

The Erdős-Hajnal Conjecture—A Survey

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Abstract

The Erdős-Hajnal conjecture states that for every graph H , there exists a constant $\delta(H) > 0$ such that every graph G with no induced subgraph isomorphic to H has either a clique or a stable set of size at least $|V(G)|^{\delta(H)}$. This paper is a survey of some of the known results on this conjecture.

1 Introduction

All graphs in this paper are finite and simple. Let G be a graph. We denote by $V(G)$ the vertex set of G . The *complement* G^c of G is the graph with vertex set $V(G)$, such that two vertices are adjacent in G if and only if they are non-adjacent in G^c . A *clique* in G is a set of vertices all pairwise adjacent. A *stable set* in G is a set of vertices all pairwise non-adjacent (thus a stable set in G is a clique in G^c). Given a graph H , we say that G is *H -free* if G has no induced subgraph isomorphic to H . For a family \mathcal{F} of graphs, we say that G is \mathcal{F} -free if G is F -free for every $F \in \mathcal{F}$.

It is a well-known theorem of Erdős [13] that there exist graphs on n vertices, with no clique or stable set of size larger than $O(\log n)$. However, in 1989 Erdős and Hajnal [15] made a conjecture suggesting that the situation is dramatically different for graphs that are H -free for some fixed graph H , the following:

Conjecture 1.1 *For every graph H , there exists a constant $\delta(H) > 0$ such that every H -free graph G has either a clique or a stable set of size at least $|V(G)|^{\delta(H)}$.*

This is the *Erdős-Hajnal conjecture*. The same paper [15] also contains a partial result toward Conjecture 1.1, showing that H -free graphs behave very differently from general graphs:

Theorem 1.1 *For every graph H , there exists a constant $c(H) > 0$ such that every H -free graph G has either a clique or a stable set of size at least $e^{c(H)} \sqrt{\log |V(G)|}$.*

However, obtaining the polynomial bound of Conjecture 1.1 seems to be a lot harder, and Conjecture 1.1 is still open. The goal of this paper is to survey some recent results on Conjecture 1.1.

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We start with some definitions. We say that a graph H has the *Erdős-Hajnal property* if there exists a constant $\delta(H) > 0$ such that every H -free graph G has either a clique or a stable set of size at least $|V(G)|^{\delta(H)}$. Clearly, H has the Erdős-Hajnal property if and only if H^c does.

Let G be a graph. For $X \subseteq V(G)$, we denote by $G[X]$ the subgraph of G induced by X . We write $G \setminus X$ for $G[V(G) \setminus X]$, and $G \setminus v$ for $G \setminus \{v\}$, where $v \in V(G)$. We denote by $\omega(G)$ the maximum size of a clique in G , by $\alpha(G)$ the maximum size of a stable set in G , and by $\chi(G)$ the chromatic number of G . The graph G is *perfect* if $\chi(H) = \omega(H)$ for every induced subgraph H of G . The Strong Perfect Graph Theorem [8] characterizes perfect graphs by forbidden induced subgraphs:

Theorem 1.2 *A graph G is perfect if and only if no induced subgraph of G or G^c is an odd cycle of length at least five.*

Thus, in a perfect graph G

$$|V(G)| \leq \chi(G)\alpha(G) = \omega(G)\alpha(G),$$

and so

Theorem 1.3 *If G is perfect, then either $\omega(G) \geq \sqrt{|V(G)|}$, or $\alpha(G) \geq \sqrt{|V(G)|}$.*

This turns out to be a useful observation in the study of Conjecture 1.1; in fact it is often convenient to work with the following equivalent version of Conjecture 1.1:

Conjecture 1.2 *For every graph H , there exists a constant $\psi(H) > 0$, such that every H -free graph G has a perfect induced subgraph with at least $|V(G)|^{\psi(H)}$ vertices.*

The equivalent of Conjecture 1.1 and Conjecture 1.2 follows from Theorem 1.3. The main advantage of Conjecture 1.2 is that instead of having two outcomes: a large clique or a large stable set, it only has one, namely a large perfect induced subgraph, thus making inductive proofs easier.

This paper is organized as follows. In Section 2 we discuss graphs and families of graphs that are known to have the Erdős-Hajnal property. Section 3 deals with weakenings of Conjecture 1.1 that are known to be true for all graphs. In Section 4 we state an analogue of Conjecture 1.1 for tournaments, and discuss related results and techniques. Sections 5 and 6 deal with special cases of Conjecture 1.1 and its tournament analogue when we restrict our attention to graphs H with a certain value of $\delta(H)$.

2 Graphs with the Erdős-Hajnal property

Obviously, Conjecture 1.1 can be restated as follows:

Conjecture 2.1 *Every graph has the Erdős-Hajnal property.*

However, at the moment, only very few graphs have been shown to have the Erdős-Hajnal property. The goal of this section is to describe all such graphs. It is clear that graphs on at most two vertices have the property. Complete graphs and their complements have the property; this follows from the famous Ramsey theorem. If H is the two-edge path, then every H -free graph G is the

disjoint union of cliques, and thus has either a clique or a stable set of size $\sqrt{|V(G)|}$, so the two-edge path has the property. By taking complements, this shows that all three-vertex graphs have the property.

Another graph for which the Erdős-Hajnal property is easily established is the three-edge path. It follows immediately from Theorem 1.2 that all graphs with no induced subgraph isomorphic to the three-edge path are perfect; this fact can also be obtained by an easy induction from the following theorem of Seinsche [24]:

Theorem 2.1 *If G is a graph with at least two vertices, and no induced subgraph of G is isomorphic to the three-edge path, then either G or G^c is not connected.*

Let us now define the *substitution* operation. Given graphs H_1 and H_2 , on disjoint vertex sets, each with at least two vertices, and $v \in V(H_1)$, we say that H is *obtained from H_1 by substituting H_2 for v* , or *obtained from H_1 and H_2 by substitution* (when the details are not important) if:

- $V(H) = (V(H_1) \cup V(H_2)) \setminus \{v\}$,
- $H|_{V(H_2)} = H_2$,
- $H|(V(H_1) \setminus \{v\}) = H_1 \setminus v$, and
- $u \in V(H_1)$ is adjacent in H to $w \in V(H_2)$ if and only if u is adjacent in H_1 to v .

A graph is *prime* if it is not obtained from smaller graphs by substitution.

In [1] Alon, Pach, and Solymosi proved that the Erdős-Hajnal property is preserved under substitution:

Theorem 2.2 *If H_1 and H_2 are graphs with the Erdős-Hajnal property, and H is obtained from H_1 and H_2 by substitution, then H has the Erdős-Hajnal property.*

This is the only operation known today that allows us to build bigger graphs with the Erdős-Hajnal property from smaller ones. The idea of the proof of Theorem 2.2 is to notice the following: since both H_1 and H_2 have the Erdős-Hajnal property, it follows that if an H -free graph G does not contain a “large” clique or stable set, then every induced subgraph of G with at least $|V(G)|^\epsilon$ vertices (where ϵ depends on the precise definition of “large”) contains an induced copy of H_1 and an induced copy of H_2 . Let $v \in V(H_1)$ be such that H is obtained from H_1 by substituting H_2 for v . Then counting shows that some copy of $H_1 \setminus v$ in G can be extended to H_1 in at least n^ϵ ways. But this guarantees that there is a copy of H_2 among the possible extensions, contrary to the fact that G is H -free, and Theorem 2.2 follows.

Since the only prime graph on four vertices is the three-edge path, using Theorem 2.2 and the fact that the three-edge-path has the Erdős-Hajnal property, it is easy to check that all graphs on at most four vertices have the Erdős-Hajnal property. Moreover, there are only four prime graphs on five vertices:

- the cycle of length five
- the four-edge path
- the complement of the four-edge path

- the bull (the graph with vertex set $\{a_1, a_2, a_3, b_1, b_2\}$ and edge set $\{a_1a_2, a_2a_3, a_1a_3, a_1b_1, a_2b_2\}$),

and Theorem 2.2 implies that all the other graphs on five vertices have the Erdős-Hajnal property.

A much harder argument [9] shows that

Theorem 2.3 *The bull has the the Erdős-Hajnal property. Moreover, every bull-free graph G has a clique or a stable set of size at least $|V(G)|^{\frac{1}{4}}$.*

The exponent $\frac{1}{4}$ is in fact best possible because of the following construction. Take a triangle-free graph T with m vertices and no stable set of size larger than $\sqrt{m \log m}$ (such graphs exist by an old theorem of Kim [22]). Let G be obtained from T by substituting a copy of T^c for every vertex of T . Then $|V(G)| = m^2$, and it is easy to check that G has no clique or stable set of size larger than $2\sqrt{m \log m}$.

The proof of Theorem 2.3 uses structural methods to show that every prime bull-free graph belongs to a certain subclass where a clique or a stable set of the appropriate size can be shown to exist. For graphs that are not prime, the result follows by induction.

Let us say that a function $f : V(G) \rightarrow [0, 1]$ is *good* if for every perfect induced subgraph P of G

$$\sum_{v \in V(P)} f(v) \leq 1.$$

For $\alpha \geq 1$, the graph G is α -*narrow* if for every good function f

$$\sum_{v \in V(G)} f(v)^\alpha \leq 1.$$

Thus perfect graphs are 1-narrow. Let G be an α -narrow graph for some $\alpha \geq 1$, and let $K = \max |V(P)|$ where the maximum is taken over all perfect induced subgraphs of G . Then the function $f(v) = \frac{1}{K}$ (for all $v \in V(G)$) is good, and so, since G is α -narrow, $\frac{|V(G)|}{K^\alpha} \leq 1$. Thus $K \geq |V(G)|^{\frac{1}{\alpha}}$. By Theorem 1.3, this implies that in order to prove that a certain graph H has the Erdős-Hajnal property, it is enough to show that there exists $\alpha \geq 1$ such that all H -free graphs are α -narrow. This conjecture was formally stated in [12]:

Conjecture 2.2 *For every graph H , there exists a constant $\alpha(H) \geq 1$ such that every H -free graph G is $\alpha(H)$ -narrow.*

Fox [18] (see [11] for details) proved that Conjecture 2.2 is in fact equivalent to Conjecture 1.1. More specifically, he showed that for every graph H with the Erdős-Hajnal property, there exists a constant $\alpha(H)$ such that every H -free graph is $\alpha(H)$ -narrow. However, at least for the purely structural approach, Conjecture 2.2 seems to be more convenient to work with than 1.1. In fact, what is really proved in [9] is the following:

Theorem 2.4 *Every bull-free graph is 2-narrow.*

And the inductive step proving Theorem 2.4 for bull-free graphs that are not prime is that α -narrowness is preserved under substitutions:

Theorem 2.5 *If H_1 and H_2 are α -narrow for $\alpha \geq 1$, and H is obtained from H_1 and H_2 by substitution, then H is also α -narrow.*

Conjecture 1.1 is still open for the four-edge path, its complement, and the five-cycle; and no prime graph on at least six vertices is known to have the Erdős-Hajnal property.

We remark that the bull is a self-complementary graph, and one might think that to be the reason for its better behavior. This philosophy is supported by the following result:

Theorem 2.6 *Every graph with no induced subgraph isomorphic to the four-edge-path or the complement of the four-edge-path is 2-narrow.*

This follows from Theorem 2.5 and from (a restatement of) a theorem of Fouquet [17]:

Theorem 2.7 *Every prime graph with no induced subgraph isomorphic to the four-edge-path or the complement of the four-edge-path is either perfect or isomorphic to the five-cycle.*

On the other hand, the five-cycle is another self-complementary graph, and yet it seems to be completely intractable. We thus propose the following conjecture that may be slightly easier than the full Conjecture 1.1 in special cases:

Conjecture 2.3 *For every graph H , there exists a constant $\epsilon(H) > 0$, such that every $\{H, H^c\}$ -free graph G has either a clique or a stable set of size at least $|V(G)|^{\epsilon(H)}$.*

In [11] Conjecture 2.3 was proved in the case when H is the five-edge path. This is a strengthening of a result of [12].

3 Approximate results

The previous section listed a few graphs for which Conjecture 2.1 is known to hold. The goal of this section is to list facts that are true for all graphs, but that do not achieve the full strength conjectured in Conjecture 2.1. The first such statement is Theorem 1.1 which we already mentioned in the Introduction.

For two disjoint subsets A and B of $V(G)$, we say that A is *complete* to B if every vertex of A is adjacent to every vertex of B , and that A is *anticomplete* to B if every vertex of A is non-adjacent to every vertex of B . If $A = \{a\}$ for some $a \in V(G)$, we write “ a is complete (anticomplete) to B ” instead of “ $\{a\}$ is complete (anticomplete) to B ”. Here is another theorem similar to Theorem 1.1, due to Erdős, Hajnal and Pach [16].

Theorem 3.1 *For every graph H , there exists a constant $\delta(H) > 0$ such that for every H -free graph G there exist two disjoint subsets $A, B \subseteq V(G)$ with the following properties:*

1. $|A|, |B| \geq |V(G)|^{\delta(H)}$, and
2. *either A is complete to B , or A is anticomplete to B .*

The idea of the proof here is to partition $V(G)$ into $|V(H)|$ equal subsets (which we call “sets of candidates”), and then try to build an induced copy of H in G , one vertex at a time, where each vertex of H is chosen from the corresponding set of candidates. In this process, sets of candidates shrink at every step, and since G is H -free, eventually we reach a situation where there do not exist enough vertices in one of the sets of candidates with the right adjacencies to another. At this stage we obtain the sets A and B , as required in Theorem 3.1.

Theorem 3.1 was recently strengthened by Fox and Sudakov in [19]:

Theorem 3.2 *For every graph H , there exists a constant $\delta(H) > 0$ such that for every H -free graph G with $\omega(G) < |V(G)|^{\delta(H)}$ there exist two disjoint subsets $A, B \subseteq V(G)$, with the following properties:*

1. $|A|, |B| \geq |V(G)|^{\delta(H)}$, and
2. A is anticomplete to B .

In [23] another weakening of Conjecture 2.1 is considered. It is shown that for every H , the proportion of H -free graphs with n vertices and no “large” cliques or stable sets tends to zero as $n \rightarrow \infty$. Let \mathcal{F}_H^n be the class of all H -free graphs on n vertices. Let $\mathcal{Q}_H^{n,\epsilon}$ be the subclass of \mathcal{F}_H^n , consisting of all graphs G that have either a clique or a stable set of size at least n^ϵ . The main result of [23] is the following:

Theorem 3.3 *For every graph H , there exists a constant $\epsilon(H) > 0$ such that $\frac{|\mathcal{Q}_H^{n,\epsilon(H)}|}{|\mathcal{F}_H^n|} \rightarrow 1$ as $n \rightarrow \infty$.*

The proof of Theorem 3.3 involves an application of Szemerédi’s Regularity Lemma [26]. Also, somewhat surprisingly, it uses Theorem 2.3.

4 Tournaments

A *tournament* is a directed graph, where for every two vertices u, v exactly one of the (ordered) pairs uv and vu is an edge. A tournament is *transitive* if it has no directed cycles (or, equivalently, no cyclic triangles). For a tournament T , we denote by $\alpha(T)$ the maximum number of vertices of a transitive subtournament of T . Transitive subtournaments seem to be a good analogue of both cliques and stable sets in graphs; furthermore, like induced perfect subgraphs in Conjecture 1.2, transitive tournaments have the advantage of being one object instead of two.

For tournaments S and T , we say that T is *S -free* if no subtournament of T is isomorphic to S . As with graphs, if \mathcal{S} is a family of tournaments, then T is *\mathcal{S} -free* if T is S -free for every $S \in \mathcal{S}$. In [1] the following conjecture was formulated, and shown to be equivalent to Conjecture 1.1:

Conjecture 4.1 *For every tournament S , there exists a constant $\delta(S) > 0$ such that every S -free tournament T satisfies $\alpha(T) \geq |V(T)|^{\delta(S)}$.*

As with graphs, let us say that a tournament S has the *Erdős-Hajnal property* if there exists $\delta(S) > 0$ such that every S -free tournament T satisfies $\alpha(T) \geq |V(T)|^{\delta(S)}$. We remark that, like in graphs, the maximum number of vertices of a transitive subtournament in a random n -vertex tournament is $O(\log n)$ [13].

A *substitution* operation can be defined for tournaments as follows. Given tournaments S_1 and S_2 , with disjoint vertex sets and each with at least two vertices, and a vertex $v \in V(S_1)$, we say that S is *obtained from S_1 by substituting S_2 for v* (or just *obtained by substitution from S_1 and S_2*) if $V(S) = (V(S_1) \cup V(S_2)) \setminus \{v\}$ and uw is an edge of S if and only if one of the following holds:

- $u, w \in V(S_1)$ and uw is an edge of S_1
- $u, w \in V(S_2)$ and uw is an edge of S_2

- $u \in S_1, w \in S_2$ and uv is an edge of S_1
- $u \in S_2, w \in S_1$, and vw is an edge of S_1 .

A tournament is *prime* if it is not obtained by substitution from smaller tournaments. Repeating the proof of Theorem 2.2 in the setting of tournaments instead of graphs, it is easy to show that

Theorem 4.1 *If S_1 and S_2 are tournaments with the Erdős-Hajnal property, and S is obtained from S_1 and S_2 by substitution, then S has the Erdős-Hajnal property.*

Clearly, all tournaments on at most three vertices have the Erdős-Hajnal property, and it is easy to check that there are no prime four-vertex tournaments. Consequently, all four-vertex tournaments have the Erdős-Hajnal property. So far this is very similar to the state of affairs in graphs, but here is a fact to which we do not have a graph analogue: we can define an infinite family of prime tournaments all with the Erdős-Hajnal property (recall that the largest prime graph known to have the property is the bull). Let us describe this family.

First we need some definitions. Let T be a tournament, and let $(v_1, \dots, v_{|T|})$ be an ordering of its vertices; denote it by θ . We say that an edge $v_j v_i$ of T is a *backward edge* under θ if $i < j$. The *graph of backward edges* under θ , denoted by $B(T, \theta)$, has vertex set $V(T)$, and $v_i v_j \in E(B(T, \theta))$ if and only if $v_i v_j$ or $v_j v_i$ is a backward edge of T under the ordering θ . For an integer $t > 0$, we call the graph $K_{1,t}$ a *star*. Let S be a star with vertex set $\{c, l_1, \dots, l_t\}$, where c is adjacent to l_1, \dots, l_t and $\{l_1, \dots, l_t\}$ is a stable set. We call c the *center of the star*, and l_1, \dots, l_t the *leaves of the star*. A *right star* in $B(T, \theta)$ is an induced subgraph with vertex set $\{v_{i_0}, \dots, v_{i_t}\}$, such that $B(T, \theta)|\{v_{i_0}, \dots, v_{i_t}\}$ is a star with center v_{i_t} , and $i_t > i_0, \dots, i_{t-1}$. A *left star* in $B(T, \theta)$ is an induced subgraph with vertex set $\{v_{i_0}, \dots, v_{i_t}\}$, such that $B(T, \theta)|\{v_{i_0}, \dots, v_{i_t}\}$ is a star with center v_{i_0} , and $i_0 < i_1, \dots, i_t$. Finally, a *star* in $B(T, \theta)$, is a left star or a right star. A tournament T is a *galaxy* if there exists an ordering θ of its vertices such that every component of $B(T, \theta)$ is a star or a singleton, and

- no center of a star appears in θ between two leaves of another star.

In [3] the following is proved:

Theorem 4.2 *Every galaxy has the Erdős-Hajnal property.*

The proof uses the directed version of Szemerédi's Regularity Lemma formulated in [2], and extensions of ideas from the proof of Theorem 3.1. Instead of starting with arbitrary sets of candidates, the way it is done in Theorem 3.1, we get a head start by using sets given by a regular partition. Let us describe the proof in a little more detail.

Let T be a tournament. For disjoint subsets A, B of $V(T)$, we say that A is *complete to* B if every vertex of A is adjacent to every vertex of B . We say that A is *complete from* B if B is complete to A . Denote by $e_{A,B}$ the number of directed edges ab , where $a \in A$ and $b \in B$. We define the *directed density from A to B* to be $d(A, B) = \frac{e_{A,B}}{|A||B|}$.

Given $\epsilon > 0$ we call a pair (X, Y) of disjoint subsets of $V(T)$ ϵ -*regular* if all $A \subseteq X$ and $B \subseteq Y$ with $|A| \geq \epsilon|X|$ and $|B| \geq \epsilon|Y|$ satisfy: $|d(A, B) - d(X, Y)| \leq \epsilon$ and $|d(B, A) - d(Y, X)| \leq \epsilon$.

Consider a partition $\{V_0, V_1, \dots, V_k\}$ of $V(T)$ in which one set V_0 has been singled out as an *exceptional set*. (This exceptional set V_0 may be empty). Such a partition is called an ϵ -*regular partition* of T if it satisfies the following three conditions:

- $|V_0| \leq \epsilon|V|$
- $|V_1| = \dots = |V_k|$
- all but at most ϵk^2 of the pairs (V_i, V_j) with $1 \leq i < j \leq k$ are ϵ -regular.

The following was proved in [2]:

Theorem 4.3 *For every $\epsilon > 0$ and every $m \geq 1$ there exists an integer $DM = DM(m, \epsilon)$ such that every tournament of order at least m admits an ϵ -regular partition $\{V_0, V_1, \dots, V_k\}$ with $m \leq k \leq DM$.*

The following is also a result from [2]; it is the directed analogue of a well-known lemma for undirected graphs [5].

Theorem 4.4 *Let $k \geq 1$ be an integer, and let $0 < \lambda < 1$. Then there exists a constant η_0 (depending on k and λ) with the following property. Let H be a tournament with vertex set $\{x_1, \dots, x_k\}$, and let T be a tournament with vertex set $V(T) = \bigcup_{i=1}^k V_i$, where the V_i 's are disjoint sets, each of order at least one. Suppose that each pair (V_i, V_j) , $1 \leq i < j \leq k$ is η -regular, that $d(V_i, V_j) \geq \lambda$ and $d(V_j, V_i) \geq \lambda$. Then there exist vertices $v_i \in V_i$ for $i \in \{1, \dots, k\}$, such that the map $x_i \rightarrow v_i$ gives an isomorphism between H and the subtournament of T induced by $\{v_1, \dots, v_k\}$ provided that $\eta \leq \eta_0$.*

Now, given a galaxy G , we start with a regular partition of a G -free tournament given by Theorem 4.3. Using Theorem 4.4 along with a few standard techniques which we will not describe here, we can find subsets V_{i_1}, \dots, V_{i_t} (for an appropriately chosen constant t), such that $d(V_{i_p}, V_{i_q}) > .999$ for every $1 \leq p < q \leq t$. This means that for every $1 \leq p < q \leq t$, vertices of V_{i_p} tend to be adjacent to a substantial proportion of the vertices of V_{i_q} . On the other hand, if a substantial subset of V_{i_p} is complete to a substantial subset of V_{i_q} , then we can apply induction to get a large transitive subtournament in T , and so we may assume that no such subsets exist. We now construct a copy of G in T , choosing at most one vertex from each of V_{i_1}, \dots, V_{i_t} , and using the fact that for $1 \leq p < q \leq t$ no substantial subset of V_{i_p} is complete to a substantial subset of V_{i_q} to obtain the backward edges in the galaxy ordering of G , thus obtaining the result of Theorem 4.2.

Obviously, every tournament obtained from a transitive tournament by adding a vertex is a galaxy. It is not difficult to check that there is only one tournament on five vertices that is not a galaxy. Here it is: its vertex set is $\{v_1, \dots, v_5\}$, and $v_i v_j$ is an edge if and only if $(j - i) \bmod 5 \in \{1, 2\}$. We call this tournament S_5 . We remark that S_5 is an example of a tournament that is obtained from a transitive tournament by adding two vertices, and that is not a galaxy.

Another result of [3] is that:

Theorem 4.5 *The tournament S_5 has the Erdős-Hajnal property.*

The proof of Theorem 4.5 uses similar ideas to the ones in the proof of Theorem 4.2. Theorem 4.2 and Theorem 4.5 together imply:

Theorem 4.6 *Every tournament on at most five vertices has the Erdős-Hajnal property.*

We finish this section with another curious corollary of Theorem 4.2. Let P_k denote a tournament of order k whose vertices can be ordered so that the graph of backward edges is a k -vertex path.

Theorem 4.7 *For every k , the tournament P_k has the Erdős-Hajnal conjecture.*

Theorem 4.7 follows from the fact that, somewhat surprisingly, P_k has a galaxy ordering.

5 Linear-size cliques, stable sets and transitive subtournaments

In Conjecture 1.1 and Conjecture 4.1, every graph (or tournament) is conjectured to have a certain constant, lying in the $(0, 1]$ interval, associated with it. A natural question is: when is this constant at its extreme? Excluding which graphs (or tournaments) guarantees a linear-size clique or stable set (or transitive subtournament)?

For undirected graphs this question turns out not to be interesting, because if for some graph H every H -free graph were to contain a linear size clique or stable set, then H would need to have at most two vertices (we explain this later). However, for tournaments the answer is quite pretty. We say that a tournament S is a *celebrity* if there exists a constant $0 < c(S) \leq 1$ such that every S -free tournament T contains a transitive subtournament on at least $c(S)|V(T)|$ vertices. So the question is to describe all celebrities. This was done in [4], but before stating the result we need some definitions.

For a tournament T and $X \subseteq V(T)$, we denote by $T|X$ the subtournament of T induced by X . Let T_k denote the transitive tournament on k vertices. If T is a tournament and X, Y are disjoint subsets of $V(T)$ such that X is complete to Y , we write $X \Rightarrow Y$. We write $v \Rightarrow Y$ for $\{v\} \Rightarrow Y$, and $X \Rightarrow v$ for $X \Rightarrow \{v\}$. If T is a tournament and (X, Y, Z) is a partition of $V(T)$ into nonempty sets satisfying $X \Rightarrow Y$, $Y \Rightarrow Z$, and $Z \Rightarrow X$, we call (X, Y, Z) a *trisection* of T . If A, B, C, T are tournaments, and there is a trisection (X, Y, Z) of T such that $T|X, T|Y, T|Z$ are isomorphic to A, B, C respectively, we write $T = \Delta(A, B, C)$. A *strongly connected component* of a tournament is a maximal subtournament that is strongly connected. One of the main results of [4] is the following:

Theorem 5.1 *A tournament is a celebrity if and only if all its strongly connected components are celebrities. A strongly connected tournament with more than one vertex is a celebrity if and only if it equals $\Delta(S, T_k, T_1)$ or $\Delta(S, T_1, T_k)$ for some celebrity S and some integer $k \geq 1$.*

Following the analogy between stable sets in graphs and transitive subtournaments in tournaments, let us define the *chromatic number* of a tournament T to be the smallest integer k , for which $V(T)$ can be covered by k transitive subtournaments of T . We denote the chromatic number of T by $\chi(T)$. Here is a related concept: let us say that a tournament S is a *hero* if there exists a constant $d(S) > 0$ such that every S -free tournament T satisfies $\chi(T) \leq d(S)$. Clearly every hero is a celebrity. Moreover, the following turns out to be true (see [4])

Theorem 5.2 *A tournament is a hero if and only if it is a celebrity.*

Another result of [4] is a complete list of all minimal non-heroes (there are five such tournaments).

Let us now get back to undirected graphs. What if instead of asking for excluding a single graph H to guarantee a linear size clique or stable set, we ask the same question for a family of graphs? Let us say that a family \mathcal{H} of graphs is *celebrated* if there exists a constant $0 < c(\mathcal{H}) \leq 1$ such that every \mathcal{H} -free graph G contains either a clique or a stable set of size at least $c(\mathcal{H})|V(G)|$. The *cochromatic number* of a graph G is the minimum number of stable sets and cliques with union $V(G)$. We denote the cochromatic number of G by $\text{cox}(G)$. Let us say that a family \mathcal{H} is *heroic* if there exists a constant $d(\mathcal{H}) > 0$ such that $\text{cox}(G) < d(\mathcal{H})$ for every \mathcal{H} -free graph G . Clearly, if \mathcal{H} is heroic, then it is celebrated. Heroic families were studied in [10].

Let G be a complete multipartite graph with m parts, each of size m . Then G has m^2 vertices, and no clique or stable set of size larger than m ; and the same is true for G^c . Thus every celebrated

family contains a complete multipartite graph and the complement of one. Recall that for every positive integer g there exist graphs with girth at least g and no linear-size stable set (this is a theorem of Erdős [14]). Consequently, every celebrated family must also contain a graph of girth at least g , and, by taking complements, a graph whose complement has girth at least g . Thus, for a finite family of graphs to be celebrated, it must contain a forest and the complement of one. In particular, if a celebrated family only contains one graph H , then $|V(H)| \leq 2$. The following conjecture is proposed in [10], stating that these necessary conditions for a finite family of graphs to be celebrated are in fact sufficient for being heroic.

Conjecture 5.1 *A finite family of graphs is heroic if and only if it contains a complete multipartite graph, the complement of a complete multipartite graph, a forest, and the complement of a forest.*

We remark that this is an extension of a well-known conjecture made independently by Gyárfás [20] and Sumner [25], that can be restated as follows in the language of heroic families:

Conjecture 5.2 *For every complete graph K and every tree T , the family $\{K, T\}$ is heroic.*

The main result of [10] is that Conjecture 5.1 and Conjecture 5.2 are in fact equivalent. Since a complete graph is a multipartite graph, the complement of one, and the complement of a forest, we deduce that Conjecture 5.1 implies Conjecture 5.2. The converse is a consequence of the following theorem of [10]:

Theorem 5.3 *Let K and J be graphs, such that both K and J^c are complete multipartite. Then there exists a constant $c(K, J)$ such that for every $\{K, J\}$ -free graph G , $V(G) = X \cup Y$, where*

- $\omega(X) \leq c(K, J)$, and
- $\alpha(Y) \leq c(K, J)$.

The situation for infinite heroic families is more complicated. Another open conjecture of Gyárfás [21] can be restated to say that a certain infinite family of graphs is heroic:

Conjecture 5.3 *For every complete graph K , and every integer $t > 0$, the family consisting of K and all cycles of length at least t is heroic.*

If Conjecture 5.3 is true, this is an example of a heroic set that does not include a minimal heroic set.

6 Near-linear transitive subtournaments

In this section we discuss an extension of the property of being a hero studied in [7]. Let us say that $\epsilon \geq 0$ is an *EH-coefficient* for a tournament S if there exists $c > 0$ such that every S -free tournament T satisfies $\alpha(T) \geq c|V(T)|^\epsilon$. (We introduce c in the definition of the Erdős-Hajnal coefficient to eliminate the effect of tournaments T with bounded number of vertices; now, whether ϵ is an EH-coefficient for S depends only on arbitrarily large S -free tournaments.) Thus, Conjecture 1.1 is equivalent to:

Conjecture 6.1 *Every tournament has a positive EH-coefficient.*

If ϵ is an EH-coefficient for S , then so is every smaller non-negative number; and thus a natural invariant is the supremum of the set of all EH-coefficients for S . We call this the *EH-supremum* for S , and denote it by $\xi(S)$. We remark that the EH-supremum for S is *not* necessarily itself an EH-coefficient for S (we will see an example later). One of the results of [7] is a characterization of all tournaments with EH-supremum 1; and not all of these tournaments turn out to be celebrities (in this language, a celebrity is a tournament for which 1 is its EH-coefficient, and not just its EH-supremum).

The following theorem from [6] suggests that EH-suprema tend to be quite small:

Theorem 6.1 *Let $\mathcal{H}^{n,c}$ be the set of all n -vertex tournaments having EH-supremum at most $\frac{c}{n}$, where c is an arbitrary constant such that $c > 4$, and let \mathcal{H}^n be the set of all n -vertex tournaments. Then*

$$\frac{|\mathcal{H}^{n,c}|}{|\mathcal{H}^n|} \rightarrow 1$$

as $n \rightarrow \infty$.

We say that a tournament S is

- a *pseudo-hero* if there exist constants $c(S), d(S) \geq 0$ such that every S -free tournament T with $|V(T)| > 1$ satisfies $\chi(T) \leq c(S)(\log(|V(T)|))^{d(S)}$; and
- a *pseudo-celebrity* if there exist constants $c(S) > 0$ and $d(S) \geq 0$ such that every S -free tournament T with $|V(T)| > 1$ satisfies $\alpha(T) \geq c(S) \frac{|V(T)|}{(\log(|V(T)|))^{d(S)}}$.

In [7] all pseudo-celebrities and pseudo-heroes are described explicitly.

Theorem 6.2 *The following statements hold:*

- *A tournament is a pseudo-hero if and only if it is a pseudo-celebrity.*
- *A tournament is a pseudo-hero if and only if all its strongly connected components are pseudo-heroes.*
- *A strongly-connected tournament with more than one vertex is a pseudo-hero if and only if either*
 - *it equals $\Delta(T_2, T_k, T_l)$ for some $k, l \geq 2$, or*
 - *it equals $\Delta(S, T_1, T_k)$ or $\Delta(S, T_k, T_1)$ for some pseudo-hero S and some integer $k > 0$.*

We remind the reader that by Theorem 5.1 the tournament $\Delta(T_2, T_2, T_2)$ is not a celebrity, and yet by Theorem 6.2 it is a pseudo-celebrity. Thus it is an example of a tournament that does not attain its EH-supremum as an EH-coefficient.

We conclude this section with another result from [7], that shows that after 1, there is a gap in the set of EH-suprema.

Theorem 6.3 *Every tournament S with $\xi(S) > 5/6$ is a pseudo-hero and hence satisfies $\xi(S) = 1$.*

The reason for Theorem 6.3 is another theorem from [7] that states that a tournament is a pseudo-hero if and only if it is \mathcal{S} -free for a certain family \mathcal{S} consisting of six tournaments (S_5 is one of them). Thus for any tournament T that is not a pseudo-hero, $\xi(T) \leq \max_{S \in \mathcal{S}} \xi(S)$, and it is shown that $\max_{S \in \mathcal{S}} \xi(S) \leq \frac{5}{6}$.

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