Excluding Induced Subdivisions of the Bull and Related Graphs

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June 27, 2011

Abstract

For any graph H, let $\mathrm{Forb}^*(H)$ be the class of graphs with no induced subdivision of H. It was conjectured in [A.D. Scott, Induced trees in graphs of large chromatic number, *Journal of Graph Theory*, 24:297–311, 1997] that, for every graph H, there is a function $f_H : \mathbb{N} \to \mathbb{R}$ such that for every graph $G \in \mathrm{Forb}^*(H)$, $\chi(G) \leq f_H(\omega(G))$. We prove this conjecture for several graphs H, namely the paw (a triangle with a pendant edge), the bull (a triangle with two vertex-disjoint pendant edges), and what we call a "necklace," that is, a graph obtained from a path by choosing a matching such that no edge of the matching is incident with an endpoint of the path, and for each edge of the matching, adding a vertex adjacent to the ends of this edge.

1 Introduction

All graphs in this paper are finite and simple. A *clique* (respectively: *stable set*) in a graph G is a set of pairwise adjacent (respectively: non-adjacent) vertices in G. Given a graph G, we denote by $\omega(G)$ the clique number of G (i.e. the maximum number of vertices in a clique in G), and we denote by $\chi(G)$ the chromatic number of G. A class $\mathcal{G}$ of graphs is said to be *hereditary* if it is closed under isomorphism and taking induced subgraphs. A hereditary class $\mathcal{G}$ is said to be χ-*bounded* if there is a non-decreasing function $f : \mathbb{N} \to \mathbb{R}$ such that $\chi(G) \leq f(\omega(G))$ for all graphs $G \in \mathcal{G}$; under such circumstances, we say that the class $\mathcal{G}$ is χ-*bounded by* f, and that f

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is a χ-*bounding function* for $\mathcal{G}$. Given a graph H, we say that a graph G is an H^* provided that G is a subdivision of the graph H (in particular, the graph H itself is an H^*). Given a graph H, we say that a graph G is H-*free* if it does not contain H as an induced subgraph, and we say that G is H^*-*free* if it does not contain any subdivision of H as an induced subgraph. We denote by $\text{Forb}(H)$ the class of all H-free graphs, and we denote by $\text{Forb}^*(H)$ the class of all H^*-free graphs. Clearly, $\text{Forb}(H)$ and $\text{Forb}^*(H)$ are hereditary classes for every graph H.

Gyárfás [4] and Sumner [16] independently conjectured that for any tree T, the class $\text{Forb}(T)$ is χ-bounded. The conjecture has been proven for trees of radius 2 and a few trees of larger radius (see [5], [6], [7], [8], [14]). Scott [14] proved a weakened ("topological") version of the conjecture: for any tree T, the class $\text{Forb}^*(T)$ is χ-bounded. (Since every forest is an induced subgraph of some tree, this result immediately implies that $\text{Forb}^*(F)$ is χ-bounded for every forest F.) Scott further conjectured that for any graph H, the class $\text{Forb}^*(H)$ is χ-bounded; this generalized a still-open conjecture of Gyárfás [5], that the class $\text{Forb}^*(C_n)$ is χ-bounded for every n, where C_n is the chordless cycle of length n (see also [15]). The aim of this paper is to investigate Scott's conjecture for several particular graphs H.

The *paw* is the graph with vertex-set $\{x_1, x_2, x_3, y\}$ and edge-set $\{x_1x_2, x_2x_3, x_3x_1, x_1y\}$. In section 2, we give a structural description of the class $\text{Forb}^*(paw)$, which we then use to compute the best possible χ-bounding function for the class (see 2.2). Together with previously known results, this theorem implies that the class $\text{Forb}^*(H)$ is χ-bounded for all graphs H on at most four vertices. Indeed, if H is a forest, then the result follows from the result of Scott [14] mentioned above. If H is the *triangle* (i.e. the complete graph on three vertices), then $\text{Forb}^*(H)$ is the class of all forests. If H is the graph with vertex-set $\{x, y, z, w\}$ and edge-set $\{xy, yz, zx\}$, then any graph G in $\text{Forb}^*(H)$ can be partitioned into a forest and a graph whose clique number is smaller than $\omega(G)$ (indeed, take any vertex v of G, and note that the subgraph of G induced v and its non-neighbors is a forest, while the subgraph of G induced by the neighbors of v has clique number smaller than $\omega(G)$), and consequently, $\text{Forb}^*(H)$ is χ-bounded by the function $f(n) = 2n$. If H is the *diamond* (i.e. the graph obtained by deleting an edge from the complete graph on four vertices), then the result follows from a theorem of Trotignon and Vušković, see [17]. If H is the complete graph on four vertices, Scott's conjecture follows from the work of several authors, see [10]. Finally, if H is the *square* (i.e. the chordless cycle on four vertices), then $\text{Forb}^*(H)$ is the famous class of *chordal* graphs, see [13].

The *bull* is the graph with vertex-set $\{x_1, x_2, x_3, y_1, y_2\}$ and edge-set

$\{x_1x_2, x_2x_3, x_3x_1, x_1y_1, x_2y_2\}$. In section 3, we prove a decomposition theorem for bull*-free graphs, see 3.1. In section 4, we use this theorem to prove that the class Forb*(*bull*) is χ-bounded by the function $f(n) = n^2$, see 4.4. We note that this is the best possible polynomial χ-bounding function for Forb*(*bull*) in the following sense: there do not exist positive constants $c, r \in \mathbb{R}$, with $r < 2$, such that Forb*(*bull*) is χ-bounded by the function $f(n) = cn^r$. As Forb*(*bull*) contains all graphs with no stable set of size three, this follows immediately from a result of Kim [9] that the Ramsey number $R(t, 3)$ has order of magnitude $\frac{t^2}{\log t}$ (in fact, it is enough that $R(t, 3) = t^{2-o(1)}$, which also follows from an earlier result of Erdős [3]).

Finally, in section 5, we consider graphs that we call "necklaces." A necklace is a graph obtained from a path by choosing a matching such that no edge of the matching is incident with an endpoint of the path, and for each edge of the matching, adding a vertex adjacent to the ends of this edge (see section 5 for a more formal definition). We prove that for any given necklace N, the class Forb*(N) is χ-bounded by an exponential function (see 5.2). We observe that the bull is a special case of a necklace, and so the results of section 5 imply that Forb*(*bull*) is χ-bounded; however, the χ-bounding function for Forb*(*bull*) from 4.4 is polynomial, whereas the one from 5.2 is exponential. Further, we note that for all positive integers m, the m-edge path, denoted by P_{m+1}, is a necklace; furthermore, since any subdivision of an m-edge path contains an m-edge path as an induced subgraph, we know that Forb(P_{m+1}) = Forb*(P_{m+1}). Thus, 5.2 implies a result of Gyárfás (see [5]) that the class Forb(P_{m+1}) is χ-bounded by an exponential function (we note, however, that our χ-bounding function is faster growing than that of Gyárfás).

We end this section with some terminology and notation that will be used throughout the paper. The vertex-set of a graph G is denoted by V_G. Given a vertex $v \in V_G$, $\Gamma_G(v)$ is the set of all neighbors of v in G. The complement of G is denoted by $\overline{G}$. Given a set $S \subseteq V_G$, the subgraph of G induced by S is denoted by $G[S]$; if $S = \{v_1, ..., v_n\}$, we sometimes write $G[v_1, ..., v_n]$ instead of $G[S]$. Given a set $S \subseteq V_G$, we denote by $G \setminus S$ the graph obtained by deleting from G all the vertices in S; if $S = \{v\}$, we often write $G \setminus v$ instead of $G \setminus S$. Given a vertex $v \in V_G$ and a set $A \subseteq V_G \setminus \{v\}$, we say that v is *complete* (respectively: *anti-complete*) to A provided that v is adjacent (respectively: non-adjacent) to every vertex in A; we say that v is *mixed* on A provided that v is neither complete nor anti-complete to A. Given disjoint sets $A, B \subseteq V_G$, we say that A is *complete* (respectively: *anti-complete*) to B provided that every vertex in A is complete (respectively: anti-complete) to B.

2 Subdivisions of the Paw

In this section, we give a structure theorem for paw*-free graphs (2.1), and then use it to derive the fact that Forb*(paw) is χ-bounded by a linear function (2.2). We first need a definition: a graph is said to be *complete multipartite* if its vertex-set can be partitioned into stable sets, pairwise complete to each other.

2.1. *A graph G is paw*-free if and only if each of its components is either a tree, a chordless cycle, or a complete multipartite graph.*

Proof. The 'if' part is established by routine checking. For the 'only if' part, suppose that G is a connected paw*-free graph. Our goal is to show that if G is both triangle-free and square-free, then G is either a tree or a chordless cycle, and otherwise G is a complete multipartite graph.

Suppose first that G is both triangle-free and square-free. If G contains no cycles, then it is a tree, and we are done. So assume that G does contain a cycle, and let $v_0 - v_1 - ... - v_{k-1} - v_0$ (with the indices in $\mathbb{Z}_k$) be a cycle in G of length as small as possible; note that the minimality of k implies that this cycle is induced, and the fact that G is triangle-free and square-free implies that $k \geq 5$. If $V_G = \{v_0, v_1, ..., v_{k-1}\}$, then G is a chordless cycle, and we are done. So assume that $\{v_0, ..., v_{k-1}\} \subsetneqq V_G$. Since G is connected, there exists a vertex $v \in V_G \smallsetminus \{v_0, ..., v_{k-1}\}$ that has a neighbor in $\{v_0, ..., v_{k-1}\}$. Note that v must have at least two neighbors in $\{v_0, v_1, ..., v_{k-1}\}$, for otherwise, $G[v, v_0, v_1, ..., v_{k-1}]$ would be a paw*. By symmetry, we may assume that for some $i \in \mathbb{Z}_k \smallsetminus \{0\}$, v is complete to $\{v_0, v_i\}$ and anti-complete to $\{v_1, ..., v_{i-1}\}$ in G. By the minimality of k, the cycle $v - v_0 - v_1 - ... - v_i - v$ is of length at least k, and so it follows that either $i = k - 2$ or $i = k - 1$. But then $v - v_i - v_{i+1} - ... - v_0 - v$ is a (not necessarily induced) cycle of length at most four in G, which contradicts the fact that G is triangle-free and square-free.

It remains to consider the case when G contains a triangle or a square. Let H be an inclusion-wise maximal complete multipartite induced subgraph of G such that H contains a cycle. (The existence of such a graph H follows from the fact that a triangle or a square is itself a complete multipartite graph that contains a cycle.) If $G = H$, then G is complete multipartite, and we are done. So assume that this is not the case. Since G is connected, there exists a vertex $v \in V_G \smallsetminus V_H$ with a neighbor in V_H.

Let $H_1, H_2, ..., H_k$ be a partition of V_H into stable sets, pairwise complete to each other. First, we claim that v is not mixed on any set among $H_1, ..., H_k$. Suppose otherwise. By symmetry, we may assume that v is adjacent to some $h_1 \in H_1$ and non-adjacent to some $h_1' \in H_1$. Then v is anti-complete to $H_2 \cup ... \cup H_k$, for if v had a neighbor $h \in H_2 \cup ... \cup H_k$, then $G[v, h, h_1, h_1']$ would be a paw. Now, since H contains a cycle, we know that $|H_2 \cup ... \cup H_k| \geq 2$; fix distinct vertices $h, h' \in H_2 \cup ... \cup H_k$. But if hh' is an edge then $G[h, h', h_1, v]$ is a paw, and if hh' is a non-edge then $G[h, h', h_1, h_1', v]$ is a paw*. This proves our claim. Now v is anti-complete to at least two sets among $H_1, ..., H_k$ (say H_1 and H_2), for otherwise, $G[V_H \cup \{v\}]$ would contradict the maximality of H. Let $h \in H_3 \cup ... \cup H_k$ be some neighbor of v, and fix $h_1 \in H_1$ and $h_2 \in H_2$. Then $G[h_1, h_2, h, v]$ is a paw, which is a contradiction. This completes the argument. □

We note that our structure theorem for paw*-free graphs (2.1) is similar to the structure theorem for paw-free graphs (due to Olariu [12]), which states that a graph G is paw-free if and only if every component of G is either triangle-free or complete multipartite. In fact, our proof of 2.1 could be slightly shortened by using [12], but in order to keep the section self-contained, we include an independent proof. We now turn to proving that the class Forb*(paw) is χ-bounded by a linear function.

2.2. Forb*(paw) *is* χ*-bounded by the function* $f : \mathbb{N} \to \mathbb{R}$ *defined by* $f(2) = 3$ *and for all* $n \neq 2$, $f(n) = n$.

Proof. Let $G \in$ Forb*(paw). We may assume that G is connected (otherwise, we consider the components of G separately). By 2.1 then, G is either a tree, or a chordless cycle, or a complete multipartite graph, and in each of these cases, we have that $\chi(G) = 3$ or $\chi(G) = \omega(G)$. □

It is easy to see that the χ-bounding function given in 2.2 is the best possible for the class Forb*(paw). Indeed, on the one hand, we have that $\omega(G) \leq \chi(G)$ for every graph G, and on the other hand, there exist paw*-free graphs with clique number 2 and chromatic number 3 (any chordless cycle of odd length greater than three is such a graph.)

3 Decomposing Bull*-Free Graphs

In this section, we prove a decomposition theorem for bull*-free graphs. We begin with some definitions. Let G be a graph. A *hole* in G is an induced cycle in G of length at least four. An *anti-hole* in G is an induced subgraph of G whose complement is a hole in $\overline{G}$. We often denote a hole (respectively: anti-hole) H in G by $h_0 - h_1 - ... - h_k - h_0$, where $V_H = \{h_0, h_1, ..., h_k\}$ and $h_0 - h_1 - ... - h_k - h_0$ is an induced cycle in G (respectively: in $\overline{G}$). The *length* of a hole or anti-hole is the number of vertices that it contains. An *odd hole* (respectively: *odd anti-hole*) is a hole (respectively: anti-hole) of odd length. Given a vertex $v \in V_G$ and a set $S \subseteq V_G \setminus \{v\}$, we say that v is a *center* (respectively: *anti-center*) for S or for $G[S]$ provided that v is complete (respectively: anti-complete) to S. We say that G is *basic* if it contains neither an odd hole with an anti-center nor an odd anti-hole with an anti-center. A non-empty set $S \subsetneqq V_G$ is said to be a *homogeneous set* in G provided that no vertex in $V_G \setminus S$ is mixed on S; a homogeneous set S in G is said to be *proper* if $|S| \geq 2$. We say that a vertex $v \in V_G$ is a *cut-vertex* of G provided that $G \setminus v$ has more components than G. Our goal in this section is to prove the following decomposition theorem.

3.1. *Let $G \in$ Forb*(bull). Then either G is basic, or it contains a proper homogeneous set or a cut-vertex.*

We will need the following result, which is an immediate consequence of 1.4 from [2].

3.2 (Chudnovsky and Safra [2])**.** *Let $G \in$ Forb*(bull). If G contains an odd hole with a center and an anti-center, or an odd anti-hole with a center and an anti-center, then G has a proper homogeneous set.*

The proof of 3.1 proceeds as follows. We assume that a graph $G \in$ Forb*(*bull*) is not basic, and then we consider two cases: when G contains an odd anti-hole of length at least seven with an anti-center; and when G contains an odd hole with an anti-center. In the former case, we show that G contains a proper homogeneous set (see 3.3 below). The latter case is more difficult, and our approach is to prove a series of lemmas that describe how vertices that lie outside of our odd hole "attach" to this odd hole and to each other, and then to use these results to prove that G contains a proper homogeneous set or a cut-vertex (see 3.8). Since an anti-hole of length five is also a hole of length five, these two results (3.3 and 3.8) imply 3.1.

3.3. *Let $G \in$ Forb*(bull), let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 7$ and the indices in $\mathbb{Z}_k$) be an odd anti-hole in G, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that G contains an anti-center for H. Then G contains a proper homogeneous set.*

Proof. We may assume that G is connected, for otherwise, G contains a proper homogeneous set and we are done. Since G is connected and contains an anti-center for H, there exist adjacent $a, a' \in V_G \setminus H$ such that a is anti-center for H and a' has a neighbor in H. Our goal is to show that a' is a center for H, for then we are done by 3.2.

First, we claim that there is no index $i \in \mathbb{Z}_k$ such that a' is anti-complete to $\{h_i, h_{i+1}\}$. Suppose otherwise. Since a' has a neighbor in H, we may assume by symmetry that a' is adjacent to h_0 and anti-complete to $\{h_1, h_2\}$. But then if $a'h_4$ is an edge, then $G[h_0, h_1, h_4, a, a']$ is a bull; and if $a'h_4$ is a non-edge, then $G[h_0, h_1, h_2, h_4, a']$ is a bull. This proves our claim.

Next, since H has an odd number of vertices, there exists some $i \in \mathbb{Z}_k$

such that a' is either complete or anti-complete to $\{h_i, h_{i+1}\}$; by what we just showed, the latter is impossible, and so the former must hold. Now, if a' is not a center for H, then we may assume by symmetry that a' is non-adjacent to h_0 and complete to $\{h_1, h_2\}$; but then $a'h_{k-1}$ is an edge (because a' is not anti-complete to $\{h_{k-1}, h_0\}$), and so $G[h_0, h_2, h_{k-1}, a, a']$ is a bull. Thus, a' is a center for H, which completes the argument. □

For the remainder of this section, we focus on graphs in Forb*(*bull*) that contain an odd-hole with an anti-center. We begin with some definitions. Let G be a graph, let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be a hole in G, let $H = \{h_0, h_1, ..., h_{k-1}\}$, and let $v \in V_G \smallsetminus H$. Then for all $i \in \mathbb{Z}_k$:

- v is a *leaf for H at h_i* if v is adjacent to h_i and anti-complete to $H \smallsetminus \{h_i\}$;
- v is a *star for H at h_i* if v is complete to $H \smallsetminus \{h_i\}$ and non-adjacent to h_i;
- v is an *adjacent clone for H at h_i* if v is complete to $\{h_{i-1}, h_i, h_{i+1}\}$ and anti-complete to $H \smallsetminus \{h_{i-1}, h_i, h_{i+1}\}$;
- v is a *non-adjacent clone for H at h_i* if v is complete to $\{h_{i-1}, h_{i+1}\}$ and anti-complete to $H \smallsetminus \{h_{i-1}, h_{i+1}\}$;
- v is a *clone for H at h_i* if v is an adjacent clone or a non-adjacent clone for H at h_i.

We say that v is a *leaf* (respectively: *star*, *adjacent clone*, *non-adjacent clone*, *clone*) for H if there exists some $i \in \mathbb{Z}_k$ such that v is a leaf (respectively: star, adjacent clone, non-adjacent clone, clone) for H at h_i. If $|H| = k$ is odd, then we say that a vertex $v \in V_G \smallsetminus H$ is *appropriate* for H or for $G[H]$ provided that one of the following holds:

- v is a center for H;
- v is an anti-center for H;
- v is a leaf for H;
- v is an adjacent clone for H;
- v is a non-adjacent clone for H and $|H| = 5$;
- v is a star for H and $|H| = 5$.

3.4. *Let $G \in$ Forb*(bull), let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in G, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Then every vertex in $V_G \smallsetminus H$ is appropriate for H.*

Proof. Fix $v \in V_G \setminus H$. We may assume that v has at least two neighbors and at least one non-neighbor in H, for otherwise, v is a center, an anti-center, or a leaf for H, and we are done.

Suppose first that v has two adjacent neighbors in H. Fix a path $h_i - h_{i+1} - ... - h_j$ of maximum length in $G[H \cap \Gamma_G(v)]$; set $P = \{h_i, h_{i+1}, ..., h_j\}$. Note first that $|P| \geq 3$, for otherwise, we would have that $j = i+1$, and then $G[v, h_{i-1}, h_i, h_{i+1}, h_{i+2}]$ would be a bull. Now, we claim that v is anti-complete to $H \setminus P$. Suppose otherwise. Fix $h_l \in H \setminus P$ such that vh_l is an edge; by the maximality of P, we know that $l \notin \{i-1, j+1\}$. Since neither $G[v, h_{i-1}, h_i, h_{i+1}, h_l]$ nor $G[v, h_{j-1}, h_j, h_{j+1}, h_l]$ is bull, we get that $l = i-2 = j+2$, and consequently, that $|H| = |P| + 3$. Since $|H|$ is odd and $|P| \geq 3$, this means that $|P| \geq 4$, and so $G[v, h_{i-1}, h_i, h_{i+1}, h_{i+3}]$ is a bull, which is a contradiction. It follows that v is anti-complete to $H \setminus P$. Now, if $|P| = 3$, then v is an adjacent clone for H at h_{i+1}, and we are done. So assume that $|P| \geq 4$. Since $G[v, h_{i-1}, h_i, h_{i+1}, h_{i+3}]$ is not a bull, h_{i+3} is adjacent to h_{i-1}, and so $|H| = 5$ and v is a star for H at h_{i-1}.

Suppose now that $H \cap \Gamma_G(v)$ is a stable set. Fix distinct $i, j \in \mathbb{Z}_k$ such that v is complete to $\{h_i, h_j\}$ and the path $h_i - h_{i+1} - ... - h_j$ is as short as possible (in particular, v is non-adjacent to the interior vertices of the path). Since the neighbors of v in H are pairwise non-adjacent, and v is complete to $\{h_i, h_j\}$, we know that v is anti-complete to $\{h_{i-1}, h_{j+1}\}$. Since $G[v, h_{i-1}, h_i, h_{i+1}, ..., h_j, h_{j+1}]$ is not a bull*, this implies that either $h_{i-1} = h_{j+1}$, or $h_{i-1}h_{j+1}$ is an edge, and in either case, v is anti-complete to $H \setminus \{h_i, h_j\}$. We now know that the path $h_j - h_{j+1} - ... - h_i$ has at most three edges and that v is adjacent to the ends of this path and non-adjacent to its interior vertices. The minimality of the path $h_i - h_{i+1} - ... - h_j$ then implies that $|H| \leq 6$. Since $|H|$ is odd and $|H| \geq 5$, it follows that $|H| = 5$. The minimality of the path $h_i - h_{i+1} - ... - h_j$ now implies that v is a non-adjacent clone for H at h_{i+1}. This completes the argument. $\square$

Given a graph G with a hole $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$), and setting $H = \{h_0, h_1, ..., h_{k-1}\}$, we let A_H denote the set of all anti-centers for H in G, and for all $i \in \mathbb{Z}_k$:

- we let L_H^i denote the set of all leaves for H at h_i;
- we let N_H^i denote the set of all non-adjacent clones for H at h_i;
- we let C_H^i denote the set of all adjacent clones for H at h_i;
- we let S_H^i denote the set of all stars for H at h_i.

3.5. *Let $G \in \text{Forb}^*(bull)$, let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in G, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that G contains an anti-center for H, and that G does not contain a proper homogeneous set. Then there exists an index $i \in \mathbb{Z}_k$ such that all of the following hold:*

(i) $L_H^i \neq \emptyset$, *and for all* $j \in \mathbb{Z}_k \setminus \{i\}$, $L_H^j = \emptyset$;

(ii) A_H *is not anti-complete to* L_H^i;

(iii) A_H *is anti-complete to* $V_G \setminus (A_H \cup L_H^i)$.

Proof. First, since G does not contain a proper homogeneous set and $|V_G| \geq 3$, we know that G is connected. Further, since G does not contain a proper homogeneous set and contains an anti-center for H, 3.2 implies that G does not contain a center for H.

Now, we claim that every vertex in $V_G \setminus (H \cup A_H)$ that has a neighbor in A_H is a leaf for H. Suppose otherwise; fix adjacent $v \in V_G \setminus (H \cup A_H)$ and $a \in A_H$ such that v is not a leaf for H. Since v is appropriate for H (by 3.4), and since v is not a leaf, or a center, or an anti-center for H, we know that v is either a star, or an adjacent clone, or a non-adjacent clone for H. Suppose first that v is a star or an adjacent clone for H. Then there exists an index $i \in \mathbb{Z}_k$ such that v is complete to $\{h_i, h_{i+1}\}$ and non-adjacent to h_{i+2}; but now $G[a, v, h_i, h_{i+1}, h_{i+2}]$ is a bull. Suppose now that v is a non-adjacent clone for H. Then there exists an index $i \in \mathbb{Z}_k$ such that v is complete to $\{h_{i-1}, h_{i+1}\}$ and anti-complete to $\{h_i, h_{i+2}\}$; but now $G[a, v, h_{i-1}, h_i, h_{i+1}, h_{i+2}]$ is a bull*. This proves our claim.

Since G is connected and A_H is non-empty, what we just showed implies that there exists an index $i \in \mathbb{Z}_k$ such that L_H^i is non-empty and is not anti-complete to A_H. The only thing left to show is that $L_H^j = \emptyset$ for all $j \in \mathbb{Z}_k \setminus \{i\}$. Suppose otherwise. Fix some $j \in \mathbb{Z}_k \setminus \{i\}$ such that $L_H^j \neq \emptyset$. First, note that L_H^j is complete to L_H^i, for if some $l_i \in L_H^i$ and $l_j \in L_H^j$ were non-adjacent, $G[H \cup \{l_i, l_j\}]$ would be a bull*. By symmetry and the fact that $|H|$ is odd, we may assume that the path $h_i - h_{i+1} - ... - h_j$ is shorter than the path $h_j - h_{j+1} - ... - h_i$; since $|H| \geq 5$, this means that $i - 1 \notin \{j, j+1\}$. Note furthermore that $j \neq i + 1$, for otherwise, we fix some $l_i \in L_H^i$ and $l_{i+1} \in L_H^{i+1}$ and note that $G[l_i, l_{i+1}, h_{i-1}, h_i, h_{i+1}, h_{i+2}]$ is a bull*. Next, fix an anti-center a for H such that a is adjacent to some $l_i \in L_H^i$. Fix $l_j \in L_H^j$. But then if al_j is an edge, $G[a, l_i, l_j, h_i, h_j]$ is a bull; and if al_j is a non-edge, then $G[a, l_i, l_j, h_{i-1}, h_i, h_{i+1}, ..., h_{j-1}, h_j]$ is a bull*. This completes the argument. $\square$

3.6. *Let $G \in \text{Forb}^*(bull)$, let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in G, and set $H = \{h_0, h_1, ..., h_{k-1}\}$.*

Assume that G contains an anti-center for H, and that G does not contain a proper homogeneous set. Then there exists an index $i \in \mathbb{Z}_k$ such that $V_G = H \cup A_H \cup L_H^i \cup S_H^i \cup \bigcup_{j\in\mathbb{Z}_k}(N_H^j \cup C_H^j)$, where L_H^i is non-empty, L_H^i is anti-complete to S_H^i, and if $k \geq 7$, then S_H^i and $\bigcup_{j\in\mathbb{Z}_k} N_H^j$ are empty.

Proof. If $k \geq 7$, then the result is immediate from 3.2, 3.4, and 3.5. So assume that $k = 5$. By 3.2, 3.4, and 3.5, we know that $V_G = H \cup A_H \cup L_H^i \cup \bigcup_{j\in\mathbb{Z}_5}(S_H^j \cup N_H^j \cup C_H^j)$, with $L_H^i \neq \emptyset$, for some $i \in \mathbb{Z}_5$. We need to show that $S_H^j = \emptyset$ for all $j \in \mathbb{Z}_5 \setminus \{i\}$, and that L_H^i is anti-complete to S_H^i.

We first show that $S_H^j = \emptyset$ for all $j \in \mathbb{Z}_5 \setminus \{i\}$. By symmetry, it suffices to show that S_H^{i+1} and S_H^{i+2} are empty. Fix some $l_i \in L_H^i$. Suppose first that $S_H^{i+1} \neq \emptyset$, and fix $s_{i+1} \in S_H^{i+1}$. But then if $s_{i+1}l_i$ is an edge, then $G[l_i, s_{i+1}, h_{i-2}, h_i, h_{i+1}]$ is a bull; and if $s_{i+1}l_i$ is a non-edge, then $G[l_i, s_{i+1}, h_{i-1}, h_i, h_{i+2}]$ is a bull. Thus, $S_H^{i+1} = \emptyset$. Suppose now that $S_H^{i+2} \neq \emptyset$, and fix $s_{i+2} \in S_H^{i+2}$. But then if $s_{i+2}l_i$ is an edge, then $G[s_{i+2}, l_i, h_{i-2}, h_{i-1}, h_{i+2}]$ is a bull; and if $s_{i+2}l_i$ is a non-edge, then $G[s_{i+2}, l_i, h_i, h_{i+1}, h_{i+2}]$ is a bull. Thus, $S_H^{i+2} = \emptyset$.

It remains to show that L_H^i is anti-complete to S_H^i. Suppose otherwise. By 3.5, A_H is not anti-complete to L_H^i, and A_H is anti-complete to $H \cup S_H^i$. We first note that every vertex in L_H^i is anti-complete to at least one of A_H and S_H^i, for otherwise, we fix some $l_i \in L_H^i$, $s_i \in S_H^i$, and $a \in A_H$ such that l_i is adjacent to both s_i and a, and we observe that $G[l_i, s_i, a, h_{i-1}, h_i, h_{i+2}]$ is a bull*. Now, fix some adjacent $l_i \in L_H^i$ and $s_i \in S_H^i$. By what we just showed, l_i is anti-complete to A_H. Since A_H is not anti-complete to L_H^i, there exist adjacent $a \in A_H$ and $l_i' \in L_H^i \setminus \{l_i\}$. Since $l_i' \in L_H^i$ has a neighbor in A_H, we know that l_i' is anti-complete to S_H^i, and in particular, that $l_i's_i$ is a non-edge. But now if l_il_i' is an edge, then $G[l_i, l_i', a, s_i, h_i]$ is a bull; and if l_il_i' is a non-edge, then $G[l_i, l_i', s_i, h_{i-1}, h_i, h_{i+2}]$ is a bull*. This completes the argument. □

3.7. *Let $G \in$ Forb*(bull), let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in G, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that G contains an anti-center for H, and that G does not contain a proper homogeneous set. Then there exists an index $i \in \mathbb{Z}_k$ such that $V_G = H \cup A_H \cup L_H^i \cup S_H^i$, where L_H^i is non-empty, L_H^i is anti-complete to S_H^i, and if $k \geq 7$, then S_H^i is empty.*

Proof. By 3.6, we just need to show that $N_H^j \cup C_H^j = \emptyset$ for all $j \in \mathbb{Z}_k$. It suffices to show that for all $j \in \mathbb{Z}_k$, $\{h_j\} \cup N_H^j \cup C_H^j$ is a homogeneous set in G, for then the fact that G contains no proper homogeneous set will imply that $\{h_j\} \cup N_H^j \cup C_H^j$ is a singleton, and therefore, that $N_H^j \cup C_H^j = \emptyset$.

Fix $j \in \mathbb{Z}_k$, and suppose that $\{h_j\} \cup N_H^j \cup C_H^j$ is not a homogeneous set in G. Fix some $v \in V_G \smallsetminus (\{h_j\} \cup N_H^j \cup C_H^j)$ such that v is mixed on $\{h_j\} \cup N_H^j \cup C_H^j$. Clearly, $v \notin H$. Fix some $c_j, c_j' \in \{h_j\} \cup N_H^j \cup C_H^j$ such that v is adjacent to c_j and non-adjacent to c_j'. Set $\hat{H} = (H \smallsetminus \{h_j\}) \cup \{c_j\}$ and $\hat{H}' = (H \smallsetminus \{h_j\}) \cup \{c_j'\}$. Then $G[\hat{H}]$ and $G[\hat{H}']$ are both odd holes of length k. Next, by 3.5, A_H is anti-complete to $\{c_j, c_j'\}$, and so since A_H is non-empty, G contains an anti-center for both $\hat{H}$ and $\hat{H}'$; thus, 3.6 applies to both $\hat{H}$ and $\hat{H}'$. This, together with the fact that v has exactly one more neighbor in $\hat{H}$ than in $\hat{H}'$, implies that either:

(a) v is a leaf for $\hat{H}$ and an anti-center for $\hat{H}'$; or

(b) $k = 5$ and one of the following holds:

- (b1) v is a non-adjacent clone for $\hat{H}$ and a leaf for $\hat{H}'$;
- (b2) v is an adjacent clone for $\hat{H}$ and a non-adjacent clone for $\hat{H}'$;
- (b3) v is a star for $\hat{H}$ and an adjacent clone for $\hat{H}'$.

Suppose that (a) holds. Since v is adjacent to c_j, v is a leaf for $\hat{H}$ at c_j. But now if c_jc_j' is an edge, then $G[v, c_j, c_j', h_{j+1}, h_{j+2}]$ is a bull; and if c_jc_j' is a non-edge, then $G[v, c_j, c_j', h_{j-1}, h_{j+1}, h_{j+2}]$ is a bull*. From now on, we assume that (b) holds, and so $k = 5$.

Suppose first that (b1) holds. Since v is a non-adjacent clone for $\hat{H}$ and is adjacent to c_j, we know that v is a non-adjacent clone for $\hat{H}$ at either h_{j-1} or at h_{j+1}; by symmetry, we may assume that v is a non-adjacent clone for $\hat{H}$ at h_{j+1}. But now if c_jc_j' is an edge, then $G[v, c_j, c_j', h_{j-2}, h_{j-1}]$ is a bull; and if c_jc_j' is a non-edge, then $G[v, c_j, c_j', h_{j-2}, h_{j-1}, h_{j+1}]$ is a bull*.

Suppose next that (b2) holds. Since v is a clone for both $\hat{H}$ and $\hat{H}'$, and since v is adjacent to c_j and non-adjacent to c_j', it is easy to see that v is an adjacent clone for $\hat{H}$ at c_j and a non-adjacent clone for $\hat{H}'$ at c_j'. But now v is a clone for H at h_j, contrary to the fact that $v \in V_G \smallsetminus (\{h_j\} \cup N_H^j \cup C_H^j)$.

Suppose finally that (b3) holds. Since v is adjacent to c_j and non-adjacent to c_j', it is easy to see that v is a star for $\hat{H}$ at either h_{j-1} or h_{j+1}; by symmetry, we may assume that v is a star for $\hat{H}$ at h_{j+1}. Since 3.6 applies to $\hat{H}$, it follows that G contains a leaf l_{j+1} for $\hat{H}$ at h_{j+1}, and that l_{j+1} is non-adjacent to v. Since l_{j+1} is appropriate for $\hat{H}'$, it is non-adjacent to c_j'. But now if c_jc_j' is an edge, then $G[v, c_j, c_j', l_{j+1}, h_{j+1}]$ is a bull; and if c_jc_j' is a non-edge, then $G[v, c_j, c_j', h_{j-1}, h_{j+2}]$ is a bull. This completes the argument. $\square$

3.8. *Let $G \in \mathrm{Forb}^*(bull)$, let $h_0 - h_1 - ... - h_{k-1} - h_0$ (with $k \geq 5$ and the indices in $\mathbb{Z}_k$) be an odd hole in G, and set $H = \{h_0, h_1, ..., h_{k-1}\}$. Assume that G contains an anti-center for H. Then G contains a proper homogeneous set or a cut-vertex.*

Proof. We assume that G does not contain a proper homogeneous set and show that it contains a cut-vertex. By 3.7, there exists an index $i \in \mathbb{Z}_k$ such that $V_G = H \cup A_H \cup L_H^i \cup S_H^i$ and L_H^i is non-empty and anti-complete to S_H^i. Now, by 3.5, A_H is anti-complete to S_H^i. Thus, $A_H \cup L_H^i$ is anti-complete to $(H \smallsetminus \{h_i\}) \cup S_H^i$. Since $V_G = H \cup A_H \cup L_H^i \cup S_H^i$, and since h_i has neighbors both in L_H^i and in $H \smallsetminus \{h_i\}$, it follows that h_i is a cut-vertex of G. □

We now restate and prove 3.1, the main result of this section.

3.1. *Let $G \in \mathrm{Forb}^*(bull)$. Then either G is basic, or it contains a proper homogeneous set or a cut-vertex.*

Proof. Since an anti-hole of length five is also a hole of length five, the result is immediate from 3.3 and 3.8. □

4 A χ-Bounding Function for $\mathrm{Forb}^*(bull)$

In this section, we use 3.1 to prove that the class $\mathrm{Forb}^*(bull)$ is χ-bounded by the function $f(n) = n^2$. We begin with some definitions. Given graphs G_1 and G_2 with $V_{G_1} \cap V_{G_2} = \{u\}$, we say that a graph G is obtained by *gluing G_1 and G_2 along u* provided that the following hold:

- $V_G = V_{G_1} \cup V_{G_2}$;
- for all $i \in \{1, 2\}$, $G[V_{G_i}] = G_i$;
- $V_{G_1} \smallsetminus \{u\}$ is anti-complete to $V_{G_2} \smallsetminus \{u\}$ in G.

We observe that if a graph G has a cut-vertex, then G is obtained by gluing smaller graphs (i.e. graphs that have strictly fewer vertices than G) along a vertex.

Given graphs G_1 and G_2 with disjoint vertex-sets, a vertex $u \in V_{G_1}$, and a graph G, we say that G is obtained by *substituting* G_2 for u in G_1 provided that the following hold:

- $V_G = (V_{G_1} \smallsetminus \{u\}) \cup V_{G_2}$;
- $G[V_{G_1} \smallsetminus \{u\}] = G_1 \smallsetminus u$;
- $G[V_{G_2}] = G_2$;

- for all $v \in V_{G_1} \setminus \{u\}$, if v is adjacent (respectively: non-adjacent) to u in G_1, then v is complete (respectively: anti-complete) to V_{G_2} in G.

Under these circumstances, we also say that G is obtained by *substitution* from G_1 and G_2. We note that if a graph G has a proper homogeneous set, then it is obtained by substitution from smaller graphs.

We say that a graph G is *perfect* if for every induced subgraph H of G, $\chi(H) = \omega(H)$. We now state two results about perfect graphs that we will need in this section.

4.1 (Chudnovsky, Robertson, Seymour, and Thomas [1])**.** *A graph G is perfect if and only if it contains no odd holes and no odd anti-holes.*

4.2 (Lovász [11])**.** *Let G_1 and G_2 be perfect graphs with disjoint vertex-sets, and let $u \in V_{G_1}$. Let G be the graph obtained by substituting G_2 for u in G_1. Then G is perfect.*

We note that 4.1 is called the strong perfect graph theorem, and 4.2 is called the replication lemma.

In this paper, a *weighted graph* is a graph G such that each vertex $v \in V_G$ is assigned a positive integer called its *weight* and denoted by w_v. The *weight* of a non-empty set $S \subseteq V_G$ is the sum of weights of the vertices in S. We denote by W_G the weight of a clique of maximum weight in G. Given an induced subgraph H of G, and a vertex $v \in V_G$, we say that H *covers* v provided that $v \in V_H$. We now prove a technical lemma, which we then use to prove the main result of this section.

4.3. *Let $G \in \text{Forb}^*(bull)$ be a weighted graph. Then there exists a family $\mathcal{P}_G$ of at most W_G perfect induced subgraphs of G such that for every vertex $v \in V_G$, at least w_v members of $\mathcal{P}_G$ cover v.*

Proof. We assume inductively that the claim holds for graphs with fewer than $|V_G|$ vertices. By 3.1, we know that either G is basic, or G contains a proper homogeneous set, or G contains a cut-vertex.

Suppose first that V_G is basic. Fix $u \in V_G$ such that w_u is maximal. Let A be the set of all neighbors of u in G, and let B be the set of all non-neighbors of u in G. Since G is basic, and u is an anti-center for B, we know that $G[B]$ contains no odd holes and no odd anti-holes. Since u is anti-complete to B, it follows that $G[B \cup \{u\}]$ contains no odd holes and no odd anti-holes, and so by the strong perfect graph theorem (4.1), $G[B \cup \{u\}]$ is perfect. Let $\mathcal{P}_B$ be the family consisting of w_u copies of the perfect graph $G[B \cup \{u\}]$. Note that by the maximality of w_u, every vertex $v \in B \cup \{u\}$ is covered by at least w_v graphs in $\mathcal{P}_B$. If $A = \emptyset$ (so that

$V_G = B \cup \{u\}$), then we set $\mathcal{P}_G = \mathcal{P}_B$, and we are done. So assume that $A \neq \emptyset$. Now by the induction hypothesis, there exists a family $\mathcal{P}_A$ of at most $W_{G[A]}$ perfect induced subgraphs of $G[A]$ such that each vertex $v \in A$ is covered by at least w_v graphs in $\mathcal{P}_A$. Since u is complete to A, we have that $w_u + W_{G[A]} \leq W_G$. Since the family $\mathcal{P}_B$ contains exactly w_u graphs, it follows that the family $\mathcal{P}_G = \mathcal{P}_A \cup \mathcal{P}_B$ contains at most W_G graphs, and by construction, every vertex $v \in V_G$ is covered by at least w_v graphs in $\mathcal{P}_G$.

Suppose now that G contains a proper homogeneous set; let S be a proper homogeneous set in G, let A be the set of all vertices in V_G that are complete to S, and let B be the set of all vertices in V_G that are anti-complete to S. Let H be the graph whose vertex-set is $\{s\} \cup A \cup B$, with $H[A \cup B] = G[A \cup B]$, and s complete to A and anti-complete to B in H. We turn H into a weighted graph by letting the vertices in $A \cup B$ have the same weights in H as they do in G, and setting $w_s = W_{G[S]}$. Clearly, $W_H = W_G$. Using the induction hypothesis, we let $\mathcal{P}_H$ be a family of at most $W_H = W_G$ perfect induced subgraphs of H such that every vertex $v \in V_H$ is covered by at least w_v graphs in $\mathcal{P}_H$, and we let $\mathcal{P}_{G[S]}$ be the family of at most $W_{G[S]} = w_s$ perfect inducted subgraphs of $G[S]$ such that every vertex $v \in S$ is covered by at least w_v graphs in $\mathcal{P}_{G[S]}$. We may assume that the vertex s is covered by exactly w_s graphs in $\mathcal{P}_H$; let $P_1, ..., P_{w_s}$ be the graphs in $\mathcal{P}_H$ covering s, and let $\mathcal{P}'_H = \mathcal{P}_H \smallsetminus \{P_1, ..., P_{w_s}\}$. We may assume that $\mathcal{P}_{G[S]}$ contains exactly $W_{G[S]} = w_s$ graphs; say $\mathcal{P}_{G[S]} = \{Q_1, ..., Q_{w_s}\}$. Now, for each $i \in \{1, ..., w_s\}$, let P'_i be the graph obtained by substituting the graph Q_i for s in P_i; by the replication lemma (4.2), the graph P'_i is perfect for all $i \in \{1, ..., w_s\}$. We then set $\mathcal{P}_G = \{P'_1, ..., P'_{w_s}\} \cup \mathcal{P}'_H$. By construction, $\mathcal{P}_G$ is a family of at most W_G perfect induced subgraphs of G such that for every vertex $v \in V_G$, at least w_v members of $\mathcal{P}_G$ cover v.

Suppose finally that G contains a cut-vertex. Then there exist $u \in V_G$ and $C_1, C_2 \subseteq V_G \smallsetminus \{u\}$ such that $V_G = \{u\} \cup C_1 \cup C_2$, where C_1 and C_2 are non-empty, disjoint, and anti-complete to each other. For $i \in \{1, 2\}$, let $G_i = G[C_i \cup \{u\}]$. (Note that G is obtained by gluing G_1 and G_2 along u.) Using the induction hypothesis, for each $i \in \{1, 2\}$, we get a family $\mathcal{P}_{G_i}$ of at most W_{G_i} perfect induced subgraphs of G_i such that each vertex $v \in V_{G_i}$ is covered by at least w_v graphs in $\mathcal{P}_{G_i}$. We may assume that for all $i \in \{1, 2\}$, $\mathcal{P}_{G_i}$ contains exactly W_{G_i} graphs, and that u_i is covered by exactly w_{u_i} graphs in $\mathcal{P}_{G_i}$. By symmetry, we may assume that $W_{G_1} \leq W_{G_2}$. For each $i \in \{1, 2\}$, let $P^i_1, ..., P^i_{w_u}$ be the graphs in $\mathcal{P}_{G_i}$ covering u, let $P^i_{w_u+1}, ..., P^i_{W_{G_1}}$ be $W_{G_1} - w_u$ graphs in $\mathcal{P}_{G_i}$ that do not cover u, and let $P^2_{W_{G_1}+1}, ..., P^2_{W_{G_2}}$ be the remaining $W_{G_2} - W_{G_1}$ graphs in $\mathcal{P}_{G_2}$. Now, for all $j \in \{1, ..., w_u\}$, let P_j be the graph obtained by gluing P^1_j and P^2_j along u; for all $j \in \{w_u + 1, ..., W_{G_1}\}$, let P_j be the disjoint union of P^1_j and

P_j^2; and for all $j \in \{W_{G_1}+1, ..., W_{G_2}\}$, let $P_j = P_j^2$. It is easy to see that P_j is perfect for all $j \in \{1, ..., W_{G_2}\}$. Now set $\mathcal{P}_G = \{P_1, ..., P_{W_{G_2}}\}$. Since $W_G = \max\{W_{G_1}, W_{G_2}\} = W_{G_2}$, $\mathcal{P}_G$ is a family of at most W_G perfect induced subgraphs of G such that for every vertex $v \in V_G$, at least w_v members of $\mathcal{P}_G$ cover v. $\square$

4.4. *The class* Forb*(*bull*) *is* χ*-bounded by the function* $f(n) = n^2$.

Proof. Let $G \in$ Forb*(*bull*). Using 4.3, we obtain a family $\mathcal{P}$ of at most $\omega(G)$ perfect induced subgraphs of G such that each vertex in V_G is covered by at least one graph in $\mathcal{P}$. Clearly, we may assume that each vertex in V_G is covered by exactly one graph in $\mathcal{P}$. Since the graphs in $\mathcal{P}$ are perfect, each graph $P \in \mathcal{P}$ can be colored with $\omega(P) \leq \omega(G)$ colors; we may assume that the sets of colors used on the graphs in $\mathcal{P}$ are pairwise disjoint. Now we take the union of the colorings of the graphs in $\mathcal{P}$ to obtain a coloring of G that uses at most $\omega(G)^2$ colors. $\square$

5 Necklaces

We begin with some definitions. Let n be a non-negative integer, and let $m_0, ..., m_n$ be positive integers. Let H be a graph whose vertex-set is $\bigcup_{i=0}^{n}\{x_{i,0}, x_{i,1}, ..., x_{i,m_i}\} \cup \{y_1, ..., y_n\}$, with adjacency as follows:

- $x_{0,0} - ... - x_{0,m_0} - x_{1,0} - ... - x_{1,m_1} - ... - x_{n,0} - ... - x_{n,m_n}$ is a chordless path;
- $\{y_1, ..., y_n\}$ is a stable set;
- for all $i \in \{1, ..., n\}$, the vertex y_i has exactly two neighbors in the set $\bigcup_{i=0}^{n}\{x_{i,0}, x_{i,1}, ..., x_{i,m_i}\}$, namely $x_{i-1,m_{i-1}}$ and $x_{i,0}$.

Under these circumstances, we say that H is an $(m_0, ..., m_n)$*-necklace with base* $x_{0,0}$ *and hook* x_{n,m_n}, or simply that H is an $(m_0, ..., m_n)$*-necklace*. If G is a subdivision of H, then we say that G is an $(m_0, ..., m_n)$*-necklace** *with base* $x_{0,0}$ *and hook* x_{n,m_n}, or simply that G is an $(m_0, ..., m_n)$*-necklace**. To simplify notation, given a non-negative integer n and a positive integer m, we often write "$(m)_n$-necklace" instead of "$\underbrace{(m, ..., m)}_{n+1}$-necklace," and "$(m)_n$-necklace*" instead of "$\underbrace{(m, ..., m)}_{n+1}$-necklace*." (We remark that a $(1)_1$-necklace is the bull, and that for all positive integers m, an $(m)_0$-necklace with base x_0 and hook x_m is a chordless m-edge path between x_0 and x_m.)

Our goal in this section is to prove that for all non-negative integers

n and positive integers $m_0, ..., m_n$, the class $\text{Forb}^*((m_0, ..., m_n) - \textit{necklace})$ is χ-bounded by an exponential function (see 5.2 below). We observe that in order to prove 5.2, it suffices to consider only the $(m)_n$-necklaces. Indeed, if $m = \max\{m_0, ..., m_n\}$, then an $(m)_n$-necklace is a subdivision of an $(m_0, ..., m_n)$-necklace, and consequently, $\text{Forb}^*((m_0, ..., m_n) - \textit{necklace}) \subseteq \text{Forb}^*((m)_n - \textit{necklace})$. Thus, it suffices to show that $\text{Forb}^*((m)_n - \textit{necklace})$ is χ-bounded by an exponential function.

We now need some more definitions. First, in this paper, the *local chromatic number* of a graph G, denoted by $\chi_l(G)$, is the number $\max_{v \in V_G} \chi(G[\Gamma_G(v)])$. Next, let n be a non-negative and m a positive integer. Let G be a graph whose vertex-set is the disjoint union of non-empty sets N and X, let x_0 and x be distinct vertices in N, and assume that the adjacency in G is as follows:

- $G[N]$ is an $(m)_n$-necklace* with base x_0 and hook x;
- $G[X]$ is connected;
- $N \setminus \{x\}$ is anti-complete to X;
- x has a neighbor in X.

Under these circumstances, we say that (G, x_0, x) is an $(m)_n$-*alloy* or simply an *alloy*. The graph G is referred to as the *base graph* of the alloy (G, x_0, x), and the ordered pair (N, X) is the *partition* of the alloy (G, x_0, x). The *potential* of the alloy (G, x_0, x) is the chromatic number of the graph $G[X]$.

We now state the main technical lemma of this section.

5.1. *Let G be a connected graph, and let $x_0 \in V_G$. Let n and β be non-negative integers, and let m and α be positive integers. Assume that $\chi_l(G) \leq \alpha$ and $\chi(G) > 2^{n+1}((m+3)\alpha + \beta)$. Then there exists an induced subgraph H of G and a vertex $x \in V_G$ such that (H, x_0, x) is an $(m)_n$-alloy of potential greater than β.*

Since the base graph of an $(m)_n$-alloy contains an $(m)_n$-necklace* as an induced subgraph, 5.1 easily implies the main result of this section (5.2), as we now show. (We note that our proof of 5.2 relies only on the special case of 5.1 when $\beta = 0$.)

5.2. *Let n be a non-negative integer, let $m_0, ..., m_n$ be positive integers, and let $m = \max\{m_0, ..., m_n\}$. Then the class* $\text{Forb}^*((m_0, ..., m_n) - \textit{necklace})$ *is χ-bounded by the exponential function $f(k) = (2^{n+1}(m+3))^{k-1}$.*

Proof. Since an $(m)_n$-necklace is a subdivision of an $(m_0, ..., m_n)$-necklace, we know that $\text{Forb}^*((m_0, ..., m_n) - \textit{necklace}) \subseteq \text{Forb}^*((m)_n - \textit{necklace})$, and so it suffices to show that $\text{Forb}^*((m)_n - \textit{necklace})$ is χ-bounded by the function f. Suppose that this is not the case, and let $k \in \mathbb{N}$ be minimal with the property that there exists a graph $G \in \text{Forb}^*((m)_n - \textit{necklace})$ such that $\omega(G) = k$ and $\chi(G) > f(k)$. Clearly, $k \geq 2$. Furthermore, we may assume that G is connected, for otherwise, instead of G, we consider a component of G with maximum chromatic number. Note that for all $v \in V_G$, we have that $\omega(G[\Gamma_G(v)]) \leq k - 1$, and so by the minimality of k, $\chi(G[\Gamma_G(v)]) \leq f(k-1)$; thus $\chi_l(G) \leq f(k-1)$. Now, set $\alpha = f(k-1)$; then $\chi_l(G) \leq \alpha$ and $\chi(G) > 2^{n+1}(m+3)\alpha$. Fix $x_0 \in V_G$. Then 5.1 implies that there exists an induced subgraph H of G and a vertex $x \in V_G$ such that (H, x_0, x) is an $(m)_n$-alloy. But then H contains an $(m)_n$-necklace* as an induced subgraph, contrary to the fact that $G \in \text{Forb}^*((m)_n - \textit{necklace})$. $\square$

The rest of the section is devoted to proving 5.1. The idea of the proof is to show that, given a connected graph G whose chromatic number is sufficiently large relative to its local chromatic number, it is possible to recursively "chisel" an $(m)_n$-alloy out of the graph G. At each recursive step, the "length" of the alloy (i.e. the number n) increases, and the potential of the alloy decreases (but in a controlled fashion, so as to allow the next recursive step). We begin with a technical lemma, which we will use many times in this section.

5.3. *Let G be a graph, let $x_0 \in V_G$, and let $S \subseteq V_G \smallsetminus \{x_0\}$ be such that $G[S]$ is connected and x_0 has a neighbor in S. Let k be a non-negative integer, let α be a positive integer, and assume that $\chi_l(G) \leq \alpha$, and that $\chi(G[S]) > k\alpha$. Then there exist vertices $x_1, ..., x_k \in S$ and a set $X \subseteq S$ such that:*

a. $x_0 - x_1 - ... - x_k$ *is an induced path in G;*

b. $G[X]$ *is connected;*

c. $x_1, ..., x_k \notin X$*;*

d. x_k *has a neighbor in X;*

e. *vertices* $x_0, ..., x_{k-1}$ *are anti-complete to X;*

f. $\chi(G[X]) \geq \chi(G[S]) - k\alpha$.

Proof. Let $i \in \{0, ..., k\}$ be maximal such that there exist vertices $x_1, ..., x_i \in S$ and a set $X \subseteq S$ such that:

- $x_0 - x_1 - ... - x_i$ is an induced path in G;
- $G[X]$ is connected;

- $x_1, ..., x_i \notin X$;
- x_i has a neighbor in X;
- vertices $x_0, ..., x_{i-1}$ are anti-complete to X;
- $\chi(G[X]) \geq \chi(G[S]) - i\alpha$.

(The existence of such an index i follows from the fact that x_0 is an induced path in G, $G[S]$ is connected, x_0 has a neighbor in S, and $\chi(G[S]) \geq \chi(G[S]) - 0 \cdot \alpha$.)

We need to show that $i = k$. Suppose otherwise, that is, suppose that $i < k$. Then:

$$\begin{array}{rcl} \chi(G[X]) & \geq & \chi(G[S]) - i\alpha \\ & > & k\alpha - i\alpha \\ & = & (k-i)\alpha \\ & \geq & \alpha, \end{array}$$

and so $\chi(G[X]) > \alpha$. Since $\chi(G[\Gamma_G(x_i)]) \leq \alpha$ (because $\chi_l(G) \leq \alpha$), it follows that x_i is not complete to X; let X' be the vertex-set of a component of $G[X \smallsetminus \Gamma_G(x_i)]$ with maximum chromatic number. Then $\chi(G[X]) \leq \chi(G[\Gamma_G(x_i)]) + \chi(G[X'])$, and so:

$$\begin{array}{rcl} \chi(G[X']) & \geq & \chi(G[X]) - \chi(G[\Gamma_G(x_i)]) \\ & \geq & (\chi(G[S]) - i\alpha) - \alpha \\ & = & \chi(G[S]) - (i+1)\alpha \end{array}$$

Fix a vertex $x_{i+1} \in X \cap \Gamma_G(x_i)$ such that x_{i+1} has a neighbor in X'. But now the sequence $x_1, ..., x_i, x_{i+1}$ and the set X' contradict the maximality of i. It follows that $i = k$, which completes the argument. $\square$

The following is an easy consequence of 5.3, and it will serve as the base for our recursive construction of an $(m)_n$-alloy.

5.4. *Let G be a connected graph, let $x_0 \in V_G$, let β be a non-negative integer, and let m and α be positive integers. Assume that $\chi_l(G) \leq \alpha$, and that $\chi(G) > (m+1)\alpha + \beta$. Then there exists a vertex $x \in V_G \smallsetminus \{x_0\}$ and an induced subgraph H of G such that (H, x_0, x) is an $(m)_0$-alloy of potential greater than β.*

Proof. Let S be the vertex-set of a component of $G \smallsetminus x_0$ of maximum chromatic number. Clearly then, $\chi(G) \leq \chi(G[S]) + 1$, and consequently, $\chi(G[S]) > m\alpha + \beta$. Since G is connected, x_0 has a neighbor in S. By 5.3 then, there exist vertices $x_1, ..., x_m \in S$ and a set $X \subseteq S$ such that:

- $x_0 - x_1 - ... - x_m$ is an induced path in G;
- $G[X]$ is connected;

- $x_1, ..., x_m \notin X$;
- x_m has a neighbor in X;
- vertices $x_0, ..., x_{m-1}$ are anti-complete to X;
- $\chi(G[X]) \geq \chi(G[S]) - m\alpha$.

The fact that $\chi(G[X]) \geq \chi(G[S]) - m\alpha$ and $\chi(G[S]) > m\alpha + \beta$ implies that $\chi(G[X]) > \beta$. Now set $H = G[\{x_0, ..., x_{m_0}\} \cup X]$ and $x = x_m$. Then (H, x_0, x) is an $(m)_0$-alloy of potential greater than β. □

Our goal now is to show that, given an $(m)_n$-alloy with large potential and small local chromatic number of the base graph, we can "chisel" out of this $(m)_n$-alloy an $(m)_{n+1}$-alloy of large potential. More formally, we wish to prove the following lemma.

5.5. *Let n and β be non-negative integers, and let m and α be positive integers. Let (G, x_0, x) be an $(m)_n$-alloy of potential greater than $2((m+3)\alpha+\beta)$, and let (N, X) be the partition of the alloy (G, x_0, x). Assume that $\chi_l(G) \leq \alpha$. Then there exist disjoint sets $N', X' \subseteq V_G$ such that $N \subseteq N'$ and $X' \subseteq X$, and a vertex $x' \in X$ such that $(G[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than β and with partition (N', X').*

We now need some definitions. Let n be a non-negative and m a positive integer, and let (G, x_0, x) be an $(m)_n$-alloy with partition (N, X). Assume that the potential of (G, x_0, x) is greater than 2β (where β is some non-negative integer). For each $i \in \mathbb{N} \cup \{0\}$, let S'_i be the set of all vertices in $\{x\} \cup X$ that are at distance i from x in $G[\{x\} \cup X]$; thus, $S'_0 = \{x\}$. Let $t \in \mathbb{N}$ be such that $\chi(G[S'_t])$ is as large as possible. As the sets $S_1, S_3, S_5, ...$ are pairwise anti-complete to each other, as are the sets $S_2, S_4, S_6, ...$, it is easy to see that $\chi(G[X]) \leq 2\chi(G[S'_t])$, and consequently, $\chi(G[S'_t]) > \beta$. Now, let S_t be the vertex-set of a component of $G[S'_t]$ with maximum chromatic number (thus, $\chi(G[S_t]) > \beta$), and for each $i \in \{0, 1, ..., t-1\}$, let S_i be an inclusion-wise minimal subset of S'_i such that every vertex in S_{i+1} has a neighbor in S_i; clearly, $S_0 = \{x\}$. Let $H = G[N \cup \bigcup_{i=1}^{t} S_i]$. We then say that (H, x_0, x) is a *reduction* of the $(m)_n$-alloy (G, x_0, x), and that $\{S_i\}_{i=0}^{t}$ is the *stratification* of (H, x_0, x). Clearly, (H, x_0, x) is itself an $(m)_n$-alloy, and $(N, \bigcup_{i=1}^{t} S_i)$ is the associated partition. Further, as $\chi(G[S_t]) > \beta$ and H is an induced subgraph of G, we know that $\chi(H[S_t]) > \beta$. Next, given vertices $a \in S_p$ and $b \in S_q$ for some $p, q \in \{0, ..., t\}$, a path P in H between a and b is said to be *monotonic* provided that it has $|p-q|$ edges. This means that if $p = q$ then $a = b$, and if $p \neq q$ then all the internal vertices of the path P lie in $\bigcup_{r=\min\{p,q\}+1}^{\max\{p,q\}-1} S_r$, with each set S_r (with $\min\{p, q\} + 1 \leq r \leq \max\{p, q\} - 1$) containing exactly one vertex of the path. Clearly, every monotonic path is induced.

We observe that for all $p \in \{0, ..., t\}$ and $a \in S_p$, there exists a monotonic path between x and a.

The idea of the proof of 5.5 is as follows. First, we let (H, x_0, x) be a reduction of the $(m)_n$-alloy (G, x_0, x), and we let $\{S_i\}_{i=0}^t$ be the associated stratification. From now on, we work only with the graph H (and not G). We find the needed vertex x' in the set S_t, and the set X' is chosen to be a suitable subset of the set S_t. Our proof splits into two cases. The first (and easier) case is when at least one of the sets $S_1, ..., S_{t-2}$ is not stable (in this case, we necessarily have $t \geq 3$); the second (and harder) case is when the sets $S_1, ..., S_{t-2}$ are all stable. We treat these two cases in two separate lemmas (the first case is treated in 5.6, and the second case in 5.7).

5.6. *Let n and β be non-negative integers, and let m and α be positive integers. Let (G, x_0, x) be an $(m)_n$-alloy of potential greater than $2(m\alpha+\beta)$, and let (N, X) be the partition of the alloy (G, x_0, x). Assume that $\chi_l(G) \leq \alpha$. Let (H, x_0, x) be a reduction of the $(m)_n$-alloy (G, x_0, x), and let $\{S_i\}_{i=0}^t$ be the associated stratification. Assume that $t \geq 3$ and that at least one of the sets $S_1, ..., S_{t-2}$ is not stable. Then there exist disjoint sets $N', X' \subseteq V_H$ such that $N \subseteq N'$ and $X' \subseteq S_t$, and a vertex $x' \in S_t$ such that $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than β and with partition (N', X').*

Proof. First, as pointed out above, we know that $\chi(H[S_t]) > m\alpha + \beta$. Now, let $r \in \{1, ..., t-2\}$ be minimal with the property that S_r is not stable; fix adjacent $a, b \in S_r$. Let $p \in \{0, ..., r-1\}$ be maximal with the property that there exists some $z \in S_p$ such that for each $d \in \{a, b\}$, there exists a monotonic path P_d between z and d (such an index p and a vertex z exist because $x_0 \in S_0$ and there exist monotonic paths between x_0 and a and between x_0 and b). Since $S_0, ..., S_{r-1}$ are all stable, this means that $H[V_{P_a} \cup V_{P_b}]$ is a chordless cycle, and by construction, $(V_{P_a} \cup V_{P_b}) \cap S_p = \{z\}$ and $(V_{P_a} \cup V_{P_b}) \cap S_r = \{a, b\}$. Next, let Q be a monotonic path between x and z. By the minimality of S_r, there exists some $s_{r+1} \in S_{r+1}$ that is adjacent to a and non-adjacent to b. Now, fix some $s_{t-1} \in S_{t-1}$ such that there exists a monotonic path R between s_{r+1} and s_{t-1} (the existence of s_{t-1} follows from the fact that for all $i \in \{0, ..., t-1\}$ and $v \in S_i$, v has a neighbor in S_{i+1}). Since s_{t-1} has a neighbor in S_t, and since $\chi(H[S_t]) > m\alpha$, we can apply 5.3 to the vertex s_{t-1} and the set S_t to obtain vertices $u_1, ..., u_m \in S_t$ and a set $X' \subseteq S_t \smallsetminus \{u_1, ..., u_m\}$ such that the following hold:

- $s_{t-1} - u_1 - ... - u_m$ is an induced path in G;
- u_m has a neighbor in X';
- vertices $s_{t-1}, u_1, ..., u_{m-1}$ are anti-complete to X';
- $H[X']$ is connected;

- $\chi(H[X']) \geq \chi(H[S_t]) - m\alpha$.

Set $N' = N \cup V_Q \cup V_{P_a} \cup V_{P_b} \cup V_R \cup \{u_1, ..., u_m\}$ and $x' = u_m$. Clearly then, $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy with partition (N', X'). Since $\chi(H[X']) \geq \chi(H[S_t]) - m\alpha$ and $\chi(H[S_t]) > m\alpha + \beta$, we get that $\chi(H[X']) > \beta$. This completes the argument. □

5.7. *Let n and β be non-negative integers, and let m and α be positive integers. Let (G, x_0, x) be an $(m)_n$-alloy of potential greater than $2((m+3)\alpha + \beta)$, and let (N, X) be the partition of the alloy (G, x_0, x). Assume that $\chi_l(G) \leq \alpha$. Let (H, x_0, x) be a reduction of the $(m)_n$-alloy (G, x_0, x), and let $\{S_i\}_{i=0}^{t}$ be the associated stratification. Assume that the sets $S_1, ..., S_{t-2}$ are all stable. Then there exist disjoint sets $N', X' \subseteq V_H$ such that $N \subseteq N'$ and $X' \subseteq S_t$, and a vertex $x' \in S_t$ such that $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than β and with partition (N', X').*

Proof. First, since the potential of the alloy (G, x_0, x) is greater than $2((m+3)\alpha + \beta)$, we know that $\chi(H[S_t]) > (m+3)\alpha + \beta$. Next, fix $a \in S_{t-1}$, and set $A = S_t \cap \Gamma_H(a)$. Note that $\chi(H[S_t]) > 2\alpha$, and so we can apply 5.3 to the vertex a and the set S_t in H to obtain vertices $u_0', u_1' \in S_t$ and a non-empty set $C \subseteq S_t \setminus \{u_0', u_1'\}$ such that $a - u_0' - u_1'$ is an induced path in H, a and u_0' are anti-complete to C (note that this implies that $C \cap A = \emptyset$), u_1' has a neighbor in C, $H[C]$ is connected, and

$$\begin{aligned} \chi(H[C]) &\geq \chi(H[S_t]) - 2\alpha \\ &> ((m+3)\alpha + \beta) - 2\alpha \\ &= (m+1)\alpha + \beta. \end{aligned}$$

Now, fix some $b \in S_{t-1}$ adjacent to u_1'; since a is not adjacent to u_1', this means that $a \neq b$. Set $B = S_t \cap \Gamma_H(b)$; clearly, $u_1' \in B$. Since $\chi(H[C]) > \alpha$ and $\chi(H[B]) \leq \alpha$, we know that $C \not\subseteq B$; let U be the vertex-set of a component of $H[C \setminus B]$ with maximum chromatic number. Then

$$\begin{aligned} \chi(H[C]) &\leq \chi(H[B]) + \chi(H[U]) \\ &\leq \alpha + \chi(H[U]), \end{aligned}$$

and so $\chi(H[U]) > m\alpha + \beta$. Note that by construction, neither A nor B intersects U.

Let us define a path of *type one* in H to be an induced path $u_0 - ... - u_p$ (with $p \geq 1$) in $H[S_t \setminus U]$ such that $u_0 \in A \cup B$, exactly one vertex among $u_1, ..., u_p$ is in $A \cup B$, u_p has a neighbor in U, and $u_0, ..., u_{p-1}$ are all anti-complete to U. We define a path of *type two* in H to be an induced path $u_0 - ... - u_p$ (with $p \geq 1$) in $H[S_t \setminus U]$ such that $u_0 = u_0'$, no vertex among $u_1, ..., u_p$ lies in $A \cup B$ (in particular, $u_1' \notin \{u_1, ..., u_p\}$), u_p has a neighbor in U, vertices $u_0, ..., u_{p-1}$ are all anti-complete to U, and u_1' is complete to $\{u_0, u_1\}$ and anti-complete to $\{u_2, ..., u_p\} \cup U$.

Our goal now is to show that H contains a path of type one or two. Suppose that there is no path of type one in H. Since $H[S_t]$ is connected, and u'_0 is anti-complete to U, there exists an induced path $u_0 - ... - u_p$ (with $p \geq 1$) in $H[S_t \setminus U]$ such that $u_0 = u'_0$, u_p has a neighbor in U, and vertices $u_0, ..., u_{p-1}$ are anti-complete to U. Note that $u_0 \in A$ (because $u_0 = u'_0$ and $u'_0 \in A$). Clearly then, $u_1, ..., u_p \notin A \cup B$, for otherwise, at least two vertices among $u_0, u_1, ..., u_p$ would lie in $A \cup B$, and then $u_{p'} - u_{p'+1} - ... - u_p$ would be a path of type one in H for $p' \in \{0, ..., p-1\}$ chosen maximal with the property that at least two vertices among $u_{p'}, u_{p'+1}, ..., u_p$ lie in $A \cup B$. Since $u_0 = u'_0$ and $u_1, ..., u_p \notin A \cup B$, we know that $u'_1 \notin \{u_0, ..., u_p\}$. Next, note that u'_1 is anti-complete to U, for otherwise, $u'_0 - u'_1$ would be a path of type one in H. Further, u'_1 is anti-complete to $\{u_2, ..., u_p\}$, for otherwise, we let $p' \in \{2, ..., p\}$ be maximal with the property that u'_1 is adjacent to $u_{p'}$, and we observe that $u'_0 - u'_1 - u_{p'} - u_{p'+1} - ... - u_p$ is a path of type one in H. Finally, u'_1 is adjacent to u_1, for otherwise, $u'_1 - u_0 - u_1 - ... - u_p$ would be a path of type one in H. Thus, $u_0 - ... - u_p$ is a path of type two in H. This proves that H contains a path of type one or two.

Let $u_0 - ... - u_p$ (with $p \geq 1$) be a path of type one or two in H. Recall that $\chi(H[U]) > m\alpha + \beta$. We now apply 5.3 to the vertex u_p and the set U in H to obtain vertices $u_{p+1}, ..., u_{p+m} \in U$ and a set $X' \subseteq U \setminus \{u_{p+1}, ..., u_{p+m}\}$ such that the following hold:

- $u_p - u_{p+1} - ... - u_{p+m}$ is an induced path in H;
- u_{p+m} has a neighbor in X';
- vertices $u_p, ..., u_{p+m-1}$ are anti-complete to X';
- $H[X']$ is connected;
- $\chi(H[X']) \geq \chi(H[U]) - m\alpha$;

note that the last condition, together with the fact that $\chi(H[U]) > m\alpha + \beta$, implies that $\chi(H[X']) > \beta$. Set $x' = u_{p+m}$. Our goal is to construct a set N' with $N \subseteq N'$ such that $(H[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy with partition (N', X'). Since $\chi(H[X']) > \beta$, the potential of any such alloy is greater than β, as desired.

First, if $u_0 - ... - u_p$ is a path of type two in H, then we let P be a monotonic path between a and x, we set $N' = N \cup V_P \cup \{u_0, ..., u_{p+m}\} \cup \{u'_1\}$, and we are done. From now on, we assume that $u_0 - ... - u_p$ is a path of type one in H. Fix $l \in \{1, ..., p\}$ such that $u_l \in A \cup B$; then by the definition of a path of type one in H, we get that $u_0, u_l \in A \cup B$, and no other vertex on the path $u_0 - ... - u_p$ lies in $A \cup B$. If some vertex $d \in \{a, b\}$ is complete to $\{u_0, u_l\}$, then we let P be a monotonic path between x and d, we set $N' = N \cup V_P \cup \{u_0, ..., u_{p+m}\}$, and we are done. From now on, we assume that neither a nor b is complete to $\{u_0, u_l\}$. Then one of a and b is adjacent to u_0 and non-adjacent to u_l, and the other is adjacent to u_l and non-adjacent to u_0. Now, fix maximal $q \in \{0, ..., t-2\}$ such that there exists a vertex $z \in S_q$ with the property that for each $d \in \{a, b\}$, there exists a monotonic path P_d between z and d. Since $S_0, ..., S_{t-2}$ are all stable, we get that if a and b are adjacent then $H[V_{P_a} \cup V_{P_b}]$ is a chordless cycle, and if a and b are non-adjacent then $H[V_{P_a} \cup V_{P_b}]$ is an induced path between a and b; in either case, we have that $(V_{P_a} \cup V_{P_b}) \cap S_{t-1} = \{a, b\}$ and $(V_{P_a} \cup V_{P_b}) \cap S_q = \{z\}$. Let Q be a monotonic path between z and x. Now, if a and b are adjacent, then we set $N' = N \cup V_Q \cup V_{P_a} \cup V_{P_b} \cup \{u_l, u_{l+1}, ..., u_{p+m}\}$; and if a and b are non-adjacent, then we set $N' = V_Q \cup V_{P_a} \cup V_{P_b} \cup \{u_0, ..., u_{p+m}\}$. This completes the argument. □

We can now prove 5.5, restated below.

5.5. *Let n and β be non-negative integers, and let m and α be positive integers. Let (G, x_0, x) be an $(m)_n$-alloy of potential greater than $2((m+3)\alpha+\beta)$, and let (N, X) be the partition of the alloy (G, x_0, x). Assume that $\chi_l(G) \leq \alpha$. Then there exist disjoint sets $N', X' \subseteq V_G$ such that $N \subseteq N'$ and $X' \subseteq X$, and a vertex $x' \in X$ such that $(G[N' \cup X'], x_0, x')$ is an $(m)_{n+1}$-alloy of potential greater than β and with partition (N', X').*

Proof. Let (H, x_0, x) be a reduction of the $(m)_n$-alloy (G, x_0, x), and let $\{S_i\}_{i=0}^{t}$ be the associated stratification. If $t \geq 3$ and at least one of the sets $S_1, ..., S_{t-2}$ is not stable, then the result follows from 5.6. Otherwise, the result follows from 5.7. □

Finally, we use 5.4 and 5.5 to prove 5.1, restated below.

5.1. *Let G be a connected graph, and let $x_0 \in V_G$. Let n and β be non-negative integers, and let m and α be positive integers. Assume that $\chi_l(G) \leq \alpha$ and $\chi(G) > 2^{n+1}((m+3)\alpha+\beta)$. Then there exists an induced subgraph H of G and a vertex $x \in V_G$ such that (H, x_0, x) is an $(m)_n$-alloy of potential greater than β.*

Proof. For all $j \in \{0, ..., n\}$, set $\beta_j = \beta + (\Sigma_{i=1}^{n-j} 2^i)((m+3)\alpha+\beta)$. Our goal is to prove inductively that for all $j \in \{0, ..., n\}$, there exist disjoint sets $N_j, X_j \subseteq V_G$ and a vertex $x^j \in V_G$ such that $(G[N_j \cup X_j], x_0, x^j)$ is an $(m)_j$-alloy of potential greater than β_j. Since $\beta_n = \beta$, the result will follow.

For the base case (when $j = 0$), we observe that

$$\begin{array}{rcl}\chi(G) & > & 2^{n+1}((m+3)\alpha+\beta)\\ & > & (\Sigma_{i=0}^{n}2^i)((m+3)\alpha+\beta)\\ & = & (m+3)\alpha+\beta+(\Sigma_{i=1}^{n}2^i)((m+3)\alpha+\beta)\\ & = & (m+3)\alpha+\beta_0\\ & > & (m+1)\alpha+\beta_0,\end{array}$$

and so 5.4 implies that there exist sets $N_0, X_0 \subseteq V_G$ and a vertex $x^0 \in V_G$ such that $(G[N_j \cup X_j], x_0, x^j)$ is an $(m)_j$-alloy of potential greater than β_0.

For the induction case, suppose that $j \in \{0, ..., n-1\}$ and that there exist disjoint sets $N_j, X_j \subseteq V_G$ and a vertex $x^j \in V_G$ such that $(G[N_j \cup X_j], x_0, x^j)$ is an $(m)_j$-alloy of potential greater than β_j. Since

$$\begin{array}{rcl}\beta_j & = & \beta+(\Sigma_{i=1}^{n-j}2^i)((m+3)\alpha+\beta)\\ & \geq & (\Sigma_{i=1}^{n-j}2^i)((m+3)\alpha+\beta)\\ & = & 2((m+3)\alpha+\beta+(\Sigma_{i=1}^{n-(j+1)}2^i)((m+3)\alpha+\beta))\\ & = & 2((m+3)\alpha+\beta_{j+1}),\end{array}$$

5.5 implies that there exist sets $N_{j+1}, X_{j+1} \subseteq V_G$ and a vertex x^{j+1} such that $(G[N_{j+1} \cup X_{j+1}], x_0, x^{j+1})$ is an $(m)_{j+1}$-alloy of potential greater than β_{j+1}. This completes the induction. □

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