Forbidden Subgraphs that Imply Hamiltonian-Connectedness

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Abstract: It is proven that if G is a 3-connected claw-free graph which is also H_1-free (where H_1 consists of two disjoint triangles connected by

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The first four authors dedicate this paper to Henk Jan Veldman, a valued colleague and beloved friend who died October 12, 1998.

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an edge), then G is hamiltonian-connected. Also, examples will be described that determine a finite family of graphs $\mathcal{L}$ such that if a 3-connected graph being claw-free and L-free implies G is hamiltonian-connected, then $L \in \mathcal{L}$.

Keywords: *hamiltonian-connected; forbidden subgraph; claw-free graph*

1. INTRODUCTION

We use Bondy and Murty [2] for terminology and notation not defined here and consider finite simple graphs only. A graph G with $n \geq 3$ vertices is *hamiltonian* if G contains a cycle of length n, and it is *hamiltonian-connected* if between each pair of vertices of G there is a Hamilton path, i.e., a path on n vertices. If H is a given graph, then a graph G is called *H-free* if G contains no induced subgraph isomorphic to H. The graph H is said to be a *forbidden* subgraph.

We first describe some graphs that will be frequently used as forbidden subgraphs. Specifically, we denote by P_k and C_k the path and the cycle on k vertices, by C the claw $K_{1,3}$, by B the bull, by D the deer, by H the hourglass, by N the net, by W the wounded, by Z_k the graph obtained by identifying a vertex of K_3 with an endvertex of P_{k+1}, and by H_k the graph obtained by joining two vertex disjoint triangles by a path of length k (see Fig. 1).

The next result was obtained in Shepherd [7], and the following one in Faudree and Gould [6]. Note that in both cases, 3-connectedness is assumed. This is natural, since the forbidden subgraph conditions, being local conditions, do not imply 3-connectedness, and any hamiltonian-connected graph (except K_1, K_2, K_3) must be 3-connected.

Theorem 1 [7]. *If a 3-connected graph G is claw-free and N-free, then G is hamiltonian-connected.*

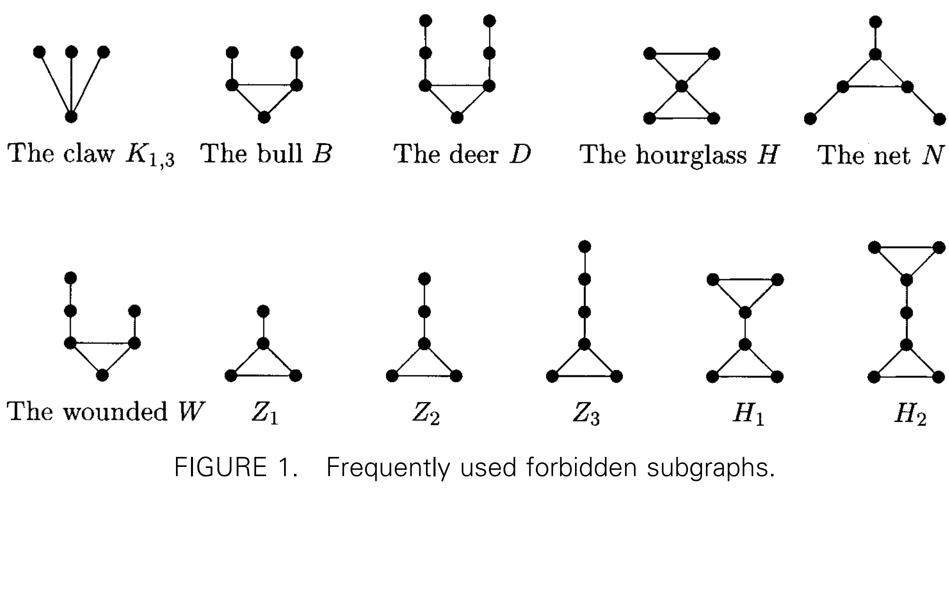

FIGURE 1. Frequently used forbidden subgraphs.

Theorem 2 [6]. *If a 3-connected graph G is claw-free and Z_2-free, then G is hamiltonian-connected.*

Recently Chen and Gould [4] extended this collection of pairs of forbidden graphs ensuring hamiltonian-connectedness of 3-connected graphs by proving the following result, which gives three new independent forbidden pairs.

Theorem 3 [5]. *If G is a 3-connected claw-free graph, then G is hamiltonian-connected if any of the following holds.*

(a) *G is Z_3-free,*
(b) *G is P_6-free,*
(c) *G is W-free.*

The cases (a) and (b) of the above result were independently proved in [3]. In Section 2, we extend the collection of forbidden pairs by proving the following result.

Theorem 4. *If G is a 3-connected claw-free H_1-free graph, then G is hamiltonian-connected.*

In Bedrossian [1], all forbidden pairs of connected graphs ensuring that a graph is hamiltonian are characterized, and the same was done for pancyclicity. The same type of characterization was done for other hamiltonian properties in Faudree and Gould [6]. A survey of results of this kind can be found in Faudree [5].

Combining their results with previous results, Chen and Gould [4] conclude that if $\{S, T\}$ is a pair of graphs such that every 2-connected $\{S, T\}$-free graph is hamiltonian then every 3-connected $\{S, T\}$-free graph is hamiltonian-connected. Theorem 4 gives a pair of forbidden graphs that implies a graph is hamiltonian-connected in the presence of 3-connectedness but does not imply a graph is hamiltonian in the presence of 2-connectedness.

Also, in [6] the following theorem was proved. It gives some context to the previous results on pairs of forbidden graphs ensuring hamiltonian-connectedness of 3-connected graphs.

Theorem 5 [6]. *Let X and Y be connected graphs with $X, Y \neq P_3$, and let G be a 3-connected graph. If G being X-free and Y-free implies G is hamiltonian-connected, then, up to symmetry, $X = K_{1,3}$, and Y satisfies each of the following conditions.*

(a) $\Delta(Y) \leq 3$,
(b) *A longest induced path in Y has at most 12 vertices,*
(c) *Y contains no cycles of length at least 4,*
(d) *All triangles in Y are vertex disjoint,*
(e) *Y is claw-free.*

One implication of Theorem 5 is that there are only a finite number of forbidden pairs of graphs implying hamiltonian-connected of 3-connected graphs. However, the gap between Theorem 5 and the positive results in Theorems 1, 2, 3, and 4 is still substantial. The following result will reduce, but not eliminate, that gap somewhat. The proof is postponed to Section 3.

Theorem 6. *Let X and Y be connected graphs with $X, Y \neq P_3$, and let G be a 3-connected graph. If G being X-free and Y-free implies G is hamiltonian-connected, then $X = K_{1,3}$, and Y satisfies each of the following conditions.*

(a) $\Delta(Y) \leq 3$,
(b) *The longest induced path in Y has at most 9 vertices,*
(c) *Y contains no cycles of length at least 4,*
(d) *The distance between two distinct triangles in Y is either 1 or at least 3,*
(e) *There are at most two triangles in Y,*
(f) *Y is claw-free.*

2. THE PROOF OF THEOREM 4

In what follows, an (x, y)-path P is said to be *maximal* if there is no (x, y)-path Q such that $V(P)$ is a proper subset of $V(Q)$.

The set up of the proof in this section will be to consider a maximal (x, y)-path P that is not a Hamilton path, between some pair of vertices x and y, and then show that P can be extended, contradicting the maximality of P. The following lemma will be useful in selecting such maximal paths.

Lemma 7. *For any pair of vertices x and y in a 3-connected claw-free graph G, there is a maximal (x, y)-path P such that $N(x) \subseteq V(P)$.*

Proof. Let $P = x_1x_2 \cdots x_m$ with $x = x_1$ and $y = x_m$ be a maximal (x, y)-path with the property that it contains a maximum number of vertices of $N(x)$. If $N(x) \subseteq V(P)$, then we are done. Hence, we may assume there is a vertex $z \in N(x) \setminus V(P)$. We will exhibit an (x, y)-path Q that contains $(N(x) \cap V(P)) \cup \{z\}$. This will give a contradiction, since any maximal path (x, y)-path Q' that contains the vertices of Q would have more vertices in $N(x)$ than P.

Since G is 3-connected, there exist three vertex disjoint (z, P)-paths, which will be denoted by Q_1, Q_2, and Q_3. We may assume that Q_1 has endvertex x_1. Let x_r and x_s (with $1 < r < s$) be the endvertices of Q_2 and Q_3, respectively. If z has more than three adjacencies on P, then select x_r and x_s to be the last two adjacencies of z on P. Let S be the set of vertices in $N(x) \cap V(P)$ that are not adjacent to z. Note that to avoid an induced claw centered at x, the vertices in S form a complete graph. Also note that $N(x) \cap N(z) \cap V(P) \subseteq x_1\overrightarrow{P}x_r \cup \{x_s\}$.

If $S \cap x_{r+1}\overrightarrow{P}x_{s-1} = \emptyset$, then $Q = x_1\overrightarrow{P}x_r\overleftarrow{Q_2}z\overrightarrow{Q_3}x_s\overrightarrow{P}x_m$ is the required path, since this path contains z as well as $N(x) \cap V(P)$.

If $S \cap x_{r+1}\overrightarrow{P}x_{s-1} \neq \emptyset$, then select i and j such that x_i is the smallest indexed vertex in $S \cap x_{r+1}\overrightarrow{P}x_{s-1}$ and x_j is the largest. It is possible that $i = j$. By the maximality of P and since G is claw-free, $x_2x_i \in E(G)$. Then $Q = x_1x_j\overleftarrow{P}x_ix_2\overrightarrow{P}x_r\overleftarrow{Q_2}z\overrightarrow{Q_3}x_s\overrightarrow{P}x_m$ is the required path. ■

In the next proof, we start with a graph G that is 3-connected and claw-free, and for which there is no Hamilton path between some pair of vertices x and y of G. By Lemma 7, we can select a maximal (x,y)-path $P = x_1x_2\cdots x_m$ with $x = x_1$ and $y = x_m$ such that $N(x) \subseteq V(P)$. Since P is not a Hamilton path, there is a vertex z not on P. Since G is 3-connected, there exist three vertex disjoint (z,P)-paths, and at least two of these paths will terminate in interior vertices of P. Let x_i, x_j, and x_k (with $1 < i < j < k \leq m$) be the endvertices on P of these paths and denote the paths by Q_i, Q_j, and Q_k, respectively. We can choose z and the paths Q_i, Q_j, Q_k in such a way that

(i) $|E(Q_i)| = 1$,
(ii) $|E(Q_j)|$ is minimum subject to (i),
(iii) $|E(Q_k)|$ is minimum subject to (i) and (ii).

For $\ell = i,j,k$, the path Q_ℓ will be denoted by $zv_\ell\cdots u_\ell x_\ell$ realizing of course that the path might be just an edge. For shortness, we will use Q to denote the path $x_i\overleftarrow{Q_i}z\overrightarrow{Q_j}x_j$. By the way the paths are chosen, we conclude that Q is an induced path except possibly for the edge x_ix_j.

The maximality of P and G being claw-free implies that $x_{i-1}x_{i+1} \in E(G)$, for otherwise there would be an induced claw centered at x_i. Likewise, $x_{j-1}x_{j+1} \in E(G)$. Note that $j - i \geq 4$, for otherwise the path P could be extended; e.g., if $j - i = 3$, then $x_1\overrightarrow{P}x_{i-1}x_{i+1}x_i\overrightarrow{Q}x_jx_{j-1}x_{j+1}\overrightarrow{P}x_m$ is such a path. Also, observe that $x_ix_{j-2} \notin E(G)$, for otherwise the path P can be extended to the path $x_1\overrightarrow{P}x_{i-1}x_{i+1}\overrightarrow{P}x_{j-2}x_i\overrightarrow{Q}x_jx_{j-1}x_{j+1}\overrightarrow{P}x_m$.

Select the smallest r_1 with $i < r_1 < j$ such that $x_ix_{r_1} \in E(G)$, but $x_ix_{r_1+1} \notin E(G)$. By the previous remarks, such an r_1 exists. Likewise, select the smallest s_1 with $j < s_1 < k$ such that $x_jx_{s_1} \in E(G)$, but $x_jx_{s_1+1} \notin E(G)$. There are no edges between $x_i\overrightarrow{P}x_{r_1+1}$ and $x_j\overrightarrow{P}x_{s_1+1}$, except possibly for x_ix_j: the existence of any of the edges gives an extension of P; e.g., if $x_{r_1+1}x_{s_1+1} \in E(G)$, then P can be extended to the path $x_1\overrightarrow{P}x_{i-1}x_{i+1}\overrightarrow{P}x_{r_1}x_i\overrightarrow{Q}x_jx_{s_1}\overleftarrow{P}x_{j+1}x_{j-1}\overleftarrow{P}x_{r_1+1}x_{s_1+1}\overrightarrow{P}x_m$. In the same way, select a largest r_2 with $i < r_2 < j$ such that $x_jx_{r_2} \in E(G)$, but $x_jx_{r_2-1} \notin E(G)$. By symmetry and the previous remarks, such an r_2 exists. Also, if $x_k \neq x_m$, in the same way an s_2 associated with the vertex x_k can be defined. Also, by a symmetry argument, we know that there are no edges between $x_{r_2-1}\overrightarrow{P}x_j$ and $x_{s_2-1}\overrightarrow{P}x_k$ except possibly for x_jx_k.

We are now ready to present the proof of Theorem 4.

Assume that G is a 3-connected, claw-free graph, and there is no Hamilton path between some pair of vertices x and y of G. We will show that G must contain an induced copy of H_1. We choose a maximal (x,y)-path $P = x_1x_2\cdots x_m$ with $x = x_1$ and $y = x_m$ subject to the condition that $N(x) \subseteq V(P)$. We choose a vertex $z \in V(G) \setminus V(P)$ and three vertex disjoint (z,P)-paths as in the general discussion. All of the notation and observations of the general discussion are assumed.

We claim that we can choose z in such a way that $|E(Q_j)| = 1$, and that $|E(Q_k)| = 1$ if $x_k \neq x_m$. Suppose $|E(Q_j)| \geq 2$, and consider z and the successor v_j of z on Q_j. By the choice of z, $x_i v_j \notin E(G)$. Since G is 3-connected, claw-free, and $zv_j^+ \notin E(G)$, there exists a triangle T containing z and v_j or there exists a triangle T containing v_j and v_j^+. We distinguish a number of cases.

Case a.1. z, v_j, and a vertex of Q_k are in a common triangle. Let $t \in V(Q_k)\backslash\{z\}$ be the third vertex of T. By the choice of Q_k, we have $t = v_k$. If $v_k \neq x_k$, then $G[\{x_{i-1}, x_{i+1}, x_i; z, v_j, v_k\}] \cong H_1$, since $x_i v_j \notin E(G)$ (otherwise v_j contradicts the choice of z) and $x_i t \notin E(G)$ (otherwise t contradicts the choice of z). Hence $v_k = x_k$.

To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, v_j, x_k\}] \cong H_1$, we must have at least one of $x_k x_{i-1}$, $x_k x_i$ and $x_{i+1} x_k$ in $E(G)$. Then, since $x_{i-1} x_k \notin E(G)$ (otherwise to avoid $G[\{x_k; x_{i-1}, z, x_{k-1}\}] \cong K_{1,3}$, we have $x_{i-1} x_{k-1} \in E(G)$ yielding a path $x_1 \overrightarrow{P} x_{i-1} x_{k-1} \overleftarrow{P} x_i z x_k \overrightarrow{P} x_m$ which contradicts the choice of P) and $x_i x_k \notin E(G)$ (otherwise to avoid $G[\{x_k; x_i, v_j, x_{k-1}\}] \cong K_{1,3}$, we have $x_i x_{k-1} \in E(G)$, also yielding a path which contradicts the choice of P), we get $x_{i+1} x_k \in E(G)$, implying also $x_{i+1} x_{k-1} \in E(G)$.

If $v_j x_j \in E(G)$ (i.e., $|E(Q_j)| = 2$), then to avoid $G[\{x_{j-1}, x_{j+1}, x_j; v_j, z, x_k\}] \cong H_1$, we similarly have that $x_{j+1} x_k \in E(G)$, and get a contradiction since $G[\{x_k; x_{i+1}, x_{j+1}, z\}] \cong K_{1,3}$. Hence we may assume $v_j x_j \notin E(G)$ and thus $v_j^+ \notin V(P)$ (where v_j^+ is the successor of v_j on Q_j). Since $v_j v_j^{++} \notin E(G)$, there exists a triangle T' containing v_j and v_j^+ or there exists a triangle T' containing v_j^+ and v_j^{++}. Note that $v_j^+ x_k \notin E(G)$ (otherwise $G[\{x_k; z, v_j^+, x_{k-1}\}] \cong K_{1,3}$).

(i) Suppose v_j and v_j^+ are in a common triangle T' with some vertex t'. Then $t' \notin \{x_i, x_j, x_k, z\}$, while also $t' \notin V(P) \setminus \{x_i, x_j, x_m\}$; otherwise if $t' \in x_1 \overrightarrow{P} x_{i-1}$, then v_j contradicts the choice of z, if $t' \in x_{i+1} \overrightarrow{P} x_{j-1}$, then the path $z v_j t'$ contradicts the choice of Q_j, and if $t' \in x_{k+1} \overrightarrow{P} x_m$, then the paths $z x_k$ and $z v_j t'$ contradict the choice of Q_j and Q_k. Hence $t' \notin V(P) \cup \{z\}$. To avoid $G[\{x_{i+1}, x_{k-1}, x_k; v_j, v_j^+, t'\}] \cong H_1$, we have $x_k t' \in E(G)$, and to avoid $G[\{x_k; x_{k-1}, z, t'\}] \cong K_{1,3}$, we have $zt' \in E(G)$. But then $G[\{x_{i-1}, x_{i+1}, x_i; z, t', v_j\}] \cong H_1$, since $x_i t' \notin E(G)$; otherwise t' contradicts the choice of z.

(ii) If v_j^+ is not in a common triangle with v_j, then there exists a triangle T' containing v_j^+ and v_j^{++}. Again let t' be the third vertex of T'. If $t' = x_k$, then $G[\{x_k; z, v_j^+, x_{k-1}\}] \cong K_{1,3}$. Hence $t' \neq x_k$ and also $t' \notin \{x_i, z\}$. If $t' \in x_1 \overrightarrow{P} x_{i-1}$ or $t' \in x_{k+1} \overrightarrow{P} x_m$, we easily get contradictions with the chosen path system. If $t' \in x_{i+1} \overrightarrow{P} x_{j-1}$, then also $v_j^{++} = x_j$, giving a contradiction, since v_j^+ contradicts the choice of z. Hence $t' \notin V(P) \cup \{z\}$. Now $G[\{t', v_j^{++}, v_j^+; v_j, z, x_k\}] \cong H_1$, unless $v_j^{++} x_k \in E(G)$ and $v_j^{++} = x_j$. But then $G[\{x_k; x_{i+1}, x_j, v_j\}] \cong K_{1,3}$.

Case a.2. z, v_j are in a common triangle T with some vertex t, and Case a.1 does not apply. Then, by the choice of z, $V(T) \cap V(P) = \emptyset$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, v_j, t\}] \cong H_1$, we have $x_i t \in E(G)$. To avoid $G[\{z; x_i, v_j, v_k\}] \cong K_{1,3}$ (with possibly $v_k = x_k$), we have $x_i v_k \in E(G)$, since $v_j v_k \notin E(G)$; otherwise we would be in Case a.1. To avoid $G[\{x_i; x_{i-1}, t, v_k\}] \cong K_{1,3}$, we have $t v_k \in E(G)$. If $v_j x_j \in E(G)$, then $G[\{x_{j-1}, x_{j+1}, x_j; v_j, z, t\}] \cong H_1$. Hence $v_j^+ \neq x_j$. We use that v_j^+ is in a triangle with v_j or with v_j^{++}.

(i) Suppose v_j^+ and v_j are in a common triangle T' with some vertex t'.

Clearly, $t' \neq z, x_i$. We easily see that $t' \notin x_1 \overrightarrow{P} x_{k-1}$. Now suppose $t' = x_k$. Then $G[\{x_k; x_{k-1}, v_j^+, u_k\}] \cong K_{1,3}$, unless $v_j^+ u_k \in E(G)$ and $u_k \neq z, v_k$. To avoid $G[\{x_k; x_{k-1}, v_j, u_k\}] \cong K_{1,3}$, we have $v_j u_k \in E(G)$. Then $G[\{x_i, v_k, t; v_j, u_k, x_k\}] \cong H_1$, unless $v_k u_k \in E(G)$. But then $G[\{z, t, v_k; u_k, v_j^+, x_k\}] \cong H_1$. Hence $t' \neq x_k$. If $t' \in x_{k+1} \overrightarrow{P} x_m$, then to avoid $G[\{x_i, v_k, t; v_j, v_j^+, t'\}] \cong H_1$, we have $v_k t' \in E(G)$. But then $v_k = x_k$ or $v_k x_k \in E(G)$. In both cases, we easily obtain path systems contradicting the chosen path system. Hence $t' \notin V(P)$.

Consider $G[\{v_j^+, t', v_j; t, x_i, v_k\}]$ (with possibly $v_k = x_k$). If $t' \notin V(Q_k)$, then to avoid an induced H_1, we have $tt' \in E(G)$. But then $G[\{x_{i-1}, x_{i+1}, x_i; t, v_j, t'\}] \cong H_1$. Hence $t' \in V(Q_k) \setminus \{z, v_k\}$. Then to avoid an H_1, we have $t' = v_k^+$. Then $v_k^+ \neq x_k$; otherwise $G[\{x_k; x_{k-1}, v_k, v_j^+\}] \cong K_{1,3}$. Considering $G[\{v_k^+; v_k, v_k^{++}, v_j\}]$, we get that $v_j v_k^{++} \in E(G)$. To avoid $G[\{v_k^+; v_k, v_k^{++}, v_j^+\}] \cong K_{1,3}$, we have $v_j^+ v_k^{++} \in E(G)$. But then $G[\{x_i, v_k, t; v_j, v_j^+, v_k^{++}\}] \cong H_1$.

(ii) If v_j^+ is not in a common triangle with v_j, then considering a triangle T with $V(T) = \{v_j^+, v_j^{++}, t'\}$, we easily obtain that $G[\{z, t, v_j; v_j^+, v_j^{++}, t'\}] \cong H_1$.

Case b. z and v_j are not in a common triangle. Hence v_j and v_j^+ are in a triangle T with some vertex t. Note that to avoid $G[\{z; x_i, v_j, v_k\}] \cong K_{1,3}$, we have $x_i v_k \in E(G)$ with possibly $v_k = x_k$.

(i) First suppose $t \notin V(P)$. Using that no induced claw is centered at x_i and that $z v_j^+ \notin E(G)$, we obtain $G[\{x_i, v_k, z; v_j, v_j^+, t\}] \cong H_1$ unless $t = v_k^+$. If $t = v_k^+$, then $v_k^+ \neq x_k$; otherwise $G[\{x_k; x_{k-1}, v_j, v_k\}] \cong K_{1,3}$ (using $v_j v_k \notin E(G)$). Considering $G[\{v_k^+; v_k, v_k^{++}, v_j^+\}]$, with possibly $x_k = v_k^{++}$, we get $v_j^+ v_k^{++} \in E(G)$. Now $G[\{x_i, z, v_k; v_k^+, v_j^+, v_k^{++}\}] \cong H_1$, unless $v_j^+ = x_j$ and $x_i x_j \in E(G)$. But then $G[\{x_i; x_{i+1}, z, x_j\}] \cong K_{1,3}$.

(ii) Now suppose $t \in V(P)$. If $t = x_k$, then $v_k \neq x_k$ (since z and v_j are not in a common triangle). No induced claw centered at x_k gives that $G[\{x_i, v_k, z; v_j, v_j^+, x_k\}] \cong H_1$, unless $v_j^+ = x_j$ and $x_i x_j \in E(G)$; in the latter case $G[\{z, v_k, x_i; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$. Hence $t \neq x_k$. If $t \in x_1 \overrightarrow{P} x_{k-1}$, then v_j contradicts the choice of z. If $t \in x_{k+1} \overrightarrow{P} x_m$ (assuming $x_k \neq x_m$), and

$v_j^{++} \neq x_j$, then to avoid $G[\{x_i, v_k, z; v_j, v_j^+, t\}] \cong H_1$, we have $v_k t \in E(G)$. But then $G[\{t; t^-, v_k, v_j\}] \cong K_{1,3}$. If $t \in x_{k+1}\overrightarrow{P}x_m$ (assuming $x_k \neq x_m$), and $v_j^{++} = x_j$, then to avoid $G[\{x_i, v_k, z; v_j, x_j, t\}] \cong H_1$ we have $x_i x_j \in E(G)$ or $x_i t \in E(G)$, both giving an induced claw as contradiction, or $v_k t \in E(G)$. In the latter case, $G[\{t; t^-, v_k, v_j\}] \cong K_{1,3}$.

We now show that $|E(Q_k)| = 1$, if $x_k \neq x_m$. This is not difficult if $x_i x_j \notin E(G)$: consider any neighbor z' of z in $V(G) \setminus V(P)$. Then, considering $G[\{z; z', x_i, x_j\}]$, to avoid an induced claw, we get that one of $z'x_i$ and $z'x_j$ is an edge. But then considering $G[\{x_{j-1}, x_{j+1}, x_j; z, z', x_i\}]$ or $G[\{x_{i-1}, x_{i+1}, x_i; z, z', x_j\}]$, we obtain both edges. This implies all vertices in the component of $G - V(P)$ containing z have x_i and x_j as neighbors. Hence, we can choose a vertex z with three neighbors on P.

Now assume $x_i x_j \in E(G)$, and assume $x_k \neq x_m$ and $|E(Q_k)| \geq 2$. Then z has no third neighbor on P. Let p denote the successor of z on Q_k. Since $\delta \geq 3$, p is in a triangle by claw-freeness. If px_i or px_j is an edge, then both edges are in; otherwise we obtain a claw induced by $\{x_i; p, x_{i+1}, x_j\}$ or $\{x_j; p, x_{j+1}, x_i\}$. But then we contradict the choice of z. Hence $px_i, px_j \notin E(G)$. We distinguish four subcases.

(i) p and z are in a common triangle with a vertex $t \notin V(P)$. Clearly, by the choice of Q_k, $t \notin V(Q_k)$. To avoid $G[\{p, t, z; x_i, x_{i+1}, x_{i-1}\}] \cong H_1$, we have $tx_i \in E(G)$, and similarly $tx_j \in E(G)$. Suppose first that $x_k = p^+$. To avoid $G[\{z, t, p; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $tx_k \in E(G)$ (note that $zx_k \notin E(G)$ by the choice of z). But then t contradicts the choice of z (since tx_i, tx_j, $tx_k \in E(G)$). Hence we may assume $p^+ \neq x_k$. We use that p^+ is in a common triangle with p or p^{++}.

(a) p and p^+ are in a common triangle with some vertex t'. Similar arguments as for p show $p^+x_i, p^+x_j \notin E(G)$. If $t' \notin V(P)$, then the choice of z implies $t'x_i, t'x_j \notin E(G)$ and $t'z \notin E(G)$; if $t' \in V(P)$, then also $t'z \notin E(G)$. Now to avoid $G[\{t', p^+, p; z, x_i, x_j\}] \cong H_1$, we conclude that $t' \in V(P)$ and that t' is adjacent to x_i or x_j. Both cases yield a claw induced by $\{x_i; z, t', x_{i+1}\}$ or $\{x_j; z, t', x_{j+1}\}$, a contradiction.

(b) p and p^+ are not in a common triangle. Hence p^+ and p^{++} are in a common triangle with some vertex t'. Using the choice of z and Q_k, to avoid $G[\{z, t, p; p^+, p^{++}, t'\}] \cong H_1$, we have $t't \in E(G)$, hence $t' \notin V(P)$. To avoid $G[\{t; t', p, x_i\}] \cong K_{1,3}$, we conclude that $x_i t' \in E(G)$, and similarly $x_j t' \in E(G)$, contradicting the choice of z.

(ii) p and z are in a common triangle with a vertex $t \in V(P)$. Together with $px_i, px_j \notin E(G)$, we contradict the assumption that z has no third neighbor on P.

(iii) p and z are not in a common triangle, but p and p^+ are in a common triangle with a vertex $t \notin V(P)$. Clearly, the assumption implies $tz \notin E(G)$, and by the choice of Q_k, $zp^+ \notin E(G)$. Hence also $tx_i, tx_j \notin E(G)$.

As before $px_i, px_j \notin E(G)$ and similarly $p^+x_i, p^+x_j \notin E(G)$, unless $p^+ = x_k$. To avoid $G[\{t, p^+, p; z, x_i, x_j\}] \cong H_1$, we conclude $p^+ = x_k$ and x_kx_i or x_kx_j is an edge. This yields a claw induced by $\{x_i; x_{i+1}, x_k, z\}$ or $\{x_j; x_{j+1}, x_k, z\}$.

(iv) p and z are not in a triangle, and p and p^+ are not in a triangle with some vertex of $V(G) \setminus V(P)$. Hence p and p^+ are in a common triangle with some vertex $t \in V(P)$. Since $px_i, px_j \notin E(G)$, the choice of Q_k implies $p^+ \in V(P)$. Consider $G[\{x_i, x_j, z; p, x_k, t\}]$. If $x_ix_k \in E(G)$, then $G[\{x_k; p, x_j, x_{j-1}\}] \cong K_{1,3}$. By similar arguments, to avoid an H_1, we conclude $t = x_m$ and tx_i or tx_j is an edge. If $tx_i \in E(G)$, we obtain $G[\{x_{i-1}, x_{i+1}, x_i; t, p, x_k\}] \cong H_1$; the case $tx_j \in E(G)$ is similar.

Case 1. $x_ix_j \notin E(G)$. Suppose first that $x_k = x_m$ and $zx_k \notin E(G)$. Then consider any neighbor z' of z in $V(Q_k) \setminus V(P)$ and $G[\{z; z', x_i, x_j\}]$. To avoid an induced claw, we get that one of $z'x_i$ and $z'x_j$ is an edge. But then considering $G[\{x_{j-1}, x_{j+1}, x_j; z, z', x_i\}]$ or $G[\{x_{i-1}, x_{i+1}, x_i; z, z', x_j\}]$, we obtain both edges. This contradicts the choice of z. Hence, we may assume $zx_i, zx_j, zx_k \in E(G)$. Since by assumption $x_ix_j \notin E(G)$, claw-freeness implies $x_ix_k \in E(G)$ or $x_jx_k \in E(G)$.

First assume $x_ix_k \in E(G)$. If also $x_jx_k \in E(G)$, then to avoid $G[\{x_k; x_i, x_j, x_{k-1}\}] \cong K_{1,3}$, we have $x_ix_{k-1} \in E(G)$ or $x_jx_{k-1} \in E(G)$, both contradicting the choice of P. So $x_jx_k \notin E(G)$. If $x_kx_{j-1} \in E(G)$, then also $x_{k-1}x_{j-1} \in E(G)$, contradicting the choice of P. Hence $x_kx_j, x_kx_{j-1} \notin E(G)$. To avoid $G[\{x_i, x_k, z; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$, we have $x_kx_{j+1} \in E(G)$, and hence also $x_{k-1}x_{j+1} \in E(G)$. Since $x_{i-1}x_{k-1} \notin E(G)$, we have $x_{i-1}x_k \notin E(G)$. Since $x_{i-1}x_k \notin E(G)$, we have $x_{i-1}x_{j+1} \notin E(G)$ (otherwise $G[\{x_{j+1}, x_{i-1}, x_j, x_k\}] \cong K_{1,3}$). If $x_{i+1}x_{k-1} \in E(G)$, then $x_1\overrightarrow{P}x_izx_j\overleftarrow{P}x_{i+1}x_{k-1}\overleftarrow{P}x_{j+1}x_k\overrightarrow{P}x_m$ contradicts the choice of P. Hence $x_{i+1}x_{k-1} \notin E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; x_k, x_{k-1}, x_{j+1}\}] \cong H_1$, we have $x_{i+1}x_k \in E(G)$. But then $G[\{x_k, x_{i+1}, z, x_{k-1}\}] \cong K_{1,3}$, a contradiction. We conclude that $x_ix_k \notin E(G)$ and $x_jx_k \in E(G)$.

To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, x_j, x_k\}] \cong H_1$, we have $x_{i+1}x_k \in E(G)$, and hence also $x_{i+1}x_{k-1} \in E(G)$. This also implies $x_k = x_m$. By the choice of P, we have $x_ix_{i+2} \notin E(G)$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_k\}] \cong K_{1,3}$, we have $x_{i+2}x_k \in E(G)$ and to avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{k-1}\}] \cong K_{1,3}$, we have $x_{i+2}x_{k-1} \in E(G)$. If $x_kx_{j+1} \in E(G)$, then $G[\{x_k; x_{i+1}, x_{j+1}, z\}] \cong K_{1,3}$. If $x_{i+1}x_{j-1} \in E(G)$, then $x_1\overrightarrow{P}x_{i+1}x_{j-1}\overleftarrow{P}x_{i+2}x_{k-1}\overleftarrow{P}x_jzx_k$ contradicts the choice of P. To avoid $G[\{x_{i+1}, x_{i+2}, x_k; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$, we have $x_{i+2}x_{j-1} \in E(G) \setminus E(P)$ (i.e., $x_{i+3} \neq x_{j-1}$). If $x_{i+1}x_{i+3} \in E(G)$, then $x_1\overrightarrow{P}x_izx_j\overrightarrow{P}x_{k-1}x_{i+2}x_{j-1}\overleftarrow{P}x_{i+3}x_{i+1}x_k$ contradicts the choice of P. Hence $x_{i+1}x_{i+3} \notin E(G)$, implying $x_{i+3}x_{j-1} \in E(G)$ (otherwise $G[\{x_{i+2}; x_{i+1}, x_{i+3}, x_{j-1}\}] \cong K_{1,3}$). If $x_ix_{i+3} \in E(G)$, then $x_1\overrightarrow{P}x_{i-1}x_{i+1}x_ix_{i+3}\overrightarrow{P}x_{j-1}x_{i+2}x_{k-1}\overleftarrow{P}x_jzx_k$ contradicts the choice of P, and if $x_{i-1}x_{i+3} \in E(G)$ so does $x_1\overrightarrow{P}x_{i-1}x_{i+3}\overrightarrow{P}x_{k-1}\,x_{i+2}x_{i+1}x_izx_k$. If $x_{i-1}x_{i+2} \in E(G)$, then, to avoid $G[\{x_{i+2}; x_{i-1}, x_{i+3}, x_{k-1}\}] \cong K_{1,3}$, we have $x_{i+3}x_{k-1} \in E(G)$ and $x_1\overrightarrow{P}x_{i+2}x_{j-1}\overleftarrow{P}x_{i+3}x_{k-1}\overleftarrow{P}x_jzx_k$ contradicts the choice of P. Hence $G[\{x_{i-1}, x_{i+1}, x_i; x_{i+2}, x_{i+3}, x_{j-1}\}] \cong K_{1,3}$.

Case 2. $x_ix_j \in E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; x_j, x_{j-1}, x_{j+1}\}] \cong H_1$, we have either $x_{i-1}x_{j+1} \in E(G)$ or $x_{i+1}x_{j-1} \in E(G)$, since the other edges are not present by standard arguments.

Case 2.1. $x_{i-1}x_{j+1} \in E(G)$. To avoid $G[\{x_{j+1}; x_j, x_{j+2}, x_{i-1}\}] \cong K_{1,3}$, we have $x_{i-1}x_{j+2} \in E(G)$, since $x_{i-1}x_j \notin E(G)$ (standard) and $x_jx_{j+2} \notin E(G)$ (otherwise $x\overrightarrow{P}x_{i-1}x_{j+1}x_{j-1}\overleftarrow{P}x_izx_jx_{j+2}\overrightarrow{P}y$ contradicts the choice of P).

We first show $zx_k \in E(G)$. Assuming the contrary we have $v_k \neq x_k$. Since $\delta \geq 3$ and G is claw-free, v_k belongs to a triangle.

Case a. There exists a triangle T containing v_k and z. Let q be the third vertex of T.

Case a.1. $q \notin V(P)$. If $x_iv_k \in E(G)$, then, to avoid $G[\{x_i; x_{i+1}, x_j, v_k\}] \cong K_{1,3}$, also $x_jv_k \in E(G)$, which contradicts the choice of z (v_k would have been a better choice). Hence, to avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, v_k, q\}] \cong H_1$, we have $x_iq \in E(G)$. But then $G[\{x_{j+1}, x_{j+2}, x_{i-1}; x_i, z, q\}] \cong H_1$.

Case a.2. $q \in V(P)$. By the way x_k was chosen, we have $q = x_i$ or $q = x_j$. If $q = x_i$, then $G[\{x_{j+1}, x_{j+2}, x_{i-1}; x_i, z, v_k\}] \cong H_1$. If $q = x_j$, then, to avoid $G[\{x_j; x_i, v_k, x_{j+1}\}] \cong K_{1,3}$, we have $x_iv_k \in E(G)$, giving the same H_1 as a contradiction.

Case b. Every triangle T containing v_k does not contain z. Let q_1 and q_2 be the two other vertices of T. If $q_1, q_2 \notin V(P)$, then $G[\{x_i, x_j, z; v_k, q_1, q_2\}] \cong H_1$; otherwise, if for example $q_1z \in E(G)$, there would be a triangle T containing v_k and z, and if $q_1x_i \in E(G)$, then $G[\{x_i; z, q_1, x_{i+1}\}] \cong K_{1,3}$. Also, if $q_1 \in V(P)$ (and/or $q_2 \in V(P)$), then $G[\{x_i, x_j, z; v_k, q_1, q_2\}] \cong H_1$; otherwise, if for example $q_1x_j \in E(G)$, then $G[\{q_1; x_j, v_k, q_1^-\}] \cong K_{1,3}$.

Case 2.1.1. $x_1 \neq x_{i-1}$. To avoid $G[\{x_{i-1}; x_{i-2}, x_i, x_{i+1}\}] \cong K_{1,3}$, we have $x_{i-2}x_{j+1} \in E(G)$, and to avoid $G[\{x_{i-1}; x_{i-2}, x_i, x_{i+2}\}] \cong K_{1,3}$, we have $x_{i-2}x_{j+2} \in E(G)$. But then $G[\{x_i, z, x_j; x_{j+1}, x_{j+2}, x_{i-2}\}] \cong H_1$.

Case 2.1.2. $x_1 = x_{i-1}$.

Case 2.1.2.1. $x_k \neq x_m$. To avoid $G[\{x_i, x_j, z; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $x_ix_k \in E(G)$ or $x_jx_k \in E(G)$. First assume $x_jx_k \in E(G)$. To avoid $G[\{x_{j-1}, x_{j+1}, x_j; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $x_{j-1}x_{k+1} \in E(G)$ or $x_{j+1}x_{k-1} \in E(G)$. However, if $x_{j+1}x_{k-1} \in E(G)$, then $x_1x_{j+2}\overrightarrow{P}x_{k-1}x_{j+1}\overleftarrow{P}x_izx_kx_{k+1}\overrightarrow{P}x_m$ contradicts the choice of P; if $x_{j-1}x_{k+1} \in E(G)$, so does $x_1x_{j+1}\overrightarrow{P}x_kzx_jx_i\overrightarrow{P}x_{j-1}x_{k+1}\overrightarrow{P}x_m$. Hence $x_ix_k \in E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $x_{i+1}x_{k-1} \in E(G)$ or $x_{i-1}x_{k+1} \in E(G)$. However, if $x_{i+1}x_{k-1} \in E(G)$, then $x_1x_{j+1}\overrightarrow{P}x_{k-1}x_{i+1}\overrightarrow{P}x_jx_izx_k\overrightarrow{P}x_m$ contradicts the choice of P; if $x_{i-1}x_{k+1} \in E(G)$, then $G[\{x_1; x_i, x_{j+1}, x_{k+1}\}] \cong K_{1,3}$.

Case 2.1.2.2. $x_k = x_m$. We distinguish between the cases that $x_jx_k \in E(G)$ and $x_jx_k \notin E(G)$.

Case 2.1.2.2.a. $x_jx_m \in E(G)$. To avoid $G[\{x_1, x_{j+2}, x_{j+1}; x_j, z, x_m\}] \cong H_1$, we have $x_{j+2}x_m \in E(G)$, since $x_1x_m \notin E(G)$ (standard) and $x_{j+1}x_m \notin E(G)$ (otherwise also $x_{j+1}x_{m-1} \in E(G)$, giving a path $x_1x_{j+2}\overrightarrow{P}x_{m-1}x_{j+1}\overleftarrow{P}x_izy$ which contradicts the choice of P) while the other possible edges are not present by standard arguments.

First assume $x_{j+3} \neq x_{m-1}$. To avoid $G[\{x_m; x_{m-1}, x_{j+2}, z\}] \cong K_{1,3}$, we have $x_{j+2}x_{m-1} \in E(G)$, and to avoid $G[\{x_{j+2}; x_1, x_{j+3}, x_{m-1}\}] \cong K_{1,3}$, we have $x_{j+3}x_{m-1} \in E(G)$. But then $G[\{x_{i+1}, x_i, x_1; x_{j+2}, x_{j+3}, x_{m-1}\}] \cong H_1$, since $x_1x_{j+3} \notin E(G)$ (otherwise $x_1x_{j+3}\overrightarrow{P}x_{m-1}x_{j+2}\overleftarrow{P}x_izx_m$ contradicts the choice of P), $x_ix_{j+3} \notin E(G)$ (otherwise $x_1x_{j+2}x_{m-1}\overleftarrow{P}x_{j+3}x_i\overrightarrow{P}x_{j-1}x_{j+1}x_jzx_m$ contradicts the choice of P), $x_{i+1}x_{j+3} \notin E(G)$ (otherwise $x_1x_{j+1}x_{j+2}x_{m-1}\overleftarrow{P}x_{j+3}x_{i+1}\overrightarrow{P}x_jx_izx_m$ contradicts the choice of P), and $x_{i+1}x_{m-1} \notin E(G)$ (otherwise $x_1x_{j+1}\overrightarrow{P}x_{m-1}x_{i+1}\overrightarrow{P}x_j\, x_izx_m$ contradicts the choice of P), while the other possible edges are not present by standard arguments.

Hence we may assume that $x_{j+3} = x_{m-1}$. Let $p \in V(G) \setminus \{x_{j+2}, x_m\}$ be a neighbor of x_{j+3}. We first show that we can choose p on P. Suppose there does not exist such a vertex p on P and let T be a triangle containing p and containing a maximum number of vertices of P. Let q_1 and q_2 be the other vertices of T. To avoid $G[\{x_{j+3}; x_{j+2}, x_m, p\}] \cong K_{1,3}$, we have $x_{j+2}y \in E(G)$.

If $V(T) \cap V(P) = \emptyset$, then $G[\{q_1, q_2, p; x_{j+3}, x_{j+2}, x_m\}] \cong H_1$.

If $|V(T) \cap V(P)| = 2$, then $q_1 \neq x_{j+3}$ (since q_2 is a neighbor of q_1, it would have been possible to choose p on P) and $q_2 \neq x_{j+3}$ (similar). But then p contradicts the choice of z.

If $|V(T) \cap V(P)| = 1$, let q_1 be the vertex not on P and let q_2 be the vertex on P. One easily shows that $q_2 \notin \{x_1, x_i, x_{i+1}, x_{j-1}, x_j, x_{j+1}, x_{j+2}, y\}$ by obtaining (x, y)-paths contradicting the choice of P. If $q_2 = x_{j+3}$, then $G[\{x_1, x_{j+1}, x_{j+2};\ q_2, q_1, p\}] \cong H_1$. If $q_2 \in x_{i+2}\overrightarrow{P}x_{j-2}$, then to avoid $G[\{q_2; q_2^-, q_2^+, q_1\}] \cong K_{1,3}$, we have $q_2^-q_2^+ \in E(G)$. However, then $G[\{q_2, q_1, p; x_{j+3}, x_{j+2}, x_m\}] \cong H_1$, since $q_2x_{j+2} \notin E(G)$ (otherwise $x_1\overrightarrow{P}q_2^-q_2^+\overrightarrow{P}x_{j+2}q_2\,px_{j+3}x_m$ contradicts the choice of P), $q_2x_{j+3} \notin E(G)$ by assumption and $q_2x_m \notin E(G)$ (otherwise also $q_2x_{j+3} \notin E(G)$ by a standard observation).

Hence we may assume that we can choose p on P, and one easily shows that $p \in x_{i+2}\overrightarrow{P}x_{j-2}$. To avoid $G[\{p; p^-, p^+, x_{j+3}\}] \cong K_{1,3}$, we have $p^-p^+ \in E(G)$, since $p^-x_{j+3} \notin E(G)$ (otherwise $x_1x_{j+2}\overleftarrow{P}px_{j+3}p^-\overleftarrow{P}x_izx_m$ contradicts the choice of P) and $p^+x_{j+3} \notin E(G)$ (similar). We may assume that $px_{j+2} \notin E(G)$ (otherwise by considering the path $x_1\overrightarrow{P}p^-p^+\overrightarrow{P}x_{j+2}px_{j+3}x_m$ we are back in the case that $x_{j+3} \neq x_{m-1}$) and $px_m \notin E(G)$ (similar). Hence, to avoid $G[\{x_{j+3}; p, x_{j+2}, x_m\}] \cong K_{1,3}$, we have $x_{j+2}x_m \in E(G)$. However, then $G[\{p^-, p^+, p; x_{j+3}, x_{j+2}, x_m\}] \cong H_1$, since $p^-x_{j+2} \notin E(G)$ (otherwise $x_1x_{j+1}\overleftarrow{P}px_{j+3}x_{j+2}p^-\overleftarrow{P}x_izx_m$ contradicts the choice of P), $p^-x_m \notin E(G)$ (otherwise also $p^-x_{j+3} \in E(G)$), $p^+x_{j+2} \notin E(G)$ (otherwise $x_1x_{j+1}x_{j+2}p^+\overrightarrow{P}x_jzx_i\overrightarrow{P}px_{j+3}x_m$ contradicts the choice of P), and $p^+x_m \notin E(G)$ (otherwise also $p^+x_{j+3} \in E(G)$).

Case 2.1.2.2.b. $x_jx_m \notin E(G)$. Let $p \in V(G) \setminus \{z, x_{m-1}\}$ be a neighbor of x_m. We first show that we can choose p on P. Suppose there does not exist such a vertex p on P. To avoid $G[\{x_m; x_{m-1}, z, p\}] \cong K_{1,3}$, we have $pz \in E(G)$. If $px_i \in E(G)$, then $G[\{p, z, x_i; x_{i-1}, x_{j+1}, x_{j+2}\}] \cong H_1$. Hence we have $px_i \notin E(G)$. Since $x_{i-1}x_{k-1} \notin$

$E(G)$, also $x_{i-1}x_k \notin E(G)$, and since $x_{i+1}x_{k-1} \notin E(G)$, also $x_{i+1}x_k \notin E(G)$. To avoid $G[\{x_{i-1}, x_{i+1}, x_i; z, p, x_k\}] \cong H_1$, we have $x_i x_k \in E(G)$. However, then $G[\{x_m, x_i, x_{m-1}, p\}] \cong K_{1,3}$.

Hence, we may assume that we can choose p on P. If $x_i x_m \in E(G)$, then to avoid $G[\{x_i, x_{i+1}, x_j, x_m\}] \cong K_{1,3}$, we have $x_{i+1}x_m \in E(G)$, and hence also $x_{m-1}x_{i+1} \in E(G)$, yielding a path $x_1 x_{j+1} \overrightarrow{P} x_{m-1} x_{i+1} \overrightarrow{P} x_j x_i z x_m$, contradicting the choice of P. Hence $x_i x_m, x_{i+1}x_m \notin E(G)$. If $x_{i-1}x_m \in E(G)$, then also $x_{i-1}x_{m-1} \in E(G)$, a contradiction. Hence $x_{i-1}x_m \notin E(G)$, and similarly $x_{j-1}x_m \notin E(G)$. If $x_{j+1}x_m \in E(G)$, then also $x_{j+1}x_{m-1} \in E(G)$, yielding a contradicting path $x_1 x_{j+2} \overrightarrow{P} x_{m-1} x_{j+1} \overleftarrow{P} x_i z x_m$. The above observations leave two cases for the location of p.

(i) $p \in x_{i+2}\overrightarrow{P}x_{j-2}$. We choose $p \in N(x_k)$ as close to x_{j-1} as possible. To avoid $G[\{x_m; p, z, x_{m-1}\}] \cong K_{1,3}$, we have $px_{m-1} \in E(G)$. To avoid $G[\{x_i, x_j, z; x_m, x_{m-1}, p\}] \cong H_1$, we have $px_i \in E(G)$ or $px_j \in E(G)$. If $px_i \in E(G)$, then also $px_1 \in E(G)$ (otherwise $G[\{x_i; x_1, p, z\}] \cong K_{1,3}$). Since $px_{m-1} \in E(G)$, the choice of P implies $p^+x_1 \notin E(G)$. To avoid $G[\{p; x_1, p^+, x_m\}] \cong K_{1,3}$, we have $p^+x_m \in E(G)$, contradicting the choice of P. Next assume $px_j \in E(G)$. Then $p^+ \neq x_{j-1}$. To avoid $G[\{p; p^+, x_j, x_m\}] \cong K_{1,3}$, we have $p^+x_j \in E(G)$, and to avoid $G[\{x_j, p, z, x_{j+1}\}] \cong K_{1,3}$, we have $p^+x_{j+1} \in E(G)$. However, then $x_1 \overrightarrow{P} p x_{m-1} \overleftarrow{P} x_{j+1} p^+ \overrightarrow{P} x_j z x_m$ contradicts the choice of P.

(ii) $p \in x_{j+2}\overrightarrow{P}x_{k-2}$. We choose $p \in N(x_k)$ as close to x_{j+1} as possible. We again have $px_{m-1} \in E(G)$ and $px_i \in E(G)$ or $px_j \in E(G)$. If $px_i \in E(G)$, then to avoid $G[\{p; x_i, p^-, x_m\}] \cong K_{1,3}$, we have $p^-x_i \in E(G)$ and $p \neq x_{j+2}$. To avoid $G[\{x_i; z, x_{i+1}, p^-\}] \cong K_{1,3}$, we have $x_{i+1}p^- \in E(G)$. But then $x_1 x_{j+1} \overrightarrow{P} p^- x_{i+1} \overrightarrow{P} x_j z x_i p \overrightarrow{P} x_m$ contradicts the choice of P.

If $px_j \in E(G)$, then also $px_{j-1}, px_{j+1} \in E(G)$. If $p^- = x_{j+1}$, then $x_1 x_{j+1} x_j z x_i \overrightarrow{P} x_{j-1} p \overrightarrow{P} x_m$ contradicts the choice of P. If $p^- \neq x_{j+1}$, then to avoid $G[\{p; x_j, p^-, x_m\}] \cong K_{1,3}$, we have $p^-x_j \in E(G)$, and to avoid $G[\{x_j; x_{j-1}, z, p^-\}] \cong K_{1,3}$, also $p^-x_{j-1} \in E(G)$. But then $x_1 x_{j+1} \overrightarrow{P} p^- x_{j-1} \overleftarrow{P} x_i z x_j p \overrightarrow{P} x_k$ contradicts the choice of P.

Case 2.2. $x_{i-1}x_{j+1} \notin E(G)$ (hence $x_{i+1}x_{j-1} \in E(G)$).

Case 2.2.1. $j - i \geq 5$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{j-1}\}] \cong K_{1,3}$, we have $x_{i+2}x_{j-1} \in E(G)$, since $x_i x_{i+2} \notin E(G)$ (contradicting path: $x_1 \overrightarrow{P} x_{i-1} x_{i+1} x_{j-1} \overleftarrow{P} x_{i+2} x_i z x_j \overrightarrow{P} x_m$). By symmetry, we also have $x_{i+1}x_{j-2} \in E(G)$. To avoid $G[\{x_{i+1}; x_i, x_{i+2}, x_{j-2}\}] \cong K_{1,3}$, we have $x_{i+2}x_{j-2} \in E(G)$. However, then $G[\{x_i, z, x_j; x_{j-1}, x_{j-2}, x_{i+2}\}] \cong H_1$.

Case 2.2.2. $j - i = 4$. We use that x_{i+2} has a neighbor $p \notin \{x_{i-1}, x_i, x_{i+1}, x_{i+2}, x_{j-1}, x_j, x_{j+1}\}$.

We first show we can choose $p \in V(P)$. Supposing this is not the case consider a triangle T containing p. Let q_1 and q_2 be the other vertices of T. First suppose $V(T) \cap V(P) = \emptyset$. If $q_1 x_{i+2} \in E(G)$, then $G[\{x_{i-1}, x_i, x_{i+1}; x_{i+2}, p, q_1\}] \cong H_1$. Hence $q_1 x_{i+2}, q_2 x_{i+2} \notin E(G)$. But then $G[\{q_1, q_2, p; x_{i+2}, x_{i+1}, x_{j-1}\}] \cong H_1$. Hence

$|V(T) \cap V(P)| \geq 1$. Let q_1 denote a neighbor of p in $(V(P) \cap V(T))\backslash\{x_{i+2}\}$. Then $x_{i+2}q_1 \notin E(G)$ by assumption. If $x_{j-1}q \in E(G)$, then also $x_{j-1}q_1^- \in E(G)$ (otherwise $G[\{q_1; q_1^-, x_{j-1}, p\}] \cong K_{1,3}$), and we easily find a path contradicting the choice of P. A similar observation shows $x_{i+1}q_1 \notin E(G)$. But then $G[\{x_{i+1}, x_{j-1}, x_{i+2}; p, q_1, q_2\}] \cong H_1$.

Hence, we can choose $p \in V(P)$. If x_{i+2} has two successive neighbors on P, it is obvious that we can find a path contradicting the choice of P. Hence, if p^- and p^+ exist, we get that $p^-p^+ \in E(G)$. We deal with the cases that $p \in \{x_1, x_m\}$ later.

To avoid $G[\{x_{i+1}, x_{j-1}, x_{i+2}; p, p^-, p^+\}] \cong H_1$, we have $x_{i+1}p \in E(G)$ or $x_{j-1}p \in E(G)$. If $x_{i+1}p \in E(G)$ and $p \in x_{j+1}\overrightarrow{P}x_{m-1}$, then by considering the path $x_1\overrightarrow{P}x_{i+1}px_{i+2}\overrightarrow{P}p^-p^+\overrightarrow{P}x_m$, we are back in Case 2.2.1. But then $G[\{x_{i+1}, x_{j-1}, x_{i+2}; p, p^-, p^+\}] \cong H_1$.

Now suppose $p = x_m$. Then $x_m \neq x_k$, since otherwise $G[\{x_m; x_{i+2}, z, x_{m-1}\}] \cong K_{1,3}$. Note that $x_k \neq x_{m-1}$ (otherwise $x\overrightarrow{P}x_{i-1}x_{i+1}x_izx_k\overleftarrow{P}x_{i+2}x_m$ contradicts the choice of P). To avoid $G[\{x_i, x_j, z; x_k, x_{k-1}, x_{k+1}\}] \cong H_1$, we have $x_ix_k \in E(G)$ or $x_jx_k \in E(G)$. First assume $x_jx_k \in E(G)$. Like in the beginning of Case 2, we have $x_{j-1}x_{k+1} \in E(G)$ or $x_{j+1}x_{k-1} \in E(G)$. If $x_{j-1}x_{k+1} \in E(G)$, also $x_{j-2}x_{k+1} \in E(G)$. However, since $x_{j-2} = x_{i+2}$ this contradicts the fact that $x_k \neq x_{m-1}$. If $x_{j+1}x_{k-1} \in E(G)$, then like in the beginning of this case, we have $k - j = 4$. To avoid $G[\{x_{i+1}, x_{i+2}, x_{j-1}; x_{j+1}, x_{j+2}, x_{j+3}\}] \cong H_1$, we have $x_{i+1}x_{j+3} \in E(G)$. But then $G[\{x_{i-1}, x_i, x_{i+1}; x_{j+3}, x_{j+1}, x_{j+2}\}] \cong H_1$. Hence we may assume that $x_jx_k \notin E(G)$ and $x_ix_k \in E(G)$. But then $G[\{x_i; x_{i-1}, x_j, x_k\}] \cong K_{1,3}$.

For the final subcase suppose $\{x_1\} = N(x_{i+2}) \setminus \{x_{i+1}, x_{j-1}\}$. By the choice of P, $N(x_1) \subseteq V(P)$ and $x_2 \neq x_{i-1}$. All neighbors of x_1, except for possibly x_{i+1}, x_{i+2}, x_{j-1}, are also neighbors of x_2, otherwise we obtain an induced claw centered at x_1. If $x_1x_i \in E(G)$, then $x_2x_i \in E(G)$ and to avoid $G[\{x_i; x_2, z, x_{i+1}\}] \cong K_{1,3}$, we have $x_2x_{i+1} \in E(G)$, contradicting the choice of P. Hence $x_1x_i \notin E(G)$ and similarly $x_1x_j \notin E(G)$.

If $x_1x_{i+1} \in E(G)$, then $G[\{x_1, x_{i+1}, x_{i+2}; x_i, z, x_j\}] \cong H_1$; if $x_1x_{j-1} \in E(G)$, then $G[\{x_1, x_{i+2}, x_{j-1}; x_j, x_i, z\}] \cong H_1$. Now assume $x_1x_{i+1}, x_1, x_{j-1} \notin E(G)$. Hence x_1 has some neighbor $q \neq x_i, x_{i+1}, x_{i+2}, x_{j-1}, x_j$ which is also a neighbor of x_2. To avoid $G[\{q, x_2, x_1; x_{i+2}, x_{i+1}, x_{j-1}\}] \cong H_1$, we have $qx_{i+1} \in E(G)$ or $qx_{j-1} \in E(G)$.

First suppose $q \in x_3\overrightarrow{P}x_{i-1}$ and $qx_{i+1} \in E(G)$. Then to avoid $G[\{x_{i+1}; q, x_i, x_{i+2}\}] \cong K_{1,3}$, we have $qx_i \in E(G)$. To avoid $G[\{x_1, x_2, q; x_i, z, x_j\}] \cong H_1$, we have $qx_j \in E(G)$. But then $G[\{q; x_2, x_{i+1}, x_j\}] \cong K_{1,3}$. Next, suppose $q \in x_3\overrightarrow{P}x_{i-1}$ and $qx_{i+1} \notin E(G)$. Then $qx_{j-1} \in E(G)$ and to avoid $G[\{x_{j-1}; q, x_{i+2}, x_j\}] \cong K_{1,3}$, we have $qx_j \in E(G)$. To avoid $G[\{x_1, x_2, q; x_j, z, x_i\}] \cong H_1$, we have $qx_i \in E(G)$. But then $G[\{q; x_2, x_i, x_{j-1}\}] \cong K_{1,3}$.

We now may assume $q \notin x_3\overrightarrow{P}x_{i-1}$, hence $q \in x_{j+1}\overrightarrow{P}x_m$. We choose q as close to x_m as possible, and deal with the subcase $qx_{j-1} \in E(G)$ first.

If $q = x_m$, then, as before, we can repeat the previous cases with x_j, x_k instead of x_i, x_j, and obtain an induced H_1, unless $x_k = x_m$; but in the latter case $G[\{x_m; x_2, u_k, x_{j-1}\}] \cong K_{1,3}$. Hence $q \neq x_m$. To avoid $G[\{x_1, x_2, q; x_{j-1}, x_j, x_{j+1}\}] \cong H_1$, we have $qx_j \in E(G)$ or $qx_{j+1} \in E(G)$, both implying $qx_{j+1} \in E(G)$. To avoid $G[\{q;$

$x_1, x_{j+1}, q^+\}] \cong K_{1,3}$, we have $x_{j+1}q^+ \in E(G)$, yielding $x_1 x_{i+2} x_{j-1} x_j z x_i x_{i+1} x_{i-1} \overleftarrow{P} x_2 q \overrightarrow{P} x_{j+1} q^+ \overrightarrow{P} x_m$, a contradiction. For the remaining case, we assume $qx_{j-1} \notin E(G)$; hence $qx_{i+1} \in E(G)$. By similar arguments as before, we may assume $q \neq x_m$. To avoid $G[\{q; q^+, x_1, x_{i+1}\}] \cong K_{1,3}$, we have $x_{i+1}q^+ \in E(G)$. If $q^+ = x_m$, then by similar arguments as before $x_m = x_k$ and $x_1 x_{i+2} \overrightarrow{P} x_{k-1} x_2 \overrightarrow{P} x_{i-1} x_{i+1} x_i z Q_k x_k$ gives a contradiction. In the final case, the path $P' = x_1 x_{i+2} \overrightarrow{P} q x_2 \overrightarrow{P} x_{i+1} q^+ \overrightarrow{P} x_m$ has the same properties as P, also with respect to the choice of z. But z has two internal vertices $x_{i'}$ and $x_{j'}$ of P' with $j' - i' \geq 5$ as neighbors, so repeating the above arguments with respect to $P', x_{i'}, x_{j'}$ we will obtain an induced H_1. This completes the proof of Theorem 4. ■

3. POSSIBLE FORBIDDEN PAIRS AND HAMILTONIAN-CONNECTEDNESS

We start by defining eight graphs which are 3-connected but not hamiltonian-connected. Let $m \geq 4$ be an integer, M_i be a K_m in which three vertices x_i, y_i, and z_i are marked and $M = \cup_{i=1}^{8} M_i$.

- $G_1 = K_{m,m}$,
- G_2 is obtained from a cycle $C = x_1 x_2 \cdots x_{2m}$, by adding the edges $x_i x_{m+i}$ $(i = 1, \ldots, m)$,
- G_3 is an arbitrary 3-connected C_4-free bipartite graph,
- G_4 is obtained from M_1 by adding two vertices a and b and all (six) edges between a, b and x_1, y_1, z_1,
- G_5 is obtained from a cycle $C = x_1 x_2 \cdots x_{6m}$ by adding the edges $x_{3i-2} x_{3i}$ $(i = 1, \ldots, 2m)$ and the edges $x_{3i-1} x_{3m+3i-1}$ $(i = 1, \ldots, m)$,
- G_6 is obtained from a cycle $C = x_1 x_2 \cdots x_{4m}$ by adding the edges $x_{2i-1} x_{2i+1}$ $(i = 1, \ldots, 2m-1)$, $x_{4m-1} x_1$, and $x_{2i} x_{2m+2i}$ $(i = 1, \ldots, m)$,
- G_7 is obtained from G_5 by replacing every triangle $x_{3i-2} x_{3i-1} x_{3i}$ $(i = 1, \ldots, 2m)$ by the graph G' of Fig. 2,

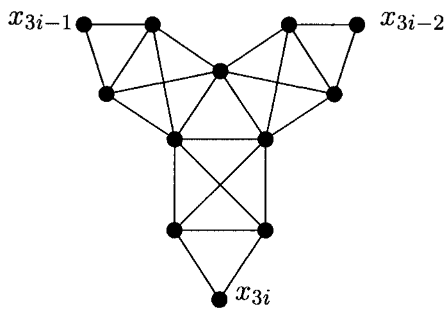

FIGURE 2. The graph G'.

- G_8 is obtained from M by identifying each vertex x_i with y_{i+1} $(i = 1, \ldots, 7)$, x_8 with y_1 and each vertex z_i with z_{i+4} $(i = 1, \ldots, 4)$.

Since the graphs $G_1, \ldots, G_8$ are not hamiltonian-connected, each of them must contain an induced copy of either X or Y. The graphs G_1, G_2, G_3, G_4 all contain a claw, but the last four graphs G_5, G_6, G_7, G_8 are all claw-free.

We will first show that one of the graphs X or Y must be $K_{1,3}$. Assume that this is not true. Assume, without loss of generality, that $X \subset G_1$. Then X must either contain an induced C_4 or it must be a generalized claw $K_{1,r}$ for $r \geq 4$. First consider the case when $C_4 \subset X$. Then Y must be an induced subgraph of both G_3 and G_4, since neither of these graphs contains an induced C_4. However, the only induced subgraph common to both G_3 and G_4 is the claw $K_{1,3}$. If $X = K_{1,r}$ for $r \geq 4$, then Y must be an induced subgraph of both G_2 and G_4, since neither of these graphs has an induced $K_{1,4}$. Again, the only induced subgraph common to both G_2 and G_4 is the claw $K_{1,3}$. Therefore, without loss of generality, we can assume that $X = K_{1,3}$.

Since G_5, G_6, G_7, G_8 are all claw-free, Y must be an induced subgraph of each of these graphs. Since G_5 is claw-free and $\Delta(G_5) = 3$, Y must satisfy both (a) and (f). There is no induced P_{10} in G_8, so (b) is satisfied. The shortest induced cycle in G_5 besides C_3 is a C_8, the longest induced cycle in G_8 is a C_8, and G_6 contains no induced C_8. Thus (c) is satisfied. In G_5, the distance between distinct triangles is either one or at least three. This implies that (d) is satisfied. The graph G_7 does not contain an induced copy of the graph S obtained from a P_5 by placing a triangle on the first and third edge (S is an H_1 with an edge attached to a vertex of degree two). Therefore, if Y contains three triangles, then each pair of triangles would have to be at distance at least three. This would imply an induced P_{10}, which is not true. Thus (e) is satisfied. This completes the proof of Theorem 6. ■

4. OPEN QUESTION

The obvious question is the following.

Question A. What is the characterization of those pairs of connected graphs X and Y such that being X-free and Y-free implies that a 3-connected graph is hamiltonian-connected?

A simpler question, but one that is critical to answering Question A is the following.

Question B. What is the largest k such that a 3-connected claw-free and P_k-free graph is hamiltonian-connected?

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