

Linear balanceable and subcubic balanceable graphs

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Abstract

In [Structural properties and decomposition of linear balanced matrices, *Mathematical Programming*, 55:129–168, 1992], Conforti and Rao conjectured that every balanced bipartite graph contains an edge that is not the unique chord of a cycle. We prove this conjecture for balanced bipartite graphs that do not contain a cycle of length 4 (also known as linear balanced bipartite graphs), and for balanced bipartite graphs whose maximum degree is at most 3. We in fact obtain results for more general classes, namely linear balanceable and subcubic balanceable graphs. Additionally, we prove that cubic balanced graphs contain a pair of twins, a result that was conjectured by Morris, Spiga and Webb in [Balanced Cayley graphs and balanced planar graphs, *Discrete Mathematics*, 310:3228–3235, 2010].

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1 Introduction

A $0,1$ matrix is *balanced* if for every square submatrix with two ones per row and column, the number of ones is a multiple of four. This notion was introduced by Berge [1], and later extended to $0, \pm 1$ matrices by Truemper [16]. A $0, \pm 1$ matrix is *balanced* if for every square submatrix with two nonzero entries per row and column, the sum of the entries is a multiple of four. These matrices have been studied extensively in literature due to their important polyhedral properties, for a survey see [8].

Given a $0,1$ matrix A , the *bipartite graph representation* of A is the bipartite graph having a vertex for every row in A , a vertex for every column of A , and an edge ij joining row i to column j if and only if the entry a_{ij} of A equals 1. We say that G is *balanced* if it is the bipartite graph representation of some balanced matrix. It is easy to see that a bipartite graph G is balanced if and only if every hole of G has length $0 \pmod{4}$, where a *hole* is a chordless cycle of length at least 4. A *signed bipartite graph* is a bipartite graph, together with an assignment of weights $+1, -1$ to the edges of G . A signed bipartite graph is *balanced* if the *weight* of every hole H of G , i.e. the sum of the weights of the edges of H , is $0 \pmod{4}$. A bipartite graph is *balanceable* if there exists a *signing* of its edges, i.e. an assignment of weights $+1, -1$ to the edges of the graphs, such that the resulting signed bipartite graph is balanced.

The following conjecture is the last unresolved conjecture about balanced (balanceable) bipartite graphs in Cornuéjols' book [11] (it is Conjecture 6.11). Note that Conjectures 9.23, 9.28 and 9.29 from [11] have been resolved by Chudnovsky and Seymour in [4].

Conjecture 1.1 (Conforti and Rao [10]) *Every balanced bipartite graph contains an edge that is not the unique chord of a cycle.*

In other words, every balanced bipartite graph contains an edge whose removal leaves the graph balanced. This is not true if the graph is balanceable, as shown by R_{10} , that is the graph defined by the cycle $x_1x_2 \dots x_{10}x_1$ (of length 10) with chords x_ix_{i+5} , $1 \leq i \leq 5$ (see Figure 1 – in all figures in this paper a solid line denotes an edge, and a dashed one a path of length greater than 1). Graph R_{10} is cubic and balanceable (a proper signing of R_{10} is to assign weight $+1$ to the edges of the cycle $x_1x_2 \dots x_{10}x_1$ and -1 to the chords), but not balanced ($x_1x_2x_3x_4x_5x_6$ is a hole of length 6). Note that in R_{10} every edge is the unique chord of some cycle. Conjecture 1.1 generalises to balanceable graphs in the following way.

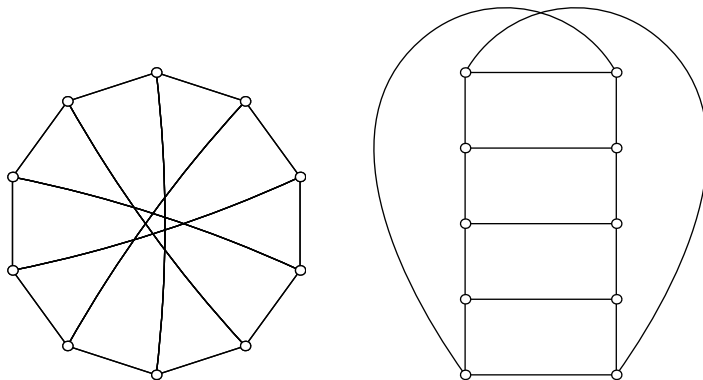


Figure 1: Two ways to draw the graph R_{10} .

Conjecture 1.2 (Conforti, Cornuéjols and Vušković [8]) *In a balanceable bipartite graph either every edge belongs to some R_{10} or there is an edge that is not the unique chord of a cycle.*

These conjectures are known to be true for several classes of graphs. A bipartite graph is *restricted balanceable* if there exists a signing of its edges so that in the resulting signed graph every cycle (induced or not) is balanced. Clearly no edge of a restricted balanceable bipartite graph can be the unique chord of a cycle. In other words, the removal of any subset of edges from a restricted balanceable graph leaves the graph restricted balanceable. A bipartite graph is *strongly balanceable* if it is balanceable and does not contain a cycle with a unique chord. Figure 2 shows that there are cubic balanceable graphs that are not strongly balanceable. This class generalizes restricted balanceable graphs, and it clearly satisfies Conjecture 1.2. On the other hand, removing any edge from a strongly balanceable graph might not leave the graph strongly balanceable. In [9] it is shown that every strongly balanceable graph has an edge whose removal leaves the graph strongly balanceable. A bipartite graph is *totally balanced* if every hole of G is of length 4. It is shown in [12] that every totally balanced bipartite graph has a bisimplicial edge (i.e. an edge uv such that the node set $N(u) \cup N(v)$ induces a complete bipartite graph). So clearly, the graph obtained by removing a bisimplicial edge from a totally balanced bipartite graph is also totally balanced.

A bipartite graph is *linear balanceable* if it is balanceable and does not contain a 4-hole (i.e. a hole of length 4). A graph G is *subcubic* if $\Delta(G) \leq 3$.

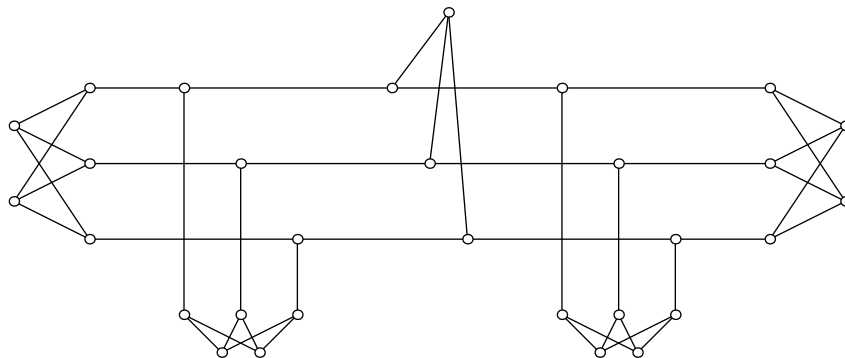


Figure 2: Cubic balanceable graph that is not strongly balanceable.

In this paper, we prove that conjectures 1.1 and 1.2 hold when restricted to linear balanceable graphs (see Corollary 4.3) and to subcubic balanceable graphs (see Corollary 5.4). For the subcubic case, our proof relies on a result conjectured by Morris, Spiga and Webb [13], stating that every cubic balanced bipartite graph contains a pair of vertices with the same neighborhood (see Corollary 5.3).

Our proofs are based on known decomposition theorems for the classes we consider, which we describe in Section 2. The decomposition theorems say that either the graph belongs to some simple subclass, that we call basic, or it has a 2-join, 6-join or star cutset. It is not straightforward to use these decomposition theorems to prove the desired results. In fact, the decomposition theorem for balanced bipartite graphs [7] has been known since the early 1990's, and still no one knows how to use it to prove the Conforti and Rao Conjecture. The key idea that makes things work for us, is the use of extreme decompositions, i.e. decompositions in which one of the blocks is basic. In Section 3 we prove that if star cutsets are excluded, then the graphs in our classes admit extreme decompositions. This is sufficient for the proof of the main result in the subcubic case in Section 5, since the induction hypothesis in this case goes through the star cutset nicely. For the linear balanceable bipartite graphs, this is not the case. Here we cannot inductively get rid of star cutsets in a straightforward manner. Furthermore, it is not true that if a (linear balanceable) graph has a star cutset, then it has a star cutset one of whose blocks of decomposition does not have a star cutset. Instead, to prove the main result for linear balanceable graphs in Section 4, we develop a new technique for finding an “extreme decomposition” with respect to star cutsets: we look for a minimally-sided double star

cutset, and show that the corresponding block of decomposition does not have a star cutset.

Terminology

We say that a graph G *contains* a graph H if H is isomorphic to an induced subgraph of G . A graph G is H -*free* if it does not contain H . For $x \in V(G)$, $N(x)$ denotes the set of neighbors of x . For $S \subseteq V(G)$, $G[S]$ denotes the subgraph of G induced by S , and $G \setminus S = G[V(G) \setminus S]$. For $S \subseteq E(G)$, $G \setminus S$ denotes the graph obtained from G by deleting edges from S .

A *path* P is a sequence of distinct vertices $p_1 p_2 \dots p_k$, $k \geq 1$, such that $p_i p_{i+1}$ is an edge for all $1 \leq i < k$. Edges $p_i p_{i+1}$, for $1 \leq i < k$, are called the *edges of* P . Vertices p_1 and p_k are the *ends* of P . A cycle C is a sequence of vertices $p_1 p_2 \dots p_k p_1$, $k \geq 3$, such that $p_1 \dots p_k$ is a path and $p_1 p_k$ is an edge. Edges $p_i p_{i+1}$, for $1 \leq i < k$, and edge $p_1 p_k$ are called the *edges of* C . Let Q be a path or a cycle. The vertex set of Q is denoted by $V(Q)$. The *length* of Q is the number of its edges. An edge $e = uv$ is a *chord* of Q if $u, v \in V(Q)$, but uv is not an edge of Q . A path or a cycle Q in a graph G is *chordless* if no edge of G is a chord of Q . The *girth* of a graph is the length of its shortest cycle.

A *cut vertex* of a connected graph G is a vertex v such that $G \setminus \{v\}$ is disconnected. A *block* of a graph is a connected subgraph that has no cut vertex and that is maximal with respect to this property. We may associate with any graph G a graph $B(G)$ on $\mathcal{B} \cup S$, where \mathcal{B} is the set of blocks of G and S the set of cut vertices of G , a block B and a cut vertex v being adjacent if and only if B contains v . It is a classical result that $B(G)$ is a tree (see [2]). The blocks that correspond to leaves of $B(G)$ are the *end blocks* of G .

2 Decomposition theorems

In this section we describe known decomposition theorems for balanceable graphs. First, we state the forbidden induced subgraph characterization of balanceable graphs. Let G be a bipartite graph. Let u, v be two nonadjacent vertices of G . A *3-path configuration connecting u and v* , is defined by three chordless paths P_1, P_2, P_3 with ends u and v , such that the vertex set $V(P_i) \cup V(P_j)$ induces a hole, for $i, j \in \{1, 2, 3\}$ and $i \neq j$. A 3-path configuration is said to be *odd* if it connects two vertices that are on opposite sides of the bipartition. A *wheel* is defined by a hole H and a vertex $x \notin V(H)$ having at least three neighbors in H , say x_1, x_2, \dots, x_n . If n is even,

then the wheel is an *even wheel*, and otherwise it is an *odd wheel*. A 3-path configuration and an odd wheel are shown in Figure 3.

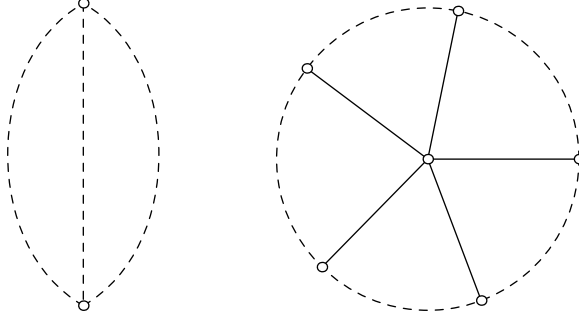


Figure 3: 3-path configuration and an odd wheel.

It is easy to see that a balanceable graph does not contain an odd 3-path configuration, nor an odd wheel. The following theorem of Truemper states that the converse is also true.

Theorem 2.1 (Truemper [16]) *A bipartite graph is balanceable if and only if it does not contain an odd wheel nor an odd 3-path configuration.*

Now, we introduce different cutsets used in the decomposition theorems that we need.

A set S of vertices (resp. edges) of a connected graph G is a *vertex cutset* (resp. *edge cutset*) if the subgraph $G \setminus S$ is disconnected.

1-join

A graph G has a *1-join* if $V(G)$ can be partitioned into sets X and Y so that the following hold:

- $|X| \geq 2$ and $|Y| \geq 2$.
- There exist sets A and B such that $\emptyset \neq A \subseteq X$ and $\emptyset \neq B \subseteq Y$; there are all possible edges between A and B ; and there are no other edges between X and Y .

We say that (X, Y, A, B) is a *split* of this 1-join.

2-join

A graph G has a *2-join* (X_1, X_2) if $V(G)$ can be partitioned into sets X_1 and X_2 so that the following hold:

- For $i = 1, 2$, X_i contains disjoint nonempty sets A_i and B_i , such that every vertex of A_1 is adjacent to every vertex of A_2 , every vertex of B_1 is adjacent to every vertex of B_2 , and there are no other adjacencies between X_1 and X_2 .
- For $i = 1, 2$, X_i contains at least one path from A_i to B_i , and if $|A_i| = |B_i| = 1$, then $G[X_i]$ is not a chordless path.

We say that $(X_1, X_2, A_1, A_2, B_1, B_2)$ is a *split* of this 2-join, and the sets A_1, A_2, B_1, B_2 are the *special sets* of this 2-join.

6-join

A graph G has a *6-join* (X_1, X_2) if $V(G)$ can be partitioned into sets X_1 and X_2 so that the following hold:

- X_1 (resp. X_2) contains disjoint nonempty sets A_1, A_3, A_5 (resp. A_2, A_4, A_6) such that, for every $i \in \{1, \dots, 6\}$, every vertex in A_i is adjacent to every vertex in $A_{i-1} \cup A_{i+1}$ (where subscripts are taken modulo 6), and these are the only adjacencies between X_1 and X_2 .
- $|X_1| \geq 4$ and $|X_2| \geq 4$.

We say that $(X_1, X_2, A_1, A_2, A_3, A_4, A_5, A_6)$ is a *split* of this 6-join.

Extended star cutset

In a connected bipartite graph G , (x, T, A, R) is an *extended star cutset* if T, A, R are disjoint subsets of $V(G)$, $x \in T$ and the following hold:

- The graph $G \setminus (T \cup A \cup R)$ is disconnected.
- $A \cup R \subseteq N(x)$
- The vertex set $T \cup A$ induces a complete bipartite graph (with vertex set T on one side of the bipartition and vertex set A on the other).
- If $|T| \geq 2$, then $|A| \geq 2$.

An extended star cutset such that $T = \{x\}$ is a *star cutset*. In this paper we will denote it as (x, R) . Note that when $|T| = 1$ and $A \cup R = \emptyset$ then $\{x\}$ is a cut vertex.

The following theorem is proved in [5], building on the decomposition theorem in [7]. We observe that the definition of 2-join in [7] and [5] is slightly different from the one we gave here. We define the 2-join and state the following theorem as in [8]. The statement is easily seen to be equivalent to the one in [5] by Lemma 2.5 below.

Theorem 2.2 (Conforti, Cornuéjols, Kapoor and Vušković [5]) *A connected balanceable bipartite graph is either strongly balanceable or is R_{10} , or it has a 2-join, a 6-join or an extended star cutset.*

Theorem 2.3 (Conforti and Rao [9]) *A strongly balanceable bipartite graph is either restricted balanceable or has a 1-join.*

A bipartite graph is *basic* if it admits a bipartition such that all the vertices in one side of the bipartition have degree at most 2.

Theorem 2.4 (Yannakakis [18]) *A restricted balanceable bipartite graph is either basic or has a cut vertex or a 2-join whose special sets are all of size 1.*

The following lemma is proved in [15] (Lemma 3.2) and a special case of it is proved in [7] (Lemma 2.4).

Lemma 2.5 *Let G be a graph that has no star cutset, and let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a 2-join of G . Then for $i = 1, 2$, the following hold:*

- (i) *Every component of $G[X_i]$ meets both A_i and B_i .*
- (ii) *Every $u \in X_i$ has a neighbor in X_i .*
- (iii) *Every vertex of A_i has a non-neighbor in B_i .*
- (iv) *Every vertex of B_i has a non-neighbor in A_i .*
- (v) $|X_i| \geq 4$.

Lemma 2.6 *Let G be a bipartite graph that has no star cutset. If G has a 1-join, then G is a 4-hole.*

PROOF — Let (X, Y, A, B) be a split of a 1-join of G . If $Y \setminus B \neq \emptyset$, then a vertex from A and set B form a star cutset, a contradiction. So $Y = B$, and by symmetry $X = A$. If $|A| \geq 3$, then a vertex from A and set B form a star cutset, a contradiction. So, by symmetry, $|A| = |B| = 2$, and therefore G is a 4-hole. \square

In 4-hole-free graphs, and also in subcubic graphs, we can reduce extended star cutset to star cutset. Indeed in a 4-hole-free graph, if (x, T, A, R) is an extended star cutset with $|T| \geq 2$, then by definition $|A| \geq 2$ and the complete bipartite graph $A \cup T$ contains a 4-hole, so $|T| = 1$ and (x, T, A, R)

is a star cutset. In a subcubic graph G if (x, T, A, R) is an extended star cutset with $|T| \geq 2$, then $|A| \geq 2$. Since each vertex of T has neighbors in at most one component of $G \setminus (T \cup A \cup R)$ (because the graph is subcubic), we see that $G \setminus (\{x\} \cup A \cup R)$ has at least as many components as $G \setminus (T \cup A \cup R)$. It follows that $(x, R \cup A)$ is a star cutset of G . So from Theorems 2.2, 2.3 and 2.4, and Lemmas 2.5 and 2.6, we get the following decomposition theorem that we will use in this paper.

Theorem 2.7 *Let G be a connected balanceable bipartite graph.*

- *If G is 4-hole-free, then G is basic, or has a 2-join, a 6-join or a star cutset.*
- *If $\Delta(G) \leq 3$, then G is basic or is R_{10} , or has a 2-join, a 6-join or a star cutset.*

We observe that a balanceable bipartite graph G with $\Delta(G) \leq 3$ is actually *matrix-regular*, as we explain now. A matrix is *totally unimodular* if every square submatrix has determinant equal to 0, +1 or -1. A 0, 1 matrix is *regular* if its nonzero entries can be signed +1 or -1 so that the resulting matrix is totally unimodular. A 0, 1 matrix A can be thought of as a vertex-vertex incidence matrix of a bipartite graph, which we denote with $G(A)$. We say that a bipartite graph G is *matrix-regular* if $G = G(A)$ for some regular 0, 1 matrix A . A graph is *eulerian* if all its vertices have even degree. By a theorem of Camion [3], a bipartite graph is matrix-regular if and only if there exists a signing of its edges with +1 or -1 so that the weight of every induced eulerian subgraph is a multiple of 4. It now clearly follows that for a bipartite graph G with $\Delta(G) \leq 3$: G is balanceable if and only if G is matrix-regular.

It is natural to ask why we use Theorem 2.7 in our proof of Conforti and Rao Conjecture in the subcubic case, instead of Seymour's decomposition theorem for matrix-regular bipartite graphs [14]. The answer is that by using Theorem 2.7 we have only to check whether three cutsets (2-join, 6-join and star cutset) go through our induction hypothesis, whereas if we used the decomposition theorem in [14] we would have to check five cutsets (1-join, 2-join, 6-join, N-join and M-join, for an explanation see [17]). Furthermore, 2-joins and 6-joins in graphs with no star cutset have special properties (given in Section 3) which are very useful for pushing the induction hypothesis through them.

3 Graphs with no star cutset

The following properties of graphs with no star cutsets will be essential in our proofs.

Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a 2-join of a graph G . The *blocks of decomposition* of G by this 2-join are graphs G_1 and G_2 defined as follows. To obtain G_i , for $i = 1, 2$, we start from $G[X_i]$, and first add a vertex a_{3-i} , adjacent to all the vertices in A_i and no other vertex of X_i , and a vertex b_{3-i} adjacent to all the vertices in B_i and no other vertex of X_i . For $i = 1, 2$, let Q_{3-i} be a path in $G[X_{3-i}]$ with smallest number of edges connecting a vertex in A_{3-i} to a vertex in B_{3-i} . For $i = 1, 2$, add to G_i a *marker path* M_{3-i} connecting a_{3-i} and b_{3-i} with length $|E(M_{3-i})| \in \{4, 5\}$ having the same parity as Q_{3-i} .

The following lemma is proved in [6]. (Note that the statement is not the same but the proof of Theorem 4.6 in [6] shows precisely what we need).

Lemma 3.1 *Let G be a bipartite graph with no star cutset. Let (X_1, X_2) be a 2-join of G , and let G_1 and G_2 be the corresponding blocks of decomposition. Then the following hold:*

- (i) *If G is balanceable, then G_1 and G_2 are balanceable.*
- (ii) *G_1 and G_2 have no star cutset.*
- (iii) *If G has no 6-join, then G_1 and G_2 have no 6-join.*

Let $(X_1, X_2, A_1, \dots, A_6)$ be a split of a 6-join of a graph G . The *blocks of decomposition* of G by this 6-join are graphs G_1 and G_2 defined as follows. For $i = 1, \dots, 6$ let a_i be any vertex of A_i . Then $G_1 = G[X_1 \cup \{a_2, a_4, a_6\}]$ and $G_2 = G[X_2 \cup \{a_1, a_3, a_5\}]$. Nodes a_2, a_4, a_6 (resp. a_1, a_3, a_5) are called the *marker nodes* of G_1 (resp. G_2).

Lemma 3.2 *Let G be a bipartite graph with no star cutset. Let $(X_1, X_2, A_1, \dots, A_6)$ be a split of a 6-join of G , and G_1 and G_2 the corresponding blocks of decomposition. Then the following hold:*

- (i) $X_1 \setminus (A_1 \cup A_3 \cup A_5) \neq \emptyset$ and $X_2 \setminus (A_2 \cup A_4 \cup A_6) \neq \emptyset$.
- (ii) *If C is a connected component of $G[X_1 \setminus (A_1 \cup A_3 \cup A_5)]$ (resp. $G[X_2 \setminus (A_2 \cup A_4 \cup A_6)]$), then a node of A_i , for every $i = 1, 3, 5$ (resp. $i = 2, 4, 6$) has a neighbor in C .*

- (iii) If G is 4-hole-free or $\Delta(G) \leq 3$, then $|A_i| = 1$ for every $i \in \{1, \dots, 6\}$, and in particular every node of $\cup_{i=1}^6 A_i$ is of degree at least 3 in G .
- (iv) If G is balanceable, then so are G_1 and G_2 .
- (v) If G is 4-hole-free, then G_1 and G_2 do not have star cutsets.

PROOF — Note that G is bipartite so there are no edges in $A_1 \cup A_3 \cup A_5$ nor in $A_2 \cup A_4 \cup A_6$.

Suppose that $X_1 \setminus (A_1 \cup A_3 \cup A_5) = \emptyset$. Then w.l.o.g. $|A_1| \geq 2$, and hence for a node $a_1 \in A_1$, $\{a_1\} \cup A_2 \cup A_6$ is a star cutset of G , a contradiction. Therefore (i) holds.

Let C be a connected component of $G[X_1 \setminus (A_1 \cup A_3 \cup A_5)]$ and suppose that no node of A_1 has a neighbor in C . Then for a node $a_4 \in A_4$, $\{a_4\} \cup A_3 \cup A_5$ is a star cutset of G separating C from the rest, a contradiction. Therefore by symmetry, (ii) holds.

If G is 4-hole-free then clearly $|A_i| = 1$ for every $i \in \{1, \dots, 6\}$, and if $\Delta(G) \leq 3$ then the same holds by (i) and (ii), therefore, (iii) holds.

Since G_1 and G_2 are induced subgraphs of G , (iv) holds.

To prove (v) assume G is 4-hole-free and w.l.o.g. G_1 has a star cutset (x, R) . Let a_2, a_4, a_6 be the marker nodes of G_1 . By (ii), $x \notin \{a_2, a_4, a_6\}$. If $x \in A_1$, then $(x, R \cup A_2 \cup A_6)$ is a star cutset of G , a contradiction. Therefore by symmetry, $x \in X_1 \setminus (A_1 \cup A_3 \cup A_5)$. Since G is 4-hole-free R may contain nodes from at most one of the sets A_1, A_3, A_5 , and hence a_2, a_4, a_6 are all contained in the same connected component of $G_1 \setminus (\{x\} \cup R)$. It follows that (x, R) is also a star cutset of G , a contradiction. Therefore (v) holds. \square

We observe that property (v) above is not true in general for balanceable graphs. On the other hand, it is true for subcubic balanceable graphs. Since we will use a different technique to prove the main result for subcubic balanceable graphs than the one we will use for linear balanceable graphs, we will not need this result.

A 2-join (X_1, X_2) of G is a *minimally-sided 2-join* if for some $i \in \{1, 2\}$ the following holds: for every 2-join (X'_1, X'_2) of G , neither $X'_1 \subsetneq X_i$ nor $X'_2 \subsetneq X_i$. In this case X_i is a *minimal side* of this minimally-sided 2-join.

Lemma 3.3 (Trotignon and Vušković [15]) *Let G be a graph with no star cutset. Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a minimally-sided 2-join of G with X_1 being a minimal side, and let G_1 and G_2 be the corresponding blocks of decomposition. Then the following hold:*

(i) $|A_1| \geq 2$, $|B_1| \geq 2$, and in particular all the vertices of $A_2 \cup B_2$ are of degree at least 3.

(ii) If G_1 and G_2 do not have star cutsets, then G_1 has no 2-join.

A partition (X_1, X_2) of $V(G)$ is a $\{2, 6\}$ -join if it is a 2-join or a 6-join of G . It is a *minimally-sided* $\{2, 6\}$ -join if for some $i \in \{1, 2\}$ the following holds: for every $\{2, 6\}$ -join (X'_1, X'_2) of G , neither $X'_1 \subsetneq X_i$ nor $X'_2 \subsetneq X_i$. In this case X_i is a *minimal side* of this minimally-sided $\{2, 6\}$ -join.

Lemma 3.4 *Let G be a 4-hole-free bipartite graph. Let (X_1, X_2) be a minimally-sided $\{2, 6\}$ -join of G , with X_1 being a minimal side. If G has no star cutset, then the block of decomposition G_1 has no $\{2, 6\}$ -join.*

PROOF — Assume the contrary, and let (X'_1, X'_2) be a $\{2, 6\}$ -join of G_1 . We now consider the following cases.

Case 1: (X_1, X_2) is a 2-join of G .

By Lemmas 3.1 (ii) and 3.3 (ii), (X'_1, X'_2) is a 6-join of G_1 , say with split $(X'_1, X'_2, A'_1, \dots, A'_6)$. Let P_2 be the marker path of G_1 . By Lemma 3.2 (iii), we may assume w.l.o.g. that $V(P_2) \subseteq X'_2$. If $V(P_2) \subseteq X'_2 \setminus (A'_2 \cup A'_4 \cup A'_6)$, then clearly $(X'_1, (X'_2 \setminus V(P_2)) \cup X_2)$ is a 6-join of G that contradicts the choice of (X_1, X_2) . So $V(P_2) \cap (A'_2 \cup A'_4 \cup A'_6) \neq \emptyset$. By Lemma 3.2 (ii), we may assume w.l.o.g. that $V(P_2) \cap (A'_4 \cup A'_6) = \emptyset$. But then $(X'_1, (X'_2 \setminus V(P_2)) \cup X_2, A'_1, A_2, A'_3, A'_4, A'_5, A'_6)$ is a split of a 6-join of G that contradicts the choice of (X_1, X_2) .

Case 2: (X_1, X_2) is a 6-join of G .

Let $(X_1, X_2, A_1, \dots, A_6)$ be the split of this 6-join, and let a_2, a_4, a_6 be the marker nodes of G_1 . We now consider the following two cases.

Case 2.1: (X'_1, X'_2) is a 6-join of G_1 .

Let $(X'_1, X'_2, A'_1, \dots, A'_6)$ be the split of this 6-join. By Lemma 3.2 (iii) we may assume w.l.o.g. that $\{a_2, a_4, a_6\} \subseteq X'_2 \setminus (A'_2 \cup A'_4 \cup A'_6)$. But then $(X'_1, X'_2 \cup X_2)$ is a 6-join of G that contradicts the choice of (X_1, X_2) .

Case 2.2: (X'_1, X'_2) is a 2-join of G_1 .

Let $(X'_1, X'_2, A'_1, A'_2, B'_1, B'_2)$ be the split of this 2-join. By Lemma 3.2 (iii), let $A_1 = \{a_1\}$, $A_3 = \{a_3\}$ and $A_5 = \{a_5\}$, and let H be the 6-hole induced by $\{a_1, \dots, a_6\}$. First suppose that both $X'_1 \setminus (A'_1 \cup B'_1)$ and $X'_2 \setminus (A'_2 \cup B'_2)$ contain a node of H . Then w.l.o.g. we may assume that $a_2 \in X'_2 \setminus (A'_2 \cup B'_2)$,

$a_4 \in B'_1$ and $a_6 \in A'_1$. Since nodes a_2, a_4 and a_6 are all of degree 2 in G_1 , it follows that $A'_2 = \{a_1\}$ and $B'_2 = \{a_3\}$, and hence by Lemma 3.2 (iii) $(a_2, \{a_1, a_3\})$ is a star cutset of G , a contradiction.

So we may assume w.l.o.g. that $(X'_2 \setminus (A'_2 \cup B'_2)) \cap V(H) = \emptyset$. By Lemma 2.5 (ii) and since a_2, a_4, a_6 are all of degree 2 in G_1 , it follows that in fact w.l.o.g. we may assume that $V(H) \cap X'_2 \subseteq A'_2$. By Lemma 2.5 (ii), every node of A'_2 has a neighbor in X'_2 , and hence (since a_2, a_4, a_6 are all of degree 2 in G_1) $\{a_2, a_4, a_6\} \subseteq X'_1$. But then $(X'_1 \cup X_2, X'_2)$ is a 2-join of G that contradicts the choice of (X_1, X_2) . \square

4 Linear balanceable graphs

A *double star cutset* of a connected graph G is a set S of vertices such that $G \setminus S$ is disconnected and S contains two adjacent vertices u and v such that every vertex of S is adjacent to at least one of u or v . Note that a star cutset is either a double star cutset or a cut vertex. If $U = (N(u) \cap S) \setminus \{v\}$ and $V = (N(v) \cap S) \setminus \{u\}$, then this double star cutset is denoted by (u, v, U, V) . Note that if G is a 4-hole-free bipartite graph, $U \cup V$ induce a stable set and $U \cap V = \emptyset$.

Let C_i , for $i = 1, 2$, be a partition of the vertex set $V(G \setminus S)$, such that there are no edges between vertices of C_1 and C_2 . Then $G_i = G[S \cup V(C_i)]$, $i = 1, 2$, are *blocks of decomposition* with respect to this double star cutset.

A double star cutset of a 2-connected graph G with blocks of decompositions G_1 and G_2 is a *minimally-sided double star cutset* if for some $i \in \{1, 2\}$ the following holds: for every double star cutset of G with blocks of decompositions G'_1 and G'_2 neither $V(G'_1) \subsetneq V(G_i)$ nor $V(G'_2) \subsetneq V(G_i)$. In this case G_i is a *minimal side* of this minimally-sided double star cutset.

Lemma 4.1 *Let G be a 2-connected 4-hole-free bipartite graph that has a star cutset. Let G_i , for some $i \in \{1, 2\}$ be a minimal side of a minimally-sided double star cutset of G . Then G_i does not have a star cutset.*

PROOF — Let (u, v, U, V) be a minimally-sided double star cutset, let G_1 be its minimal side, and let $S = \{u, v\} \cup U \cup V$. Observe that every vertex of $U \cup V$ has a neighbor in $G_1 \setminus S$. In particular, G_1 is 2-connected. Let us assume by way of contradiction that (x, R) is a star cutset of G_1 . Since G_1 is 2-connected, $R \neq \emptyset$.

Case 1: $x \notin S$.

Since G is 4-hole-free and bipartite, x has at most one neighbor in S . If $R \cap \{u, v\} = \emptyset$, then vertices of $S \setminus R$ are in the same connected component of $G_1 \setminus (\{x\} \cup R)$, and therefore $(x, y, R \setminus \{y\}, \emptyset)$, for a vertex $y \in R$, is a double star cutset of G that contradicts the minimality of G_1 . So w.l.o.g. $u \in R$. Let C be a connected component of $G_1 \setminus (\{x\} \cup R)$ that does not contain a node of $\{v\} \cup V$. If $V(C) \setminus U \neq \emptyset$, then $(x, u, R \setminus \{u\}, U)$ is a double star cutset of G that contradicts the minimality of G_1 . So $V(C) \setminus U = \emptyset$. But then some vertex $u' \in U$ is of degree 1 in G_1 (since G_1 is 4-hole-free and bipartite), contradicting the fact that G_1 is 2-connected.

Case 2: $x \in S$.

First, let us assume that $x \in \{u, v\}$, say $x = u$. Since G is 4-hole-free and bipartite, every connected component of $G_1 \setminus (\{x\} \cup R)$ that contains a vertex from U or a vertex from V contains a vertex from $G_1 \setminus S$. Therefore, $(x, v, (U \cup R) \setminus \{v\}, V)$ is a double star cutset of G that contradicts the minimality of G_1 . So, $x \in U \cup V$, and w.l.o.g. we may assume that $x \in U$. Then the nodes of $\{v\} \cup V$ are all contained in the same connected component of $G_1 \setminus (\{x\} \cup R)$. Again, since G is 4-hole-free and bipartite, every connected component of $G_1 \setminus (\{x\} \cup R)$ that contains a vertex from U contains a vertex from $G_1 \setminus S$. Therefore, $(x, u, R \setminus \{u\}, U \setminus \{x\})$ is a double star cutset of G that contradicts the minimality of G_1 . \square

Our main result about linear balanceable graphs is the following.

Theorem 4.2 *If G is a linear balanceable graph on at least two vertices, then G contains at least two vertices of degree at most 2.*

PROOF — We prove the theorem by induction on $|V(G)|$. If $|V(G)| = 2$, then the theorem trivially holds. So, let G be a linear balanceable graph such that $|V(G)| > 2$. We may assume that G is connected, else we are done by induction.

Let u be a cut vertex of G , and let $\{C_1, C_2\}$ be a partition of $V(G) \setminus \{u\}$, such that there are no edges between vertices of C_1 and C_2 . Then, by induction applied to graphs $G[C_i \cup \{u\}]$ for $i = 1, 2$, there is a vertex $c_i \in C_i \setminus \{u\}$, for $i = 1, 2$, that is of degree at most 2 in $G[C_i \cup \{u\}]$. But then c_1 and c_2 are also of degree at most 2 in G . So, we may assume that G is 2-connected.

Now suppose that G admits a star cutset. By Lemma 4.1, there is a double star cutset (u, v, U, V) of G , such that a block of decomposition w.r.t. this cutset, say G' , has no star cutset. Let $S = \{u, v\} \cup U \cup V$ and note

that all vertices from U and V have a neighbor in $G' \setminus S$. By Theorem 2.7 G' is basic or has a $\{2, 6\}$ -join.

Case 1: G' is basic.

Let (X, Y) be a bipartition of G' such that all vertices of Y are of degree 2. Vertices u and v are adjacent, so we may assume w.l.o.g. that $\{v\} \cup U \subseteq Y$ and $\{u\} \cup V \subseteq X$. In particular, $|V| \leq 1$.

Suppose $V = \{v'\}$. All the neighbors of v' in $G' \setminus S$ are of degree 2 in G' and in G , so we may assume that v' has a unique neighbor w in $G' \setminus S$. Let w' be the unique neighbor of w in $G' \setminus v'$. Since G' is 4-hole-free and bipartite, $w' \in V(G') \setminus S$. If w' is of degree 2 in G' (and hence in G), then w' and w are the desired two vertices. So we may assume that w' has at least three neighbors in G' . But then, since G' is 4-hole-free and bipartite, w' must have a neighbor $w'' \in V(G') \setminus (S \cup \{w\})$, and hence w and w'' are the desired two vertices.

Now suppose that $V = \emptyset$ and let v' be the neighbor of v in $V(G') \setminus S$. Since G is 4-hole-free and bipartite, v' has no neighbors in $U \cup \{u\}$. So, either $\deg_{G'}(v') \geq 3$, in which case v' has at least two neighbors in $V(G') \setminus S$ of degree 2 in G' , and hence in G , or $\deg_{G'}(v') = 2$, in which case v' and the neighbor of v' in $V(G') \setminus S$ are both of degree 2 in G' , and hence in G . Therefore G has at least two vertices of degree 2.

Case 2: G' has a $\{2, 6\}$ -join.

Let (X'_1, X'_2) be a $\{2, 6\}$ -join of G' . W.l.o.g. we may assume that $|X'_1 \cap \{u, v\}| \leq 1$. Let (X_1, X_2) be a minimally-sided $\{2, 6\}$ -join of G' such that $X_1 \subseteq X'_1$, and let G_1 be the corresponding block of decomposition. Clearly G_1 is 4-hole-free and $|X_1 \cap \{u, v\}| \leq 1$. By Lemmas 3.1 and 3.2, G_1 is linear balanceable and has no star cutset. By Lemma 3.4, G_1 has no $\{2, 6\}$ -join, and hence by Theorem 2.7, G_1 is basic. We now consider the following two cases.

Case 2.1: (X_1, X_2) is a 6-join of G' .

Let $(X_1, X_2, A_1, \dots, A_6)$ be the split of this 6-join. By Lemma 3.2, $A_1 = \{a_1\}$, $A_3 = \{a_3\}$, $A_5 = \{a_5\}$, and all these nodes are of degree at least 3 in G_1 . Since G_1 is 4-hole-free, nodes a_1, a_3, a_5 do not have common neighbors in X_1 . Since $|X_1 \cap \{u, v\}| \leq 1$, we may assume w.l.o.g. that $(X_1 \setminus \{a_1\}) \cap \{u, v\} = \emptyset$. Let a'_3 (resp. a'_5) be a neighbor of a_3 (resp. a_5) in X_1 . Then $a'_3 \neq a'_5$ and $\{a'_3, a'_5\} \cap S = \emptyset$. Since G_1 is basic, a'_3 and a'_5 are of degree 2 in G_1 , and hence in G' . Since $\{a'_3, a'_5\} \cap S = \emptyset$, they are also of degree 2 in G .

Case 2.2: (X_1, X_2) is a 2-join of G' .

Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be the split of this 2-join, and let P_2 be the marker path of G_1 . By Lemma 3.3, $|A_1| \geq 2$, $|B_1| \geq 2$ and the ends of P_2 are of degree at least 3 in G_1 . Since G_1 is basic, it follows that the nodes of $A_1 \cup B_1$ are all of degree 2 in G_1 , and on the same side of bipartition of G_1 , and hence of G' as well. In particular, it is not possible that both u and v are in $A_2 \cup B_2$. Since G' is 4-hole-free and bipartite, it follows that $|A_2| = |B_2| = 1$, and hence the nodes of $A_1 \cup B_1$ are of degree 2 in G' . Since $|X_1 \cap \{u, v\}| \leq 1$, w.l.o.g. $B_1 \cap S = \emptyset$, and hence the nodes of B_1 are also of degree 2 in G .

So, we may assume that G does not admit a star cutset. Thus, by Theorem 2.7 G is basic or has a $\{2, 6\}$ -join. So the theorem holds by the same proof as in Cases 1 and 2 above. \square

Corollary 4.3 *Let G be a linear balanceable graph that has at least one edge. Then there is an edge of G that is not the unique chord of a cycle.*

PROOF — Follows immediately from Theorem 4.2 since an edge incident to a degree 2 vertex cannot be the unique chord of a cycle. \square

5 Subcubic balanceable graphs

A *branch vertex* is a vertex of degree at least 3. A *branch* is a path connecting two branch vertices and containing no other branch vertices. Two branches are *non incident* if the sets of ends of the corresponding paths are disjoint. Note that a 2-connected graph that is not a cycle is edgewise partitioned into its branches. A pair of vertices (u, v) of G is a *pair of twins* in G if $N(u) = N(v)$ and $|N(u)| \geq 3$. Note that a cubic bipartite graph has a pair of twins if and only if it contains a $K_{2,3}$ as a subgraph. Note that R_{10} does not have a pair of twins.

Our main result on subcubic balanceable graphs is the following theorem.

Theorem 5.1 *Let G be a 2-connected balanceable bipartite graph with $\Delta(G) \leq 3$. If G is not equal to R_{10} and has at least three branch vertices, then one of the following holds:*

- (i) G has two vertices of degree 2 that are in non incident branches.
- (ii) G has a pair of twins and a vertex of degree 2.
- (iii) G has two disjoint pairs of twins.

In the previous theorem, if G has at least three branch vertices, then it has in fact at least four branch vertices (because 2-connected graphs have no vertex of degree 1).

The following lemma settles the case in which G does not admit a star cutset nor a 6-join. We treat this case separately because it does not need induction.

Lemma 5.2 *Let G be a 2-connected balanceable bipartite graph with $\Delta(G) \leq 3$, that is not equal to R_{10} and has at least three branch vertices. If G does not have a star cutset nor a 6-join, then G has two vertices of degree 2 that are in non incident branches.*

PROOF — By Theorem 2.7, G is either basic or has a 2-join, so we consider the following two cases. Note that every vertex of G is of degree at least 2.

Case 1: G is basic.

Since G is basic, no two branch vertices are adjacent, and hence every branch of G contains a vertex of degree 2. Let a, b, c be distinct vertices of degree 3, such that there is a branch from a to b . There are three branches in G with end c . If one of the other ends of these branches is not a or b , the proof is complete. So we may assume w.l.o.g. that we have two branches between a and c and one branch between b and c . But then there is a branch from b with an end not in $\{a, c\}$, and hence the result follows.

Case 2: G has a 2-join.

Let $(X_1, X_2, A_1, A_2, B_1, B_2)$ be a split of a minimally-sided 2-join of G with X_1 being a minimal side. Let G_1 be the corresponding block of decomposition. By Lemma 3.1, G_1 is balanceable and it does not have a star cutset nor a 6-join. By Lemmas 3.3 and 2.5, G_1 has no 2-join, $|A_1| = |B_1| = 2$, and all vertices of $A_2 \cup B_2$ are of degree 3. So by Theorem 2.7 G_1 is basic.

Claim: $X_1 \setminus (A_1 \cup B_1)$ contains a vertex of degree 2.

Proof of Claim: Assume not. Let (X, Y) be a bipartition of G_1 such that all vertices of X are of degree 2. Let a_2, \dots, b_2 be the marker path of G_1 , with a_2 complete to A_1 and b_2 complete to B_1 . Then a_2 and b_2 are in Y and hence $A_1 \cup B_1 \subseteq X$. In particular, there are no edges in $G[A_1 \cup B_1]$. So by Lemma 2.5, $X_1 \setminus (A_1 \cup B_1)$ is not empty. By our assumption $X_1 \setminus (A_1 \cup B_1) \subseteq Y$. So for every $u \in X_1 \setminus (A_1 \cup B_1)$, $N(u) \subseteq A_1 \cup B_1$. But then since $|A_1 \cup B_1| = 4$, $|N(u)| \leq 3$ and the fact that each vertex of $A_1 \cup B_1$ is of degree 2 in G_1 , we have a contradiction. This completes the proof of the claim.

By the claim let $c_1 \in X_1 \setminus (A_1 \cup B_1)$ be of degree 2 (in G_1 , and hence in G as well). Let $(X'_1, X'_2, A'_1, A'_2, B'_1, B'_2)$ be a split of a minimally-sided 2-join of G with X'_2 being a minimal side and $X'_2 \subseteq X_2$. Then, as before, $|A'_2| = |B'_2| = 2$, and hence all the vertices of $A'_1 \cup B'_1$ are of degree 3. By the claim, there is a vertex $c_2 \in X'_2 \setminus (A'_2 \cup B'_2)$ that is of degree 2 in G .

Since $|A_1| = |B_1| = |A'_2| = |B'_2| = 2$, we see that no branch of G may overlap the three following sets: $A_1 \cup B_1$, $X_1 \setminus (A_1 \cup B_1)$ and $A'_2 \cup B'_2$ (resp. $A'_2 \cup B'_2$, $X'_2 \setminus (A'_2 \cup B'_2)$ and $A_1 \cup B_1$). It follows that c_1 and c_2 are in non incident branches. \square

Proof of Theorem 5.1: We proceed by induction on $|V(G)|$. If $|V(G)| = 1$, then the theorem is vacuously true. By Theorem 2.7 and Lemma 5.2, we may assume that G has a star cutset or a 6-join.

Proof when G has a star cutset.

Let (x, R) be a star cutset of G such that $|R|$ is minimum. Since G is 2-connected, $|R| \geq 1$, and by the choice of (x, R) and since G is subcubic, every vertex of R has neighbors in every connected component of $G \setminus (\{x\} \cup R)$, every vertex of R is of degree 3 and $G \setminus (\{x\} \cup R)$ has exactly two connected components, say C_1 and C_2 . Let G_i be the block of decomposition w.r.t. this cutset that contains C_i , for $i = 1, 2$. Note that every vertex of R is of degree 2 in G_i . Note also that both G_1, G_2 are 2-connected.

Claim: *If x is of degree 2 in G_i , for some $i \in \{1, 2\}$, then C_i contains a vertex u of degree 2, or a pair of twins. Furthermore, if G_i has at least two branch vertices, then u can be chosen so that x and u are not in the same branch of G_i .*

Proof of Claim: If G_i has no branch vertices, then C_i contains a vertex of degree 2. If G_i has exactly two branch vertices, both are in C_i . Since these vertices can have at most one branch of length 1 connecting them, there must be a branch between them that is fully contained in C_i and is of length at least 2, and therefore there is a vertex of degree 2 in C_i that is not in the same branch as x . If G_i has at least 3 branch vertices, then, by the induction hypothesis, C_i contains a vertex of degree 2 that is not in the same branch as x , or C_i contains a pair of twins. This completes the proof of Claim.

We now consider the following cases.

Case 1: $|R| = 1$.

Note that since G is 2-connected, x has a neighbor in both C_1 and C_2 , and in particular, x is of degree 2 in both G_1 and G_2 . Since G has at least three

branch vertices, at least one of G_1 or G_2 has at least two branch vertices, so, by Claim applied for $i = 1$ and $i = 2$, G satisfies the theorem.

Case 2: $|R| = 2$.

Let $R = \{y_1, y_2\}$. Suppose that $\deg(x) = 2$. Then at least one of G_1 or G_2 has at least two branch vertices (since neither can have exactly one), w.l.o.g. say G_1 does. By Claim applied to G_1 , there is a degree 2 vertex u in C_1 that is not in the same branch of G_1 as x . Since y_1 and y_2 have degree 3 in G , x and u are degree 2 vertices of G that are contained in non incident branches of G , a contradiction. So $\deg(x) = 3$, and w.l.o.g. x has a neighbor in C_1 and does not in C_2 . If G_1 has exactly two branch vertices and they are adjacent, then for a shortest path P from y_1 to y_2 in $G_2 \setminus \{x\}$, the set $V(G_1) \cup V(P)$ induces an odd wheel with centre x , contradicting Theorem 2.1. So, if G_1 has exactly two branch vertices, then there is a vertex of degree 2 in G_1 in a branch that does not contain y_1 nor y_2 , and therefore, by Claim applied to G_2 , G satisfies the theorem, a contradiction. So G_1 must have at least three branch vertices, and hence by induction hypothesis, G_1 has a pair of twins or a vertex of degree 2 in a branch that has both of its ends in C_1 . But then by Claim applied to G_2 , G satisfies the theorem.

Case 3: $|R| = 3$.

Let $R = \{y_1, y_2, y_3\}$. First, let us suppose that both G_1 and G_2 have exactly two branch vertices, and that v_i is a branch vertex of G_i different from x , for $i = 1, 2$. If G_i , for $i = 1, 2$, does not have a vertex of degree 2 other than y_j , for $j = 1, 2, 3$, then G is a $K_{3,3}$, and hence it satisfies (iii) of the theorem. So, we may assume that there is a vertex of degree 2 (in G) in a branch of G_1 containing y_1 . If y_2v_2 or y_3v_2 is not an edge, then G satisfies (i) of the theorem, so we may assume that y_2v_2 and y_3v_2 are edges. If y_1v_2 is also an edge, then x and v_2 form a pair of twins, and therefore G satisfies (ii) of the theorem. When y_1v_2 is not an edge, then by symmetry v_1y_2 and v_1y_3 are edges. But then y_2 and y_3 form a pair of twins, and therefore G satisfies (ii) of the theorem.

Observe that if G_i has at least three branch vertices, then, by induction hypothesis, there is a vertex u_i of degree 2 in a branch of G_i not having x as its end, or G_i has a pair of twins that does not contain x (since G_i has at least three branch vertices). So if both G_1 and G_2 have at least three branch vertices, then the theorem holds. Therefore we may assume that G_1 has at least three and G_2 exactly two branch vertices. If G_2 has a vertex u_2 of degree 2 not in $\{y_1, y_2, y_3\}$, then G satisfies (i) or (ii) of the theorem. So we may assume that the only vertices of G_2 of degree 2 are y_1, y_2 and y_3 , and therefore x and the other branch vertex of G_2 form a pair of twins,

hence G satisfies (ii) or (iii). This completes the proof when G has a star cutset.

Proof when G has a 6-join.

We may assume that G has no star cutset. In particular, G does not contain a pair of twins (for if u, v is a pair of twins of G , since G has at least three branch vertices, $V(G) \setminus (N(u) \cup \{u, v\}) \neq \emptyset$, and hence $N(u) \cup \{u\}$ is a star cutset). Let $(X_1, X_2, A_1, A_2, A_3, A_4, A_5, A_6)$ be a split of a 6-join of G and let $A = \cup_{i=1}^6 A_i$. By Lemma 3.2 (iii), $|A_i| = 1$ for every $i \in \{1, \dots, 6\}$ and all nodes of A are of degree 3 in G . It follows that both blocks of decomposition G_1 and G_2 have at least three branch vertices. By the choice of G , each of them has a vertex of degree 2 not in A , and hence G satisfies (i) of the theorem. This completes the proof. \square

As a consequence of Theorem 5.1 we have the following corollary, a special case of which was conjectured in [13].

Corollary 5.3 *If G is a cubic balanceable graph that is not R_{10} , then G has a pair of twins none of whose neighbors is a cut vertex of G .*

PROOF — Let G' be an end block of G . Then G' has at most one vertex of degree 2, and all the other vertices of degree 3. If G' does not have a vertex of degree 2, then let $G'' = G'$, and otherwise let G'' be the graph obtained from G' by subdividing twice an edge incident to the degree 2 vertex. Clearly G'' is 2-connected balanceable and not equal to R_{10} . Note that G'' has at most one branch of length greater than 1. By Theorem 5.1 G'' has a pair of twins $\{u_1, u_2\}$. Note that none of the neighbors of u_1 and u_2 in G'' can be of degree 2 in G'' , and hence $\{u_1, u_2\}$ is the desired pair of twins of G . \square

As was noticed in [13] (for the special case of cubic balanced graphs), Corollary 5.3 implies the following.

Corollary 5.4 *Let G be a cubic balanceable graph. Then the following hold:*

- (i) G has girth four.
- (ii) If $G \neq R_{10}$ then G contains an edge that is not the unique chord of a cycle.
- (iii) G is not planar.

PROOF — It is easy to see that if $G = R_{10}$ then (i) and (iii) hold. So we may assume that $G \neq R_{10}$. By Corollary 5.3, let $\{u_1, u_2\}$ be a pair of twins

of G , and $\{v_1, v_2, v_3\}$ the set of neighbors of u_1 and u_2 . Then $u_1v_1u_2v_2$ is a cycle of length 4, and hence (i) holds. Suppose that u_1v_1 is a unique chord of a cycle C in G . Then all neighbors of u_1 and v_1 belong to C , and in particular, u_2 belongs to C and has three neighbors in C , a contradiction. Hence (ii) holds.

By Corollary 5.3 we may assume that none of v_1, v_2, v_3 is a cut vertex of G . So there is a connected component C of $G \setminus \{u_1, u_2, v_1, v_2, v_3\}$ such that all of v_1, v_2, v_3 have a neighbor in C . Let C' be a minimal induced subgraph of C that is connected and all of v_1, v_2, v_3 have a neighbor in C' . Since G is cubic, it is easy to see that $V(C') \cup \{u_1, u_2, v_1, v_2, v_3\}$ induces a subdivision of $K_{3,3}$. Therefore, by Kuratowski's Theorem (see for example [2]), G is not planar. \square

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