Cycle-saturated graphs with minimum number of edges *

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Abstract

A graph G is called H-saturated if it does not contain any copy of H, but for any edge e in the complement of G the graph G + e contains some H. The minimum size of an n-vertex H-saturated graph is denoted by sat(n, H). We prove

 $sat(n, C_k) = n + n/k + O((n/k^2) + k^2)$

holds for all $n \ge k \ge 3$, where C_k is a cycle with length k. We have a similar result for semi-saturated graphs

 $ssat(n, C_k) = n + n/(2k) + O((n/k^2) + k).$

We conjecture that our three constructions are optimal.

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1 A short history

A graph G is said to be H-saturated if

— it does not contain H as a subgraph, but

— the addition of any new edge (from $E(\overline{G})$) creates a copy of H.

Let $\operatorname{sat}(n, H)$ denote the *minimum* size of an *H*-saturated graph on *n* vertices. Given *H*, it is difficult to determine $\operatorname{sat}(n, H)$ because this function is not necessarily monotone in *n*, or in *H*. Recent surveys are by J. Faudree, Gould, and Schmitt [11], and by Pikhurko [19] on the hypergraph case. It is known [17] that for every graph *H* there exists a constant c_H such that

$$\operatorname{sat}(n, H) < c_H n$$

holds for all n. However, it is not known if the $\lim_{n\to\infty} \operatorname{sat}(n, H)/n$ exists; Pikhurko [19] has an example of a four graph set \mathcal{H} when $\operatorname{sat}(n, \mathcal{H})/n$ oscillates, it does not tend to a limit.

Since the classical theorem of Erdős, Hajnal, and Moon [9] (they determined $\operatorname{sat}(n, K_p)$ for all n and p), and its generalization for hypergraphs by Bollobás [5], there have been many interesting hypergraph results (e.g., Kalai [16], Frankl [14], Alon [1], using Lovász' algebraic method) but here we only discuss the graph case.

Remarkable asymptotics were given by Alon, Erdős, Holzman, and Krivelevich [2, 10] (saturation and degrees). Bohman, Fonoberova, and Pikhurko [4] determined the sat-function asymptotically for a class of complete multipartite graphs. More recently, for multiple copies of K_p Faudree, Ferrara, Gould, and Jacobson [12] determined sat (tK_p, n) for $n \ge n_0(p, t)$.

2 Cycle-saturated graphs

What is the saturation number for the cycle, C_k ? This has been considered by various authors, however, in most cases it has remained unsolved. Here relatively tight bounds are given.

Theorem 2.1. For all $k \ge 7$ and $n \ge 2k-5$ $\left(1+\frac{1}{k+2}\right)n-1 < \operatorname{sat}(n,C_k) < \left(1+\frac{1}{k-4}\right)n + \binom{k-4}{2}.$ (1)

The construction giving the upper bound is presented at the end of this section, the proof of the lower bound (which works for all $n, k \ge 5$) is postponed to Section 10.

The case of $\operatorname{sat}(n, C_3) = n - 1$ is trivial; the cases k = 4 and k = 5 were established by Ollmann [18] in 1972 and by Ya-Chen [7] in 2009, resp.

$$\operatorname{sat}(n, C_4) = \lfloor \frac{3n-5}{2} \rfloor \quad \text{for } n \ge 5.$$

$$\operatorname{sat}(n, C_5) = \lceil \frac{10(n-1)}{7} \rceil \quad \text{for } n \ge 21.$$
(2)

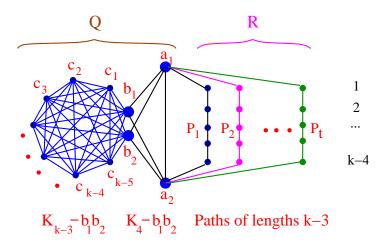
Actually, (2) was conjectured by Fisher, Fraughnaugh, Langley [13]. Later Ya-Chen [8] determined $\operatorname{sat}(n, C_5)$ for all n, as well as all extremal graphs.

The best previously known general lower bound came from Barefoot, Clark, Entringer, Porter, Székely, and Tuza [3], and the best upper bound (a clever, complicated construction resembling a bicycle wheel) came from Gould, Luczak, and Schmitt [15]

$$\left(1 + \frac{1}{2k+8}\right)n \le \operatorname{sat}(n, C_k) \le \left(1 + \frac{2}{k-\varepsilon(k)}\right)n + O(k^2)$$
(3)

where $\varepsilon(k) = 2$ for k even ≥ 10 , $\varepsilon(k) = 3$ for k odd ≥ 17 . Although there is still a gap, Theorem 2.1 supersedes all earlier results for $k \ge 6$ except the construction giving sat $(n, C_6) \le \frac{3}{2}n$ for $n \ge 11$ from [15].

Our new construction for a k-cycle saturated graph for n = (k - 1) + t(k - 4) can be read from the picture below.



To be precise, define the graph $H := H_{k,n}$ on n vertices, for arbitrary $n > k \ge 7$ as follows. Write n in the form

$$n = (k - 1) + r + t(k - 4)$$

where $t \ge 1$ is an integer and $0 \le r \le k-5$. The vertex set V(H) consists of the pairwise disjoint sets A, B, C, D, and R_i for $1 \le i \le t$, $V(H) = A \cup B \cup C \cup D \cup R_1 \cup R_2 \cup \cdots \cup R_t$ where |A| = |B| = 2, |C| = k-5, |D| = r, and $|R_1| = |R_2| = \cdots = |R_t| = k-4$ and $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}, C = \{c_1, c_2, \cdots, c_{k-5}\}, D = \{d_1, d_2, \cdots, d_r\}, R_\alpha = \{r_{\alpha,1}, r_{\alpha,2}, \ldots, r_{\alpha,k-4}\}$. We also denote $A \cup B \cup C \cup D$ by Q and $R_1 \cup \cdots \cup R_t$ by R.

The edge set of H does not contain b_1b_2 and it consists of an almost complete graph K_{k-3} minus an edge on $C \cup B$, a K_4 minus an edge on $B \cup A$, r pending edges connecting c_i and d_i , and t paths P_{α} of length k-3 with vertex sets $A \cup R_{\alpha}$ with endpoints a_1 and a_2 . The number of edges

$$|E(G)| = \binom{k-3}{2} + 4 + r + t(k-3).$$

It is not difficult to check that, indeed, H is C_k -saturated (See details in Section 3). After which, a little calculation yields the upper bound in (1).

We strongly believe that this construction is essentially optimal.

Conjecture 2.2. There exists a k_0 such that $\operatorname{sat}(n, C_k) = \left(1 + \frac{1}{k-4}\right)n + O(k^2)$ holds for each $k > k_0$.

3 The graph $H_{k,n}$ is C_k -saturated, the proof of the upper bound for $sat(n, C_k)$

First we check that $H := H_{k,n}$ is C_k -free. If a cycle with vertex set Y is entirely in Q, then it is contained in $A \cup B \cup C$, so $|Y| \le k - 1$. If Y contains a vertex $r_{\alpha,i}$ then $A \cup R_{\alpha} \subset Y$, the k - 3edges of the path P_{α} are part of the cycle. However, it is impossible to join a_1 and a_2 by a path of length 3, so $|Y| \ne k$.

The key observation to know that H is C_k -saturated is that a_1 and a_2 are connected inside Qby a path T_{ℓ} of any other lengths ℓ except for 3

$$\exists \text{ path } T_{\ell} \subset Q : \ell \in \{1, 2, 4, 5, \dots, k-3, k-2\} \text{ with endpoints } a_1, a_2.$$
(4)

For example, T_1 is a_1a_2 , $T_2 = a_1b_1a_2$, $T_4 = a_1b_1c_1b_2a_2$, etc. Also the vertices a_i (i = 1, 2) and $q \in Q \setminus \{a_i\}$ are connected by a path $U^i(m)$ of length m inside Q for $\lceil (k+1)/2 \rceil \le m \le k-2$.

$$\exists \text{ path } U^{i}(m) \subset Q : m \in \{ \lceil (k+1)/2 \rceil, \dots, k-3, k-2 \} \text{ with endpoints } a_{i}, q \in Q.$$
 (5)

Note that this is true for any $m \ge 4$ but we will apply (5) only for $\lceil (k+1)/2 \rceil \ge 4$.

Now add an edge e to H from its complement. We distinguish four disjoint cases.

Case 1. If e is contained in the induced cycle $A \cup R_{\alpha}$ then we get a path connecting a_1 and a_2 in $A \cup R_{\alpha}$ of length t, where t is at least two and at most k - 4. This path with T_{k-t} form a k-cycle. Case 2. If the endpoints of e are $r_{\alpha,i}$ and $r_{\beta,j}$ with $\alpha \neq \beta$ then we may suppose that $1 \leq i \leq j \leq k - 4$. The vertex $r_{\alpha,i}$ splits the path P_{α} into two parts, P_{α}^1 and P_{α}^2 , where P_{α}^1 starts at a_1 and has length i, and P_{α}^2 ends at a_2 and has length k - 3 - i. Consider the path $\pi := P_{\alpha}^1 e P_{\beta}^2$, its length is k - 2 - j + i. This length is between 3 and k - 2 so we can apply (4) to add an appropriate T_{j-i+2} to complete π to a k-cycle unless j - i + 2 = 3. In the latter, the edge a_1a_2 together with P_{β}^1 , e, and P_{α}^2 form a C_k .

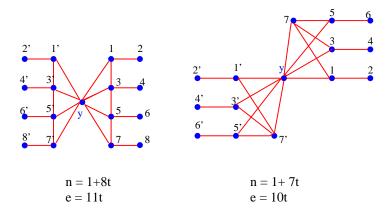
Case 3. If the endpoints of e are $r_{\alpha,i}$ and $q \in B \cup C \cup D$, then again by symmetry, we may suppose that $i \leq (k-3)/2$, so the length of P_{α}^1 is at most $\lfloor (k-3)/2 \rfloor$. Then, by (5) there is an $U^1(m)$ so that P_{α}^1 , e and $U^1(m)$ form a k-cycle.

Case 4. Finally, e is contained in Q. For $e = a_1c_1$ we use P_1 to get the k-cycle $a_1c_1b_1a_2P_1$, for $e = a_1d_1$ we have the k-cycle $d_1c_1c_2 \dots c_{k-5}b_2a_2b_1a_1$, for $e = b_1b_2$ we have to use P_1 , i.e., here we need again that $t \ge 1$, for $e = b_1d_1$ we have the k-cycle $d_1c_1c_2 \dots c_{k-5}b_2a_2a_1b_1$, for $e = c_1d_2$ we have the k-cycle $c_1d_2c_2 \dots c_{k-5}b_2a_2a_1b_1$, finally for $e = d_1d_2$ we have the k-cycle $c_1d_1d_2c_2 \dots c_{k-5}b_2a_2b_1$.

4 Semisaturated graphs

A graph G is H-semisaturated (formerly called *strongly* saturated) if G + e contains more copies of H than G does for $\forall e \in E(\overline{G})$. Let $\operatorname{ssat}(n, H)$ be the minimum size of an H-semisaturated graph. Obviously, $\operatorname{ssat}(n, H) \leq \operatorname{sat}(n, H)$.

It is known that $\operatorname{ssat}(n, K_p) = \operatorname{sat}(n, K_p)$ (it follows from Frankl/Alon/Kalai generalizations of Bollobás set pair theorem) and $\operatorname{ssat}(n, C_4) = \operatorname{sat}(n, C_4)$ (Tuza [20]). Below we have a C_5 semisaturated graph on 1 + 8t vertices and 11t edges. Every vertex can be reached by a path of length 2 from y. Joining one, two or three triangles to the central vertex y one obtains C_5



semisaturated graphs with 8t + 3, 8t + 5, or 8t + 7 vertices and 11t + 3, 11t + 6, or 11t + 9 edges, resp. Leaving out a pendant edge, we can extend these constructions for even values of n

$$\operatorname{ssat}(n, C_5) \le \left\lceil \frac{11}{8}(n-1) \right\rceil \text{ for all } n \ge 5.$$
 (6)

The picture on the right is the extremal C_5 -saturated graph by (2).

Conjecture 4.1. ssat $(n, C_5) = \frac{11}{8}n + O(1)$. Maybe equality holds in (6) for $n > n_0$. Since 11/8 = 1.375 < 10/7 = 1.42... inequalities (2) and (6) imply that

 $\operatorname{ssat}(n, C_5) < \operatorname{sat}(n, C_5)$ for all $n \ge 21$.

Our next Theorem shows that a similar statement holds for every cycle C_k with k > 12 (and probably for $k \in \{6, 7, ..., 12\}$, too).

Theorem 4.2. For all $n \ge k \ge 6$

$$\left(1 + \frac{1}{2k - 2}\right)n - 2 < \operatorname{ssat}(n, C_k) < \left(1 + \frac{1}{2k - 10}\right)n + k - 1.$$
(7)

The proof of the lower bound is postponed to Section 9. The construction yielding the upper bound is presented in the next two sections where we describe a way to improve the O(k) term as well as give better constructions for k = 6. We believe that our constructions are essentially optimal.

Conjecture 4.3. There exists a k_0 such that $\operatorname{ssat}(n, C_k) = \left(1 + \frac{1}{2k - 10}\right)n + O(k)$ holds for each $k > k_0$.

5 Constructions of sparse C_k -semisaturated graphs

In this section we define an infinite class of C_k -semisaturated graphs, $H^2_{k,n}$ (more precisely $H^2_{k,n}(G)$).

Call a graph G k-suitable with special vertices a_1 and a_2 if

- (S1) G is C_k -semisaturated,
- (S2) \exists a path T_{ℓ} in G with endpoints a_1 and a_2 and of length ℓ for all $1 \leq \ell \leq k-2$, and
- (S3) for every $q \in V(G) \setminus \{a_1, a_2\}$, and integers m_1 and m_2 with $m_1 + m_2 = k$ and $2 \le m_i \le k 2$ \exists an $i \in \{1, 2\}$ and a path $U(a_i, q, m_i)$ of length m_i and with endpoints a_i and q.

For example, it is easy to see, that a **wheel** with r spikes W_k^r is such a graph, $k \ge r$, $k \ge 4$. It is defined by the (k + r)-element vertex set $\{a_1, a_2, \ldots, a_k, d_1, \ldots, d_r\}$ and by 2k - 2 + r edges joining a_1 to all other a_i 's, forming a cycle $a_2a_3 \ldots a_k$ of length k - 1, and joining each d_i to a_i .

Define the graph $H^2_{k,n}(G)$ as follows, when n is in the form

$$n = |V(G)| + t(k-3)$$

where $t \ge 0$ is an integer. The vertex set V(H) consists of the pairwise disjoint sets Q and R_i for $1 \le i \le t$, $V(H) = Q \cup R_1 \cup \cdots \cup R_t$ where |Q| = |V(G)|, $|R_1| = |R_2| = \cdots = |R_t| = k - 3$ and

 $A := \{a_1, a_2\} \subset Q$. The edge set of H consists of a copy of G with vertex set Q, and t paths with endpoints a_1 and a_2 and vertex sets $A \cup R_{\alpha}$. The number of edges is

$$|E(H)| = |E(G)| + t(k-2).$$

It is not difficult to check that, indeed, H is C_k -semisaturated, the details are similar (but much simpler) to those in Section 3, so we do not repeat that proof.

Finally, considering $H^2_{k,n}(W^r_k)$ (where now $4 \le r \le k$) we obtain that for all $n \ge k+4$

$$\operatorname{ssat}(n, C_k) \le n + \left\lfloor \frac{n-7}{k-3} \right\rfloor + k - 3.$$
(8)

Corollary 5.1. $\operatorname{ssat}(n, C_6) \leq \lceil \frac{4}{3}n \rceil.$

6 Thinner constructions of sparse C_k -semisaturated graphs

In this section we define another infinite class of C_k -semisaturated graphs, $H^3_{k,n}$ (more precisely $H^3_{k,n}(G)$) yielding the upper bound (7) in Theorem 4.2.

Call a graph $G \{k.k+2\}$ -suitable with special vertices a_1 and a_2 if (S1) and (S2) hold but (S3) is replaced by the following

 $(S3)^+$ for every $q \in V(G) \setminus \{a_1, a_2\}$, and integers m_1, m_2 either there exists a path $U(a_1, q, m_1)$ (of length m_1 and with endpoints a_1 and q) or a path $U(a_2, q, m_2)$ in the following cases $m_1 + m_2 = k$ and $3 \le m_i \le k - 3$,

 $m_1 + m_2 = k + 2$ and $4 \le m_i \le k - 4$.

It is easy to see, that the wheel W_k^r with r spikes is such a graph, $k \ge r \ge 0, k \ge 4$.

Define the graph $H^3_{k,n}(G)$ as follows, when n is in the form

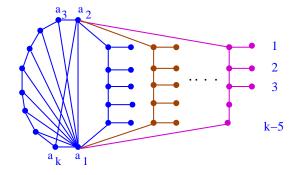
$$n = |V(G)| + t(2k - 10) - r \tag{9}$$

where $t \ge 2$ is an integer and $0 \le r < 2k - 10$. The vertex set V(H) consists of the pairwise disjoint sets Q, R_i and D for $1 \le i \le t$, $V(H) = Q \cup R_1 \cup \cdots \cup R_t \cup D$ where |Q| = |V(G)|, $|R_1| = |R_2| = \cdots = |R_t| = k - 5$, |D| = t(k - 5) - r and $A := \{a_1, a_2\} \subset Q$. The edge set of Hconsists of a copy of G with vertex set Q, and t paths with endpoints a_1 and a_2 and vertex sets $A \cup R_\alpha$ and finally |D| spikes, a matching with edges from $\cup R_\alpha$ to D.

The number of edges is

$$|E(H)| = |E(G)| + t(2k - 9) - r.$$
(10)

It is not difficult to check that H is C_k -semisaturated, the details are similar (but simpler) to those in Section 3. As an example we present one case.



Add the edge qd to H where $q \in V(G) \setminus \{a_1, a_2\}$ and $d \in D$. Let us denote the (unique) neighbor of d by $x, x \in R_{\alpha}$. The distance of x to a_1 is denoted by ℓ . Then the length of the $qdx \ldots a_1$ path is $\ell + 2 \ge 3$ and the length of the $qdx \ldots a_2$ path is $(k - 4 - \ell) + 2 \ge 3$ and one can find a C_k through qd using property $(S3)^+$.

Considering $H^3_{k,n}(W_k)$ (with $t \ge 2$) we obtain from (9) and (10) that for all $n \ge 3k - 9$

$$\operatorname{ssat}(n, C_k) \le \left\lceil \left(1 + \frac{1}{2k - 10}\right)(n - k) \right\rceil + 2k - 2.$$
(11)

Using $H^2(k, n)$, it is easy to see that (11) holds for all $n \ge k$, leading to the upper bound in (7).

One can slightly improve (8) and (11) if there are special graphs thinner than the wheel W_k .

Problem 6.1. Determine s(k), the minimum size of a k-vertex k-special graph (i.e., one satisfying (S1)–(S3)). Determine s'(k), the minimum size of a k-vertex $\{k, k+2\}$ -special graph (i.e., one satisfying (S1), (S2) and (S3)⁺).

7 Degree one vertices in (semi)saturated graphs

Suppose that G is a C_k -semisaturated graph where $k \ge 5$, $|V(G)| = n \ge k$. Obviously, G is connected. Let X be the set of vertices of degree one, $X := \{v \in V(G) : \deg_G(v) = 1\}$, its size is s and its elements are denoted as $X = \{x_1, x_2, \dots, x_s\}$. Denote the neighbor of x_i by y_i , $Y := \{y_1, \dots, y_s\}$ and let $Z := V(G) \setminus (X \cup Y)$. We also denote the neighborhood of any vertex v by $N_G(v)$ or briefly by N(v).

Lemma 7.1. (The neighbors of degree one vertices.)

- (i) $y_i \neq y_j$ for $1 \le i \ne j \le s$, so |Y| = |X|.
- (ii) $\deg(y) \ge 3$ for every $y \in Y$,
- (iii) if $\deg_G(x) = 1$, then $G \{x\}$ is also a C_k -semisaturated graph.

Proof. If $y_i = y_j$, then the addition of $x_i x_j$ to G does not create a new k-cycle. If $\deg(y_i) = 2$ and $N(y_i) = \{x_i, w\}$, the addition of $x_i w$ to G does not create a new k-cycle. Finally, (iii) is obvious.

Split Y and Z according to the degrees of their vertices. Thus, divide V(G) into five parts $\{X, Y_3, Y_{4+}, Z_2, Z_{3+}\},\$

 $Y_3 := \{ v \in Y : \deg_G(v) = 3 \} \text{ and } Y_{4+} := \{ v \in Y : \deg_G(v) \ge 4 \},$ $Z_2 := \{ v \in Z : \deg_G(v) = 2 \} \text{ and } Z_{3+} := \{ v \in Z : \deg_G(v) \ge 3 \}.$

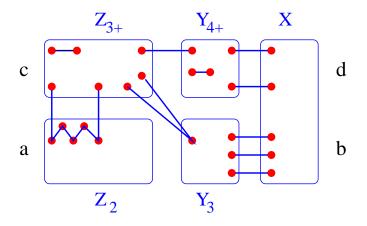
Lemma 7.2. (The structure of C_k -saturated graphs. See [3]).

Suppose that G is a C_k -saturated graph (and $k \ge 5$). Then

(iv) if $x_i y_i w$ is a path in G (with $x_i \in X$, $y_i \in Y$), then $\deg(w) \ge 3$. So there are no edges from Z_2 to Y (or to X).

(v) If y_iy_j is an edge of G (with $y_i, y_j \in Y$), then $\deg(y_i) \ge 4$. So there are no edges in Y_3 , no edges from Y_3 to Y_4 . In other words, every $y \in Y_3$ has one neighbor in X and two in Z_{3+} .

(vi) The induced graph $G[Z_2]$ consists of paths of length at most k-2.



8 Semisaturated graphs without pendant edges

Claim 8.1. Suppose that G is a C_k -semisaturated graph on n vertices with minimum degree at least 2, $k \ge 5$. Then every vertex w is contained in some cycle of length at most k + 1.

Proof. Consider two arbitrary vertices z_1, z_2 in the neighborhood N(w). If $z_1z_2 \in E(G)$, then w is contained in a triangle. If $z_1z_2 \notin E(G)$, then $G + z_1z_2$ contains a new k-cycle; there is a path P of length (k-1) in G with endpoints z_1 and z_2 . If P avoids w, then P together with z_1wz_2 form a k+1 cycle. If w splits P into two paths L_1, L_2 , where L_i starts in z_i and ends in w, then either $L_1 + z_1w$, or $L_2 + z_2w$, or both form a proper cycle of length at most k-1.

Note that Claim 8.1 itself (and the connectedness of G) immediately imply

$$e(G) \ge (n-1)\frac{k+2}{k+1}$$

We can do a bit better repeatedly using the semisaturatedness of G.

Lemma 8.2. Suppose that G is a C_k -semisaturated graph on n vertices with minimum degree at least 2, $k \ge 5$. Then

$$e(G) \ge \frac{k}{k-1} n - \frac{k+1}{k-1}$$

Proof. We define an increasing sequence of subgraphs $G_1, G_2, \ldots, G_t = G$ with vertex sets $V_1 \subseteq V_2 \subseteq \cdots \subseteq V_t = V(G)$ such that G_i is a subgraph of G_{i+1} and

$$|E(G_{i+1}) \setminus E(G_i)| \ge \frac{k}{k-1} \left(|V_{i+1}| - |V_i| \right)$$
(12)

(for $i = 1, 2, \ldots, t - 1$). This, together with

$$e(G_1) \ge \frac{k}{k-1} |V_1| - \frac{k+1}{k-1}$$
(13)

imply the claim.

 G_1 is the shortest cycle in the graph G. Its length is at most k + 1 so (13) obviously holds.

If G_i is defined and one can find a path P of length at most k with endpoints in V_i but $E(P) \setminus E(G_i) \neq \emptyset$, then we can take $E(G_{i+1}) = E(G_i) \cup E(P)$. From now on, we suppose that such a *short returning* path does not exist. Our procedure stops if $V(G_i) = V(G) =: V$.

In the case of $V \setminus V_i \neq \emptyset$, the connectedness of G implies that there exists an xy edge with $x \in V_i$ and $y \in V \setminus V_i$. Since $|N(y)| \ge 2$ we have another edge $yz \in E(G), z \ne x$.

We have $N(y) \cap V_i = \{x\}$, otherwise one gets a path xyz of length smaller than k with endpoints in V_i but going out of G_i , contradicting our earlier assumption. Similarly, we obtain that N(y)contains no edge, otherwise we can define $E(G_{i+1})$ as either $E(G_i)$ plus the three edges of a triangle xy, yz, xz or we add four edges xy, yz_1, yz_2 , and z_1z_2 but only three vertices (namely y, z_1 , and z_2). The obtained G_{i+1} obviously satisfies (12) in both cases. Similarly, if there is a cycle C of length at most k - 1 containn y, then we can define $E(G_{i+1})$ as $E(G_i)$ plus E(C) and xy. From now on, we suppose that such a short cycle through y does not exist.

Fix a neighbor z of $y, z \neq x$. Since $zx \notin E(G)$, G contains a path P of length k-1 with endvertices x and z. Since G does not contain a short returning path nor a short cycle through y, we obtain that P avoids y and $V(P) \cap V_i = \{x\}$.

If the cycle C := P + xy + yz of length k+1 has any diagonal edge then G_{i+1} is obtained by adding C together with its diagonals. From now on, we suppose that C does not have any diagonals. More generally, if there is any *diagonal path* P of length $\ell \leq k-1$ with edges disjoint from $E(G_i) \cup E(C)$ but with endpoints in $V_i \cup V(C)$ then we can define $E(G_{i+1}) := E(G_i) \cup E(C) \cup E(P)$ and have added $k + \ell - 1$ vertices and $k + \ell + 1$ edges, obviously satisfying (12).

However, such a diagonal path exists. Let $w \neq y$ be the other neighbor of x in C. Since $wz \notin E(G)$, there is a path P' of length k-1 with endpoints w and z. This P' must have edges outside $E(G_i) \cup E(C)$ so it can be shortened to a diagonal path P of length at most k-1. This completes the proof of the Lemma.

9 A lower bound for the number of edges of semisaturated graphs

In this section we finish the proof of Theorem 4.2. Let G be a C_k -semisaturated graph on n vertices with minimum number of edges, $k \ge 5$. Let X be the set of degree one vertices, x := |X|. By Lemma 7.1 $|X| \le n/2$, and for $G' := G \setminus X$ we have e(G') = e(G) - x and G' is a C_k -semisaturated graph on n - x vertices with minimum degree at least 2. Then Lemma 8.2 can be applied to e(G'). We obtain

$$\operatorname{ssat}(n, C_k) = e(G) \geq x + (n - x)\frac{k}{k - 1} - \frac{k + 1}{k - 1}$$
$$\geq \frac{n}{2} + \frac{n}{2}\frac{k}{k - 1} - \frac{k + 1}{k - 1} = n\left(1 + \frac{1}{2k - 2}\right) - \frac{k + 1}{k - 1}$$

Since $\operatorname{sat}(n, C_k) \ge \operatorname{ssat}(n, C_k)$, this is already a better lower bound than the one in (3) from [3].

10 A lower bound for the number of edges of C_k -saturated graphs

In this section we finish the proof of Theorem 2.1. Let G be a C_k -saturated graph on n vertices, $k \ge 5$. Let us consider the partition of $V(G) = X \cup Y_3 \cup Y_{4+} \cup Z_2 \cup Z_{3+}$ defined in Section 7, where X is the set of degree one vertices, Y is their neighbors. By Lemma 7.1 |X| = |Y|. To simplify notations we use $a := |Z_2|, b := |Y_3|, c := |Z_{3+}|$, and $d := |Y_{4+}|$. We have

$$n = a + 2b + c + 2d.$$

By definition of the parts we have the lower bound

$$2e(G) = \sum_{v \in V} \deg(v) \ge |X| + 2|Z_2| + 3|Y_3| + 3|Z_{3+}| + 4|Y_{4+}|.$$

This yields

$$2e \ge 2n + c + d. \tag{14}$$

Now we estimate the number of edges by considering four disjoint subsets of E(G). The part X is adjacent to |X| edges, and according to Lemma 7.2, Z_2 is adjacent to at least $\frac{k}{k-1}|Z_2|$ edges,

 Y_3 is adjacent to exactly $3|Y_3|$ edges from which $|Y_3|$ has already been counted at X, and finally Y_{4+} is adjacent to at least another $\frac{3}{2}|Y_{4+}|$ edges. We obtain

$$e(G) \ge |X| + \frac{k}{k-1}|Z_2| + 2|Y_3| + \frac{3}{2}|Y_{4+}|.$$

Therefore we get

$$e \ge n + \frac{1}{k-1}a + b - c + \frac{1}{2}d.$$
(15)

By Lemma 7.1 $G \setminus X$ is also C_k -semisaturated. Apply Lemma 8.2 to estimate $e(G \setminus X) = e - b - d$, multiply by (k - 1) and rearrange, we get

$$(k-1)e \ge kn - b - d - (k+1).$$
(16)

Adding up the above three inequalities (14), (15), and (16) we obtain

$$(k+2)e \ge (k+3)n + \frac{1}{k-1}a + \frac{1}{2}d - (k+1).$$

This implies the desired lower bound in (1).

Remark. We can do slightly better if we multiply (14), (15), and (16) by k, k-1, and k-3, resp., then adding up and simplifying we get

$$e(G) > \frac{k^2}{k^2 - k + 2} n - 1.$$
(17)

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