# Cycle-saturated graphs with minimum number of edges * 

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#### Abstract

A graph $G$ is called $H$-saturated if it does not contain any copy of $H$, but for any edge $e$ in the complement of $G$ the graph $G+e$ contains some $H$. The minimum size of an $n$-vertex $H$-saturated graph is denoted by $\operatorname{sat}(n, H)$. We prove $$
\operatorname{sat}\left(n, C_{k}\right)=n+n / k+O\left(\left(n / k^{2}\right)+k^{2}\right)
$$ holds for all $n \geq k \geq 3$, where $C_{k}$ is a cycle with length $k$. We have a similar result for semi-saturated graphs $$
\operatorname{ssat}\left(n, C_{k}\right)=n+n /(2 k)+O\left(\left(n / k^{2}\right)+k\right)
$$

We conjecture that our three constructions are optimal.

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## 1 A short history

A graph $G$ is said to be $H$-saturated if

- it does not contain $H$ as a subgraph, but
- the addition of any new edge (from $E(\bar{G})$ ) creates a copy of $H$.

Let $\operatorname{sat}(n, H)$ denote the minimum size of an $H$-saturated graph on $n$ vertices. Given $H$, it is difficult to determine $\operatorname{sat}(n, H)$ because this function is not necessarily monotone in $n$, or in $H$. Recent surveys are by J. Faudree, Gould, and Schmitt [11], and by Pikhurko [19] on the hypergraph case. It is known [17] that for every graph $H$ there exists a constant $c_{H}$ such that

$$
\operatorname{sat}(n, H)<c_{H} n
$$

holds for all $n$. However, it is not known if the $\lim _{n \rightarrow \infty} \operatorname{sat}(n, H) / n$ exists; Pikhurko [19] has an example of a four graph set $\mathcal{H}$ when $\operatorname{sat}(n, \mathcal{H}) / n$ oscillates, it does not tend to a limit.

Since the classical theorem of Erdős, Hajnal, and Moon [9] (they determined sat $\left(n, K_{p}\right)$ for all $n$ and $p$ ), and its generalization for hypergraphs by Bollobás [5], there have been many interesting hypergraph results (e.g., Kalai [16], Frankl [14], Alon [1], using Lovász' algebraic method) but here we only discuss the graph case.

Remarkable asymptotics were given by Alon, Erdős, Holzman, and Krivelevich [2, 10] (saturation and degrees). Bohman, Fonoberova, and Pikhurko [4] determined the sat-function asymptotically for a class of complete multipartite graphs. More recently, for multiple copies of $K_{p}$ Faudree, Ferrara, Gould, and Jacobson [12] determined sat $\left(t K_{p}, n\right)$ for $n \geq n_{0}(p, t)$.

## 2 Cycle-saturated graphs

What is the saturation number for the cycle, $C_{k}$ ? This has been considered by various authors, however, in most cases it has remained unsolved. Here relatively tight bounds are given.

Theorem 2.1. For all $k \geq 7$ and $n \geq 2 k-5$

$$
\begin{equation*}
\left(1+\frac{1}{k+2}\right) n-1<\operatorname{sat}\left(n, C_{k}\right)<\left(1+\frac{1}{k-4}\right) n+\binom{k-4}{2} \tag{1}
\end{equation*}
$$

The construction giving the upper bound is presented at the end of this section, the proof of the lower bound (which works for all $n, k \geq 5$ ) is postponed to Section 10 .

The case of $\operatorname{sat}\left(n, C_{3}\right)=n-1$ is trivial; the cases $k=4$ and $k=5$ were established by Ollmann [18] in 1972 and by Ya-Chen [7] in 2009, resp.

$$
\begin{align*}
& \operatorname{sat}\left(n, C_{4}\right)=\left\lfloor\frac{3 n-5}{2}\right\rfloor \quad \text { for } n \geq 5 \\
& \operatorname{sat}\left(n, C_{5}\right)=\left\lceil\frac{10(n-1)}{7}\right\rceil \quad \text { for } n \geq 21 \tag{2}
\end{align*}
$$

Actually, (2) was conjectured by Fisher, Fraughnaugh, Langley [13]. Later Ya-Chen [8] determined $\operatorname{sat}\left(n, C_{5}\right)$ for all $n$, as well as all extremal graphs.

The best previously known general lower bound came from Barefoot, Clark, Entringer, Porter, Székely, and Tuza [3], and the best upper bound (a clever, complicated construction resembling a bicycle wheel) came from Gould, Łuczak, and Schmitt [15]

$$
\begin{equation*}
\left(1+\frac{1}{2 k+8}\right) n \leq \operatorname{sat}\left(n, C_{k}\right) \leq\left(1+\frac{2}{k-\varepsilon(k)}\right) n+O\left(k^{2}\right) \tag{3}
\end{equation*}
$$

where $\varepsilon(k)=2$ for $k$ even $\geq 10, \varepsilon(k)=3$ for $k$ odd $\geq 17$. Although there is still a gap, Theorem [2.1] supersedes all earlier results for $k \geq 6$ except the construction giving $\operatorname{sat}\left(n, C_{6}\right) \leq \frac{3}{2} n$ for $n \geq 11$ from (15].

Our new construction for a $k$-cycle saturated graph for $n=(k-1)+t(k-4)$ can be read from the picture below.


To be precise, define the graph $H:=H_{k, n}$ on $n$ vertices, for arbitrary $n>k \geq 7$ as follows. Write $n$ in the form

$$
n=(k-1)+r+t(k-4)
$$

where $t \geq 1$ is an integer and $0 \leq r \leq k-5$. The vertex set $V(H)$ consists of the pairwise disjoint sets $A, B, C, D$, and $R_{i}$ for $1 \leq i \leq t, V(H)=A \cup B \cup C \cup D \cup R_{1} \cup R_{2} \cup \cdots \cup R_{t}$ where $|A|=|B|=2,|C|=k-5,|D|=r$, and $\left|R_{1}\right|=\left|R_{2}\right|=\cdots=\left|R_{t}\right|=k-4$ and $A=\left\{a_{1}, a_{2}\right\}$, $B=\left\{b_{1}, b_{2}\right\}, C=\left\{c_{1}, c_{2}, \cdots, c_{k-5}\right\}, D=\left\{d_{1}, d_{2}, \cdots, d_{r}\right\}, R_{\alpha}=\left\{r_{\alpha, 1}, r_{\alpha, 2}, \ldots, r_{\alpha, k-4}\right\}$. We also denote $A \cup B \cup C \cup D$ by $Q$ and $R_{1} \cup \cdots \cup R_{t}$ by $R$.

The edge set of $H$ does not contain $b_{1} b_{2}$ and it consists of an almost complete graph $K_{k-3}$ minus an edge on $C \cup B$, a $K_{4}$ minus an edge on $B \cup A, r$ pending edges connecting $c_{i}$ and $d_{i}$, and
$t$ paths $P_{\alpha}$ of length $k-3$ with vertex sets $A \cup R_{\alpha}$ with endpoints $a_{1}$ and $a_{2}$. The number of edges

$$
|E(G)|=\binom{k-3}{2}+4+r+t(k-3)
$$

It is not difficult to check that, indeed, $H$ is $C_{k}$-saturated (See details in Section 3). After which, a little calculation yields the upper bound in (1).

We strongly believe that this construction is essentially optimal.
Conjecture 2.2. There exists a $k_{0}$ such that $\operatorname{sat}\left(n, C_{k}\right)=\left(1+\frac{1}{k-4}\right) n+O\left(k^{2}\right)$ holds for each $k>k_{0}$.

## 3 The graph $H_{k, n}$ is $C_{k}$-saturated, the proof of the upper bound for $\operatorname{sat}\left(n, C_{k}\right)$

First we check that $H:=H_{k, n}$ is $C_{k}$-free. If a cycle with vertex set $Y$ is entirely in $Q$, then it is contained in $A \cup B \cup C$, so $|Y| \leq k-1$. If $Y$ contains a vertex $r_{\alpha, i}$ then $A \cup R_{\alpha} \subset Y$, the $k-3$ edges of the path $P_{\alpha}$ are part of the cycle. However, it is impossible to join $a_{1}$ and $a_{2}$ by a path of length 3 , so $|Y| \neq k$.

The key observation to know that $H$ is $C_{k}$-saturated is that $a_{1}$ and $a_{2}$ are connected inside $Q$ by a path $T_{\ell}$ of any other lengths $\ell$ except for 3

$$
\begin{equation*}
\exists \text { path } T_{\ell} \subset Q: \ell \in\{1,2,4,5, \ldots, k-3, k-2\} \text { with endpoints } a_{1}, a_{2} \tag{4}
\end{equation*}
$$

For example, $T_{1}$ is $a_{1} a_{2}, T_{2}=a_{1} b_{1} a_{2}, T_{4}=a_{1} b_{1} c_{1} b_{2} a_{2}$, etc. Also the vertices $a_{i}(i=1,2)$ and $q \in Q \backslash\left\{a_{i}\right\}$ are connected by a path $U^{i}(m)$ of length $m$ inside $Q$ for $\lceil(k+1) / 2\rceil \leq m \leq k-2$.

$$
\begin{equation*}
\exists \operatorname{path} U^{i}(m) \subset Q: m \in\{\lceil(k+1) / 2\rceil, \ldots, k-3, k-2\} \text { with endpoints } a_{i}, q \in Q \tag{5}
\end{equation*}
$$

Note that this is true for any $m \geq 4$ but we will apply (5) only for $\lceil(k+1) / 2\rceil \geq 4$.
Now add an edge $e$ to $H$ from its complement. We distinguish four disjoint cases.
Case 1. If $e$ is contained in the induced cycle $A \cup R_{\alpha}$ then we get a path connecting $a_{1}$ and $a_{2}$ in $A \cup R_{\alpha}$ of length $t$, where $t$ is at least two and at most $k-4$. This path with $T_{k-t}$ form a $k$-cycle. Case 2. If the endpoints of $e$ are $r_{\alpha, i}$ and $r_{\beta, j}$ with $\alpha \neq \beta$ then we may suppose that $1 \leq i \leq j \leq$ $k-4$. The vertex $r_{\alpha, i}$ splits the path $P_{\alpha}$ into two parts, $P_{\alpha}^{1}$ and $P_{\alpha}^{2}$, where $P_{\alpha}^{1}$ starts at $a_{1}$ and has length $i$, and $P_{\alpha}^{2}$ ends at $a_{2}$ and has length $k-3-i$. Consider the path $\pi:=P_{\alpha}^{1} e P_{\beta}^{2}$, its length is $k-2-j+i$. This length is between 3 and $k-2$ so we can apply (4) to add an appropriate $T_{j-i+2}$ to complete $\pi$ to a $k$-cycle unless $j-i+2=3$. In the latter, the edge $a_{1} a_{2}$ together with $P_{\beta}^{1}$, e,
and $P_{\alpha}^{2}$ form a $C_{k}$.
Case 3. If the endpoints of $e$ are $r_{\alpha, i}$ and $q \in B \cup C \cup D$, then again by symmetry, we may suppose that $i \leq(k-3) / 2$, so the length of $P_{\alpha}^{1}$ is at most $\lfloor(k-3) / 2\rfloor$. Then, by (5) there is an $U^{1}(m)$ so that $P_{\alpha}^{1}, e$ and $U^{1}(m)$ form a $k$-cycle.
Case 4. Finally, $e$ is contained in $Q$.
For $e=a_{1} c_{1}$ we use $P_{1}$ to get the $k$-cycle $a_{1} c_{1} b_{1} a_{2} P_{1}$,
for $e=a_{1} d_{1}$ we have the $k$-cycle $d_{1} c_{1} c_{2} \ldots c_{k-5} b_{2} a_{2} b_{1} a_{1}$,
for $e=b_{1} b_{2}$ we have to use $P_{1}$, i.e., here we need again that $t \geq 1$,
for $e=b_{1} d_{1}$ we have the $k$-cycle $d_{1} c_{1} c_{2} \ldots c_{k-5} b_{2} a_{2} a_{1} b_{1}$,
for $e=c_{1} d_{2}$ we have the $k$-cycle $c_{1} d_{2} c_{2} \ldots c_{k-5} b_{2} a_{2} a_{1} b_{1}$, finally
for $e=d_{1} d_{2}$ we have the $k$-cycle $c_{1} d_{1} d_{2} c_{2} \ldots c_{k-5} b_{2} a_{2} b_{1}$.

## 4 Semisaturated graphs

A graph $G$ is $H$-semisaturated (formerly called strongly saturated) if $G+e$ contains more copies of $H$ than $G$ does for $\forall e \in E(\bar{G})$. Let ssat $(n, H)$ be the minimum size of an $H$-semisaturated graph. Obviously, $\operatorname{ssat}(n, H) \leq \operatorname{sat}(n, H)$.

It is known that $\operatorname{ssat}\left(n, K_{p}\right)=\operatorname{sat}\left(n, K_{p}\right)$ (it follows from Frankl/Alon/Kalai generalizations of Bollobás set pair theorem) and $\operatorname{ssat}\left(n, C_{4}\right)=\operatorname{sat}\left(n, C_{4}\right)$ (Tuza [20]). Below we have a $C_{5^{-}}{ }^{-}$ semisaturated graph on $1+8 t$ vertices and $11 t$ edges. Every vertex can be reached by a path of length 2 from $y$. Joining one, two or three triangles to the central vertex $y$ one obtains $C_{5}$


$$
\begin{aligned}
& \mathrm{n}=1+8 \mathrm{t} \\
& \mathrm{e}=11 \mathrm{t}
\end{aligned}
$$


$\mathrm{n}=1+7 \mathrm{t}$
$\mathrm{e}=10 \mathrm{t}$
semisaturated graphs with $8 t+3,8 t+5$, or $8 t+7$ vertices and $11 t+3,11 t+6$, or $11 t+9$ edges, resp. Leaving out a pendant edge, we can extend these constructions for even values of $n$

$$
\begin{equation*}
\operatorname{ssat}\left(n, C_{5}\right) \leq\left\lceil\frac{11}{8}(n-1)\right\rceil \text { for all } n \geq 5 \tag{6}
\end{equation*}
$$

The picture on the right is the extremal $C_{5}$-saturated graph by (2).
Conjecture 4.1. $\quad \operatorname{ssat}\left(n, C_{5}\right)=\frac{11}{8} n+O(1)$. Maybe equality holds in (6) for $n>n_{0}$.
Since $11 / 8=1.375<10 / 7=1.42 \ldots$ inequalities (2) and (6) imply that

$$
\operatorname{ssat}\left(n, C_{5}\right)<\operatorname{sat}\left(n, C_{5}\right) \text { for all } n \geq 21
$$

Our next Theorem shows that a similar statement holds for every cycle $C_{k}$ with $k>12$ (and probably for $k \in\{6,7, \ldots, 12\}$, too).

Theorem 4.2. For all $n \geq k \geq 6$

$$
\begin{equation*}
\left(1+\frac{1}{2 k-2}\right) n-2<\operatorname{ssat}\left(n, C_{k}\right)<\left(1+\frac{1}{2 k-10}\right) n+k-1 . \tag{7}
\end{equation*}
$$

The proof of the lower bound is postponed to Section 9. The construction yielding the upper bound is presented in the next two sections where we describe a way to improve the $O(k)$ term as well as give better constructions for $k=6$. We believe that our constructions are essentially optimal.

Conjecture 4.3. There exists a $k_{0}$ such that $\operatorname{ssat}\left(n, C_{k}\right)=\left(1+\frac{1}{2 k-10}\right) n+O(k)$ holds for each $k>k_{0}$.

## 5 Constructions of sparse $C_{k}$-semisaturated graphs

In this section we define an infinite class of $C_{k}$-semisaturated graphs, $H_{k, n}^{2}$ (more precisely $\left.H_{k, n}^{2}(G)\right)$.
Call a graph $G k$-suitable with special vertices $a_{1}$ and $a_{2}$ if
(S1) $G$ is $C_{k}$-semisaturated,
(S2) $\exists$ a path $T_{\ell}$ in $G$ with endpoints $a_{1}$ and $a_{2}$ and of length $\ell$ for all $1 \leq \ell \leq k-2$, and
(S3) for every $q \in V(G) \backslash\left\{a_{1}, a_{2}\right\}$, and integers $m_{1}$ and $m_{2}$ with $m_{1}+m_{2}=k$ and $2 \leq m_{i} \leq k-2$ $\exists$ an $i \in\{1,2\}$ and a path $U\left(a_{i}, q, m_{i}\right)$ of length $m_{i}$ and with endpoints $a_{i}$ and $q$.
For example, it is easy to see, that a wheel with $r$ spikes $W_{k}^{r}$ is such a graph, $k \geq r, k \geq 4$. It is defined by the $(k+r)$-element vertex set $\left\{a_{1}, a_{2}, \ldots, a_{k}, d_{1}, \ldots, d_{r}\right\}$ and by $2 k-2+r$ edges joining $a_{1}$ to all other $a_{i}$ 's, forming a cycle $a_{2} a_{3} \ldots a_{k}$ of length $k-1$, and joining each $d_{i}$ to $a_{i}$.

Define the graph $H_{k, n}^{2}(G)$ as follows, when $n$ is in the form

$$
n=|V(G)|+t(k-3)
$$

where $t \geq 0$ is an integer. The vertex set $V(H)$ consists of the pairwise disjoint sets $Q$ and $R_{i}$ for $1 \leq i \leq t, V(H)=Q \cup R_{1} \cup \cdots \cup R_{t}$ where $|Q|=|V(G)|,\left|R_{1}\right|=\left|R_{2}\right|=\cdots=\left|R_{t}\right|=k-3$ and
$A:=\left\{a_{1}, a_{2}\right\} \subset Q$. The edge set of $H$ consists of a copy of $G$ with vertex set $Q$, and $t$ paths with endpoints $a_{1}$ and $a_{2}$ and vertex sets $A \cup R_{\alpha}$. The number of edges is

$$
|E(H)|=|E(G)|+t(k-2) .
$$

It is not difficult to check that, indeed, $H$ is $C_{k}$-semisaturated, the details are similar (but much simpler) to those in Section 33, so we do not repeat that proof.

Finally, considering $H_{k, n}^{2}\left(W_{k}^{r}\right)$ (where now $4 \leq r \leq k$ ) we obtain that for all $n \geq k+4$

$$
\begin{equation*}
\operatorname{ssat}\left(n, C_{k}\right) \leq n+\left\lfloor\frac{n-7}{k-3}\right\rfloor+k-3 \tag{8}
\end{equation*}
$$

Corollary 5.1. $\quad \operatorname{ssat}\left(n, C_{6}\right) \leq\left\lceil\frac{4}{3} n\right\rceil$.

## 6 Thinner constructions of sparse $C_{k}$-semisaturated graphs

In this section we define another infinite class of $C_{k}$-semisaturated graphs, $H_{k, n}^{3}$ (more precisely $\left.H_{k, n}^{3}(G)\right)$ yielding the upper bound (7) in Theorem 4.2,

Call a graph $G\{k . k+2\}$-suitable with special vertices $a_{1}$ and $a_{2}$ if (S1) and (S2) hold but (S3) is replaced by the following
$(\mathrm{S} 3)^{+}$for every $q \in V(G) \backslash\left\{a_{1}, a_{2}\right\}$, and integers $m_{1}, m_{2}$ either there exists a path $U\left(a_{1}, q, m_{1}\right)$ (of length $m_{1}$ and with endpoints $a_{1}$ and $q$ ) or a path $U\left(a_{2}, q, m_{2}\right)$ in the following cases
$m_{1}+m_{2}=k$ and $3 \leq m_{i} \leq k-3$,
$m_{1}+m_{2}=k+2$ and $4 \leq m_{i} \leq k-4$.
It is easy to see, that the wheel $W_{k}^{r}$ with $r$ spikes is such a graph, $k \geq r \geq 0, k \geq 4$.
Define the graph $H_{k, n}^{3}(G)$ as follows, when $n$ is in the form

$$
\begin{equation*}
n=|V(G)|+t(2 k-10)-r \tag{9}
\end{equation*}
$$

where $t \geq 2$ is an integer and $0 \leq r<2 k-10$. The vertex set $V(H)$ consists of the pairwise disjoint sets $Q, R_{i}$ and $D$ for $1 \leq i \leq t, V(H)=Q \cup R_{1} \cup \cdots \cup R_{t} \cup D$ where $|Q|=|V(G)|$, $\left|R_{1}\right|=\left|R_{2}\right|=\cdots=\left|R_{t}\right|=k-5,|D|=t(k-5)-r$ and $A:=\left\{a_{1}, a_{2}\right\} \subset Q$. The edge set of $H$ consists of a copy of $G$ with vertex set $Q$, and $t$ paths with endpoints $a_{1}$ and $a_{2}$ and vertex sets $A \cup R_{\alpha}$ and finally $|D|$ spikes, a matching with edges from $\cup R_{\alpha}$ to $D$.

The number of edges is

$$
\begin{equation*}
|E(H)|=|E(G)|+t(2 k-9)-r . \tag{10}
\end{equation*}
$$

It is not difficult to check that $H$ is $C_{k}$-semisaturated, the details are similar (but simpler) to those in Section 3, As an example we present one case.


Add the edge $q d$ to $H$ where $q \in V(G) \backslash\left\{a_{1}, a_{2}\right\}$ and $d \in D$. Let us denote the (unique) neighbor of $d$ by $x, x \in R_{\alpha}$. The distance of $x$ to $a_{1}$ is denoted by $\ell$. Then the length of the $q d x \ldots a_{1}$ path is $\ell+2 \geq 3$ and the length of the $q d x \ldots a_{2}$ path is $(k-4-\ell)+2 \geq 3$ and one can find a $C_{k}$ through $q d$ using property (S3) ${ }^{+}$.

Considering $H_{k, n}^{3}\left(W_{k}\right)$ (with $t \geq 2$ ) we obtain from (9) and (10) that for all $n \geq 3 k-9$

$$
\begin{equation*}
\operatorname{ssat}\left(n, C_{k}\right) \leq\left\lceil\left(1+\frac{1}{2 k-10}\right)(n-k)\right\rceil+2 k-2 . \tag{11}
\end{equation*}
$$

Using $H^{2}(k, n)$, it is easy to see that (11) holds for all $n \geq k$, leading to the upper bound in (77).
One can slightly improve (8) and (11) if there are special graphs thinner than the wheel $W_{k}$.
Problem 6.1. Determine $s(k)$, the minimum size of a $k$-vertex $k$-special graph (i.e., one satisfying (S1)-(S3)). Determine $s^{\prime}(k)$, the minimum size of a $k$-vertex $\{k, k+2\}$-special graph (i.e., one satisfying (S1), (S2) and (S3) ${ }^{+}$.

## 7 Degree one vertices in (semi)saturated graphs

Suppose that $G$ is a $C_{k}$-semisaturated graph where $k \geq 5,|V(G)|=n \geq k$. Obviously, $G$ is connected. Let $X$ be the set of vertices of degree one, $X:=\left\{v \in V(G): \operatorname{deg}_{G}(v)=1\right\}$, its size is $s$ and its elements are denoted as $X=\left\{x_{1}, x_{2}, \cdots, x_{s}\right\}$. Denote the neighbor of $x_{i}$ by $y_{i}$, $Y:=\left\{y_{1}, \ldots, y_{s}\right\}$ and let $Z:=V(G) \backslash(X \cup Y)$. We also denote the neighborhood of any vertex $v$ by $N_{G}(v)$ or briefly by $N(v)$.

Lemma 7.1. (The neighbors of degree one vertices.)
(i) $y_{i} \neq y_{j}$ for $1 \leq i \neq j \leq s$, so $|Y|=|X|$.
(ii) $\operatorname{deg}(y) \geq 3$ for every $y \in Y$,
(iii) if $\operatorname{deg}_{G}(x)=1$, then $G-\{x\}$ is also a $C_{k}$-semisaturated graph.

Proof. If $y_{i}=y_{j}$, then the addition of $x_{i} x_{j}$ to $G$ does not create a new $k$-cycle. If $\operatorname{deg}\left(y_{i}\right)=2$ and $N\left(y_{i}\right)=\left\{x_{i}, w\right\}$, the addition of $x_{i} w$ to $G$ does not create a new $k$-cycle. Finally, (iii) is obvious. $\square$

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Split $Y$ and $Z$ according to the degrees of their vertices. Thus, divide $V(G)$ into five parts $\left\{X, Y_{3}, Y_{4+}, Z_{2}, Z_{3+}\right\}$,
$Y_{3}:=\left\{v \in Y: \operatorname{deg}_{G}(v)=3\right\}$ and $Y_{4+}:=\left\{v \in Y: \operatorname{deg}_{G}(v) \geq 4\right\}$,
$Z_{2}:=\left\{v \in Z: \operatorname{deg}_{G}(v)=2\right\}$ and $Z_{3+}:=\left\{v \in Z: \operatorname{deg}_{G}(v) \geq 3\right\}$.
Lemma 7.2. (The structure of $C_{k}$-saturated graphs. See [3]).
Suppose that $G$ is a $C_{k}$-saturated graph (and $k \geq 5$ ). Then
(iv) if $x_{i} y_{i} w$ is a path in $G$ (with $x_{i} \in X, y_{i} \in Y$ ), then $\operatorname{deg}(w) \geq 3$. So there are no edges from $Z_{2}$ to $Y$ (or to $X$ ).
(v) If $y_{i} y_{j}$ is an edge of $G$ (with $y_{i}, y_{j} \in Y$ ), then $\operatorname{deg}\left(y_{i}\right) \geq 4$. So there are no edges in $Y_{3}$, no edges from $Y_{3}$ to $Y_{4}$. In other words, every $y \in Y_{3}$ has one neighbor in $X$ and two in $Z_{3+}$.
(vi) The induced graph $G\left[Z_{2}\right]$ consists of paths of length at most $k-2$.


## 8 Semisaturated graphs without pendant edges

Claim 8.1. Suppose that $G$ is a $C_{k}$-semisaturated graph on $n$ vertices with minimum degree at least $2, k \geq 5$. Then every vertex $w$ is contained in some cycle of length at most $k+1$.

Proof. Consider two arbitrary vertices $z_{1}, z_{2}$ in the neighborhood $N(w)$. If $z_{1} z_{2} \in E(G)$, then $w$ is contained in a triangle. If $z_{1} z_{2} \notin E(G)$, then $G+z_{1} z_{2}$ contains a new $k$-cycle; there is a path $P$ of length $(k-1)$ in $G$ with endpoints $z_{1}$ and $z_{2}$. If $P$ avoids $w$, then $P$ together with $z_{1} w z_{2}$ form a $k+1$ cycle. If $w$ splits $P$ into two paths $L_{1}, L_{2}$, where $L_{i}$ starts in $z_{i}$ and ends in $w$, then either $L_{1}+z_{1} w$, or $L_{2}+z_{2} w$, or both form a proper cycle of length at most $k-1$.

Note that Claim 8.1 itself (and the connectedness of $G$ ) immediately imply

$$
e(G) \geq(n-1) \frac{k+2}{k+1}
$$

We can do a bit better repeatedly using the semisaturatedness of $G$.

Lemma 8.2. Suppose that $G$ is a $C_{k}$-semisaturated graph on $n$ vertices with minimum degree at least $2, k \geq 5$. Then

$$
e(G) \geq \frac{k}{k-1} n-\frac{k+1}{k-1}
$$

Proof. We define an increasing sequence of subgraphs $G_{1}, G_{2}, \ldots, G_{t}=G$ with vertex sets $V_{1} \subseteq$ $V_{2} \subseteq \cdots \subseteq V_{t}=V(G)$ such that $G_{i}$ is a subgraph of $G_{i+1}$ and

$$
\begin{equation*}
\left|E\left(G_{i+1}\right) \backslash E\left(G_{i}\right)\right| \geq \frac{k}{k-1}\left(\left|V_{i+1}\right|-\left|V_{i}\right|\right) \tag{12}
\end{equation*}
$$

(for $i=1,2, \ldots, t-1$ ). This, together with

$$
\begin{equation*}
e\left(G_{1}\right) \geq \frac{k}{k-1}\left|V_{1}\right|-\frac{k+1}{k-1} \tag{13}
\end{equation*}
$$

imply the claim.
$G_{1}$ is the shortest cycle in the graph $G$. Its length is at most $k+1$ so (13) obviously holds.
If $G_{i}$ is defined and one can find a path $P$ of length at most $k$ with endpoints in $V_{i}$ but $E(P) \backslash E\left(G_{i}\right) \neq \emptyset$, then we can take $E\left(G_{i+1}\right)=E\left(G_{i}\right) \cup E(P)$. From now on, we suppose that such a short returning path does not exist. Our procedure stops if $V\left(G_{i}\right)=V(G)=: V$.

In the case of $V \backslash V_{i} \neq \emptyset$, the connectedness of $G$ implies that there exists an $x y$ edge with $x \in V_{i}$ and $y \in V \backslash V_{i}$. Since $|N(y)| \geq 2$ we have another edge $y z \in E(G), z \neq x$.

We have $N(y) \cap V_{i}=\{x\}$, otherwise one gets a path $x y z$ of length smaller than $k$ with endpoints in $V_{i}$ but going out of $G_{i}$, contradicting our earlier assumption. Similarly, we obtain that $N(y)$ contains no edge, otherwise we can define $E\left(G_{i+1}\right)$ as either $E\left(G_{i}\right)$ plus the three edges of a triangle $x y, y z, x z$ or we add four edges $x y, y z_{1}, y z_{2}$, and $z_{1} z_{2}$ but only three vertices (namely $y, z_{1}$, and $z_{2}$ ). The obtained $G_{i+1}$ obviously satisfies (12) in both cases. Similarly, if there is a cycle $C$ of length at most $k-1$ contaning $y$, then we can define $E\left(G_{i+1}\right)$ as $E\left(G_{i}\right)$ plus $E(C)$ and $x y$. From now on, we suppose that such a short cycle through $y$ does not exist.

Fix a neighbor $z$ of $y, z \neq x$. Since $z x \notin E(G), G$ contains a path $P$ of length $k-1$ with endvertices $x$ and $z$. Since $G$ does not contain a short returning path nor a short cycle through $y$, we obtain that $P$ avoids $y$ and $V(P) \cap V_{i}=\{x\}$.

If the cycle $C:=P+x y+y z$ of length $k+1$ has any diagonal edge then $G_{i+1}$ is obtained by adding $C$ together with its diagonals. From now on, we suppose that $C$ does not have any diagonals. More generally, if there is any diagonal path $P$ of length $\ell \leq k-1$ with edges disjoint from $E\left(G_{i}\right) \cup E(C)$ but with endpoints in $V_{i} \cup V(C)$ then we can define $E\left(G_{i+1}\right):=E\left(G_{i}\right) \cup E(C) \cup E(P)$ and have added $k+\ell-1$ vertices and $k+\ell+1$ edges, obviously satisfying (12).

However, such a diagonal path exists. Let $w \neq y$ be the other neighbor of $x$ in $C$. Since $w z \notin E(G)$, there is a path $P^{\prime}$ of length $k-1$ with endpoints $w$ and $z$. This $P^{\prime}$ must have edges outside $E\left(G_{i}\right) \cup E(C)$ so it can be shortened to a diagonal path $P$ of length at most $k-1$. This completes the proof of the Lemma.

## 9 A lower bound for the number of edges of semisaturated graphs

In this section we finish the proof of Theorem 4.2. Let $G$ be a $C_{k}$-semisaturated graph on $n$ vertices with minimum number of edges, $k \geq 5$. Let $X$ be the set of degree one vertices, $x:=|X|$. By Lemma $7.1|X| \leq n / 2$, and for $G^{\prime}:=G \backslash X$ we have $e\left(G^{\prime}\right)=e(G)-x$ and $G^{\prime}$ is a $C_{k}$-semisaturated graph on $n-x$ vertices with minimum degree at least 2. Then Lemma 8.2 can be applied to $e\left(G^{\prime}\right)$. We obtain

$$
\begin{aligned}
\operatorname{ssat}\left(n, C_{k}\right)=e(G) & \geq x+(n-x) \frac{k}{k-1}-\frac{k+1}{k-1} \\
& \geq \frac{n}{2}+\frac{n}{2} \frac{k}{k-1}-\frac{k+1}{k-1}=n\left(1+\frac{1}{2 k-2}\right)-\frac{k+1}{k-1}
\end{aligned}
$$

Since $\operatorname{sat}\left(n, C_{k}\right) \geq \operatorname{ssat}\left(n, C_{k}\right)$, this is already a better lower bound than the one in (3) from (3).

## 10 A lower bound for the number of edges of $C_{k}$-saturated graphs

In this section we finish the proof of Theorem [2.1. Let $G$ be a $C_{k}$-saturated graph on $n$ vertices, $k \geq 5$. Let us consider the partition of $V(G)=X \cup Y_{3} \cup Y_{4+} \cup Z_{2} \cup Z_{3+}$ defined in Section 7, where $X$ is the set of degree one vertices, $Y$ is their neighbors. By Lemma $7.1|X|=|Y|$. To simplify notations we use $a:=\left|Z_{2}\right|, b:=\left|Y_{3}\right|, c:=\left|Z_{3+}\right|$, and $d:=\left|Y_{4+}\right|$. We have

$$
n=a+2 b+c+2 d .
$$

By definition of the parts we have the lower bound

$$
2 e(G)=\sum_{v \in V} \operatorname{deg}(v) \geq|X|+2\left|Z_{2}\right|+3\left|Y_{3}\right|+3\left|Z_{3+}\right|+4\left|Y_{4+}\right| .
$$

This yields

$$
\begin{equation*}
2 e \geq 2 n+c+d \tag{14}
\end{equation*}
$$

Now we estimate the number of edges by considering four disjoint subsets of $E(G)$. The part $X$ is adjacent to $|X|$ edges, and according to Lemma 7.2, $Z_{2}$ is adjacent to at least $\frac{k}{k-1}\left|Z_{2}\right|$ edges,
$Y_{3}$ is adjacent to exactly $3\left|Y_{3}\right|$ edges from which $\left|Y_{3}\right|$ has already been counted at $X$, and finally $Y_{4+}$ is adjacent to at least another $\frac{3}{2}\left|Y_{4+}\right|$ edges. We obtain

$$
e(G) \geq|X|+\frac{k}{k-1}\left|Z_{2}\right|+2\left|Y_{3}\right|+\frac{3}{2}\left|Y_{4+}\right| .
$$

Therefore we get

$$
\begin{equation*}
e \geq n+\frac{1}{k-1} a+b-c+\frac{1}{2} d \tag{15}
\end{equation*}
$$

By Lemma $7.1 G \backslash X$ is also $C_{k}$-semisaturated. Apply Lemma 8.2 to estimate $e(G \backslash X)=e-b-d$, multiply by $(k-1)$ and rearrange, we get

$$
\begin{equation*}
(k-1) e \geq k n-b-d-(k+1) \tag{16}
\end{equation*}
$$

Adding up the above three inequalities (14), (15), and (16) we obtain

$$
(k+2) e \geq(k+3) n+\frac{1}{k-1} a+\frac{1}{2} d-(k+1)
$$

This implies the desired lower bound in (11).
Remark. We can do slightly better if we multiply (14), (15), and (16) by $k, k-1$, and $k-3$, resp., then adding up and simplifying we get

$$
\begin{equation*}
e(G)>\frac{k^{2}}{k^{2}-k+2} n-1 . \tag{17}
\end{equation*}
$$

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