# A structure theorem for graphs with no cycle with a unique chord and its consequences 

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#### Abstract

We give a structural description of the class $\mathcal{C}$ of graphs that do not contain a cycle with a unique chord as an induced subgraph. Our main theorem states that any connected graph in $\mathcal{C}$ is either in some simple basic class or has a decomposition. Basic classes are chordless cycles, cliques, bipartite graphs with one side containing only nodes of degree two and induced subgraphs of the famous Heawood or Petersen graph. Decompositions are node cutsets consisting of one or two nodes and edge cutsets called 1 -joins. Our decomposition theorem actually gives a complete structure theorem for $\mathcal{C}$, i.e. every graph in $\mathcal{C}$ can be built from basic graphs that can be explicitly constructed, and gluing them together by prescribed composition operations; and all graphs built this way are in $\mathcal{C}$.

This has several consequences: an $\mathcal{O}(n m)$-time algorithm to decide whether a graph is in $\mathcal{C}$, an $\mathcal{O}(n+m)$-time algorithm that finds a maximum clique of any graph in $\mathcal{C}$ and an $\mathcal{O}(n m)$-time coloring algorithm for graphs in $\mathcal{C}$. We prove that every graph in $\mathcal{C}$ is either 3 -colorable or has a coloring with $\omega$ colors where $\omega$ is the size of a largest clique. The problem of finding a maximum stable set for a graph in $\mathcal{C}$ is known to be NP-hard.


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## 1 Motivation

In this paper all graphs are simple. We give a structural characterization of graphs that do not contain a cycle with a unique chord as an induced

[^0]subgraph. For the sake of conciseness we call $\mathcal{C}$ this class of graph. Our main result, Theorem 2.1, states that every connected graph in $\mathcal{C}$ is either in some simple basic class or has a particular decomposition. Basic classes are chordless cycles, cliques, bipartite graphs with one side containing only nodes of degree two and graphs that are isomorphic to an induced subgraph of the famous Heawood or Petersen graph. Our decompositions are node cutsets consisting of one or two nodes or an edge cutset called a 1-join. The definitions and the precise statement are given in Section 2. The proof is given in Section 3. Both Petersen and Heawood graphs were discovered at the end of the XIXth century in the research on the four color conjecture, see [20] and [13]. It is interesting to us to have them both as sporadic basic graphs. Note that our theorem works in two directions: a graph is in $\mathcal{C}$ if and only if it can be constructed by gluing basic graphs along our decompositions (this is proved in Section (4). Such structure theorems are stronger than the usual decomposition theorems and there are not so many of them (see [3] for a survey). This is our first motivation.

Our structural characterization allows us to prove properties of classical invariants. We prove in Section 6 that every graph $G$ in $\mathcal{C}$ satisfies either $\chi(G)=\omega(G)$ or $\chi(G) \leq 3$ (where $\chi(G)$ denotes the chromatic number and $\omega(G)$ denotes the size of a maximum clique). This is a strengthening of the classical Vizing bound $\chi(G) \leq \omega(G)+1$. So this class of graphs belongs to the family of $\chi$-bounded graphs, introduced by Gyárfás [12] as a natural extension of perfect graphs: a family of graphs $\mathcal{G}$ is $\chi$-bounded with $\chi$-binding function $f$ if, for every induced subgraph $G^{\prime}$ of $G \in \mathcal{G}, \chi\left(G^{\prime}\right) \leq f\left(\omega\left(G^{\prime}\right)\right)$. A natural question to ask is: what choices of forbidden induced subgraphs guarantee that a family of graphs is $\chi$-bounded? Much research has been done in this area, for a survey see [22. We note that most of that research has been done on classes of graphs obtained by forbidding a finite number of graphs. Since there are graphs with arbitrarily large chromatic number and girth [11], in order for a family of graphs defined by forbidding a finite number of graphs (as induced subgraphs) to be $\chi$-bounded, at least one of these forbidden graphs needs to be acyclic. Vizing's Theorem [25] states that for a simple graph $G, \Delta(G) \leq \chi^{\prime}(G) \leq \Delta(G)+1$ (where $\Delta(G)$ denotes the maximum vertex degree of $G$, and $\chi^{\prime}(G)$ denotes the chromatic index of $G$, i.e. the minimum number of colors needed to color the edges of $G$ so that no two adjacent edges receive the same color). This implies that the class of line graphs of simple graphs is a $\chi$-bounded family with $\chi$-binding function $f(x)=x+1$. This special upper bound for the chromatic number is called the Vizing bound. We obtain the Vizing bound for the chromatic number by forbidding a family of graphs none of which is acyclic. Our result is algorithmic: we provide an $\mathcal{O}(\mathrm{nm})$ algorithm that computes an optimal coloring of every graph in $\mathcal{C}$. Furthermore, it is easy to see that there exists an $\mathcal{O}(n+m)$ algorithm that computes a maximum clique for every graph


Figure 1: Some s-graphs
in $\mathcal{C}$; and it follows from a construction of Poljak [21 that finding a maximum stable set of a graph in $\mathcal{C}$ is NP-hard (see Section (7). All this is our second motivation.

A third motivation is the detection of induced subgraphs. A subdivisible graph (s-graph for short) is a triple $B=(V, D, F)$ such that $(V, D \cup F)$ is a graph and $D \cap F=\emptyset$. The edges in $D$ are said to be real edges of $B$ while the edges in $F$ are said to be subdivisible edges of $B$. A realisation of $B$ is a graph obtained from $B$ by subdividing edges of $F$ into paths of arbitrary length (at least one). The problem $\Pi_{B}$ is the decision problem whose input is a graph $G$ and whose question is "Does $G$ contain a realisation of $B$ as an induced subgraph?". In the discussion below, by "detection problem", we mean "problem $\Pi_{B}$ for some fixed s-graph $B$ ". This is restrictive since a lot of detection problems of great interest (such as the detection of odd holes, where a hole is an induced cycle of length at least four) are not of that kind.

Let $H_{1 \mid 1}$ be the s-graph on nodes $a, b, c, d$ with real edges $a b, a c, a d$ and subdivisible edges $b d, c d$. We also define for $k, l \geq 1$ the s-graph $H_{k \mid l}$ obtained from $H_{1 \mid 1}$ by subdividing the edge $a b$ into a path of length $k$ and the edge $a c$ into a path of length $l$. See Fig. [ where real edges are represented as straight lines and subdivisible edges as dashed lines. The question in Problem $\Pi_{H_{1 \mid 1}}$ can be rephrased as "Does $G$ contain a cycle with a unique chord?" or "Is $G$ not in $\mathcal{C}$ ?". The existence of a polynomial time algorithm was an open question. A consequence of our structural description of $\mathcal{C}$ is an $\mathcal{O}(n m)$-time algorithm for $\Pi_{H_{111}}$ (see Section (5). This is a solution to the recognition problem for the class $\mathcal{C}$ and it is interesting for reasons explained below.

Several problem $\Pi_{B}$ 's can be solved in polynomial time by non-trivial algorithms (such as detecting pyramids in [2] and thetas in [4) and others


Figure 2: Some s-graphs
that may look similar at first glance are NP-complete (see [1, [18, and 16] for a survey). A general criterion on an s-graph that decides whether the related decision problem is NP-complete or polynomial would be of interest. Our solution of $\Pi_{H_{1 \mid 1}}$ gives some insight in the quest for such a criterion.

A very powerful tool for solving detection problems is the algorithm three-in-a-tree of Chudnovsky and Seymour (see [4). This algorithm decides in time $\mathcal{O}\left(n^{4}\right)$ whether three given nodes of a given graph $G$ are in an induced tree of $G$. In [4] and [16] it is observed that every detection problem $\Pi_{B}$ for which a polynomial time algorithm is known can be solved easily by a brute force enumeration or by using three-in-a-tree. But as far as we can see, three-in-a-tree cannot be used to solve $\Pi_{H_{1 \mid 1}}$, so our solution of $\Pi_{H_{1 \mid 1}}$ yields the first example of a detection problem that does not fall under the scope of three-in-a-tree. Is there a good reason for that? We claim that a polynomial time algorithm for $\Pi_{H_{1 \mid 1}}$ exists thanks to what we call degeneracy. Let us explain this. Every statement that we give from here on to the end of the section is under the assumption that $\mathrm{P} \neq \mathrm{NP}$.

Degeneracy has to deal with the following question: does putting bounds on the lengths of the paths in realisations of an s-graph affect the complexity of the related detection problem? For upper bounds, the answer can be found in previous research. First, putting upper bounds may turn the complexity from NP-complete to polynomial. This follows from a simple observation: let $B$ be any s-graph. A realisation of $B$, where the lengths of the paths arising from the subdivisions of subdivisible edges are bounded by an integer $N$, has a number of nodes bounded by a fixed integer $N^{\prime}$ (that depends only on $N$ and the size of $B$ ). So, such a realisation can be detected in time $\mathcal{O}\left(n^{N^{\prime}}\right)$ by a brute force enumeration. But surprisingly, putting upper bounds in another way may also turn the complexity from polynomial to NP-complete: in [2], a polynomial time algorithm for $\Pi_{K}$ is given, while in [19] it is proved that $\Pi_{K^{\prime}}$ is NP-complete, where $K, K^{\prime}$ are the s-graphs represented in Figure 2, Note that $\Pi_{K}$ is usually called the pyramid (or $3 \mathrm{PC}(\Delta, \cdot))$ detection problem.

Can putting lower bounds turn the complexity from polynomial to NPcomplete? Our recognition algorithm for $\mathcal{C}$ shows that the answer is yes since in Section $\square$ we also prove that the problem $\Pi_{H_{3 \mid 3}}$ is NP-complete. A realisation of $H_{3 \mid 3}$ is simply a realisation of $H_{1 \mid 1}$ where every subdivisible
edge is subdivided into a path of length at least three. We believe that a satisfactory structural description of the class $\mathcal{C}^{\prime}$ of graphs that do not contain a realisation of $H_{3 \mid 3}$ is hopeless because $\Pi_{H_{3 \mid 3}}$ is NP-complete. So why is there a decomposition theorem for $\mathcal{C}$ ? Simply because degenerate small graphs like the diamond (that is the cycle on four nodes with exactly one chord) are forbidden in $\mathcal{C}$, not in $\mathcal{C}^{\prime}$, and this helps a lot in our proof of Theorem 2.1 (the decomposition theorem for $\mathcal{C}$ ). This is what we call the degeneracy of the class $\mathcal{C}$. It is clear that degeneracy can help in solving detection problems, and our results give a first example of this phenomenon.

So the last question is: can putting lower bounds turn the complexity from NP-complete to polynomial? We do not know the answer. Also, we were not able to solve the following questions: what is the complexity of the problems $\Pi_{H_{2 \mid 1}}, \Pi_{H_{3 \mid 1}}, \Pi_{H_{2 \mid 2}}$ and $\Pi_{H_{3 \mid 2}}$ ? The related classes of graphs are not degenerate enough to allow us to decompose, and they are too degenerate to allow us to find an NP-completeness proof.

A fourth motivation is that our class $\mathcal{C}$ is related to well studied classes. It is a generalization of strongly balanceable graphs, see [7] for a survey. A bipartite graph is balanceable if there exists a signing of its edges with +1 and -1 so that the weight of every hole is a multiple of 4 . A bipartite graph is strongly balanceable if it is balanceable and it does not contain a cycle with a unique chord. There is an excluded induced subgraph characterization of balanceable bipartite graphs due to Truemper [24]. A wheel in a graph consists of a hole $H$ and a node $v$ that has at least three neighbors in $H$, and the wheel is odd if $v$ has an odd number of neighbors in $H$. In a bipartite graph $G$, a 3-odd-path configuration consists of two nonadjacent nodes $u$ and $v$ that are on opposite sides of the bipartition of $G$, together with three internally node-disjoint $u v$-paths, such that there are no other edges in $G$ among the nodes of the three paths. A bipartite graph is balanceable if and only if it does not contain an odd wheel nor a 3-odd-path configuration [24]. So a bipartite graph is strongly balanceable if and only if it does not contain a 3-odd-path configuration nor a cycle with a unique chord.

A bipartite graph is restricted balanceable if there exists a signing of its edges with +1 and -1 so that the weight of every cycle is a multiple of 4. Conforti and Rao [8] show that a strongly balanceable graph is either restricted balanceable or has a 1-join, which enables them to recognize the class of strongly balanceable graphs (they decompose along 1-joins, and then directly recognize restricted balanceable graphs). A bipartite graph is 2-bipartite if all the nodes in one side of the bipartition have degree at most 2. Yannakakis [26] shows that a restricted balanceable graph is either 2-bipartite or has a 1-cutset or a 2-join consisting of two edges (this is an edge cutset that consists of two edges that have no common endnode), and hence obtains a linear time recognition algorithm for restricted balanceable graphs.

We note that the basic graphs from our decomposition theorem that do not have any of our cutsets, and are balanceable, are in fact 2-bipartite.

Class $\mathcal{C}$ is contained in another well studied class of graphs, the cap-free graphs (where a cap is a graph that consists of a hole and a node that has exactly two neighbors on this hole, and these two neighbors are adjacent) [6]. In [6] cap-free graphs are decomposed with 1 -amalgams (a generalization of a 1-join) into triangulated graphs and biconnected triangle-free graphs together with at most one additional node that is adjacent to all other nodes of the graph. This decomposition theorem is then used to recognize strongly even-signable and strongly odd-signable graphs in polynomial time, where a graph is strongly even-signable if its edges can be signed with 0 and 1 so that every cycle of length $\geq 4$ with at most one chord has even weight and every triangle has odd weight, and a graph is strongly odd-signable if its edges can be signed with 0 and 1 so that cycles of length 4 with one chord are of even weight and all other cycles with at most one chord are of odd weight.

## 2 The main theorem

We say that a graph $G$ contains a graph $H$ if $H$ is isomorphic to an induced subgraph of $G$. A graph $G$ is $H$-free if it does not contain $H$. For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of $G$ induced by $S$. A cycle $C$ in a graph $G$ is a sequence of nodes $v_{1} v_{2} \ldots v_{n} v_{1}$, that are distinct except for the first and the last node, such that for $i=1, \ldots, n-1, v_{i} v_{i+1}$ is an edge and $v_{n} v_{1}$ is an edge (these are the edges of $C$ ). An edge of $G$ with both endnodes in $C$ is called a chord of $C$ if it is not an edge of $C$. One can similarly define a path and a chord of a path. In this paper we will only use what is in literature known as chordless paths, so for the convenience, in this paper (like in [5]) we define a path as follows: a path $P$ in a graph $G$ is a sequence of distinct nodes $v_{1} \ldots v_{n}$ such that for $i=1, \ldots, n-1, v_{i} v_{i+1}$ is an edge and these are the only edges of $G$ that have both endnodes in $\left\{v_{1}, \ldots, v_{n}\right\}$. Such a path $P$ is also called a $v_{1} v_{n}$-path. A hole is a chordless cycle of length at least four. A triangle is a cycle of length 3. A square is a hole of length 4. A cycle in a graph is Hamiltonian is every node of the graph is in the cycle. Let us define our basic classes:

The Petersen graph is the graph on nodes $\left\{a_{1}, \ldots a_{5}, b_{1}, \ldots, b_{5}\right\}$ so that $\left\{a_{1}, \ldots, a_{5}\right\}$ and $\left\{b_{1}, \ldots, b_{5}\right\}$ both induce a $C_{5}$ with nodes in their natural order, and such that the only edges between the $a_{i}$ 's and the $b_{i}$ 's are $a_{1} b_{1}$, $a_{2} b_{4}, a_{3} b_{2}, a_{4} b_{5}, a_{5} b_{3}$. See Fig. 3.

The Heawood graph is the graph on $\left\{a_{1}, \ldots, a_{14}\right\}$ so that $\left\{a_{1}, \ldots, a_{14}\right\}$ is a Hamiltonian cycle with nodes in their natural order, and such that the only other edges are $a_{1} a_{10}, a_{2} a_{7}, a_{3} a_{12}, a_{4} a_{9}, a_{5} a_{14}, a_{6} a_{11}, a_{8} a_{13}$. See Fig. (4.

It can be checked that both Petersen and Heawood graph are in $\mathcal{C}$. Note that since the Petersen graph and the Heawood graph are both vertex-


Figure 3: Four ways to draw the Petersen graph
transitive, and are not themselves a cycle with a unique chord, to check that they are in $\mathcal{C}$, it suffices to delete one node, and then check that there is no cycle with a unique chord. Also the Petersen graph has girth 5 so a cycle with a unique chord in it must contain at least 8 nodes. The Heawood graph has girth 6 so a cycle with a unique chord in it must contain at least 10 nodes. For the Petersen graph, deleting a node yields an Hamiltonian graph, and it is easy to check that it does not contain a cycle with a unique chord. For the Heawood graph, it is useful to notice that deleting one node yields the Petersen graph with edges $a_{1} b_{1}, b_{3} b_{4}, a_{3} a_{4}$ subdivided.

Let us define our last basic class. A graph is strongly 2-bipartite if it is square-free and bipartite with bipartition $(X, Y)$ where $X$ is the set of all degree 2 nodes of $G$ and $Y$ is the set of all nodes of $G$ with degree at least 3 . A strongly 2-bipartite graph is clearly in $\mathcal{C}$ because any chord of a cycle is an edge linking two nodes of degree at least three, so every cycle in a strongly 2-bipartite graph is chordless.

We now define cutsets used in our decomposition theorem:

- A 1 -cutset of a connected graph $G$ is a node $v$ such that $V(G)$ can be


Figure 4: Four ways to draw the Heawood graph
partitioned into non-empty sets $X, Y$ and $\{v\}$, so that there is no edge between $X$ and $Y$. We say that $(X, Y, v)$ is a split of this 1-cutset.

- A proper 2-cutset of a connected graph $G$ is a pair of non-adjacent nodes $a, b$, both of degree at least three, such that $V(G)$ can be partitioned into non-empty sets $X, Y$ and $\{a, b\}$ so that: $|X| \geq 2,|Y| \geq 2$; there are no edges between $X$ and $Y$; and both $G[X \cup\{a, b\}]$ and $G[Y \cup\{a, b\}]$ contain an $a b$-path. We say that $(X, Y, a, b)$ is a split of this proper 2-cutset.
- A 1-join of a graph $G$ is a partition of $V(G)$ into sets $X$ and $Y$ such that there exist sets $A, B$ satisfying:
$-\emptyset \neq A \subseteq X, \emptyset \neq B \subseteq Y$;
$-|X| \geq 2$ and $|Y| \geq 2$;
- there are all possible edges between $A$ and $B$;
- there are no other edges between $X$ and $Y$.

We say that $(X, Y, A, B)$ is a split of this 1 -join. The sets $A, B$ are special sets with respect to this 1-join.
1-Joins were first introduced by Cunningham [9]. In our paper we will use a special type of a 1-join called a proper 1-join: a 1-join such that $A$ and $B$ are stable sets of $G$ of size at least two. Note that a square admits a proper 1-join.

Our main result is the following decomposition theorem:
Theorem 2.1 Let $G$ be a connected graph that does not contain a cycle with a unique chord. Then either $G$ is strongly 2-bipartite, or $G$ is a hole of length at least 7, or $G$ is a clique, or $G$ is an induced subgraph of the Petersen or the Heawood graph, or G has a 1-cutset, a proper 2-cutset, or a proper 1-join.

The following intermediate results are proved in the next section. Theorem 2.1 follows from Theorems 2.3 and 2.4 (more precisely, it follows from 2.4 for square-free graphs, and from 2.3 for graphs that contain a square).

Theorem 2.2 Let $G$ be a connected graph that does not contain a cycle with a unique chord. If $G$ contains a triangle then either $G$ is a clique, or one node of the maximal clique that contains this triangle is a 1-cutset of $G$.

Theorem 2.3 Let $G$ be a connected graph that does not contain a cycle with a unique chord. Suppose that $G$ contains either a square, the Petersen graph or the Heawood graph. Then either $G$ is the Petersen graph or $G$ is the Heawood graph or G has a 1-cutset or a proper 1-join.

Theorem 2.4 Let $G$ be a connected square-free graph that does not contain a cycle with a unique chord. Then either $G$ is strongly 2-bipartite, or $G$ is a hole of length at least 7, or $G$ is a clique or $G$ is an induced subgraph of the Petersen or the Heawood graph, or G has a 1-cutset or a proper 2-cutset.

## 3 Proof of Theorems [2.2, 2.3 and 2.4

We first need two lemmas:

Lemma 3.1 Let $G$ be a graph in $\mathcal{C}, H$ a hole of $G$ and $v$ a node of $G \backslash H$. Then $v$ has at most two neighbors in $H$, and these two neighbors are not adjacent.

Proof - If $v$ has at least three neighbors in $H$, then $H$ contains a subpath $P$ with exactly three neighbors of $v$ and $V(P) \cup\{v\}$ induces a cycle of $G$ with a unique chord, a contradiction. If $v$ has two neighbors in $H$, they
must be non-adjacent for otherwise $H \cup\{v\}$ is a cycle with a unique chord.

In a connected graph $G$ two nodes $a$ and $b$ form a 2-cutset if $G \backslash\{a, b\}$ is disconnected.

Lemma 3.2 Let $G$ be a connected graph that has no 1-cutset. If $\{a, b\}$ is a 2-cutset of $G$ and $a b$ is an edge, then $G \notin \mathcal{C}$.

Proof - Suppose $\{a, b\}$ is a 2-cutset of $G$, and $a b$ is an edge. Let $C_{1}, \ldots, C_{k}$ be the connected components of $G \backslash\{a, b\}$. Since $G$ is connected and has no 1 -cutset, for every $i \in\{1, \ldots, k\}$, both $a$ and $b$ have a neighbor in $C_{i}$. Let $G^{\prime}$ be the graph obtained from $G$ by removing the edge $a b$. So for every $i \in\{1, \ldots, k\}$, there is an $a b$-path $P_{i}$ in $G^{\prime}$ whose interior nodes are contained in $C_{i}$. Then $G\left[V\left(P_{1}\right) \cup V\left(P_{2}\right)\right]$ is a cycle with a unique chord, and hence $G \notin \mathcal{C}$.

If $H$ is any induced subgraph of $G$ and $D$ is a subset of nodes of $G \backslash H$, the attachment of $D$ over $H$ is the set of all nodes of $H$ that have at least one neighbor in $D$. When clear from context we do not distinguish between a graph and its node set, so we also refer to the attachment of $G[D]$ over $H$.

## Proof of Theorem 2.2

Suppose $G$ contains a triangle, and let $C$ be a maximal clique of $G$ that contains this triangle. In fact, $C$ is unique or otherwise $G$ contains a diamond. If $G \neq C$ and if no node of $C$ is a 1-cutset of $G$ then let $D$ be a connected induced subgraph of $G \backslash C$, whose attachment over $C$ contains at least two nodes, and that is minimal with respect to this property. So, $D$ is a path with one end adjacent to $a \in C$, the other end adjacent to $b \in C \backslash\{a\}$ and $D \cup\{a, b\}$ induces a chordless cycle. If $D$ has length zero, then its unique node (say $u$ ) must have a non-neighbor $c \in C$ since $C$ is maximal. Hence, $\{u, a, b, c\}$ induces a diamond, a contradiction. If $D$ has length at least one then let $c \neq a, b$ be any node of $C$. Then the hole induced by $D \cup\{a, b\}$ and node $c$ contradict Lemma 3.1. This proves Theorem 2.2.

## Proof of Theorem 2.3

Claim 1 We may assume that $G$ is triangle-free.
Proof - Clear by Theorem 2.2 (note that $G$ cannot be a clique).

Claim 2 We may assume that $G$ is square-free.
Proof - Assume $G$ contains a square. Then $G$ contains disjoint sets of nodes $A$ and $B$ such that $G[A]$ and $G[B]$ are both stable graphs, $|A|,|B| \geq 2$
and every node of $A$ is adjacent to every node of $B$. Let us suppose that $A \cup B$ is chosen to be maximal with respect to this property. If $V(G)=A \cup B$ then $(A, B)$ is a proper 1 -join of $G$, so we may assume that there are nodes in $G \backslash(A \cup B)$.
(1) Every component of $G \backslash(A \cup B)$ has neighbors only in $A$ or only in $B$.

Else, let us take a connected induced subgraph $D$ of $G \backslash(A \cup B)$, whose attachment over $A \cup B$ contains nodes of both $A$ and $B$, and that is minimal with respect to this property. So $D=u \ldots v$ is a path, no interior node of which has a neighbor in $A \cup B$ and there exists $a \in A, b \in B$ such that $u a, v b \in E(G)$. By Claim 亿 $u \neq v, u$ has no neighbor in $B$ and $v$ has no neighbor in $A$. By maximality of $A \cup B, u$ has a non-neighbor $a^{\prime} \in A$ and $v$ has a non-neighbor $b^{\prime} \in B$. Now, $D \cup\left\{a, b, a^{\prime}, b^{\prime}\right\}$ is a cycle with a unique chord (namely $a b$ ), a contradiction. This proves (1).

From (1), it follows that $G$ has a proper 1 -join with special sets $A, B$.
Now, we just have to prove the following two claims:
Claim 3 If $G$ contains the Petersen graph then the theorem holds.
Proof - Let $\Pi=\left\{a_{1}, \ldots a_{5}, b_{1}, \ldots, b_{5}\right\}$ be a set of ten nodes of $G$ so that $G[\Pi]$ has adjacencies like in the definition of the Petersen graph. We may assume that there are some other nodes in $G$ for otherwise the theorem holds.
(1) $A$ node of $G \backslash \Pi$ has at most one neighbor in $\Pi$.

Otherwise $G$ contains a triangle or a square, contrary to Claims 1. 2, This proves (1).

Here below, we use symmetries in the Petersen graph to shorten the list of cases. First, the Petersen graph is edge-transitive, so up to an automorphism, all edges are equivalent. But also, it is "distance-two-transitive", meaning that every induced $P_{3}$ is equivalent to every other induced $P_{3}$. To see this, it suffices to check that every induced $P_{3}$ is included in an induced $C_{5}$ and that removing any $P_{3}$ always yields the same graph.
(2) The attachment of any component of $G \backslash \Pi$ over $\Pi$ contains at most one node.

Else, let $D$ be a connected induced subgraph of $G \backslash \Pi$ whose attachment over $\Pi$ contains at least two nodes, and that is minimal with respect to this property. By minimality and up to symmetry, $D$ is a path with one end adjacent to $a_{1}$ (and to no other node of $\Pi$ by (11)), one end adjacent to $x \in\left\{a_{2}, a_{3}\right\}$ (and to no other node of $\Pi$ ). Moreover, no interior node of $D$ has a neighbor in $\Pi$. If $x=a_{2}$ then $D \cup\left\{a_{1}, \ldots, a_{5}\right\}$ is a cycle with a unique
chord, a contradiction. If $x=a_{3}$ then $D \cup\left\{a_{1}, a_{5}, b_{3}, b_{4}, b_{5}, a_{4}, a_{3}\right\}$ is a cycle with a unique chord, a contradiction again. This proves (2).

From (2) it follows that $G$ has a 1-cutset.

Claim 4 If $G$ contains the Heawood graph then the theorem holds.
Proof - Let $\Pi=\left\{a_{1}, \ldots, a_{14}\right\}$ be a set of fourteen nodes of $G$ so that $G[\Pi]$ has adjacencies like in the definition of the Heawood graph. We may assume that there are some other nodes in $G$ for otherwise the theorem holds.
(1) A node of $G \backslash \Pi$ has at most two neighbors in $\Pi$.

Suppose that some node $v$ in $G \backslash \Pi$ has at least two neighbors in $\Pi$. Since the Heawood graph is vertex-transitive we may assume $v a_{1} \in E(G)$. By Claims 1 and 2, $v$ cannot be adjacent to a node at distance 1 or 2 from $v$, namely to any of $a_{2}, a_{14}, a_{10}, a_{3}, a_{13}, a_{9}, a_{11}, a_{5}, a_{7}$. So, the only other possible neighbors are $a_{4}, a_{6}, a_{8}, a_{12}$. But these four nodes are pairwise at distance two in $\Pi$, so by Claim 2, $v$ can be adjacent to at most one of them. This proves (1).
(2) The attachment of any component of $G \backslash \Pi$ over $\Pi$ contains at most one node.

Else, let $D$ be a connected induced subgraph of $G \backslash \Pi$ whose attachment over $\Pi$ contains at least two nodes, and is minimal with respect to this property. By minimality and up to symmetry, $D$ is a path, possibly of length zero, with one end adjacent to $a_{1}$, one end adjacent to $a_{i}$ where $i \neq 1$ and no interior node of $D$ has neighbors in $\Pi$. Note that by assumption, if $D$ is of length at least one then no end of $D$ can have more than one neighbor in $\Pi$, because such an end would contradict the minimality of $D$. So, $a_{1}, a_{i}$ are the only nodes of $\Pi$ that have neighbors in $D$ (when $D$ is of length zero this holds by (11).

If $i=2$ then $D \cup\left\{a_{1}, a_{2}, a_{7}, a_{8}, a_{9}, a_{10}\right\}$ is a cycle with a unique chord. If $i \in\{3,4,5\}$ then $D \cup\left\{a_{i}, a_{i+1}, \ldots, a_{8}, a_{13}, a_{14}, a_{1}\right\}$ is a cycle with a unique chord. If $i=6$ then $D \cup\left\{a_{6}, a_{5}, a_{4}, a_{9}, a_{8}, a_{13}, a_{14}, a_{1}\right\}$ is a cycle with a unique chord. If $i \in\{7,8\}$ then $D \cup\left\{a_{1}, a_{2}, \ldots, a_{i}\right\}$ is a cycle with a unique chord. If $i=9$ then $D \cup\left\{a_{9}, \ldots, a_{14}\right\}$ is a cycle with a unique chord. If $i \in\{10,11\}$ then $D \cup\left\{a_{1}, a_{2}, a_{3}, a_{4}, a_{9}, a_{10}, a_{i}\right\}$ is a cycle with a unique chord. If $i \in\{12,13\}$ then $D \cup\left\{a_{1}, a_{2}, a_{3}, a_{4}, a_{5}, a_{6}, a_{11}, a_{12}, a_{i}\right\}$ is a cycle with a unique chord. If $i=14$ then $D \cup\left\{a_{1}, a_{2}, a_{7}, a_{8}, a_{13}, a_{14}\right\}$ is a cycle with a unique chord. In every case, there is a contradiction. This proves (2).

From (2) it follows that $G$ has a 1-cutset.
This proves Theorem 2.3.

## Proof of Theorem 2.4

We consider a graph $G$ containing no cycle with a unique chord and no square. So:

Claim $1 G$ is square-free.
Our proof now goes through thirteen claims, most of them of the same kind: if some basic graph $H$ is an induced subgraph of $G$, then either $G=H$ and so $G$ itself is basic, or some nodes of $G \backslash H$ must be attached to $H$ in a way that entails a proper 2 -cutset. At the end of this process there are so many induced subgraphs forbidden in $G$ that we can prove that $G$ is strongly 2 -bipartite.

Claim 2 We may assume that $G$ is triangle-free.
Proof - Clear by Theorem 2.2.

Claim 3 We may assume that $G$ does not contain the Petersen graph.
Proof - Clear by Theorem [2.3. Note that $G$ cannot admit a proper 1-join since it is square-free.

Claim 4 We may assume that $G$ does not contain the Heawood graph.
Proof - Clear by Theorem 2.3,

Claim 5 We may assume that $G$ does not contain the following configuration: three node-disjoint paths $X=x \ldots x^{\prime}, Y=y \ldots y^{\prime}$ and $Z=z \ldots z^{\prime}$, of length at least two and with no edges between them. There are four more nodes $a, b, c, d$. The only edges except those from the paths are $a x, a y, a z, b x^{\prime}, b y, b z^{\prime}, c x^{\prime}, c y^{\prime}, c z, d x, d y^{\prime}, d z^{\prime}$.

Proof - Let $\Pi=X \cup Y \cup Z \cup\{a, b, c, d\}$. Nodes $a, b, c, d, x, x^{\prime}, y, y^{\prime}, z, z^{\prime}$ are called here the branch nodes of $\Pi$. It is convenient to notice that $G[\Pi]$ can be obtained by subdividing the edges of any induced matching of size three of the Petersen graph. Note also that either $G[\Pi]$ is the Heawood graph with one node deleted (when $X, Y, Z$ all have length two), or $G[\Pi]$ has a proper 2-cutset (when one of the paths is of length at least three, the proper 2 -cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \backslash \Pi$.
(1) A node of $G \backslash \Pi$ has neighbors in at most one of the following sets: $X, Y, Z,\{a\},\{b\},\{c\},\{d\}$.

Let $u$ be a node of $G \backslash \Pi$. Note that $a, b, c, d$ are pairwise at distance two in $\Pi$, so by Claims 1 and 2, $u$ can be adjacent to at most one of them.

Suppose first that $u x \in E(G)$. Then $u$ can be adjacent to none of $a, d, y, y^{\prime}, z, z^{\prime}$ by Claims 1 and 2. If $u$ is adjacent to some other branch-node of $G[\Pi]$ distinct from $x^{\prime}$, then we may assume that $u$ is adjacent to one of $b, c$ (say $b$ up to symmetry), but then $u$ is not adjacent to $c$ so $u x d y^{\prime} c z a y b u$ is a cycle with a unique chord, a contradiction. Hence we may assume that $u$ has a neighbor $v$ in the interior of $Y$ or $Z$ (say $Z$ up to symmetry) for otherwise the claim holds. By Lemma 3.1 and since $x d z^{\prime} Z z c x^{\prime} X x$ is a hole, we note that $u$ has exactly two neighbors in $X \cup Z$, namely $x$ and $v$. If $v z^{\prime} \notin E(G)$ then $x u v Z z a y b z^{\prime} d x$ is a cycle with a unique chord, a contradiction. So $v z^{\prime} \in E(G)$ and $x u v Z z c x^{\prime} b z^{\prime} d x$ is a cycle with a unique chord (namely $\left.v z^{\prime}\right)$, a contradiction again. Hence we may assume that $u x \notin E(G)$, and symmetrically $u x^{\prime}, u y, u y^{\prime}, u z, u z^{\prime} \notin E(G)$.

Suppose now that $u a \in E(G)$. Then $u$ cannot be adjacent to any other branch-node of $\Pi$ by the discussion above. So we may assume that $u$ has a neighbor in the interior of $X, Y, Z$ (say $Z$ up to symmetry) for otherwise our claim holds. Now we define $v$ to be the neighbor of $u$ along $Z$ closest to $z$ and observe that $u v Z z c x^{\prime} b y a u$ is a cycle with a unique chord, a contradiction. Therefore we may assume that $u$ is not adjacent to any branch-node of $\Pi$.

We may suppose now that $u$ has neighbors in the interior of at least two of the paths among $X, Y, Z$ (say $X, Z$ w.l.o.g.) for otherwise our claim holds. Let $v($ resp. $w$ ) be a neighbor of $u$ in $X$ (resp. $Z$ ). Since $X \cup Z \cup$ $\{d, c\}$ induces a hole, by Lemma 3.1, $N(u) \cap(X \cup Z)=\{v, w\}$. If $v x \notin$ $E(G)$ then $v u w Z z^{\prime} d x a y b x^{\prime} X v$ is a cycle with unique chord. So $v x \in E(G)$ and symmetrically, $v x^{\prime}, w z, w z^{\prime} \in E(G)$. If $u$ has no neighbor in $Y$ then $v u w z^{\prime} d y^{\prime} Y y b x^{\prime} v$ is a cycle with a unique chord, so $u$ must have at least one neighbor $w^{\prime}$ in $Y$. By the same discussion that we have done above on $X, Z$ we can prove that $w^{\prime} y, w^{\prime} y^{\prime} \in E(G)$. Now we observe that $G[\Pi \cup\{u\}]$ is the Heawood graph, contradicting Claim 4. This proves (1).
(2) The attachment of any component of $G \backslash \Pi$ is included in one of the sets $X, Y, Z,\{a\},\{b\},\{c\},\{d\}$.

Else let $D$ be a connected induced subgraph of $G \backslash \Pi$ whose attachment overlaps two of the sets, and is minimal with respect this property. By (1), $D$ is a path of length at least one, with ends $u, v$ and $u$ (resp. $v$ ) has neighbors in exactly one set $S_{u}$ (resp. $S_{v}$ ) of $X, Y, Z,\{a\},\{b\},\{c\},\{d\}$. Moreover, no interior node of $D$ has a neighbor in $\Pi$. If $S_{u}=X$ and $S_{v}=\{a\}$ then let $w$ be the neighbor of $u$ closest to $x$ along $X$. We observe that $a y Y y^{\prime} d x X w u D v a$ is a cycle with a unique chord, a contradiction. Every case where there is an edge between $S_{u}$ and $S_{v}$ is symmetric, so we may assume that there is no edge between $S_{u}$ and $S_{v}$. If $S_{u}=X$ and $S_{v}=Z$ then let $w$ (resp. $w^{\prime}$ ) be the neighbor of $u$ (resp. of $v$ ) closest to $x^{\prime}$ along $X$ (resp. to $z^{\prime}$
along $Z$ ). If $w=x$ and $w^{\prime}=z$ then $x d y^{\prime} Y y a z v D u x$ is a cycle with a unique chord, a contradiction. So up to symmetry we may assume $w^{\prime} \neq z$. Hence $w X x^{\prime} c y^{\prime} Y y b z^{\prime} Z w^{\prime} v D u w$ is a cycle with a unique chord, a contradiction. So up to symmetry we may assume that $S_{u}=\{a\}$ and $S_{v}=\{c\}$. But then $a u D v c x^{\prime} b z^{\prime} Z z a$ is a cycle with a unique chord, a contradiction. This proves (2).

By (2), either some component of $G \backslash \Pi$ attaches to a node of $\Pi$ and there is a 1-cutset, or some component attaches to one of $X, Y, Z$ (say $X$ up to symmetry), and $\left\{x, x^{\prime}\right\}$ is a proper 2 -cutset.

Claim 6 We may assume that $G$ does not contain the following configuration: three node-disjoint paths $X=x \ldots x^{\prime}, Y=y \ldots y^{\prime}$ and $Z=z \ldots z^{\prime}$, of length at least two, and such that the only edges between them are $x y, y z$, $z x^{\prime}, x^{\prime} y^{\prime}, y^{\prime} z^{\prime}$ and $z^{\prime} x$.

Proof - Note that $G\left[x, y, z, x^{\prime}, y^{\prime}, z^{\prime}\right]$ is a hole on six nodes. Also either $G[X \cup Y \cup Z]$ is the Petersen graph with one node deleted (when $X, Y, Z$ have length two), or $G[X \cup Y \cup Z]$ has a proper 2-cutset (when one of the paths is of length at least three, the 2-cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \backslash(X \cup Y \cup Z)$.
(1) A node of $G \backslash(X \cup Y \cup Z)$ has neighbors in at most one of the sets $X, Y, Z$.

Let $u$ be a node of $G \backslash(X \cup Y \cup Z)$. Note that $u$ has neighbors in at most one of the following sets: $\left\{x, x^{\prime}\right\},\left\{y, y^{\prime}\right\},\left\{z, z^{\prime}\right\}$, for otherwise $G$ contains a triangle or a square, contradicting Claims 1 and 2 .

If $u$ has at least two neighbors among $x, x^{\prime}, y, y^{\prime}, z, z^{\prime}$ then we may assume by the paragraph above $u x, u x^{\prime} \in E(G)$. Since $X \cup Y$ and $X \cup Z$ both induce holes, every node in $X \cup Y \cup Z$ is in a hole going through $x, x^{\prime}$. So, by Lemma $3.1 u$ has no other neighbors in $X \cup Y \cup Z$. Hence, from here on, we assume that $u$ has at most one neighbor among $x, x^{\prime}, y, y^{\prime}, z, z^{\prime}$.

If $u x \in E(G)$ then we may assume that $u$ has neighbors in one of $Y, Z$, say $Z$ up to symmetry. Let $v \in Z$ be a neighbor of $u$. Then by Lemma 3.1, since $X \cup Z$ induces a hole, $v$ and $x$ are the only neighbors of $u$ in $X \cup Z$. So, $x u v Z z^{\prime} y^{\prime} x^{\prime} X x$ is a cycle with a unique chord, a contradiction. Hence we may assume that $u$ has no neighbors among $x, x^{\prime}, y, y^{\prime}, z, z^{\prime}$.

If $u$ has neighbors in the interior of at most one of $X, Y, Z$ our claim holds, so let us suppose that $u$ has neighbors in the interior of $X$ and the interior of $Y$. Since $X \cup Y$ induces a hole, by Lemma 3.1, $u$ has a unique neighbor $v \in X$ and a unique neighbor $w \in Y$. If $u$ has no neighbor in $Z$ then $x X v u w Y y z Z z^{\prime} x$ is a cycle with a unique chord, a contradiction. So $u$ has a neighbor $w^{\prime} \in Z$ that is unique by Lemma 3.1. If $v x, w y \notin E(G)$ then $x^{\prime} z y x z^{\prime} y^{\prime} Y w u v X x^{\prime}$ is a cycle with a unique chord, a con-
tradiction. So, up to a symmetry we may assume $v x \in E(G)$. If $w y^{\prime} \notin E(G)$ then $x^{\prime} y^{\prime} z^{\prime} x y Y w u v X x^{\prime}$ is a cycle with a unique chord, a contradiction, so $w y^{\prime} \in E(G)$. By the same argument, we can prove that $w^{\prime} z \in E(G)$. If $v x^{\prime}, w y, w^{\prime} z^{\prime} \in E(G)$ then we observe that $G[X \cup Y \cup Z \cup\{u\}]$ is the Petersen graph, contradicting Claim3. If $v x^{\prime}, w y, w^{\prime} z^{\prime} \notin E(G)$ then we observe that the three paths $v X x^{\prime}, w Y y, w^{\prime} Z z^{\prime}$ and nodes $u, z, y^{\prime}, x$ have the same configuration as those in Claim [5, a contradiction. So, we may assume that $v x^{\prime} \in E(G)$ and $w y \notin E(G)$. But then, $w u v x y z x^{\prime} y^{\prime} w$ is a cycle with a unique chord, a contradiction. This proves (1).
(2) The attachment of any component of $G \backslash(X \cup Y \cup Z)$ is included in one of the sets $X, Y, Z$.

Else let $D$ be a connected induced subgraph of $G \backslash(X \cup Y \cup Z)$, whose attachment overlaps two of the sets, and is minimal with this property. By (11), $D$ is a path of length at least one, with ends $u, v$ and no interior node of $D$ has a neighbor in $X \cup Y \cup Z$. We may assume that $u$ has neighbors only in $X$ and $v$ only in $Y$. Let $u^{\prime}$ (resp. $v^{\prime}$ ) be the neighbor of $u$ (resp. of $v$ ) closest to $x$ along $X$ (resp. to $y$ along $Y$ ). If $u^{\prime} \neq x^{\prime}$ and $v^{\prime} \neq y^{\prime}$ then $z y Y v^{\prime} v D u u^{\prime} X x z^{\prime} Z z$ is a cycle with a unique chord, a contradiction, so we may assume $v^{\prime}=y^{\prime}$. Let $u^{\prime \prime}$ be the neighbor of $u$ closest to $x^{\prime}$ along $X$. If $u^{\prime \prime} \neq x$ then $u u^{\prime \prime} X x^{\prime} z Z z^{\prime} y^{\prime} v D u$ is a cycle with a unique chord a contradiction. So, $u^{\prime \prime}=x$ and $u x z^{\prime} Z z x^{\prime} y^{\prime} v D u$ is a cycle with a unique chord, a contradiction. This proves (2).

By (2) one of $\left\{x, x^{\prime}\right\},\left\{y, y^{\prime}\right\},\left\{z, z^{\prime}\right\}$ is a proper 2-cutset of $G$.

Claim 7 We may assume that $G$ does not contain the following configuration: four node-disjoint paths $X=a_{1} \ldots a_{5}, Y=a_{2} \ldots a_{6}, Z=a_{3} \ldots a_{7}$ and $T=a_{4} \ldots a_{8}$, of length at least two, and such that the only edges between them are $a_{1} a_{2}, a_{2} a_{3}, a_{3} a_{4}, a_{4} a_{5}, a_{5} a_{6}, a_{6} a_{7}, a_{7} a_{8}$ and $a_{8} a_{1}$.

Proof - Either $G[X \cup Y \cup Z \cup T]$ is obtained from the Heawood graph by deleting two adjacent nodes (when $X, Y, Z, T$ have length two), or $G[X \cup$ $Y \cup Z \cup T]$ has a proper 2-cutset (when one of the paths is of length at least three, the 2-cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \backslash(X \cup Y \cup Z \cup T)$.
(1) A node of $G \backslash(X \cup Y \cup Z \cup T)$ has at most two neighbors in $X \cup Y \cup Z \cup T$.

For suppose that a node $u$ of $G \backslash(X \cup Y \cup Z \cup T)$ has at least three neighbors in $X \cup Y \cup Z \cup T$. Since every pair of path from $X, Y, Z, T$ can be embedded in a hole (for instance, $X \cup Y$ or $X \cup Z \cup\left\{a_{2}, a_{6}\right\}$ are holes, and the other cases are symmetric), by Lemma 3.1, the neighbors of $u$ lie on three or four paths and every path contains at most one neighbor of $u$.

Suppose $u$ is adjacent to one of the $a_{i}$ 's, say $a_{1}$. Then by Lemma 3.1 $u$ has at most one neighbor in $Y \cup T$ since $Y \cup T \cup\left\{a_{1}, a_{5}\right\}$ is a hole. So up to symmetry we assume that $u$ has a neighbor $v$ in $Y$, no neighbor in $T$, and so $u$ must have a neighbor $w$ in $Z$. By Lemma 3.1 applied to the hole $X \cup Z \cup\left\{a_{2}, a_{6}\right\}$ and node $u, v$ must be in the interior of $Y$. If $w \neq a_{3}$ then $w Z a_{7} a_{8} T a_{4} a_{5} a_{6} Y v u w$ is a cycle with a unique chord. So $w=a_{3}$ and $G$ contains a square, a contradiction to Claim (1) Hence we may assume that $u$ has no neighbors among the $a_{i}$ 's.

Up to symmetry we assume that $u$ has neighbors $x \in X, y \in Y, z \in$ $Z$. These neighbors are unique and are in the interior of their respective paths. So $u x X a_{5} a_{6} Y a_{2} a_{3} Z z u$ is a cycle with a unique chord (namely $u y$ ), a contradiction. This proves (1).
(2) The attachment of any component of $G \backslash(X \cup Y \cup Z \cup T)$ is included in one of the sets $X, Y, Z, T$.

Else let $D$ be a connected induced subgraph of $G \backslash(X \cup Y \cup Z \cup T)$, whose attachment overlaps two of the sets, and is minimal with respect this property. By the choice of $D$, the following hold. $D$ is a path, possibly of length zero, with ends $u, v$, and we may assume up to symmetry that $u$ has neighbors in $X$ and that $v$ has neighbors in $Y$ or in $Z$. No interior node of $D$ has neighbors in $X \cup Y \cup Z \cup T$. If $u \neq v$ then $u$ has neighbors only in $X$ and $v$ only in $Y$ or in $Z$. If $u=v$ then by (1) $u$ has neighbors only in $X \cup Y$ or only in $X \cup Z$.

If $v$ has neighbors in $Y$ then let $x$ be the neighbor of $u$ closest to $a_{5}$ along $X$ and $y$ be the neighbor of $v$ closest to $a_{6}$ along $Y$. If $x=a_{1}$ and $y=a_{2}$ then $D \cup\left\{a_{1}, \ldots, a_{8}\right\}$ is a cycle with a unique chord, so up to symmetry we may assume $x \neq a_{1}$. But then, $u x X a_{5} a_{4} T a_{8} a_{7} a_{6} Y y v D u$ is a cycle with a unique chord, a contradiction.

So $v$ has neighbors in $Z$. We claim that $v$ has a unique neighbor $z$ in $Z$, that is in the interior of $Z$, and that $Z$ has length two. Else, up to the symmetry between $a_{3}$ and $a_{7}$ we may assume that the neighbor $z$ of $v$ closest to $a_{3}$ along $Z$ is not $a_{7}$ and is not adjacent to $a_{7}$. Let $x$ be the neighbor of $u$ closest to $a_{5}$ along $X$. If $x=a_{1}$ then $a_{1} X a_{5} a_{6} Y a_{2} a_{3} Z z v D u a_{1}$ is a cycle with a unique chord. So $x \neq a_{1}$. If $v$ is not adjacent to $a_{7}$, then $a_{5} a_{6} a_{7} a_{8} T a_{4} a_{3} Z z v D u x X a_{5}$ is a cycle with unique chord, a contradiction. So $v$ is adjacent to $a_{7}$. In particular, $u \neq v$. By (1), $a_{7}$ and $z$ are the only neighbors of $v$ in $Z$. But then $a_{5} a_{6} a_{7} Z z v D u x X a_{5}$ is a cycle with unique chord, a contradiction. So, our claim is proved. Similarly, it can be proved that $u$ has a unique neighbor $x$ in $X$, that this neighbor is in the interior of $X$, and that $X$ has length two. We observe that the three paths $x u D v z, Y$, $T$ and nodes $a_{1}, a_{3}, a_{7}, a_{5}$ have the same configuration as those in Claim [5, a contradiction. This proves (2).

By (2), one of $\left\{a_{1}, a_{5}\right\},\left\{a_{2}, a_{6}\right\},\left\{a_{3}, a_{7}\right\},\left\{a_{4}, a_{8}\right\}$ is a proper 2-cutset.

Claim 8 We may assume that $G$ does not contain the following configuration: five paths $P_{13}=a_{1} \ldots a_{3}, P_{15}=a_{1} \ldots a_{5}, P_{48}=a_{4} \ldots a_{8}$, $P_{37}=a_{3} \ldots a_{7}, P_{57}=a_{5} \ldots a_{7}$, node disjoint except for their ends, of length at least two, and such that $G\left[P_{13} \cup P_{15} \cup P_{37} \cup P_{57}\right]$ is a hole and the only edges between this hole and $P_{48}$ are $a_{3} a_{4}, a_{4} a_{5}, a_{7} a_{8}$ and $a_{8} a_{1}$.

Proof - We put $\Pi=P_{13} \cup P_{15} \cup P_{48} \cup P_{37} \cup P_{57}$. Either $G[\Pi]$ is the Heawood graph with three nodes inducing a $P_{3}$ deleted (when the five paths have length two), or $G[\Pi]$ has a proper 2 -cutset (when one of the paths is of length at least three, the 2-cutset is formed by the ends of that path). Hence we may assume that there are nodes in $G \backslash \Pi$.
(1) A node of $G \backslash \Pi$ has at most two neighbors in $\Pi$.

Let $u$ be a node of $G \backslash \Pi$ and suppose that $u$ has more than two neighbors in $\Pi$. By Lemma 3.1 and since $P_{13} \cup P_{15} \cup P_{37} \cup P_{57}$ is a hole, $u$ has at most two neighbors among these paths. So, $u$ must have one neighbor in $P_{48}$, and this neighbor is unique since the union of $P_{48}$ with any of the other paths yields a hole. For the same reason, $u$ has a unique neighbor in exactly two paths among $P_{13}, P_{15}, P_{37}, P_{57}$. So there are two cases up to symmetry: either $u$ has neighbors in two paths among $P_{13}, P_{15}, P_{37}, P_{57}$ that have a common end, or $u$ has neighbors in two paths among $P_{13}, P_{15}, P_{37}, P_{57}$ that have no common ends.

In the first case, we may assume that $u$ has neighbors $x \in P_{37}, y \in$ $P_{48}$ and $z \in P_{13}$. Note that $x \neq a_{3}$ and $z \neq a_{3}$, for otherwise $P_{13}$ or $P_{37}$ would contain two neighbors of $u$. Suppose $y \neq a_{4}$. If $z \neq a_{1}$ then xuy $P_{48} a_{8} a_{1} P_{15} a_{5} P_{57} a_{7} P_{37} x$ is a cycle with a unique chord, a contradiction. If $z=a_{1}$ then by Claim 1, $x \neq a_{7}$ and hence $x u y P_{48} a_{8} a_{1} P_{13} a_{3} P_{37} x$ is a cycle with a unique chord, a contradiction. So $y=a_{4}$. But then, since $G$ does not contain a square, $x a_{3} \notin E(G)$ and hence $u x P_{37} a_{7} a_{8} a_{1} P_{13} a_{3} a_{4} u$ is a cycle with a unique chord, a contradiction.

In the second case, we may assume that $u$ has neighbors $x$ in $P_{13}, y$ in $P_{48}$ and $z$ in $P_{57}$. If $x=a_{1}$ then the previous case applies. Hence we may assume $x \neq a_{1}$ and symmetrically $z \neq a_{5}$. So, $u x P_{13} a_{3} a_{4} P_{48} a_{8} a_{7} P_{57} z u$ is a cycle with a unique chord (namely $u y$ ). This proves (1).
(2) The attachment of any component of $G \backslash \Pi$ is included in one of the sets $P_{13}, P_{15}, P_{37}, P_{57}, P_{48}$.

Else let $D$ be a connected induced subgraph of $G \backslash \Pi$, whose attachment is not contained in one of the sets, and is minimal with respect to this property. By the choice of $D$ the following hold. $D$ is a path, possibly of length zero, with
ends $u, v$, where $u$ has neighbors in one of the sets $P_{13}, P_{15}, P_{37}, P_{57}, P_{48}$ that we denote by $X_{u}$, and $v$ has neighbors in another one, say $X_{v}$. No interior node of $D$ has neighbors in $\Pi$. If $u \neq v$ then $u$ has neighbors only in $X_{u}$ and $v$ only in $X_{v}$. If $u=v$ then by (11) $u$ has neighbors only in $X_{u} \cup X_{v}$.

If $X_{u}=P_{48}$ then up to symmetry we may assume $X_{v}=P_{57}$. Let $x$ be the neighbor of $u$ closest to $a_{8}$ along $P_{48}$. If $x \neq a_{4}$ then let $y$ be the neighbor of $v$ closest to $a_{7}$ along $P_{57}$. Then $u D v y P_{57} a_{7} P_{37} a_{3} P_{13} a_{1} a_{8} P_{48} x u$ is a cycle with a unique chord. If $x=a_{4}$ then let $y$ be the neighbor of $v$ closest to $a_{5}$ along $P_{57}$. Then $u D v y P_{57} a_{5} P_{15} a_{1} P_{13} a_{3} a_{4} u$ is a cycle with a unique chord. So, $X_{u} \neq P_{48}$, and symmetrically $X_{v} \neq P_{48}$.

If $X_{u}, X_{v}$ are paths with a common end then we may assume $X_{u}=P_{37}$ and $X_{v}=P_{57}$. Let $x$ be the neighbor of $u$ closest to $a_{3}$ along $P_{37}$ and $y$ the neighbor of $v$ closest to $a_{5}$ along $P_{57}$. We note that $x, y \neq a_{7}$ for otherwise the attachment of $D$ is a single path $P_{37}$ or $P_{57}$ contrary to the definition of $D$. So, $x u D v y P_{57} a_{5} a_{4} P_{48} a_{8} a_{1} P_{13} a_{3} P_{37} x$ is a cycle with a unique chord.

If $X_{u}, X_{v}$ are paths with no common end then we may assume $X_{u}=P_{13}$ and $X_{v}=P_{57}$. We claim that $u$ has a unique neighbor in $P_{13}$, that is in the interior of $P_{13}$, and that $P_{13}$ has length two. Else, up to the symmetry between $a_{1}$ and $a_{3}$ we may assume that the neighbor $x$ of $u$ closest to $a_{3}$ along $P_{13}$ is not $a_{1}$ and is not adjacent to $a_{1}$. Let $y$ be the neighbor of $v$ closest to $a_{5}$ along $P_{57}$. If $y=a_{7}$ then the previous case applies, i.e. the neighbors of $u$ and $v$ in $\Pi$ are contained in the paths with a common end. So $y \neq a_{7}$. If $u$ is not adjacent to $a_{1}$, then $x u D v y P_{57} a_{5} P_{15} a_{1} a_{8} P_{48} a_{4} a_{3} P_{13} x$ is a cycle with a unique chord. So $u$ is adjacent to $a_{1}$. By (11), $a_{1}$ and $x$ are the only neighbors of $u$ in $\Pi$. But then $x u D v y P_{57} a_{5} P_{15} a_{1} P_{13} x$ is a cycle with a unique chord. Our claim is proved, and similarly we can prove that $v$ has a unique neighbor in $P_{57}$, that this neighbor is in the interior of $P_{57}$ and that $P_{57}$ has length two. Now we observe that the paths $x u D v y, P_{15}, P_{37}, P_{48}$ have the same configuration as those in Claim 7, a contradiction. This proves (2).

By (21), one of $\left\{a_{1}, a_{5}\right\},\left\{a_{1}, a_{3}\right\},\left\{a_{4}, a_{8}\right\},\left\{a_{5}, a_{7}\right\}\left\{a_{3}, a_{7}\right\}$ is a proper 2-cutset.

Claim 9 We may assume that $G$ does not contain a cycle with exactly two chords.

Proof - For let $C$ be a cycle in $G$ with exactly two chords $a b, c d$. We may assume up to the symmetry between $c$ and $d$ that $a, c, b, d$ appear in this order along $C$ for otherwise there is a cycle with a unique chord. We denote by $P_{a c}$ the unique path in $C$ from $a$ to $c$ that does not go through $b, d$. We define similarly $P_{c b}, P_{b d}, P_{d a}$. We assume that $C$ is a cycle with exactly two chords in $G$ that has the fewest number of nodes.

If $P_{c b}$ has length one then $P_{a c} \cup P_{b d}$ is a cycle with a unique chord unless $P_{a d}$ has also length one. But then $G[a, b, c, d]$ is a square or contains a triangle, a contradiction to Claims 耳 and 2. So $P_{c b}$ has length at least two and symmetrically, $P_{a c}, P_{b d}, P_{d a}$ have all length at least two.

Note that either $C$ is the Petersen graph with two adjacent nodes deleted (when $C$ is on eight nodes), or $C$ has a proper 2-cutset (when $C$ is on at least nine nodes). Hence we may assume that there are nodes in $G \backslash C$.
(1) A node of $G \backslash C$ has at most two neighbors in $C$, and if it has two neighbors in $C$ then these two neighbors are not included in one of the sets $P_{a c}, P_{c b}, P_{b d}, P_{d a}$.

Let $u$ be a node of $G \backslash C$ that has at least three neighbors in $C$. Note that by Lemma 3.1, $u$ has at most one neighbor in each of $P_{a c}, P_{c b}, P_{b d}, P_{d a}$ because the union of any two of them forms a hole. So, up to symmetry we may assume that $u$ has neighbors $x \in P_{a d}, y \in P_{a c}, z \in P_{b d}$ (and possibly one more in $P_{c b}$ ). If $y=c$ then $x \neq d$ and $x d \notin E(G)$ for otherwise $G$ contains a square or a triangle, contradicting Claims 1 and 2, and hence $u c d P_{b d} b a P_{a d} x u$ has a unique chord (namely $u z$ ). So $c \neq y$ and symmetrically, $b \neq z$. Hence $y u z P_{b d} d P_{a d} a P_{a c} y$ is a cycle with a unique chord (namely $u x$ ), a contradiction. So $u$ has at most two neighbors in $C$.

Let $x$ and $y$ be two neighbors of $u$ in $C$, and suppose that they both belong to the same path, say $P_{b d}$. W.l.o.g. $x$ is closer to $b$ on $P_{b d}$. By Claims 1 and 2, the $x y$-subpath $P$ of $P_{b d}$ is of length greater than 2. Let $C^{\prime}$ be the cycle induced by $(C \backslash P) \cup\{x, y\}$. Then $C^{\prime}$ is a cycle with exactly two chords that has fewer nodes than $C$, contradicting our choice of $C$. This proves (1).
(2) The attachment of any component of $G \backslash C$ is included in one of the sets $P_{a c}, P_{c b}, P_{b d}, P_{d a}$.
Else let $D$ be a connected induced subgraph of $G \backslash C$, whose attachment is not contained in one of the sets and is minimal with respect to this property. By the choice of $D$ the following hold. $D$ is a path possibly of length zero, with ends $u, v$, where $u$ has neighbors in one of the sets $P_{a c}, P_{c b}, P_{b d}, P_{d a}$ that we denote by $X_{u}$, and $v$ has neighbors in another one, say $X_{v}$. No interior node of $D$ has neighbors in $C$. If $u \neq v$ then $u$ has neighbors only in $X_{u}$ and $v$ only in $X_{v}$. If $u=v$, then by (1) $u$ has neighbors only in $X_{u} \cup X_{v}$. Let $x$ be a neighbor of $u$ in $X_{u}$, and $y$ a neighbor of $v$ in $X_{v}$. By (1), $(N(u) \cup N(v)) \cap C=\{x, y\}$.

If $X_{u}$ and $X_{v}$ share a common end then up to symmetry we assume $X_{u}=P_{a c}, X_{v}=P_{a d}$. Neither $x$ nor $y$ coincides with $a$ for otherwise the attachment of $D$ over $C$ is in $P_{a c}$ or $P_{a d}$, contrary to the definition of $D$. So, $u D v y P_{a d} d P_{b d} b P_{b c} C P_{a c} x u$ is a cycle with a unique chord.

So $X_{u}$ and $X_{v}$ do not share a common end, hence up to symmetry we assume $X_{u}=P_{a c}, X_{v}=P_{b d}$. By the previous paragraph, we may assume $x \notin\{a, c\}$ and $y \notin\{b, d\}$. If $x a, y b \notin E(G)$ then $x P_{a c} c P_{c b} b a P_{a d} d P_{b d} y v D u x$ is a cycle with a unique chord. So, up to symmetry we assume $x a \in E(G)$. If $y d \notin E(G)$ then $x u D v y P_{b d} b a P_{a d} d c P_{a c} x$ is a cycle with a unique chord, a contradiction. So, $y d \in E(G)$. Since $x a b P_{b c} c d y v D u x$ cannot be a cycle with a unique chord, $x c, y b$ are either both in $E(G)$ or both not in $E(G)$. In the first case, the three paths $x u D v y, P_{a d}, P_{b c}$ have the same configuration as those in Claim 6. In the second case, the five paths $x u D v y, P_{a d}, P_{b c}, x P_{a c} c, y P_{b d} b$ have the same configuration as those in Claim [8. This proves (2).

By (2) one of $\{a, c\},\{c, b\},\{b, d\},\{d, a\}$ is a proper 2 -cutset.

Claim 10 We may assume that $G$ does not contain a cycle with exactly three chords.

Proof - Let $C$ be a cycle in $G$ with exactly three chords $a b, c d$, ef say. Up to symmetry we may assume that $a, c, e, b, d, f$ appear in this order along the cycle and are pairwise distinct for otherwise $C$ contains a cycle with a unique chord. We denote by $P_{a c}$ the unique path from $a$ to $c$ in $C$ that does not go through $e, b, d, f$. We define similarly $P_{c e}, P_{e b}, P_{b d}, P_{d f}, P_{f a}$. If $G[\{a, b, c, d, e, f\}]$ contains only three edges then $P_{a f} \cup P_{f d} \cup P_{c e} \cup P_{e b}$ is a cycle with a unique chord (namely $f e$ ), a contradiction. Hence, up to symmetry we may assume $a c \in E(G)$. Now $P_{a f} \cup P_{d b} \cup P_{c e}$ is a cycle with one, two or three chords : ac and possibly $f d$ and $e b$. By Claim (9, this cycle must have three chords, so $e b, f d \in E(G)$. Note that $b d \notin E(G)$ since $G$ contains no square by Claim प, and similarly $a f$, ce $\notin E(G)$. Now we observe that the paths $P_{a f}, P_{c e}, P_{b d}$ have the same configuration as those in Claim 6, a contradiction.

Claim 11 We may assume that $G$ does not contain a cycle with at least one chord.

Proof - Let $C$ be a cycle in $G$ with at least one chord $a b$. We choose $C$ minimal with this property. Cycle $C$ must have another chord $c d$, and we may assume that $a, d, b, c$ are pairwise distinct and in this order along $C$ for otherwise $C$ contains a cycle with at least one chord that contradicts the minimality of $C$. By Claim $9, C$ must have another chord $e f$, and again we may assume that $a, e, d, b, f, c$ are pairwise distinct and in this order along $C$ because of the minimality of $C$. By Claim 10, $C$ must have again another chord $g h$, and again we may assume that $a, g, e, d, b, h, f, c$ are pairwise distinct and in this order along $C$ because of the minimality of $C$. Now, the path from $a$ to $f$ along $C$ that goes through $c$ and the path from $e$
to $b$ along $C$ that goes through $d$ form a cycle smaller than $C$ with at least one chord (namely $c d$ ), a contradiction.

A non-induced path $P$ in a graph $G$ is a sequence of distinct nodes $v_{1} \ldots v_{n}$ such that for $i=1, \ldots, n-1, v_{i} v_{i+1}$ is an edge (these are the edges of the path). There might be other edges: the chords of the path.

Claim 12 We may assume that $G$ does not contain the following configuration: five non-induced paths $P=a \ldots c, Q=a \ldots c, R=b \ldots d, S=b \ldots d$, $X=c \ldots d$, node-disjoint except for their ends, of length at least one, except for $X$ that can be of length zero or more, together with edge ab.

Proof - Let $\Pi=P \cup Q \cup R \cup S \cup X$. We suppose that $\Pi$ is chosen subject to the minimality of $X$. Note that the only edges of $G[\Pi]$ are the edges of the non-induced paths $P, Q, R, S, X$ and $a b$. Indeed, every pair of nodes $\{x, y\} \subseteq \Pi$ can be embedded into a cycle containing only edges of the non-induced paths and $a b$, so if $x y$ is an edge of $G[\Pi]$ that contradicts our statement then there is a cycle $C$ of $G[\Pi]$ such that $x y$ is a chord of $C$, and this contradicts Claim 11. In particular, the five non-induced paths have no chords, so they are in fact paths. Note that $P, Q, R, S$ are all of length at least two for if $P$ (say) is of length one, then the unique edge of $P$ is a chord of a cycle of $G[\Pi]$, and this contradicts Claim 11,

We now show that $\{a, c\}$ is a proper 2-cutset of $G$. Assume not. Then there is a path $D=u \ldots v$ in $G \backslash \Pi$ such that $u$ has a neighbor $x$ in $(P \cup$ $Q) \backslash\{a, c\}$ (say in $Q \backslash\{a, c\}$ ) and $v$ has a neighbor $y$ in $(X \cup R \cup S) \backslash\{c\}$.

Suppose that $y \in X \backslash\{c\}$. Then the five non-induced paths $a Q x u D v y$, $a P c X y, R, S$ and $y X d$ form a configuration that contradicts the minimality of $X$.

So $y \notin X \backslash\{c\}$, and hence w.l.o.g. $y \in R \backslash\{d\}$. So $u x Q a P c X d S b R y v D u$ is a cycle with at least one chord (namely $a b$ ), contradicting Claim 11,

Claim 13 We may assume that $G$ does not contain the following induced subgraph (that we call I): six nodes $a, b, c, d, e, f$ with the following edges: $a b, a c, a d, b e, b f$.

Proof - We may assume that $G$ has no 1-cutset. Hence, by Lemma 3.2, $\{a, b\}$ is not a cutset of $G$. So there exists a path in $G \backslash\{a, b\}$ with an end having neighbors in $\{c, d\}$ and an end having neighbors in $\{e, f\}$. We choose a minimal such path $D=u \ldots v$. Up to symmetries and since $G$ contains no square and no triangle by Claims 1 and 2, we may assume that $u$ has a unique neighbor in $\{c, d\}$, and that $v$ has a unique neighbor in $\{e, f\}$. W.l.o.g. $d u, v f \in E(G)$. Note that from the minimality of $D, c, e$ have no neighbor in $D$.

Since $a$ is not a 1-cutset there is a path $F$ in $G \backslash a$ with one end $y$ adjacent to $c$ and an end $x$ adjacent to some node $w$ in $D \cup\{e, b, f, d\}$. We choose
such a path $F$ minimal with respect to this property. If $x$ is adjacent to $e$ or $b$ then $G$ contains a cycle with at least one chord (namely $a b$ ), contradicting Claim 11. So, $w \in D^{\prime}=d u D v f$. Note that from the minimality of $F$, $e$ has no neighbor in $F$.

Since $b$ is not a 1-cutset, there is in $G \backslash b$ a path $H$ with an end $z$ adjacent to $e$ and an end $t$ adjacent to some node $s$ in $D \cup F \cup\{f, d, a, c\}$. Let $Q=a c y F x$. If $s \in Q$ then ezHtsQadD $D^{\prime} f b e$ is a cycle with at least one chord (namely $a b$ ), contradicting Claim 11, So $s \notin Q$. Let $Q^{\prime}=a d D^{\prime} w \backslash w$ (note that if $w=d$ then $Q^{\prime}=a$ ). If $s \in Q^{\prime}$ then $s Q^{\prime} a c y F x w D^{\prime}$ fbezHts is a cycle with at least one chord (namely $a b$ ), contradicting Claim 11. So $s \notin Q^{\prime}$. Hence, $s \in f D^{\prime} w$.

Now we observe that the five non-induced paths bezHts, bf $D^{\prime} s, a c y F x w$, $a d D^{\prime} w$ and $w D^{\prime} s$ together with edge $a b$ have the same configuration as those in Claim 12, a contradiction.

We can now prove that $G$ is strongly 2-bipartite. Indeed, we may assume that $G$ has no 1 -cutset and $G$ contains no square by Claim 1 . We may assume that $G$ is not a chordless cycle because $C_{3}$ is a clique, $C_{4}$ is a square, $C_{5}$, $C_{6}$ are induced subgraphs of the Petersen graph and $C_{k}$ where $k \geq 7$ is an output of our theorem. We may also assume that $G$ is not a clique. Let us call a branch of a graph $G$ any path of length at least one, whose ends are of degree at least 3 , and whose interior nodes are of degree 2 . Since $G$ is not a chordless cycle, has no 1-cutset and is not a clique of size one or two, it is edge-wise partitioned into its branches. In particular, every node of $G$ is of degree at least two. No branch of $G$ is of length one, because such a branch is an edge of $G$ that has both ends of degree at least three, and then $G$ contains either a triangle, a square or an $I$, and this contradicts Claim 2, 1 or 13. We may also assume that $G$ has no branch of length at least 3 because the ends of such a branch, say $a$ and $b$, form a proper 2-cutset (note that $a$ and $b$ cannot be adjacent since there is no branch of length 1 , and since there is no 1 -cutset, for every connected component $C$ of $G \backslash\{a, b\}$ there is an $a b$-path in $G[C \cup\{a, b\}])$.

So we proved that every branch of $G$ is of length exactly 2 . This implies that the set $X$ of all nodes of $G$ of degree 2 and the set $Y$ of all nodes of $G$ with degree at least 3 are stable sets. So $G$ is strongly 2-bipartite. This proves Theorem 2.4.

## 4 Structure theorem

The block $G_{X}$ (resp. $G_{Y}$ ) of a graph $G$ with respect to a 1-cutset with split $(X, Y, v)$ is $G[X \cup\{v\}]$ (resp. $G[Y \cup\{v\}]$ ).

The block $G_{X}$ (resp. $G_{Y}$ ) of a graph $G$ with respect to a 1-join with split $(X, Y, A, B)$ is the graph obtained by taking $G[X]$ (resp. $G[Y]$ ) and adding


Figure 5: A graph in $\mathcal{C}$ with no extremal decomposition
a node $y$ complete to $A$ (resp. $x$ complete to $B$ ). Nodes $x, y$ are called the markers of their respective blocks.

The block $G_{X}$ (resp. $G_{Y}$ ) of a graph $G$ with respect to a proper 2-cutset with split ( $X, Y, a, b$ ) is the graph obtained by taking $G[X \cup\{a, b\}]$ (resp. $G[Y \cup\{a, b\}])$ and adding a node $c$ adjacent to $a, b$. Node $c$ is a called the marker of the block $G_{X}$ (resp. $G_{Y}$ ).

A graph is basic if it is connected and it is either a clique, a hole of length at least 7, a strongly 2-bipartite graph, or an induced subgraph of the Petersen graph or the Heawood graph. Note that the square is not basic (but has a proper 1 -join). Every $C_{k}, k \geq 3, k \neq 4$ is basic.

It is sometime useful to prove that every graph in a class has an extremal decomposition, that is a decomposition such that one of the blocks is basic. With our basic classes, decompositions and blocks, this is false for graphs in $\mathcal{C}$. The graph in Fig. $5^{5}$ is a counter-example. This graph has no proper 2 -cutset and a unique 1 -join. No block with respect to this proper 1 -join is basic, but both blocks have a proper 2 -cutset.

Lemma 4.1 Let $G_{X}$ and $G_{Y}$ be the blocks of decomposition of $G$ w.r.t. a 1-cutset, a proper 1-join or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$.

Proof - Suppose $G_{X}$ and $G_{Y}$ are the blocks of decomposition of $G$ w.r.t. a 1-cutset or a proper 1-join. Then $G_{X}$ and $G_{Y}$ are induced subgraphs of $G$, and hence if $G \in \mathcal{C}$ then $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$. Conversely, suppose that $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$. If they are blocks w.r.t. a 1-cutset, then every cycle of $G$ belongs to $G_{X}$ or $G_{Y}$, and hence $G \in \mathcal{C}$. Assume they are blocks w.r.t. a proper 1-join. Then every cycle $C$ that has at least two nodes in $X$ and at least two nodes in $Y$, has at least two nodes in $A$ and at least two nodes in $B$. Since $A$ and $B$ are stable sets, $C$ is either a square or it has at least two chords. It follows that every cycle of $G$ with a unique chord is contained in $G_{X}$ or $G_{Y}$ (where possibly the marker node plays the role of one of the nodes of the cycle). Hence $G \in \mathcal{C}$.

Now suppose that $G_{X}$ and $G_{Y}$ are the blocks of decomposition of $G$ w.r.t. a proper 2 -cutset, with split $(X, Y, a, b)$. Suppose $G \in \mathcal{C}$. Suppose w.l.o.g. that $G_{X}$ contains a cycle $C$ with a unique chord. Then $C$ must contain $c$. Let $P$ be an ab-path in $G[Y \cup\{a, b\}]$. Then $G[(V(C) \backslash\{c\}) \cup V(P)]$ is a cycle with a unique chord, a contradiction. So $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$.

To prove the converse, assume that $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$, and $G$ contains a cycle $C$ with a unique chord. Since $C$ cannot be contained in $G_{X}$ nor $G_{Y}$, it must contain a node of $X$ and a node of $Y$, and hence it contains $a$ and $b$. Let $P_{X}$ (resp. $P_{Y}$ ) be the section of $C$ in $G[X \cup\{a, b\}]$ (resp. $G[Y \cup\{a, b\}])$. Since $C$ contains a unique chord, w.l.o.g. $P_{Y}$ is a path and $P_{X}$ has a unique chord. But then $G_{X}\left[V\left(P_{X}\right) \cup\{c\}\right]$ is a cycle with a unique chord, a contradiction.

Theorem 2.1 and Lemma 4.1 actually give us a complete structure theorem for class $\mathcal{C}$, i.e. every graph in $\mathcal{C}$ can be built starting from basic graphs, that can be explicitly constructed, and gluing them together by prescribed composition operations, and all graphs built this way are in $\mathcal{C}$.

Cliques, holes and induced subgraphs of the Petersen graph or the Heawood graph can clearly be explicitly constructed. Also strongly 2-bipartite graphs can be constructed as follows. Let $X$ and $Y$ be node sets. We construct a bipartite graphs with bipartition $(X, Y)$ by making every node of $X$ adjacent to two nodes of $Y$. Every strongly 2-bipartite graph can be constructed this way, and every graph constructed this way belongs to $\mathcal{C}$. Indeed, a graph so constructed does not have an edge both of whose endnodes are of degree at least 3 , whereas a chord of a cycle has endnodes that are both of degree at least 3 .

The composition operations we need are just the reverse of our decompositions, and the union of two graphs. Each operation takes as input two node disjoint graphs $G_{1}$ and $G_{2}$, and outputs a third graph $G$.

Operation $\mathcal{O}_{0}$ is the operation of taking the disjoint union of two graphs, i.e. $V(G)=V\left(G_{1}\right) \cup V\left(G_{2}\right)$ and $E(G)=E\left(G_{1}\right) \cup E\left(G_{2}\right)$.

Operation $\mathcal{O}_{1}$ is the operation that is the reverse of 1-cutset decomposition. For some node $u$ of $G_{1}$ and some node $w$ of $G_{2}, G$ is obtained from the disjoint union of $G_{1} \backslash\{u\}$ and $G_{2} \backslash\{w\}$, by adding a new node $v$ and all the edges between $v$ and $N_{G_{1}}(u) \cup N_{G_{2}}(w)$.

Operation $\mathcal{O}_{2}$ is the operation that is the reverse of proper 1-join decomposition. For some node $u$ (resp. $v$ ) of $G_{1}$ (resp. $G_{2}$ ) such that $N_{G_{1}}(u)$ (resp. $N_{G_{2}}(v)$ ) is a stable set of size at least $2, G$ is obtained from the disjoint union of $G_{1} \backslash\{u\}$ and $G_{2} \backslash\{v\}$ by adding all edges between $N_{G_{1}}(u)$ and $N_{G_{2}}(v)$.

Operation $\mathcal{O}_{3}$ is the operation that is the reverse of proper 2-cutset decomposition. For some degree 2 node $u$ (resp. $v$ ) of $G_{1}$ (resp. $G_{2}$ )
which is not a 1 -cutset, with neighbors $u_{1}$ and $u_{2}$ (resp. $v_{1}$ and $v_{2}$ ) such that $u_{1}$ and $u_{2}$ (resp. $v_{1}$ and $v_{2}$ ) are nonadjacent, and $\left(d_{G_{1}}\left(u_{1}\right)-1\right)+\left(d_{G_{2}}\left(v_{1}\right)-1\right) \geq 3$ and $\left(d_{G_{1}}\left(u_{2}\right)-1\right)+\left(d_{G_{2}}\left(v_{2}\right)-1\right) \geq$ 3, $G$ is obtained from the disjoint union of $G_{1} \backslash\left\{u, u_{1}, u_{2}\right\}$ and $G_{2} \backslash\left\{v, v_{1}, v_{2}\right\}$ by adding new nodes $w_{1}$ and $w_{2}$ and all edges between $w_{1}$ and $\left(N_{G_{1}}\left(u_{1}\right) \backslash\{u\}\right) \cup\left(N_{G_{2}}\left(v_{1}\right) \backslash\{v\}\right)$ and between $w_{2}$ and $\left(N_{G_{1}}\left(u_{2}\right) \backslash\{u\}\right) \cup\left(N_{G_{2}}\left(v_{2}\right) \backslash\{v\}\right)$.

Theorem 4.2 If $G \in \mathcal{C}$ then either $G$ is basic or can be obtained starting from basic graphs by repeated applications of operations $\mathcal{O}_{0}, \ldots, \mathcal{O}_{3}$. Conversely, every graph obtained in this way is in $\mathcal{C}$.

Proof - Follows from Theorem 2.1 and Lemma 4.1.

## 5 Constructing a decomposition tree

We will now construct a decomposition tree for an input graph $G$, and then use this tree to obtain an $\mathcal{O}(n m)$ recognition algorithm for class $\mathcal{C}$ (in descriptions of algorithms, $n$ stands for the number of nodes and $m$ for the number of edges). An $\mathcal{O}\left(n^{5}\right)$ or a slightly more involved $\mathcal{O}\left(n^{4}\right)$ algorithm could be obtained from first principles, but we use sophisticated algorithms from other authors, namely Dahlhaus [10, Hopcroft and Tarjan [14, 23, 15] to get our algorithm to run in $\mathcal{O}(n m)$-time. Note that we do not use the full strength of the works of these authors since they are able to decompose fully a graph in linear time using 1-joins, or using 1-cutsets, or using 2cutsets. But their notions of decompositions differ slightly from what we need so we just use their algorithms to find the cutsets in linear time. We leave as an open question whether it is possible to recognize graphs in $\mathcal{C}$ in $\mathcal{O}(n+m)$-time.

We could use the definition of blocks of decomposition from Section 4 to construct a decomposition tree, and use it to obtain the recognition algorithm, but such a tree cannot be used for our coloring algorithm, because for coloring we need the blocks of a square-free graph with respect to a proper 2 -cutset to be also square-free. So in this section, blocks of decomposition w.r.t. a 1-cutset and a proper 1-join stay the same as in Section 4 but the blocks of decomposition w.r.t. a proper 2-cutset are redefined here below.

The block $G_{X}$ (resp. $G_{Y}$ ) of a graph $G$ with respect to a proper 2-cutset with split $(X, Y, a, b)$ is the graph obtained as follows:

- if there exists a node $c$ of $G$ such that $N(c)=\{a, b\}$, then take such a node $c$, and let $G_{X}=G[X \cup\{a, b, c\}]$ and $G_{Y}=G[Y \cup\{a, b, c\}]$;
- else $G_{X}$ (resp. $G_{Y}$ ) is the block defined in Section 4 that is the graph obtained by taking $G[X \cup\{a, b\}]$ (resp. $G[Y \cup\{a, b\}]$ ) and adding a node $c$ adjacent to $a, b$.

Node $c$ is called the marker of the block $G_{X}$ (resp. $G_{Y}$ ).
Lemma 5.1 Let $G \in \mathcal{C}$ and suppose that $G_{X}$ and $G_{Y}$ are the blocks of decomposition of $G$ w.r.t. a 1-cutset, a proper 1-join or a proper 2-cutset. If $G$ is connected and triangle-free then $G_{X}$ and $G_{Y}$ are connected and trianglefree.

Proof - The blocks of $G$ are clearly connected. The blocks of $G$ with respect to a 1-cutset or a proper 1-join are induced subgraphs of $G$ so they are triangle-free. If one of the block of $G$ w.r.t. a proper 2-cutset $\{a, b\}$ contains a triangle then this triangle must contain the marker $c$. So the triangle must be $a b c$ and this contradicts $a b \notin E(G)$.

Lemma 5.2 Let $G \in \mathcal{C}$ and suppose that $G_{X}$ and $G_{Y}$ are the blocks of decomposition of $G$ w.r.t. a proper 2-cutset with split $(X, Y, a, b)$. If $G$ is connected, triangle-free, square-free, Petersen-free, has no 1-cutset and no proper 1-join, then $G_{X}$ and $G_{Y}$ have the same property.

Proof - For connectivity and triangles, the lemma follows from Lemma 5.1.

For squares, suppose that w.l.o.g. $G_{X}$ contains a square $C$. Since $G$ is square-free, $C$ contains the marker node $c$ (that is not a real node of $G$ ), and hence $C=c a z b c$, for some node $z \in X$. Since $c$ is not a real node of $G, d_{G}(z)>2$ for otherwise, $z$ would have been chosen to serve as a marker. Let $z^{\prime}$ be a neighbor of $z$ that is distinct from $a$ and $b$. Note that since $G$ is triangle-free, $a z^{\prime}$ and $b z^{\prime}$ are not edges. Since $z$ is not a 1-cutset, there exists a path $P$ in $G[X \cup\{a, b\}]$ from $z^{\prime}$ to $\{a, b\}$. We choose $z^{\prime}$ and $P$ subject to the minimality of $P$. So, w.l.o.g. $z^{\prime} P a$ is a path. Note that $b$ is not adjacent to the neighbor of $a$ along $P$ since $z$ is the unique common neighbor of $a, b$ because $G$ is square-free. So by minimality of $P, b$ does not have a neighbor in $P$. Now let $Q$ be a path from $a$ to $b$ whose interior is in $Y$. So, $b z z^{\prime} P a Q b$ is a cycle with a unique chord (namely $a z$ ), a contradiction.

For the Petersen graph, it suffices to notice that if a block of $G$ contains it, then the marker $c$ must be in it, and this is a contradiction since $c$ is of degree two.

For 1-cutsets, suppose w.l.o.g. that $G_{X}$ has a 1-cutset with split $(A, B, v)$. Since $G$ is connected and $G[X \cup\{a, b\}]$ contains an $a b$-path, $v \neq c$ (where $c$ is the marker node of $G_{X}$ ). Suppose $v=a$. Then w.l.o.g. $b \in B$, and hence $(A, B \cup Y, a)$ is a split of a 1-cutset of $G$ (with possibly $c$ removed from $B \cup Y$, if $c$ is not a real node of $G$ ), a contradiction. So $v \neq a$ and by symmetry $v \neq b$. So $v \in X \backslash\{c\}$. W.l.o.g. $\{a, b, c\} \subseteq B$. Then $(A, B \cup Y, v)$ is a split of a 1-cutset of $G$ (with possibly $c$ removed from $B \cup Y$, if $c$ is not a real node of $G$ ), a contradiction.

For proper 1-joins, it suffices to notice that the blocks of $G$ are squarefree, so they cannot have a proper 1-join.

Lemma 5.3 Let $G_{X}$ and $G_{Y}$ be the blocks of decomposition of $G$ w.r.t. a 1-cutset, a proper 1-join or a proper 2-cutset. Then $G \in \mathcal{C}$ if and only if $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$.

Proof - By Lemma 4.1 we may assume that $G_{X}$ and $G_{Y}$ are the blocks of decomposition of $G$ w.r.t. a proper 2-cutset, with $\operatorname{split}(X, Y, a, b)$ and that the marker $c$ is a real node of $G$. So $G_{X}$ and $G_{Y}$ are induced subgraphs of $G$. Hence, $G \in \mathcal{C}$ implies $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$.

To prove the converse, assume that $G_{X} \in \mathcal{C}$ and $G_{Y} \in \mathcal{C}$, and $G$ contains a cycle $C$ with a unique chord. Since $C$ cannot be contained in $G_{X}$ nor $G_{Y}$, it must contain a node of $X$ and a node of $Y$, and hence it contains $a$ and $b$. Let $P_{X}$ (resp. $\left.P_{Y}\right)$ be the section of $C$ in $G[X \cup\{a, b\}]$ (resp. $G[Y \cup\{a, b\}]$ ). Since $C$ contains a unique chord, w.l.o.g. $P_{Y}$ is a path and $P_{X}$ has a unique chord. Note that $c \notin V\left(P_{X}\right)$, since $c$ is of degree 2 in $G$ and it is adjacent to both $a$ and $b$. Hence $G_{X}\left[V\left(P_{X}\right) \cup\{c\}\right]$ is a cycle with a unique chord, a contradiction.

An algorithm of Hopcroft and Tarjan [14, 23] finds in linear time a 1cutset of $G$ (if any). An algorithm of Dahlhaus [10] finds in linear time a 1-join of $G$ if any. The next lemma shows how to use this algorithm to find a proper 1 -join or determine that $G \notin \mathcal{C}$.

Lemma 5.4 Let $G$ be a graph that is not a clique and has no 1-cutset. Assume $G$ has a 1-join. If this 1-join is not proper, then $G \notin \mathcal{C}$.

Proof - Let $(X, Y, A, B)$ be the split of a 1-join of $G$ that is not proper. If $|A|=1$ then $A$ is a 1 -cutset of $G$, a contradiction. So $|A| \geq 2$, and by symmetry $|B| \geq 2$. Since the 1 -join is not proper, w.l.o.g. there is an edge with both ends in $A$. This edge together with any node of $B$ forms a triangle, and so by Theorem 2.2 (and since $G$ is not a clique and has no 1 -cutset) $G \notin \mathcal{C}$.

Recall that in a graph $G$ two nodes $a$ and $b$ form a 2-cutset if $G \backslash\{a, b\}$ is disconnected. Hopcroft and Tarjan [15] give an algorithm that finds a 2-cutset in a graph (if any) in linear time. This 2-cutset is not necessarily a proper 2-cutset (which is what we need). We now show how to find a proper 2-cutset in linear time.

Recall that if $H$ is an induced subgraph of $G$ and $D$ is a set of nodes of $G \backslash H$, the attachment of $D$ over $H$ is the set of all nodes of $H$ that have a neighbor in $D$.

Lemma 5.5 There is an algorithm with the following specifications.

Input: A connected graph $G$ that has no 1-cutset nor a proper 1-join, and is not basic.

Output: $G$ is correctly identified as not belonging to $\mathcal{C}$, or a proper 2cutset of $G$.

Running time: $\mathcal{O}(n+m)$.
Proof - Consider the following algorithm.
Step 1: Let $G_{2}$ be the subgraph of $G$ induced by the degree 2 nodes of $G$. Since $G$ is connected, has no 1-cutset and is not a chordless cycle (because $C_{k}, 3 \leq k \neq 4$, is basic and $C_{4}$ admits a proper 1-join), the connected components of $G_{2}$ are paths, and for every such path $P$, the attachment of $P$ over $G \backslash P$ consists of two distinct nodes of $G$ that are both of degree at least 3 in $G$. If there exists a path $P$ in $G_{2}$ whose attachment $\{a, b\}$ over $G \backslash P$ is such that $a b$ is an edge, then output $G \notin \mathcal{C}$ and stop.

Step 2: If there is a path $P$ in $G_{2}$ of length at least 1 then let $\{a, b\}$ be the attachment of $P$ over $G \backslash P$. Output $\{a, b\}$ as a proper 2-cutset of $G$ and stop.

Step 3: Now all paths of $G_{2}$ are of length 0 . Create the graph $G^{\prime}$ from $G \backslash V\left(G_{2}\right)$ as follows: for every path $P$ of $G_{2}$ put an edge between the pair of nodes that are the attachment of $P$ over $G \backslash P$. Note that if $G_{2}$ is empty, then $G=G^{\prime}$. If $G^{\prime}$ has no 2-cutset, output $G \notin \mathcal{C}$ and stop.

Step 4: Find a 2-cutset $\{a, b\}$ of $G^{\prime}$. Note that $\{a, b\}$ is also a 2-cutset of $G$. If $a b$ is an edge of $G$, then output $G \notin \mathcal{C}$ and stop. Otherwise, output $\{a, b\}$ as a proper 2 -cutset of $G$ and stop.

Since 2-cutsets in Step 3 and 4 can be found in time $\mathcal{O}(n+m)$ by the Hopcroft and Tarjan algorithm [15], it is clear that the above algorithm can be implemented to run in time $\mathcal{O}(n+m)$. We now prove the correctness of the algorithm.

First note that since $G$ is not a clique and it does not have a 1-cutset, all nodes of $G$ have degree at least 2. Suppose the algorithm stops in Step 1. So there exists a path $P$ in $G_{2}$ whose attachment over $G \backslash P$ induces an edge $a b$. Since $d_{G}(a) \geq 3$, it follows that $V(G) \backslash(V(P) \cup\{a, b\}) \neq \emptyset$, and hence $\{a, b\}$ is a 2 -cutset of $G$. So by Lemma 3.2, the algorithm correctly identifies $G$ as not belonging to $\mathcal{C}$.

Suppose the algorithm stops in Step 2. By Step 1, $a b$ is not an edge. Since $d_{G}(a) \geq 3,|V(G) \backslash(V(P) \cup\{a, b\})| \geq 2$, and since $P$ is of length at least $1,|V(P)| \geq 2$. Since $G$ has no 1-cutset, there is an $a b$-path in $G \backslash P$. Hence $\{a, b\}$ is a proper 2 -cutset of $G$.

Suppose the algorithm stops in Step 3. This means that $G^{\prime}$ has no 2cutset. Since the output is $G \notin \mathcal{C}$, the only problem is when $G \in \mathcal{C}$, so let us suppose for a contradiction $G \in \mathcal{C}$. Then by Theorem [2.1, $G$ has a proper 2-cutset $\{a, b\}$ with split $(X, Y, a, b)$. Since $d_{G}(a) \geq 3$ and $d_{G}(b) \geq 3$, $\{a, b\} \subseteq V\left(G^{\prime}\right)$. If $\left|X \cap V\left(G^{\prime}\right)\right| \geq 1$ and $\left|Y \cap V\left(G^{\prime}\right)\right| \geq 1$, then $\{a, b\}$ is a 2-cutset of $G^{\prime}$, so we may assume w.l.o.g. that $X \cap V\left(G^{\prime}\right)=\emptyset$. Since $\{a, b\}$ is a proper 2 -cutset of $G,|X| \geq 2$. So $X$ contains two nodes $u_{1}$ and $u_{2}$ that are both of degree 2 in $G$. By Step 2, $u_{1}$ and $u_{2}$ are paths of $G_{2}$ of length 0 . Since $\{a, b\}$ is a cutset of $G$, and $G$ is connected and has no 1-cutset, it follows that both $u_{1}$ and $u_{2}$ are adjacent to both $a$ and $b$. So $a u_{1} b u_{2}$ is a square of $G$, so by Theorem 2.3, $G$ must have a proper 1-join, a 1-cutset or must be basic, in either case a contradiction.

Suppose the algorithm stops in Step 4. Let $\{a, b\}$ be a 2 -cutset of $G^{\prime}$. Then clearly $\{a, b\}$ is also a 2-cutset of $G$. If $a b$ is an edge of $G$, then by Lemma 3.2 the algorithm correctly identifies $G$ as not belonging to $\mathcal{C}$. So assume $a b$ is not an edge of $G$. Note that for every $u \in V\left(G^{\prime}\right), d_{G}(u) \geq 3$. In particular, since $a, b \in V\left(G^{\prime}\right), d_{G}(a) \geq 3$ and $d_{G}(b) \geq 3$. Let $C^{\prime}$ be a connected component of $G^{\prime} \backslash\{a, b\}$. Let $u$ be a node of $C^{\prime}$, and let $C$ be the connected component of $G \backslash\{a, b\}$ that contains $u$. Since $d_{G}(u) \geq 3$, it follows that $|V(C)| \geq 2$. This is true of every connected component of $G^{\prime} \backslash\{a, b\}$. Also, since $G$ is connected and has no 1-cutset, for every connected component $C$ of $G \backslash\{a, b\}$ there is an $a b$-path in $G[C \cup\{a, b\}]$. Therefore $\{a, b\}$ is a proper 2 -cutset of $G$.

A decomposition tree of a graph $G$ is a rooted tree $T_{G}$ such that the following hold:

1. $G$ is the root of $T_{G}$.
2. For every non-leaf node $H$ of $T_{G}$, the children of $H$ are the blocks of decomposition of $H$ w.r.t. a 1-cutset, a proper 1-join or a proper 2-cutset of $H$.

Lemma 5.6 Let $G$ be any graph and let $T$ be a decomposition tree of $G$. Then $T$ has size $\mathcal{O}(n)$.

Proof - Note that $T$ is finite since the children of a graph are smaller than its parent. Let $T^{\prime}$ be the subtree of $T$ on the nodes that are graphs on at least five nodes. For any graph $G$ we define $\varphi(G)=|V(G)|-4$. It is easily seen that when $G_{X}, G_{Y}$ are the blocks of $G$ with respect to some decomposition, then $\varphi(G) \geq \varphi\left(G_{X}\right)+\varphi\left(G_{Y}\right)$. Indeed, for a 2 -cutset with split ( $X, Y, a, b$ ) where the marker $c$ is not a real node of $G$ the inequality follows from $\varphi(G)=|X|+|Y|-2, \varphi\left(G_{X}\right)=|X|-1$ and $\varphi\left(G_{Y}\right)=|Y|-1$. For the other decompositions, the proof is similar.

Since in $T^{\prime}$ every node is a graph on at least five nodes, every node $F$ of $T^{\prime}$ is such that $\varphi(F) \geq 1$. So the number of leaves of $T^{\prime}$ is at most $\varphi(G)$.

Hence the size of $T^{\prime}$ is $\mathcal{O}(n)$. It follows that the size of $T$ is also $\mathcal{O}(n)$, since the decomposition of the graphs that have fewer than 5 nodes is bounded by a constant.

Decomposition trees would be sufficient for a recognition algorithm, but for coloring we need a more sophisticated kind of tree. A proper decomposition tree of a connected graph $G \in \mathcal{C}$ is a rooted tree $T_{G}$ such that the following hold:

1. $G$ is the root of $T_{G}$.
2. Every node of $T_{G}$ is a connected graph.
3. Every leaf of $T_{G}$ is basic.
4. Every non-leaf node $H$ of $T_{G}$ is of one of the following type:

Type 1: the children of $H$ in $T_{G}$ are the blocks of decomposition w.r.t. a 1-cutset or a proper 1-join;

Type 2: $H$ and all its descendants are Petersen-, triangle-, square-free and have no 1-cutset and no proper 1-join. Moreover the children of $H$ in $T_{G}$ are the blocks of decomposition w.r.t. a proper 2-cutset and every non-leaf descendant of $H$ is of type 2 .
5. If a node of $T_{G}$ is a triangle-free graph then all its descendants are triangle-free graphs.

Theorem 5.7 There is an algorithm with the following specifications.
Input: A connected graph $G$.
Output: $G$ is correctly identified as not belonging to $\mathcal{C}$, or if $G \in \mathcal{C}$, a proper decomposition tree for $G$.

Running time: $\mathcal{O}(m n)$.
Proof - Consider the following algorithm.

Step 1: Let $G$ be the root of $T_{G}$.
Step 2: If all the leaves of $T_{G}$ have been declared as LEAF NODE, then output $T_{G}$ and stop. Otherwise, let $H$ be a leaf of $T_{G}$ that has not been declared a LEAF NODE.

Step 3: If $H$ is basic, declare $H$ to be a LEAF NODE and go to Step 2.
Step 4: If $H$ has a 1-cutset, then let the children of $H$ be the blocks of decomposition w.r.t. this 1-cutset, and go to Step 2.

STEP 5: If $H$ has a 1-join, then check whether this 1-join is proper. If it is, then let the children of $H$ be the blocks of decomposition by this proper 1-join, and go to Step 2. If it is not, then output $G \notin \mathcal{C}$ and stop.

Step 6: Apply algorithm from Lemma 5.5 to $H$. If the output is that $G \notin \mathcal{C}$ then output the same and stop. Otherwise a proper 2 -cutset is found. Then let the children of $H$ be the blocks of decomposition by this proper 2-cutset, and go to Step 2.

Note that this algorithm stops, because the children of a graph are smaller than its parent. We first prove the correctness of the algorithm. If the algorithm stops in Step 5, then by Lemma 5.4, $G$ is correctly identified as not belonging to $\mathcal{C}$. If the algorithm stops in Step 6 , then by Lemma 5.5, $G$ is correctly identified as not belonging to $\mathcal{C}$. So we may assume that the algorithm stops in Step 2. This means that the algorithm outputs a decomposition tree $T_{G}$. By Lemma [5.3, it follows that $G \in \mathcal{C}$ since every leaf of $T_{G}$ is basic. Let us check that $T_{G}$ is proper.

Clearly, $G$ is the root of $T_{G}$ and every leaf of $T_{G}$ is basic. Since $G$ is connected, and by the construction of blocks of decomposition, all nodes of $T_{G}$ are connected graphs. Let $H$ be a non-leaf node of $T_{G}$. Note that $H$ is not basic because of Step 3 .

If $H$ is Petersen-, triangle- and square-free, has no 1-cutset and no proper 1-join, then by Theorem [2.4, $H$ has a proper 2 -cutset and is decomposed along a proper 2-cutset because of Step 6. Also, by Lemma 5.2 the children of $H$ are also connected, Petersen-, triangle-, square-free, and have no 1cutset and no proper 1-join. So by induction, every non-leaf descendant of $H$ is decomposed along proper 2 -cutsets and $H$ is of type 2 .

Else, $H$ contains a triangle, a square, the Petersen graph or has a 1cutset or a proper 1-join. By Theorems 2.2 and 2.3, $H$ must have a 1-cutset or a proper 1-join. Note that this 1-cutset or proper 1-join is discovered by the algorithm rather than a possible proper 2 -cutset. So, $H$ is of type 1.

So every non-leaf node of $T_{G}$ is of type 1 or 2 , and by Lemma 5.1, if a node of $T_{G}$ is a triangle-free graph then all its descendants are triangle-free graphs. We have proved that $T_{G}$ is a proper decomposition tree.

We now show that the algorithm can be implemented to run in time $\mathcal{O}(n m)$. Testing whether a graph is a clique in Step 3 relies only on a check of the degrees: $H$ is a clique if and only if every node has degree $n-1$, so this can be done in time $\mathcal{O}(n+m)$. To decide whether a graph is strongly 2-bipartite, we also check the degrees to be sure that nodes of degree 2 and nodes of degree at least 3 form stable sets. We still have to check that $H$ is square-free, but this can be done by running the $\mathcal{O}(n+m)$ algorithm of Dahlhaus [10] for 1-joins because at this step, $H$ contains a square if and only if $H$ has a 1-join.

To find a 1-cutset in Step 4, we use the $\mathcal{O}(n+m)$ algorithm of Hopcroft and Tarjan [14, 23]. To find a 1-join in Step 5, we use the $\mathcal{O}(n+m)$ algorithm of Dahlhaus [10]. By Lemma 5.5. Step 6 can be implemented to run in time $\mathcal{O}(n+m)$. Now we note that when the algorithm stops, it has computed a decomposition tree (that will be output or not when $G \notin \mathcal{C}$ ), and the numbers of steps processed by the algorithm is bounded by the size of this tree. By Lemma [5.6 the size of the tree is $\mathcal{O}(n)$, so we have to run $\mathcal{O}(n)$ times each of the steps, and hence the overall complexity is $\mathcal{O}(n m)$.

Theorem 5.8 There exists an $\mathcal{O}(n m)$-time algorithm that decides whether a graph is in $\mathcal{C}$.

Proof - Apply the $\mathcal{O}(\mathrm{nm})$ algorithm from Theorem 5.7. If the output is $G \notin \mathcal{C}$ then output the same. Else $G$ has a proper decomposition tree and $G \in \mathcal{C}$ by Lemma 5.3.

## 6 Coloring

Let us call third color of a graph any stable set that contains at least one node of every odd cycle. Any graph that admits a third color $S$ is 3-colorable: give color 3 to the third color; since $G \backslash S$ contains no odd cycle, it is bipartite: color it with colors 1,2 . We shall prove by induction that any triangle-free graph in $\mathcal{C}$ has a third color. But for the sake of induction, we need to prove a stronger statement.

Let us call strong third color of a graph any stable set that contains at least one node of every cycle (odd or even). By $N[v]$ we denote $\{v\} \cup N(v)$. When $v$ is a node of a graph $G$, a pair of disjoint subsets $(R, T)$ of $V(G)$ is admissible with respect to $G$ and $v$ if one of the following holds (see Fig. [6):

- $T=N(v)$ and $R=\{v\}$;
- $T=\emptyset$ and $R=N[v] ;$
- $v$ is of degree two, $N(v)=\{u, w\}, T=\{u\}, R=\{v, w\}$;
- $v$ is of degree two, $N(v)=\{u, w\}, T=\{u\}, R=N[w]$;
- $v$ is of degree two, $N(v)=\{u, w\}, T=\emptyset, R=\{u\} \cup N[w]$.

We say that a pair of disjoint subsets $(R, T)$ is an admissible pair of $G$ if for some $v \in V(G),(R, T)$ is admissible w.r.t. $G, v$. An admissible pair ( $R, T$ ) should be seen as a constraint for coloring: we will look for third colors (sometimes strong, sometimes not) that must contain every node of $T$ and no node of $R$. We will do this first in basic graphs, and then by induction in all triangle-free graphs of $\mathcal{C}$, thus proving that they are 3 -colorable.


Figure 6: Five examples of admissible pairs (nodes of $T$ are white, nodes of $R$ are black)

Lemma 6.1 Let $G$ be a triangle-free basic graph that is not the Petersen graph. Let $(R, T)$ be an admissible pair of $G$. Then $G$ admits a strong third color $S$ such that $T \subseteq S$ and $S \cap R=\emptyset$. Furthermore, $S$ can be found in time $\mathcal{O}(n+m)$.

Proof - Note that squares may fail to admit strong third color that satisfies our constraints (because when $G$ is a square $R=V(G)$ is possible). But squares are not basic. The proof follows from the following claims, since it will be clear that all $S$ 's found in them can be found in time $\mathcal{O}(n+m)$.
(1) The lemma holds when $G$ is a chordless cycle of length at least 7 .

Because then any non-empty set of nodes is a strong third color. Since $R=V(G)$ is impossible because any path in $G[R]$ is of length at most 3, it is always possible to pick a node of $G$ not in $R$. This proves (1).
(2) The lemma holds when $G$ is a clique, a strongly 2-bipartite graph or is an induced subgraph of the Heawood graph.

Note that $G$ is bipartite (for cliques, because it is triangle free). Let $A, B$ be a bipartition of $G$. Note that $A, B$ can be computed in linear time. Up to symmetry between $A, B$ we may assume $T \subset A$ and $|A \cap R| \leq 2$. Let $S=A \backslash R$. So $T \subset S$ and $S \cap R=\emptyset$. Moreover, $|A \backslash S| \leq 2$. So every cycle in $G \backslash S$ contains at most two nodes of $A$, and since $G$ is square-free, there is no such cycle. This proves (2).
(3) The lemma holds when $G$ is a proper induced subgraph of the Petersen graph.

Note that by assumption, $G$ is not the Petersen graph. We use our notation for the Petersen graph $\Pi$. So $V(G) \subsetneq V(\Pi)=\left\{a_{1}, \ldots, a_{5}, b_{1}, \ldots, b_{5}\right\}$. Let $v$ be a node of $G$ and $(R, T)$ be admissible with respect to $G, v$. We may assume $v=a_{1}$ since the Petersen graph is vertex-transitive. Note that $a_{1} \in V(G)$.

Suppose $T=N\left(a_{1}\right)$ and $R=\left\{a_{1}\right\}$. Then we put $Q=\left\{a_{2}, a_{5}, b_{1}\right\}$ and we observe that $\Pi \backslash Q$ is a $C_{6}$ plus an isolated node. But some node $z \neq a_{1}$
of $\Pi$ is not a node of $G$. If $z$ is in the $C_{6}$, then $S=Q \cap V(G)$ is a strong third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$. Else $z$ must be a neighbor of $a_{1}$, say $a_{5}$ up to symmetry. So, $S=\left(Q \cup\left\{a_{4}\right\}\right) \cap V(G)$ is a strong third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

Suppose $T=\emptyset$ and $R=N[v]$. Then we put $Q=\left\{a_{3}, b_{3}, b_{5}\right\}$ and we observe that $\Pi \backslash Q$ is a tree. So $S=Q \cap V(G)$ is a strong third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

From here on we may assume that $v$ is a node of degree two of $G$. So up to symmetry we may assume $b_{1} \notin V(G)$ and $N_{G}(v) \subseteq\left\{a_{2}, a_{5}\right\}$.

Suppose $N(v)=\{u, w\}, T=\{u\}, R=\{v, w\}$. So w.l.o.g. $T=\left\{a_{2}\right\}$ and $R=\left\{a_{1}, a_{5}\right\}$. Then we put $Q=\left\{a_{2}, b_{3}\right\}$ and we observe that $\Pi \backslash Q$ is a tree. So $S=Q \cap V(G)$ is a strong third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

Suppose $N(v)=\{u, w\}, T=\{u\}, R=N[w]$. So up to symmetry we may assume $T=\left\{a_{2}\right\}, R=\left\{a_{1}, a_{5}, a_{4}, b_{3}\right\} \cap V(G)$. Then we put $Q=\left\{a_{2}, b_{2}, b_{5}\right\}$ and we observe that like in the previous case that $\Pi \backslash Q$ is a tree. So $S=Q \cap V(G)$ is a strong third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

Suppose $N(v)=\{u, w\}, T=\emptyset, R=\{u\} \cup N[w]$. So up to symmetry we may assume $R=\left\{a_{1}, a_{2}, a_{5}, b_{3}, a_{4}\right\} \cap V(G)$. We put $Q=\left\{b_{1}, a_{3}, b_{4}\right\}$ and we observe that $\Pi \backslash Q$ is a tree. So $S=Q \cap V(G)$ is a strong third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$. This proves (3).

Lemma 6.2 Let $G$ be the Petersen graph and $(R, T)$ be an admissible pair of $G$. Then $G$ admits a third color $S$ (possibly not strong) such that $T \subseteq S$ and $S \cap R=\emptyset$. Furthermore, $S$ can be found in time $\mathcal{O}(1)$.

Proof - We use our notation for the Petersen graph: $V(G)=\left\{a_{1}, \ldots, a_{5}\right.$, $\left.b_{1}, \ldots, b_{5}\right\}$. Let $v$ be a node of $G$ and $(R, T)$ be admissible with respect to $G, v$. We may assume $v=a_{1}$ since the Petersen graph is vertex-transitive. Since $v$ has degree three, we just have to study the following two cases:

Suppose $T=N\left(a_{1}\right)$ and $R=\left\{a_{1}\right\}$. Then we put $S=T$ and we observe that $G \backslash S$ is a $C_{6}$ plus an isolated node. So $S$ is a third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$. Note that in this case there exists no strong third color that satisfies our constraints.

Suppose $T=\emptyset$ and $R=N[v]$. Then we put $S=\left\{a_{3}, b_{3}, b_{5}\right\}$ and we observe that $G \backslash S$ is a tree. So $S$ is a third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

Lemma 6.3 Let $G$ be a non-basic, connected, triangle-free, square-free and Petersen-free graph in $\mathcal{C}$ that has no 1-cutset and no proper 1-join. Let $(R, T)$ be an admissible pair of $G$. Then $G$ admits a strong third color $S$ such that $T \subseteq S$ and $S \cap R=\emptyset$. Furthermore, $S$ is obtained in time $\mathcal{O}(1)$
from well chosen strong third colors of blocks of $G$ w.r.t. a proper 2-cutset of $G$.

Proof - By Theorem [2.4, $G$ has a proper 2 -cutset. Let $(X, Y, a, b)$ be a split of a proper 2-cutset of $G$. Let $v$ be a node of $G$ and $(R, T)$ an admissible pair with respect to $G, v$. We now show that $G$ admits a strong third color $S$ such that $T \subseteq S$ and $S \cap R=\emptyset$. We use induction on the blocks of decomposition $G_{X}$ and $G_{Y}$ w.r.t. this proper 2-cutset, as defined in Section5. Note that by Lemma 5.2, $G_{X}$ and $G_{Y}$ are connected, triangle-free, square-free, Pertersen-free, contain no 1-cutset and no proper 1-join.

Here below, when we write "by induction", we mean that either we use inductively Lemma 6.3 for a smaller graph (when this smaller graph is not basic), or that we use Lemma 6.1 (when this smaller graph is basic). By symmetry it is enough to consider the following three cases.
Case 1: $v=a$.
Since $a$ is not of degree two, either $T=N(a)$ and $R=\{a\}$, or $T=\emptyset$ and $R=N[a]$.

Suppose that $T=N(a)$. By induction there is a strong third color $S_{X}$ of $G_{X}$ (resp. $S_{Y}$ of $G_{Y}$ ) such that $N_{G_{X}}(a) \subseteq S_{X}$ (resp. $N_{G_{Y}}(a) \subseteq S_{Y}$ ). So marker node $c \in S_{X} \cap S_{Y}$, and hence neither $a$ nor $b$ belongs to $S_{X} \cup S_{Y}$. Therefore $S=S_{X} \cup S_{Y}$ with possibly $c$ removed if $c$ is not a real node of $G$, is a stable set of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$ (since $R=\{a\}$ ). Let $H$ be a cycle of $G$. If $H$ contains $a$, then it must contain a node of $N(a)=T$, and hence it contains a node of $S$. So assume that $H$ does not contain $a$. Since $H$ does not contain $a$, w.l.o.g. $V(H) \subseteq X \cup\{b\}$ and does not contain $c$. Hence $H$ is a cycle of $G_{X}$ that does not contain $c$. Since $S_{X}$ is a strong third color of $G_{X}$, a node of $H$ belongs to $S_{X} \backslash\{c\}$, and hence to $S$. Therefore $S$ is a strong third color of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

Now suppose that $R=N[a]$. Note that since $c$ is of degree two in $G_{X}$, $\left(N_{G_{X}}[a],\{b\}\right)$ is an admissible pair w.r.t. $G_{X}, c$, and hence by induction, there exists a strong third color $S_{X}$ of $G_{X}$ such that $N_{G_{X}}[a] \cap S_{X}=\emptyset$ (in particular, $\left.c \notin S_{X}\right)$ and $b \in S_{X}$. Similarly, there exists a strong third color $S_{Y}$ of $G_{Y}$ such that $N_{G_{Y}}[a] \cap S_{Y}=\emptyset$ (in particular, $c \notin S_{Y}$ ) and $b \in S_{Y}$. Clearly $S=S_{X} \cup S_{Y}$ is a stable set such that $\emptyset=T \subseteq S$ and $S \cap R=\emptyset$. Let us check that every cycle of $G$ contains a node of $S$. Let $H$ be a cycle of $G$. If $H$ contains $b$ then we are done since $b \in S$, so w.l.o.g. $V(H) \subseteq X \cup\{a\}$, i.e. $H$ is a cycle of $G_{X}$, and hence, since $S_{X}$ is a strong third color of $G_{X}$, $S_{X}$ contains a node of $H$, and so does $S$.
Case 2: $v$ is of degree two, $N(v)=\{u, w\}, v \in X, w=a$ and $N[w] \subseteq R$.
Note that either $u=b$ or $u \in X$, and either $u \in R$ or $u \in T$. By induction there exists a strong third color $S_{X}$ of $G_{X}$ such that $N_{G_{X}}[w] \cap S_{X}=\emptyset$ and $u \in S_{X}$ if and only if $u \in T$. By induction, since $c$ is of degree 2 (and hence both $\left(N_{G_{Y}}[w],\{b\}\right)$ and $\left(N_{G_{Y}}[w] \cup\{b\}, \emptyset\right)$ are admissible w.r.t. $\left.G_{Y}, c\right)$, there exists a strong third color $S_{Y}$ of $G_{Y}$ such that $N_{G_{Y}}[w] \cap S_{Y}=\emptyset$ and $b \in S_{Y}$
if and only if $b \in S_{X}$. Clearly $S=S_{X} \cup S_{Y}$ is a stable set of $G$ such that $T \subseteq S$ and $R \cap S=\emptyset$. Since $c \notin S_{X}$ and $c \notin S_{Y}$, it is easy to see that $S$ contains a node of every cycle of $G$, i.e. $S$ is a strong third color of $G$.
Case 3: $T \cup R \subseteq X \cup\{a, b\}$.
By induction there exists a strong third color $S_{X}$ of $G_{X}$ such that $T \subseteq S_{X}$ and $R \cap S_{X}=\emptyset$. If $c \in S_{X}$ and $c$ is a real node of $G$, then let $T_{Y}=N_{G_{Y}}(a)$ and $R_{Y}=\{a\}$. Note that $\left(R_{Y}, T_{Y}\right)$ is an admissible pair w.r.t. $G_{Y}, a$. In all other cases, let $T_{Y}=S_{X} \cap\{a, b\}$ and $R_{Y}=\{c\} \cup\left(\{a, b\} \backslash S_{X}\right)$. Note that $\left(R_{Y}, T_{Y}\right)$ is an admissible pair w.r.t. $G_{Y}, c$. By induction there exists a strong third color $S_{Y}$ of $G_{Y}$ such that $T_{Y} \subseteq S_{Y}$ and $R_{Y} \cap S_{Y}=\emptyset$. Note that $S_{X} \cap\{a, b\}=S_{Y} \cap\{a, b\}$. Furthermore, if $c \in S_{X}$ and $c$ is a real node of $G$, then $c \in S_{Y}$, and in all other cases $c \notin S_{Y}$ because $c \in R_{Y}$. If $c \in S_{X}$ and $c$ is a real node of $G$, then let $S=S_{X} \cup S_{Y}$, and otherwise let $S=\left(S_{X} \cup S_{Y}\right) \backslash\{c\}$. Clearly $S$ is a stable set of $G$ such that $T \subseteq S$ and $R \cap S=\emptyset$.

Let $H$ be a cycle of $G$. We now show that $S$ contains a node of $H$. If $H$ is a cycle of $G_{X}$ (resp. $G_{Y}$ ) then $S_{X}$ (resp. $S_{Y}$ ) contains a node of $H$, and hence so does $S$. So we may assume that $H$ is not a cycle of $G_{X}$ nor $G_{Y}$. In particular $H$ contains both $a$ and $b$, a node of $X$ and a node of $Y$. Let $H_{Y}$ be the $a b$-subpath of $H$ whose intermediate nodes belong to $Y$. Note that $H_{Y} \neq a c b$, since otherwise $H$ belongs to $G_{X}$. So $V\left(H_{Y}\right) \cup\{c\}$ induces a cycle of $G_{Y}$. Since $S_{Y}$ is a strong third color of $G_{Y}$, it contains a node $h$ of $V\left(H_{Y}\right) \cup\{c\}$. If $h \neq c$ then $h \in S \cap V(H)$. So assume that $h=c$. But then $c \notin R_{Y}$ so $c \in S_{Y}$, and hence $c \in S_{X}$ and it is a real node of $G$. Therefore, $c \in S \cap V(H)$.

Lemma 6.3 implies a weaker statement: the existence of a third color (possibly not strong). But we do not know how to prove this weaker result with the weaker induction hypothesis. An attempt fails at the proof of Case 3 .

Lemma 6.4 Let $G$ be a non-basic connected triangle-free graph in $\mathcal{C}$ and $(R, T)$ be an admissible pair of $G$. Then $G$ admits a third color $S$ such that $T \subseteq S$ and $S \cap R=\emptyset$. Furthermore, $S$ is obtained in time $\mathcal{O}(1)$ from well chosen third colors of blocks of $G$ w.r.t. a 1-cutset, a proper 1-join or a proper 2-cutset of $G$.

Proof - Here below, we use the fact that every strong third color is a third color with no explicit mention. So, we may assume that $G$ contains a square or the Petersen graph, or has a 1-cutset or a proper 1-join, for otherwise the result follows from Lemma 6.3. Hence, by Theorem [2.3, the proof follows from the following two claims.

Here below, when we write "by induction", we mean that either we use inductively Lemma 6.4 for a smaller graph (when this smaller graph is not
basic), or that we use Lemma 6.1 or 6.2 (when this smaller graph is basic).
(1) The lemma holds when $G$ has a 1-cutset.

Let $(X, Y, z)$ be a split of a 1-cutset of $G$. Let $G_{X}$ and $G_{Y}$ be the blocks of decomposition w.r.t. this 1-cutset. Note that $G_{X}$ and $G_{Y}$ are triangle-free by Lemma 5.1 .
Case 1: $\quad X \cap(R \cup T)$ and $Y \cap(R \cup T)$ are both non-empty.
Then, $z \in R \cup T$. We put $R_{X}=R \cap(X \cup\{z\}), R_{Y}=R \cap(Y \cup\{z\})$, $T_{X}=T \cap(X \cup\{z\}), T_{Y}=T \cap(Y \cup\{z\})$. We observe that $\left(R_{X}, T_{X}\right)$ and $\left(R_{Y}, T_{Y}\right)$ are admissible with respect to $G_{X}$ and $G_{Y}$ respectively. So by induction there exists a third color $S_{X}$ of $G_{X}$ such that $S_{X} \subseteq T_{X}$, $R_{X} \cap S_{X}=\emptyset$, and a third color $S_{Y}$ of $G_{Y}$ such that $T_{Y} \subseteq S_{Y}, R_{Y} \cap S_{Y}=\emptyset$. So, $S=S_{X} \cup S_{Y}$ is a third color of $G$ such that $T \subseteq S$ and $R \cap S=\emptyset$.
Case 2: One of $X \cap(R \cup T), Y \cap(R \cup T)$ is empty.
We assume w.l.o.g. that $Y \cap(R \cup T)=\emptyset$. Hence, $R \cup T \subseteq X \cup\{z\}$. Let $S_{X}$ be a third color of $G_{X}$ such that $S_{X} \subseteq T, R \cap S_{X}=\emptyset$. If $z \in S_{X}$, let $S_{Y}$ be a third color of $G_{Y}$ such that $z \in S_{Y}$. Else, let $S_{Y}$ be a third color of $G_{Y}$ such that $z \notin S_{Y}$. In either case, $S=S_{X} \cup S_{Y}$ is a third color of $G$ such that $S \subseteq T$ and $R \cap S=\emptyset$. This proves (1).

## (2) The lemma holds when $G$ has a proper 1-join.

Let $(X, Y, A, B)$ be a split of a proper 1-join of $G$. We show that $G$ admits a third color $S$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

Suppose that $v$ is of degree at least three. Then we assume w.l.o.g. $v \in X$. So $v \in V\left(G_{X}\right), T \cap Y=B$ or $\emptyset$, and $R \cap Y=B$ or $\emptyset$.

Suppose that $v$ is of degree 2 and $N(v)=\{u, w\}$. If $\{u, v, w\}$ is contained in $X$ or $Y$, then we assume w.l.o.g. that it is contained in $X$. Otherwise, $v$ must be contained in $A \cup B$, and we assume w.l.o.g. that $v \in B$, which implies that $A=\{u, w\}$ (since $|A| \geq 2$ ). If $v \in B$, we assume that the marker $y$ of block $G_{X}$ is $v$.

So in all cases $v \in V\left(G_{X}\right), T \cap Y=B$ or $\emptyset, R \cap Y=B$ or $\emptyset$, and if $N(v)=\{u, w\}$ then $v \notin A$ and $u, w \in X$. If $T \cap Y=B$ then let $T_{X}=(T \backslash B) \cup\{y\}$, and if $T \cap Y=\emptyset$ then let $T_{X}=T$. If $R \cap Y=B$ then let $R_{X}=(R \backslash B) \cup\{y\}$, and if $R \cap Y=\emptyset$ then let $R_{X}=R$. Note that $\left(R_{X}, T_{X}\right)$ is an admissible pair w.r.t. $G_{X}, v$. By induction, there exists a third color $S_{X}$ of $G_{X}$ such that $T_{X} \subseteq S_{X}$ and $R_{X} \cap S_{X}=\emptyset$. By induction, there exists a third color $S_{Y}^{\prime}$ of $G_{Y}$ such that $N(x)=B \subseteq S_{Y}^{\prime}$, and a third color $S_{Y}^{\prime \prime}$ of $G_{Y}$ such that $S_{Y}^{\prime \prime} \cap N[x]=\emptyset$. If $y \in S_{x}$ then let $S_{Y}=S_{Y}^{\prime}$, and otherwise let $S_{Y}=S_{Y}^{\prime \prime}$. Note that $x \notin S_{Y}$, i.e. $S_{Y} \subseteq Y$. Let $S=\left(S_{X} \cap X\right) \cup S_{Y}$. Note that only one of $S_{Y}^{\prime}$, $S_{Y}^{\prime \prime}$ needs to be computed once $S_{X}$ is known.

Clearly $S$ is a stable set. If $T \cap Y=\emptyset$ then $T_{X}=T$, and hence, since $T_{X} \subseteq S_{X}, T \subseteq S$. If $T \cap Y=B$ then $y \in T_{X}$, and hence, since $T_{X} \subseteq S_{X}$ (and in particular $y \in S_{X}$ ), $B \subseteq S$, and therefore $T \subseteq S$. If $R \cap Y=\emptyset$
then $R_{X}=R$, and hence, since $R_{X} \cap S_{X}=\emptyset, R \cap S=\emptyset$. If $R \cap Y=B$ then $R_{X}=(R \backslash B) \cup\{y\}$, and hence, since $R_{X} \cap S_{X}=\emptyset, y \notin S_{X}$ and so $S_{Y} \cap N[x]=\emptyset$, implying that $R \cap S=\emptyset$.

So it only remains to show that $S$ contains a node of every odd cycle of $G$. Let $H$ be an odd cycle of $G$. If $V(H) \subseteq X$, then since $S_{X}$ is a third color of $G_{X}, S_{X}$ contains a node of $H$, and hence so does $S$. If $V(H) \subseteq Y$, then since $S_{Y}$ is a third color of $G_{Y}, S_{Y}$ contains a node of $H$, and hence so does $S$. So we may assume that $H$ contains both a node of $X$ and a node of $Y$. Hence, $H$ is node-wise partitioned into a path of $X$ and a path of $Y$ of different parity. Hence if we suppose that $H$ is minimal with respect to the property of overlapping $X, Y$ and being odd, then either $V(H) \cap X=\left\{h_{X}\right\} \subseteq A$ or $V(H) \cap Y=\left\{h_{Y}\right\} \subseteq B$. Suppose that $V(H) \cap X=\left\{h_{X}\right\}$. Then $\left(V(H) \backslash\left\{h_{X}\right\}\right) \cup\{x\}$ induces an odd cycle $H^{\prime}$ of $G_{Y}$. Since $S_{Y}$ is a third color of $G_{Y}, S_{Y}$ contains a node $h$ of $H^{\prime}$. Since $x \notin S_{Y}, h$ is a node of $V(H) \cap S$. Finally assume that $V(H) \cap Y=\left\{h_{Y}\right\}$. Then $\left(V(H) \backslash\left\{h_{Y}\right\}\right) \cup\{y\}$ induces an odd cycle $H^{\prime}$ of $G_{X}$. Since $S_{X}$ is a third color of $G_{X}, S_{X}$ contains a node $h$ of $H^{\prime}$. If $h \neq y$ then $h$ is a node of $V(H) \cap S$. So assume $h=y$. Then $y \in S_{X}$ and hence $B \subseteq S_{Y}$, and in particular $h_{Y} \in V(H) \cap S$. This proves (2).

Our proof of Lemmas 6.3, 6.4 suggests that for every triangle-free graph in $\mathcal{C}$ there might exist a stable set that intersects every cycle. Such a property might be of use for stronger notions of coloring (list coloring, ...). It holds for every basic graph (even for the Petersen graph), for every square-and-Petersen-free graph by a slight variant of Lemma 6.3 and we almost proved it in general. But it is false. Let us build a counter-example $G$, obtained from four disjoint copies $\Pi_{1}, \ldots, \Pi_{4}$ of the Petersen graph minus one node. So $\Pi_{i}$ contains a set $X_{i}$ of three nodes of degree two $(i=1, \ldots, 4)$. We add all edges between $X_{1}, X_{2}$, between $X_{2}, X_{3}$, between $X_{3}, X_{4}$ and between $X_{4}, X_{1}$. Note that $G$ can be obtained by gluing one square $S=s_{1} s_{2} s_{3} s_{4}$ and four disjoint copies $\Pi_{1}, \Pi_{2}, \Pi_{3}, \Pi_{4}$ of the Petersen graph along Operation $\mathcal{O}_{2}$ of Theorem 4.2 as follows: let $G=S$ and for $i=1$ to $i=4$, replace $G$ by itself glued through $s_{i}$ with $\Pi_{i}$. So $G \in \mathcal{C}$ by Theorem4.2,

We claim now that $G$ does not contain a stable set that intersects every cycle. Indeed, if $S$ is such a stable set then $S$ must contain all nodes in one of the $X_{i}$ 's for otherwise we build a $C_{4}$ of $G \backslash S$ by choosing a node in every $X_{i}$. So $X_{1} \subseteq S$ say. We suppose that $\Pi_{1}$ has nodes $\left\{a_{2}, \ldots, a_{5}, b_{1}, \ldots, b_{5}\right\}$ with our usual notation. So $X_{1}=\left\{b_{1}, a_{2}, a_{5}\right\} \subseteq S$ and we observe that every node in $C=V\left(\Pi_{1} \backslash S\right)$ has a neighbor in $X_{1}$. Hence $C \cap S=\emptyset$ while $G[C]$ is a cycle on six nodes, a contradiction. Note that for any node $v$ of $G, G \backslash v$ contains a stable set that intersects every cycle. So, a characterisation by forbidding induced subgraphs of the class of graphs that admit a stable set intersecting every cycle needs to consider $G$ somehow. For this reason, we
believe that such a characterisation must be complicated.
Theorem 6.5 If $G \in \mathcal{C}$, then either $\chi(G)=\omega(G)$ or $\chi(G) \leq 3$. In particular, $\chi(G) \leq \omega(G)+1$.

Proof - Clearly we may assume that $G$ is connected. If $\omega(G) \leq 2$ then $\chi(G) \leq 3$ since $G$ contains a third color by Lemma 6.1, 6.2 or 6.4 (indeed, every non-empty graph has an admissible pair: $(N[v], \emptyset)$ ). If $\omega(G) \geq 3$ then by Theorem [2.2, $G$ admits a 1-cutset. So every 2-connected component of $G$ is either a clique or is 3 -colorable. Hence $\chi(G)=\omega(G)$.

Theorem 6.6 There exists an algorithm that computes an optimal coloring of any graph in $\mathcal{C}$ in time $\mathcal{O}(n m)$.

Proof - Let $G$ be a graph in $\mathcal{C}$. When $G$ has a 1-cutset, then it is easy to obtain an optimal coloring of $G$ from optimal colorings of its blocks. So by Theorem 2.2 we may assume that our input graph is triangle-free. We may also assume that $G$ is connected and not bipartite. Now we show how to 3 -color $G$ by finding a third color of $G$.

We first construct a proper decomposition tree $T_{G}$ of the input graph $G$ in time $\mathcal{O}(\mathrm{nm})$, by Theorem 5.7. Let $v$ be any node of $G$. We associate with node $G$ of $T_{G}$ an admissible pair $(R, T)$, say $R=N[v]$ and $T=\emptyset$, and we use Lemmas 6.1, 6.2, 6.3 and 6.4 to recursively find a third color $S$ of $G$ such that $T \subseteq S$ and $S \cap R=\emptyset$.

First note that all the leaves of $T_{G}$ are basic, and since $G$ is trianglefree, they are one of the graphs described in Lemma 6.1 or 6.2, So once the appropriate admissible pairs have been associated with a given leaf of $T_{G}$, Lemma 6.1 or 6.2 shows how to find the appropriate third color (or if needed, strong third colors), each in linear time in the size of the leaf.

For a non-leaf node $H$ of type 1 of $T_{G}$, Lemma 6.4 shows how to proceed (in linear time) to find a third color of $H$ by asking recursively for appropriately chosen third colors of its children (i.e. choosing appropriate admissible pairs to associate with its children, and once the appropriate third colors of its children are found how to put them together to find the desired third color of $H$ ). Note that in several cases, the algorithm has to compute the third colors of the children $H_{1}, H_{2}$ of $H$ in a prescribed order, that is wait for the answer for the coloring of $H_{1}$ before knowing what admissible pair is needed for the coloring of $\mathrm{H}_{2}$.

For a non-leaf node $H$ of type 2 of $T_{G}$, to recursively find a third color of $H$, we actually need to find a strong third color. This we can do by proceeding as in the proofs of Lemma 6.3. Note that since $H$ is of type 2, every non-leaf descendant of $H$ is of type 2 . Also, no descendant of $H$ contains the Petersen graph, so all leaves under $H$ will have a strong third color computed by Lemma 6.1.

So the processing time at each non-leaf node of $T_{G}$ is $\mathcal{O}(1)$. Since by Lemma 5.6 the size of $T_{G}$ is $\mathcal{O}(n)$, the sum of processing times at the leaves of $T_{G}$ is $\mathcal{O}(n+m)$. So the time needed to process the tree is $\mathcal{O}(n+m)$.

Hence, the total computation time is $\mathcal{O}(n m)$.
We note that the algorithm above has complexity $\mathcal{O}(n+m)$ once the proper decomposition tree is given. So, if one can find an $\mathcal{O}(n+m)$-time algorithm for constructing a proper decomposition tree, then one gets an $\mathcal{O}(n+m)$-time coloring algorithm for graphs in $\mathcal{C}$.

## 7 Cliques and stable sets

The problems of finding a maximum clique and a maximum stable set for a graph in $\mathcal{C}$ have a complexity that is easy to establish. Note first that any graph in $\mathcal{C}$ contains no diamond. So any edge of a graph in $\mathcal{C}$ is contained in a unique maximal clique. Hence, to find a maximum clique in polynomial time, it is enough to look for the common neighborhood of every edge. A simple implementation of this leads to an $\mathcal{O}(n m)$-time algorithm and no faster algorithm seems to be known for diamond-free graphs. Here below we give a linear time algorithm restricted to our class $\mathcal{C}$. Note in contrast that, as proved by Maffray and Preissmann, the coloring problem is NP-hard even when restricted to triangle-free graphs [17].

Theorem 7.1 There exists a linear time algorithm whose input is a graph in $\mathcal{C}$ and whose output is a maximum clique of $G$.

Proof - It is trivial to decide in linear time whether $\omega(G)=1$. So, we assume $\omega(G) \geq 2$. Then the 2 -connected components of $G$ can be found in linear time (see [14] and [23]). By Theorem 2.2, every 2-connected component is either a clique or is triangle-free. So, to find a maximum clique it suffices for each 2-connected component to test whether it is a clique or not, and to output a largest such clique (if any). If no 2 -connected component is a clique then output any edge.

A 2-subdivision is a graph obtained from any graph by subdividing twice every edge. More precisely, every edge $u v$ of a graph $G$ is replaced by an induced path $u a b v$ where $a$ and $b$ are of degree two. Let $F$ be the resulting graph. It is easy to see that $\alpha(F)=\alpha(G)+|E(G)|$. This construction, due to Poljak, easily yields:

Theorem 7.2 (Poljak [21]) The problem whose instance is a 2subdivision $G$ and an integer $k$ and whose question is "Does $G$ contain a stable set of size at least $k "$ is NP-complete.

Since every 2-subdivision is in $\mathcal{C}$, a direct consequence is:

Theorem 7.3 Finding a maximum stable set of a graph in $\mathcal{C}$ is NP-hard.

## 8 NP-completeness of $\Pi_{H_{3 \mid 3}}$

It is mentioned in Section 1 that the problem $\Pi_{H_{3 \mid 3}}$ is NP-complete. To prove this, we need a slight variation on a theorem proved in [16] using a refinement of a construction due to Bienstock [1. We remind the reader that $I$ denotes the graph on nodes $a, b, c, d, e, f$ with the following edges: $a b$, $a c, a d, b e, b f$.

For any integer $k \geq 1$ we define the problem $\Pi_{k}$ whose instance is described below:

1. a graph $G$ that does not contain $I$;
2. two nodes of degree two $x$ and $y$;
3. and these are such that there exists a path $P_{x}$ (resp. $P_{y}$ ), on $2 k-1$ nodes, all interior nodes of which are of degree 2, and $x$ (resp. $y$ ) is the middle of $P_{x}$ (resp. $P_{y}$ ). Moreover, $P_{x}$ and $P_{y}$ are node-disjoint.

The question of $\Pi_{k}$ is "does there exist an induced cycle of $G$ that goes through $x, y$ ?".

Theorem 8.1 (Lévêque, Lin, Maffray and Trotignon [16]) For all integers $k \geq 1$ the problem $\Pi_{k}$ is NP-complete.

Proof - The theorem is not stated this way in [16], so we explain here how to obtain a proof from [16]. In this article, Theorem 2.3 states that Problem $\Pi$, that is Problem $\Pi_{k}$ where Condition 3 is forgoten, is NP-complete. The proof is obtained by a reduction from 3-SAT. For every integer $k$ and every instance $f$ of 3 -SAT, an instance denoted by $G_{f}(k, 1,1,1,3,1,1)$ of $\Pi$ is built. It is shown that $f$ has a truth assignment if and only if $G_{f}(k, 1,1,1,3,1,1)$ contains an induced cycle that goes through $x, y$. It turns out that luckily $G_{f}(k, 1,1,1,3,1,1)$ is an instance of $\Pi_{k}$ (even if unluckily this is not explicitly stated in [16]), because $G_{f}(k, 1,1,1,3,1,1)$ satisfies Condition 3 ,

Theorem 8.2 Problem $\Pi_{H_{3 \mid 3}}$ is NP-complete.
Proof - We use the fact that every realisation of $H_{3 \mid 3}$ contains $I$ (note that this is false for $H_{1 \mid 1}, H_{2 \mid 1}, H_{3 \mid 1}, H_{2 \mid 2}$ and $H_{3 \mid 2}$ ). Let $G, x, y$ be an instance of the NP-complete problem $\Pi_{3}$. So, $G$ is a graph that does not contain $I$ and $x, y$ are two nodes of $G$ of degree two.

We prepare now an instance $G^{\prime}$ of $\Pi_{H_{3 \mid 3}}$ by adding an edge between $x, y$. We note that $G^{\prime}$ contains a unique $I$, induced by $N(x) \cup N(y)$. So, any
realisation of $H_{3 \mid 3}$ in $G^{\prime}$ must contain $P_{x}$ and $P_{y}$. It follows that $G$ contains a hole passing through $x, y$ if and only if $G^{\prime}$ contains a realisation of $H_{3 \mid 3}$.

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