# On the Complexity of the Circular Chromatic Number

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#### Abstract

Circular chromatic number,  $\chi_c$  is a natural generalization of chromatic number. It is known that it is **NP**-hard to determine whether or not an arbitrary graph G satisfies  $\chi(G) = \chi_c(G)$ . In this paper we prove that this problem is **NP**-hard even if the chromatic number of the graph is known. This answers a question of Xuding Zhu. Also we prove that for all positive integers  $k \ge 2$  and  $n \ge 3$ , for a given graph G with  $\chi(G) = n$ , it is **NP**-complete to verify if  $\chi_c(G) \le n - \frac{1}{k}$ .

#### 1 Introduction

We follow [4] for terminology and notation not defined here, and we consider finite undirected simple graphs. Given a graph G, an edge e = xy of G and a triple (H; a, b) where a and b are distinct vertices of the graph H, by replacing the edge eby (H; a, b), we mean taking the disjoint union of G - e and H, and identifying xwith a and y with b. For our purposes, it does not matter whether x is identified with a or with b.

For two positive integers p and q, a (p, q)-coloring of a graph G is a vertex coloring c of G with colors  $\{0, 1, 2, \ldots, p-1\}$  such that

$$(x,y) \in E(G) \Longrightarrow q \le |c(x) - c(y)| \le p - q.$$

The circular chromatic number is defined as

 $\chi_c(G) = \inf \{ p/q : G \text{ is } (p,q) \text{-circular colorable} \}.$ 

So for a positive integer k, a (k, 1)-coloring of a graph G is just an ordinary kcoloring of G. The circular chromatic number of a graph was introduced by Vince [3] as "the star-chromatic number" in 1988. He proved that for every finite graph G, the infimum in the definition of the circular chromatic number is attained, so the circular chromatic number  $\chi_c(G)$  is always rational. He also proved, among other things, that  $\chi - 1 < \chi_c \leq \chi$ , and  $\chi_c(K_n) = n$ .

For a (p,q)-coloring  $\phi$  of a graph G, let  $D_{\phi}(G)$  be the digraph with vertex set V(G) and for every edge xy in G there is a directed edge (x,y) in  $D_{\phi}(G)$ , if  $\phi(y) - \phi(x) = q \pmod{p}$ .

**Lemma A.** [1] For a graph G,  $\chi_c(G) < p/q$  if and only if  $D_c(G)$  is acyclic for some (p,q)-coloring c of G.

The question determining which graphs have  $\chi_c = \chi$  was raised by Vince [3]. It was shown by Guichard [1] that it is **NP**-hard to determine whether or not an arbitrary graph G satisfies  $\chi_c(G) = \chi(G)$ . In [5] X. Zhu surveyed many results on circular chromatic number and posed some open problems on this topic, among them the following problem ([5], Question 8.23).

**Problem 1** What is the complexity of determining whether or not  $\chi_c(G) = \chi(G)$ , if the chromatic number  $\chi(G)$  is known?

We answer this question, using the following theorem.

**Theorem A.** [2] It is **NP**-hard to determine whether a graph is 3-colorable or any coloring of it requires at least 5 colors.

#### 2 Complexity

Consider the graph  $K^-$  which is obtained from a copy of  $K_4$  with vertices  $v_1, v_2, v_3$ , and  $v_4$ , by removing the edge  $\{v_1, v_2\}$ . In the following trivial lemma all equalities are in  $\mathcal{Z}_4$ .

**Lemma 1** In every (4, 1)-coloring c of  $K^-$ ,

- (a) if  $c(v_1) = c(v_2)$ , then  $D_c(K^-)$  is acyclic and has no directed path between  $v_1$ and  $v_2$ .
- (b) if  $c(v_1) c(v_2) = 1$ , then  $D_c(K^-)$  is acyclic and has a directed path from  $v_1$  to  $v_2$ .
- (c) if  $c(v_1) c(v_2) = 2$ , then  $D_c(K^-)$  has a cycle.

Consider the graph H shown in Figure 1. One can easily check that  $\chi(H) = 4$  and we have the following Lemma.

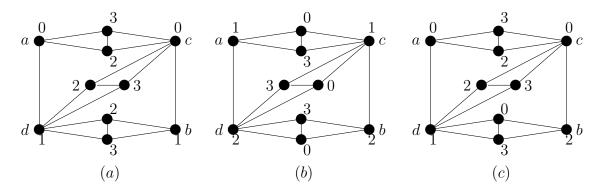


Figure 1: The graph H and its desired colorings

**Lemma 2** Consider the graph H shown in Figure 1.

- (a) For every (4,1)-coloring c of H, if c(a) = c(b), then  $D_c(H)$  has a cycle.
- (b) For every  $0 \le x < y \le 2$ , there is a coloring c for H such that c(a) = x, c(b) = y, and  $D_c(H)$  is acyclic and has no directed path from b to a.

**Proof.** (a) Without loss of generality assume that c(a) = c(b) = 0. For all cases except when c(c) = c(d) = 1 and c(c) = c(d) = 3, one can easily check by Lemma 1(c) that  $D_c(H)$  has a cycle. Without loss of generality assume that c(c) = c(d) = 1. Now by Lemma 1(b) there are directed paths from d to b, b to c, c to a and a to d. Thus  $D_c(H)$  has a cycle.

(b) Such colorings are given in Figures 1(a), 1(b), 1(c).

**Theorem 1** Given a graph G and its chromatic number, the problem of determining whether or not  $\chi_c(G) = \chi(G)$  is **NP**-hard.

**Proof.** For every graph G', we construct a graph G such that  $\chi(G) = 4$ , and if G' is 3-colorable, then  $\chi_c(G) < 4$ , and if G' is not 4-colorable, then  $\chi_c(G) = 4$ . Thus by Theorem A the result is proven.

Construct the graph G by replacing every edge of G' by (H; a, b). Obviously, for every nontrivial graph G',  $\chi(G) = 4$ .

First suppose that G' is 3-colorable. So we can properly color the vertices of G' with 0, 1, and 2. Now by Lemma 2(b), this coloring can be expanded to a (4, 1)-coloring c of G such that in  $D_c(G)$  the copies of H are acyclic, and also for every two vertices u and v of G', there is no path from u to v in  $D_c(G)$  if c(u) > c(v). This implies that  $D_c(G)$  is acyclic. So  $\chi_c(G) < 4$ .

Next suppose that G' is not 4-colorable. So in any (4, 1)-coloring c of G there are two adjacent vertices u and v of G such that c(u) = c(v). So by Lemma 2(a) for the copy of H which is between u and v there exists a cycle in  $D_c(H)$ . Hence  $\chi_c(G) = 4$ .

Now we prove that it is **NP**-complete to verify that the difference between chromatic number and circular chromatic number of a given graph is greater than or equal to  $\frac{1}{k}$ , when  $k \geq 2$  is an arbitrary positive integer is **NP**-complete. Let K be a graph with vertex set  $\{a, b, v_1, \ldots, v_{n-1}\}$  in which each  $v_i$  is adjacent to every other  $v_j$ , a is adjacent to  $v_1, \ldots, v_{n-2}$ , and b is adjacent to  $v_{n-1}$ .

**Lemma 3** For all integers  $0 \le x, y \le kn - 1$ , K has a (kn - 1, k)-coloring c with c(a) = x and c(b) = y if and only if  $x \ne y$ .

**Proof.** If x = y, then a (kn-1, k)-coloring of K can be transformed to a (kn-1, k)coloring of  $K_n$  by identifying a and b. And this is impossible because  $\chi_c(K_n) = n$ .
If  $x \neq y$  without loss of generality we can assume that x = 0 and  $0 < y \leq \frac{kn-1}{2}$ .

First suppose that  $y \ge k$ . In this case define a desired (kn - 1, k)-coloring c by  $c(a) = 0, c(b) = y, c(v_i) = ik$  for  $1 \le i \le n - 2$  and  $c(v_{n-1}) = 0$ .

Next suppose that y < k. In this case define a desired (kn - 1, k)-coloring c by  $c(a) = 0, c(b) = y, c(v_i) = ik$  for  $1 \le i \le n - 2$ , and  $c(v_{n-1}) = y - k$ .

**Theorem 2** For all positive integers  $k \ge 2$  and  $n \ge 3$ , the following problem is **NP**-complete. A graph G is given where  $\chi(G) = n$ , and it is asked whether  $\chi_c(G) \le n - \frac{1}{k}$ ?

**Proof.** Clearly, the problem is in **NP**. We reduce VERTEX COLORING to this problem. Consider a graph G' as an instance of VERTEX COLORING. It is asked whether the vertices of G' can be colored with kn - 1 colors. We construct a new graph G with the property that  $\chi_c(G) \leq n - \frac{1}{k}$  if and only if the vertices of G' can be colored with kn - 1 colors.

Construct a graph G by replacing every edge uv of  $G' \sqcup K_n$ , the disjoint union of G' and a copy of  $K_n$ , by (K; a, b). Obviously,  $\chi(G) \leq n$ . Since in every (n-1)coloring of K the vertices a and b must have different colors, thus  $\chi(G) = n$ . We know that  $\chi_c(G) \leq n - \frac{1}{k}$  if and only if there exists a (kn - 1, k)-coloring c for G.

First suppose that  $\chi(G') \leq kn - 1$ , and c is a (kn - 1)-coloring of  $G' \sqcup K_n$ . For all copies of K in G, we have  $c(a) \neq c(b)$ . By Lemma 3, c can be extended to a (kn - 1, k)-coloring of G. Thus  $\chi_c(G) \leq n - \frac{1}{k}$ .

Next suppose that  $\chi(G') > kn - 1$  and c is a (kn - 1, k)-coloring of G. There exist two adjacent vertices u and v in G' such that c(u) = c(v). But by Lemma 3, the copy of K between u and v has no (kn - 1, k)-coloring. This is a contradiction. Thus  $\chi_c(G) > n - \frac{1}{k}$ .

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