# A Note on Ramsey Numbers for Books 

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#### Abstract

The book with $n$ pages $B_{n}$ is the graph consisting of $n$ triangles sharing an edge. The book Ramsey number $r\left(B_{m}, B_{n}\right)$ is the smallest integer $r$ such that either $B_{m} \subset G$ or $B_{n} \subset \bar{G}$ for every graph $G$ of order $r$. We prove that there exists a positive constant $c$ such that $r\left(B_{m}, B_{n}\right)=2 n+3$ for all $n \geq c m$.


## 1 Introduction

The graph $B_{n}=K_{1}+K_{1, n}$, consisting of $n$ triangles sharing a common edge, is known as the book with $n$ pages. The book Ramsey number $r\left(B_{m}, B_{n}\right)$ is the smallest integer $r$ such that either $B_{m} \subset G$ or $B_{n} \subset \bar{G}$ for every graph $G$ of order $r$. The study of Ramsey numbers for books was initiated in [7] and continued in [5]. The following results are known.

Theorem 1 (Rousseau, Sheehan). For all $n>1, r\left(B_{1}, B_{n}\right)=2 n+3$.
Theorem 2 (Parsons, Rousseau, Sheehan). If $2(m+n+1)>(n-m)^{3} / 3$ then $r\left(B_{m}, B_{n}\right) \leq 2(m+n+1)$. More generally,

$$
\begin{equation*}
r\left(B_{m}, B_{n}\right) \leq m+n+2+\left\lfloor\frac{2}{3} \sqrt{3\left(m^{2}+m n+n^{2}\right)}\right\rfloor . \tag{1}
\end{equation*}
$$

If $4 n+1$ is a prime power, then $r\left(B_{n}, B_{n}\right)=4 n+2$. If $m \equiv 0(\bmod 3)$ then $r\left(B_{m}, B_{m+2}\right) \leq$ $4 m+5$.

The more general upper bound (11) was noted by Parsons in [6]. In looking for cases where equality holds in (11) or in other cases covered by Theorem 1, it is natural to consider the class of strongly regular graphs. A $(v, k, \lambda, \mu)$ strongly regular graph (SRG) is a graph with $v$ vertices that is regular of degree $k$ in which any two distinct vertices have $\lambda$ common neighbors if they are adjacent and $\mu$ common neighbors if they are nonadjacent. Thus if a $(v, k, \lambda, \mu)$ graph exists then

$$
r\left(B_{\lambda+1}, B_{v-2 k+\mu-1}\right) \geq v+1 .
$$

Inspection of a table known strongly regular graphs [4] yields a number of exact values for book Ramsey numbers.

Corollary 1. In addition those cases where $4 n+1$ is a prime power and $r\left(B_{n}, B_{n}\right)=$ $4 n+2(n=1,2,3,4,6, \ldots, 69)$, Theorem 2 gives the following exact values for $r\left(B_{m}, B_{n}\right)$ in which the lower bound comes from a strongly regular graph of order at most 280.

| $(m, n)$ | $r\left(B_{m}, B_{n}\right)$ | $(v, k, \lambda, \mu)$ |
| :---: | :---: | :---: |
| $(2,5)$ | 16 | $(15,6,1,3)$ |
| $(3,5)$ | 17 | $(16,6,2,2)$ |
| $(4,6)$ | 22 | $(21,10,3,6)$ |
| $(7,10)$ | 36 | $(35,16,6,8)$ |
| $(11,11)$ | 46 | $(45,22,10,11)$ |
| $(14,17)$ | 64 | $(63,30,13,15)$ |
| $(23,26)$ | 100 | $(99,48,22,24)$ |
| $(22,37)$ | 120 | $(119,54,21,27)$ |
| $(29,38)$ | 136 | $(135,64,28,32)$ |
| $(34,37)$ | 144 | $(143,70,33,35)$ |
| $(47,50)$ | 196 | $(195,96,46,48)$ |
| $(46,58)$ | 210 | $(209,100,45,50)$ |
| $(56,56)$ | 226 | $(225,112,55,56)$ |
| $(38,82)$ | 244 | $(243,110,37,60)$ |
| $(62,65)$ | 256 | $(255,126,61,63)$ |
| $(69,71)$ | 281 | $(280,135,70,60)$ |

The starting point for this paper is Theorem 1 together with the following pair of results from (5).

## Theorem 3 (Faudree, Rousseau, Sheehan).

$$
r\left(B_{2}, B_{n}\right) \leq \begin{cases}2 n+6, & 2 \leq n \leq 11 \\ 2 n+5, & 12 \leq n \leq 22 \\ 2 n+4, & 23 \leq n \leq 37 \\ 2 n+3, & n \geq 38\end{cases}
$$

Theorem 4 (Faudree, Rousseau, Sheehan). If $m>1$ and

$$
n \geq(m-1)\left(16 m^{3}+16 m^{2}-24 m-10\right)+1
$$

then $r\left(B_{m}, B_{n}\right)=2 n+3$.

From these results, we see that for each $m$ there exists a smallest positive integer $f(m)$ such that $r\left(B_{m}, B_{n}\right)=2 n+3$ for all $n \geq f(m)$. Moreover $f(1)=2$ and $f(2) \leq 38$. Our main purpose here is to prove the following strengthening of Theorem 4.

Theorem 5. There exists a positive constant $c$ such that $r\left(B_{m}, B_{n}\right)=2 n+3$ for all $n \geq c m$.

## 2 Proofs

For standard terminology and notation, see [2]. For $v \in V(G)$ we denote the neighborhood of $v$ by $N(v)$ and the degree of $v$ by $\operatorname{deg}(v)$. If needed, we shall use a subscript to identify the graph in question; for example, $N_{G}(v)$ denotes the neighborhood of $v$ in $G$. Given two disjoint sets $U, W \subset V(G)$, let $e(U, W)=|\{u w \in E(G) \mid u \in U, w \in W\}|$. The subgraph of $G$ induced by $X \subset V(G)$ will be denoted by $G[X]$. Given graphs $G$ and $H$, let $M_{G}(H)$ denote the number of induced subgraphs of $G$ that are isomorphic to $H$. The number of pages in the largest book contained in $G$ will be called the book size of $G$ and this will be denoted by $b s(G)$. It is convenient to identify the graph and its complement in terms of edge colorings of a complete graph. In this framework, $r\left(B_{m}, B_{n}\right)$ is the smallest $r$ such that in every $(R, B)=$ (red, blue) coloring of $E\left(K_{r}\right)$, either $b s(R) \geq m$ or $b s(B) \geq n$. Theorem 5 clearly follows from the following fact.

Theorem 6. Suppose $m$ and $n$ are positive integers satisfying $n \geq 10^{6} m$. If $(R, B)$ is any two-coloring of $E\left(K_{n}\right)$ then either $b s(R)>m$ or $b s(B) \geq n / 2-2$.

In view of the case $R=K(\lfloor n / 2\rfloor,\lceil n / 2\rceil)$ the conclusion $b s(B) \geq n / 2-2$ is best possible. The proof of Theorem 6 uses the following counting result.

Lemma 1. Let $G$ be a graph with $p$ vertices and $q$ edges that satisfies $b s(G) \leq m$. For $0<\lambda<1$ suppose that $\delta(G) \geq \lambda p$ so $q \geq \lambda p^{2} / 2$. If $p>5(2 \lambda+1) / \lambda^{2}$ then

$$
M_{G}\left(C_{4}\right)>\left(\frac{\lambda^{3} p^{2}}{5}-\frac{m^{2}}{2}\right) q
$$

Proof. For distinct vertices $u, v \in V(G)$ let $c(u, v)=\left|N_{G}(u) \cap N_{G}(v)\right|$. Then

$$
\begin{aligned}
& \sum_{\{u, v\}}\binom{c(u, v)}{2}=2 M_{G}\left(C_{4}\right)+6 M_{G}\left(K_{4}\right)+2 M_{G}\left(B_{2}\right) \quad \text { and } \\
& \sum_{u v \in E}\binom{c(u, v)}{2}=6 M_{G}\left(K_{4}\right)+M_{G}\left(B_{2}\right)
\end{aligned}
$$

from which we get

$$
\begin{equation*}
M_{G}\left(C_{4}\right)=\frac{1}{2} \sum_{\{u, v\}}\binom{c(u, v)}{2}-\sum_{u v \in E}\binom{c(u, v)}{2}+3 M_{G}\left(K_{4}\right) . \tag{2}
\end{equation*}
$$

Since $b s(G) \leq m$,

$$
\sum_{u v \in E}\binom{c(u, v)}{2} \leq q\binom{m}{2}<\frac{q m^{2}}{2}
$$

Note that

$$
\sum_{\{u, v\}} c(u, v)=\sum_{v \in V(G)}\binom{\operatorname{deg}_{G}(v)}{2} \geq p\binom{2 q / p}{2}=q(2 q / p-1) \geq q(\lambda p-1):=x
$$

so by convexity,

$$
\sum_{\{u, v\}}\binom{c(u, v)}{2} \geq\binom{ p}{2}\binom{x /\binom{p}{2}}{2}=\frac{x}{2}\left(\frac{x}{\binom{p}{2}}-1\right) .
$$

Since

$$
\frac{x}{\binom{p}{2}}>\frac{2 x}{p^{2}}=\frac{2 q(\lambda p-1)}{p^{2}} \geq \lambda(\lambda p-1)
$$

we have

$$
\sum_{\{u, v\}}\binom{c(u, v)}{2}>\frac{q(\lambda p-1)(\lambda(\lambda p-1)-1)}{2}>\frac{2 \lambda^{3} p^{2} q}{5}
$$

Note. The last inequality is clear if $\lambda^{2} p-10 \lambda-5 \geq 0$, and hence it holds since we have required $p \geq 5(2 \lambda+1) / \lambda^{2}$. In view of (2) we have

$$
M_{G}\left(C_{4}\right)>\left(\frac{\lambda^{3} p^{2}}{5}-\frac{m^{2}}{2}\right) q
$$

as claimed.

Proof of Theorem 6. Suppose $n \geq 10^{6} m$ and that $(R, B)$ is a two-coloring of $E\left(K_{n}\right)$ such that $b s(R) \leq m$. We shall prove that $b s(B) \geq n / 2-2$. Let $\mathcal{H}=C_{4} \cup K_{1}$.

Claim 1. If $b s(B) \leq n / 2-2$ then $M_{R}(\mathcal{H}) \leq 4 m M_{R}\left(C_{4}\right)$.

Note. The hypothesis $b s(G) \leq n / 2-2$ rather than, as one might naturally expect, $b s(G)<n / 2-2$, is made for convenience.

Proof. Suppose $M_{R}(\mathcal{H})>4 m M_{R}\left(C_{4}\right)$. Then there exists an induced $C_{4}=(u, v, w, z)$ such that

$$
\left|N_{B}(u) \cap N_{B}(v) \cap N_{B}(w) \cap N_{B}(z)\right| \geq 4 m+1
$$

Since $b s(B) \leq n / 2-2$ we have

$$
\left|N_{B}(u) \cap N_{B}(w)\right| \leq n / 2-2 \quad \text { and } \quad\left|N_{B}(v) \cap N_{B}(z)\right| \leq n / 2-2
$$

It then follows that there are at least $4 m+1$ vertices outside of $\{u, v, w, z\}$ that are adjacent in $R$ to at least one of $u, w$ and at least one of $v, z$. This gives $m+1$ or more red triangles on at least one of the four edges $u v, v w, w z, z u$, and thus the desired contradiction.

It is known that for any graph $G$ of order $n$,

$$
M_{G}\left(C_{4}\right) \leq\binom{\lfloor n / 2\rfloor}{ 2}\binom{\lceil n / 2\rceil}{ 2}<\frac{n^{4}}{64}
$$

See [3] for a proof of the more general result

$$
M_{G}\left(K_{m, m}\right) \leq\binom{\lfloor n / 2\rfloor}{ m}\binom{\lceil n / 2\rceil}{ m}
$$

Hence by Claim 1,

$$
M_{R}(\mathcal{H})<\frac{m n^{4}}{16}
$$

or else $b s(B)>n / 2-2$.
Claim 2. If $b s(B) \leq n / 2-2$ then $R$ has at most $n / 20$ vertices of degree $9 n / 20$ or less.

Proof. Let $v$ be any vertex of degree $9 n / 20$ or less in $R$ and let $X=N_{B}(v)$. Then $B[X]$ has maximum degree at most $n / 2-2$ so $G=G(v)=R[X]$ has minimum degree $\delta$ satisfying

$$
\delta \geq|X|-1-\frac{n}{2}+2 \geq n-1-\frac{9 n}{20}-1-\frac{n}{2}+2=\frac{n}{20}
$$

and (since $|X|+1 \geq 11 n / 20$ )

$$
\delta \geq|X|+1-\frac{n}{2} \geq|X|+1-\frac{1}{2}\left(\frac{20(|X|+1)}{11}\right)=\frac{|X|+1}{11}
$$

Let us check that Lemma 1 applies to $G$. Take $\lambda=1 / 11$ and $p=|X| \geq 11 n / 20$. Then $p>5(2 \lambda+1) / \lambda^{2}$ holds provided $n \geq 1302$. This is certainly the case since $n \geq 10^{6} \mathrm{~m}$. Using $m \leq n / 10^{6}$, Lemma 1 gives

$$
M_{G}\left(C_{4}\right)>\left(\frac{1}{5} \cdot \frac{1}{11^{3}}\left(\frac{11 n}{20}\right)^{2}-\frac{1}{2}\left(\frac{n}{10^{6}}\right)^{2}\right) \frac{1}{2}\left(\frac{11 n}{20}\right)\left(\frac{n}{20}\right) \approx \frac{n^{4}}{640,000}
$$

Suppose more than $n / 20$ vertices in $R$ have degree $9 n / 20$ or less. Then

$$
\frac{m n^{4}}{16}>M_{R}(\mathcal{H})=\sum_{v} M_{G(v)}\left(C_{4}\right)>\sum_{\operatorname{deg}(v) \leq 9 n / 20} M_{G(v)}\left(C_{4}\right)>\frac{n}{20} \cdot \frac{n^{4}}{640,000}
$$

so $n<8 \cdot 10^{5} m$, a contradiction.

Let $S=\left\{v \mid \operatorname{deg}_{R}(v)>9 n / 20\right\}$. From Lemma 3 we know that $|S|>19 n / 20$, so the minimum degree of $R[S]$ satisfies

$$
\delta \geq \frac{9 n}{20}-(n-|S|)>\frac{2 n}{5} \geq \frac{2|S|}{5}
$$

Now we use the following result of Andrásfai, Erdős and Sós [1].
Theorem 7 (Andrásfai, Erdős, Sós). Suppose $r \geq 3$. For any graph $G$ of order $n$, at most two of the following properties can hold:

$$
\text { (i) } K_{r} \nsubseteq G, \quad \text { (ii) } \delta(G)>\frac{3 r-7}{3 r-4} n, \quad \text { (iii) } \chi(G) \geq r
$$

Note. In particular, a triangle-free graph $G$ with $\delta(G)>2|V(G)| / 5$ is bipartite.
Now we are prepared to complete the proof of Theorem 6. It is easy to see that $R[S]$ has no triangle. If $T=\{u, v, w\}$ is a triangle in $R[S]$ and $U$ is the set of $n-3$ vertices outside $T$, then

$$
3(9 n / 20-2)<e_{R}(T, U) \leq 3(m-1) \cdot 2+(n-3(m-1))=n+3(m-1)
$$

or $7 n / 20<3 m+3$, which is false. Hence $R[S]$ is bipartite by Theorem 7. Let $S_{1}$ and $S_{2}$ denote the two color classes of $R[S]$. Put $v \in T_{1}$ if $v$ is adjacent in $B$ to every vertex of $S_{1}$.

Then for the remaining vertices put $v \in T_{2}$ if $v$ is adjacent in $B$ to every vertex of $S_{2}$. Let $W_{1}=S_{1} \cup T_{1}, W_{2}=S_{2} \cup T_{2}$, and let $X$ denote the set of vertices in neither $W_{1}$ nor $W_{2}$. If $X=\varnothing$ then we may assume that $\left|W_{1}\right| \geq n / 2$. In this case it is clear that $b s(B) \geq n / 2-2$.

We are left to consider the case $X \neq \varnothing$. For $u \in S$ let $Z(u)=N_{B}(u) \cap X$. For distinct vertices $u, v \in S_{1}$, consideration of the blue book on $u v$ shows that

$$
\begin{aligned}
b s(B) & \geq\left|S_{1}\right|-2+\left|T_{1}\right|+|Z(u) \cap Z(v)| \\
& \geq\left|S_{1}\right|-2+\left|T_{1}\right|+|Z(u)|+|Z(v)|-|X|
\end{aligned}
$$

Summing over all pairs $u, v \in S_{1}$ and computing the average, we find

$$
\begin{aligned}
b s(B) & \geq\left|S_{1}\right|-2+\left|T_{1}\right|+\frac{2\left(\left|S_{1}\right||X|-e_{R}\left(S_{1}, X\right)\right)}{\left|S_{1}\right|}-|X| \\
& =\left|S_{1}\right|+\left|T_{1}\right|+|X|-2-\frac{2 e_{R}\left(S_{1}, X\right)}{\left|S_{1}\right|} .
\end{aligned}
$$

Similarly,

$$
b s(B) \geq\left|S_{2}\right|+\left|T_{2}\right|+|X|-2-\frac{2 e_{R}\left(S_{2}, X\right)}{\left|S_{2}\right|}
$$

Note that $\left|S_{1}\right|<n / 2$ or else we are done at the outset; similarly $\left|S_{2}\right|<n / 2$. Hence

$$
\left|S_{1}\right|=|S|-\left|S_{2}\right|>\frac{19 n}{20}-\frac{n}{2}=\frac{9 n}{20}
$$

and likewise $\left|S_{2}\right|>9 n / 20$. Consequently

$$
\begin{aligned}
& b s(B)>\left|S_{1}\right|+\left|T_{1}\right|+|X|-2-\frac{40 e_{R}\left(S_{1}, X\right)}{9 n} \\
& b s(B)>\left|S_{2}\right|+\left|T_{2}\right|+|X|-2-\frac{40 e_{R}\left(S_{2}, X\right)}{9 n}
\end{aligned}
$$

Addition then gives

$$
2 b s(B)>n-4+|X|-\frac{40 e_{R}(S, X)}{9 n}
$$

Hence $e_{R}(S, X)>9 n|X| / 40$ or else the proof is complete.
Thus we assume $e_{R}(S, X)>9 n|X| / 40$ and now seek a companion bound on $e_{R}(S, X)$. For each $x \in X$ there is at least one $v \in S_{1}$ such that $x v \in R$. Since $\left|N_{R}(v) \cap S_{2}\right| \geq 2|S| / 5$,
consideration of the red book on $x v$ shows that

$$
\begin{aligned}
b s(R) & \geq\left|N_{R}(x) \cap N_{R}(v) \cap S_{2}\right| \\
& =\left|N_{R}(x) \cap S_{2}\right|+\left|N_{R}(v) \cap S_{2}\right|-\left|S_{2}\right| \\
& \geq\left|N_{R}(x) \cap S_{2}\right|+\frac{2|S|}{5}-\left|S_{2}\right| .
\end{aligned}
$$

Taking the average over $x \in X$, we obtain

$$
b s(R) \geq \frac{e_{R}\left(S_{2}, X\right)}{|X|}+\frac{2|S|}{5}-\left|S_{2}\right| .
$$

In exactly the same way,

$$
b s(R) \geq \frac{e_{R}\left(S_{1}, X\right)}{|X|}+\frac{2|S|}{5}-\left|S_{1}\right| .
$$

Hence

$$
2 m \geq 2 b s(R) \geq \frac{e_{R}(S, X)}{|X|}-\frac{|S|}{5}
$$

Thus

$$
e_{R}(S, X) \geq 2 m|X|+\frac{|S||X|}{5}
$$

From the two bounds for $e_{R}(S, X)$, we obtain

$$
\frac{9 n|X|}{40}<e_{R}(S, X) \leq \frac{|S||X|}{5}+2 m|X|<\frac{n|X|}{5}+2 m|X| .
$$

By assumption $|X|>0$, so

$$
\frac{9 n}{40}<\frac{n}{5}+2 m
$$

which is false.

## 3 Concluding Remarks

The determination of the best constant $c$ in Theorem 5 is open, as are other basic problems on book Ramsey numbers stated in [5]. In particular, it is unknown whether or not there exists a constant $C$ such that $r\left(B_{m}, B_{n}\right) \leq 2(m+n)+C$ for all $m, n$.

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