$.

Definition 5. We define the rewriting system \mathcal{P} by taking sets of pq-permutations as terms together with the following six rules:

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 \begin{array}{l} - \text{ gluing: } \varSigma, \{W, (w, a)\}_{\langle p, q \rangle}, \{V, (v, \bar{a})\}_{\langle p', q' \rangle} \to \varSigma, \{W, V, (w, v)\}_{\langle p+p', q+q' \rangle} \\ - \text{ invert: } \varSigma, \{(w_1), \ldots, (w_n)\}_{\langle p, q \rangle} \to \varSigma, \{(\bar{w}_1), \ldots, (\bar{w}_n)\}_{\langle p, q \rangle} \\ - \text{ cylinder: } \varSigma, \{W, (w, a, v, \bar{a})\}_{\langle p, q \rangle} \to \varSigma, \{W, (w), (v)\}_{\langle p, q \rangle} \\ - \text{ torus: } \varSigma, \{W, (w, a), (\bar{a}, v)\}_{\langle p, q \rangle} \to \varSigma, \{W, (w, v)\}_{\langle p+1, q \rangle} \\ - \text{ M\"obius: } \varSigma, \{W, (w, a, v, a)\}_{\langle p, q \rangle} \to \varSigma, \{W, (w, \bar{v})\}_{\langle p, q+1 \rangle} \\ - \text{ Klein: } \varSigma, \{W, (w, a), (a, v)\}_{\langle p, q \rangle} \to \varSigma, \{W, (w, \bar{v})\}_{\langle p, q+2 \rangle} \\ - \text{ sieve: } \varSigma, \{W\}_{\langle p, q \rangle} \to \varSigma, \{W\}_{\langle 0, 2p+q \rangle}. \end{array}
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Gluing and invert rules are nothing else but the conterpart of their homonymous rules in W. Cylinder says that the effect of identifying two opposite edges occurring on the same piece of boundary, is that one of decomposing this boundary-component into two components. As far as the torus rule is concerned, if opposite paired edges occur on two different boundary-components, for identifying them we have to merge the two pieces into one by forming a new handle (one torus added to the connected sum). The Möbius rule comes straightforwardly from Lemma 1, whereas the Klein rule should be interpreted as a kind of "non-orientable torus" whose effect, as its own name suggests, consists in producing a Klein bottle (two projective planes added to the connected sum). Finally, the rule of sieve just expresses the property stated in Lemma 3.

Definition 6. We associate with any pq-permutation α a word w_{α} defined as follows:

$$\alpha = \{(w_1), \dots, (w_n)\}_{\langle p, q \rangle} \mapsto w_\alpha = \mathsf{tor}_p \mathsf{pjp}_q x_1 w_1 \bar{x}_1 \dots x_n w_n \bar{x}_n.$$

If $\Sigma = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, then $W_{\Sigma} = \{w_{\alpha_1}, w_{\alpha_2}, \dots, w_{\alpha_n}\}$; so, the equivalence relation " \sim " can be extended to sets of pq-permutations in a very natural way: $\Sigma \sim \Xi \Leftrightarrow W_{\Sigma} \sim W_{\Xi}$.

Theorem 3. Given two pq-permutations α and β , if $\alpha \to_{\mathcal{P}} \beta$, then $\alpha \sim \beta$.

Proof. The proof consists in showing that any chain of pq-permutations $\Xi \leadsto_{\mathcal{P}} \Xi'$ has a precise counterpart in terms of words $W_{\Xi} \leadsto_{\mathcal{W}} W_{\Xi'}$. Just a preliminary remark on notation: when a set of cycles $W = \{(u_1), \ldots, (u_n)\}$ occurring in a

pq-permutation is "translated" into a word, its notation is kept unchanged but it is meant to be $W = x_1 u_1 \bar{x}_1 \dots x_n u_n \bar{x}_n$. Thus, it is clear that a segment like W can be freely moved inside a word throughout a series of *shift* rules.

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- Gluing: The set \Sigma, \{W, (w, a)\}_{\langle p, q \rangle}, \{V, (v, \bar{a})\}_{\langle p', q' \rangle} becomes
      W_{\Sigma}, \operatorname{tor}_{p}\operatorname{pjp}_{q}Wx_{1}wa\bar{x}_{1}, \operatorname{tor}_{p'}\operatorname{pjp}_{q'}Vx_{2}v\bar{a}\bar{x}_{2}. Then we have:
      W_{\Sigma}, \operatorname{tor}_{p}\operatorname{pjp}_{a}Wx_{1}wa\bar{x}_{1}, \operatorname{tor}_{p'}\operatorname{pjp}_{a'}Vx_{2}v\bar{a}\bar{x}_{2} \to_{\operatorname{glue}}
      \rightarrow_{\mathrm{glue}} W_{\Sigma}, \mathtt{tor}_{p}\mathtt{pjp}_{q}Wx_{1}w\bar{x}_{2}\mathtt{tor}_{p'}\mathtt{pjp}_{q'}Vx_{2}v\bar{x}_{1} \leadsto_{\mathrm{Lemma2}}
      \leadsto_{\operatorname{Lemma2}} W_{\Sigma}, \operatorname{tor}_p \operatorname{tor}_{p'} \operatorname{pjp}_a \operatorname{pjp}_{a'} W x_1 w \bar{x}_2 V x_2 v \bar{x}_1 \sim
     \sim W_{\Sigma}, \mathsf{tor}_{p+p'}\mathsf{pjp}_{q+q'}Wx_1w\bar{x}_2Vx_2v\bar{x}_1 \leadsto_{\mathrm{shift}}
      \leadsto_{\mathrm{shift}} W_{\Sigma}, \mathtt{tor}_{p+p'}\mathtt{pjp}_{q+q'}WVx_1w\bar{x}_2x_2v\bar{x}_1 \rightarrow_{\mathrm{cut}}
      \rightarrow_{\mathrm{cut}} W_{\Sigma}, \mathsf{tor}_{p+p'}\mathsf{pjp}_{q+q'}WVx_1wv\bar{x}_1;
     in terms of pq-permutations: \Sigma, \{W, V, (w, v)\}_{(p+p', q+q')}.
Invert: easy.
- Cylinder: \Sigma, \{W, (w, a, v, \bar{a})\}_{\langle p, q \rangle} \to \Sigma, \{W, (w), (v)\}_{\langle p, q \rangle}. Two cases.
         -v is not the empty word.
               \Sigma, \{W, (w, a, v, \bar{a})\}_{\langle p, q \rangle} becomes W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_q Wxwav\bar{a}\bar{x} and so:
               W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a Wxwav \bar{a} \bar{x} \to_{\mathsf{shift}} W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a Wxw \bar{x} av \bar{a},
              namely \Sigma, \{W, (w), (v)\}_{\langle p,q\rangle}.
         -v is the empty word: instead of a shift rule, we apply a cutting-out.
- Torus: \Sigma, \{W, (w, a), (\bar{a}, v)\}_{\langle p, q \rangle} corresponds to W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_q W x_1 w a \bar{x}_1 x_2 \bar{a} v \bar{x}_2.
     W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a W x_1 w a \bar{x}_1 x_2 \bar{a} v \bar{x}_2 \to_{\mathsf{shift}} W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a W x_1 w v \bar{x}_2 a \bar{x}_1 x_2 \bar{a}
      \rightarrow_{\text{shift}} W_{\Sigma}, \text{tor}_{p} \text{pjp}_{a} W x_{1} w v \bar{a} \bar{x}_{2} a \bar{x}_{1} x_{2} \rightarrow_{\text{shift}}
      \rightarrow_{\mathrm{shift}} W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a W x_1 w v \bar{x}_1 x_2 \bar{a} \bar{x}_2 a \sim W_{\Sigma}, x_2 \bar{a} \bar{x}_2 a \mathsf{tor}_p \mathsf{pjp}_a W x_1 w v \bar{x}_1 \sim 0
      \sim W_{\Sigma}, tor<sub>p+1</sub>pjp<sub>q</sub>Wx_1wv\bar{x}_1, in terms of pq-permutations:
      \Sigma, \{W, (w, v)\}_{\langle p+1, q\rangle}.
- Möbius: \Sigma, \{W, (w, a, v, a)\}_{\langle p, q \rangle} becomes W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a W x w a v a \bar{x}.
      W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a Wxwava\bar{x} \to_{\mathsf{Lemma1}} W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a Wxw\bar{v}aa\bar{x} \to_{\mathsf{Lemma2}}
      \rightarrow_{\operatorname{Lemma2}} W_{\Sigma}, \operatorname{tor}_{p} \operatorname{pjp}_{q} aaWxw \bar{v} \bar{x} \sim W_{\Sigma}, \operatorname{tor}_{p} \operatorname{pjp}_{q+1} Wxw \bar{v} \bar{x},
     namely \Sigma, \{W, (w, \bar{v})\}_{\langle p, q+1 \rangle}.
- Klein: \Sigma, \{W, (w, a), (a, v)\}_{\langle p, q \rangle} becomes W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_q W x_1 w a \bar{x}_1 x_2 v a \bar{x}_2.
     W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a W x_1 w a \bar{x}_1 x_2 v a \bar{x}_2 \to_{\mathsf{Lemma1}} W_{\Sigma}, \mathsf{tor}_p \mathsf{pjp}_a W x_1 w \bar{v} \bar{x}_2 x_1 a a \bar{x}_2
      \rightarrow_{\operatorname{Lemma1}} W_{\Sigma}, \operatorname{tor}_{p} \operatorname{pjp}_{q} W x_{1} w \bar{v} \bar{a} \bar{a} \bar{x}_{1} \bar{x}_{2} \bar{x}_{2} \rightarrow_{\operatorname{Lemma2}}
      \rightarrow_{\operatorname{Lemma2}} W_{\Sigma}, \operatorname{tor}_{p} \operatorname{pjp}_{a} \bar{a} \bar{a} W x_{1} w \bar{v} \bar{x}_{1} \bar{x}_{2} \bar{x}_{2} \rightarrow_{\operatorname{Lemma2}}
      \rightarrow_{\operatorname{Lemma2}} W_{\Sigma}, \operatorname{tor}_{p} \operatorname{pjp}_{q} \bar{a} \bar{a} \bar{x}_{2} \bar{x}_{2} W x_{1} w \bar{v} \bar{x}_{1} \sim W_{\Sigma}, \operatorname{tor}_{p} \operatorname{pjp}_{q+2} W x_{1} w \bar{v} \bar{x}_{1}.
     In terms of pq-permutations: \Sigma, \{W, (w, v)\}_{\langle p, q+2\rangle}
- Sieve: immediately by applying Lemma 3.
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In the next definitions we recall same fundamental properties of rewriting systems (for more details the reader can consult [2]).

Definition 7 (normal form, weak and strong normalization properties). In a rewriting system, terms which cannot be written any further are called normal forms. A rewriting system enjoys the weak normalization property if, for every term t of the system, there exists a rewriting sequence which transforms t into a normal form. If any rewriting strategy is able to carry t into a normal form, our system is said to be strongly normalizing.

Remark 2. According to the previous definition, we remark that a pq-permutation α is in normal form if $|\alpha|$ does not contain paired edges and at least one of the two indices is null (three admitted situations: $\langle p, 0 \rangle$, $\langle 0, q \rangle$ and $\langle 0, 0 \rangle$).

Theorem 4. The rewriting system \mathcal{P} strongly normalizes.

Proof. For proving this property, one usually attaches a convenient size to terms and shows that it decreases at each single step of reduction. In case of papermutations, we associate to each α a size $[\alpha] = i - j$, where i is the number of paired edges occurring in $|\alpha|$ and j the number of stored tori (namely, the first index of α). Now it is sufficient to remark that, if $\alpha \to_{\mathcal{S}} \alpha'$, then $[\alpha'] < [\alpha]$.

Definition 8 (confluence, strict strong confluence). A rewriting system \mathcal{R} is said to be confluent if, for any three terms $a,b,c\in\mathcal{R}$ such that $a\leadsto_{\mathcal{R}} b$ and $a\leadsto_{\mathcal{R}} c$, there exists a fourth term $d\in\mathcal{R}$ such that $b\leadsto_{\mathcal{R}} d$ and $c\leadsto_{\mathcal{R}} d$. \mathcal{R} enjoys the strict strong confluence property if in definition of confluence the arrow " \leadsto " can be replaced everywhere by the single step arrow " \Longrightarrow ".

Lemma 4. If we consider pq-permutations modulo sieve rule, then \mathcal{P} is strictly strongly confluent.

Proof. With $\alpha \to_a \alpha'$ we mean that the pq-permutation α' has been obtained from α by identifying edges labeled with "a". By considering all the possible cases it is (long but) easy to see that, if $\alpha \to_a \beta$ and $\alpha \to_b \gamma$, then there exists a $\delta \in \mathcal{P}$ such that $\beta \to_b \delta$ and $\gamma \to_a \delta$.

Remark 3. Strict strong confluence implies both confluence and the uniqueness of normal forms (namely, any pq-permutation has exactly one normal form) [2]. In other words, \mathcal{P} is shown to be a deterministic system, not only in terms of outputs, but also in terms of computations. Strict strong confluence extends in fact determinism to computational processes by asserting their equivalence modulo permutation of rules (in case of pq-permutations, modulo permutations of identified edges).

Given all the technical details about \mathcal{P} , here we propose some example in which this system is used for computing the surface presented by a certain polygonal presentation.

Example 3. We show that the connected sum of a torus with a projective plane is effectively homeomorphic to a connected sum of three projective planes. The polygon denoting the surface at issue has perimeter: $ab\bar{a}b\bar{c}c$. Any polygon is

topologically nothing else but a disk, so we start rewriting the pq-permutation $\{(a, b, \bar{a}, \bar{b}, c, c)\}_{(0,0)}$:

$$\begin{split} &\{(a,b,\bar{a},\bar{b},c,c)\}_{\langle 0,0\rangle} \to_{\text{cyl.}} \{(b),(\bar{b},c,c)\}_{\langle 0,0\rangle} \to_{\text{Mobius}} \\ &\to_{\text{Mobius}} \{(b),(\bar{b})\}_{\langle 0,1\rangle} \to_{\text{torus}} \varnothing_{\langle 1,1\rangle} \to_{\text{sieve}} \varnothing_{\langle 0,3\rangle}. \end{split}$$

Or, alternatively:

$$\begin{split} &\{(a,b,\bar{a},\bar{b},c,c)\}_{\langle 0,0\rangle} \to_{\mathrm{Mobius}} \{(a,b,\bar{a},\bar{b})\}_{\langle 0,1\rangle} \to_{\mathrm{cyl.}} \\ &\to_{\mathrm{cyl.}} \{(b),(\bar{b})\}_{\langle 0,1\rangle} \to_{\mathrm{torus}} \varnothing_{\langle 1,1\rangle} \to_{\mathrm{sieve}} \varnothing_{\langle 0,3\rangle}. \end{split}$$

Example 4. We show that the polygon $ab\bar{a}b$ presents a Klein bottle:

$$\{(a,b,\bar{a},b)\}_{\langle 0,0\rangle} \to_{\text{cyl.}} \{(b),(b)\}_{\langle 0,0\rangle} \to_{\text{Klein}} \emptyset_{\langle 0,2\rangle}.$$

Or, alternatively:

$$\{(a,b,\bar{a},b)\}_{\langle 0,0\rangle} \to_{\text{Mobius}} \{(a,a)\}_{\langle 0,1\rangle} \to_{\text{Mobius}} \emptyset_{\langle 0,2\rangle}.$$

Corollary 1. The rewriting system W enjoys the weak normalization property and the uniqueness of the normal form.

Proof. In case of words, we make coincide normal and canonical forms. Given any presentation W, the proof of Theorem 3 clearly provides a strategy for reducing W into a normal form. The uniqueness of normal forms in \mathcal{P} yields the same property for W.

Proof of the classification theorem. By considering canonical form as synonym of normal form, the already mentioned proof of the classification theorem provided by Massey can be seen as a proof of weak normalization (for a set of transformations on polygons which is here essentially that one expressed by W^3), together with the uniqueness of the normal form. But, it is exactly what Corollary 1 affirms.

5 Future Work

Many directions of research are opened, not necessarily in convergent directions. In the final lines of this contribution, the classification theorem has been recovered inside the specific framework here proposed and other standard achievements in topology of 2-dimensional manifolds are expected to be reinterpreted throughout pq-permutations: in primis, a combinatorial treatment of the Jordan curve theorem for closed surfaces. More generally, we uphold the idea that pq-permutations provide an optimal context for studying the decomposition of surfaces, especially in presence of specific constraints. Moreover, because of the

³ The *shift* rule constitutes an exception: the algorithm reported by Massey in [8] works in fact on polygons presenting surfaces without boundary. Surfaces with boundary are later classified by introducing some supplementary remarks on the triangularization theorem, essentially aside from the inner mechanism of the procedure.

strict locality of computation, the system \mathcal{P} turns out to be particularly adapt for studying the possibilities of relaxing sequentiality during the processes of forming surfaces.

In order to improve its computer-scientific *status*, a categorical formalization of \mathcal{P} should be provided. In this direction a possible way could be that one of considering pq-permutations as objects, surfaces as morphisms and their composition as composition (gluing) of surfaces.

Finally, we hint at some possible applications in the framework of process calculi applied to biological systems. In particular, we are thinking to Brane Calculi introduced by Cardelli and their later developments [3,4], in which a topological context is imposed by the fact that membranes are two-dimensional fluids which interact embedded in a three-dimensional fluid. The structure of pq-permutations recall that one of membranes (trivially, in case of cyclic permutations) and some transformations considered by the system $\mathcal P$ would seem to be very close to Cardelli's bitonal interactions.

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