

The Game Saturation Number of a Graph

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Abstract

Given a family \mathcal{F} and a host graph H , a graph $G \subseteq H$ is \mathcal{F} -saturated relative to H if no subgraph of G lies in \mathcal{F} but adding any edge from $E(H) - E(G)$ to G creates such a subgraph. In the \mathcal{F} -saturation game on H , players *Max* and *Min* alternately add edges of H to G , avoiding subgraphs in \mathcal{F} , until G becomes \mathcal{F} -saturated relative to H . They aim to maximize or minimize the length of the game, respectively; $\text{sat}_g(\mathcal{F}; H)$ denotes the length under optimal play (when Max starts).

Let \mathcal{O} denote the family of all odd cycles and \mathcal{T} the family of n -vertex trees, and write F for \mathcal{F} when $\mathcal{F} = \{F\}$. Our results include $\text{sat}_g(\mathcal{O}; K_{2k}) = k^2$, $\text{sat}_g(\mathcal{T}; K_n) = \binom{n-2}{2} + 1$ for $n \geq 6$, $\text{sat}_g(K_{1,3}; K_n) = 2 \lfloor n/2 \rfloor$ for $n \geq 8$, $\text{sat}_g(K_{1,r+1}; K_n) = \frac{rn}{2} - \frac{r^2}{8} + O(1)$, and $|\text{sat}_g(P_4; K_n) - \frac{4n-1}{5}| \leq 1$. We also determine $\text{sat}_g(P_4; K_{m,n})$; with $m \geq n$, it is n when n is even, m when n is odd and m is even, and $m + \lfloor n/2 \rfloor$ when mn is odd. Finally, we prove the lower bound $\text{sat}_g(C_4; K_{n,n}) \geq \frac{1}{10.4}n^{13/12} - O(n^{35/36})$. The results are very similar when Min plays first, except for the P_4 -saturation game on $K_{m,n}$.

1 Introduction

The archetypal question in extremal graph theory asks for the maximum number of edges in an n -vertex graph that does not contain a specified graph F as a subgraph. The answer is called the *extremal number* of F , denoted $\text{ex}(F; n)$. The celebrated theorem of Turán [27] gives the answer when F is the complete graph K_r and determines the largest n -vertex graphs not containing K_r (the *size* of a graph is the number of edges).

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We consider maximal graphs not containing F . The concept extends to a family \mathcal{F} of graphs. A graph G is \mathcal{F} -saturated if no subgraph of G belongs to \mathcal{F} but $G + e$ contains a graph in \mathcal{F} whenever $e \in E(\overline{G})$. The extremal number $\text{ex}(\mathcal{F}; n)$ is the maximum size (number of edges) of an \mathcal{F} -saturated n -vertex graph. (In all notation involving families of graphs, we write \mathcal{F} as F when \mathcal{F} consists of a single graph F .)

One may also ask for the minimum size of an \mathcal{F} -saturated n -vertex graph; this is the *saturation number* of \mathcal{F} , denoted $\text{sat}(\mathcal{F}; n)$. Erdős, Hajnal, and Moon [8] initiated the study of graph saturation by determining $\text{sat}(K_r; n)$.

Generalizing further, a subgraph G of a host graph H is \mathcal{F} -saturated relative to H if no subgraph of G lies in \mathcal{F} but adding any edge of $E(H) - E(G)$ to G completes a subgraph belonging to \mathcal{F} . The extremal number and saturation number concern saturation relative to K_n , but saturation has also been studied relative to other graphs. For example, Zarankiewicz's Problem involves saturation relative to $K_{n,n}$. When two agents have opposing interests in creating a large or a small \mathcal{F} -saturated graph, we obtain the "saturation game".

Definition 1.1. The \mathcal{F} -saturation game on a host graph H has players *Max* and *Min*. The players jointly construct a subgraph G of H by iteratively adding one edge of H , constrained by G having no subgraph that lies in \mathcal{F} . The game ends when G becomes \mathcal{F} -saturated relative to H . Max aims to maximize the length of the game, while Min aims to minimize it. When both players play optimally, the length of the game is the *game \mathcal{F} -saturation number* of H , denoted $\text{sat}_g(\mathcal{F}; H)$ when Max starts the game and by $\text{sat}'_g(\mathcal{F}; H)$ when Min starts it. For clarity and for consistency with the extremal and saturation numbers, we write the values as $\text{sat}_g(\mathcal{F}; n)$ and $\text{sat}'_g(\mathcal{F}; n)$ when playing on K_n .

The saturation game generalizes to any hereditary family of sets. Let D be a family of subsets of a set X such that every subset of a member of D also belongs to D . The saturated subsets are the maximal elements of D . Max and Min alternately add elements of X to a set that always lies in D . The game ends when a saturated set is reached, with Max and Min having the same goals as before. In the \mathcal{F} -saturation game on H , we have $X = E(H)$, and avoiding subgraphs in \mathcal{F} defines the hereditary family D .

Patkós and Vizer [26] introduced this general model and studied the case where X_n is the family of k -element subsets of $\{1, \dots, n\}$ and D is the set of intersecting families of k -sets. View X_n as the n -vertex complete k -uniform hypergraph $K_n^{(k)}$. Letting M be the forbidden subgraph consisting of two disjoint edges, the game becomes $\text{sat}_g(M; X_n)$. The Erdős–Ko–Rado Theorem [9] then states $\text{ex}(M; K_n^{(k)}) = \binom{n-1}{k-1} \sim \frac{1}{(k-1)!} n^{k-1}$. Füredi [15] proved that $\text{sat}(M; K_n^{(k)}) \leq \frac{3}{4}k^2$ when a projective plane of order $r/2$ exists. For $k \geq 2$, Patkós and Vizer [26] proved $\Omega(n^{\lfloor k/3 \rfloor - 5}) \leq \text{sat}_g(M; K_n^{(k)}) \leq O(n^{k - \sqrt{k}/2})$.

The saturation game is also related to other well-studied graph games. In a *Maker-Breaker* game, the players Maker and Breaker take turns choosing edges of a host graph

H , typically K_n . Maker wins by claiming all of the edges in a subgraph of H having some specified property \mathcal{P} , and Breaker wins by preventing this. For example, Hefetz, Krivelevich, Stojaković, and Szabó [19] studied Maker-Breaker games played on K_n in which Maker seeks to build non-planar graphs, non- k -colorable graphs, or K_t -minors. Several papers have considered the minimum number of turns needed for Maker to win (see [12, 21]). In this context, Breaker behaves like Max in the saturation game, making the game last as long as possible. In the saturation game both players contribute edges, but here Maker cannot use the edges taken by Breaker.

In an *Avoider-Enforcer* game, again two players alternately choose edges of a fixed host graph. Avoider wants to avoid creating any subgraph satisfying \mathcal{P} ; Enforcer wants to force Avoider to build such a subgraph. Hefetz, Krivelevich, and Szabó [22] introduced such games, establishing general results and studying the cases where Avoider seeks to avoid spanning trees or spanning cycles of H . In Avoider-Enforcer games winnable by Enforcer, one may ask how quickly Enforcer can win. Here Enforcer behaves like Min in the saturation game, but again the moves by Enforcer are not part of Avoider's subgraph (see [1, 2, 3, 20]).

The \mathcal{F} -saturation game on H is also related to the \mathcal{F} -free process on H , equivalent to both players moving randomly. The length of the process is the number of moves to reach a graph that is \mathcal{F} -saturated relative to H . Usually $H = K_n$ (see [5, 6, 10, 25]), but [4] is more general. For the C_4 -free process on $K_{n,n}$, the lower bound of [4] specializes to $\Omega(n^{4/3}(2 \log n)^{1/3})$.

The saturation game on graphs was introduced by Füredi, Reimer, and Seress [17]; they studied $\text{sat}_g(K_3; n)$, calling it “a variant of Hajnal’s triangle-free game”. In Hajnal’s original “triangle-free game”, the players aim only to avoid creating triangles, and the loser is the player first forced to create one (Ferrara, Jacobson, and Harris [13] considered the generalization of Hajnal’s loser criterion to arbitrary \mathcal{F} and G). Since the F -saturation game always produces an F -saturated graph, $n - 1 = \text{sat}(K_3; n) \leq \text{sat}_g(K_3; n) \leq \text{ex}(K_3; n) = \lfloor n^2/4 \rfloor$; hence $\text{sat}_g(K_3; n) \in \Omega(n) \cap O(n^2)$. Füredi et al. [17] proved $\text{sat}_g(K_3; n) \in \Omega(n \lg n)$. Erdős (unpublished) stated $\text{sat}_g(K_3; n) \leq n^2/5$. The correct order of growth remains unknown.

The P_3 -saturation game was studied by Cranston, Kinnersley, O, and West [7]; here P_k denotes the k -vertex path. The subgraphs of H that are P_3 -saturated relative to H are precisely the maximal matchings in H . Thus the game P_3 -saturation number is just the *game matching number*, with $\alpha'_g(G)$ and $\hat{\alpha}'_g(G)$ denoting the values of the Max-start and Min-start games since $\alpha'(G)$ denotes the maximum size of a matching in G . They proved $\alpha'_g(G) \geq \frac{2}{3}\alpha'(G)$ for every graph G (with equality for some split graphs) and $\alpha'_g(G) \geq \frac{3}{4}\alpha'(G)$ when G is a forest (with equality for some trees). The minimum of $\alpha'_g(G)$ over n -vertex 3-regular graphs is between $n/3$ and $7n/18$.

We have mentioned bounds on α'_g but not $\hat{\alpha}'_g$ because the two parameters never differ by more than 1 (see [7]). This does not hold for \mathcal{F} -saturation in general. For example, when the host graph is obtained from a star with m edges by subdividing one edge, the

Max-start $2K_2$ -saturation number is m , but the Min-start $2K_2$ -saturation number is 2. As a less artificial example, we will show that $|\text{sat}_g(P_4; K_{m,n}) - \text{sat}'_g(P_4; K_{m,n})|$ can be large, where $K_{m,n}$ is the complete bipartite graph with part-sizes m and n . In most instances that we study, the choice of the starting player does not affect the outcome by much.

In Section 2, we study the \mathcal{F} -saturation games on K_n for $\mathcal{F} \in \{\mathcal{O}, \mathcal{T}_n, \{K_{1,r+1}\}, \{P_4\}\}$, where \mathcal{O} is the family of all odd cycles and \mathcal{T}_n is the family of n -vertex trees. We first prove $\text{sat}_g(\mathcal{O}; 2k) = \text{sat}'_g(\mathcal{O}; 2k) = k^2$, achieving the trivial upper bound $\text{ex}(\mathcal{O}; 2k)$. For $n \geq 3$, we prove $\text{sat}_g(\mathcal{T}_n; n) = \text{sat}'_g(\mathcal{T}_n; n) = \binom{n-2}{2} + 1$, except $\text{sat}_g(\mathcal{T}_5; 5) = 6$ and $\text{sat}'_g(\mathcal{T}_4; 4) = 3$; note $\text{ex}(\mathcal{T}_n; n) = \binom{n-1}{2}$. Hefetz et al. [18] have since studied more general versions of both of these problems. They studied $\text{sat}_g(\mathcal{C}_k; n)$ and $\text{sat}_g(\mathcal{X}_k)$ where \mathcal{C}_k is the family of k -connected graphs with n vertices and \mathcal{X}_k is the family of non- k -colorable graphs. In both cases, the value is close to the extremal number. Lee and Riet [24] have generalized the tree problem in a different direction, studying $\text{sat}_g(\mathcal{T}_k; n)$.

Always $\text{sat}_g(K_{1,3}; n)$ and $\text{sat}'_g(K_{1,3}; n)$ lie in $\{n, n-1\}$. Except for $n \in \{2, 3, 4, 7\}$, they are unequal, with $\text{sat}_g(K_{1,3}; n)$ being the even value and $\text{sat}'_g(K_{1,3}; n)$ being the odd value. That is, $\text{sat}_g(K_{1,3}; n) = 2 \lfloor n/2 \rfloor$ and $\text{sat}'_g(K_{1,3}; n) = 2 \lceil n/2 \rceil - 1$ when $n \geq 8$. Note that $\text{ex}(K_{1,3}; n) = n$. For $n > r > 2$, it has been checked by computer that $\text{sat}_g(K_{1,r+1}; n) = \lfloor \frac{rn-1}{2} \rfloor$ when $n \leq 8$. We ask whether this holds for larger n ; note that $\text{ex}(K_{1,r+1}; n) = \lfloor \frac{rn}{2} \rfloor$. Kászonyi and Tuza [23] proved $\text{sat}(K_{1,r+1}) = \left\lceil \frac{rn}{2} - \frac{(r+1)^2}{8} \right\rceil$ for $n \geq 3r/2$. Lee and Riet [24] proved $\text{sat}_g(K_{1,r+1}; n) \geq (rn/2) - k + 1$.

For the P_4 -saturation game on K_n , the value is not asymptotic to the extremal number. We prove $|\text{sat}_g(P_4; n) - \frac{4n-1}{5}| \leq 1$ and $|\text{sat}'_g(P_4; n) - \frac{4n}{5}| \leq .6$, while $\text{ex}(P_4; n) \in \{n, n-1\}$. Lee and Riet [24] proved $n-1 \leq \text{sat}_g(P_5; n) \leq n+2$.

In Section 3, we study the P_4 -saturation game on $K_{m,n}$; we may assume $m \geq n$. The choice of who starts the game can matter a lot, as do the parities of m and n . The value of $\text{sat}_g(P_4; K_{m,n})$ is n when n is even (equaling $\text{sat}(P_4; K_{m,n})$), m when m is even and n is odd, and $m + \lfloor n/2 \rfloor$ when mn is odd. The value of $\text{sat}'_g(P_4; K_{m,n})$ is m when $n \leq 2$ and $m + \lfloor n/2 \rfloor - \epsilon$ when $n > 2$, where $\epsilon = 0$ when mn is even and $\epsilon = 1$ when mn is odd.

Note that the difference is $m-2$ when $n=2$, and for larger n the difference is $(n-1)/2$ when m is even and n is odd. Note also that $\text{sat}(P_4; K_{m,n}) = \min\{m, n\}$, so when $\min\{m, n\}$ is even we obtain an example where $\text{sat}_g(P_4; G) = \text{sat}(P_4, G)$. We ask whether there are other interesting examples where $\text{sat}_g(\mathcal{F}; n)$ or $\text{sat}'_g(\mathcal{F}; n)$ equals $\text{sat}(\mathcal{F}; n)$; [11] provides a survey of saturation numbers as of 2009.

In Section 4, we study the C_4 -saturation game on $K_{n,n}$. This game is the natural bipartite analogue of the triangle-saturation game on K_n studied by Füredi, Reimer, and Seress [17]. Every subgraph that is C_4 -saturated relative to $K_{n,n}$ is connected, so $\text{sat}_g(C_4; K_{n,n}) = \Omega(n)$. On the other hand, Füredi [16] proved $\text{ex}(C_4; K_{n,n}) = n^{3/2} + O(n^{4/3})$, so $\text{sat}_g(C_4; K_{n,n}) = O(n^{3/2})$. Our main result is a polynomial improvement over the natural lower bound:

$$\text{sat}_g(C_4; K_{n,n}) \geq \frac{1}{10.4}n^{13/12} - O(n^{35/36}).$$

Our results leave many open questions. The most interesting specific question is the order of growth of $\text{sat}_g(C_4; K_{n,n})$. One would also like to understand the conditions under which $\text{sat}_g(\mathcal{F}; n)$, $\text{sat}_g(\mathcal{F}; K_{m,n})$, or $\text{sat}_g(\mathcal{F}; H)$ does not differ much from the value of the corresponding Min-start game.

2 Saturation games on complete graphs

We begin with saturation games on the complete graph K_n . A graph is *nontrivial* if it has at least one edge.

Theorem 2.1. $\text{sat}_g(\mathcal{O}; 2k) = \text{sat}'_g(\mathcal{O}; 2k) = k^2 = \text{ex}(\mathcal{O}; 2k)$.

Proof. An \mathcal{O} -saturated graph is a complete bipartite graph. With $2k$ vertices, the largest has parts of equal size. It therefore suffices to give Max a strategy ensuring that after each turn by Max the bipartition of each nontrivial component is balanced. Whether Max or Min starts, the first move by Max ensures this (yielding two isolated edges if Min moves first).

Subsequently, a move by Min can connect two nontrivial components, lie within a component, connect two isolated vertices, or connect an isolated vertex to a nontrivial component. In the last case, since $2k$ is even, Max can connect another isolated vertex to the same nontrivial component, keeping the bipartition balanced. In the other cases, Max can play an edge within a nontrivial component or, if they are all complete bipartite (and balanced), connect two nontrivial components or, if there is just one nontrivial component and it is balanced, connect two isolated vertices. If no move is available, then the game has ended, with k^2 moves played. \square

The disjoint union of graphs G and H is denoted $G + H$. The largest subgraph of K_n containing no spanning tree of K_n is $K_{n-1} + K_1$, with $\binom{n-1}{2}$ edges.

Theorem 2.2. *If $n \geq 3$, then $\text{sat}_g(\mathcal{T}_n; n) = \text{sat}'_g(\mathcal{T}_n; n) = \binom{n-2}{2} + 1$, except that $\text{sat}_g(\mathcal{T}_5; 5) = 6$ and $\text{sat}'_g(\mathcal{T}_4; 4) = 3$.*

Proof. Every \mathcal{T}_n -saturated subgraph of K_n has the form $K_r + K_{n-r}$ for some r . Throughout the game there are some number of components, and a move either joins two components or adds an edge within a component.

If some move by Max leaves at least two nontrivial components, then Min can maintain this condition after each subsequent move until all vertices are in nontrivial components, ensuring the upper bound. Min connects two isolated vertices if two isolated vertices remain, increasing the number of nontrivial components to at least 3, and Max then cannot reduce it below 2. When only one isolated vertex remains, Min connects it to a nontrivial component.

To exceed the upper bound, Max must therefore always leave only one nontrivial component. If the move by Max leaves an even number of isolated vertices, then Min makes an isolated edge, and Max must connect the nontrivial components. This repeats until Min connects the last two isolated vertices to make a second nontrivial component that Max cannot absorb.

If the number of isolated vertices is odd after the first move by Max (and the number of nontrivial components is 1), then Min works to fix the parity. If Max starts, then n is odd. Min creates P_3 . Max now must enlarge the component to P_4 or $K_{1,3}$ to keep the number of isolates odd. Because K_4 has an even number of edges, Max eventually must reduce the number of isolates by 1 or create a second nontrivial component, unless $n = 5$.

If Min starts, then Max must create P_3 , and n is even. Now Min completes the triangle, and again Max must reduce the number of isolates by 1 or create a second nontrivial component, unless $n = 4$.

Max can enforce the lower bound by always leaving only one nontrivial component. Only when Min connects the last two isolated vertices will a second component survive. \square

Let kG denote the disjoint union of k copies of G .

Theorem 2.3.

$$\text{sat}_g(K_{1,3}; n) = \begin{cases} n & \text{when } n \in \{3, 7\} \cup 2\mathbb{N} - \{2\} \\ n - 1 & \text{otherwise} \end{cases}$$

$$\text{sat}'_g(K_{1,3}; n) = \begin{cases} n - 1 & \text{when } n \in 2\mathbb{N} - \{4\} \\ n & \text{otherwise} \end{cases}$$

Proof. All $K_{1,3}$ -saturated graphs are disjoint unions of cycles plus possibly one isolated vertex or isolated edge (not both). Hence the only possible outcomes are n (call this *Max wins*) or $n - 1$ (call this *Min wins*). Let $X(n)$ and $Y(n)$ denote the Max-start and Min-start $K_{1,3}$ -saturation games on K_n , respectively.

For $n \geq 5$, our claim is that the first player wins when n is even and the second player wins when n is odd, except that Max wins $X(7)$. After giving specific strategies for $n \leq 7$, we provide general strategies for $n \geq 8$ that reduce the problem to the cases $n \in \{5, 6\}$.

When $n \leq 3$, there is no claw, so Min wins when $n \leq 2$ and Max wins when $n = 3$, no matter who starts. When $n = 4$, Max can create $2K_2$ or P_4 to win, no matter who starts.

In $Y(5)$, Max creates $2K_2$ and can then force C_5 . In $X(5)$, Min creates $2K_2$ and can then close a cycle on the next turn to win.

In $X(6)$, Max completes a triangle if Min makes P_3 , reducing to $Y(3)$, which Max wins. If Min makes $2K_2$, then Max makes $3K_2$ and next P_6 to win. In $Y(6)$, Min makes P_4 on the second move and can then close a 4-cycle or 5-cycle to win.

In $X(7)$, if Min makes P_3 , then Max closes the 3-cycle and wins $Y(4)$. If Min makes $2K_2$, then Max makes $3K_2$. Whether Min next makes P_3 or P_4 , Max closes the cycle and wins. In $Y(7)$, Max makes P_3 and will later win a game played on three or four vertices.

Now assume $n \geq 8$. Let W be the first player when n is even and the second player when n is odd; we give a winning strategy for W . Player W always leaves the components being one nontrivial path, an even number of isolated vertices, and some number of cycles, until the number of isolated vertices is 6. By making P_2 or P_3 in the first round, W initiates this process. If the other player V closes the cycle, then W starts a new path, while if V extends the path or makes an isolated edge the path is left longer by two edges. In either case, the number of isolated vertices decreases by 2.

When six isolated vertices remain, if V closes the cycle or makes an isolated edge and lets W close the cycle, then the remaining game is the game on six vertices started by W . If V extends the path, then W closes the cycle to leave the game on five vertices started by V . We have shown that when $n = 6$ the game is won by the first player, and when $n = 5$ the game is won by the second player. \square

Because there are only two possible (consecutive) lengths of the $K_{1,3}$ -saturation game on K_n , the outcome is determined by who plays last. Ferrara, Jacobson, and Harris [13] studied that question explicitly; in their game the player who moves last wins. Although their analysis is similar to ours due to the structure of $K_{1,3}$ -saturated graphs, their result is different: in their game, for $n \geq 5$, the first player wins if and only if n is even, except $n = 7$. In particular, under their criterion for winning, the number of moves played will always be $n - 1$ (except $n = 7$).

Our final game on K_n is the P_4 -saturation game. Note that during the game, all components of the built subgraph must be stars or triangles. Since Max seeks a large ratio of number of edges to number of vertices, triangles and large stars are beneficial to Max, while small stars are beneficial to Min. However, stars with two edges are dangerous for Min, since Max can turn them into triangles. This intuition motivates the strategies for the players.

Theorem 2.4. *For $n \geq 4$,*

$$\frac{4n - 6}{5} \leq \text{sat}_g(P_4; n) \leq \frac{4n + 4}{5}$$

$$\frac{4n - 3}{5} \leq \text{sat}'_g(P_4; n) \leq \frac{4n + 3}{5}.$$

Proof. During the game, let the *value* of the current position count a contribution for each component: 0 for an isolated vertex or triangle, $\frac{1}{2}$ for P_2 or P_3 , and 1 for a larger star. The only way to decrease the value is to turn a copy of P_3 into a triangle. When we speak of “making” or “creating” a subgraph, we mean producing it as a component of G .

Upper bound: Min strategy. While two isolated vertices are available, Min never makes P_3 , and if Max makes P_3 , then Min responds by converting it to $K_{1,3}$. Otherwise, Min makes

P_2 , except that when exactly three isolated vertices remain Min enlarges an existing star with at least two edges (if one exists). If only one isolated vertex remains, then Min attaches it to a largest existing star.

With this strategy, each move by Min increases the value by $\frac{1}{2}$, except possibly the last when one isolated vertex remains, or the next-to-last when exactly three isolated vertices remain. This strategy ensures that no triangles are created, unless Max stupidly makes isolated edges and the final graph is $K_3 + \frac{n-3}{2}P_2$ with $\frac{n+3}{2}$ edges. Hence the components are all stars, and the number of them is $n - m$, where m is the final number of edges. Since the strategy also prevents Max from decreasing the value (unless Max makes isolated edges), the value reaches at least $\frac{m-4}{4}$, where m is the final number of edges. Also the final value is at most the number of components. We obtain $\frac{m-4}{4} \leq n - m$, which simplifies to $m \leq \frac{4n+4}{5}$ (the same computation yields $m \leq \frac{4n+3}{5}$ in the Min-start game).

Lower bound: Max strategy. While an isolated vertex is available, Max never makes P_2 , except on the first turn of the Max-start game. If Min makes P_2 , then Max turns it into P_3 . If there is no isolated edge, then Max adds an edge to a star with at least three edges or completes a triangle if no such star exists.

With this strategy, Max never increases the value, except on the first turn of the Max-start game. With each Min move increasing it by at most $\frac{1}{2}$, the upper bound on the value is $\frac{m+2}{4}$ (or $\frac{m+1}{4}$ in the Min-start game). Also Max ensures that no isolated edge remains, except possibly the initial move in the Max-start game and an edge joining the last two isolated vertices. Except for those one or two components, the number of edges in a component is its number of vertices minus its contribution to the value. Hence the final value is at least $n - m - \frac{1}{2}$ in the Min-start game, or $n - m - 1$ in the Max-start game (an isolated edge contributes $\frac{1}{2}$ to the value but 1 to $n - m$). We obtain $\frac{m+2}{4} \geq n - m - 1$ in the Max-start game and $\frac{m+1}{4} \geq n - m - \frac{1}{2}$ in the Min-start game, simplifying to $m \geq \frac{4n-6}{5}$ and $m \geq \frac{4n-3}{5}$, respectively. \square

3 The P_4 -saturation game on $K_{m,n}$

Now we study the P_4 -saturation game on the complete bipartite graph $K_{m,n}$. Since $K_{m,n}$ contains no triangles, during the game all components are stars. Throughout this section, X and Y are the partite sets of $K_{m,n}$, with $|X| = m \geq n = |Y|$. Let an X -star or Y -star be a star having at least two leaves in X or in Y , respectively. Recall that $\alpha'(G)$ denotes the maximum size of a matching in G .

Lemma 3.1. *A graph G that is P_4 -saturated relative to $K_{m,n}$ has at most $m + n - \alpha'(G)$ edges. If it contains both an X -star and a Y -star (or an isolated edge), then equality holds.*

Proof. Any even cycle contains P_4 , so G is a forest. To avoid P_4 , edges of a matching must

lie in distinct components. Since G is a forest, $E(G)$ is the number of vertices minus the number of components, so $|E(G)| \leq m + n - \alpha'(G)$.

A saturated subgraph containing both an X -star and a Y -star (or an isolated edge) cannot have isolated vertices. The components are then nontrivial stars, so there are $\alpha'(G)$ of them. Hence there are exactly $m + n - \alpha'(G)$ edges. \square

Call a P_4 -saturated subgraph that contains both an X -star and a Y -star a *full subgraph*. A P_4 -saturated subgraph that is not full has stars of only one of these types (plus isolated edges, possibly) and thus has m or n edges. Hence Max wants to make a full subgraph. When $\min\{m, n\}$ is even, Min can prevent this in the Max-start game, and we obtain $\text{sat}_g(P_4; K_{m,n}) = \text{sat}(P_4; K_{m,n})$ in that case. When Max can make a full subgraph, Lemma 3.1 encourages Min to create a large matching.

Theorem 3.2. *For $m \geq n \geq 1$, the P_4 -saturation numbers of $K_{m,n}$ are given by*

$$\text{sat}_g(P_4; K_{m,n}) = \begin{cases} n & \text{when } n \text{ is even,} \\ m & \text{when } n \text{ is odd and } m \text{ is even,} \\ m + \lfloor \frac{n}{2} \rfloor & \text{when } mn \text{ is odd.} \end{cases}$$

and

$$\text{sat}'_g(P_4; K_{m,n}) = \begin{cases} m & \text{when } n \leq 2, \\ m + \lfloor \frac{n}{2} \rfloor & \text{when } n > 2 \text{ and } mn \text{ is even,} \\ m + \lfloor \frac{n}{2} \rfloor - 1 & \text{when } n > 2 \text{ and } mn \text{ is odd.} \end{cases}$$

Proof. We will consider cases based on who moves first and the parity of m and n . Let G denote the P_4 -saturated subgraph built during the game. Again “making” a subgraph means producing it as a component of the current graph.

Upper bounds. We give strategies for Min. If Max moves first and m or n is even, then Min ensures that only X -stars or Y -stars are created, respectively, by immediately extending isolated edges made by Max to such stars and otherwise enlarging such stars. The final number of edges is then $|X|$ or $|Y|$, respectively.

In the other cases, Min just ensures a large matching. If Max moves first and mn is odd, or Min moves first and mn is even, then Min makes isolated edges until a matching of size $\lfloor \frac{n}{2} \rfloor$ is built, later playing any legal move. By Lemma 3.1, at most $m + \lfloor \frac{n}{2} \rfloor$ moves are played.

If Min moves first and mn is odd, then Min can do slightly better. If Max responds to the first move by making an X -star or Y -star, then the parity allows Min to ensure that only X -stars or Y -stars, respectively, will be played, yielding an outcome of $|X|$ or $|Y|$. Hence Max must immediately make another isolated edge. The moves by Min still yield a matching of size $\lfloor \frac{n}{2} \rfloor$, and with the extra edge made by Max the bound improves by 1.

Lower bounds. We give strategies for Max. Since the game cannot leave an isolated vertex in each part, at least $\min\{m, n\}$ moves are played. If an X -star is made, then no

isolated vertex can be left in X , and at least m moves are made. In the Max-start game with n odd and m even, Min can prevent an X -star only by leaving only Y -stars after each move. After $n - 1$ moves, Max makes K_2 using the last isolated vertex of Y , and then Min is forced to make an X -star. In the Min-start game with $n \leq 2$, Max makes an X -star immediately.

In the other cases, we may assume $n \geq 3$. Max wants to force a full subgraph and keep $\alpha'(G)$ small. In the Min-start game with mn even, Max responds to the first move by making a Y -star if n is even or an X -star if n is odd and m is even. In the Max-start game with mn odd, Max makes a Y -star on the third move if Min made K_2 on the second; otherwise Max adds to the X -star or Y -made by Min.

In each of these cases, Max continues enlarging the original X -star or Y -star. If the graph has not become full by the time X or Y , respectively, has only one isolated vertex remaining, then every move has created a leaf in that part. By the parity of the size of that part, it is Max's turn. Max makes K_2 , and now Min must make the graph full.

Hence the graph becomes full, so Max takes advantage of Lemma 3.1 by making the initial star large. Max can play at least $\lfloor n/2 \rfloor$ edges in the initial star. Max can play one more such edge on the n th move unless Min has also played edges into stars in the same direction. Hence $\alpha'(G) \leq n - \lfloor n/2 \rfloor + 1$. By Lemma 3.1, the final number of edges is $m + n - \alpha'(G)$, which is at least $m + \lfloor n/2 \rfloor$.

For the Min-start game with mn odd, Max cannot do quite as well. As noted when discussing upper bounds, if Max makes an X -star or Y -star on move 2, then Min can limit the final number of edges to m or n , respectively. Hence Max makes K_2 on move 2. If Min makes $K_{1,2}$, then Max makes the other type of star. If Min makes K_2 , then Max makes an X -star and can make a Y -star on the next round.

Hence the graph becomes full. Max subsequently enlarges Y -stars until Y has no more isolated vertices. All moves by Max to that point except the first two enlarge Y -stars, and there is also one such edge among the first four moves (played by Max or Min). Letting a maximum matching consist of the first edge from each component, we thus have $\alpha'(G) \leq n - 1 - \lfloor \frac{n-4}{2} \rfloor = \lceil \frac{n}{2} \rceil + 1$, so the final number of edges is at least $m + \lfloor \frac{n}{2} \rfloor - 1$. \square

4 The C_4 -saturation game on $K_{n,n}$

In this section, we study the C_4 -saturation game on $K_{n,n}$, the natural bipartite analogue of the Füredi-Reimer-Seress problem. As we have noted, the trivial lower bound and the result of [16] yield $\Omega(n) \leq \text{sat}_g(C_4, K_{n,n}) \leq O(n^{3/2})$.

Our main result is a polynomial improvement of the lower bound: $\text{sat}_g(C_4, K_{n,n}) = \Omega(n^{13/12})$. We first prove a technical lemma giving a lower bound on the size of a restricted type of graph that is also C_4 -saturated relative to $K_{n,n}$. Here our interest is the exponent on n ; we make no attempt to optimize lower-order terms or the leading coefficient.

Lemma 4.1. *Let G be C_4 -saturated relative to $K_{n,n}$, and let c and d be positive constants. If there exists $S \subseteq V(G)$ with at least cn vertices in each partite set such that $|N(v) \cap S| \leq d\sqrt{n}$ for all $v \in V(G)$, then $|E(G)| \geq an^{13/12} - O(n^{35/36})$, where $a = \min\{\frac{1}{2}(\frac{c^2}{2d^2})^{2/3}, \frac{c^2}{2d}\}$.*

Proof. Let S_X and S_Y be the subsets of S in the two partite sets. Consider $x \in S_X$ and $y \in S_Y$ such that $xy \notin E(G)$. Since G is C_4 -saturated relative to $K_{n,n}$, it contains a copy of P_4 with endpoints x and y . Each vertex in S_X has at most $d\sqrt{n}$ neighbors and hence at least $cn - d\sqrt{n}$ nonneighbors in S_Y . Thus G contains at least $c^2n^2 - cdn^{3/2}$ copies of P_4 with endpoints in S_X and S_Y ; call such paths *essential paths*. Since each essential path has endpoints in S_X and S_Y , and since no vertex has more than $d\sqrt{n}$ neighbors in S , no edge is the central edge of more than d^2n essential paths.

Let T be the set of vertices of G with degree at least $n^{5/12}$, and let $b = (\frac{c^2}{2d^2})^{2/3}$. If $|T| \geq bn^{2/3}$, then $\sum_{v \in T} d(v) \geq bn^{13/12}$, which yields $|E(G)| \geq \frac{b}{2}n^{13/12}$. Otherwise, let H be the subgraph of G induced by T . Since H is C_4 -free, a result of Füredi [16] yields $|E(H)| \leq (bn^{2/3})^{3/2} + O((bn^{2/3})^{4/3})$, which simplifies to $|E(H)| \leq \frac{c^2}{2d^2}n + O(n^{8/9})$. Multiplying by d^2n , we conclude that at most $\frac{c^2}{2}n^2 + O(n^{17/9})$ essential paths have central edges in H .

Thus at least $\frac{c^2}{2}n^2 - O(n^{17/9})$ essential paths have central edges incident to a vertex with degree less than $n^{5/12}$. Each such edge is the central edge of at most $dn^{11/12}$ essential paths; hence G has at least $\frac{c^2}{2d}n^{13/12} - O(n^{35/36})$ such edges. \square

Though the hypotheses of Lemma 4.1 seem technical, they apply whenever $\Delta(G) \leq d\sqrt{n}$. Hence we obtain a corollary for ordinary saturation (using $c = 1$).

Corollary 4.2. *If G is C_4 -saturated relative to $K_{n,n}$ and $\Delta(G) \leq d\sqrt{n}$, then $|E(G)| \geq an^{13/12} - O(n^{35/36})$, where $a = \min\{\frac{1}{2}(\frac{1}{2d^2})^{2/3}, \frac{1}{2d}\}$. (If $d \leq \frac{1}{4}$, then $a = \frac{1}{2d}$.)*

Our main result for the C_4 -saturation game on $K_{n,n}$ follows easily from Lemma 4.1.

Theorem 4.3. $\text{sat}_g(C_4, K_{n,n}) \geq \frac{1}{10.4}n^{13/12} - O(n^{35/36})$, and similarly for $\text{sat}'_g(C_4; K_{n,n})$.

Proof. We provide a strategy for Max that forces the final subgraph of $K_{n,n}$ to satisfy the hypotheses of Lemma 4.1. This strategy governs almost the first $2n/3$ moves for Max, after which Max plays arbitrarily.

Let $k = \lfloor \sqrt{n/3} \rfloor - 1$. Max arranges to give degree k to k specified vertices in each partite set. Each move by Max makes an isolated vertex adjacent to a vertex with growing degree; hence it cannot complete a 4-cycle. Fewer than $n/3$ vertices are needed by Max in each part, so Min cannot exhaust the isolated vertices in either part with fewer than $2n/3$ moves. After this phase, Max may play any legal move.

In the final subgraph G , let S be the set of leaves of the $2k$ specified stars constructed by Max. By construction, the stars are disjoint, so S has about $n/3 - 2\sqrt{n/3}$ vertices in each part. Moreover, no vertex in G has more than $\sqrt{n/3}$ neighbors in S , since each vertex other than the center of a star is adjacent to at most one leaf of the star.

Thus G satisfies the hypotheses of Lemma 4.1 with c being any constant less than $1/3$ and $d = \sqrt{1/3}$, from which the claim follows. \square

While Theorem 4.3 does establish a nontrivial asymptotic lower bound for $\text{sat}_g(C_4; K_{n,n})$, the correct order of growth remains undetermined. Lemma 3.1 suggests the following question, which would yield improved lower bounds for $\text{sat}_g(C_4; K_{n,n})$: What is the minimum number of edges in a graph with maximum degree D that is C_4 -saturated relative to $K_{n,n}$?

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