

Finite subgraphs of uncountably chromatic graphs

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October 30, 2018

Abstract

It is consistent that for every function $f : \omega \rightarrow \omega$ there is a graph with size and chromatic number \aleph_1 in which every n -chromatic subgraph contains at least $f(n)$ vertices ($n \geq 3$). This solves a \$ 250 problem of Erdős. It is consistent that there is a graph X with $\text{Chr}(X) = |X| = \aleph_1$ such that if Y is a graph all whose finite subgraphs occur in X then $\text{Chr}(Y) \leq \aleph_2$ (so the Taylor conjecture may fail). It is also consistent that if X is a graph with chromatic number at least \aleph_2 then for every cardinal λ there exists a graph Y with $\text{Chr}(Y) \geq \lambda$ all whose finite subgraphs are induced subgraphs of X .

1 Introduction

In [8] Erdős and Hajnal determined those finite graphs which appear as subgraphs in every uncountably chromatic graph: the bipartite graphs. In fact, not just that any odd circuit can be omitted, for every natural number $n \geq 1$ and infinite cardinal κ there is a graph with cardinality and chromatic number κ such that it omits all odd circuits up to length $2n + 1$. They observed that the so-called r -shift graph construction has all but one of these properties; the vertex set of $\text{Sh}_r(\kappa)$ is the set of all r -tuples from κ , with $\{x_0, x_1, \dots, x_{r-1}\}_<$ joined to $\{x_1, x_2, \dots, x_r\}_<$, then this graph omits odd circuits of length $3, 5, \dots, 2r+1$ and the Erdős-Rado theorem asserts that the chromatic number of $\text{Sh}_r(\exp_{r-1}(\kappa)^+)$ is at least κ^+ .

The problem of determining the classes of finite graphs that occur in uncountably chromatic graphs seems to be much harder, and its investigation was strongly pushed by Erdős and Hajnal.

An early conjecture for example was the following. Every uncountably chromatic graph contains all odd circuits from some length onward. This was then

*Research partially supported by Hungarian National Research Grant T 032455.

†This research was supported by the Israel Science Foundation. Publication 788.

proved by Erdős, Hajnal, and Shelah [11], and independently, by Thomassen [17].

In [19] and later in [11] the following problem was posed (the Taylor conjecture). If κ , λ are uncountable cardinals and X is a κ -chromatic graph, is there a λ -chromatic graph Y such that every finite subgraph of Y appears as a subgraph of X . Notice that the above shift graphs give some evidence for this conjecture—the finite subgraphs of $\text{Sh}_r(\kappa)$ do not depend on the parameter κ . The authors of [11] remarked that even the following much stronger conjecture seemed possible. If X is uncountably chromatic, then for some r it contains all finite subgraphs of $\text{Sh}_r(\omega)$. This conjecture was then disproved in [13].

One easy remark, the so called Hanf number argument gives that there is a cardinal κ with the property that if the chromatic number of some graph X is at least κ then there are arbitrarily large chromatic graphs with all finite subgraphs appearing in X . This argument, however, does not give any reasonable bound on κ .

Another conjecture of Erdős and Hajnal if the maximal chromatic number of n -element subgraphs of an uncountably chromatic graph as a function of n can converge to infinity arbitrarily slowly as n tends to infinity. It was mentioned in several problem papers, for example in [2], [5], [6], [10], [12]. See also [1], [14]. The relevance of the above examples is that the chromatic number of the n -vertex subgraphs of (any) $\text{Sh}_r(\kappa)$ grows roughly as the $r - 1$ times iterated logarithm of n . Perhaps it was this fact that led Erdős and Hajnal to the above problem.

Erdős also tirelessly popularized the Taylor conjecture, he mentioned it e.g., in [3], [4], [7], [9], [10], [11]. It is also mentioned in [1], the book collecting Erdős' conjectures on graphs. In [15] we gave some results when the additional hypotheses $|X| = \kappa$, $|Y| = \lambda$ was imposed. We described countably many different classes $\mathcal{K}_{n,e}$ of finite graphs and proved that if $\lambda^{\aleph_0} = \lambda$ then every λ^+ -chromatic graph of cardinal λ^+ contains, for some n , e , all elements of $\mathcal{K}_{n,e}$ as subgraphs. On the other hand, it is consistent for every regular infinite cardinal κ that there is a κ^+ -chromatic graph on κ^+ that contains finite subgraphs only from $\mathcal{K}_{n,e}$. We got, therefore, some models of set theory, where the finite subgraphs of graphs with $|X| = \text{Chr}(X) = \kappa^+$ for regular uncountable cardinals κ were completely described.

Notice that the class of regular cardinals on which the above result operated excludes ω_1 , and in this paper we show the reason, by resolving the above Erdős-Hajnal conjecture: it is consistent that for every monotonically increasing function $f : \omega \rightarrow \omega$ there is a graph with size and chromatic number \aleph_1 in which every n -chromatic subgraph has at least $f(n)$ elements ($n \geq 3$). The possibility of transforming the proof into a ZFC argument will be checked in the forthcoming [16], Chapter 9. An application of the method presented here gives the consistent existence of a graph X with $\text{Chr}(X) = |X| = \aleph_1$ such that if Y is a graph (of any size) all whose subgraphs are subgraphs of X then $\text{Chr}(Y) \leq \aleph_2$. This gives a consistent negative answer to the Taylor conjecture. As for the positive direction we prove that it is consistent that if X is a graph with chromatic number at least \aleph_2 then there are arbitrarily large chromatic

graphs all whose finite subgraphs being induced subgraphs of X .

Theorems 1 and 2 were proved by S. Shelah and then P. Komjáth proved Theorems 3 and 4.

Notation. We use the standard axiomatic set theory notation. If A is a set of ordinals, β is an ordinal, then $\beta < A$, means that $\beta < \alpha$ holds for every $\alpha \in A$. Similarly for $\beta \leq A$, $A < \beta$, etc. If f is a function, A a set, then we let $f[A] = \{f(x) : x \in A\}$. If S is a set, κ is a cardinal, $[S]^\kappa = \{X \subseteq S : |X| = \kappa\}$, $[S]^{<\kappa} = \{X \subseteq S : |X| < \kappa\}$. A *graph* is an ordered pair (V, X) where V is some set (the set of *vertices*) and $X \subseteq [V]^2$ (the set of *edges*). In some cases we identify the graph and X . The *chromatic number* of a graph (V, X) is the least cardinal μ such that there exists a function $f : V \rightarrow \mu$ with $f(x) \neq f(y)$ for $\{x, y\} \in X$ (a good coloring). A *path* of length n is a sequence $\{v_0, \dots, v_n\}$ of distinct vertices such that $\{v_i, v_{i+1}\} \in X$ holds for $i < n$. A *cycle* of length n is a sequence $\{v_1, \dots, v_n\}$ of vertices such that for $1 \leq i < n$ we have $\{v_i, v_{i+1}\} \in X$ and $\{v_n, v_1\} \in X$ also holds. If the vertices are distinct then we call it a *circuit*.

2 A large chromatic graph with small chromatic finite subgraphs

Let $f : \omega \rightarrow \omega$ be a strictly increasing function. Fix a sequence $\{C_\alpha : \alpha < \omega_1, \text{ limit}\}$ such that C_α is an ω -sequence converging to α , and the whole sequence is a club guessing sequence, that is, if $C \subseteq \omega_1$ is a closed unbounded set, then $C_\alpha \subseteq C$ holds for some α . Notice that the existence of a guessing sequence is an easy consequence of the diamond principle.

We are going to define the notion of forcing (Q^f, \leq) . Every condition $p \in Q^f$ will be of the form $p = (s, u, g, m, h_i, c)$ where $s \in [\omega_1]^{<\omega}$, $u \subseteq s$, consisting of limit ordinals, g is a graph on u , for every $\alpha \in u$ we have an $m(\alpha) < \omega$, and then the ordinals $h_0(\alpha) < \dots < h_{m(\alpha)-1}(\alpha) < \alpha$ which are C_α -separated, that is, $\min(C_\alpha) < h_0(\alpha)$ and between $h_i(\alpha)$ and $h_{i+1}(\alpha)$ there is an element of C_α ($h_i(\alpha)$ is undefined if $i \geq m(\alpha)$ or $\alpha \notin u$). $c : g \rightarrow \omega$ satisfies that if $\{x, \alpha\} \in g$, $\{y, \alpha\} \in g$ and $x, y < \alpha$ then $c(x, \alpha) \neq c(y, \alpha)$.

Given $p \in Q^f$ we define

$$Y_r^p = \{e \in g : c(e) \geq r\}$$

for any natural number r .

We add the following stipulations.

- (1.) If $\alpha \in u$ is incident to some $e \in g$ with $c(e) \geq r$ then $m(\alpha) \geq r$.
- (2.) If x, y are connected in Y_r^p in i steps then $h_{j-i}(x) < h_j(y) < h_{j+i}(x)$ holds for $i \leq j \leq f(r) - i$.
- (3.) Y_r^p does not contain odd circuits of length $\leq f(r)$.

We notice that although a condition $p = (s, u, g, m, h_i, c)$ is infinite (perhaps $p = (s, u, g, m, (h_i)_{i < \omega}, c)$ or $p = (s, u, g, m, h_i, c)_{i < \omega}$ would be better notation) as all but finitely many of the partial functions $\{h_i : i < \omega\}$ are the empty function, every condition is really a finite object.

The partial order on Q^f is defined the natural way. $p' = (s', u', g', m', h'_i, c')$ extends $p = (s, u, g, m, h_i, c)$ iff the following hold. $s' \supseteq s$, $u = u' \cap s$, $g = g' \cap [s]^2$. $m'(\alpha) \geq m(\alpha)$ holds for $\alpha \in u$, and $h'_i(\alpha) = h_i(\alpha)$ for $i < m(\alpha)$. Finally, $c'(x, \alpha) = c(x, \alpha)$ holds for $\{x, \alpha\} \in g$.

We call two conditions $p = (s, u, g, m, h_i, c)$ and $p' = (s', u', g', m', h'_i, c')$ *isomorphic* iff $|s| = |s'|$ and the unique order preserving mapping $\pi : s \rightarrow s'$ satisfies $u' = \pi[u]$, $g' = \pi[g]$, $m(\alpha) = m'(\pi(\alpha))$ for $\alpha \in u$, $h'_i(\pi(\alpha)) = h_i(\alpha)$ and $c'(\pi(x), \pi(\alpha)) = c(x, \alpha)$ hold whenever the right hand sides are defined. Notice that, as our conditions are finite structures, we have only countably many isomorphism types.

From a generic $G \subseteq Q^f$ we define the following graphs on ω_1 :

$$X = \bigcup \{g : (s, u, g, m, h_i, c) \in G\},$$

$$X_n = \{\{x, y\} : \{x, y\} \in g, c(x, y) = n, (s, u, g, m, h_i, c) \in G\},$$

$$Y_r = X_r \cup X_{r+1} \cdots$$

and notice that $Y_r = \cup \{Y_r^p : p \in G\}$.

Lemma 1. *For $\alpha < \omega_1$ the set $\{(s, u, g, m, h_i, c) \in Q^f : \alpha \in s\}$ is dense.*

Proof. Straightforward. \square

Lemma 2. *If $\alpha < \omega_1$ is limit, $(s, u, g, m, h_i, c) \in Q^f$, $\alpha \notin s$, then there is an extension $(s', u', g', m', h'_i, c') \leq (s, u, g, m, h_i, c)$ with $\alpha \in u'$.*

Proof. Straightforward. \square

Lemma 3. *If $(s, u, g, m, h_i, c) \in Q^f$, $\alpha \in u$, $n < \omega$ then there is an extension $(s', u', g', m', h'_i, c') \leq (s, u, g, m, h_i, c)$ with $m(\alpha) \geq n$.*

Proof. It suffices to show that $m(\alpha)$ can be incremented by one. Given $h_{m(\alpha)-1}(\alpha) < \alpha$ if we choose $h'_{m(\alpha)}(\alpha) < \alpha$ large enough the condition on C_α will surely be satisfied. \square

Lemma 4. (Q^f, \leq) is ccc.

Proof. Modulo standard arguments we have to show that $p = (s, u, g, m, h_i, c)$ and $p' = (s', u', g', m', h'_i, c')$ are compatible, assuming that they are isomorphic, and $s \cap s'$ is an initial segment of both s and s' . Let $\pi : s \rightarrow s'$ be the order preserving structure isomorphism. We let $p'' = (s'', u'', g'', m'', h''_i, c'')$ where we take unions in all coordinates.

In order to show that p'' is a condition we have to check properties (1.–3.). (1.) is obvious.

For (3.) assume that C is an odd circuit of length $\leq f(r)$ in $Y_r^{p''}$. If we replace every $e \in C$ that contains at least one vertex from $s - s'$ with $\pi(e)$ then we get an odd cycle C' in $Y_r^{p'}$. C' splits into circuits, at least one of them odd, so we get a contradiction.

For (2.) notice that it holds if x, y are joined via a path going entirely in Y_r^p or $Y_r^{p'}$. It suffices, therefore, to show that if some path P between x and y of length i is split by an inner point z into the paths P_0 and P_1 between x and z , and z and y , respectively, and of the respective lengths i_0 and i_1 (so $i_0 + i_1 = i$) and the statement holds for P_0 and P_1 then it holds for P , as well. Indeed, for $i \leq j \leq f(r) - i$ we have

$$h_{j-(i_0+i_1)}(x) < h_{j-i_1}(z) < h_j(y) < h_{j+i_1}(z) < h_{j+(i_0+i_1)}(x).$$

□

Lemma 5. $\text{Chr}(X) = \omega_1$.

Proof. Assume that some \bar{p} forces that \underline{F} is a good coloring of X with the elements of ω . Select an increasing, continuous sequence of countable elementary submodels $\bar{p}, \underline{F}, Q^f, \Vdash \in N_0 \prec N_1 \prec \dots \prec N_\alpha \prec H(\lambda)$ with some large enough regular cardinal λ , for $\alpha < \omega_1$, such that $\gamma_\alpha = N_\alpha \cap \omega_1$ is an ordinal. The set $C = \{\gamma_\alpha : \alpha < \omega_1\}$ will be a closed, unbounded set, so by the guessing property there is some $\delta = \gamma_\alpha$ such that $C_\delta \subseteq C$. Notice that all points of \bar{p} are smaller than δ .

Extend \bar{p} to a p' using Lemma 2., adding δ to the u -part, then let p^* be a condition extending p' such that $p^* \Vdash \underline{F}(\delta) = i$ holds for some $i < \delta$.

Let n be some natural number that $n \notin \{c^*(x, \delta) : \{x, \delta\} \in g^*, x < \delta\}$ where $p^* = (s^*, u^*, g^*, m^*, h_i^*, c^*)$. Extend p^* using Lemma 3., to some condition p with $p = (s, u, g, m, h_i, c)$ such that $m = m(\delta) \geq f(n)$ holds.

In p we have the values

$$h_0(\delta) < h_1(\delta) < \dots < h_{m-1}(\delta) < \delta$$

and by our requirements on conditions there are elements $\delta_0, \dots, \delta_{m-1}$ of C_δ such that

$$\delta_0 < h_0(\delta) < \delta_1 < h_1(\delta) < \dots < \delta_{m-1} < h_{m-1}(\delta) < \delta$$

holds.

The values $\delta_0, \dots, \delta_{m-1}, \delta$ break s into disjoint parts: $s = s_0 \cup \dots \cup s_m \cup s_{m+1}$ with

$$s_0 < \delta_0 \leq s_1 < \delta_1 \leq s_2 < \dots < \delta_{m-1} \leq s_m < \delta \leq s_{m+1}$$

(some of them may be empty).

Sublemma. *There is a condition p' on some $s' = s_0 \cup s'_1 \cup \dots \cup s'_m \cup s'_{m+1}$ isomorphic to p with $|s'_i| = |s_i|$, $\delta' = \min(s'_{m+1})$, $p' \Vdash \underline{F}(\delta') = i$ and*

$$s_0 < s'_1 < \delta_0 \leq s_1 < s'_2 < \delta_1 \leq s_2 < \dots < s'_m < \delta_{m-1} \leq s_m < s'_{m+1} < \delta \leq s_{m+1}$$

holds.

Proof. Let θ be the isomorphism type of p . For the ordered finite sets $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{m+1}$ let $\psi(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{m+1})$ denote the statement that $\bar{x}_0 < \bar{x}_1 < \dots < \bar{x}_{m+1}$, $|\bar{x}_i| = |s_i|$ and for the (unique) condition p on $\bar{x}_0 \cup \bar{x}_1 \cup \dots \cup \bar{x}_{m+1}$ of type θ $p \Vdash \underline{F}(\delta) = i$ where $\delta = \min(\bar{x}_{m+1})$.

Set

$$\varphi_{m+1}(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{m+1}) = \psi(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{m+1})$$

and for $0 \leq i \leq m$

$$\varphi_i(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_i) = \exists^* \bar{x}_{i+1} \varphi_{i+1}(\bar{x}_0, \bar{x}_1, \dots, \bar{x}_{i+1})$$

where the quantifier \exists^* denotes “there exist unboundedly many” which is expressible in the first order language of $(\omega_1, <)$.

Claim. For $0 \leq i \leq m+1$ the sentence $\varphi_i(s_0, s_1, \dots, s_i)$ holds.

Proof. We prove the statement by reverse induction on $0 \leq i \leq m+1$. We certainly have $\varphi_{m+1}(s_0, s_1, \dots, s_{m+1})$. If for some $0 \leq i \leq m$ we had that $\varphi_{i+1}(s_0, s_1, \dots, s_{i+1})$ holds yet

$$\varphi_i(s_0, s_1, \dots, s_i) = \exists^* \bar{x}_{i+1} \varphi_{i+1}(s_0, s_1, \dots, s_i, \bar{x}_{i+1})$$

fails, then there was a bound, computable from s_0, s_1, \dots, s_i for the minima of those sets \bar{x}_{i+1} for which $\varphi_{i+1}(s_0, s_1, \dots, s_i, \bar{x}_{i+1})$ holds. Then this bound was smaller than δ_i (δ for $i = m$) as there is an elementary submodel containing the ordinals $< \delta_i$ (or $< \delta$) but this contradicts the fact that $\varphi_{i+1}(s_0, s_1, \dots, s_{i+1})$ holds and $\delta_i \leq s_i$. \square

Using the Claim we can inductively select the sets s'_1, \dots, s'_{m+1} such that for every $1 \leq i \leq m+1$ we have $\varphi_i(s_0, s'_1, \dots, s'_i)$ and $s_i < s'_{i+1} \leq \delta_i$, as required. \square

Using the Sublemma we create the following one-edge amalgamation p'' of p and p' . $p'' = (s'', u'', g'', m'', h''_i, c'')$ where $s'' = s \cup s'$, $u'' = u \cup u'$, $g'' = g \cup g' \cup \{\{\delta, \delta'\}\}$, $m'' = m \cup m'$, $h''_i = h_i \cup h'_i$, $c'' = c \cup c'$.

We have to show that p'' is indeed a condition, that is, we have to check if the properties (1.-3.) hold.

(1.) is obvious.

For (2.) we argue as in the proof of Lemma 4; every path in question is the union of paths for which this condition holds, plus possibly the path $\{\delta, \delta'\}$ but (2.) also holds for this.

Assume finally, that C is a circuit of length $2t+1 \leq f(r)$ in $Y_r^{p''}$. Unless C contains $\{\delta, \delta'\}$, we can argue as in Lemma 4. So we are left with the case that C contains $\{\delta, \delta'\}$ and $r \leq n$. That is, $\delta \in s$ and $\delta' \in s'$ are joined in $Y_r^p \cup Y_r^{p'}$ in $2t$ steps, and this is only possible if the connecting path has vertices in $s \cap s'$. So we get that δ can be connected in Y_r^p with some point in $s \cap s'$ in $\leq t$ steps. But this is impossible: if $x \in s \cap s'$ is such a point then $h_0(\delta) < h_t(x) < h_{2t}(\delta)$ by condition (2.) and also $x < h_0(\delta)$, a contradiction.

As p'' forces that $F(\delta) = F(\delta') = i$ yet they are joined in X , we are finished.

□

Theorem 1. *The forcing Q^f adds an uncountably chromatic graph X on ω_1 such that every subgraph on at most $f(r)$ vertices is at most 2^{r+1} -chromatic.*

Proof. As every X_n is a circuitfree graph, it can be colored with two colors. Consider now a subgraph of X induced by a set S of at most $f(r)$ vertices. On S all the graphs X_0, \dots, X_{r-1} are bipartite, and so is $Y_r = X_r \cup \dots$ (as it has no odd circuits of length $\leq f(r)$). So their union, X restricted to S , can be colored by at most 2^{r+1} colors. □

Theorem 2. *It is consistent with CH that for every function $f : \omega \rightarrow \omega$ there is an uncountably chromatic graph X on ω_1 such that every subgraph of X on $f(r)$ vertices is at most r -chromatic ($r \geq 2$).*

Proof. Assume that \diamond holds in the ground model. Then we have CH and there is a club guessing sequence $\{C_\alpha : \alpha < \omega_1, \text{limit}\}$ as required for Theorem 1. We force with a finite support iteration $P = \{P_\alpha, Q_\alpha : \alpha < \omega_1\}$. In step $\alpha < \omega_1$ we add $Q_\alpha = Q^{f_\alpha}$ for some increasing function $f_\alpha : \omega \rightarrow \omega$. As this will be a ccc forcing that preserves CH it is possible by bookkeeping to make sure that every suitable $f : \omega \rightarrow \omega$ occurs as some f_α . Also, as P_α , the iteration up to α is ccc, every closed, unbounded set C in V^{P_α} contains a ground model closed, unbounded set D , and as there is some element C_α of the club guessing system that $C_\alpha \subseteq D$ we have $C_\alpha \subseteq C$, that is the club guessing system retains its property in V^{P_α} .

Call a condition $p \in P$ *determined* if for every $\alpha < \omega_1$ the condition $p|_\alpha$ completely determines $p(\alpha)$, that is, for every α coordinate $p(\alpha)$ is not just a name for a finite structure but it is actually a finite structure.

Lemma 6. *The determined conditions form a dense set in P .*

Proof. We prove by induction on $\alpha < \omega_1$ that the determined conditions form a dense subset of P_α . This is obvious if α is limit, as we are considering finite supports. Assume that we have the statement for some $\alpha < \omega_1$ and we try to handle the case of $\alpha + 1$. Let $(p, q) \in P_{\alpha+1} = P_\alpha * Q_\alpha$ be arbitrary. Extend p to some p' that completely determines q , that is, there is a finite structure h that $p' \Vdash q = h$. Then extend p' to a determined $p^* \in P_\alpha$. Now (p^*, h) is a determined extension of (p, q) . □

Lemma 7. *For every $\alpha < \omega_1$, $\text{Chr}(X_\alpha) = \omega_1$ holds in V^P .*

Proof. By moving to V^{P_α} we can assume that $\alpha = 0$. We imitate the proof of Lemma 5. By Lemma 6 we can work with determined conditions. We consider every such condition as a finite structure on some finite subset s of ω_1 , here s contains all points of all graphs $p(\beta)$ where β is an arbitrary element of the support of p , and we also add the elements of the support to s . Assume that some $\bar{p} \in P$ forces that \underline{F} is a good coloring of X_0 with the elements of ω . With an argument like in Lemma 5 we get some natural number n , ordinals $\delta' < \delta < \omega_1$, and also $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_t, \beta'_1, \dots, \beta'_t$ and two isomorphic determined

conditions p and p' with the respective supports $\{0, \alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_t\}$ and $\{0, \alpha_1, \dots, \alpha_m, \beta'_1, \dots, \beta'_t\}$ that $p \Vdash \underline{F}(\delta) = n$, $p' \Vdash \underline{F}(\delta') = n$ hold, $p(0)$ and $p'(0)$ are isomorphic conditions that behave like p and p' in Lemma 5, for every α_i the structures $p(\alpha_i)$ and $p'(\alpha_i)$ are isomorphic with the common part preceding the tails, that is, it is possible to make a non-edge amalgamation. We can, therefore, take the union of p and p' , and add the edge $\{\delta, \delta'\}$ with color n in coordinate 0. \square

The above Lemma concludes the proof of the Theorem. \square

3 The Taylor conjecture

Theorem 3. *It is consistent that there is a graph X with $\text{Chr}(X) = |X| = \aleph_1$ such that if Y is a graph with all finite subgraphs occurring in X then $\text{Chr}(Y) \leq \aleph_2$, that is, the Taylor conjecture fails.*

Proof. Let V be a model of GCH and \diamond . Let P be the notion of forcing that adds a Cohen real. It is well known that P adds an undominated real, that is, a function $f : \omega \rightarrow \omega$ such that for no $g : \omega \rightarrow \omega$ in V does $f(n) \leq g(n)$ hold for every $n < \omega$. Let V' be the forced model. Notice that V' still has GCH and club guessing (by the argument in the proof of Theorem 2.). Now force over V' with the partial order Q^f , and get a graph X with $\text{Chr}(X) = |X| = \aleph_1$ such that every n -chromatic subgraph of X has at least $f(n)$ elements ($n \geq 3$). This X will be our graph. To show the property stated, assume that Y is a graph in V^{P, Q^f} whose every finite subgraph is a subgraph of X . We assume that the vertex set of Y is some cardinal λ . We notice that every n -chromatic subgraph of Y has at least $f(n)$ elements.

Lemma 8. *If $Z \subseteq Y$ is a subgraph with $Z \in V$ then Z is finitely chromatic.*

Proof. Otherwise for every $n < \omega$ we can let $g(n)$ be the minimal size of an n -chromatic subgraph of Z . Now notice that $g \in V$ and also by the absoluteness of the set of finite subsets of λ and the absoluteness of the chromatic number of a finite graph, g denotes the same thing in V and V^{P*Q^f} . This implies that $g(n) \geq f(n)$ holds for every n , but that obviously contradicts the fact that f cannot be dominated by the the ground model $\omega \rightarrow \omega$ functions. \square

We finally need the following Lemma.

Lemma 9. *If R is a notion of forcing over some model V , Y is a graph in the extended model on some ordinal λ then Y is the union of at most $|R|$ subgraphs which are elements of V .*

Proof. Let τ be a name for (the edge set of) Y . Set

$$Z_p = \{e \in [\lambda]^2 : p \Vdash e \in \tau\}$$

for $p \in R$, then $Y = \bigcup \{Z_p : p \in G\}$ where $G \subseteq R$ is a generic set. \square

To finish the proof of the Theorem we remark that by Lemma 9 Y decomposes into the union of $|P * Q^f| = \aleph_1$ graphs each being in V , therefore finitely chromatic, so we get $\text{Chr}(Y) \leq 2^{\aleph_1} = \aleph_2$. \square

The followig argument gives that the Hanf number mentioned in the Introduction can be as small as \aleph_2 .

Theorem 4. *It is consistent that if X is a graph with $\text{Chr}(X) \geq \aleph_2$ then for every cardinal λ there exists a graph Y with $\text{Chr}(Y) \geq \lambda$ all whose finite subgraphs are induced subgraphs of X .*

Proof. Let V be a model of GCH. Choose the regular cardinal κ so large that the following holds. If X is a graph with $\text{Chr}(X) \geq \kappa$ and λ is a cardinal then there is a graph with $\text{Chr}(Y) \geq \lambda$ all whose finite subgraphs occur as subgraphs of X . Clearly, such a κ exists.

Let $P = \text{Col}(\omega, \kappa)$ be the collapse of κ to \aleph_0 , that is, the elements of P are those functions of the form $p : n \rightarrow \kappa$ for some $n < \omega$ with $p \leq q$ iff p extends q as a function. Our claim is that if $G \subseteq P$ is generic then $V[G]$ models the statement of the Theorem. Notice that $|\kappa| = \aleph_0$ holds there and calculation shows that GCH still holds in $V[G]$.

Assume that X is a graph in $V[G]$ with chromatic number at least \aleph_2 (that is, $\aleph_2^{V[G]}$). By Lemma 9., X is the union of $|G| = \aleph_0$ ground model graphs. As $\aleph_1^{\aleph_0} = \aleph_1$, one of them, say Y must have chromatic number at least \aleph_2 . In V , Y has chromatic number at least $\aleph_2^{V[G]} = \kappa^{++}$. Assume that we are given some $\lambda > \kappa$. By the choice of κ , there is a graph Z with $\text{Chr}(Z) \geq \lambda$ such that every finite induced subgraph of Z is an induced subgraph of Y .

Lemma 10. $\text{Chr}(Z) \geq \lambda$ holds in $V[G]$.

Proof. Otherwise let \underline{F} be a name for a coloring with the ordinals less than $\tau < \lambda$. Then the coloring $x \mapsto (p, \xi)$ is a good coloring of the vertices of Z with $\kappa + \tau < \lambda$ colors, where $p \in P$ is some element of P with $p \Vdash \underline{F}(x) = \xi$. \square

We are almost finished, the only problem is that the finite induced subgraphs of Z are not induced subgraphs of X , they only are (edge-)subgraphs of induced subgraphs of X . The following Lemma is what we need.

Lemma 11. *There is a graph Z' on the vertex set of Z with $Z' \supseteq Z$ and such that every induced subgraph of Z' is an induced subgraph of X .*

Proof. Let S be the vertex set of Z . For every finite subset s of S there are some graphs on s , which on the one hand are isomorphic to induced subgraphs of X , on the other hand they are supergraphs of the graph Z restricted to s . Call these graphs *appropriate* for s . Notice that there are finitely many appropriate graphs for every given s , and if T is an appropriate graph for s and s' is a subset of s then T restricted to s' is a graph appropriate for s' . We can therefore apply the Rado selection principle (or the compactness theorem of model theory) and get a graph Z' on S every induced subgraph of which is appropriate, so Z' is a required. \square

As $\text{Chr}(Z') \geq \text{Chr}(Z) \geq \lambda$ holds we are done. \square

References

- [1] F. Chung, R. Graham: *Erdős on graphs*, A K Peters, 1998.
- [2] P. Erdős: Problems and results on finite and infinite combinatorial analysis, *Infinite and finite sets (Colloq. Keszthely 1973; dedicated to P. Erdős on his 60th birthday)*, Vol. I. *Colloq. Math. Soc. J. Bolyai*, Vol. **10**, North Holland, Amsterdam, 1975, 403–424.
- [3] P. Erdős: On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42.
- [4] P. Erdős: Problems and results on chromatic numbers in finite and infinite graphs, *Graph theory with applications to algorithms and computer science (Kalamazoo, Mich. 1984)*, *Wiley-Interscience Publ.*, Wiley, New York, 1985, 201–213,
- [5] P. Erdős: Some of my favourite unsolved problems, *A tribute to Paul Erdős*, *Cambridge Univ. Press*, Cambridge, 1990., 467–478.
- [6] P. Erdős: On some of my favourite problems in various branches of combinatorics, in: *Proc. Fourth Czechoslovak Symp. on Combinatorics, Graphs and Complexity (Prachatice, 1990)*, *Ann. Discrete Math.* (eds.: J. Nešetřil and M. Fiedler), **51**(1992), North-Holland, 69–79.
- [7] P. Erdős: On some of my favourite theorems, *Combinatorics. Paul Erdős is Eighty, II*, (ed. D. Miklós, V. T. Sós, T. Szőnyi), 97–132.
- [8] P.Erdős, A.Hajnal: On chromatic number of graphs and set-systems, *Acta Math. Acad. Sci. Hung.* **17** (1966), 61–99.
- [9] P. Erdős, A. Hajnal: Unsolved and solved problems in set theory, *Proceedings of the Tarski Symposium (Proc. Symp. Pure Math., Vol. XXV, Univ. of California, Berkeley, Calif., 1971)*, Amer.Math.Soc., Providence, R.I., 1974, 269–287.
- [10] P. Erdős, A. Hajnal: Chromatic number of finite and infinite graphs and hypergraphs. (French summary) Special volume on ordered sets and their applications (L’Arbresle, 1982). *Discrete Math.* **53**(1985), 281–285.
- [11] P.Erdős, A.Hajnal, S.Shelah: On some general properties of chromatic number, in: *Topics in Topology, Keszthely (Hungary), 1972*, *Coll. Math. Soc. J. Bolyai* **8**, 243–255.
- [12] P. Erdős, A. Hajnal, E. Szemerédi: On almost bipartite large chromatic graphs, *Annals of Discrete Math.* **12**(1982), 117–123.
- [13] A. Hajnal, P.Komjáth: What must and what need not be contained in a graph of uncountable chromatic number ? *Combinatorica* **4** (1984), 47–52.

- [14] T. R. Jensen, B. Toft: *Graph coloring problems*, John Wiley and Sons, 1995.
- [15] P.Komjáth, S. Shelah: On Taylor's problem, *Acta Math. Hung.*, **70** (1996), 217–225.
- [16] S. Shelah: *Non-structure Theory*, Oxford University Press, to appear.
- [17] C. Thomassen: Cycles in graphs of uncountable chromatic number, *Combinatorica* **3** (1983), 133–134.
- [18] W.Taylor: Atomic compactness and elementary equivalence, *Fund. Math.* **71** (1971), 103–112.
- [19] W.Taylor: Problem 42, Comb. Structures and their applications, *Proc. of the Calgary International Conference*, 1969.

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