# On the Complexity of the Circular Chromatic Number 

H. Hatami and R. Tusserkani<br>Department of Computer Engineering<br>Sharif University of Technology<br>P.O. Box 11365-9517, Tehran, I.R. Iran


#### Abstract

Circular chromatic number, $\chi_{c}$ is a natural generalization of chromatic number. It is known that it is NP-hard to determine whether or not an arbitrary graph $G$ satisfies $\chi(G)=\chi_{c}(G)$. In this paper we prove that this problem is NP-hard even if the chromatic number of the graph is known. This answers a question of Xuding Zhu. Also we prove that for all positive integers $k \geq 2$ and $n \geq 3$, for a given graph $G$ with $\chi(G)=n$, it is NP-complete to verify if $\chi_{c}(G) \leq n-\frac{1}{k}$.


## 1 Introduction

We follow [4] for terminology and notation not defined here, and we consider finite undirected simple graphs. Given a graph $G$, an edge $e=x y$ of $G$ and a triple ( $H ; a, b$ ) where $a$ and $b$ are distinct vertices of the graph $H$, by replacing the edge $e$ by $(H ; a, b)$, we mean taking the disjoint union of $G-e$ and $H$, and identifying $x$ with $a$ and $y$ with $b$. For our purposes, it does not matter whether $x$ is identified with $a$ or with $b$.

For two positive integers $p$ and $q$, a $(p, q)$-coloring of a graph $G$ is a vertex coloring $c$ of $G$ with colors $\{0,1,2, \ldots, p-1\}$ such that

$$
(x, y) \in E(G) \Longrightarrow q \leq|c(x)-c(y)| \leq p-q
$$

The circular chromatic number is defined as

$$
\chi_{c}(G)=\inf \{p / q: G \text { is }(p, q) \text {-circular colorable }\} .
$$

So for a positive integer $k$, a $(k, 1)$-coloring of a graph $G$ is just an ordinary $k$ coloring of $G$. The circular chromatic number of a graph was introduced by Vince [3]
as "the star-chromatic number" in 1988. He proved that for every finite graph $G$, the infimum in the definition of the circular chromatic number is attained, so the circular chromatic number $\chi_{c}(G)$ is always rational. He also proved, among other things, that $\chi-1<\chi_{c} \leq \chi$, and $\chi_{c}\left(K_{n}\right)=n$.

For a $(p, q)$-coloring $\phi$ of a graph $G$, let $D_{\phi}(G)$ be the digraph with vertex set $V(G)$ and for every edge $x y$ in $G$ there is a directed edge $(x, y)$ in $D_{\phi}(G)$, if $\phi(y)-\phi(x)=q(\bmod p)$.

Lemma A. [1] For a graph $G, \chi_{c}(G)<p / q$ if and only if $D_{c}(G)$ is acyclic for some $(p, q)$-coloring $c$ of $G$.

The question determining which graphs have $\chi_{c}=\chi$ was raised by Vince [3]. It was shown by Guichard [1] that it is NP-hard to determine whether or not an arbitrary graph $G$ satisfies $\chi_{c}(G)=\chi(G)$. In [5] X. Zhu surveyed many results on circular chromatic number and posed some open problems on this topic, among them the following problem ([5], Question 8.23).

Problem 1 What is the complexity of determining whether or not $\chi_{c}(G)=\chi(G)$, if the chromatic number $\chi(G)$ is known?

We answer this question, using the following theorem.
Theorem A. [2] It is NP-hard to determine whether a graph is 3-colorable or any coloring of it requires at least 5 colors.

## 2 Complexity

Consider the graph $K^{-}$which is obtained from a copy of $K_{4}$ with vertices $v_{1}, v_{2}, v_{3}$, and $v_{4}$, by removing the edge $\left\{v_{1}, v_{2}\right\}$. In the following trivial lemma all equalities are in $\mathcal{Z}_{4}$.

Lemma 1 In every (4,1)-coloring c of $K^{-}$,
(a) if $c\left(v_{1}\right)=c\left(v_{2}\right)$, then $D_{c}\left(K^{-}\right)$is acyclic and has no directed path between $v_{1}$ and $v_{2}$.
(b) if $c\left(v_{1}\right)-c\left(v_{2}\right)=1$, then $D_{c}\left(K^{-}\right)$is acyclic and has a directed path from $v_{1}$ to $v_{2}$.
(c) if $c\left(v_{1}\right)-c\left(v_{2}\right)=2$, then $D_{c}\left(K^{-}\right)$has a cycle.

Consider the graph $H$ shown in Figure 1. One can easily check that $\chi(H)=4$ and we have the following Lemma.


Figure 1: The graph $H$ and its desired colorings

Lemma 2 Consider the graph $H$ shown in Figure 1.
(a) For every $(4,1)$-coloring $c$ of $H$, if $c(a)=c(b)$, then $D_{c}(H)$ has a cycle.
(b) For every $0 \leq x<y \leq 2$, there is a coloring c for $H$ such that $c(a)=x$, $c(b)=y$, and $D_{c}(H)$ is acyclic and has no directed path from $b$ to $a$.

Proof. (a) Without loss of generality assume that $c(a)=c(b)=0$. For all cases except when $c(c)=c(d)=1$ and $c(c)=c(d)=3$, one can easily check by Lemma (c) that $D_{c}(H)$ has a cycle. Without loss of generality assume that $c(c)=c(d)=1$. Now by Lemma (b) there are directed paths from $d$ to $b, b$ to $c, c$ to $a$ and $a$ to $d$. Thus $D_{c}(H)$ has a cycle.
(b) Such colorings are given in Figures 1 (a), 1(b), 1 (c).

Theorem 1 Given a graph $G$ and its chromatic number, the problem of determining whether or not $\chi_{c}(G)=\chi(G)$ is NP-hard.

Proof. For every graph $G^{\prime}$, we construct a graph $G$ such that $\chi(G)=4$, and if $G^{\prime}$ is 3-colorable, then $\chi_{c}(G)<4$, and if $G^{\prime}$ is not 4-colorable, then $\chi_{c}(G)=4$. Thus by Theorem A the result is proven.

Construct the graph $G$ by replacing every edge of $G^{\prime}$ by $(H ; a, b)$. Obviously, for every nontrivial graph $G^{\prime}, \chi(G)=4$.

First suppose that $G^{\prime}$ is 3 -colorable. So we can properly color the vertices of $G^{\prime}$ with 0,1 , and 2 . Now by Lemma $2(b)$, this coloring can be expanded to a $(4,1)$ coloring $c$ of $G$ such that in $D_{c}(G)$ the copies of $H$ are acyclic, and also for every two vertices $u$ and $v$ of $G^{\prime}$, there is no path from $u$ to $v$ in $D_{c}(G)$ if $c(u)>c(v)$. This implies that $D_{c}(G)$ is acyclic. So $\chi_{c}(G)<4$.

Next suppose that $G^{\prime}$ is not 4-colorable. So in any (4,1)-coloring $c$ of $G$ there are two adjacent vertices $u$ and $v$ of $G$ such that $c(u)=c(v)$. So by Lemma 2(a) for the copy of $H$ which is between $u$ and $v$ there exists a cycle in $D_{c}(H)$. Hence $\chi_{c}(G)=4$.

Now we prove that it is NP-complete to verify that the difference between chromatic number and circular chromatic number of a given graph is greater than or
equal to $\frac{1}{k}$, when $k \geq 2$ is an arbitrary positive integer is NP-complete. Let $K$ be a graph with vertex set $\left\{a, b, v_{1}, \ldots, v_{n-1}\right\}$ in which each $v_{i}$ is adjacent to every other $v_{j}, a$ is adjacent to $v_{1}, \ldots, v_{n-2}$, and $b$ is adjacent to $v_{n-1}$.

Lemma 3 For all integers $0 \leq x, y \leq k n-1$, K has a ( $k n-1, k$ )-coloring c with $c(a)=x$ and $c(b)=y$ if and only if $x \neq y$.

Proof. If $x=y$, then a $(k n-1, k)$-coloring of $K$ can be transformed to a $(k n-1, k)$ coloring of $K_{n}$ by identifying $a$ and $b$. And this is impossible because $\chi_{c}\left(K_{n}\right)=n$. If $x \neq y$ without loss of generality we can assume that $x=0$ and $0<y \leq \frac{k n-1}{2}$.

First suppose that $y \geq k$. In this case define a desired $(k n-1, k)$-coloring $c$ by $c(a)=0, c(b)=y, c\left(v_{i}\right)=i k$ for $1 \leq i \leq n-2$ and $c\left(v_{n-1}\right)=0$.

Next suppose that $y<k$. In this case define a desired $(k n-1, k)$-coloring $c$ by $c(a)=0, c(b)=y, c\left(v_{i}\right)=i k$ for $1 \leq i \leq n-2$, and $c\left(v_{n-1}\right)=y-k$.

Theorem 2 For all positive integers $k \geq 2$ and $n \geq 3$, the following problem is NP-complete. A graph $G$ is given where $\chi(G)=n$, and it is asked whether $\chi_{c}(G) \leq$ $n-\frac{1}{k}$ ?

Proof. Clearly, the problem is in NP. We reduce Vertex Coloring to this problem. Consider a graph $G^{\prime}$ as an instance of Vertex Coloring. It is asked whether the vertices of $G^{\prime}$ can be colored with $k n-1$ colors. We construct a new graph $G$ with the property that $\chi_{c}(G) \leq n-\frac{1}{k}$ if and only if the vertices of $G^{\prime}$ can be colored with $k n-1$ colors.

Construct a graph $G$ by replacing every edge $u v$ of $G^{\prime} \sqcup K_{n}$, the disjoint union of $G^{\prime}$ and a copy of $K_{n}$, by $(K ; a, b)$. Obviously, $\chi(G) \leq n$. Since in every $(n-1)$ coloring of $K$ the vertices $a$ and $b$ must have different colors, thus $\chi(G)=n$. We know that $\chi_{c}(G) \leq n-\frac{1}{k}$ if and only if there exists a $(k n-1, k)$-coloring $c$ for $G$.

First suppose that $\chi\left(G^{\prime}\right) \leq k n-1$, and $c$ is a $(k n-1)$-coloring of $G^{\prime} \sqcup K_{n}$. For all copies of $K$ in $G$, we have $c(a) \neq c(b)$. By Lemma 3, $c$ can be extended to a ( $k n-1, k)$-coloring of $G$. Thus $\chi_{c}(G) \leq n-\frac{1}{k}$.

Next suppose that $\chi\left(G^{\prime}\right)>k n-1$ and $c$ is a $(k n-1, k)$-coloring of $G$. There exist two adjacent vertices $u$ and $v$ in $G^{\prime}$ such that $c(u)=c(v)$. But by Lemma 3, the copy of $K$ between $u$ and $v$ has no ( $k n-1, k$ )-coloring. This is a contradiction. Thus $\chi_{c}(G)>n-\frac{1}{k}$.

## Acknowledgements

The authors wish to thank Hossein Hajiabolhassan who drew their attention to this subject, and for suggesting the problem.

## References

[1] D.R. Guichard. Acyclic graph coloring and the complexity of the star chromatic number. J. Graph Theory, 17:129-134, 1993.
[2] S. Khanna, N. Linial, and S. Safra. On the hardness of approximating the chromatic number. In Proc. 2nd Israel Symp. on Theory of Computing and Systems, pages 250-260, 1993.
[3] A. Vince. Star chromatic number. J. Graph Theory, 12:551-559, 1988.
[4] D.B. West. Introduction to Graph Theory. Prentice-Hall, Inc, United States of America, 2001. 2nd Edition.
[5] X. Zhu. Circular chromatic number: a survey. Discrete Math., 229:371-410, 2001.

