# Maximal and Maximum Independent Sets In Graphs With At Most $r$ Cycles 

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#### Abstract

Let $m(G)$ denote the number of maximal independent sets of vertices in a graph $G$ and let $c(n, r)$ be the maximum value of $m(G)$ over all connected graphs with $n$ vertices and at most $r$ cycles. A theorem of Griggs, Grinstead, and Guichard gives a formula for $c(n, r)$ when $r$ is large relative to $n$, while a theorem of Goh, Koh, Sagan, and Vatter does the same when $r$ is small relative to $n$. We complete the determination of $c(n, r)$ for all $n$ and $r$ and characterize the extremal graphs. Problems for maximum independent sets are also completely resolved.


## 1 Introduction and preliminary lemmas

Let $G=(V, E)$ be a simple graph. A subset $I \subseteq V$ is independent if there is no edge of $G$ between any two vertices of $I$. Also, $I$ is maximal if it is not properly contained in any other independent set. We let $m(G)$ be the number of maximal independent sets of $G$. Several previous authors have been interested in the problem of maximizing $m(G)$ over different families of graphs.

[^0]In [2] the authors studied two families of graphs: the family of all graphs with at most $r$ cycles, and the family of all connected graphs with at most $r$ cycles. For the family of all graphs, they were able to completely settle the problem, by using the result of Moon and Moser [7] (Theorem 1.1 below) when $n$ is small relative to $r$ and providing new arguments for all values of $(n, r)$ to which the Moon-Moser Theorem does not apply (see Theorem 1.5 (I), also below).

For the family of connected graphs, [2] only characterizes the extremal graphs when $n \geq 3 r$ (Theorem [1.5(II)) while the connected analogue of the Moon-Moser Theorem (the Griggs-Grinstead-Guichard Theorem, Theorem 1.2 below) settles the problem for $n$ small relative to $r$, leaving a gap between the values where these two theorems apply. This gap is filled in Section 2 by a careful analysis of the possible endblocks of extremal graphs.

In the later sections we turn our attention to maximum independent sets (independent sets of maximum cardinality) in these two families of graphs. Like with maximal independent sets, we start with the case where $n$ is large relative to $r$ in Section 3 and then consider the gap in Section 4

For the remainder of this section we briefly recount the results we will need. These results appear in [2] (with the single exception of Proposition 1.8] which occurs here in a strengthened form), and we refer the reader to that paper for examples and proofs.

For any two graphs $G$ and $H$, let $G \uplus H$ denote the disjoint union of $G$ and $H$, and for any nonnegative integer $t$, let $t G$ stand for the disjoint union of $t$ copies of $G$.

Let

$$
G(n):= \begin{cases}\frac{n}{3} K_{3} & \text { if } n \equiv 0(\bmod 3), \\ 2 K_{2} \uplus \frac{n-4}{3} K_{3} & \text { if } n \equiv 1(\bmod 3), \\ K_{2} \uplus \frac{n-2}{3} K_{3} & \text { if } n \equiv 2(\bmod 3) .\end{cases}
$$

Further, let

$$
G^{\prime}(n):=K_{4} \uplus \frac{n-4}{3} K_{3} \text { if } n \equiv 1(\bmod 3) .
$$

Also define

$$
g(n):=m(G(n))= \begin{cases}3^{\frac{n}{3}} & \text { if } n \equiv 0(\bmod 3) \\ 4 \cdot 3^{\frac{n-4}{3}} & \text { if } n \equiv 1(\bmod 3) \\ 2 \cdot 3^{\frac{n-2}{3}} & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

Note that $m\left(G^{\prime}(n)\right)=m(G(n))$ when $n \equiv 1(\bmod 3)$.
Theorem 1.1 (Moon and Moser [7]) Let $G$ be a graph with $n \geq 2$ vertices. Then

$$
m(G) \leq g(n)
$$

with equality if and only if $G \cong G(n)$ or, for $n \equiv 1(\bmod 3), G \cong G^{\prime}(n)$.
The extremal connected graphs were found by Griggs, Grinstead, and Guichard. To define these graphs we need one more piece of notation. Let $G$ be a graph all of whose
components are complete and let $K_{m}$ be a complete graph disjoint from $G$. Construct the graph $K_{m} * G$ by picking a vertex $v_{0}$ in $K_{m}$ and connecting it to a single vertex in each component of $G$. If $n \geq 6$ then let

$$
C(n):= \begin{cases}K_{3} * \frac{n-3}{3} K_{3} & \text { if } n \equiv 0(\bmod 3) \\ K_{4} * \frac{n-4}{3} K_{3} & \text { if } n \equiv 1(\bmod 3) \\ K_{4} *\left(K_{4} \uplus \frac{n-8}{3} K_{3}\right) & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

It can be calculated that

$$
c(n):=m(C(n))= \begin{cases}2 \cdot 3^{\frac{n-3}{3}}+2^{\frac{n-3}{3}} & \text { if } n \equiv 0(\bmod 3) \\ 3^{\frac{n-1}{3}}+2^{\frac{n-4}{3}} & \text { if } n \equiv 1(\bmod 3) \\ 4 \cdot 3^{\frac{n-5}{3}}+3 \cdot 2^{\frac{n-8}{3}} & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

Theorem 1.2 (Griggs, Grinstead, and Guichard [4]) Let $G$ be a connected graph with $n \geq 6$ vertices. Then

$$
m(G) \leq c(n)
$$

with equality if and only if $G \cong C(n)$.
The study of $m(G)$ for graphs with a restricted number of cycles began with Wilf. Let

$$
t(n):= \begin{cases}2^{\frac{n-2}{2}}+1 & \text { if } n \text { is even } \\ 2^{\frac{n-1}{2}} & \text { if } n \text { is odd }\end{cases}
$$

Theorem 1.3 (Wilf [9]) If $G$ is a tree with $n \geq 1$ vertices then $m(G) \leq t(n)$.
Sagan [8] gave another proof of this theorem in which he also characterized the extremal graphs, but we will not need them.

Now let

$$
f(n):=2^{\left\lfloor\frac{n}{2}\right\rfloor} .
$$

From Theorem 1.3, one can easily solve the problem for forests.
Theorem 1.4 If $G$ is a forest with $n \geq 1$ vertices then $m(G) \leq f(n)$.
To move from trees to a bounded number of cycles, suppose that $n, r$ are positive integers with $n \geq 3 r$. Define

$$
G(n, r):= \begin{cases}r K_{3} \uplus \frac{n-3 r}{2} K_{2} & \text { if } n \equiv r(\bmod 2), \\ (r-1) K_{3} \uplus \frac{n-3 r+3}{2} K_{2} & \text { if } n \not \equiv r(\bmod 2) .\end{cases}
$$



Figure 1: The exceptional graph $E$

Again, it can be computed that

$$
g(n, r):=m(G(n, r))= \begin{cases}3^{r} \cdot 2^{\frac{n-3 r}{2}} & \text { if } n \equiv r(\bmod 2) \\ 3^{r-1} \cdot 2^{\frac{n-3 r+3}{2}} & \text { if } n \not \equiv r(\bmod 2)\end{cases}
$$

It is also convenient to define $G(n, r):=G(n)$ and $g(n, r):=g(n)$ when $n<3 r$. The extremal connected graphs where $n \geq 3 r$ are given by

$$
C(n, r):= \begin{cases}K_{3} *\left((r-1) K_{3} \uplus \frac{n-3 r}{2} K_{2}\right) & \text { if } n \equiv r(\bmod 2), \\ K_{1} *\left(r K_{3} \uplus \frac{n-3 r-1}{2} K_{2}\right) & \text { if } n \not \equiv r(\bmod 2) .\end{cases}
$$

As usual, we let

$$
c(n, r):=m(C(n, r))= \begin{cases}3^{r-1} \cdot 2^{\frac{n-3 r+2}{2}}+2^{r-1} & \text { if } n \equiv r(\bmod 2), \\ 3^{r} \cdot 2^{\frac{n-3 r-1}{2}} & \text { if } n \not \equiv r(\bmod 2) .\end{cases}
$$

Theorem 1.5 ([2]) Let $G$ be a graph with $n$ vertices and at most $r$ cycles where $r \geq 1$.
(I) If $n \geq 3 r-1$ then $m(G) \leq g(n, r)$ with equality if and only if $G \cong G(n, r)$.
(II) If $n \geq 3 r$ then for all such graphs that are connected we have $m(G) \leq c(n, r)$. Equality occurs if and only if $G \cong C(n, r)$, or if $G$ is one of the exceptional cases listed in the following table.

| $n$ | $r$ | possible $G \not \approx C(n, r)$ |
| :---: | :---: | :---: |
| 4 | 1 | $P_{4}$ |
| 5 | 1 | $C_{5}$ |
| 7 | 2 | $C(7,1), E$ |

(Here $P_{4}$ and $C_{5}$ are the path and cycle on 4 and 5 vertices, respectively, and $E$ is the graph shown in Figure 1.)

We have a list of inequalities that will be useful in our proofs. Here and elsewhere it will be convenient to let $g(n, 0)=f(n)$ and $c(n, 0)=t(n)$.

Lemma 1.6 ([2]) We have the following monotonicity results.
(1) If $r \geq 1$ and $n>m \geq 3 r-1$ then

$$
g(n, r)>g(m, r)
$$

(2) If $r \geq 1$ and $n>m \geq 3 r$ then

$$
c(n, r)>c(m, r) .
$$

(3) If $r>q \geq 0$ and $n \geq 3 r-1$ then

$$
g(n, r) \geq g(n, q)
$$

with equality if and only if $n$ and $r$ have different parity and $q=r-1$.
(4) If $r>q \geq 0$ and $n \geq 3 r$ then

$$
c(n, r) \geq c(n, q)
$$

with equality if and only if $(n, r, q)=(4,1,0)$ or $(7,2,1)$.
We also need two results about $m(G)$ for general graphs $G$. In what follows, if $v \in V$ then the open and closed neighborhoods of $v$ are $N(v)=\{u \in V \mid u v \in E\}$ and $N[v]=$ $\{v\} \cup N(v)$, respectively. We also call a block an endblock of $G$ if it has at most one cutvertex in the graph as a whole.

Proposition 1.7 The invariant $m(G)$ satisfies the following.
(1) If $v \in V$ then $m(G) \leq m(G-v)+m(G-N[v])$.
(2) If $G$ has an endblock $B$ that is isomorphic to a complete graph then

$$
m(G)=\sum_{v \in V(B)} m(G-N[v])
$$

In fact, the same equality holds for any complete subgraph $B$ having at least one vertex that is adjacent in $G$ only to other vertices of $B$.

We will refer to the formulas in parts (1) and (2) of this proposition as the $m$-bound and $m$-recursion, respectively.

Using the fact that the blocks and cutvertices of a graph have a tree structure (1) Proposition 3.1.1], one obtains the following result.

Proposition 1.8 Every graph has an endblock which intersects at most one non-endblock. Furthermore, if a graph is not 2-connected itself, then it contains at least two such endblocks.

Note that any block with at least 3 vertices is 2-connected. Our analysis of the possible endblocks of the extremal graphs will rely upon Whitney's Ear Decomposition Theorem from [11].

Theorem 1.9 (Ear Decomposition Theorem) A graph B is 2-connected if and only if there is a sequence

$$
B_{0}, B_{1}, \ldots, B_{l}=B
$$

such that $B_{0}$ is a cycle and $B_{i+1}$ is obtained by taking a nontrivial path and identifying its two endpoints with two distinct vertices of $B_{i}$.

Proofs of the Ear Decomposition Theorem may also be found in Diestel [1, Proposition 3.1.2] and West [10, Theorem 4.2.8].

## 2 Filling the gap

For any graph $G$, it will be convenient to let

$$
r(G)=\text { number of cycles of } G \text {. }
$$

For the family of all graphs on $n$ vertices we have already seen the maximum value of $m(G)$ for all possible $r(G)$. Now consider the family of connected graphs. If $n \equiv 0(\bmod 3)$ then Theorems 1.2 and 1.5 (II) characterize the maximum for all possible $r(G)$. So for the rest of this section we will concentrate on connected graphs with $n \equiv 1,2(\bmod 3)$.

Let

$$
r_{0}:=\lfloor n / 3\rfloor=\text { the largest value of } r \text { for which Theorem } 1.5 \text { (II) is valid. }
$$

Also let

$$
r_{1}:=r(C(n))= \begin{cases}r_{0}+6 & \text { if } n \equiv 1(\bmod 3) \\ r_{0}+12 & \text { if } n \equiv 2(\bmod 3)\end{cases}
$$

To characterize the extremal graphs in the gap $r_{0}<r<r_{1}$ we will need an extension of the star operation. Let $G$ and $H$ be graphs all of whose components are complete and such that each component of $H$ has at least 2 vertices. Construct $K_{m} *[G, H]$ by picking a vertex $v_{0}$ of $K_{m}$ and connecting it to a single vertex in each component of $G$ and to two vertices in each component of $H$. If $n=3 r_{0}+1$ then define

$$
C(n, r):= \begin{cases}K_{1} *\left[\left(r_{0}-1\right) K_{3}, K_{3}\right] & \text { if } r=r_{0}+2 \\ K_{1} *\left[\left(r_{0}-2\right) K_{3}, 2 K_{3}\right] & \text { if } r=r_{0}+4\end{cases}
$$

Note that $C(4,5)$ is not well-defined because then $r_{0}-2=-1<0$, and we will leave this graph undefined. We also need the exceptional graph

$$
C(7,3):=K_{1} *\left[3 K_{2}, \emptyset\right] .
$$



Figure 2: Examples of $C(n, r)$ for $n<3 r$

For the case $n=3 r_{0}+2$, let

$$
C(n, r):=K_{1} *\left[\left(r_{0}-1\right) K_{3}, 2 K_{2}\right] \quad \text { if } r=r_{0}+1 .
$$

These will turn out to be the new extremal graphs in the gap. Two examples may be found in Figure 2

Note that for all graphs just defined we have $m(C(n, r))=m\left(C\left(n, r_{0}\right)\right)$. We let

$$
m_{0}:=m\left(C\left(n, r_{0}\right)\right)= \begin{cases}3^{r_{0}} & \text { if } n=3 r_{0}+1 \\ 4 \cdot 3^{r_{0}-1}+2^{r_{0}-1} & \text { if } n=3 r_{0}+2\end{cases}
$$

It will also be convenient to extend the domain of $c(n, r)$ to all $n$ and $r$ by defining

$$
\begin{aligned}
& c(n, r)=c(n) \quad \text { when } r \geq r(C(n)), \\
& c(n, r)=m_{0} \quad \text { when } r_{0}<r<r_{1}
\end{aligned}
$$

We extend the definition of $C(n, r)$ similarly by defining $C(n, r):=C(n)$ when $r \geq r(C(n))$.
We will need a special case of the Moon-Moser transformation [4, 7] which, in conjunction with the Ear Decomposition Theorem, will prove useful in cutting down on the number of cases to consider. Suppose the graph $G$ contains a path tuvw such that
(a) $\operatorname{deg} u=\operatorname{deg} v=2$,
(b) the edge $u v$ lies on a cycle, and
(c) $u$ is in at least as many maximal independent sets as $v$.

Then construct the (connected) graph $G_{u, v}$ where $V\left(G_{u, v}\right)=V(G)$ and

$$
E\left(G_{u, v}\right)=E(G) \cup\{t v\}-\{v w\} .
$$

The edge $u v$ lies on a unique cycle (a 3-cycle) in $G_{u, v}$, and so $r\left(G_{u, v}\right) \leq r(G)$ by (b).

Lemma 2.1 Suppose $G$ contains a path tuvw satisfying (a)-(c). Then $m\left(G_{u, v}\right) \geq m(G)$, with equality only if $N_{G}(w)-\{v\} \subseteq N_{G}(t)$.

Proof: Let $M$ be a maximal independent set in $G$. Then there are three mutually exclusive possibilities for $M$, namely $u, v \notin M ; u \in M$ and $v \notin M$; or $v \in M$ and $u \notin M$. In the first two cases, $M$ gives rise to distinct maximal independent $\operatorname{set}(\mathrm{s})$ in $G_{u, v}$ as in the following chart.

| type of MIS $M$ in $G$ | corresponding $\operatorname{MIS(s)~in~} G_{u, v}$ |
| :---: | :---: |
| $u, v \notin M$ | $M$ |
| $u \in M, v \notin M$ | $M, M \cup\{v\}-\{u\}$ |

Therefore by (c) we have $m\left(G_{u, v}\right) \geq m(G)$ without even considering the third case where $v \in M$ and $u \notin M$. In this case, if there is some vertex $x \in N_{G}(w)-N_{G}(t)-\{v\}$, then there is a maximal independent set $M$ in $G$ with $\{t, x, v\} \subseteq M$. Hence $M-\{v\}$ is a maximal independent set in $G_{u, v}$ so $m\left(G_{u, v}\right)>m(G)$, as desired. If such a vertex $x$ does not exist, we have $N_{G}(w)-\{v\} \subseteq N_{G}(t)$.

Given a graph $G$, we let $\mathcal{T}(G)$ denote the set of all graphs that can be obtained from $G$ by applying a maximal sequence of these special Moon-Moser transformations. By Lemma 2.1, every graph in $\mathcal{T}(G)$ has at least as many cycles and at least as many maximal independent sets as $G$. Furthermore, since all of these graphs are formed by maximal sequences of transformations, if $H \in \mathcal{T}(G)$ then $H$ cannot contain a path tuvw satisfying (a)-(c) above. Another way to state this is that every $H \in \mathcal{T}(G)$ has the following property:
$(\Delta)$ If $u v \in E(H)$ lies on a cycle and $\operatorname{deg} u=\operatorname{deg} v=2$, then $u v$ lies on a 3 -cycle.
Before closing the gap, we wish to mention a result which we will need to rule out some graphs from the list of possible extremals. To state this lemma, we say that a vertex $v \in V(G)$ is duplicated if there is a vertex $w \in V(G)$ such that $v$ and $w$ have the same neighbors, that is, $N(v)=N(w)$.

Lemma 2.2 Let $G$ be a graph with $n$ vertices and a vertex $v$ that is duplicated.
(1) We have $m(G)=m(G-v)$.
(2) If $n \geq 7, n \equiv 1$ or $2(\bmod 3)$, and $G$ is connected with less than $r_{1}$ cycles then $m(G)<m_{0}$.

Proof: If $u$ and $v$ are duplicated vertices, then they lie in the same maximal independent sets and neither is a cutvertex. So $m(G)=m(G-v)$ and under the hypotheses of (2), $m(G-v) \leq c(n-1)<m_{0}$.

We now finish our characterization of the extremal graphs.
Theorem 2.3 Let $G$ be a connected graph with $n \geq 7$ vertices, $n \equiv 1$ or $2(\bmod 3)$, and less than $r_{1}$ cycles. Then

$$
m(G) \leq m_{0}
$$

with equality if and only if $G \cong C(n, s)$ for some $s$ with $r_{0} \leq s<r_{1}$.

Proof: We prove the theorem by induction on $n$. The cases where $n \leq 10$ have been checked by computer, so let $G$ be an extremal connected graph with $n>10$ vertices and less than $r_{1}$ cycles.

First note that it suffices to prove the theorem for graphs that satisfy $(\Delta)$ : If $G$ satisfies the hypotheses of the theorem then every graph $H \in \mathcal{T}(G)$ also satisfies these hypotheses and satisfies $(\Delta)$. Since $G$ is extremal, then in Lemma 2.1 we would always have equality and thus the given subset relation, when replacing $G$ by $G_{u, v}$. Since none of our candidate extremal graphs can be generated by this transformation if such a condition is imposed, we must have that $\mathcal{T}(G)=\{G\}$ and so $G$ satisfies $(\Delta)$.

Pick an endblock $B$ of $G$ satisfying the conclusion of Proposition 1.8 with $|V(B)|$ maximum among all such endblocks. If $G \neq B$ then we will use $x$ to denote the cutvertex of $G$ in $B$. The argument depends on the nature of $B$.

If $B \cong K_{2}$ then the argument used in the proof of Theorem 1.5 (II) can be easily adapted for use in this context. Let $V(B)=\{x, v\}$ so that $\operatorname{deg} v=1$ and $\operatorname{deg} x \geq 2$. By the choice of $B, G-N[v]$ is the union of some number of $K_{1}$ 's and a connected graph with at most $n-2$ vertices and at most $r$ cycles. Also, $G-N[x]$ has at most $n-3$ vertices and at most $r$ cycles, so the $m$-recursion and monotonicity give, for $n \geq 11$,

$$
m(G) \leq c(n-2, r)+g(n-3, r) \leq c(n-2)+g(n-3) \leq m_{0},
$$

with equality if and only if $n=3 r_{0}+2$ and $G \cong C\left(n, r_{0}\right)$.
All other possible endblocks must be 2-connected and so we will use the Ear Decomposition Theorem to organize the cases to consider based on $l$, the number of paths that are added to the initial cycle.

If $l=0$ then $(\Delta)$ guarantees that $B \cong K_{3}$. Let $V(B)=\{x, v, w\}$ where $x$ denotes the cutvertex. Let $i$ denote the number of other $K_{3}$ endblocks containing $x$. If the graph consists entirely of $K_{3}$ endblocks which intersect at $x$, then

$$
m(G)=2^{\frac{n-1}{2}}+1,
$$

which shows that $G$ is not extremal. Thus $x$ is adjacent to at least one vertex which does not lie in a $K_{3}$ endblock. Since $B$ was chosen with $|V(B)|$ maximal, it follows that $G-N[v]=G-N[w]$ has some number of trivial components, $i$ components isomorphic to $K_{2}$, and at most one other component, $H$, with at most $n-2 i-3$ vertices at at most $r-i-1 \leq r_{1}-i-2$ cycles. Since $x$ is adjacent to at least one vertex not in the $K_{3}$ endblocks, $G-N[x]$ has at most $n-2 i-4$ vertices. This gives us the upper bound

$$
\begin{equation*}
m(G) \leq 2^{i+1} c\left(n-2 i-3, r_{1}-i-2\right)+g(n-2 i-4) \tag{1}
\end{equation*}
$$

To show that this bound is always at most $m_{0}$, we consider the two values $n=3 r_{0}+1$ and $n=3 r_{0}+2$ as well as the three possible congruence classes of $i$ modulo 3 separately. So let $j=\lfloor i / 3\rfloor$. Considering the number of vertices in $G-N[x]$ gives $n-2 i-4 \geq 0$ and translating this into a bound involving $r_{0}$ and $j$ gives $r_{0} \geq 2 j+k_{0}$ where $1 \leq k_{0} \leq 3$ depending on which of the six cases we are in. We now wish to show that the right-hand side of (11) is a strictly decreasing function of $j$ for any fixed but sufficiently large $n$. This


Figure 3: $D$, the multigraph for $l=1$
is clearly true of the $g(n-2 i-4)$ term, so let $f\left(r_{0}, j\right)=2^{i+1} c\left(n-2 i-3, r_{1}-i-2\right)$ where the right side has been converted to a function of $r_{0}$ and $j$. In all cases, we get that

$$
f\left(r_{0}, j\right)=a\left(r_{0}\right) 2^{j}+b\left(r_{0}\right)(8 / 9)^{j}
$$

for certain functions $a\left(r_{0}\right), b\left(r_{0}\right)$. It follows that $f\left(r_{0}, j\right)-f\left(r_{0}, j+1\right)>0$ if and only if $b\left(r_{0}\right) / a\left(r_{0}\right)>9(3 / 2)^{2 j}$. Solving for $r_{0}$ shows that we have a decreasing function of $j$ for $r_{0} \geq 2 j+k_{1}$ where $4 \leq k_{1} \leq 7$. So it suffices to check that the right-hand side of (11) is at most $m_{0}$ for $2 j+k_{0} \leq r_{0} \leq 2 j+k_{1}$. This is done by substituting each value of $r_{0}$ in turn to get a function of $j$ alone, noting that this function is decreasing for all $j$ sufficiently large to make $r_{0} \geq 3$, and then verifying that this function is bounded by $m_{0}$ when $j$ is at this minimum value. The only cases where we get equality are when $n=3 r_{0}+2$ and $G \cong C\left(n, r_{0}\right)$.

If $l=1$, then $B$ must be a subdivision of the multigraph $D$ in Figure 3, i.e., it must be obtained from $D$ by inserting vertices of degree 2 into the edges of $D$. By $(\Delta)$, we can insert at most one vertex into an edge, unless one of the inserted vertices is the cutvertex $x$ in which case it is possible to insert a vertex before and after $x$ as well. To turn this multigraph into a graph, it is necessary to subdivide at least two of the edges. If all three edges are subdivided, or two edges are subdivided and $x$ is one of the original vertices of $D$, then $G$ has a duplicated vertex and so is not extremal by Lemma [2.2. In the only remaining case, the following lemma applies.

Lemma 2.4 (Triangle Lemma) Suppose $G$ contains three vertices $\{u, v, w\}$ satisfying the following restrictions.
(a) These vertices form a $K_{3}$ with $\operatorname{deg} u=2$ and $\operatorname{deg} v, \operatorname{deg} w \geq 3$.
(b) The graph $G-\{u, v, w\}$ is connected.

Then $m(G) \leq m_{0}$ with equality only if $n=3 r_{0}+1$ and $G \cong C(n, s)$ for some $s$ with $r_{0} \leq s<r_{1}$.

Proof: Because of (a), the $K_{3}$ satisfies the alternative hypothesis in the $m$-recursion. Using induction to evaluate the $c$ and $g$ functions, we get

$$
\begin{aligned}
m(G) & =m(G-N[u])+m(G-N[v])+m(G-N[w]) \\
& \leq c(n-3, r-1)+2 g(n-4, r-1) \\
& \leq m_{0}
\end{aligned}
$$



Figure 4: The multigraphs for $l=2$
with equality only if $n=3 r_{0}+1, G-N[v] \cong G-N[w] \cong G(n-4)$, and $G-N[u] \cong C(n-3, s)$ for some $s \leq r_{0}+4$. These easily imply the conclusion of the lemma.

For $l \geq 2$, we must consider the two congruence classes for $n$ separately. First consider $n=3 r_{0}+1$. The following lemma will help eliminate many cases. In it, we use $r(v)$ to denote the number of cycles of $G$ containing the vertex $v$.

Lemma 2.5 Let $n=3 r_{0}+1$ and let $v$ be a non-cutvertex with $\operatorname{deg} v \geq 3$ and $r(v) \geq 6$. Then $G$ is not extremal.

Proof: Using the $m$-bound, we have

$$
\begin{aligned}
m(G) & \leq c(n-1, r-6)+g(n-4, r-6) \\
& \leq c\left(3 r_{0}, r_{0}-1\right)+g\left(3 r_{0}-3, r_{0}-1\right) \\
& =2 \cdot 3^{r_{0}-1}+3^{r_{0}-1} \\
& =m_{0}
\end{aligned}
$$

However, if we have equality then this forces us to have $G-v \cong C\left(3 r_{0}, r_{0}-1\right)$ and $G-N[v] \cong G\left(3 r_{0}-3, r_{0}-1\right)$. The only way this can happen is if $G \cong C\left(3 r_{0}+1, r_{0}+2\right)$ where $v$ is one of the degree 3 vertices in the 4 -vertex block. But then $v$ is in only 3 cycles, contradicting our hypothesis that $r(v) \geq 6$.

When $l=2, B$ must be a subdivision of one of the multigraphs in Figure 4 Lemma 2.5 shows that $B$ cannot be a subdivision of $E_{1}, E_{4}$, or any block formed by a sequence of length $l \geq 3$ since in all these cases there are at least two vertices having degree at least 3 and lying in at least 6 cycles. So even if $B$ has a cutvertex, there will still be a non-cutvertex in $B$ satisfying the hypotheses of the lemma.

If $B$ is formed by subdividing $E_{2}$, the same lemma shows that we need only consider the case where the vertex of degree 4 in $E_{2}$ is a cutvertex, $x$, of $G$. Also, since $B$ can't have duplicated vertices and must satisfy $(\Delta)$, each pair of doubled edges has a vertex inserted in exactly one edge. This means there are only two possibilities for $B$, depending on whether the non-doubled edge is subdivided or not, and it is easy to check that in both cases $G$ is not extremal by using the $m$-bound on the vertex $x$.

Finally, if $B$ is a subdivision of $E_{3}$ then, because of the pair of disjoint doubled edges, there will always be one doubled edge which does not contain a cutvertex. In $B$ that pair
will give rise to either a duplicated vertex or a $K_{3}$ satisfying the hypotheses of the Triangle Lemma, and thus in either case we will be done. This ends the proof for $n=3 r_{0}+1$.

Now we look at the case where $n=3 r_{0}+2$. The analogue of Lemma 2.5 in this setting is as follows and since the proof is similar, we omit it.

Lemma 2.6 Let $n=3 r_{0}+2$ and let $v$ be a non-cutvertex that satisfies either
(1) $\operatorname{deg} v \geq 3$ and $r(v) \geq 12$, or
(2) $\operatorname{deg} v \geq 4$ and $r(v) \geq 6$.

Then $G$ is not extremal.
The ideas used to rule out $E_{3}$ for $n=3 r_{0}+1$ will be used many times in the current case, so we codify them in the lemma below.

Lemma 2.7 Suppose $n=3 r_{0}+2$ and the block $B$ is a subdivision of a multigraph having two disjoint submultigraphs each of which is of one of the following forms:
(i) a doubled edge, or
(ii) a vertex $v$ satisfying $\operatorname{deg} v \geq 3$ and $r(v) \geq 12$, or
(iii) a vertex $v$ satisfying $\operatorname{deg} v \geq 4$ and $r(v) \geq 6$.

Then $G$ is not extremal.
Proof: If any set of doubled edges has both edges subdivided exactly once, then $G$ is not extremal by Lemma 2.2. Otherwise, since $B$ has at most one cutvertex $x$ in $G$, either the hypotheses of the Triangle Lemma or of the previous lemma will be satisfied.

Finally, we will need a way to eliminate blocks that only have vertices of degree at most 3 , but not sufficiently many cycles to satisfy Lemma 2.6(1). One way would be to make sure that $G-N[v]$ is connected. Since a given multigraph $M$ has many possible subdivisions, we also need a criterion on $M$ that will guarantee that most of the subdivisions will have the desired connectivity.

Lemma 2.8 Let $n=3 r_{0}+2$.
(1) Suppose that $G$ contains a non-cutvertex $v$ such that $\operatorname{deg} v \geq 3$ and $r(v) \geq 6$. Suppose further that $G-N[v]$ contains at most two nontrivial components and that if there are two, then one of them is a star (a complete bipartite graph of the form $K_{1, s}$ ). Then $G$ is not extremal.
(2) Suppose $G$ comes from subdivision of a multigraph $M$ that contains a vertex $v$ with $\operatorname{deg} v=3, r(v) \geq 6$, and such that all vertices in $N_{M}[v]$ are non-cutvertices in $M$ and $M-N_{M}[v]$ is connected. Suppose further that there are at most two edges of $M$ between the elements of $N_{M}(v)$. Then $G$ and $v$ satisfy the hypotheses of (1).

Proof: For (1), first assume that there is only one nontrivial component in $G-N[v]$. Then, using the induction hypothesis about the behavior of graphs in the gap,

$$
\begin{aligned}
m(G) & \leq c(n-1, r-6)+c(n-4, r-6) \\
& =c\left(3 r_{0}+1, r_{0}+5\right)+c\left(3 r_{0}-2, r_{0}+5\right) \\
& =c\left(3 r_{0}+1, r_{0}\right)+c\left(3 r_{0}-2\right) \\
& =3^{r_{0}}+3^{r_{0}-1}+2^{r_{0}-2} \\
& <m_{0} .
\end{aligned}
$$

In the case with a star component, we use $m(G) \leq c(n-1, r-6)+2 c(n-6, r-6)$ to obtain the same result.

For (2) we will break the proof into several cases depending on how the edges of $M$ at $v$ are subdivided in $G$, noting that by our hypotheses each can be subdivided at most once and that the same is true of any edge between elements of $N_{M}(v)$. Let $N_{M}(v)=\{s, t, u\}$ and let $H$ be the subdivision of $L=M-N_{M}[v]$ induced by $G$. Note that $H$ is connected by assumption. If none of $v s, v t, v u$ are subdivided in $G$, then $G-N_{G}[v]$ is just $H$ together, possibly, with some vertices of degree one attached (if any edges from $s$, $t$, or $u$ to $L$ were subdivided) and some trivial components (if any edges between $s, t$, and $u$ were subdivided). If exactly one of the three edges is subdivided, suppose it is $v s$. Then it is possible that $s$ is in a different nontrivial component of $G-N_{G}[v]$ than $H$. But since there are at most two edges from $s$ to $t$ and $u$, the component of $s$ is a star. Now suppose that $v s$ and $v t$ are subdivided, but not $v u$. At least one of $s, t$ are connected to $H$ otherwise $u$ becomes a cutvertex. So again the only possibility for a nontrivial component other than $H$ is a star containing either $s$ or $t$, but not both. Finally, if all three edges are subdivided, then $G-N_{G}[v]$ is connected because $v$ is not a cutvertex in $M$.

We need a little terminology before we handle the $n=3 r_{0}+2$ case. Let $L$ and $M$ be 2 -connected multigraphs with no vertices of degree two. We say that $M$ is a child of $L$ if there is some sequence $B_{0}, B_{1}, \ldots, B_{l}$ formed as in the Ear Decomposition Theorem with $B_{l-1}=L$ and $B_{l}=M$. We will use words like "descendant," "parent," and so on in a similar manner.

We now pick up the proof for $n=3 r_{0}+2$ where we left off, namely with $l=2$. Lemma 2.7 (iii) shows that $B$ cannot be a subdivision of $E_{1}$ or any of its descendants. Also, if $B$ is a subdivision of $E_{2}$, then by Lemma 2.6 (2) we need only consider the case where the vertex of degree 4 is a cutvertex $x$, and the same argument we used in the $3 r_{0}+1$ case shows that such graphs are not extremal.

Next we consider the children of $E_{2}$. The only multigraphs not ruled out by Lemma 2.7 are the first three listed in Figure 5 As before, we need only consider the case where there is a cutvertex $x$ at the vertex as indicated. It can be checked that $F_{1}$ can't lead to an extremal graph by using the vertex marked $v$ in Lemma [2.8. For $F_{2}$, first note that if any of the edges containing $x$ is subdivided, and it doesn't matter which one by symmetry, then taking $v$ to be the other endpoint of that edge (after subdivision) in Lemma 2.8 shows that $G$ is not extremal. If none of the edges containing $x$ are subdivided then $b:=|V(B)|$ satisfies $5 \leq b \leq 9$ because $G$ has Property ( $\Delta$ ), and applying the $m$-bound to the vertex


Figure 5: Three children and one grandchild of $E_{2}$
$x$ gives

$$
m(G) \leq \max _{5 \leq b \leq 9}\{c(b-1,1) g(n-b)+g(n-b-1)\}<m_{0}
$$

Finally, $F_{3}$ is treated the same way as $F_{2}$, noting that the two pairs of doubled edges must both be subdivided in the same manner, the only edge containing $x$ which can be subdivided further is the vertical one in the diagram, and the maximum is now taken over $6 \leq b \leq 9$.

The only grandchild of $E_{2}$ not thrown out by either Lemma [2.6 or Lemma 2.7 is the multigraph $G_{1}$ in Figure 5 a child of $F_{3}$. It is handled in the same way as $F_{2}$ and $F_{3}$ and the reader should be able to fill in the details at this point. It is easy to check that the children of $G_{1}$ are all eliminated, and so we have finished with the descendants of $E_{2}$.

Lemma 2.7 rules out subdivisions of $E_{3}$ directly as well as, in conjunction with Lemma 2.6, many of its children and all of its grandchildren. The only surviving multigraphs not previously considered are those children listed in Figure 6. In $F_{4}$, we are reduced in the usual manner to the case where the vertex of degree at least four is a cutvertex $x$. But then we can take $v$ as indicated in Lemma 2.8 and so this child is not extremal. In $F_{5}$ we need only consider when there is a cutvertex $x$ in the doubled edge. But then either $v_{1}$ or $v_{2}$ (depending on the placement of $x$ ) can be used in Lemma 2.8 to take care of this child. Similarly, in $F_{6}$ it is easy to see by symmetry that no matter where the cutvertex is placed, there is a $v$ for Lemma 2.8,

Finally we come to $E_{4} \cong K_{4}$. Lemma 2.8 shows that if $B$ is a subdivision of $E_{4}$ then we need only consider when one of the degree 3 vertices is a cutvertex $x$. If one of the edges $x v$ is subdivided then $v$ satisfies Lemma 2.8. Therefore if there is a degree 2 vertex in $B$, it must be formed by subdividing an edge between two non-cutvertices of $E_{4}$, say $u$ and $v$. Hence $G-\{u, v\}$ contains only one nontrivial connected component with $n-3$ vertices, and we get

$$
\begin{aligned}
m(G) & \leq m(G-u)+m(G-N[u]) \\
& \leq m(G-\{u, v\})+m(G-u-N[v])+m(G-N[u]) \\
& \leq c(n-3)+g(n-5)+g(n-4)
\end{aligned}
$$



Figure 6: Three children of $E_{3}$

$$
<m_{0}
$$

so these graphs are not extremal.
We are reduced to considering the case where $B \cong K_{4}$, that is, when no edges are subdivided. By our choice of $B$ and the cases we have disposed of so far, we can assume that all other endblocks containing the cutvertex $x$ are isomorphic to $K_{2}, K_{3}$, or $K_{4}$ Assume that there are $i$ copies of $K_{3}$ and $j$ copies of $K_{4}$ other than $B$. If these are the only blocks of $G$, then $m(G)=2^{i} 3^{j+1}+1$ where $2 i+3 j+4=n=3 r_{0}+2$. This quantity is maximized when $i=2$ and $j=r_{0}-2$, giving $m(G)=4 \cdot 3^{r_{0}-1}+1<m_{0}$.

Hence we may assume that $G$ has other blocks. We subdivide this case into three subcases. First, if $j \geq 2$ (in other words, if $x$ lies in at least three $K_{4}$ endblocks) then applying $m$-bound shows that

$$
m(G) \leq 27 g(n-10, r-21)+g(n-11, r-21)<m_{0}
$$

Now consider $j=1$. Here our upper bound is

$$
\begin{aligned}
m(G) & \leq 9 \cdot 2^{i} c(n-2 i-7, r-i-14)+g(n-2 i-8) \\
& \leq 9 \cdot 2^{i} c\left(3 r_{0}-2 i-5, r_{0}-i-3\right)+g\left(3 r_{0}-2 i-6\right)
\end{aligned}
$$

This is a decreasing function of $i$ within each congruence class modulo 2 , and using this fact it is routine to check that $m(G)<m_{0}$.

We are left with the case where $j=0$. The $m$-bound gives

$$
\begin{aligned}
m(G) & \leq 3 \cdot 2^{i} c(n-2 i-4, r-i-7)+g(n-2 i-5) \\
& \leq 3 \cdot 2^{i} c\left(3 r_{0}-2 i-2, r_{0}-i+4\right)+g\left(3 r_{0}-2 i-3\right)
\end{aligned}
$$

Note that $3 r_{0}-2 i-2 \geq 3\left(r_{0}-i+4\right)$ only for $i \geq 14$. For these values of $i$, the upper bound is a decreasing function of $i$ within each congruence class modulo 2 , so we only need to verify that $m(G)<m_{0}$ for $i \leq 15$. These cases can all be routinely checked, although when $i \equiv 1,2(\bmod 3)$ the desired inequality will hold only for sufficiently large $r_{0}$, and


Figure 7: $F_{7}$, a child of $E_{4}$
one must note that these cases can arise only for such sufficiently large $r_{0}$. In particular, $n \geq 2 i+5$ by our assumptions, and this implies that $r_{0} \geq 2 i / 3+1$.

The only child of $E_{4}$ that is not a child of any other $E_{k}$ and is not ruled out by Lemma 2.6 is $F_{7}$, shown in Figure 7. One can verify by considering several cases that whether or not there is a cutvertex in $B$, there is a vertex $v$ satisfying the hypotheses of Lemma [2.8, Finally, the grandchildren of $E_{4}$ all fall under the purview of Lemma 2.6.

We have now considered all the cases and so completed the proof of the theorem.

## 3 Maximum independent sets

We now turn to the consideration of maximum independent sets. An independent set $I$ if $G$ is maximum if it has maximum cardinality over all independent sets of $G$. We let $m^{\prime}(G)$ denote the number of maximum independent sets of $G$. Since every maximum independent set is also maximal we have $m^{\prime}(G) \leq m(G)$, so for any finite family of graphs,

$$
\begin{equation*}
\max _{G \in \mathcal{F}} m^{\prime}(G) \leq \max _{G \in \mathcal{F}} m(G) \tag{2}
\end{equation*}
$$

We say that $G$ is well covered if every one of its maximal independent sets is also maximum. Then we have equality in (2) if and only if some graph with a maximum number of maximal independent sets is well covered. The graphs $G(n), G^{\prime}(n), C(n)$, and $G(n, r)$ are well covered for all pairs $n, r$ and $C(n, r)$ is well covered when $n \equiv r(\bmod 2)$ and $n \geq 3 r$, so we immediately have the following result.
Theorem 3.1 Let $G$ be a graph with $n$ vertices and at most $r \geq 1$ cycles.
(I) For all such graphs,

$$
m^{\prime}(G) \leq g(n, r)
$$

with equality if and only if $G \cong G(n, r)$.
(II) If $n \geq 3 r$ and $n \equiv r(\bmod 2)$, or if $r \geq r(C(n))$, then for all such graphs that are connected,

$$
m^{\prime}(G) \leq c(n, r)
$$

with equality if and only if $G \cong C(n, r)$.

This leaves only the case where $G$ is connected and $n \not \equiv r(\bmod 2)$. To state and motivate the result in this case, we recall the work done on trees and graphs with at most one cycle.

When $n$ is even, the tree $K_{1} *\left(K_{1} \uplus \frac{n-2}{2} K_{2}\right)$ has $t(n)$ maximum independent sets (in fact, by the upcoming Theorem [3.2, it is the only such tree). When $n$ is odd there is only one extremal tree for the maximal independent set problem, and it is not well-covered. Define a family of trees by

$$
T^{\prime}(n):= \begin{cases}K_{1} *\left(K_{1} \uplus \frac{n-2}{2} K_{2}\right) & \text { if } n \equiv 0(\bmod 2), \\ K_{1} *\left(2 K_{1} \uplus \frac{n-3}{2} K_{2}\right) & \text { if } n \equiv 1(\bmod 2),\end{cases}
$$

and let

$$
t^{\prime}(n):=m^{\prime}\left(T^{\prime}(n)\right)= \begin{cases}2^{\frac{n-2}{2}}+1 & \text { if } n \equiv 0(\bmod 2) \\ 2^{\frac{n-3}{2}} & \text { if } n \equiv 1(\bmod 2)\end{cases}
$$

Zito [12] proved the following result.
Theorem 3.2 (Zito [12]) If $T$ is a tree on $n \geq 2$ vertices then

$$
m^{\prime}(T) \leq t^{\prime}(n)
$$

with equality if and only if $T \cong T^{\prime}(n)$.
In [6], Jou and Chang gave a short proof of this theorem and considered graphs with at most one cycle.

Theorem 3.3 (Jou and Chang [6]) Let $G$ be a graph with at most one cycle and $n$ vertices where $n \equiv 0(\bmod 2)$. Then

$$
m^{\prime}(G) \leq t(n)
$$

with equality if and only if $G$ is a tree, and thus, by Theorem 3.2, if and only if $G \cong T^{\prime}(n)$.
The rest of this section will be devoted to the proof of the maximum independent set version of Theorem 1.5

Theorem 3.4 Let $G$ be a connected graph with $n \geq 3 r$ vertices and at mostr cycles, where $r \geq 1$ and $n \not \equiv r(\bmod 2)$. Then

$$
m^{\prime}(G) \leq c(n, r-1)
$$

with equality if and only if $G$ has precisely $r-1$ cycles, and thus, by Theorem 1.5, if and only if $G \cong C(n, r-1)$ or if $G$ is isomorphic to one of the exceptional graphs listed there.

Before proving Theorem 3.4 we present two lemmas. The first is an analogue the $m$ bound and $m$-recursion that proved useful for maximal independent sets. Its proof is similar and so is omitted.

Lemma 3.5 The invariant $m^{\prime}(G)$ satisfies the following inequalities.
(1) If $v \in V$ then

$$
m^{\prime}(G) \leq m^{\prime}(G-v)+m^{\prime}(G-N[v])
$$

(2) If $G$ has a complete subgraph $B$ with at least one vertex adjacent only to other vertices of $B$ then

$$
m^{\prime}(G) \leq \sum_{v \in V(B)} m^{\prime}(G-N[v])
$$

We will refer to parts (1) and (2) of this lemma as the $m^{\prime}$-bound and $m^{\prime}$-recursion, respectively.

Our second lemma will be useful for eliminating the cases where a vertex serves as a cutvertex for more than one endblock.

Lemma 3.6 Suppose that the graph $G$ contains a vertex $x$ and a set of at least two other vertices $U$ such that the induced graph $G[U]$ is not complete and for all $u \in U$,

$$
\begin{equation*}
x \in N(u) \subseteq U \cup\{x\} \tag{3}
\end{equation*}
$$

Then $m^{\prime}(G)=m^{\prime}(G-x)$.
Proof: It suffices to show that no maximum independent set of $G$ contains $x$. Suppose not, and let $I$ be a maximum independent set with $x \in I$. Then by (3) we have $I \cap U=\emptyset$. Also, since $G[U]$ is not complete there is an independent set $A \subset U$ containing at least two vertices. But then $I \cup A-x$ is a larger independent set than $I$, a contradiction.
Proof (of Theorem (3.4): We will use induction on $r$. The base case of $r=1$ is precisely Theorem 3.3, so we will assume $r \geq 2$. Note that since $n \geq 3 r$ and $n \not \equiv r(\bmod 2)$ we have $n \geq 3 r+1$. Let $G$ be an extremal graph satisfying the hypothesis of the theorem.

If $G$ has less than $r$ cycles then we are done by Theorem [1.5, so we may assume that $G$ has exactly $r$ cycles. Since we have chosen $G$ to be extremal, we may assume that

$$
m^{\prime}(G) \geq c(n, r-1)=3^{r-2} \cdot 2^{\frac{n-3 r+5}{2}}+2^{r-2}
$$

Let $B$ be an endblock of $G$. First, if $B$ has intersecting cycles, then the Ear Decomposition Theorem shows that $B$ must contain a subdivision of the multigraph $D$ (shown in Figure (3). This implies that $B$ contains a non-cutvertex of degree at least 3 that lies in at least 3 cycles, from which $m^{\prime}$-bound gives the contradiction

$$
m^{\prime}(G) \leq c(n-1, r-3)+g(n-4, r-3)<c(n, r-1) .
$$

Hence $B$ is either $K_{2}, K_{3}$, or $C_{p}$ for some $p \geq 4$. Since these possibilities have at most one cycle and we are assuming that $G$ has $r \geq 2$ cycles, $G$ cannot be a single block. Hence $B$ must contain a cutvertex $x$ of $G$.

First suppose that $B \cong C_{p}$ for some $p \geq 4$. Label the vertices of $B$ as $x, u, v, w, \ldots$ so that they read one of the possible directions along the cycle. Since $G-v$ is connected
with $n-1$ vertices and $r-1$ cycles, induction applies to give $m^{\prime}(G-v) \leq c(n-1, r-2)$. Furthermore, $G-v$ has exactly $r-1$ cycles, so by induction we cannot have equality. Similarly, $G-N[v]$ has $n-3$ vertices and $r-1$ cycles, so $m^{\prime}(G-N[v])<c(n-3, r-2)$. An application of the $m^{\prime}$-bound gives the contradiction

$$
m^{\prime}(G) \leq m^{\prime}(G-v)+m^{\prime}(G-N[v])<c(n-1, r-2)+c(n-3, r-2)=c(n, r-1)
$$

We now know that all endblocks of $G$ must be copies of either $K_{2}$ or $K_{3}$. We claim that such endblocks must be disjoint. Suppose to the contrary that two endblocks share a vertex, which must therefore be the cutvertex $x$. Considering the two cases when at least one endblock is a $K_{2}$ (so that $G-x$ has an isolated vertex which must be in each of its maximum independent sets) or when both are copies of $K_{3}$, we can use Lemma 3.6 and the fact that $n \geq 3 r+1$ to get

$$
\begin{aligned}
m^{\prime}(G) & =m^{\prime}(G-x) \\
& \leq \begin{cases}g(n-2, r) & \text { if one endblock is a } K_{2}, \\
g(n-1, r-2) & \text { if both endblocks are } K_{3} \text { 's }\end{cases} \\
& <c(n, r-1)
\end{aligned}
$$

This contradiction proves the claim.
Now let $B$ be an endblock of $G$ satisfying Proposition 1.8, so that it intersects at most one non-endblock. By what we have just shown, $B$ intersects precisely one other block and that block is not an endblock. We claim that this block is isomorphic to $K_{2}$. Suppose not. Then the cutvertex $x$ of $B$ lies in at least one cycle not contained in $B$ and is adjacent to at least two vertices not in $B$. Again, we consider the cases $B \cong K_{2}$ and $B \cong K_{3}$ separately to obtain

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-x)+m^{\prime}(G-N[x]) \\
& \leq \begin{cases}c(n-2, r-1)+g(n-4, r-1) & \text { if } B \cong K_{2}, \\
2 c(n-3, r-2)+g(n-5, r-2) & \text { if } B \cong K_{3}\end{cases} \\
& <c(n, r-1)
\end{aligned}
$$

proving our claim.
Since $G$ is not itself a block, Proposition 1.8 shows that $G$ contains at least two endblocks, say $B$ and $B^{\prime}$, that each intersect at most one non-endblock. Let the cutvertices of these endblocks be labeled $x$ and $x^{\prime}$, respectively. We have shown that $B$ and $B^{\prime}$ are disjoint and that they each intersect precisely one other block, which must be isomorphic to $K_{2}$. We claim that there is vertex $v_{0}$ such that these two copies of $K_{2}$ have vertices $\left\{x, v_{0}\right\}$ and $\left\{x^{\prime}, v_{0}\right\}$. If this is not the case then

$$
\begin{equation*}
\left|(N(x)-B) \cup\left(N\left(x^{\prime}\right)-B^{\prime}\right)\right|=2 . \tag{4}
\end{equation*}
$$

There are now three cases to consider depending on the nature of $B$ and $B^{\prime}$. First, suppose $B \cong B^{\prime} \cong K_{2}$. Using the $m^{\prime}$-recursion twice, we get

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-B)+m^{\prime}(G-N[x]) \\
& \leq m^{\prime}(G-B)+m^{\prime}\left(G-N[x]-B^{\prime}\right)+m^{\prime}\left(G-N[x]-N\left[x^{\prime}\right]\right)
\end{aligned}
$$

Since the three graphs in this last expression may still have $r$ cycles, we will also have to use induction on $n$. We consider two cases, depending on whether or not $G-N[x]-N\left[x^{\prime}\right]$ has parameters lying in the range of Theorem 1.5. Clearly $G-B$ is connected with $n-2$ vertices and $r$ cycles, so if $n \geq 3 r+5$ then $n-6 \geq 3 r-1$. Also, $n-2$ and $r$ are of different parity with $n-2 \geq 3 r$. By assumption (4), $G-N[x]-N\left[x^{\prime}\right]$ has $n-6$ vertices and at most $r$ cycles yielding $m^{\prime}\left(G-N[x]-N\left[x^{\prime}\right]\right) \leq g(n-6, r)$. Secondly, $G-N[x]-B^{\prime}$ has $n-5$ vertices and at most $r$ cycles so $m^{\prime}\left(G-N[x]-B^{\prime}\right) \leq g(n-5, r)$. Finally, we can apply induction to conclude that $m^{\prime}(G-B) \leq c(n-2, r-1)$. Putting everything together we get

$$
\begin{aligned}
m^{\prime}(G) & \leq \begin{cases}c(n-2, r-1)+g(n-5, r)+g(n-6, r) & \text { if } n \geq 3 r+5, \\
c(n-2, r)+g(n-5)+g(n-6) & \text { if } n=3 r+1 \text { or } n=3 r+3\end{cases} \\
& <c(n, r-1)
\end{aligned}
$$

a contradiction.
Now suppose that $B \cong K_{2}$ and $B^{\prime} \cong K_{3}$. Proceeding in much the same manner as before gives

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-B)+2 m^{\prime}\left(G-N[x]-B^{\prime}\right)+m^{\prime}\left(G-N[x]-N\left[x^{\prime}\right]\right) \\
& \leq \begin{cases}c(n-2, r-1)+2 g(n-6, r-1)+g(n-7, r-1) & \text { if } n \geq 3 r+3 \\
c(n-2, r)+2 g(n-6)+g(n-7) & \text { if } n=3 r+1\end{cases} \\
& \leq c(n, r-1)
\end{aligned}
$$

and equality cannot occur because $G-B$ has exactly $r$ cycles.
The third case is when $B \cong K_{3}$ and $B^{\prime} \cong K_{3}$. Trying the same technique we obtain

$$
\begin{aligned}
m^{\prime}(G) & \leq 2 m^{\prime}(G-B)+2 m^{\prime}\left(G-N[x]-B^{\prime}\right)+m^{\prime}\left(G-N[x]-N\left[x^{\prime}\right]\right) \\
& \leq 2 c(n-3, r-2)+2 g(n-7, r-2)+g(n-8, r-2) \\
& \leq c(n, r-1)
\end{aligned}
$$

Again we cannot have equality throughout because $G-B$ has exactly $r-1$ cycles.
Now that we have established the existence of $v_{0}$, we are almost done. Observe that there is at most one block $C$ other than the $K_{2}$ 's connecting $v_{0}$ to endblocks and those endblocks themselves. (If there were more than one such block, then since endblocks can't intersect this would force the existence of another $K_{2}$ and corresponding endblock which we hadn't considered.) So $C$, if it exists, must be an endblock containing $v_{0}$. By our characterization of endblocks, this leaves only three possibilities, namely $C \cong \emptyset, K_{2}, K_{3}$. It is easy to check the corresponding graphs $G$ either do not exist because of parity considerations or satisfy $m^{\prime}(G)<c(n, r-1)$. We have now shown that no graph with exactly $n$ vertices and $r$ cycles has as many maximum independent sets as $C(n, r-1)$, and thus finished the proof of Theorem 3.4.

## 4 The gap revisited

We now need to look at maximum independent sets in the gap. Consider first the case when $G$ is connected with $n=3 r_{0}+2$ vertices and less than $r_{1}$ cycles. Then $n \equiv r_{0}(\bmod 2)$
and by Theorem 2.3 we have $m^{\prime}(G) \leq m(G) \leq c\left(n, r_{0}\right)$, with the second inequality reducing to an equality if and only if $G \cong C\left(n, r_{0}\right)$ or $C\left(n, r_{0}+1\right)$. Since the former graph is well covered but the latter is not, we have the following result.

Theorem 4.1 Let $G$ be a connected graph with $n$ vertices, $n=3 r_{0}+2$ where $n \geq 7$, and less than $r_{1}$ cycles. Then

$$
m^{\prime}(G) \leq c\left(n, r_{0}\right)
$$

with equality if and only if $G \cong C\left(n, r_{0}\right)$.
For the $n=3 r_{0}+1$ case we need to adapt the proof of Theorem 2.3. The result mirrors the trend exhibited by Theorem 3.4.

Theorem 4.2 Let $G$ be a connected graph with $n$ vertices, $n=3 r_{0}+1$ where $n \geq 7$, and less than $r_{1}$ cycles. Then

$$
m^{\prime}(G) \leq c\left(n, r_{0}-1\right)
$$

with equality if and only if $G \cong C\left(n, r_{0}-1\right)$.
Proof: We will use induction on $n$. The $n=7$ case has been checked by computer and so we assume $G$ is a graph satisfying the hypotheses of the theorem where $n \geq 10$, or equivalently, $r_{0} \geq 3$. We will begin as in the proof of Theorem [2.3, considering possible endblocks produced by the inductive procedure in the Ear Decomposition Theorem.

Our first order of business will be to show that any endblock of an extremal $G$ must be isomorphic to $K_{i}$ for $2 \leq i \leq 4$ or the graph $D_{1}$ shown in Figure 8 . Note that, unlike in the proof of Theorem [2.3, here we will consider all endblocks of $G$, not just those that satisfy the conclusion of Proposition 1.8. As the proof of Lemma 2.1 no longer holds for maximum independent sets, we need the following result to replace Property ( $\Delta$ ). In it, and in the future, it will be convenient to use the notation

$$
m_{0}^{\prime}=c\left(n, r_{0}-1\right)=8 \cdot 3^{r_{0}-2}+2^{r_{0}-2}
$$

Lemma 4.3 (Path Lemma) Suppose there is a path $P=v_{1} v_{2} v_{3}$ in a block of $G$ satisfying the following three conditions.
(1) $\operatorname{deg} v_{1} \geq 3$, $\operatorname{deg} v_{2}=2$, and $v_{1} v_{3} \notin E(G)$,
(2) $G-P$ is connected,
(3) One of the following two subconditions hold
(a) $G-N\left[v_{1}\right]$ is connected, or
(b) $G-v_{1}-N\left[v_{3}\right]=G_{1} \uplus G_{2}$ where $G_{1}$ is connected and $\left|V\left(G_{2}\right)\right| \leq 2$.

Then $m^{\prime}(G)<m_{0}^{\prime}$.

Proof: Using the $m^{\prime}$-bound twice gives

$$
\begin{equation*}
m^{\prime}(G) \leq m^{\prime}\left(G-v_{1}-v_{3}\right)+m^{\prime}\left(G-v_{1}-N\left[v_{3}\right]\right)+m^{\prime}\left(G-N\left[v_{1}\right]\right) \tag{5}
\end{equation*}
$$

Conditions (1) and (2) of the current lemma imply that $G-v_{1}-v_{3}=H \uplus\left\{v_{2}\right\}$ where $H$ is connected. Furthermore, since $P$ is in a block, $v_{2}$ must lie in at least one cycle of $G$. Hence $H$ has $n-3$ vertices and less than $r_{1}-1$ cycles with these two parameters satisfying the hypotheses of the theorem. By induction,

$$
m^{\prime}\left(G-v_{1}-v_{3}\right) \leq c\left(n-3, r_{0}-2\right)
$$

Now suppose (3a) holds. Then $m^{\prime}\left(G-N\left[v_{1}\right]\right) \leq c(n-4)$. Also $\operatorname{deg} v_{3} \geq 2$ (since it is in a non- $K_{2}$ block) and $v_{1} \notin N\left[v_{3}\right]$ by condition (1), so $m^{\prime}\left(G-v_{1}-N\left[v_{3}\right]\right) \leq g(n-4)$. Putting all these bounds into (5) gives

$$
m^{\prime}(G) \leq c\left(n-3, r_{0}-2\right)+c(n-4)+g(n-4) \leq m_{0}^{\prime} .
$$

Equality can only be achieved if $r_{0}=3, H \cong C(7,1)$, and $G-v_{1}-N\left[v_{3}\right] \cong G(6)$. But $C(7,1)$ has only one cycle while $G(6)$ has two, contradicting the fact that $G-v_{1}-N\left[v_{3}\right] \subset H$, so we must have $m^{\prime}(G)<m_{0}^{\prime}$ in this case.

Next we look at (3b). Considering the cases where $\left|V\left(G_{2}\right)\right|=0$, 1 , or 2 gives

$$
m^{\prime}\left(G-v_{1}-N\left[v_{3}\right]\right) \leq \max \{c(n-4), c(n-5), 2 c(n-6)\}=c(n-4),
$$

while $m^{\prime}\left(G-N\left[v_{1}\right]\right) \leq g(n-4)$. Thus we get the same bound on $m^{\prime}(G)$ as in the case where (3a) held. Equality implies $H \cong C(7,1)$ and $G_{1} \cong C(6)$ or $K_{4}$ since $c(6)=2 c(4)$, but then we have the same problem with cycles. This final contradiction ends the proof of Lemma 4.3

In all of our applications of the Path Lemma we will set up the notation so that $v_{1}=v$.
Lemma 4.4 Let $B$ be a block of $G$ which comes from subdividing a multigraph $M$, and suppose that $M$ and $B$ satisfy either
(1) $M$ contains an edge uv such that
(a) $v$ is a non-cutvertex in $G$,
(b) $\operatorname{deg}_{M} v\left(=\operatorname{deg}_{B} v\right)=3$, and
(c) $u v$ is subdivided more than twice in $B$ and none of these inserted vertices are cutvertices,
or
(2) $M$ contains a doubled edge where both edges are subdivided exactly once and neither of these inserted vertices is a cutvertex.

Then $G$ is not extremal.

Proof: For (1), suppose such an edge $v w$ is subdivided three or more times. Then the hypotheses of the Path Lemma are satisfied with condition (3b) and $v_{1}=v$, so $m^{\prime}(G)<m_{0}^{\prime}$.

Part (2) of the lemma follows from the fact that extremal graphs cannot have duplicated vertices: if both edges are subdivided exactly once then we have a pair $s, t$ of duplicated vertices, and Lemma 2.2 shows that

$$
m^{\prime}(G) \leq m(G)=m(G-s) \leq c(n-1)<m_{0}^{\prime}
$$

another contradiction.
We are now ready to begin restricting the type of endblocks an extremal graph may possess. Let $B$ denote an endblock of $G$. First we consider the case where we have $B \cong C_{p}$. As $c(n, 1)<m_{0}^{\prime}$ for all $n \geq 10, G$ may not itself be a block, and thus we may assume that there is a cutvertex, say $x$, in $B$. The $p=4$ case is ruled out by Lemma 4.4 (2). If $p=5$ then considering the cases where $x$ is adjacent to exactly one or more than one vertex outside $B$ gives

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-x)+m^{\prime}(G-N[x]) \\
& \leq \begin{cases}3 c(n-5)+2 g(n-6) & \text { if } x \text { has exactly one neighbor outside } B, \\
3 g(n-5)+2 g(n-7) & \text { if } x \text { has more than one neighbor outside } B\end{cases} \\
& <m_{0}^{\prime}
\end{aligned}
$$

If $p \geq 6$, then label the non-cutvertices of $B$ by $v_{1}, v_{2}, \ldots$ so that $x v_{1} v_{2} \ldots v_{p-1}$ is a path. Using the $m^{\prime}$-bound twice gives

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}\left(G-v_{2}\right)+m^{\prime}\left(G-N\left[v_{2}\right]\right) \\
& \leq m^{\prime}\left(G-v_{2}-v_{4}\right)+m^{\prime}\left(G-v_{2}-N\left[v_{4}\right]\right)+m^{\prime}\left(G-N\left[v_{2}\right]\right)
\end{aligned}
$$

We can apply induction to $m^{\prime}\left(G-v_{2}-v_{4}\right)$ since it consists of an isolated vertex together with a connected graph with $n-3$ vertices and one less cycle than $G$, and similarly to $m^{\prime}\left(G-N\left[v_{2}\right]\right)$. This gives

$$
m^{\prime}(G) \leq 2 c\left(n-3, r_{0}-2\right)+c(n-4) \leq m_{0}^{\prime}
$$

with equality if and only if $r_{0}=3$. But then $G-v_{2}-N\left[v_{4}\right] \cong C(6)$, which contradicts the fact that the former graph has a vertex of degree 1 while the latter does not. So if $B$ is a cycle in an extremal graph then $B \cong K_{3}$.

We now consider the case where $B$ comes from subdividing the graph $D$ in Figure 3 If $G$ is itself a subdivision of $D$ then Lemma 4.4 (1) shows that $G$ has at most 8 vertices, which falls within the range of the computer calculations we have performed. Therefore $B$ must contain a cutvertex $x$.

First suppose that $x$ is a vertex of $D$ and let $v$ be the other vertex of degree 3. If all of the edges of $D$ are subdivided then at most one of them can be subdivided once by Lemma 4.4 (2), and so $G$ is not extremal because of option (3a) in the Path Lemma. If one of the edges is not subdivided, then the only two possibilities for the number of subdividing


Figure 8: Three special subdivisions of $D$
vertices in the three edges are $(2,1,0)$ and $(2,2,0)$ by Lemma 4.4 (1). It is then easy to check that

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-x)+m^{\prime}(G-N[x]) \\
& \leq\left\{\begin{array}{cl}
3 g(n-5)+g(n-6) & \text { in the }(2,1,0) \text { case } \\
g(n-6)+g(n-7) & \text { in the }(2,2,0) \text { case }
\end{array}\right. \\
& <m_{0}^{\prime}
\end{aligned}
$$

This leaves the case where $x$ is interior to an edge of $D$. Neither of the other edges of $D$ may be subdivided more than twice by Lemma 4.4 (1), so if one of them is subdivided twice then $G$ is not extremal by the Path Lemma (3b). Lemma 4.4 (2) rules out the case where both edges not containing $x$ are subdivided once. Therefore we may assume that one of these edges is subdivided once and the other is not subdivided. Furthermore, the Path Lemma (3a) shows that the edge containing $x$ may be subdivided at most once to either side of $x$. Thus we are reduced to considering the three endblocks $D_{1}, D_{2}$, and $D_{3}$ shown in Figure 8

Both $D_{2}$ and $D_{3}$ can be eliminated by using the $m^{\prime}$-bound on the vertex $x$ :

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-x)+m^{\prime}(G-N[x]) \\
& \leq \begin{cases}2 g(n-5)+2 g(n-6) & \text { if } B \cong D_{2}, \\
g(n-6)+3 g(n-7) & \text { if } B \cong D_{3}\end{cases} \\
& <m_{0}^{\prime} .
\end{aligned}
$$

This leaves us with the case $B \cong D_{1}$, which we postpone until later.
We now need to go through the $E$ graphs from Figure 4. It will be useful to have an analogue of the large degree and large number of cycles results (Lemmas 2.5 and 2.6) used in proving the first Gap Theorem.

Lemma 4.5 Suppose that $v \in V(G)$ is a non-cutvertex satisfying either
(1) $\operatorname{deg} v \geq 3$ and $r(v) \geq 7$, or
(2) $\operatorname{deg} v \geq 4$ and $r(v) \geq 6$.

Then $m^{\prime}(G)<m_{0}^{\prime}$.
Proof: Since $r \leq r_{0}+5$, we have

$$
\begin{array}{rlr}
m^{\prime}(G) & \leq m^{\prime}(G-v)+m^{\prime}(G-N[v]), \\
& \leq \begin{cases}c\left(n-1, r_{0}-2\right)+g\left(n-4, r_{0}-2\right) & \text { if } \operatorname{deg} v \geq 3 \text { and } r(v) \geq 7 \\
c\left(n-1, r_{0}-1\right)+g(n-5) & \text { if } \operatorname{deg} v \geq 4 \text { and } r(v) \geq 6\end{cases} \\
& <m_{0}^{\prime} &
\end{array}
$$

proving the lemma.
No subdivision of $E_{1}$ can be an endblock in an extremal graph by part (2) of the previous lemma. The same reasoning shows that $B$ may only be a subdivision of $E_{2}$ if the vertex of degree 4 in $E_{2}$ is the cutvertex $x$. Label the other two vertices $v$ and $w$. Lemma 4.4 shows that the number of subdividing vertices for the pair of $v x$ edges must be one of $(1,0)$, $(2,0),(2,1)$, or $(2,2)$. The last two cases are eliminated by (3a) of the Path Lemma. The same can be said of the $w x$ edges and that $v w$ can be subdivided at most once by the Path Lemma (3b). So $5 \leq|V(B)| \leq 8$ and

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-x)+m^{\prime}(G-N[x]) \\
& \leq \max _{5 \leq b \leq 8}\left\{t^{\prime}(b-1) g(n-b)\right\}+g(n-6) \\
& <m_{0}^{\prime}
\end{aligned}
$$

where we remind the reader that $t^{\prime}(n)$ denotes the maximum number of maximum independent sets in a tree with $n$ vertices.

To deal with $E_{3}$, we need an analogue of the Triangle Lemma in this setting.
Lemma 4.6 (Strict Triangle Lemma) Suppose $G$ contains three vertices $\left\{v_{1}, v_{2}, v_{3}\right\}$ satisfying the following two restrictions.
(1) These vertices form a $K_{3}$ with $\operatorname{deg} v_{2}=2$ and $\operatorname{deg} v_{1}, \operatorname{deg} v_{3} \geq 3$.
(2) $G-N\left[v_{2}\right]$ is connected and at least one of $G-N\left[v_{1}\right]$ or $G-N\left[v_{3}\right]$ is connected.

Then $m^{\prime}(G)<m_{0}^{\prime}$.
Proof: Since $G-N\left[v_{2}\right]$ has $n-3$ vertices and less than $r_{1}-1$ cycles, we can use induction to conclude $m^{\prime}\left(G-N\left[v_{2}\right]\right) \leq c\left(n-3, r_{0}-2\right)$. Now using the $m^{\prime}$-recursion

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}\left(G-N\left[v_{2}\right]\right)+m^{\prime}\left(G-N\left[v_{1}\right]\right)+m^{\prime}\left(G-N\left[v_{3}\right]\right) \\
& \leq c\left(n-3, r_{0}-2\right)+c(n-4)+g(n-4) \\
& \leq m_{0}^{\prime}
\end{aligned}
$$

Equality forces $r_{0}=3, G-N\left[v_{2}\right] \cong C(7,1)$, and $G-N\left[v_{1}\right] \cong C(6)$ or $G(6)$. Considering numbers of cycles and containments gives a contradiction.

Now consider subdivisions of $E_{3}$. Let $v w$ be one of the doubled edges. By symmetry, we can assume that the cutvertex of $B$ in $G$ (if there is one) is neither in one of the $v w$ edges nor adjacent to $v$. If one of the $v w$ edges is subdivided more than once then $G$ is not extremal by (3b) of the Path Lemma and Lemma 4.4 (1). By Lemma 4.4 (2) the only other option is to have one edge subdivided once and the other not subdivided at all. But then the Strict Triangle Lemma shows that $G$ is not extremal.

Finally we come to $E_{4} \cong K_{4}$. First we claim that any edge $v w$ of $K_{4}$ that does not contain a cutvertex of $G$ cannot be subdivided. By the Path Lemma (3b) such an edge cannot be subdivided more than once. If it is subdivided exactly once, then we can use the $m^{\prime}$-bound twice and induction to get

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-v-w)+m^{\prime}(G-v-N[w])+m^{\prime}(G-N[v]) \\
& \leq c\left(n-3, r_{0}-2\right)+g(n-5)+g(n-4) \\
& <m_{0}^{\prime}
\end{aligned}
$$

So in order for $G$ to have at least 10 vertices this $K_{4}$ must contain a cutvertex $x$ of $G$. Suppose first that $x$ is a vertex of $K_{4}$ (before subdivision). Let $u, v, w$ be the other three vertices of $K_{4}$. Since none of the edges between these three vertices are subdivided, we can use (3a) of the Path Lemma to conclude that the edge $v x$ is subdivided at most once. If $v x$ is subdivided exactly once, then

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-v)+m^{\prime}(G-N[v]) \\
& \leq c\left(n-1, r_{0}-1\right)+c\left(n-4, r_{0}-2\right) \\
& <m_{0}^{\prime}
\end{aligned}
$$

If $x$ is interior to an edge of $K_{4}$, then taking $v$ to be a vertex of $K_{4}$ which is not adjacent to $x$ gives the same inequality, so $G$ is not extremal in this case either. This shows that if $B$ comes from subdividing $K_{4}$ then we must have $B \cong K_{4}$ and one of the vertices of $B$ is a cutvertex. We will return to eliminate this case at the end of the proof.

Like in the $n \equiv 1(\bmod 3)$ case of the first Gap Theorem, one can use Lemma 4.5 to rule out all descendants of the $E$ graphs. So now we know that all endblocks are copies of $K_{i}, 2 \leq i \leq 4$, or $D_{1}$.

The rest of our proof will parallel the last part of the demonstration of Theorem 3.4 There we were able to show that any two endblocks are disjoint. Here we will have to settle for showing that only copies of $D_{1}$ may intersect.

Suppse that the endlbocks $B$ and $B^{\prime}$ both contain the cutvertex $x$. If $B \cong K_{i}$ and $B^{\prime} \cong K_{j}$ where $2 \leq i, j \leq 4$ then Lemma 3.6 shows that

$$
m^{\prime}(G)=m^{\prime}(G-x) \leq(i-1)(j-1) g(n-i-j+1)<m_{0}^{\prime}
$$

unless $i=j=4$. But in that case removing $B$ and $B^{\prime}$ destroys 14 of the at most $r_{0}+5$ cycles and we can use the bound $m^{\prime}(G-x) \leq 9 g\left(n-7, r_{0}-9\right)<m_{0}^{\prime}$ instead.

Next consider the case where $B \cong K_{i}, 2 \leq i \leq 4$, and $B^{\prime} \cong D_{1}$ where $D_{1}$ is labeled as in Figure [8, Then using the $m^{\prime}$-bound, Lemma 3.6, and induction, we have

$$
m^{\prime}(G) \leq m^{\prime}(G-u)+m^{\prime}(G-N[u])
$$



Figure 9: Three graphs to eliminate

$$
\begin{aligned}
& \leq \begin{cases}2 g(n-5)+c\left(n-3, r_{0}-2\right) & \text { if } B \cong K_{2}, \\
4 g(n-6)+c\left(n-3, r_{0}-2\right) & \text { if } B \cong K_{3}, \\
6 g\left(n-7, r_{0}-5\right)+c\left(n-3, r_{0}-2\right) & \text { if } B \cong K_{4},\end{cases} \\
& <m_{0}^{\prime} .
\end{aligned}
$$

Hence we know that if two or more endblocks intersect at a cutvertex $x$ then they must all be copies of $D_{1}$. Now choose $B$ so that it intersects at most one other non-endblock (by Proposition (1.8). We can eliminate $K_{4}$ as a possibility for $B$ by labeling the cutvertex $x$ and counting cycles:

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-x)+m^{\prime}(G-N[x]) \\
& \leq 3 c\left(n-4, r_{0}-2\right)+g\left(n-5, r_{0}-2\right) \\
& <m_{0}^{\prime}
\end{aligned}
$$

If $B \cong D_{1}$ doesn't intersect another $D_{1}$ endblock then the Strict Triangle Lemma can be used to show that $G$ is not extremal. If $B \cong D_{1}$ intersects with $i$ other endblocks isomorphic to $D_{1}$ then we can adapt the proof of the Strict Triangle Lemma to show that $G$ is not extremal. Label the vertices of $B$ as in Figure 8, Using the $m^{\prime}$-recursion we get

$$
m^{\prime}(G) \leq m^{\prime}(G-N[u])+m^{\prime}(G-N[v])+m^{\prime}(G-N[w])
$$

If $i \geq 2$, then both $m^{\prime}(G-N[v])$ and $m^{\prime}(G-N[w])$ lie in the range of Theorem 1.5, and by applying induction to $G-N[u]$ we have

$$
m^{\prime}(G) \leq c\left(n-3, r_{0}-2\right)+2 \cdot 3^{i} c\left(n-3 i-4, r_{0}-3 i+2\right) .
$$

This is a decreasing function of $i$, and it is strictly less that $m_{0}^{\prime}$ when $i=2$.
This leaves the $i=1$ case. Here we use the bound

$$
m^{\prime}(G-N[v])=m^{\prime}(G-N[w]) \leq 3 \cdot c(n-7)
$$

and induction to get

$$
m^{\prime}(G) \leq 20 \cdot 3^{r_{0}-3}+7 \cdot 2^{r_{0}-3}
$$

This quantity is strictly less than $m_{0}^{\prime}$ for $r_{0} \geq 4$. As we are assuming $r_{0} \geq 3$, we have only to eliminate the case $r_{0}=3$. In this case $G$ has 10 vertices, of which 7 are accounted for by the two intersecting copies of $D_{1}$. This leaves 3 vertices unaccounted for, and since we have limited the types of endblocks that can occur, there are only three possible graphs of this description. These graphs are depicted in Figure 9 It is easy to check that $m^{\prime}\left(G_{1}\right)=2$, $m^{\prime}\left(G_{2}\right)=10$, and $m^{\prime}\left(G_{3}\right)=11$, which are all less than $c(10,2)=26$.

Hence if $B$ intersects precisely one non-endblock then $B$ is isomorphic to either $K_{2}$ or $K_{3}$, and thus by our previous work $B$ intersects precisely one other block, say $A$. We claim that $A \cong K_{2}$ in the other two cases. If not, then the cutvertex $x$ is adjacent to two vertices of a cycle in $A$. Using the $m^{\prime}$-bound as well as induction

$$
\begin{array}{rlr}
m^{\prime}(G) & \leq m^{\prime}(G-x)+m^{\prime}(G-N[x]) \\
& \leq \begin{cases}c(n-2)+g(n-4) & \text { if } B \cong K_{2}, \\
2 c\left(n-3, r_{0}-2\right)+g(n-5) & \text { if } B \cong K_{3}\end{cases} \\
& <m_{0}^{\prime} &
\end{array}
$$

for $r_{0} \geq 3$, proving our claim that $A \cong K_{2}$. Thus if an endblock $B$ intersects at most one non-endblock $A$ then $B$ is the only endblock intersecting $A, B \cong K_{2}$ or $K_{3}$, and $A \cong K_{2}$.

Suppose $B$ and $B^{\prime}$ are endblocks of the type considered in the previous paragraph with cutvertices $x$ and $x^{\prime}$, respectively. We now claim that the associated $K_{2}$ blocks must have vertex sets $\left\{x, v_{0}\right\}$ and $\left\{x^{\prime}, v_{0}\right\}$ for some $v_{0}$. Suppose not and consider first the case $B \cong B^{\prime} \cong K_{2}$. The same argument as in Theorem 3.4 shows that

$$
\begin{aligned}
m^{\prime}(G) & \leq m^{\prime}(G-B)+m^{\prime}\left(G-N[x]-B^{\prime}\right)+m^{\prime}\left(G-N[x]-N\left[x^{\prime}\right]\right) \\
& \leq c(n-2)+g(n-5)+g(n-6) \\
& <m_{0}^{\prime}
\end{aligned}
$$

for $r_{0} \geq 3$, so such graphs are not extremal. Now suppose that $B \cong K_{3}$ and $B^{\prime} \cong K_{2}$. In order to apply induction, it is important to use the $m^{\prime}$-recursion first on $B$ and then on $B^{\prime}$ to get

$$
\begin{aligned}
m^{\prime}(G) & \leq 2 m^{\prime}(G-B)+m^{\prime}\left(G-N[x]-B^{\prime}\right)+m^{\prime}\left(G-N[x]-N\left[x^{\prime}\right]\right) \\
& \leq 2 c\left(n-3, r_{0}-2\right)+g(n-6)+g(n-7) \\
& <m_{0}^{\prime}
\end{aligned}
$$

again resulting in a non-extremal graph. Finally, if $B \cong B^{\prime} \cong K_{3}$ then

$$
\begin{aligned}
m^{\prime}(G) & \leq 2 m^{\prime}(G-B)+2 m^{\prime}\left(G-N[x]-B^{\prime}\right)+m^{\prime}\left(G-N[x]-N\left[x^{\prime}\right]\right) \\
& \leq 2 c\left(n-3, r_{0}-2\right)+2 g(n-7)+g(n-8) \\
& =m_{0}^{\prime}
\end{aligned}
$$

Equality can only be achieved if $G-B \cong C\left(n-3, r_{0}-2\right)$. But then $r(G-B)=r_{0}-2$ and so $r(G)=r_{0}-1$, and then Theorem 3.1] implies that $G \cong C\left(n, r_{0}-1\right)$ as desired.

Now that $v_{0}$ must exist, the possibilities for other blocks in $G$ are severely limited: $G$ can have no other blocks, or a $K_{2}, K_{3}$, or $K_{4}$ endblock containing $v_{0}$, or any number of $D_{1}$ endblocks which intersect at $v_{0}$. Checking the cases where $G$ contains either a complete block or no other block gives us either a graph which is either not extremal or isomorphic to $C\left(n, r_{0}-1\right)$ if that block is isomorphic to $K_{3}$. Now suppose that $G$ contains one or more copies of $D_{1}$ endblocks intersecting at $v_{0}$. Let $u_{1}, u_{2}, \ldots, u_{i}$ denote the vertices of these blocks that are not adjacent to $v_{0}$. Every maximum independent set in $G$ must contain $\left\{v_{0}, u_{1}, u_{2}, \ldots, u_{i}\right\}$, and from this it is easy to see that such graphs are not extremal. This completes the proof of Theorem 4.1

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