# Outerplanar and planar oriented cliques 

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#### Abstract

The clique number of an undirected graph $G$ is the maximum order of a complete subgraph of $G$ and is a well-known lower bound for the chromatic number of $G$. Every proper $k$-coloring of $G$ may be viewed as a homomorphism (an edge-preserving vertex mapping) of $G$ to the complete graph of order $k$. By considering homomorphisms of oriented graphs (digraphs without cycles of length at most 2), we get a natural notion of (oriented) colorings and oriented chromatic number of oriented graphs. An oriented clique is then an oriented graph whose number of vertices and oriented chromatic number coincide. However, the structure of oriented cliques is much less understood than in the undirected case.

In this paper, we study the structure of outerplanar and planar oriented cliques. We first provide a list of 11 graphs and prove that an outerplanar graph can be oriented as an oriented clique if and only if it contains one of these graphs as a spanning subgraph. Klostermeyer and MacGillivray conjectured that the order of a planar oriented clique is at most 15 , which was later proved by Sen. We show that any planar oriented clique on 15 vertices must contain a particular oriented graph as a spanning subgraph, thus reproving the above conjecture. We also provide tight upper bounds for the order of planar oriented cliques of girth $k$ for all $k \geq 4$.


## 1 Introduction and statement of results

An oriented graph is a digraph with no cycle of length 1 or 2 . By replacing each edge of a simple graph $G$ with an arc (ordered pair of vertices) we obtain an oriented graph $\vec{G}$; we say that $\vec{G}$ is an orientation of $G$ and that $G$ is the underlying graph of $\vec{G}$. We denote by $V(\vec{G})$ and $A(\vec{G})$ the set of vertices and arcs of $\vec{G}$, respectively. An $\operatorname{arc}(u, v)$ (where $u$ and $v$ are vertices) is denoted by $\overrightarrow{u v}$. Two arcs $\overrightarrow{u v}$ and $\overrightarrow{v w}$ of an oriented graph are together called a directed 2-path, or a 2-dipath, where $u$ and $w$ are terminal vertices and $v$ is an internal vertex.

Colorings of oriented graphs first appeared in the work of Courcelle [3] on the monadic second order logic of graphs. Since then it has been considered by many researchers, following the work of Raspaud and Sopena [9] on oriented colorings of planar graphs.

An oriented $k$-coloring [12] of an oriented graph $\vec{G}$ is a mapping $\phi$ from $V(\vec{G})$ to the set $\{1,2, \ldots, k\}$ such that:
(i) $\phi(u) \neq \phi(v)$ whenever $u$ and $v$ are adjacent, and
(ii) if $\overrightarrow{u v}$ and $\overrightarrow{w x}$ are two arcs in $\vec{G}$, then $\phi(u)=\phi(x)$ implies $\phi(v) \neq \phi(w)$.

We say that an oriented graph $\vec{G}$ is $k$-colorable whenever it admits an oriented $k$-coloring. The oriented chromatic number $\chi_{o}(\vec{G})$ of an oriented graph $\vec{G}$ is the smallest integer $k$ such that $\vec{G}$ is $k$-colorable.

Alternatively, one can define the oriented chromatic number by means of homomorphisms of oriented graphs. Let $\vec{G}$ and $\vec{H}$ be two oriented graphs. A homomorphism of $\vec{G}$ to $\vec{H}$ is a mapping $\phi: V(\vec{G}) \rightarrow V(\vec{H})$ which preserves the arcs, that is, $u v \in A(\vec{G})$ implies $\phi(u) \phi(v) \in A(\vec{H})$. The oriented chromatic number $\chi_{o}(\vec{G})$ of an oriented graph $\vec{G}$ is then the minimum order (number of vertices) of an oriented graph $\vec{H}$ such that $\vec{G}$ admits a homomorphism to $\vec{H}$.

Notice that the terminal vertices of a 2-dipath must receive distinct colors in every oriented coloring because of the second condition of the definition. In fact, for providing an oriented coloring of an oriented graph, only the pairs of vertices which are either adjacent or connected by a 2-dipath must receive distinct colors (that is, for every two non-adjacent vertices $u$ and $v$ which are not linked by a 2-dipath, there exists an oriented coloring which assigns the same color to $u$ and $v$ ). Motivated by this observation, the following definition was proposed.

An absolute oriented clique, or simply an oclique - a term coined by Klostermeyer and MacGillivray in [6] , is an oriented graph $\vec{G}$ for which $\chi_{o}(\vec{G})=|V(\vec{G})|$. Note that ocliques can hence be characterized as those oriented graphs whose any two distinct vertices are at (weak) directed distance at most 2 from each other, that is, either adjacent or connected by a 2-dipath in either direction. Note that an oriented graph with an oclique of order $n$ as a subgraph has oriented chromatic number at least $n$. The absolute oriented clique number $\omega_{a o}(\vec{G})$ of an oriented graph $\vec{G}$ is the maximum order of an oclique contained in $\vec{G}$ as a subgraph.

The oriented chromatic number $\chi_{o}(G)$ (resp. absolute oriented clique number $\omega_{a o}(G)$ ) of a simple graph $G$ is the maximum of the oriented chromatic numbers (resp. absolute oriented clique numbers) of all the oriented graphs with underlying graph $G$. The oriented chromatic number $\chi_{o}(\mathcal{F})$ (resp. absolute oriented clique number $\omega_{a o}(\mathcal{F})$ ) of a family $\mathcal{F}$ of graphs is the maximum of the oriented chromatic numbers (resp. absolute oriented clique numbers) of the graphs from the family $\mathcal{F}$.

From the definitions, clearly we have the following:
Lemma 1.1. For any oriented graph $\vec{G}, \omega_{a o}(\vec{G}) \leq \chi_{o}(\vec{G})$.
One of the first major results proved regarding the oriented chromatic number of planar graphs is the following by Raspaud and Sopena [9].

Theorem 1.2 (Raspaud and Sopena, 1994). Every planar graph has oriented chromatic number at most 80 .

In the same paper, they also proved that every oriented forest is 3 -colorable.
Theorem 1.3 (Raspaud and Sopena, 1994). Every forest has oriented chromatic number at most 3 .

Later, Sopena [11] proved that every oriented outerplanar graph is 7-colorable and provided an example of an outerplanar oclique of order 7 (Figure 1) to prove the tightness of the result.

Theorem 1.4 (Sopena, 1997). Every outerplanar graph has oriented chromatic number at most 7 .

The structure of ocliques is much less understood than in the undirected case, where a clique is nothing but a complete graph. For instance, the exact value of the minimum number of arcs


Figure 1: The outerplanar oclique $\vec{O}$ of order 7.
in an oclique of order $k$ is not known yet. Füredi, Horak, Pareek and Zhu [4, and Kostochka, Łuczak, Simonyi and Sopena [7], independently proved that this number is $(1+o(1)) k \log _{2} k$.

The questions related to the absolute oriented clique number of planar graphs have been first asked by Klostermeyer and MacGillivray [6 in 2002. In their paper they asked: "what is the maximum order of a planar oclique?", which is equivalent to asking "what is the absolute oriented clique number of planar graphs?". In order to find the answer to this question, Sopena [13] found a planar oclique of order 15 (Figure (3) while Klostermeyer and MacGillivray [6] showed that there is no planar oclique of order more than 36 , improving the upper bound of 80 which can be obtained by using Lemma 1.1 and Theorem 1.2, and conjectured that the maximum order of a planar oclique is 15 . Later in 2011, the conjecture was positively settled [10] and we will state it as Theorem 1.8(a) in this article.

Klostermeyer and MacGillivray also showed that any outerplanar oclique of order 7 must contain a particular unique spanning subgraph (oriented).

Theorem 1.5 (Klostermeyer and MacGillivray, 2002). An oriented outerplanar graph of order at least 7 is an oclique if and only if it contains the outerplanar oclique $\vec{O}$ depicted in Figure $\mathbb{1}$ as a spanning subgraph.

Bensmail, Duvignau and Kirgizov [1 showed that given an undirected graph $G$ it is NP-hard to decide if $G$ has an orientation $\vec{G}$ such that $\vec{G}$ is an oclique (the similar problem, but using the directed distance instead of the weak directed distance, was shown to be also NP-complete by Chvátal and Thomassen in [2]). Now it is easy to notice from Theorem 1.5 that an undirected outerplanar graph of order at least 7 can be oriented as an oclique if and only if it contains the graph $O$ (the underlying undirected graph of the oriented graph $\vec{O}$ from Figure (1) as a spanning subgraph. We extend this idea to characterize every outerplanar graph that can be oriented as an oclique in the following result.

Theorem 1.6. An undirected outerplanar graph can be oriented as an oclique if and only if it contains one of the graphs depicted in Figure as a spanning subgraph.

We also prove a result similar to Theorem 1.5) for planar graphs which implies Theorem 1.8(a) (that is, the absolute oriented clique number of the family of planar graphs is 15).

Theorem 1.7. A planar oclique has order at most 15 and every planar oclique of order 15 contains the planar oclique $\vec{P}$ depicted in Figure 扄 as a spanning subgraph.

The question regarding the upper bound for the absolute oriented clique number of the families of planar graphs with given girth (length of the smallest cycle in a graph) is also of interest and was asked by Klostermeyer and MacGillivray in [6]. We answer these questions and provide tight bounds.

Let $\mathcal{P}_{k}$ denote the family of planar graphs with girth at least $k$. We will prove the following.


Figure 2: List of edge-minimal oclique spanning subgraphs of all outerplanar ocliques

## Theorem 1.8.

(a) $\omega_{a o}\left(\mathcal{P}_{3}\right)=15$.
(b) $\omega_{a o}\left(\mathcal{P}_{4}\right)=6$.
(c) $\omega_{a o}\left(\mathcal{P}_{5}\right)=5$.
(d) $\omega_{a o}\left(\mathcal{P}_{k}\right)=3$ for $k \geq 6$.

In Section 2 we fix the notation to be used in this article and state some useful results. We also define the relative oriented clique number of an oriented graph, which will be used later in a proof. In Section 3, 4 and 5 we prove Theorem 1.6, 1.7 and 1.8, respectively. Finally, we mention in Section 6 some future directions for research on this topic.

## 2 Preliminaries

For an oriented graph $\vec{G}$, every parameter we introduce below is denoted using $\vec{G}$ as a subscript. In order to simplify notation, this subscript will be dropped whenever there is no chance of confusion.

The set of all adjacent vertices of a vertex $v$ in an oriented graph $\vec{G}$ is called its set of neighbors and is denoted by $N_{\vec{G}}(v)$. If $\overrightarrow{u v}$ is an arc, then $u$ is an in-neighbor of $v$ and $v$ is an out-neighbor of $u$. The set of all in-neighbors and the set of all out-neighbors of $v$ are denoted by $N_{\vec{G}}^{-}(v)$ and $N_{\vec{G}}^{+}(v)$, respectively. The degree of a vertex $v$ in an oriented graph $\vec{G}$, denoted by $\operatorname{deg}_{\vec{G}}(v)$, is the number of neighbors of $v$ in $\vec{G}$. Naturally, the in-degree (resp. out-degree)


Figure 3: The planar oclique $\vec{P}$ of order 15.
of a vertex $v$ in an oriented graph $\vec{G}$, denoted by $\operatorname{deg}_{\vec{G}}^{-}(v)$ (resp. $\operatorname{deg}_{\vec{G}}^{+}(v)$ ), is the number of in-neighbors (resp. out-neighbors) of $v$ in $\vec{G}$. The order $|\vec{G}|$ of an oriented graph $\vec{G}$ is the cardinality of its set of vertices $V(\vec{G})$.

We say that two vertices $u$ and $v$ of an oriented graph agree on a third vertex $w$ of that graph if $w \in N^{\alpha}(u) \cap N^{\alpha}(v)$ for some $\alpha \in\{+,-\}$ and that they disagree on $w$ if $w \in N^{\alpha}(u) \cap N^{\beta}(v)$ for $\{\alpha, \beta\}=\{+,-\}$.

A directed path of length $k$, or a $k$-dipath, from $v_{0}$ to $v_{k}$ is an oriented graph with vertices $v_{0}, v_{1}, \ldots, v_{k}$ and $\operatorname{arcs} \overrightarrow{v_{0} v_{1}}, \overrightarrow{v_{1}} \overrightarrow{v_{2}}, \ldots, \overrightarrow{v_{k-1} v_{k}}$ where $v_{0}$ and $v_{k}$ are the terminal vertices and $v_{1}, \ldots, v_{k-1}$ are internal vertices. A 2-dipath with arcs $\overrightarrow{u v}$ and $\overrightarrow{v w}$ is denoted by $\overrightarrow{u v \vec{w}}$. More generally, a 2-dipath with terminal vertices $u, w$ and internal vertex $v$ is denoted by $u v w$ (this denotes either the 2-dipath $\overrightarrow{u v w}$ or the 2-dipath $\overrightarrow{w v u})$. A directed cycle of length $k$, or a directed $k$-cycle, is an oriented graph with vertices $v_{1}, v_{2}, \ldots, v_{k}$ and $\operatorname{arcs} \overrightarrow{v_{1} v_{2}}, \overrightarrow{v_{2} v_{3}}, \ldots, \overrightarrow{v_{k-1} v_{k}}$ and $\overrightarrow{v_{k} v_{1}}$.

The directed distance $\vec{d} \vec{G}(u, v)$ between two vertices $u$ and $v$ of an oriented graph $\vec{G}$ is the smallest length of a directed path from $u$ to $v$ in $\vec{G}$. We let $\vec{d} \vec{G}(u, v)=\infty$ if no such directed path exists. The weak directed distance $\bar{d}_{\vec{G}}(u, v)$ between $u$ and $v$ is then given by $\bar{d}_{\vec{G}}(u, v)=\min \left\{\vec{d}_{\vec{G}}(u, v), \vec{d}_{\vec{G}}(v, u)\right\}$.

Let now $G$ be an undirected graph. A path of length $k$, or a $k$-path, from $v_{0}$ to $v_{k}$ is a graph with vertices $v_{0}, v_{1}, \ldots, v_{k}$ and edges $\overrightarrow{v_{0} v_{1}}, \overrightarrow{v_{1} v_{2}}, \ldots, \overrightarrow{v_{k-1} v_{k}}$. The distance $d_{G}(x, y)$ between


Figure 4: List of all triangle-free planar graphs with diameter 2 (Plesník (1975)).
two vertices $x$ and $y$ of $G$ is the smallest length of a path connecting $x$ and $y$. The diameter $\operatorname{diam}(G)$ of a graph $G$ is the maximum distance between pairs of vertices of the graph.

Triangle-free graphs with diameter 2 have been characterized by Plesník in [8].
Theorem 2.1 (Plesník, 1975). The triangle-free graphs with diameter 2 are precisely the graphs listed in Figure 4.

The graphs depicted in Figure 4 are the stars, the complete bipartite graphs $K_{2, n}$ for some natural number $n$, and the graph obtained by adding copies of two non-adjacent vertices of the 5-cycle.

A vertex subset $D$ is a dominating set of a graph $G$ if every vertex of $G$ is either in $D$ or adjacent to a vertex of $D$. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set of $G$.

We now define a new parameter for oriented graphs which will be used in our proof. A relative oriented clique of an oriented graph $\vec{G}$ is a set $R \subseteq V(\vec{G})$ of vertices such that any two vertices from $R$ are at weak directed distance at most 2 in $\vec{G}$. The relative oriented clique number $\omega_{r o}(\vec{G})$ of an oriented graph $\vec{G}$ is the maximum order of a relative oriented clique of $\vec{G}$.

The relative oriented clique number $\omega_{r o}(G)$ of a simple graph $G$ is the maximum of the relative oriented clique numbers of all the oriented graphs with underlying graph $G$. The relative oriented clique number $\omega_{r o}(\mathcal{F})$ of a family $\mathcal{F}$ of graphs is the maximum of the relative oriented clique numbers of the graphs from the family $\mathcal{F}$.

From the definitions, we clearly have the following extension of Lemma 1.1.
Lemma 2.2. For any oriented graph $\vec{G}, \omega_{a o}(\vec{G}) \leq \omega_{r o}(\vec{G}) \leq \chi_{o}(\vec{G})$.
The relative oriented clique number of outerplanar graphs is at most 7 and this bound is tight.

Theorem 2.3. Let $\mathcal{O}$ be the family of outerplanar graphs. Then, $\omega_{r o}(\mathcal{O})=7$.
Proof. Note that the oriented outerplanar graph depicted in Figure $\mathbb{1}$ is an oclique. Hence, by Theorem 1.4 and Lemma 2.2 the result follows.

## 3 Proof of Theorem 1.6

It is easy to verify that each graph in Figure 2 is outerplanar and can be oriented as an oclique. If $G$ is an outerplanar graph which contains any of the graphs from Figure 2 as a spanning subgraph, then we can orient the edges of that spanning subgraph to obtain an oclique and orient all the other edges arbitrarily. Note that with such an orientation $G$ is an outerplanar oclique. This proves the "if" part.

For the "only if" part, first note that a graph cannot be oriented as an oclique if it is disconnected. Now there are only two connected graphs with at most two vertices, namely, the complete graphs $K_{1}$ (single vertex) and $K_{2}$ (an edge). Both of them are outerplanar and can be oriented as ocliques. Hence, any outerplanar graph on at most two vertices that can be oriented as an oclique must contain one of the graphs depicted in Figure 2(a) and Figure 2(b) as a spanning subgraph.

If $G$ is connected and has 3 vertices it must contain a 2 -path as a spanning subgraph. We know that a 2-path is outerplanar and can be oriented as an oclique. Hence, any outerplanar graph on three vertices that can be oriented as an oclique must contain a 2-path (Figure 2(c)) as a spanning subgraph.

If $G$ has at least 4 vertices and is a minimal (with respect to spanning subgraph inclusion) outerplanar graph that can be oriented as an oclique, then $\Delta(G) \geq 2$ as every oriented tree is 3 -colorable. Now we will do a case analysis to prove the remaining part.

For the remainder of the proof assume that $V(G)=\left\{v_{1}, v_{2}, v_{3}, \ldots, v_{|G|}\right\}$ where $G$ is a minimal (with respect to spanning subgraph inclusion) outerplanar graph of order at least 4 that can be oriented as an oclique.
(i) For $|G|=4$ and $\Delta(G)=2$ : A triangle would force one vertex to have degree zero and hence $G$ can not be oriented as an oclique. So, $G$ must contain a 4-cycle (Figure 2 $(d)$ ).
(ii) For $|G|=4$ and $\Delta(G)=3: G$ can not be $K_{1,3}$ since it is 3 -colorable (as it is a tree). So we need to add at least one more edge to it. By adding one more edge to it, without loss of generality, we obtain the graph depicted in Figure 2(e).
(iii) For $|G|=5$ and $\Delta(G)=2$ : $G$ must contain a $C_{5}$ (Figure $2(e)$ ) as any other connected graph on five vertices is either a tree or has maximum degree greater than 2 .
(iv) For $|G|=5$ and $\Delta(G)=3$ : Without loss of generality assume that $\operatorname{deg}\left(v_{1}\right)=3$ and $N\left(v_{1}\right)=\left\{v_{2}, v_{3}, v_{4}\right\}$. If $\left|N\left(v_{1}\right) \cap N\left(v_{5}\right)\right|<2$, then there cannot be a 2-dipath between $v_{5}$ and a vertex from $N\left(v_{1}\right) \backslash N\left(v_{5}\right)$. If $\left|N\left(v_{1}\right) \cap N\left(v_{5}\right)\right|>2$, that is, $N\left(v_{1}\right)=N\left(v_{5}\right)$, then $G$ will contain $K_{2,3}$ as a subgraph which contradicts the outerplanarity of $G$. Hence we must have $\left|N\left(v_{1}\right) \cap N\left(v_{5}\right)\right|=2$.
Without loss of generality assume that $N\left(v_{5}\right)=\left\{v_{2}, v_{3}\right\}$. Since $\Delta(G)=3, v_{4}$ must be adjacent to either $v_{2}$ or $v_{3}$ to allow a 2-dipath between $v_{4}$ and $v_{5}$. But then $C_{5}$ is a spanning subgraph of $G$ contradicting the minimality of $G$.
(v) For $|G|=5$ and $\Delta(G)=4$ : First note that $G$ cannot have four edges as then it will be a tree and hence 3 -colorable. Assume first that $G$ has five edges. Without loss of generality assume that $N\left(v_{1}\right)=\left\{v_{2}, v_{3}, v_{4}, v_{5}\right\}$ and that the fifth edge in $G$ is $v_{2} v_{3}$. Now assume the arc $\overrightarrow{v_{2} v_{1}}$ without loss of generality. This implies the arcs $\overrightarrow{v_{1} v_{4}}, \overrightarrow{v_{1} v_{5}}$ to have a 2-dipath between the pair of vertices $\left\{v_{2}, v_{4}\right\}$ and $\left\{v_{2}, v_{5}\right\}$ respectively. But then it is not possible to connect $v_{4}$ and $v_{5}$ by a 2 -dipath. Therefore, $G$ has at least six edges. If $G$ has at least six edges, then $G$ will contain the graph depicted either in Figure 2 $2(g)$ or in Figure $2(h)$ as a spanning subgraph.
(vi) For $|G|=6$ and $\Delta(G)=2$ : The only two connected graphs on six vertices with $\Delta(G)=2$ are the cycle $C_{6}$ on six vertices and the path on six vertices. None of them can be oriented as an oclique as the oriented chromatic number of trees and cycles is bounded above by 3 and 5 , respectively.
(vii) For $|G|=6$ and $\Delta(G)=3$ : Let $N\left(v_{1}\right)=\left\{v_{2}, v_{3}, v_{4}\right\}$. Since $G$ is an outerplanar graph we have $\left|N\left(v_{1}\right) \cap N\left(v_{i}\right)\right| \leq 2$ for $i \in\{5,6\}$. Now let $\left|N\left(v_{1}\right) \cap N\left(v_{5}\right)\right|=0$. Then $v_{2}, v_{3}$ and $v_{4}$ must be connected by 2-dipaths to $v_{5}$ through $v_{6}$ contradicting $\left|N\left(v_{1}\right) \cap N\left(v_{6}\right)\right| \leq 2$. Hence $\left|N\left(v_{1}\right) \cap N\left(v_{5}\right)\right| \in\{1,2\}$.
First assume that $\left|N\left(v_{1}\right) \cap N\left(v_{5}\right)\right|=1$ and without loss of generality let $N\left(v_{1}\right) \cap N\left(v_{5}\right)=$ $\left\{v_{2}\right\}$. Since $\Delta(G)=3, v_{2}$ can be adjacent to at most one of the vertices from $\left\{v_{3}, v_{4}\right\}$. If $v_{2}$ is adjacent to exactly one vertex from $\left\{v_{3}, v_{4}\right\}$, say $v_{3}$ without loss of generality, then $v_{4}$ must be connected to $v_{5}$ by a 2 -dipath through $v_{6}$. Now for having weak directed distance at most 2 between $v_{3}$ and $v_{6}$ we must either have the edge $v_{3} v_{4}$ or have the edge $v_{3} v_{6}$ (it is not possible to have the edge $v_{2} v_{6}$ as $\Delta(G)=3$ ) creating a $K_{4}$-minor or a $K_{2,3}$ respectively, contradicting the outerplanarity of $G$. Hence, $v_{2}$ is non-adjacent to both $v_{3}$ and $v_{4}$. In that case, $v_{5}$ must be connected to $v_{3}$ and $v_{4}$ by 2 -dipaths through $v_{6}$. This will create a $K_{2,3}$-minor in $G$ contradicting its outerplanarity.
Now assume that $\left|N\left(v_{1}\right) \cap N\left(v_{5}\right)\right|=2$ and let $N\left(v_{1}\right) \cap N\left(v_{5}\right)=\left\{v_{2}, v_{3}\right\}$. Since $G$ is outerplanar, $v_{4}$ cannot be connected to $v_{5}$ by a 2 -dipath through $v_{6}$ as that will create a $K_{2,3}$-minor. So $v_{4}$ must be connected to $v_{5}$ by a 2 -dipath through either $v_{2}$ or $v_{3}$. Note that $v_{4}$ cannot be adjacent to both $v_{2}$ and $v_{3}$ as it will create a $K_{4}$-minor contradicting the outerplanarity of $G$. Without loss of generality assume that $v_{4}$ is connected to $v_{5}$ by a 2 -dipath through $v_{3}$. Now $v_{6}$ cannot be adjacent to $v_{3}$ as $\Delta(G)=3$. To have weak directed distance from $v_{6}$ to $v_{2}$ and $v_{4}$ at most 2 we must have the edges $v_{4} v_{6}$ and $v_{5} v_{6}$. This will create a $K_{2,3}$-minor contradicting the outerplanarity of $G$. Hence we are done with this case as well.
(viii) For $|G|=6$ and $\Delta(G)=4$ : Let $N\left(v_{1}\right)=\left\{v_{2}, v_{3}, v_{4}, v_{5}\right\}$. Then we have $\left|N\left(v_{1}\right) \cap N\left(v_{6}\right)\right| \leq 2$ since $G$ is outerplanar. Let $N\left(v_{1}\right) \cap N\left(v_{6}\right)=N\left(v_{6}\right)=\left\{v_{2}\right\}$. Then $v_{2}$ can be a neighbor of at most two vertices from $\left\{v_{3}, v_{4}, v_{5}\right\}$ in order to preserve outerplanarity of $G$ and hence the remaining vertex will not have any 2-dipath connecting it to $v_{6}$. So we must have $\left|N\left(v_{6}\right)\right|=2$. Without loss of generality assume that $N\left(v_{6}\right)=\left\{v_{3}, v_{4}\right\}$. Now $v_{6}$ must be connected by 2 -dipaths to $v_{2}$ and $v_{5}$. These two 2 -dipaths must go through $v_{3}$ or $v_{4}$. Both 2-dipaths cannot go through the same vertex $v_{3}$ (or $v_{4}$ ) as it will create a $K_{2,3}$-minor contradicting the outerplanarity of $G$. Therefore, one of the two 2-dipaths must go through $v_{3}$ while the other must go through $v_{4}$. This will force the graph depicted in Figure 2(i) to be a spanning subgraph of $G$ contradicting its minimality.
(ix) For $|G|=6$ and $\Delta(G)=5$ : Assume that $N\left(v_{1}\right)=\left\{v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right\}$ and let the vertices $v_{2}, v_{3}, v_{4}, v_{5}$ and $v_{6}$ be arranged in clockwise order around $v_{1}$ in a fixed planar embedding of $G$. Since $G$ is outerplanar, the induced subgraph $G\left[v_{2}, v_{3}, v_{4}, v_{5}, v_{6}\right]$ cannot have a cycle as it will create a $K_{4}$-minor, contradicting the outerplanarity of $G$.
Now without loss of generality assume that $\left|N^{+}\left(v_{1}\right)\right|>\left|N^{-}\left(v_{1}\right)\right|$. Note that if $\left|N^{+}\left(v_{1}\right)\right| \geq 4$ then we cannot have weak directed distance at most 2 between all the vertices of $N^{+}\left(v_{1}\right)$ keeping the graph outerplanar. Hence we must have $\left|N^{+}\left(v_{1}\right)\right|=3$ and $\left|N^{-}\left(v_{1}\right)\right|=2$. Now to have weak directed distance at most 2 between all the vertices of $N^{+}\left(v_{1}\right)$ and between all the vertices of $N^{-}\left(v_{1}\right)$, keeping the graph outerplanar, we must have the graph depicted in Figure $2(j)$ as a spanning subgraph of $G$.
(x) For $|G|=7$ : It has been proved by Klostermeyer and MacGillivray in [6] that $G$ must contain the graph depicted in Figure $2(k)$ as a spanning subgraph.

This concludes the proof.

## 4 Proof of Theorem 1.7

Goddard and Henning [5] proved that every planar graph of diameter 2 has domination number at most 2 except for a particular graph on nine vertices.

Let $\vec{B}$ be a planar oclique dominated by the vertex $v$. Sopena [12] showed that any oriented outerplanar graph has an oriented 7 -coloring (see Theorem (1.4). Hence let $c$ be an oriented 7-coloring of the oriented outerplanar graph obtained from $\vec{B}$ by deleting the vertex $v$. For $u \in N^{\alpha}(v)$ let us assign the color $(c(u), \alpha)$ to $u$ for $\alpha \in\{+,-\}$ and the color 0 to $v$. It is easy to check that this is an oriented 15 -coloring of $\vec{B}$. Hence any planar oclique dominated by one vertex has order at most 15 .
Lemma 4.1. Let $\vec{H}$ be a planar oclique of order 15 dominated by one vertex. Then $\vec{H}$ contains the planar oclique depicted in Figure $\mathbf{Q}^{3}$ as a spanning subgraph.
Proof. Suppose $\vec{H}$ is a triangulated planar oclique of order 15 dominated by one vertex $v$. Note that $N^{\alpha}(v)$ is a relative oriented clique in $\vec{H}[N(v)]$ (that is, the oriented subgraph of $\vec{H}$ induced by the neighbors of $v$, which is actually the oriented graph obtained by deleting the vertex $v$ from $\vec{H}$ ) for any $\alpha \in\{+,-\}$. Also note that $\vec{H}[N(v)]$ is an outerplanar graph. Hence, by Theorem [2.3], we have $\left|N^{\alpha}(v)\right| \leq 7$ for any $\alpha \in\{+,-\}$. But we also have

$$
\left|N^{+}(v)\right|+\left|N^{-}(v)\right|=14
$$

Hence we get

$$
\left|N^{+}(v)\right|=\left|N^{-}(v)\right|=7 .
$$

Now assume $N(v)=\left\{x_{1}, x_{2}, \ldots, x_{14}\right\}$. Moreover, fix a planar embedding of $\vec{H}$ and, without loss of generality, assume that the vertices $x_{1}, x_{2}, \ldots, x_{14}$ are arranged in a clockwise order around $v$. The triangulation of $\vec{H}$ forces the edges $x_{1} x_{2}, x_{2} x_{3}, \ldots, x_{13} x_{14}$ and $x_{14} x_{1}$. We know from the above discussion that there should be two disjoint relative oriented cliques $N^{+}(v)$ and $N^{-}(v)$, each of order 7 , in the outerplanar graph $\vec{H}[N(v)]$. We already have the cycle $x_{1} x_{2} \ldots x_{14}$ forced in the outerplanar graph $\vec{H}[N(v)]$. We will now prove some more structural properties of $\vec{H}[N(v)]$.

As $\vec{H}[N(v)]$ is an outerplanar graph, it must have at least two vertices of degree at most 2 . As every vertex of the graph is part of a cycle, there is no vertex of degree at most 1 . Hence there are at least two vertices of degree exactly 2 in $\vec{H}[N(v)]$.

Without loss of generality, assume that $\operatorname{deg}_{\vec{H}[N(v)]}\left(x_{2}\right)=2$ and $x_{2} \in N^{\alpha}(v)$ for some fixed $\alpha \in\{+,-\}$ and let $\{\alpha, \bar{\alpha}\}=\{+,-\}$. Since $\vec{H}$ is triangulated, we must have the edge $x_{1} x_{3}$. The vertices of $N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ must then be connected to $x_{2}$ by 2-dipaths with internal vertex either $x_{1}$ or $x_{3}$.

Let four vertices of $N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ be connected to $x_{2}$ by 2-dipaths with internal vertex $x_{1}$. Then there will be two vertices, among the above mentioned four vertices, at weak directed distance at most 3 which is a contradiction. So at most three vertices of $N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ can be connected to $x_{2}$ by 2-dipaths with internal vertex $x_{1}$. Similarly we can show that at most three vertices of $N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ can be connected to $x_{2}$ by 2-dipaths with internal vertex $x_{3}$.

Now suppose there are at least two vertices $x_{i}, x_{j} \in N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ that are connected to $x_{2}$ by 2-dipaths with internal vertex $x_{1}$ and there are at least two vertices $x_{k}, x_{l} \in N^{\alpha}(v) \backslash$ $\left\{x_{1}, x_{2}, x_{3}\right\}$ that are connected to $x_{2}$ by 2-dipaths with internal vertex $x_{3}$.


Figure 5: Structure of $\vec{G}$ (not a planar embedding)
Notice that, as the graph $\vec{H}$ is planar, with the given planar embedding of $\vec{H}$ we must have $i, j>k, l$. Now, without loss of generality, we can assume that $i>j$ and $k>l$. But it will be impossible to have weak directed distance at most 2 between $x_{i}$ and $x_{l}$ keeping the graph $\vec{H}$ planar. So, at least one of the vertices $x_{1}$ or $x_{3}$ must be the internal vertex of at most one 2-dipath connecting a vertex of $N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ to $x_{2}$.

If at least one of the vertices $x_{1}$ or $x_{3}$ belongs to $N^{\bar{\alpha}}(v)$, then we have

$$
\left|N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}\right| \geq 5
$$

But then, by the above discussion, we will have a contradiction (there will be at least two vertices of $N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ connected by 2-dipaths with internal vertex $x_{1}$ and least two vertices of $N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ connected by 2-dipaths with internal vertex $x_{3}$ ).

Hence we must have $x_{1}, x_{3} \in N^{\alpha}(v)$. Without loss of generality, we have three vertices $x_{i}, x_{j}, x_{k} \in N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ connected by 2-dipaths with internal vertex $x_{1}$ and one vertex $x_{l} \in N^{\alpha}(v) \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ connected by a 2-dipath with internal vertex $x_{3}$. Without loss of generality, we can assume $i>j>k>l$.

Now, to have $\bar{d}\left(x_{i}, x_{s}\right) \leq 2$ for $s \in\{2,3, l\}$, the vertices $x_{2}, x_{3}, x_{l}$ must disagree with $x_{i}$ on $x_{1}$. Also, to have $\bar{d}\left(x_{i}, x_{k}\right) \leq 2$ and $\bar{d}\left(x_{2}, x_{l}\right) \leq 2$, we must have the 2-dipaths $x_{i} x_{j} x_{k}$ and $x_{2} x_{3} x_{l}$. But then the induced oriented graph $\vec{H}\left[N^{\alpha}(v)\right]$ contains the oriented graph induced by $N^{\alpha}\left(a_{0}\right)$ of the planar oclique depicted in Figure 3,

Further notice that no vertex of $N^{\alpha}(v)$, other than $x_{2}$, has degree 2 in $\vec{H}[N(v)]$. Hence we can infer that a vertex of $N^{\bar{\alpha}}(v)$ has degree 2 in $\vec{H}[N(v)]$. That will imply that the induced oriented graph $\vec{H}\left[N^{\bar{\alpha}}(v)\right]$ contains the oriented graph induced by $N^{\bar{\alpha}}\left(a_{0}\right)$ of the planar oclique depicted in Figure 3.

Hence the planar oclique depicted in Figure 3 is a subgraph of $\vec{H}$. It is easy to check that, regardless of the choice of $\vec{H}$ (it is a triangulation of the planar oclique depicted in Figure 3), if we delete one arc of the oriented subgraph, isomorphic to the planar oclique depicted in Figure 3, of $\vec{H}$, the oriented graph $\vec{H}$ does no longer remain an oclique.

Now, to prove Theorem 1.7, it will be enough to prove that every planar oclique of order at least 15 must have domination number 1. In other words, it will be enough to prove that any planar oclique with domination number 2 must have order at most 14 . More precisely, we need to prove the following lemma.
Lemma 4.2. Let $\vec{H}$ be a planar oclique with domination number 2. Then $|\vec{H}| \leq 14$.


Figure 6: A planar embedding of $\operatorname{und}(\vec{H})$
Let $\vec{G}$ be a planar oclique with $|\vec{G}|>14$. Assume that $\vec{G}$ is triangulated and has domination number 2.

We define a partial order $\prec$ on the set of all dominating sets of order 2 of $\vec{G}$ as follows: for any two dominating sets $D=\{x, y\}$ and $D^{\prime}=\left\{x^{\prime}, y^{\prime}\right\}$ of order 2 of $\vec{G}, D^{\prime} \prec D$ if and only if $\left|N\left(x^{\prime}\right) \cap N\left(y^{\prime}\right)\right|<|N(x) \cap N(y)|$.

Let $D=\{x, y\}$ be a maximal dominating set of order 2 of $\vec{G}$ with respect to $\prec$. Also, for the remainder of this section, let $t, t^{\prime}, \alpha, \bar{\alpha}, \beta, \bar{\beta}$ be variables satisfying $\left\{t, t^{\prime}\right\}=\{x, y\}$ and $\{\alpha, \bar{\alpha}\}=\{\beta, \bar{\beta}\}=\{+,-\}$.

Let us fix the following notations (see Figure (5):

$$
\begin{array}{r}
C=N(x) \cap N(y), C^{\alpha \beta}=N^{\alpha}(x) \cap N^{\beta}(y), C_{t}^{\alpha}=N^{\alpha}(t) \cap C, \\
S_{t}=N(t) \backslash C, S_{t}^{\alpha}=S_{t} \cap N^{\alpha}(t), S=S_{x} \cup S_{y} .
\end{array}
$$

Hence we have

$$
\begin{equation*}
15 \leq|\vec{G}|=|D|+|C|+|S| . \tag{1}
\end{equation*}
$$

Let $\vec{H}$ be the oriented graph obtained from the induced subgraph $\vec{G}[D \cup C]$ of $\vec{G}$ by deleting all the arcs between the vertices of $D$ and all the arcs between the vertices of $C$. Note that it is possible to extend the planar embedding of und $(\vec{H})$ given in Figure 6 to a planar embedding of $\operatorname{und}(\vec{G})$ for some particular ordering of the elements of, say $C=\left\{c_{0}, c_{1}, \ldots, c_{k-1}\right\}$.

Notice that und $(\vec{H})$ has $k$ faces, namely the unbounded face $F_{0}$ and the faces $F_{i}$ bounded by edges $x c_{i-1}, c_{i-1} y, y c_{i}$ and $c_{i} x$ for $i \in\{1, \ldots, k-1\}$. Geometrically, und $(\vec{H})$ divides the plane into $k$ connected components. The region $R_{i}$ of $\vec{G}$ is the $i^{t h}$ connected component (corresponding to the face $F_{i}$ ) of the plane. The boundary points of a region $R_{i}$ are $c_{i-1}$ and $c_{i}$ for $i \in$ $\{1, \ldots, k-1\}$, and $c_{0}$ and $c_{k-1}$ for $i=0$. Two regions are adjacent if they have at least one common boundary point (hence, a region is adjacent to itself).

Now, for the different possible values of $|C|$, we want to show that und $(\vec{H})$ cannot be extended to a planar oclique of order at least 15 . Note that, for extending $\operatorname{und}(\vec{H})$ to $\vec{G}$, we can add new vertices only from $S$. Any vertex $v \in S$ will be inside one of the regions $R_{i}$. If there is at least one vertex of $S$ in a region $R_{i}$, then $R_{i}$ is non-empty and empty otherwise. In fact, when there is no chance of confusion, $R_{i}$ might represent the set of vertices of $S$ contained in the region $R_{i}$.

As any two distinct non-adjacent vertices of $\vec{G}$ must be connected by a 2-dipath, we have the following three lemmas:


Figure 7: For $|C|=1$ while $x$ and $y$ are non-adjacent

Lemma 4.3. (a) If $(u, v) \in S_{x} \times S_{y}$ or $(u, v) \in S_{t}^{\alpha} \times S_{t}^{\alpha}$, then $u$ and $v$ are in adjacent regions.
(b) If $(u, c) \in S_{t}^{\alpha} \times C_{t}^{\alpha}$, then $c$ is a boundary point of a region adjacent to the region containing $u$.

Lemma 4.4. Let $R, R^{1}$ and $R^{2}$ be three distinct regions such that $\underline{R}$ is adjacent to $R^{i}$ with common boundary point $c^{i}$ while the other boundary point of $R^{i}$ is $\overline{c^{i}}$, for all $i \in\{1,2\}$. If $v \in S_{t}^{\alpha} \cap R$ and $u^{i} \in\left(\left(S_{t}^{\alpha} \cup S_{t^{\prime}}\right) \cap R^{i}\right) \cup\left(\left\{\overline{c^{i}}\right\} \cap C_{t}^{\alpha}\right)$, then $v$ disagrees with $u^{i}$ on $c^{i}$, for all $i \in\{1,2\}$. Moreover, if both $u^{1}$ and $u^{2}$ exist, then $\left|S_{t}^{\alpha} \cap R\right| \leq 1$.

Lemma 4.5. For any arc $\overrightarrow{u v}$ in $\vec{G},\left|N^{\alpha}(u) \cap N^{\beta}(v)\right| \leq 3$.
Now we ask the question "How small $|C|$ can be?" and try to prove possible lower bounds of $|C|$. The first result regarding the lower bound of $|C|$ is proved below.

Lemma 4.6. $|C| \geq 2$.
Proof. We know that $x$ and $y$ are either connected by a 2 -dipath or by an arc. If $x$ and $y$ are adjacent then, as $\vec{G}$ is triangulated, we have $|C| \geq 2$. If $x$ and $y$ are non-adjacent, then $|C| \geq 1$. Hence it is enough to show that we cannot have $|C|=1$ while $x$ and $y$ are non-adjacent.

If $|C|=1$ and $x$ and $y$ are non-adjacent, then the triangulation of $\vec{G}$ will force the configuration depicted in Figure 7 as a subgraph of $\operatorname{und}(\vec{G})$, where $C=\left\{c_{o}\right\}, S_{x}=\left\{x_{1}, \ldots, x_{n_{x}}\right\}$ and $S_{y}=\left\{y_{1}, \ldots, y_{n_{y}}\right\}$. Without loss of generality, we may assume $\left|S_{y}\right| \geq\left|S_{x}\right|$. Then, by equation (11), we have

$$
n_{y}=\left|S_{y}\right| \geq\lceil(15-2-1) / 2\rceil=6 .
$$

Clearly $n_{x}=\left|S_{x}\right| \geq 3$, as otherwise $\left\{c_{0}, y\right\}$ would be a dominating set with at least two common neighbors $\left\{y_{1}, y_{n_{y}}\right\}$, which contradicts the maximality of $D$.

For $n_{x}=3$, we know that $c_{0}$ is not adjacent to $x_{2}$ as otherwise $\left\{c_{0}, y\right\}$ would be a dominating set with at least two common neighbors $\left\{y_{1}, y_{n_{y}}\right\}$, contradicting the maximality of $D$. But then $x_{2}$ should be adjacent to $y_{i}$ for some $i \in\left\{1, \ldots, n_{y}\right\}$, as otherwise $d\left(x_{2}, y\right)>2$. Now the triangulation of $\vec{G}$ forces $x_{2}$ and $y_{i}$ to have at least two common neighbors. Also, $x_{2}$ cannot be adjacent to $y_{j}$ for any $j \neq i$, as it would create a dominating set $\left\{x_{2}, y\right\}$ with at least two common neighbors $\left\{y_{i}, y_{j}\right\}$, contradicting the maximality of $D$. Hence, $x_{2}$ and $y_{i}$ are adjacent to both $x_{1}$ and $x_{3}$. Note that $t_{\ell_{t}}$ and $t_{\ell_{t}+k}$ are adjacent if and only if $k=1$, as otherwise $d\left(t_{\ell_{t}+1}, t^{\prime}\right)>2$ for $1 \leq \ell_{t}<\ell_{t}+k \leq n_{t}$. In this case, by equation (11), we have

$$
n_{y}=\left|S_{y}\right| \geq 15-2-1-3=9
$$

Assume $i \geq 5$. Hence, $c_{0}$ is adjacent to $y_{j}$ for all $j=1,2,3$, as otherwise $d\left(y_{j}, x_{3}\right)>2$. This implies $d\left(y_{2}, x_{2}\right)>2$, a contradiction. Similarly $i<5$ would also force a contradiction. Hence $n_{x} \geq 4$.

For $n_{x}=4, c_{0}$ cannot be adjacent to both $x_{3}$ and $x_{n_{x}-2}=x_{2}$ as it would create a dominating set $\left\{c_{0}, y\right\}$ with at least two common neighbors $\left\{y_{1}, y_{n_{y}}\right\}$, contradicting the maximality of $D$. For $n_{x} \geq 5, c_{0}$ is adjacent to $x_{3}$ implies, either for all $i \geq 3$ or for all $i \leq 3, x_{i}$ is adjacent to $c_{0}$, as otherwise $d\left(x_{i}, y\right)>2$. Either of these cases would force $c_{0}$ to become adjacent to $y_{j}$, as otherwise we would have either $d\left(x_{1}, y_{j}\right)>2$ or $d\left(x_{n_{x}}, y_{j}\right)>2$ for all $j \in\left\{1,2, \ldots, n_{y}\right\}$. But then we would have a dominating set $\left\{c_{0}, x\right\}$ with at least two common vertices, contradicting the maximality of $D$. Hence, for $n_{x} \geq 5, c_{0}$ is not adjacent to $x_{3}$. Similarly we can show, for $n_{x} \geq 5$, that $c_{0}$ is neither adjacent to $x_{3}$ nor to $x_{n_{x}-2}$.

So, for $n_{x} \geq 4$, we can assume without loss of generality that $c_{0}$ is not adjacent to $x_{3}$. We know that $d\left(y_{1}, x_{3}\right) \leq 2$. We have already noted that $t_{l_{t}}$ and $t_{l_{t}+k}$ are adjacent if and only if $k=1$ for any $0 \leq l_{t}<l_{t}+k \leq n_{t}$. Hence, to have $d\left(y_{1}, x_{3}\right) \leq 2$, we must have one of the following edges: $y_{1} x_{2}, y_{1} x_{3}, y_{1} x_{4}$ or $y_{2} x_{3}$.

The first edge would imply the edges $x_{2} y_{j}$ as otherwise $d\left(x_{1}, y_{j}\right)>2$ for all $j=3,4,5$. These three edges would then imply $d\left(x_{4}, y_{3}\right)>2$. Hence we do not have the edge $y_{1} x_{2}$.

The other three edges, assuming we cannot have the edge $y_{1} x_{2}$, would force the edges $x_{2} c_{0}$ and $x_{1} c_{0}$ for having $d\left(x_{2}, y\right) \leq 2$ and $d\left(x_{1}, y\right) \leq 2$. This would imply $d\left(x_{1}, y_{4}\right)>2$, a contradiction. Therefore, we cannot have the other three edges too.

Hence we are done.
We now prove that, for $2 \leq|C| \leq 5$, at most one region of $\vec{G}$ can be non-empty. Later, using this result, we will improve the lower bound of $|C|$.

Lemma 4.7. If $2 \leq|C| \leq 5$, then at most one region of $\vec{G}$ is non-empty.
Proof. (For pictorial help, refer to Figure 6) If $|C|=2$ and $x$ and $y$ are adjacent, then the region that contains the edge $x y$ is empty, as otherwise the triangulation of $\vec{G}$ would force $x$ and $y$ to have a common neighbor other than $c_{0}$ and $c_{1}$. So, for the rest of the proof, we can assume $x$ and $y$ are non-adjacent if $|C|=2$.

Step 0. We first show that it is not possible to have either $S_{x}=\emptyset$ or $S_{y}=\emptyset$ and have at least two non-empty regions. Without loss of generality, assume that $S_{x}=\emptyset$. Then $x$ and $y$ are non-adjacent, as otherwise $y$ would be a dominating vertex which is not possible.

For $|C|=2$, if both $S_{y} \cap R_{0}$ and $S_{y} \cap R_{1}$ are non-empty, then the triangulation of $\vec{G}$ forces either two parallel edges $c_{0} c_{1}$ (one in each region) or a common neighbor of $x$ and $y$ other than $c_{0}, c_{1}$, a contradiction.

For $|C|=3,4$ and 5 , the triangulation of $\vec{G}$ implies the edges $c_{0} c_{1}, \ldots, c_{k-2} c_{k-1}$ and $c_{k-1} c_{0}$. Hence every $v \in S_{y}$ must be connected to $x$ by a 2-dipath through $c_{i}$ for some $i \in\{1,2, \ldots, k-1\}$. Now assume $\left|S_{y}^{\alpha}\right| \geq\left|S_{y}^{\bar{\alpha}}\right|$ for some $\alpha \in\{+,-\}$. Then, by equation (1), we have

$$
\left|S_{y}^{\alpha}\right| \geq\lceil(15-2-5) / 2\rceil=4
$$

By Lemma 4.3, we know that the vertices of $S_{y}^{\alpha}$ will be contained in two adjacent regions for $|C|=4,5$. For $|C|=3, S_{y}^{\alpha} \cap R_{i} \neq \emptyset$ for all $i \in\{0,1,3\}$ implies $\left|S_{y}^{\alpha}\right| \leq 3$ by Lemma 4.4. Hence, without loss of generality, we may assume $S_{y}^{\alpha} \subseteq R_{1} \cup R_{2}$. If both $S_{y}^{\alpha} \cap R_{1}$ and $S_{y}^{\alpha} \cap R_{2}$ are non-empty then, by Lemma 4.4, each vertex of $S_{y}^{\alpha} \cap R_{1}$ disagrees with each vertex of $S_{y}^{\alpha} \cap R_{2}$ on $c_{1}$. But then, $\left\{c_{1}, y\right\}$ becomes a dominating set with at least six common neighbors (namely $c_{0}, c_{2}$, and four vertices from $S_{y}^{\alpha}$ ), which contradicts the maximality of $D$.

Hence, all the vertices of $S_{y}^{\alpha}$ must be contained in one region, say $R_{1}$. Each of them should then be connected to $x$ by a 2-dipath with internal vertex either $c_{0}$ or $c_{1}$. However, the vertices that are connected to $x$ by a 2-dipath with internal vertex $c_{0}$ should have weak directed distance at most 2 with the vertices connected to $x$ by a 2 -dipath with internal vertex $c_{1}$. But it is not possible to connect them unless they are all adjacent to either $c_{0}$ or $c_{1}$, in which case it would contradict the maximality of $D$.

Hence both $S_{x}$ and $S_{y}$ are non-empty.
Step 1. We now prove that at most four sets out of the $2 k$ sets $S_{t} \cap R_{i}$ can be non-empty, for all $t \in\{x, y\}$ and $i \in\{0,1, \ldots, k-1\}$. It is immediate for $|C|=2$. For $|C|=4$ and 5 , the statement follows from Lemma 4.3, For $|C|=3$, we consider the following two cases:
(i) Assume $S_{t} \cap R_{i} \neq \emptyset$ for all $t \in\{x, y\}$ and for all $i \in\{0,1,2\}$. Then, by Lemma 4.4, we have $\left|S_{t} \cap R_{i}\right| \leq 1$ for all $t \in\{x, y\}$ and for all $i \in\{0,1,2\}$. By equation (1), we have

$$
15 \leq|\vec{G}|=2+3+4=9
$$

This is a contradiction.
(ii) Assume that five out of the six sets $S_{t} \cap R_{i}$ are non-empty and that the other one is empty, where $t \in\{x, y\}$ and $i \in\{0,1,2\}$. Without loss of generality, we can assume $S_{x} \cap R_{0}=\emptyset$. By Lemma 4.4, we have $\left|S_{t} \cap R_{i}\right| \leq 1$ for all $(t, i) \in\{(x, 1),(x, 2),(y, 0)\}$. In particular, $\left|S_{x}\right| \leq 2$.
Now, all vertices of $S_{t} \cap R_{i}$ are adjacent to $c_{1}$ for $i \in\{1,2\}$, for being at weak directed distance at most 2 from each other, by Lemma 4.4. That means every vertex of $S_{x}$ is adjacent to $c_{1}$. Hence, there can be at most three vertices in $\left(S_{y} \cap R_{1}\right) \cup\left(S_{y} \cap R_{2}\right)$ as otherwise the dominating set $\left\{c_{1}, y\right\}$ would contradict the maximality of $D$. Hence, $\left|S_{y}\right| \leq 4$.
Therefore, by equation (1) we have

$$
15 \leq|\vec{G}|=2+3+(2+4)=11 .
$$

This is a contradiction.
Hence, at most four sets out of the $2 k$ sets $S_{t} \cap R_{i}$ can be non-empty, where $t \in\{x, y\}$ and $i \in\{0,1, \ldots, k-1\}$.

Step 2. Assume now that exactly four sets out of the sets $S_{t} \cap R_{i}$ are non-empty, for all $t \in\{x, y\}$ and $i \in\{0, \ldots, k-1\}$. Without loss of generality, we have the following three cases (by Lemma 4.3):
(i) Assume the four non-empty sets are $S_{x} \cap R_{1}, S_{y} \cap R_{0}, S_{y} \cap R_{1}$ and $S_{y} \cap R_{2}$ (only possible for $|C| \geq 3$ ). The triangulation of $\vec{G}$ then forces the edges $c_{0} c_{k-1}$ and $c_{1} c_{2}$. Lemma 4.4 implies that $S_{x} \cap R_{1}=\left\{x_{1}\right\}$ and that the vertices of $S_{y} \cap R_{0}$ and the vertices of $S_{y} \cap R_{2}$ disagree with $x_{1}$ on $c_{0}$ and $c_{1}$, respectively.
For $|C|=3$, if every vertex from $S_{y} \cap R_{1}$ is adjacent to either $c_{0}$ or $c_{1}$, then $\left\{c_{0}, c_{1}\right\}$ will be a dominating set with at least four common neighbors $\left\{x, y, x_{1}, c_{2}\right\}$, contradicting the maximality of $D$. If not, then the triangulation of $\vec{G}$ will force $x_{1}$ to be adjacent to at least two vertices, from $S_{y}$, say $y_{1}$ and $y_{2}$. But then, $\left\{x_{1}, y\right\}$ would be a dominating set with at least four common neighbors $\left\{y_{1}, y_{2}, c_{0}, c_{1}\right\}$, contradicting the maximality of $D$.

For $|C|=4$ and 5 , Lemma 4.3 implies that vertices of $S_{y} \cap R_{0}$ and vertices of $S_{y} \cap R_{2}$ disagree with each other on $y$. Now, by Lemma 4.4, any vertex of $S_{y} \cap R_{1}$ is adjacent either to $c_{0}$ (if it agrees with the vertices of $S_{y} \cap R_{0}$ on $y$ ) or to $c_{1}$ (if it agrees with the vertices of $S_{y} \cap R_{2}$ on $y$ ). Also, the vertices of $S_{y} \cap R_{0}$ and $S_{y} \cap R_{2}$ are connected to $x_{1}$ by a 2 -dipath through $c_{0}$ and $c_{1}$ respectively. Hence, by Lemma 4.5, we have $\left|S_{y} \cap R_{0}\right|,\left|S_{y} \cap R_{2}\right| \leq 3$.
Now, by equation (1), we have

$$
\left|S_{y}\right| \geq(15-2-5-1)=7
$$

Hence, without loss of generality, at least four vertices $y_{1}, y_{2}, y_{3}, y_{4}$ of $S_{y}$ are adjacent to $c_{0}$. But, in that case, $\left\{c_{0}, y\right\}$ is a dominating set with at least five common neighbors $\left\{y_{1}, y_{2}, y_{3}, y_{4}, c_{k-1}\right\}$, contradicting the maximality of $D$ for $|C|=4$.
For $|C|=5$, each vertex of $S_{y} \cap R_{1}$ disagrees with $c_{3}$ on $y$ by Lemma 4.3 and therefore, without loss of generality, all of them are adjacent to $c_{0}$. Then, vertices of $S_{y} \cap R_{1}$ disagrees with vertices of $S_{y} \cap R_{2}$ on $y$ as well. This implies vertices of $S_{y} \cap R_{2}$ agrees with $c_{3}$ on $y$ and must be connected to $c_{3}$ by 2-dipaths with internal vertex $c_{2}$. Now, by Lemma 4.4, $\left|S_{y} \cap R_{2}\right| \leq 1$. So, $\left|S_{y}\right| \geq 7$ implies $\left|S_{y} \cap\left(R_{0} \cup R_{1}\right)\right| \geq 6$. But every vertex of $S_{y} \cap\left(R_{0} \cup R_{1}\right)$ are adjacent to $c_{0}$. In that case, $\left\{c_{0}, y\right\}$ is a dominating set with at least six common neighbors, contradicting the maximality of $D$ for $|C|=5$.
(ii) Assume the four non-empty sets are $S_{x} \cap R_{0}, S_{x} \cap R_{1}, S_{y} \cap R_{0}$ and $S_{y} \cap R_{1}$. For $|C|=2$, every vertex in $S$ is adjacent either to $c_{0}$ or to $c_{1}$ (by Lemma 4.4). So, $\left\{c_{0}, c_{1}\right\}$ is a dominating set. Hence, no vertex $w \in S$ can be adjacent to both $c_{0}$ and $c_{1}$ since otherwise $\left\{c_{0}, c_{1}\right\}$ would be a dominating set with at least three common neighbors $\{x, y, w\}$, contradicting the maximality of $D$. By equation (1), we have

$$
|S| \geq 15-2-2=11
$$

Hence, without loss of generality, we may assume $\left|S_{x} \cap R_{0}\right| \geq 3$. Suppose $\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq$ $S_{x} \cap R_{0}$. In that case, all vertices of $S_{x} \cap R_{0}$ must be adjacent to $c_{0}$ (or to $c_{1}$ ), as otherwise it would force all vertices of $S_{y} \cap R_{1}$ to be adjacent to both $c_{0}$ and $c_{1}$ (by Lemma 4.4). Without loss of generality, assume that all vertices of $S_{x} \cap R_{0}$ are adjacent to $c_{0}$. Then, all vertices $w \in S_{y}$ will be adjacent to $c_{0}$, as otherwise $d\left(w, x_{i}\right)>2$, for some $i \in\{1,2,3\}$. But then $\left\{c_{0}, x\right\}$ would be a dominating set with at least three common vertices $\left\{x_{1}, x_{2}, x_{3}\right\}$, contradicting the maximality of $D$.
For $|C|=3,4$, every vertex of $S$ will be adjacent to $c_{0}$ (by Lemma 4.4). By equation (11), we have

$$
|S| \geq(15-2-4)=9
$$

Hence, without loss of generality, $\left|S_{x}\right| \geq 5$. In that case, $\left\{c_{o}, x\right\}$ is a dominating set with at least five common neighbors $S_{x} \cup\{y\}$, contradicting the maximality of $D$ for $|C|=3,4$.
For $|C|=5$, every vertex of $S_{t} \cap R_{i}$ disagrees with $c_{i+2}$ on $t$ and, therefore, $\left|S_{t} \cap R_{i}\right| \leq$ 3 for $i \in\{0,1\}$ by Lemma 4.3. Assume $\left|S_{x} \cap R_{0}\right|=3$ and $S_{x} \cap R_{0}=\left\{x_{1}, x_{2}, x_{3}\right\}$. Moreover, assume without loss of generality that $c_{2} \in N^{\alpha}(x)$. In that case, we must have $\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq N^{\bar{\alpha}}(x)$.
Note that $x_{1}, x_{2}$ and $x_{3}$ must agree on $c_{0}$ in order to be at weak directed distance at most 2 with the vertices of $S_{y} \cap R_{1}$. Further, assume that $\left\{x_{1}, x_{2}, x_{3}\right\} \subseteq N^{\beta}\left(c_{0}\right)$. But then, as all
the three vertices $\left\{x_{1}, x_{2}, x_{3}\right\}$ are adjacent to both $x$ and $c_{0}$, the only way each of them can be at weak directed distance 2 from $c_{3}$ is through a 2 -dipath with internal vertex $x$. Hence, we have $c_{3} \in N^{\alpha}(x)$. This implies $x_{4} \in N^{\bar{\alpha}}(x)$ for any vertex $x_{4} \in S_{x} \cap R_{1}$. But then, the vertices of $S_{x} \cap R_{1}$ must disagree with vertices of $S_{x} \cap R_{0}$ on $c_{0}$, making it impossible for the vertices of $S_{y} \cap R_{0}$ to be at weak directed distance at most 2 from $x_{1}, x_{2}, x_{3}$ and from the vertices of $S_{x} \cap R_{1}$. Therefore, we must have $\left|S_{x} \cap R_{0}\right| \leq 2$.
Similarly, we can prove $\left|S_{t} \cap R_{i}\right| \leq 2$ for $i \in\{0,1\}$.
We now show that it is not possible to have $\left|S_{t} \cap R_{i}\right|=2$ for all $(t, i) \in\{x, y\} \times\{0,1\}$. Suppose on the contrary that this is the case. Then, clearly, the vertices of $S_{t} \cap R_{i}$ disagree with $c_{i+2}$ and $c_{i+3}$ on $t$. Hence, the vertices of $S_{t} \cap R_{0}$ agree with the vertices of $S_{t} \cap R_{1}$ on $t$. Therefore, the vertices of $S_{t} \cap R_{0}$ must disagree with the vertices of $S_{t} \cap R_{1}$ on $c_{0}$.
Then it will not be possible to have both the vertices of $S_{x} \cap R_{0}$ at weak directed distance at most 2 from all the four vertices of $S_{y}$.
Therefore, we have $|S| \leq 7$. Hence, by equation (1), we have

$$
15 \leq|\vec{G}| \leq 2+5+7=14
$$

This is a contradiction and we are done.
(iii) Assume the four non-empty sets are $S_{x} \cap R_{1}, S_{x} \cap R_{2}, S_{y} \cap R_{0}$ and $S_{y} \cap R_{1}$ (only possible for $|C|=3)$. In that case, Lemma 4.4 implies that every vertex of $\left(S_{x} \cap R_{1}\right) \cup\left(S_{y} \cap R_{0}\right)$ is adjacent to $c_{0}$ and that every vertex of $\left(S_{x} \cap R_{2}\right) \cup\left(S_{y} \cap R_{1}\right)$ is adjacent to $c_{1}$.
Moreover, the triangulation of $\vec{G}$ forces the edges $c_{0} c_{2}$ and $c_{1} c_{2}$. It also forces some vertex $v_{1} \in S_{y} \cap R_{1}$ to be adjacent to $c_{0}$. But this would create the dominating set $\left\{c_{0}, c_{1}\right\}$ with at least four common neighbors $\left\{x, y, v_{1}, c_{2}\right\}$ contradicting the maximality of $D$.

Hence at most three sets out of the $2 k$ sets $S_{t} \cap R_{i}$ can be non-empty, where $t \in\{x, y\}$ and $i \in\{0,1, \ldots, k-1\}$.

Step 3. Now assume that exactly three sets out of the sets $S_{t} \cap R_{i}$ are non-empty, where $t \in\{x, y\}$ and $i \in\{0, \ldots, k-1\}$. Without loss of generality we have the following two cases (by Lemma 4.3):
(i) Assume the three non-empty sets are $S_{x} \cap R_{0}, S_{y} \cap R_{0}$ and $S_{y} \cap R_{1}$. The triangulation of $\vec{G}$ implies that the edge $c_{0} c_{1}$ lies inside the region $R_{1}$.

For $|C|=2$, there exists $u \in S_{y} \cup R_{1}$ such that $u$ is adjacent to both $c_{0}$ and $c_{1}$, by the triangulation of $\vec{G}$. Now, if $\left|S_{y} \cup R_{1}\right| \geq 2$, then some other vertex $v \in S_{y} \cup R_{1}$ must be adjacent to either $c_{0}$ or $c_{1}$. Without loss of generality, we may assume that $v$ is adjacent to $c_{0}$. Then, every vertex $w \in S_{x} \cap R_{0}$ will be adjacent to $c_{0}$, in order to have $d(v, w) \leq 2$. But in that case $\left\{c_{0}, y\right\}$ would be a dominating set with at least three common neighbors $\left\{c_{1}, u, v\right\}$, contradicting the maximality of $D$.
So we must have $\left|S_{y} \cup R_{1}\right|=1$. Assume that $S_{y} \cup R_{1}=\{u\}$. Then, any vertex $w \in S_{x} \cap R_{0}$ is adjacent to either $c_{0}$ or $c_{1}$. If $\left|S_{x}\right| \geq 5$ then, without loss of generality, we can assume that at least three vertices of $S_{x}$ are adjacent to $c_{0}$. Now, to have weak directed distance at most 2 from all those three vertices, every vertex of $S_{y}$ must be adjacent to $c_{0}$. This would create the dominating set $\left\{c_{0}, x\right\}$ with at least three common neighbors, contradicting the maximality of $D$.

Also $\left|S_{x}\right|=1$ clearly creates the dominating set $\left\{c_{0}, y\right\}$ (as $x_{1}$ is adjacent to $c_{0}$ by the triangulation of $\vec{G}$ ) with at least three common neighbors (a vertex from $S_{y} \cap R_{0}$ by the triangulation of $\vec{G}, u$ and $c_{1}$ ), contradicting the maximality of $D$.
For $2 \leq\left|S_{x}\right| \leq 4, c_{0}$ (or $c_{1}$ ) can be adjacent to at most two vertices of $S_{y} \cap R_{0}$ since otherwise there would be one vertex $v \in S_{y} \cap R_{0}$ which would force $c_{0}$ (or $c_{1}$ ) to be adjacent to all vertices of $w \in S_{x}$ (in order to satisfy $d(v, w) \leq 2$ ) and create a dominating set $\left\{c_{0}, y\right\}$ that contradicts the maximality of $D$.
Also, not all vertices of $S_{x}$ can be adjacent to $c_{0}$ (or $c_{1}$ ), as otherwise $\left\{c_{o}, y\right\}$ (or $\left\{c_{1}, y\right\}$ ) would be a dominating set with at least three common neighbors ( $u, c_{1}$ (or $c_{0}$ ) and a vertex from $S_{y} \cap R_{0}$ ), contradicting the maximality of $D$.
Note that, by equation (1), we have

$$
\left|S_{y} \cap R_{0}\right| \geq 10-S_{x} .
$$

Assume $S_{x}=\left\{x_{1}, \ldots, x_{n}\right\}$, with the triangulation of $\vec{G}$ forcing the edges $c_{0} x_{1}, x_{1} x_{2}, \ldots$, $x_{n-1} x_{n}$ and $x_{n} c_{1}$ for $n \in\{2,3,4\}$.

For $\left|S_{x}\right|=2$, at most four vertices of $S_{y} \cap R_{0}$ can be adjacent to $c_{0}$ or $c_{1}$. Hence, there will be at least four vertices of $S_{y} \cap R_{0}$ each connected to $x$ by a 2-dipath through $x_{1}$ or $x_{2}$. Without loss of generality, $x_{1}$ will be adjacent to at least 2 vertices of $S_{y}$, and hence $\left\{x_{1}, y\right\}$ will be a dominating set contradicting the maximality of $D$.

For $\left|S_{x}\right|=3$, without loss of generality, assume that $x_{2}$ is adjacent to $c_{0}$. To satisfy $\bar{d}\left(x_{1}, v\right) \leq 2$ for all $v \in S_{y} \cap R_{0}$, at least four vertices of $S_{y}$ will be connected to $x_{1}$ by a 2-dipath through $x_{2}$ (as, according to previous discussions, at most two vertices of $S_{y}$ can be adjacent to $c_{0}$ ). This would create the dominating set $\left\{x_{2}, y\right\}$, contradicting the maximality of $D$.

For $\left|S_{x}\right|=4$ we have the edges $x_{2} c_{0}$ and $x_{3} c_{1}$, as otherwise at least three vertices of $S_{x}$ would be adjacent to either $c_{0}$ or $c_{1}$, which is not possible (because it forces all vertices of $S_{y}$ to be adjacent to $c_{0}$ or $\left.c_{1}\right)$. Now, each vertex $v \in S_{y} \cap R_{0}$ must be adjacent either to $c_{0}$ or to $x_{2}$ (to satisfy $\bar{d}\left(v, x_{1}\right) \leq 2$ ) and also either to $c_{1}$ or to $x_{3}$ (to satisfy $\bar{d}\left(v, x_{4}\right) \leq 2$ ), which is not possible due to the planarity of $\vec{G}$.
For $|C|=3,4,5$, by Lemma 4.4, each vertex of $S_{x}$ disagrees with each vertex of $S_{y} \cap R_{1}$ on $c_{0}$. We also have the edge $x_{1} c_{2}$ for some $x_{1} \in S_{x}$ by the triangulation of $\vec{G}$. By equation (1), we have

$$
|S| \geq(15-2-|C|)=13-|C| .
$$

Hence, $\left|S_{x}\right| \leq 2$ for $|C|=3,4$, as otherwise every vertex $u \in S_{y}$ would be adjacent to $c_{0}$, creating a dominating set $\left\{c_{0}, t\right\}$ with at least $(|C|+1)$ common neighbors $S_{t} \cup\left\{c_{1}\right\}$ for some $t \in\{x, y\}$, contradicting the maximality of $D$. For $|C|=5$, since all the vertices in $S_{x} \cap R_{0}$ agree with each other on $x$ (as they all must disagree with $c_{2}$ on $x$ ) and on $c_{0}$ (as they all disagree with vertices of $S_{y} \cap R_{1}$ on $c_{0}$ ), by Lemma 4.5, we have $\left|S_{x} \cap R_{0}\right| \leq 3$. But if $\left|S_{x} \cap R_{0}\right|=3$ then every vertex of $S_{y}$ will be adjacent to $c_{0}$, creating a dominating set $\left\{c_{0}, y\right\}$ with at least six common neighbors $S_{y} \cup\left\{c_{1}\right\}$, contradicting the maximality of D.

Hence $\left|S_{x}\right| \leq 2$ for $|C|=3,4,5$.

Now for $|C|=3$, we can assume that $x$ and $y$ are non-adjacent as otherwise $\left\{c_{0}, y\right\}$ would be a dominating set with at least four common neighbors ( $x, c_{1}$ and two other vertices each from the sets $S_{y} \cap R_{0}, S_{y} \cap R_{1}$ by triangulation) contradicting the maximality of $D$. Hence triangulation will imply the edge $c_{1} c_{2}$. Now for $\left|S_{x}\right| \leq 2$, either $\left\{c_{0}, c_{2}\right\}$ is a dominating set with at least four common neighbors $\left\{x, y, c_{1}, x_{1}\right\}$ contradicting the maximality of $D$ or $x_{1}$ is adjacent to at least two vertices $y_{1}, y_{2} \in S_{y} \cap R_{0}$ creating a dominating set $\left\{x_{1}, y\right\}$ (the other vertex in $S_{x}$ must be adjacent to $x_{1}$ by triangulation) with at least four common neighbors $\left\{y_{1}, y_{2}, c_{0}, c_{2}\right\}$ contradicting the maximality of $D$.
For $|C|=4$ we have $\left|S_{y} \cap R_{1}\right| \leq 2$ as otherwise we will have the dominating set $\left\{c_{0}, y\right\}$ with at least five common neighbors ( $c_{1}$, vertices of $S_{y} \cap R_{1}$ and one vertex of $S_{y} \cap R_{0}$ by the triangulation of $\vec{G}$ ), contradicting the maximality of $D$. By equation (1), we have

$$
\begin{aligned}
\left|S_{y} \cap R_{0}\right| & \geq\left(15-|D|-|C|-\left|S_{x}\right|-\left|S_{y} \cap R_{1}\right|\right) \\
& \geq(15-2-4-2-2)=5 .
\end{aligned}
$$

Now, at most two vertices of $S_{y} \cap R_{0}$ can be adjacent to $c_{0}$ as otherwise $\left\{c_{0}, y\right\}$ would be a dominating set with at least five common neighbors ( $c_{1}$, vertices of $S_{y} \cap R_{0}$ and one vertex of $S_{y} \cap R_{1}$ by the triangulation of $\vec{G}$ ), contradicting the maximality of $D$.
Also, by the triangulation of $\vec{G}$, in $R_{3}$ we have either the edge $x y$ or the edge $c_{2} c_{3}$. But, if we have the edge $x y$, then $\left|S_{y} \cap R_{1}\right|=1$ as otherwise the dominating set $\left\{c_{0}, y\right\}$ would contradict the maximality of $D$. Hence, by the triangulation of $\vec{G}$, and in order to have weak directed distance at most 2 from the vertices of $S_{x} \cup\{x\}$, each vertex of $S_{y} \cap R_{0}$ will be adjacent either to $c_{3}$ or to $x_{1}$. This will create a dominating set $\left\{x_{1}, y\right\}$ or $\left\{c_{3}, y\right\}$ that contradicts the maximality of $D$. Hence, we do not have the edge $x y$ (not even in other regions) and we thus have the edge $c_{2} c_{3}$.
For $\left|S_{x}\right| \leq 2$, the vertices of $S_{y} \cap R_{0}$ will be adjacent to either $c_{3}, c_{0}$ or $x_{1}$ in order to have weak directed distance at most 2 from $x$. But then, the triangulation of $\vec{G}$ will force at least two vertices of $S_{y} \cap R_{0}$ to be common neighbors of $c_{3}$ and $x_{1}$, or to have the edge $c_{0} c_{3}$. It is not difficult to check, casewise (drawing a picture for individual cases will help in understanding the scenario), that one of the sets $\left\{c_{0}, y\right\},\left\{c_{3}, y\right\}$ or $\left\{x_{1}, y\right\}$ would then be a dominating set contradicting the maximality of $D$.
For $|C|=5$, by Lemma 4.3, each vertex of $S_{y} \cap R_{i}$ must disagree with $c_{i+2}$ on $y$. If the vertices of $S_{y} \cap R_{0}$ and the vertices of $S_{y} \cap R_{1}$ agree with each other on $y$, then they must disagree with each other on $c_{0}$, which implies $\left|S_{y} \cap R_{i}\right| \leq 3$ for all $i \in\{0,1\}$. If the vertices of $S_{y} \cap R_{0}$ and the vertices of $S_{y} \cap R_{1}$ disagree with each other on $y$, then the vertices of $S_{y} \cap R_{i}$ must agree with $c_{3-i}$ on $y$. In that case, by Lemma 4.4, each vertex of $S_{y} \cap R_{i}$ must be connected to $c_{3-i}$ by a 2-dipath through $c_{4-3 i}$, which implies $\left|S_{y} \cap R_{i}\right| \leq 3$ for all $i \in\{0,1\}$.
Assume $\left|S_{y} \cap R_{0}\right|=3$ and $\left|S_{y} \cap R_{1}\right|=3$. Then, each vertex of $S_{y} \cap R_{i}$ must disagree with both $c_{i+2}$ and $c_{i+3}$ on $y$. This would imply that the vertices of $S_{y} \cap R_{0}$ and the vertices of $S_{y} \cap R_{1}$ disagree with each other on $c_{0}$. Now, there would be no way to have weak directed distance at most 2 between a vertex of $S_{x}$ and all the six vertices of $S_{y}$.
Hence we must have $\left|S_{y}\right| \leq 5$. Then, by equation (1), we have

$$
15 \leq|\vec{G}| \leq 2+5+(2+5)=14
$$

This is a contradiction, and this concludes this particular subcase.
(ii) Assume the three non-empty sets are $S_{x} \cap R_{1}, S_{y} \cap R_{0}$ and $S_{y} \cap R_{2}$ (only possible for $|C| \geq 3$ ). By Lemma 4.4, we have $S_{x}=\left\{x_{1}\right\}$ and the fact that each vertex of $S_{y} \cap R_{i}$ disagrees with $c_{i^{2} / 4}$ on $x_{1}$ for $i \in\{0,2\}$. Moreover, the triangulation of $\vec{G}$ implies the edges $x_{1} c_{0}, x_{1} c_{1}, c_{k-1} c_{0}, c_{0} c_{1}$ and $c_{1} c_{2}$.
For $|C|=3,\left\{c_{0}, c_{1}\right\}$ is a dominating set with at least four common neighbors $\left\{x, y, c_{2}, x_{1}\right\}$, contradicting the maximality of $D$. For $|C|=4,5$, every vertex of $S_{y} \cap R_{0}$ disagrees with every vertex of $S_{y} \cap R_{2}$ on $y$. Hence, by Lemma 4.5, we have $\left|S_{y} \cap R_{i}\right| \leq 3$ for all $i \in\{0,2\}$. By equation (1), we then have

$$
\begin{aligned}
15 \leq|\vec{G}| & =|D|+|C|+|S| \\
& \leq[2+5+(1+3+3)]=14 .
\end{aligned}
$$

This is a contradiction.

Step 4. Hence, at most two sets out of the $2 k$ sets $S_{t} \cap R_{i}$ can be non-empty, where $t \in\{x, y\}$ and $i \in\{0,1, \ldots, k-1\}$.

Assume that exactly two sets out of the sets $S_{t} \cap R_{i}$ are non-empty, where $t \in\{x, y\}$ and $i \in\{0, \ldots, k-1\}$, yet there are two non-empty regions. Without loss of generality, assume that the two non-empty sets are $S_{x} \cap R_{0}$ and $S_{y} \cap R_{1}$.

The triangulation of $\vec{G}$ would force $x$ and $y$ to have a common neighbor other than $c_{0}$ and $c_{1}$ for $|C|=2$ which is a contradiction. For $|C|=3,4,5$ the triangulation of $\vec{G}$ forces the edges $c_{k-1} c_{0}$ and $c_{0} c_{1}$. By Lemma 4.4, we know that each vertex of $S$ is adjacent to $c_{0}$. By equation (1), we have

$$
|S| \geq(15-2-5)=8
$$

Hence, without loss of generality, we may assume $\left|S_{x}\right| \geq 4$. But then $\left\{c_{0}, x\right\}$ would be a dominating set with at least six common neighbors $S_{x} \cup\left\{c_{k-1}, c_{1}\right\}$, contradicting the maximality of $D$.

This concludes the proof.
The lemma proved above was one of the key steps to prove the theorem. Now we will improve the lower bound on $|C|$.

Lemma 4.8. $|C| \geq 6$.
Proof. For $|C|=2,3,4,5$, without loss of generality by Lemma 4.7, we may assume $R_{1}$ to be the only non-empty region. The triangulation of $\vec{G}$ will then force the configuration depicted in Figure 8 as a subgraph of $\operatorname{und}(\vec{G})$, where $C=\left\{c_{o}, \ldots, c_{k-1}\right\}, S_{x}=\left\{x_{1}, \ldots, x_{n_{x}}\right\}$ and $S_{y}=$ $\left\{y_{1}, \ldots, y_{n_{y}}\right\}$. Without loss of generality, we may assume

$$
\left|S_{y}\right|=n_{y} \geq n_{x}=\left|S_{x}\right| .
$$

Then, by equation (1), we have

$$
\begin{equation*}
n_{y}=\left|S_{y}\right| \geq\left(15-2-|C|-\left|S_{x}\right|\right)=13-|C|-\left|S_{x}\right| \tag{2}
\end{equation*}
$$



Figure 8: The only non-empty region is $R_{1}$

First of all, assume $n_{x}=0$. Then $x$ is not adjacent to $y$, as otherwise $y$ would dominate the whole graph. So we have the edges $c_{0} c_{1}, c_{1} c_{2}, \ldots, c_{k-1} c_{0}$ by the triangulation of $\vec{G}$. Then, by equation 2, we have

$$
\left|S_{y}\right| \geq 13-5=8
$$

Now, to have $\bar{d}\left(x, y_{i}\right) \leq 2$, every $y_{i}$ must be connected to $x$ by a 2-dipath with internal vertex either $c_{0}$ or $c_{1}$. Hence, at least four vertices of $S_{y}$ must be adjacent to either $c_{0}{ }_{0} c_{1}$. Note that $c_{0}$ is also adjacent to $c_{k-1}, c_{1}$ and that $c_{1}$ is also adjacent to $c_{0}, c_{2}$. But then, the dominating set $\left\{c_{0}, y\right\}$ or $\left\{c_{1}, y\right\}$ will contradict the maximality of $D$. Hence $n_{x} \geq 1$.

The proof will now directly follow from the four claims below.
Claim 1: $|C|=5$ is not possible.
Proof of claim 1: Assume that $|C|=5$. Then, by equation 2, we have

$$
\left|S_{y}\right| \geq 13-5-n_{x}=8-n_{x} .
$$

Therefore, as $n_{y} \geq n_{x}$, we have $n_{y} \geq 4$. Now, every vertex of $S_{y}$ disagrees with $c_{3}$ on $y$. They must also disagree with $y$ on $c_{2}$, as otherwise all of them would be connected to $c_{2}$ by 2 -dipaths with internal vertex $c_{1}$, which would imply $\bar{d}\left(y_{1}, y_{4}\right)>2$. For similar reasons, the vertices of $S_{y}$ must disagree with $c_{4}$ on $y$.

Moreover, the edge $c_{0} c_{1}$ does not exist since it would force each vertex of $S_{y}$ to be connected to vertices of $S_{x}$ by 2 -dipaths with internal vertex either $c_{0}$ or $c_{1}$. In fact, for $n_{x} \geq 2$, as not all vertices of $S_{x}$ can be adjacent to both $c_{0}$ and $c_{1}$, every vertex of $S_{y}$ would be connected to the vertices of $S_{x}$ by 2-dipaths with internal vertex being exactly one of $c_{0}, c_{1}$, thus implying $\bar{d}\left(y_{1}, y_{4}\right)>2$. For $n_{x}=1$, as $n_{y} \geq 7$, at least four vertices of $S_{y}$ would be connected to the vertices of $S_{x}$ by 2-dipaths with internal vertex being exactly one of $c_{0}, c_{1}$, implying $\bar{d}\left(y_{i}, y_{i+3}\right)>2$ for some $i \in\left\{1,2, \ldots, n_{y}\right\}$. Hence, the edge $c_{0} c_{1}$ does not exist.

Also, if we have the edge $y_{1} y_{4}$ and, without loss of generality, the edge $y_{1} y_{3}$ by the triangulation of $\vec{G}$, then every vertex of $S_{x}$ must be connected to $y_{2}$ by 2-dipaths with internal vertex $y_{1}$. In this case, $\left\{y_{1}, y\right\}$ is a dominating set with at least $n_{y}$ common neighbors ( $c_{0}$ and $n_{y}-1$ common neighbors from $S_{y}$ ). Hence, to avoid a contradiction with the maximality of $D$, we must have $n_{y} \leq 5$. We must also have $n_{x} \geq 3$. But then, as every vertex of $S_{x}$ agree with each other on $y_{1}$ and on $x$ (as they all disagree with $c_{3}$ on $x$ ), they must all disagree with $c_{1}$ and $c_{4}$
on $x$ to have weak directed distance at most 2 with them. Also the vertices of $S_{y}$ must disagree with $c_{1}$ and $c_{4}$ on $y$ to have directed distance at most 2 with them. So, $c_{1}$ and $c_{4}$ agrees with each other on both $x$ and $y$, Therefore, to have weak directed distance at most 2 between $c_{1}$ and $c_{4}$ we must have the 2 -dipath connecting $c_{4}$ and $c_{1}$ with internal vertex $c_{0}$. But this is a contradiction as we can not have the edge $c_{0} c_{1}$.

Similarly, we cannot have the edge $y_{1} y_{3}$ also. Therefore, $y_{1}$ and $y_{4}$ must be connected by a 2-dipath with an internal vertex $x_{j}$ from $S_{x}$ for some $j \in\left\{1,2, \ldots, n_{x}\right\}$. As we cannot have the edge $y_{1} y_{4}$, this implies that every vertex of $S \backslash\left\{x_{j}\right\}$ is adjacent to $x_{j}$ to be at weak directed distance at most 2 from each other. We can then reach a contradiction, exactly as in the case described in the paragraph above.

This proves the claim.
Claim 2: $|C|=4$ is not possible.
Proof of claim 2: Assume that $|C|=4$. Then, by equation 2, we have

$$
\left|S_{y}\right| \geq 13-4-n_{x}=9-n_{x} .
$$

Therefore, as $n_{y} \geq n_{x}$, we have $n_{y} \geq 5$.
We now show that every vertex of $S_{y}$ disagrees with $c_{2}$ and $c_{3}$ on $y$. First note that no vertex can agree with both $c_{2}$ and $c_{3}$ on $y$ as otherwise it would be adjacent to both $c_{0}$ and $c_{1}$, which is impossible since $n_{y} \geq 5$. So, if the claim is not true, then some vertices of $S_{y}$ will agree with $c_{2}$ on $y$ and the other vertices of $S_{y}$ will agree with $c_{3}$ on $y$.

Also at most three vertices of $S_{y}$ can agree with $c_{2}$ (or $c_{3}$ ) on $y$. So, $n_{y} \leq 6$. Hence, $n_{x} \geq 3$.
Now, three vertices of $S_{y}$ agree on $y$ with, say, $c_{2}$. Then they will all disagree with $c_{2}$ on $c_{1}$ and every vertex (there are at least three such vertices) of $S_{x}$ will disagree with those three vertices on $c_{1}$. Then, to have weak directed distance at most 2 between the vertices of $S_{x}$, the other vertices (there are at least two such vertices) of $S_{y}$ should be adjacent to $c_{1}$, which is not possible as they are already connected to $c_{3}$ with 2 -dipaths with internal vertex $c_{0}$.

The rest of the proof is similar to the proof of Claim 1. Using similar arguments, it is possible to show that the edge $c_{0} c_{1}$ does not exist, that the edge $y_{1} y_{4}$ does not exist and that it is not possible to have a 2-dipath with internal vertex from $S_{x}$ connecting $y_{1}$ and $y_{4}$.

Claim 3: $|C|=3$ is not possible.
Proof of claim 3: Assume that $|C|=3$. Then, by equation 2, we have

$$
\left|S_{y}\right| \geq 13-3-n_{x}=10-n_{x} .
$$

Therefore, as $n_{y} \geq n_{x}$, we have $n_{y} \geq 5$.
First note that it is not possible to have the edge $c_{0} c_{1}$, as this will force some three vertices of $S_{y}$ to be connected to vertices of $S_{x}$ by 2-dipaths with internal vertex $c_{0}$ (or $c_{1}$ ), making $\left\{c_{0}, y\right\}$ (or $\left\{c_{1}, y\right\}$ ) a dominating set that contradicts the maximality of $D$.

For $n_{y} \geq 7$, there are at least four vertices in $S_{y}$ that agree with each other on $y$. We need to have weak directed distance at most 2 between them. Let those four vertices be $y_{i}, y_{j}, y_{k}, y_{l}$ with $i>j>k>l$.

Now, assume that we have the edge $y_{i} y_{l}$. Then, every vertex of $S_{x}$ will be adjacent to either $y_{i}$ or $y_{l}$. Without loss of generality, assume that every vertex of $S_{x}$ is adjacent to $y_{i}$. But then, $\left\{y_{i}, y\right\}$ would be a dominating set with at least four common neighbors, contradicting the maximality of $D$. Hence $n_{y} \leq 6$ and, therefore, we must have $n_{x} \geq 4$.

For $n_{y}=5,6$, one can show that these cases are not possible without creating a dominating set that contradicts the maximality of $D$. If one just tries to have weak directed distance at most 2 between the vertices of $S$, the proof will follow. The proof of this part is also similar to the ones done before and, though a bit tedious, is not difficult to check.

Claim 4: $|C|=2$ is not possible.
Proof of claim 4: Assume that $|C|=2$. Then, by equation 2, we have

$$
\left|S_{y}\right| \geq 13-2-n_{x}=11-n_{x}
$$

Therefore, as $n_{y} \geq n_{x}$, we have $n_{y} \geq 6$.
This is actually the easiest of the four claims. The case $n_{y} \geq 7$ can be argued as in the previous proof. For $n_{y}=6$, we must have $n_{x} \geq 5$. If one just tries to have directed distance at most 2 between the vertices of $S$, the proof will follow. The proof of this part is also similar to the ones done before and, though a bit tedious, is again not difficult to check.

This completes the proof of the lemma.

Up to now, we have proved that the value of $|C|$ is at least 6 . This is an answer to our question "how small $|C|$ can be?". We will now consider the question "How big $|C|$ can be?" and try to provide upper bounds for the value of $|C|$. The following lemma will help us to do so.

Lemma 4.9. If $|C| \geq 6$, then the following holds:
(a) $\left|C^{\alpha \beta}\right| \leq 3,\left|C_{t}^{\alpha}\right| \leq 6,|C| \leq 12$. Moreover, if $\left|C^{\alpha \beta}\right|=3$, then $\vec{G}\left[C^{\alpha \beta}\right]$ is a 2-dipath.
(b) $\left|C_{t}^{\alpha}\right| \geq 5$ (respectively $4,3,2,1,0$ ) implies $\left|S_{t}^{\alpha}\right| \leq 0$ (respectively $1,3,4,5,6$ ).

Proof. (a) If $\left|C^{\alpha \beta}\right| \geq 4$, then there will be two vertices $u, v \in C^{\alpha \beta}$ with $d(u, v)>2$, which is a contradiction. Hence we have the first inequality, which implies the other two.

If $\left|C^{\alpha \beta}\right|=3$, then the only way to connect the two non-adjacent vertices $u, v$ of $C^{\alpha \beta}$ is to connect them with a 2-dipath through the other vertex (other than $u, v$ ) of $C^{\alpha \beta}$.
(b) Lemma4.3(b) implies that if all the elements of $C_{t}^{\alpha}$ do not belong to the set of four boundary points of any three consecutive regions (like $R, R^{1}, R^{2}$ in Lemma 4.4), then $\left|S_{t}^{\alpha}\right|=0$. Hence, we have $\left|C_{t}^{\alpha}\right| \geq 5$ implies $\left|S_{t}^{\alpha}\right| \leq 0$.

By Lemma 4.4, if all the elements of $C_{t}^{\alpha}$ belong to the set of four boundary points $c^{1}, c^{2}, \overline{c^{1}}, \overline{c^{2}}$ of three consecutive regions $R, R^{1}, R^{2}$ (like in Lemma 4.4) and contains both $\overline{c^{1}}, \overline{c^{2}}$, then $\left|S_{t}^{\alpha}\right| \leq 1$. Also $S_{t}^{\alpha} \subseteq R$ by Lemma 4.4. Hence we have

$$
\left|C_{t}^{\alpha}\right| \geq 4 \text { implies }\left|S_{t}^{\alpha}\right| \leq 1
$$

Assume now that all the elements of $C_{t}^{\alpha}$ belong to the set of three boundary points $c^{1}, c^{2}, \overline{c^{1}}$ of two adjacent regions $R, R^{1}$ (like in Lemma 4.4) and contain both $\overline{c^{1}}, c^{2}$. Then, by Lemma 4.3, $v \in S_{t}^{\alpha}$ implies $v$ is in $R$ or $R^{1}$.

Now, if both $S_{t}^{\alpha} \cap R$ and $S_{t}^{\alpha} \cap R^{1}$ are non-empty, then each vertex of $\left(S_{t}^{\alpha} \cap R\right) \cup\left\{c^{2}\right\}$ disagrees with each vertex of $\left(S_{t}^{\alpha} \cap R^{1}\right) \cup\left\{\overline{c^{1}}\right\}$ on $c^{1}$ (by Lemma 4.4).

Hence, by Lemma 4.5, we have

$$
\left|\left(S_{t}^{\alpha} \cap R\right) \cup\left\{\overline{c^{1}}\right\}\right|,\left|\left(S_{t}^{\alpha} \cap R^{1}\right) \cup\left\{c^{2}\right\}\right| \leq 3
$$

This clearly implies

$$
\left|S_{t}^{\alpha} \cap R\right|,\left|S_{t}^{\alpha} \cap R^{1}\right| \leq 2 \text { and }\left|S_{t}^{\alpha}\right| \leq 4
$$

Suppose now that we have $\left|S_{t}^{\alpha}\right|=4$ and, hence, also $\left|S_{t}^{\alpha} \cap R\right|,\left|S_{t}^{\alpha} \cap R^{1}\right|=2$. Then, $S_{t^{\prime}}=\emptyset$ as the only way for a vertex of $S_{t^{\prime}}$ to be at weak directed distance at most 2 from every vertex of $S_{t}$ is by being connected by a 2 -dipath with internal vertex $c_{1}$, which is impossible as the vertices of $S_{t}^{\alpha} \cap R$ disagree with the vertices of $S_{t}^{\alpha} \cap R^{1}$ on $c_{1}$.

In fact, for the same reason, it is impossible to have weak directed distance at most 2 between all the vertices of $S_{t}$ and $t^{\prime}$ unless we have the edge $t t^{\prime}$ (that is the edge $x y$ ). But then, the edge $t t^{\prime}$ makes $t$ a vertex that dominates the whole graph, contradicting the domination number of the graph being 2. Therefore, it is not possible to have $\left|S_{t}^{\alpha}\right|=4$. Hence, we have $\left|S_{t}^{\alpha}\right| \leq 3$ in this case.

Also, if one of $S_{t}^{\alpha} \cap R$ and $S_{t}^{\alpha} \cap R^{1}$ is empty then we must have $\left|S_{t}^{\alpha}\right| \leq 3$ by Lemmas 4.4 and 4.5. Hence, we have

$$
\left|C_{t}^{\alpha}\right| \geq 3 \text { implies }\left|S_{t}^{\alpha}\right| \leq 3 .
$$

Let $R, R^{1}, R^{2}, c^{1}, c^{2}, \overline{c^{1}}, \overline{c^{2}}$ be as in Lemma 4.4 and assume $C_{t}^{\alpha}=\left\{c^{1}, c^{2}\right\}$. By Lemma 4.3, $v \in S_{t}^{\alpha}$ implies $v$ is in $R, R^{1}$ or $R^{2}$, and also that both $S_{t}^{\alpha} \cap R^{1}$ and $S_{t}^{\alpha} \cap R^{2}$ cannot be non-empty. Hence, without loss of generality, assume $S_{t}^{\alpha} \cap R^{2}=\emptyset$.

By Lemma 4.4, the vertices of $S_{t}^{\alpha} \cap R^{1}$ disagree with the vertices of $\left(S_{t}^{\alpha} \cap R\right) \cup\left\{c^{2}\right\}$ on $c^{1}$. Hence, by Lemma 4.5, we have

$$
\left|S_{t}^{\alpha} \cap R^{1}\right|,\left|\left(S_{t}^{\alpha} \cap R\right) \cup\left\{c^{2}\right\}\right| \leq 3
$$

This implies $\left|S_{t}^{\alpha}\right| \leq 5$.
Now, if $S_{t}^{\alpha} \cap R^{1}=\emptyset$ then $S_{t}^{\alpha}=S_{t}^{\alpha} \cap R$. Let $\left|S_{t}^{\alpha} \cap R\right| \geq 6$ and consider the induced graph $\vec{O}=\vec{G}\left[(S \cap R) \cup\left\{c^{1}, c^{2}\right\}\right]$. In this graph, the vertices of $\left(S_{t}^{\alpha} \cap R\right) \cup\left\{c^{1}, c^{2}\right\}$ are at weak directed distance at most 2 from each other. Hence, $\chi_{o}(\vec{O}) \geq 8$. But this is a contradiction since $\vec{O}$ is an outerplanar graph and every outerplanar graph has an oriented 7 -coloring [12]. Hence,

$$
\left|C_{t}^{\alpha}\right| \geq 2 \text { implies }\left|S_{t}^{\alpha}\right| \leq 5 .
$$

Suppose now that we have $\left|S_{t}^{\alpha}\right|=5$. Then we must have $S_{t^{\prime}}=\emptyset$ as otherwise it is not possible to have weak directed distance at most 2 between the vertices of $S$.

We also do not have the edge $x y$ as it would contradict the domination number of the graph being $2\left(t\right.$ will dominate the graph). So, by the triangulation of $\vec{G}$, we have the edges $c^{1} c^{2}$ and $c^{\overline{1}} c^{1}$. Hence, each vertex of $S_{t}$ must be connected to $t^{\prime}$ with a 2 -dipath with internal vertices from $\left\{c^{\overline{1}}, c^{1}, c^{2}\right\}$. But then, it will not be possible to have weak directed distance at most 2 between the five vertices of $S_{t}$.

Hence,

$$
\left|C_{t}^{\alpha}\right| \geq 2 \text { implies }\left|S_{t}^{\alpha}\right| \leq 4 .
$$

In general, $S_{t}^{\alpha}$ is contained in two distinct adjacent regions by Lemma 4.3. Without loss of generality, assume $S_{t}^{\alpha} \subseteq R_{1} \cup R_{2}$. If both $S_{t}^{\alpha} \cap R_{1}$ and $S_{t}^{\alpha} \cap R_{2}$ are non-empty then, by Lemma 4.4, we know that the vertices of $S_{t}^{\alpha} \cap R_{1}$ disagree with the vertices of $S_{t}^{\alpha} \cap R_{2}$ on $c_{1}$. Hence, $\left|S_{t}^{\alpha} \cap R_{1}\right|,\left|S_{t}^{\alpha} \cap R_{2}\right| \leq 3$, which implies $\left|S_{t}^{\alpha}\right| \leq 6$.

Assume now that only one of the two sets $S_{t}^{\alpha} \cap R_{1}$ and $S_{t}^{\alpha} \cap R_{2}$ is non-empty. Without loss of generality, assume $S_{t}^{\alpha} \cap R_{1} \neq \emptyset$. If $c_{0}, c_{1} \notin C_{t}^{\alpha}$ and $\left|C_{t}^{\alpha}\right|=1$ then we have $\left|S_{t}^{\alpha} \cap R_{1}\right| \leq 3$ by Lemmas 4.4 and 4.5. In the induced outerplanar graph $\vec{O}=\vec{G}\left[\left(S \cap R_{1}\right) \cup\left\{c_{1}, c_{2}\right\}\right]$, the vertices of $S_{t}^{\alpha} \cup\left(c_{t}^{\alpha} \cap\left\{c_{1}, c_{2}\right\}\right)$ are at weak directed distance at most 2 from each other.

Hence, $7 \geq \chi_{o}(\vec{O}) \geq\left|S_{t}^{\alpha} \cup\left(c_{t}^{\alpha} \cap\left\{c_{1}, c_{2}\right\}\right)\right|$. Therefore,

$$
\left|C_{t}^{\alpha}\right| \geq 1 \text { (respectively } 0 \text { ) implies }\left|S_{t}^{\alpha}\right| \leq 6 \text { (respectively } 7 \text { ). }
$$

Now, when both the equalities hold, we must have $S_{t^{\prime}}=\emptyset$ as otherwise $C \cup S_{t} \cup S_{t^{\prime}}$ would contain an oriented outerplanar graph with oriented chromatic number at least 8 , which is not possible, in order to have all the vertices of $S$ at weak directed distance at most 2 from each other.

Now, $S_{t^{\prime}}=\emptyset$ would imply that the edge $x y$ is not there, as otherwise $t$ would dominate the whole graph. Hence, each vertex of $S_{t}^{\alpha}$ must be connected to $t^{\prime}$ by a 2-dipath with internal vertex $c_{i}$ for some $i \in\{0,1,2\}$. But this would force $\left|S_{t}^{\alpha} \cup C_{t}^{\alpha}\right| \leq 6$ as otherwise the vertices of $S_{t}^{\alpha} \cup C_{t}^{\alpha}$ would no longer be at weak directed distance at most 2 from each other.

Hence,

$$
\left|C_{t}^{\alpha}\right| \geq 1(\text { respectively } 0) \text { implies }\left|S_{t}^{\alpha}\right| \leq 5(\text { respectively } 6)
$$

and we are done.
We now prove that the value of $|C|$ can be at most 5 , which contradicts our previously proven lower bound on $|C|$. That actually proves Lemma 4.2.

Lemma 4.10. $|C| \leq 5$.
Proof. Without loss of generality, we can suppose $\left|C_{x}^{\alpha}\right| \geq\left|C_{y}^{\beta}\right| \geq\left|C_{y}^{\bar{\beta}}\right| \geq\left|C_{x}^{\bar{\alpha}}\right|$ (the last inequality is forced). We know that $|C| \leq 12$ and that $\left|C_{x}^{\alpha}\right| \leq 6$ (Lemma 4.9(a)). Therefore, it is enough to show that $|S| \leq 12-|C|$ for all possible values of $\left(|C|,\left|C_{x}^{\alpha}\right|,\left|C_{y}^{\beta}\right|\right)$, since it contradicts (1).

For $\left(|C|,\left|C_{x}^{\alpha}\right|,\left|C_{y}^{\beta}\right|\right)=(12,6,6),(11,6,6),(10,6,6),(10,6,5),(10,5,5),(9,5,5),(8,4,4)$ we have $|S| \leq 12-|C|$, using Lemma 4.9(b).

For $\left(|C|,\left|C_{x}^{\alpha}\right|,\left|C_{y}^{\beta}\right|\right)=(8,6,6),(7,6,6),(7,6,5),(6,6,6),(6,6,5),(6,6,4),(6,5,5)$ we are forced to have

$$
\left|C^{\alpha \beta}\right|>3
$$

This is a contradiction by Lemma 4.9(a).
So, $\left(|C|,\left|C_{x}^{\alpha}\right|,\left|C_{y}^{\beta}\right|\right) \neq(12,6,6),(11,6,6),(10,6,6),(10,6,5),(10,5,5),(9,5,5),(8,4,4)$, $(8,6,6),(7,6,6),(7,6,5),(6,6,6),(6,6,5),(6,6,4),(6,5,5)$.

We will be done if we prove that $\left(|C|,\left|C_{x}^{\alpha}\right|,\left|C_{y}^{\beta}\right|\right)$ cannot take the other possible values also. That leaves us checking a lot of cases. We will check just a few cases and observe that the other cases can be checked using similar arguments.
Case 1: Assume $\left(|C|,\left|C_{x}^{\alpha}\right|,\left|C_{y}^{\beta}\right|\right)=(9,6,6)$.
We are then forced to have $\left|C^{\alpha \beta}\right|=\left|C^{\alpha \bar{\beta}}\right|=\left|C^{\bar{\alpha} \beta}\right|=3$ in order to satisfy the first inequality of Lemma 4.9 (a). So, $\vec{G}\left[C^{\alpha \beta}\right], \vec{G}\left[C^{\alpha^{\bar{\beta}}}\right]$ and $\vec{G}\left[C^{\bar{\alpha} \beta}\right]$ are 2-dipaths by Lemma 4.9 (a). Without loss of generality, we can assume $C^{\alpha \bar{\beta}}=\left\{c_{0}, c_{1}, c_{2}\right\}$ and $C^{\bar{\alpha} \beta}=\left\{c_{3}, c_{4}, c_{5}\right\}$. Hence, by Lemma 4.3, we have $u \in R_{1} \cup R_{2}$ and $v \in R_{4} \cup R_{5}$ for any $(u, v) \in S_{y}^{\bar{\beta}} \times S_{x}^{\bar{\alpha}}$. Now, by Lemma 4.3, either $S_{y}^{\bar{\beta}}$ or $S_{x}^{\bar{\alpha}}$ is empty. Without loss of generality, assume $S_{y}^{\bar{\beta}}=\emptyset$. Therefore, we have $|S|=\left|S_{x}\right|=\left|S_{x}^{\bar{\alpha}}\right| \leq 3$ (by Lemma 4.9(b)). So this case is not possible.


Figure 9: Planar targets with girth at least 4

Case 2: Assume $\left(|C|,\left|C_{x}^{\alpha}\right|,\left|C_{y}^{\beta}\right|\right)=(7,6,4)$.
So, without loss of generality, we can assume that $\vec{G}\left[C^{\alpha \beta}\right]$ and $\vec{G}\left[C^{\alpha \bar{\beta}}\right]$ are 2-dipaths, and $C^{\alpha \beta}=\left\{c_{0}, c_{1}, c_{2}\right\}, C^{\alpha \bar{\beta}}=\left\{c_{3}, c_{4}, c_{5}\right\}$ and $C^{\bar{\alpha} \beta}=\left\{c_{6}\right\}$.

By Lemma 4.9, we have $\left|S_{x}\right| \leq 5$ and $\left|S_{y}\right| \leq 3+1=4$. So we are done if either $S_{x}=\emptyset$ or $S_{y}=\emptyset$.

So assume both $S_{x}$ and $S_{y}$ are non-empty. First assume that $S_{y}^{\beta} \neq \emptyset$. Then, by Lemma 4.3, we have $S_{y}^{\beta} \subseteq R_{5}, S_{x}^{\bar{\alpha}} \subseteq R_{5} \cup R_{6}$ and hence $S_{y}^{\bar{\beta}}=\emptyset$. By Lemma4.4, the vertices of $S_{y}^{\beta}$ and the vertices of $S_{x}^{\bar{\alpha}} \cap R_{5}$ must disagree with $c_{6}$ on $c_{5}$ while disagreeing with each other on $c_{5}$, which is not possible. Hence, $S_{x}^{\bar{\alpha}} \cap R_{5}=\emptyset$. Also, $\left|S_{x}^{\bar{\alpha}} \cap R_{6}\right| \leq 3$ as they all disagree on $c_{5}$ with the vertices of $S_{y}^{\beta}$. Hence, $|S| \leq 4$ when $S_{y}^{\beta} \neq \emptyset$.

Now assume $S_{y}^{\beta}=\emptyset$ hence $S_{y}^{\bar{\beta}} \neq \emptyset$. Then, by Lemma4.3, we have $S_{y}^{\bar{\beta}} \subseteq R_{1} \cup R_{2}, S_{x}^{\bar{\alpha}} \subseteq R_{0} \cup R_{1}$ and hence $S_{y}^{\beta}=\emptyset$. Assume $S_{y}^{\bar{\beta}} \cap R_{2}=\emptyset$, as otherwise the vertices of $S_{x}^{\bar{\alpha}}$ would be adjacent to both $c_{0}$ and $c_{1}$ (to be connected to $c_{6}$ and to vertices of $S_{y}^{\bar{\beta}} \cap R_{2}$ by a 2-dipath), implying $\left|S_{x}^{\bar{\alpha}}\right| \leq 1$, implying $|S| \leq 5$. If $S_{x}^{\bar{\alpha}} \cap R_{0} \neq \emptyset$ then $\left|S_{y}^{\bar{\beta}} \cap R_{1}\right|=1,\left|S_{y}^{\bar{\alpha}} \cap R_{1}\right| \leq 1$ and $\left|S_{y}^{\bar{\alpha}} \cap R_{0}\right| \leq 3$, by Lemma 4.4, and hence $|S| \leq 5$. If $S_{x}^{\bar{\alpha}} \cap R_{0}=\emptyset$ then $\left|S_{y}^{\bar{\beta}} \cap R_{1}\right| \leq 2,\left|S_{y}^{\bar{\alpha}} \cap R_{1}\right| \leq 3$ and hence $|S| \leq 5$. So also this case is not possible.

Similarly one can handle the remaining cases.
From the above lemmas, we get that every planar oclique of order at least 15 is dominated by a single vertex. Moreover, we also proved that a planar oclique dominated by one vertex can have order at most 15. Hence, there is no planar oclique of order more than 15 . We also proved that every oclique of order 15 must contain the planar oclique depicted in Figure 3 as a spanning subgraph.

This concludes the proof of Theorem 1.7.

## 5 Proof of Theorem 1.8

(a) The proof directly follows from Theorem 1.7
(b) In 1975, Plesník [8] characterized and listed all triangle-free planar graphs with diameter 2 (see Theorem [2.1). They are precisely the graphs depicted in Figure 4. Now note that every orientation of those graphs admits a homomorphism to the graphs depicted in Figure 9, respectively (that is, any oriented graph with underlying graph from the first, second and third family of graphs described in Figure 4 admits a homomorphism to the first, second and third oriented graph depicted in Figure 9, respectively).

To prove the homomorphisms, we map the vertices $w, u, v$ and $a$ from Figure 4 to the corresponding vertices $\phi(w), \phi(u), \phi(v)$ and $\phi(a)$ in Figure 9, respectively. The vertices $b$ and $c$ are mapped to the vertices $\phi(b)$ (or $\phi(c)$ ) and $\phi(c)$ (or $\phi(b)$ ) depending on the orientation of the edge $b c$. Without loss of generality, we can assume the edge to be oriented as $\overrightarrow{b c}$ and assume that the vertices $b$ and $c$ map to the vertices $\phi(b)$ and $\phi(c)$, respectively.

Now, to complete the first homomorphism, map the vertices of $N^{\alpha}(w)$ to the unique vertex in $N^{\alpha}(\phi(w))$ for $\alpha \in\{+,-\}$.

To complete the second homomorphism, map the vertices of $N^{\alpha}(u) \cap N^{\beta}(u)$ to the unique vertex in $N^{\alpha}(\phi(u)) \cap N^{\beta}(\phi(v))$ for $\alpha, \beta \in\{+,-\}$.

To complete the third homomorphism, map the vertices of $N^{\alpha}(a) \cap N^{\beta}(t)$ to the unique vertex in $N^{\alpha}(\phi(a)) \cap N^{\beta}(\phi(t))$ for $\alpha, \beta \in\{+,-\}$ and $t \in\{b, c\}$.

Now, note that the first two oriented graphs depicted in Figure 9 are ocliques of order 3 and 6 , respectively, while the third graph is not an oclique but clearly has absolute oriented clique number 5 .

Hence, there is no triangle-free planar oclique of order more than 6 . Also, the only example of a trianlge-free oclique of order 6 is the second graph depicted in Figure 9 ,
(c) From the proof above, we know that there is no triangle-free planar oclique of order more than 6 , and the only example of a triangle-free oclique of order 6 is the second graph depicted in Figure 9, which is a graph with girth 4 . Hence, there is no planar oclique with girth at least 5 on more than 5 vertices, while the directed cycle of length 5 is clearly a planar oclique with girth 5 .
(d) The 2-dipath is an oclique of order 3. From Plesník's characterization, the rest of the proof follows easily.

## 6 Conclusion

In this paper we proved three main results regarding the order of planar ocliques, that is oriented planar graphs with weak directed diameter (that is, the maximum weak directed distance between two vertices of an oriented graph) at most two. We provided an exhaustive list of spanning subgraphs of outerplanar graphs that admits an orientation with weak directed diameter at most two. Now the question is, can a similar result be proved for planar graphs also?

Question 6.1. Characterize the set $L$ of graphs such that a planar graph can be oriented as an oclique if and only if it contains one of the graphs from $L$ as a spanning subgraph.

We partially answer the question by proving that every planar oclique of order 15 must contain a particular oclique as a spanning subgraph. As the order of a planar oclique can at most be 15, a similar study for planar ocliques of order less than 15 will answer the question. We also proved tight upper bounds for the order of planar ocliques of girth at least $k$ for all $k \geq 4$.

We defined the parameter oriented relative oclique number and used it for proving Theorem 1.7. Determining oriented relative clique number for different families of graphs, such as the family of planar graphs, seems to be an interesting direction of research.

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