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A Density Turán Theorem

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Abstract

Let F be a graph which contains an edge whose deletion reduces its chromatic number. For such a graph F, a classical result of Simonovits from 1966 shows that every graph on $n > n_0(F)$ vertices with more than $\frac{\chi(F)-2}{\chi(F)-1} \cdot \frac{n^2}{2}$ edges contains a copy of F. In this paper we derive a similar theorem for multipartite graphs.

For a graph H and an integer $\ell \geq v(H)$, let $d_\ell(H)$ be the minimum real number such that every ℓ-partite graph whose edge density between any two parts is greater than $d_\ell(H)$ contains a copy of H. Our main contribution in this paper is to show that $d_\ell(H) = \frac{\chi(H)-2}{\chi(H)-1}$ for all $\ell \geq \ell_0(H)$ sufficiently large if and only if H admits a vertex-colouring with $\chi(H) - 1$ colours such that all colour classes but one are independent sets, and the exceptional class induces just a matching. When H is a complete graph, this recovers a result of Pfender [Complete subgraphs in multipartite graphs, *Combinatorica* **32** (2012), 483–495]. We also consider several extensions of Pfender's result.

1 Introduction

Extremal graph theory has enjoyed tremendous growth in recent decades. One of the central questions from which the theory originated can be described as follows. Given a *forbidden graph* H, the Turán problem asks to determine $\mathrm{ex}(n, H)$, the maximum possible number of edges in a graph on n vertices without a copy of H. This number is called the *Turán number* of H. Instances of this problem have many connections and applications to other areas. In this paper we consider a multipartite version of the problem, suggested by Bollobás [1]. Before stating the problem at hand and presenting our contributions, we begin with a brief survey of relevant results.

1.1 Background

The fundamental Turán theorem of 1941 [24] completely determined the Turán numbers of a clique: the Turán graph $T_{k-1}(n)$, the complete $(k-1)$-partite graph on n vertices with parts as equal as possible, has the largest number of edges among all K_k-free n-vertex graphs. Thus, we have $\mathrm{ex}(n, K_k) = t_{k-1}(n)$, where $t_{k-1}(n)$ is the number of edges in $T_{k-1}(n)$. This theorem generalises a previous result by Mantel [15] from 1907, which states that $\mathrm{ex}(n, K_3) = \lfloor \frac{n^2}{4} \rfloor$.

A large and important class of graphs for which the Turán numbers are well-understood is formed by *colour-critical graphs*, that is, graphs whose chromatic number can be decreased by removing

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an edge. Simonovits [22] introduced the stability method to show that $\mathrm{ex}(n,H)=t_{k-1}(n)$ for all $n\geq n_0(H)$ sufficiently large, provided H is a colour-critical graph with $\chi(H)=k$; furthermore, $T_{k-1}(n)$ is the unique extremal graph. As the cliques are colour-critical, Simonovits' theorem implies Turán's theorem for large n.

For general graphs H we still do not know how to compute the Turán numbers $\mathrm{ex}(n,H)$ exactly; but if we are satisfied with an approximate answer the theory becomes quite simple: it is enough to know the chromatic number of H. The important and deep theorem of Erdős and Stone [12] together with an observation of Erdős and Simonovits [10] shows that $\mathrm{ex}(n,H)=\left(\frac{\chi(H)-2}{\chi(H)-1}+o(1)\right)\frac{n^2}{2}$, where the $o(1)$ term tends to 0 as n tends to infinity. In the literature, this result is usually referred as the Erdős–Stone–Simonovits theorem.

In the years since these seminal theorems appeared, great efforts have been made to extend them, some of which are discussed in Nikiforov's survey [18]. We are particularly interested in the following two extensions.

For every integer $s\geq 2$, let $K_{k-1}(s)$ denote the complete $(k-1)$-partite graph $K_{k-1}(s,\dots,s)$, and let $K^+_{k-1}(s)$ be the graph obtained from $K_{k-1}(s)$ by adding an edge to the first class. Nikiforov [17] and Erdős [7] (for $k=3$) proved that for all $k\geq 3$ and all sufficiently small $c>0$, every graph of sufficiently large order n with $t_{k-1}(n)+1$ edges contains not only a K_k but a copy of $K^+_{k-1}\big(\lfloor c\ln n\rfloor\big)$. For fixed k, the Erdős-Rényi random graph $G_{n,p}$ shows that the lower bound $c\ln n$ on the size of the subgraph in this result is optimal up to a constant factor.

Seeking an extension of Turán's theorem, Erdős [9] asked how many K_k sharing a common edge must exist in a graph on n vertices with $t_{k-1}(n)+1$ edges. Bollobás and Nikiforov [3] sharpened Erdős's result [9] showing that for large enough n, every graph of order n with $t_{k-1}(n)+1$ edges has an edge that is contained in $k^{-k-4}n^{k-2}$ copies of K_k. This result is best possible, up to a $\mathrm{poly}(k)$ factor.

In this paper we shall study analogues of these results for multipartite graphs. For a graph H and an integer $\ell\geq v(H)$, let $d_\ell(H)$ be the minimum real number such that every ℓ-partite graph $G=(V_1\cup\ldots\cup V_\ell,E)$ with $d(V_i,V_j):=\frac{e(V_i,V_j)}{|V_i||V_j|}>d_\ell(H)$ for all $i\neq j$ contains a copy of H. The problem of determining the exact value of $d_\ell(H)$ was suggested by Bollobás (see the discussion after the proof of Theorem VI.2.15 in [1]). However, it was first studied systematically by Bondy, Shen, Thomassé and Thomassen [4]. Amongst other things Bondy et.al. showed that for every graph H the sequence $d_\ell(H)$ decreases to $\frac{\chi(H)-2}{\chi(H)-1}$ as ℓ tends to infinity. To show the lower bound $d_\ell(H)\geq\frac{\chi(H)-2}{\chi(H)-1}$, they observed that the ℓ-partite graph G obtained from the empty graph on $\{1,\dots,\ell\}$ by splitting each vertex v of $\{1,\dots,\ell\}$ into $\chi(H)-1$ vertices $v_1,v_2,\dots,v_{\chi(H)-1}$, and joining two vertices x_i and y_j if and only if $x\neq y$ and $i\neq j$, has all edge densities equal to $\frac{\chi(H)-2}{\chi(H)-1}$. Since G is $(\chi(H)-1)$-colourable (with vertex classes $V_i=\{v_i : v\in\{1,\dots,\ell\}\}$ for $1\leq i\leq\chi(H)-1$), it does not contain a copy of H. For the opposite inequality $\lim\limits_{\ell\to\infty}d_\ell(H)\leq\frac{\chi(H)-2}{\chi(H)-1}$, they used the Erdős–Stone–Simonovits theorem together with an averaging argument.

When $H=K_3$, the aforementioned result of Bondy et. al. [4] implies that $d_\ell(K_3)$ decreases to $\frac{1}{2}$ as ℓ tends to infinity. They also showed that $d_3(K_3)=\frac{-1+\sqrt{5}}{2}\approx 0.61$, $d_4(K_3)>0.51$, and speculated that $d_\ell(K_3)>\frac{1}{2}$ for all $\ell\geq 3$. Refuting this conjecture, Pfender [19] proved that $d_\ell(K_k)=\frac{k-2}{k-1}$ for large enough ℓ. He also described the family $\mathcal{G}^k_\ell$ of extremal graphs; we shall define this family later in Section 2.2.

Theorem 1.1 (Pfender [19]). *For every integer $k \geq 3$ there exists a constant $C = C(k)$ such that the following holds for every integer $\ell \geq C$. If $G = (V_1 \cup \ldots \cup V_\ell, E)$ is an ℓ-partite graph with*

$$d(V_i, V_j) \geq \tfrac{k-2}{k-1} \text{ for } i \neq j,$$

then either G contains a K_k or G is isomorphic to a graph in $\mathcal{G}_\ell^k$. In particular, $d_\ell(K_k) = \frac{k-2}{k-1}$ for every $\ell \geq C$.

This theorem can be seen as a multipartite version of the Turán theorem. For an arbitrary graph H, Pfender suggested that $d_\ell(H)$ should be equal to $\frac{\chi(H)-2}{\chi(H)-1}$ for every $\ell \geq \ell_0(H)$ sufficiently large.

1.2 Our results

In this paper we shows that Pfender's suggestion is not quite true. In fact, we characterise those graphs for which the sequence $d_\ell(H)$ is eventually constant, calling them *almost colour-critical*.

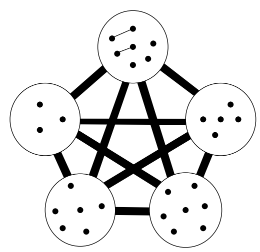

Figure 1: An almost colour-critical graph.

Definition 1.2. A graph H is called almost colour-critical if there exists a map ϕ from $V(H)$ to $\{1, 2, \ldots, \chi(H) - 1\}$ such that

(i) The induced subgraph of H on $\phi^{-1}(1)$ has maximum degree at most 1,

(ii) For $2 \leq i \leq \chi(H) - 1$, $\phi^{-1}(i)$ is an independent set of H.

In other words, an almost colour-critical graph H has a vertex-colouring with $\chi(H) - 1$ colours that is almost proper: all colour classes but one are independent sets, and the exceptional class induces just a matching (see Figure 1). For example, cliques, or, more generally colour-critical graphs, are almost colour-critical while the complete k-partite graphs $K_k(s_1, \ldots, s_k)$ are not for every $s_1 \geq 1, s_2 \geq 2, \ldots, s_k \geq 2$.

Our main result shows that almost colour-critical graphs are exactly those for which the sequence $d_\ell(H)$ is eventually constant.

Theorem 1.3. *The following statement holds for every graph H.*

(1) *If H is not almost colour-critical, then $d_\ell(H) \geq \frac{\chi(H)-2}{\chi(H)-1} + \frac{1}{(\chi(H)-1)^2(\ell-1)^2}$ for every $\ell \geq v(H)$.*

(2) *If H is an almost colour-critical graph, then there exists a positive integer $C = C(H)$ so that $d_\ell(H) = \frac{\chi(H)-2}{\chi(H)-1}$ for every $\ell > C$.*

Note that the estimate in the first statement is tight for $H = K_{1,2}$, and the second statement implies Pfender's result since cliques are almost colour-critical. This result can be viewed as a multipartite version of the Simonovits theorem. Since the proof uses the graph removal lemma, the resulting constant $C(H)$ is fairly large.

The rest of the paper deals with various extensions of Pfender's result. More precisely, we investigate the extensions of Turán's theorem discussed in Section 1.1 for balanced multipartite graphs. An ℓ-partite graph G on non-empty independent sets $V_1, \ldots, V_\ell$ is *balanced* if the vertex classes $V_1, \ldots, V_\ell$ are of the same size.

A multipartite version of the extension considered by Nikiforov [17] and Erdős [7] can be stated as follows.

Theorem 1.4. *Let k and ℓ be integers with $k \geq 3$ and $\ell \geq e^{4k^{(k+6)k}}$, and let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph on n vertices such that*

$$d(V_i, V_j) \geq \tfrac{k-2}{k-1} \quad \text{for } i \neq j.$$

Then, either G is isomorphic to a graph in $\mathcal{G}_\ell^k$ or G contains a copy of $K_{k-1}^{+}\big(\lfloor c \ln n \rfloor\big)$, where $c = k^{-(k+6)k}/2$.

For fixed k, the random graph $G_{n,p}$ shows that the lower bound $c \ln n$ on the size of the subgraph in this theorem is tight up to a constant factor.

The extension of Turán's theorem studied by Bollobás and Nikiforov [3] has the following multipartite version.

Theorem 1.5. *Let k and ℓ be integers with $k \geq 3$ and $\ell \geq k^{12k}$, and let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph on n vertices such that*

$$d(V_i, V_j) \geq \tfrac{k-2}{k-1} \quad \text{for } i \neq j.$$

Then, G either contains a family of $k^{-2k^2} n^{k-2}$ cliques of order k sharing a common edge or is isomorphic to a graph in $\mathcal{G}_\ell^k$.

With some minor modifications, this result follows from our proof of Theorem 1.4. For the sake of clarity we sketch these modifications after detailing the proof of Theorem 1.4.

1.3 Organisation

The remainder of this paper is organised as follows. In Section 2 we introduce some notation and definitions. In Section 3 we extend ideas developed in [19] to prove Theorem 1.3. A proof of Theorem 1.4 is given in Section 4. We sketch how to modify the proof of Theorem 1.4 to get Theorem 1.5 in Section 5, and close with some further remarks and open problems in Section 6.

2 Preliminaries

2.1 Notation

All graphs in this paper are finite, simple and undirected. Given a graph G, we denote its vertex and edge sets by $V(H)$ and $E(H)$, and the cardinalities of these two sets by $v(H)$ and $e(H)$, respectively. The minimum degree of G will be denoted by $\delta(G)$. For a set $U \subseteq V(G)$, we write $G[U]$ for the subgraph of G induced by U. The *common neighbourhood* $N(U)$ of U is the set of all vertices of G that are adjacent to every vertex in U. Given a vertex $v \in V(G)$, let $\deg(v, U)$ stand for the number of vertices in U adjacent to v. For pairwise disjoint vertex sets $W_1, \ldots, W_r \subseteq V(G)$, we write $G[W_1, \dots, W_r]$ for the r-colourable graph which can be obtained from $G[W_1 \cup \dots \cup W_r]$ by deletion of edges in $G[W_i]$ for all $i \leq r$.

Let G be an ℓ-partite graph on non-empty independent sets $V_1, \dots, V_\ell$. For $X \subseteq V(G)$ and $i \leq \ell$, write $X_i = X \cap V_i$. The edge density between V_i and V_j is $d_{ij} := d(V_i, V_j) := \frac{e(V_i,V_j)}{|V_i||V_j|}$.

For $r \geq 2$ and $t_1 \geq 1, \dots, t_r \geq 1$, let $K_r(t_1, \dots, t_r)$ be the complete r-partite graph with classes of sizes $t_1, \dots, t_r$. If $t_1 = \dots = t_r = t$, we simply write $K_r(t)$ instead of $K_r(t_1, \dots, t_r)$. For $r \geq 2$, $s \geq 1$ and $t_1 \geq 2s$, $t_2 \geq 1, \dots, t_r \geq 1$, we denote by $K_r^{+s}(t_1, \dots, t_r)$ the graph obtained from $K_r(t_1, \dots, t_r)$ by adding a matching of size s to the first vertex class. If $s = 1$, we omit the upper index s. In particular, $K_r^{+s}(t)$ is the short form for $K_r^{+s}(t, \dots, t)$ and $K_r^{+}(t)$ is nothing but $K_r^{+1}(t, \dots, t)$.

For $a, b, c \in \mathbb{R}$, we write $a = b \pm c$ if $b - c \leq a \leq b + c$. In order to simplify the presentation, we omit floors and ceilings and treat large numbers as integers whenever this does not affect the argument. Unless stated otherwise, all logarithms are base e.

The set $\{1, 2, \dots, n\}$ of the first n positive integers is denoted by $[n]$. For $k \in \mathbb{N}$, we define $\binom{X}{k} := \{A \subseteq X : |A| = k\}$. We use the symbol $\dot\bigcup$ for union of disjoint sets.

2.2 Extremal graphs

In this section we shall recall the definition of the family $\mathcal{G}_\ell^k$ of extremal graphs given by Pfender [19]. For $k \geq 3$ and $\ell \geq (k-1)!$, a graph G is in $\bar{\mathcal{G}}_\ell^k$ if it can be constructed as follows. Let $\{\pi_1, \pi_2, \dots, \pi_{(k-1)!}\}$ be the set of all permutations of $\{1, \dots, k-1\}$. For $1 \leq i \leq \ell$ and $1 \leq s \leq k-1$, pick non-negative integers n_i^s such that

$$n_i^{\pi_i(1)} \geq n_i^{\pi_i(2)} \geq \dots \geq n_i^{\pi_i(k-1)} \text{ for } 1 \leq i \leq (k-1)!,$$
$$n_i^1 = n_i^2 = \dots = n_i^{k-1} > 0 \text{ for } (k-1)! < i \leq \ell, \text{ and}$$
$$\sum_s n_i^s > 0 \text{ for } 1 \leq i \leq \ell.$$

Vertex and edge sets of G are defined as (see Figure 2)

$$V(G) = \{(i, s, t) : 1 \leq i \leq \ell, 1 \leq s \leq k-1, 1 \leq t \leq n_i^{(s)}\},$$
$$E(G) = \{(i, s, t)(i', s', t') : i \neq i', s \neq s'\}.$$

It is not hard to see that G is an $(k-1)$-colourable ℓ-partite graph with parts $V_i = \{(i, s, t) : 1 \leq s \leq k-1, 1 \leq t \leq n_i^s\}$ for $1 \leq i \leq \ell$, and colour classes $V^{(s)} = \{(i, s, t) : 1 \leq i \leq \ell, 1 \leq t \leq n_i^s\}$ for $1 \leq s \leq k-1$. Moreover, if all n_i^s are equal, we get $d_{ij} = \frac{k-2}{k-1}$ for every $i \neq j$. Note that other weights $n_i^{(s)}$ can be used to achieve the inequality $d_{ij} \geq \frac{k-2}{k-1}$ for every $i \neq j$.

Let $\mathcal{G}_\ell^k$ be the family of graphs which can be obtained from graphs in $\bar{\mathcal{G}}_\ell^k$ by removal of some edges in $\{(i, s, t)(i', s', t') : 1 \leq i < i' \leq (k-1)!\}$. The following simple observation by Pfender [19] will be useful for our investigation.

Lemma 2.1. *Let $k \geq 3$ and $\ell \geq (k-1)!$ be integers. If $G = (V_1 \cup \dots \cup V_\ell, E)$ is a $(k-1)$-colourable ℓ-partite graph with $d(V_i, V_j) \geq \frac{k-2}{k-1}$ for $i \neq j$, then it is isomorphic to a graph in $\mathcal{G}_\ell^k$.*

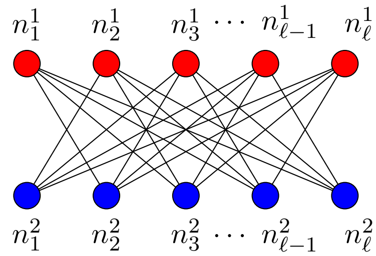

Figure 2: A graph in $\bar{\mathcal{G}}^3_\ell$, all edges between different colours in different parts exists.

2.3 Infracolourable structures

The following notation will play a key role in our investigation.

Definition 2.2. Given a real number $\eta \geq 0$, and integers $k \geq 3$ and $\ell \geq 2$, an (η, k, ℓ)-*infracolourable structure* is an ℓ-partite graph $G = (V_1 \cup \ldots \cup V_\ell, E)$ together with pairs $(D_i^{(s)}, Y_i^{(s)})_{s \leq k-1, i \leq \ell}$ satisfying:

(i) For every $i \leq \ell$, $V_i = \dot{\bigcup}_{s \leq k-1} Y_i^{(s)}$ and $\left|Y_i^{(1)}\right| \geq \left|Y_i^{(2)}\right| \geq \ldots \geq \left|Y_i^{(k-1)}\right|$;

(ii) For every $i \leq \ell$ and every $s \leq k-1$, $D_i^{(s)} \subseteq Y_i^{(s)}$ and $\bigcup_{i \leq \ell} Y_i^{(s)} \setminus D_i^{(s)}$ is an independent set;

(iii) For every $s \leq k-1$, each vertex $v \in \bigcup_{i \leq \ell} D_i^{(s)}$ has at most $\eta \cdot \frac{v(G)}{k-1}$ neighbours in $\bigcup_{i \leq \ell} Y_i^{(s)}$ and at least $3\eta \cdot \frac{v(G)}{k-1}$ non-neighbours in $\bigcup_{i \leq \ell} V_i \setminus Y_i^{(s)}$.

The graph G is called the *base graph* of the infracolourable structure.

Infracolourable structures are useful for us mainly because theirs base graphs break the density conditions in our theorems.

Lemma 2.3. *Let η be a positive real number, and let $k \geq 3$ and $\ell \geq 2$ be integers. Suppose that an ℓ-partite graph $G = (V_1 \cup \ldots \cup V_\ell, E)$ together with a system of pairs $(D_i^{(s)}, Y_i^{(s)})_{s \leq k-1, i \leq \ell}$ of vertex sets form an (η, k, ℓ)-infracolourable structure. Then*

$$e(G) \leq \tfrac{k-2}{k-1} \cdot \sum_{i<j} |V_i|\,|V_j|\,.$$

In particular, there exist two different indices i and j such that $d(V_i, V_j) \leq \frac{k-2}{k-1}$. Furthermore, the equality occurs if and only if there exists $i_0 \in \{0, 1, \ldots, \ell\}$ such that $D_i^{(s)} = \emptyset$ for all s and all i, $\left|Y_i^{(s)}\right| = \frac{1}{k-1} \cdot |V_i|$ for all s and all $i \neq i_0$, and $d(Y_i^{(s)}, Y_j^{(t)}) = 1$ for all $s \neq t$ and $i \neq j$.

Proof. It follows from the assumption that

$$\begin{aligned} e(G) &\leq \sum_{\substack{i<j \\ s \neq t}} \left|Y_i^{(s)}\right| \left|Y_j^{(t)}\right| + \left|\bigcup_{i,s} D_i^{(s)}\right| \cdot \left(\eta \cdot \frac{v(G)}{k-1} - \tfrac{1}{2} \cdot 3\eta \cdot \frac{v(G)}{k-1}\right) \\ &\leq \sum_{\substack{i<j \\ s \neq t}} \left|Y_i^{(s)}\right| \left|Y_j^{(t)}\right| = \sum_{i<j} |V_i|\,|V_j| - \sum_{\substack{i<j \\ s \leq k-1}} \left|Y_i^{(s)}\right| \left|Y_j^{(s)}\right| \leq \tfrac{k-2}{k-1} \cdot \sum_{i<j} |V_i|\,|V_j|\,, \end{aligned}$$

where in the last inequality we use Chebyshev's sum inequality. ■

To find an infracolourable structure in host graphs we shall need the following technical lemma. It was implicitly stated in [19]. We include a proof here for the sake of completeness.

Lemma 2.4. *Let $k \geq 3$ and $\ell \geq 2$ be integers, and let ε be a real number with $0 < \varepsilon < \frac{1}{4}$. Suppose that $G = (V_1 \cup \ldots \cup V_\ell, E)$ is an ℓ-partite graph with $d(V_i, V_j) \geq \frac{k-2}{k-1}$ for all $i \neq j$. Assume that $X_i^{(s)}$ and T_i be subsets of $V(G)$ for $i \leq \ell$ and $s \leq k-1$ with the following three properties:*

(i) *For every $i \leq \ell$, $V_i = X_i^{(1)} \dot\cup \ldots \dot\cup X_i^{(k-1)} \dot\cup T_i$;*

(ii) *For every $i \leq \ell$, $\left|X_i^{(1)}\right| \geq \ldots \geq \left|X_i^{(k-1)}\right|$ and $|T_i| \leq \varepsilon |V_i|$;*

(iii) *For every $s \leq k-1$, $\bigcup_{i \leq \ell} X_i^{(s)}$ is an independent set.*

Then there exists a subset $I_0 \in \binom{\mathbb{N}}{k-1}$ so that $\left|X_i^{(s)}\right| = \left(\frac{1}{k-1} \pm k\sqrt{\varepsilon}\right)|V_i|$ for $s \leq k-1$ and $i \notin I_0$.

Proof. It suffices to show that for each $s \leq k-1$ there is at most one index $i \leq \ell$ such that $\frac{\left|X_i^{(s)}\right|}{|V_i|} > \frac{1}{k-1} + \sqrt{\varepsilon}$. Assume to the contrary that $\frac{\left|X_i^{(s)}\right|}{|V_i|} \geq \frac{\left|X_j^{(s)}\right|}{|V_j|} > \frac{1}{k-1} + \sqrt{\varepsilon}$ for some s and $i \neq j$. We first prove that $\frac{\left|X_i^{(s)}\right|}{|V_i|} \leq 1 - \varepsilon$. Otherwise, if $\frac{\left|X_i^{(s)}\right|}{|V_i|} > 1 - \varepsilon$, then

$$d(V_i, V_j) \leq 1 - \frac{\left|X_i^{(s)}\right|}{|V_i|} \cdot \frac{\left|X_j^{(s)}\right|}{|V_j|} \leq 1 - (1-\varepsilon)\left(\frac{1}{k-1} + \sqrt{\varepsilon}\right) < \frac{k-2}{k-1}$$

for $k \geq 3$ and $\varepsilon < \frac{1}{4}$, as $X_i^{(s)} \cup X_j^{(s)}$ is an independent set by (iii). But this contradicts the density condition that $d(V_i, V_j) \geq \frac{k-2}{k-1}$.

We shall get a contradiction by proving that $d(V_i, V_j) < \frac{k-2}{k-1}$. Indeed, we can infer from Chebyschev's sum inequality that

$$\begin{aligned} d(V_i, V_j) &\overset{(iii)}{\leq} 1 - \frac{1}{|V_i|\,|V_j|} \cdot \sum_t \left|X_i^{(t)}\right| \left|X_j^{(t)}\right| \\ &\leq 1 - \frac{\left|X_i^{(s)}\right| \left|X_j^{(s)}\right|}{|V_i|\,|V_j|} - \frac{1}{(k-2)\,|V_i|\,|V_j|} \cdot \left(|V_i| - |T_i| - \left|X_i^{(s)}\right|\right)\left(|V_j| - |T_j| - \left|X_j^{(s)}\right|\right) \\ &= 1 - x_i x_j - \tfrac{1}{k-2}\left(1 - t_i - x_i\right)\left(1 - t_j - x_j\right), \end{aligned}$$

where $x_i = \frac{\left|X_i^{(s)}\right|}{|V_i|}$, $x_j = \frac{\left|X_j^{(s)}\right|}{|V_j|}$, $t_i = \frac{|T_i|}{|V_i|}$ and $t_j = \frac{|T_j|}{|V_j|}$. Since both x_i and x_j are bounded from below by $\frac{1}{k-1}$, the expression $f(x_i, x_j, t_i, t_j) := 1 - x_i x_j - \frac{1}{k-2}\left(1 - t_i - x_i\right)\left(1 - t_j - x_j\right)$ is decreasing with respect to both x_i and x_j. Therefore, the density $d(V_i, V_j)$ is bounded from above by

$$f(x_i, x_j, t_i, t_j) \leq f\left(\tfrac{1}{k-1} + \sqrt{\varepsilon}, \tfrac{1}{k-1} + \sqrt{\varepsilon}, t_i, t_j\right) \leq f\left(\tfrac{1}{k-1} + \sqrt{\varepsilon}, \tfrac{1}{k-1} + \sqrt{\varepsilon}, \varepsilon, \varepsilon\right) < \tfrac{k-2}{k-1},$$

where the second inequality follows from the assumption that $t_i, t_j \in [0, \varepsilon]$. However, this contradicts the assumption that $d(V_i, V_j) \geq \frac{k-2}{k-1}$. ■

3 Proof of Theorem 1.3

In this section we will prove Therem 1.3. We begin with a proof of the first assertion.

Proof of Theorem 1.3(1). We prove by contradiction. Assume that $d_\ell(H) < \frac{\chi(H)-2}{\chi(H)-1} + \frac{1}{(\chi(H)-1)^2(\ell-1)^2}$. Let $r = \chi(H) - 1$, and let $V_1, \ldots, V_\ell$ be ℓ disjoint sets of size $(\ell - 1)r$. For $i \leq \ell$, we partition V_i into r subsets $V_i^{(1)}, \ldots, V_i^{(r)}$ of size $(\ell - 1)$ each. We form a complete bipartite graph between $V_i^{(s)}$ and $V_j^{(t)}$ for $i < j$ and $s \neq t$. We then create a perfect matching in $V_1^{(1)} \cup \ldots \cup V_\ell^{(1)}$ such that there is exactly one edge between $V_i^{(1)}$ and $V_j^{(1)}$ for every $i \neq j$. The resulting graph G satisfies

$$d(V_i, V_j) = \frac{\chi(H) - 2}{\chi(H) - 1} + \frac{1}{(\chi(H) - 1)^2(\ell - 1)^2} > d_\ell(H) \quad \text{for } i \neq j.$$

Thus, by the definition of $d_\ell(H)$, G must contain a copy of H. From the construction of G, we can see that H is an almost colour-critical graph. This finishes our proof of Theorem 1.3(1). ∎

Remark 3.1. The estimate in Theorem 1.3(1) is tight for $K_{1,2}$, that is $d_\ell(K_{1,2}) = \frac{1}{(\ell-1)^2}$ for $\ell \geq 3$. Indeed, let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be an ℓ-partite graph with $d(V_i, V_j) > \frac{1}{(\ell-1)^2}$ for every $i \neq j$. We wish to show that G contains a copy of $K_{1,2}$. Suppose to the contrary that G is $K_{1,2}$-free. For $i \neq j$, we write $V_{i,j}$ for the set of vertices in V_i with at least one neighbour in V_j. Since G is $K_{1,2}$-free, we see that

(i) the edges between V_i and V_j form a perfect matching between $V_{i,j}$ and $V_{j,i}$ for every $i \neq j$;

(ii) $V_{i,j}$ and $V_{i,j'}$ are disjoint for all distinct indices i, j and j'.

Notice that $V_{i,j}$ is non-empty for every $i \neq j$ as $d(V_i, V_j) > 0$. Combining this with property (ii), we conclude that

$$|V_i| \geq \sum_{j \in [\ell] \setminus \{i\}} |V_{i,j}| \geq \ell - 1 \text{ for } i \leq \ell. \tag{1}$$

Hence

$$\sum_{1 \leq i < j \leq \ell} \left(\frac{|V_{i,j}|}{|V_i|} + \frac{|V_{j,i}|}{|V_j|} \right) = \sum_{1 \leq i \leq \ell} \left(\sum_{j' \neq i} \frac{|V_{i,j'}|}{|V_i|} \right) \leq \ell.$$

Consequently, there exist $1 \leq i < j \leq \ell$ with $\frac{|V_{i,j}|}{|V_i|} + \frac{|V_{j,i}|}{|V_j|} \leq \frac{\ell}{\binom{\ell}{2}} = \frac{2}{\ell-1}$. By appealing to the AM-GM inequality, we thus get $\sqrt{|V_{i,j}|\,|V_{j,i}|} \leq \frac{1}{\ell-1} \cdot \sqrt{|V_i|\,|V_j|}$. This forces

$$d(V_i, V_j) \overset{(i)}{=} \frac{|V_{i,j}|}{|V_i|\,|V_j|} \overset{(i)}{=} \frac{\sqrt{|V_{i,j}|\,|V_{j,i}|}}{|V_i|\,|V_j|} \leq \frac{1}{(\ell - 1)\sqrt{|V_i|\,|V_j|}} \overset{(1)}{\leq} \frac{1}{(\ell - 1)^2},$$

contradicting the assumption that $d(V_i, V_j) > \frac{1}{(\ell-1)^2}$.

To handle the second statement of Theorem 1.3, we shall prove a stronger result.

Theorem 3.2. *Let H be an almost colour-critical graph. Then, there exists a constant $C = C(H)$ such that for every integer $\ell > C$, every ℓ-partite graph $G = (V_1 \cup \ldots \cup V_\ell, E)$ with*

$$d(V_i, V_j) > \frac{\chi(H) - 2}{\chi(H) - 1} \quad \textit{for } i \neq j$$

contains a copy of H whose vertices are in different parts of G.

Remark 3.3. Suppose that H is almost colour-critical. Let $k = \chi(H)$ and $q = v(H)$. From the definition of almost colour-critical graphs, H is a subgraph $K_{k-1}^{+q}(2q)$. Moreover, it is easy to see that $\chi(K_{k-1}^{+q}(2q)) = k = \chi(H)$ and $K_{k-1}^{+q}(2q)$ is almost colour-critical. Therefore, if Theorem 3.2 holds for $K_{k-1}^{+q}(2q)$, it will hold for H as well.

The main idea of the proof of Theorem 3.2 is as follows. Let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a counterexample. We first apply a stability result (Lemma 3.4) to obtain an induced $(\chi(H)-1)$-colourable subgraph of G which almost spans $V(G)$. Using embedding results (Lemmas 3.8 and 3.6) we can then show that there exists a subset $I \subseteq [\ell]$ such that $G[\bigcup_{i\in I} V_i]$ is the base graph of an $(\eta, k, |I|)$-infracolourable structure. But according to Lemma 2.3, this forces $d(V_i, V_j) \le \frac{k-2}{k-1}$ for some $i, j \in I$, violating the density condition.

Our first step in the proof of Theorem 3.2 will be to show that a counterexample G must contain an induced $(\chi(H)-1)$-colourable subgraph which almost spans $V(G)$. For that we shall need the following stability result.

Lemma 3.4. *Given integers $k \ge 3$ and $q \ge 1$ and a real number $0 < \varepsilon < \frac{1}{8k^2q}$, there exists a constant $C = C(k, q, \varepsilon)$ such that the following holds for $\ell \ge C$. Let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph on n vertices with $d(V_i, V_j) \ge \frac{k-2}{k-1}$ for all $i \ne j$. Suppose G contains no copy of $K_{k-1}^{+q}(2q)$ whose vertices lie in different parts of G. Then, G contains an induced $(k-1)$-colourable subgraph F whose vertex classes $X^{(1)}, \ldots, X^{(k-1)}$ satisfy the following properties*

(i) *For $s \le k-1$, $\left|X^{(s)}\right| = \left(\frac{1}{k-1} \pm \varepsilon\right) n$;*

(ii) *For $s \le k-1$ and $v \in \bigcup_{t \ne s} X^{(t)}$, $\deg(v, X^{(s)}) \ge \left|X^{(s)}\right| - \varepsilon n$.*

To prove Lemma 3.4 we require the following result whose proof can be found in Section 5.

Proposition 3.5. *For every graph H and every $\varepsilon > 0$, there exist positive constants $\gamma = \gamma(H, \varepsilon)$ and $C = C(H, \varepsilon)$ such that the following holds for $n \ge C$. Suppose that G is an n-vertex graph with $e(G) \ge \left(\frac{\chi(H)-2}{\chi(H)-1} - \gamma\right)\binom{n}{2}$ containing at most $\gamma n^{v(H)}$ copies of H. Then, G contains a $(\chi(H)-1)$-colourable subgraph of order at least $(1-\varepsilon)n$ and minimum degree at least $\left(\frac{\chi(H)-2}{\chi(H)-1} - \varepsilon\right) n$.*

Another tool that will be used in the proof of Lemma 3.4 and Theorem 3.2 is an embedding result. Before stating it, we shall introduce the necessary terminology. Let $G[W^{(1)}, \ldots, W^{(r)}]$ be an r-colourable graph such that $W^{(s)} = \dot{\bigcup}_{i\ge 1} W_i^{(s)}$ for every $s \le r$. We call an embedding $f : K_r(a_1, \ldots, a_r) \to G$ *good* if the sth vertex class of $K_r(a_1, \ldots, a_r)$ is mapped to $W^{(s)}$ for every $s \le r$, and for each index i there is at most one vertex $v \in K_r(a_1, \ldots, a_r)$ with $f(v) \in \bigcup_{s \le r} W_i^{(s)}$.

Lemma 3.6. *Suppose that $r \ge 2$ and $q \ge 1$ are integers, and let $G[W^{(1)}, \ldots, W^{(r)}]$ be an r-colourable graph which satisfies the following properties*

(i) *For $s \le r$, $W^{(s)} = \dot{\bigcup}_i W_i^{(s)}$ and $\left|W_i^{(s)}\right| < \frac{1}{2rq} \cdot \left|W^{(s)}\right|$ for all i,*

(ii) *For $s \le r$ and $v \in \bigcup_{t \ne s} W^{(t)}$, $\deg(v, W^{(s)}) > (1 - \frac{1}{2rq}) \cdot \left|W^{(s)}\right|$.*

Then, for every r-tuple of integers $a_1, \ldots, a_r \in [0, q]$, every good embedding from $K_r(a_1, \ldots, a_r)$ to G can be extended to a good embedding from $K_r(q)$ to G.

Proof. Suppose f is a good embedding from $K_r(a_1, \ldots, a_r)$ to G. To prove the lemma, it suffices to show that f can be extended to a good embedding g from $K_r(a_1, \ldots, a_s + 1, \ldots, a_r)$ to G whenever

$a_s \leq q-1$. Let v be the vertex of $K_r(a_1,\dots,a_s+1,\dots,a_r)$ which is not in $K_r(a_1,\dots,a_r)$, and let X denote the set of vertices of $K_r(a_1,\dots,a_r)$ which are not in the sth vertex class. By property (ii), we see that each vertex of X has at most $\frac{1}{2rq}\cdot\left|W^{(s)}\right|$ non-neighbours in $W^{(s)}$, and thus $\left|N(X)\cap W^{(s)}\right| \geq \left|W^{(s)}\right| - |X|\cdot\frac{\left|W^{(s)}\right|}{2rq} \geq \frac{1}{2}\left|W^{(s)}\right|$. Note that, by property (i), each vertex of X can forbid at most $\frac{1}{2rq}\cdot\left|W^{(s)}\right|$ vertices of $W^{(s)}$ from being the image of v. Therefore, the number of possible images of v under g is at least $\left|N(X)\cap W^{(s)}\right| - |X|\cdot\frac{\left|W^{(s)}\right|}{2rq} \geq \frac{1}{2}\left|W^{(s)}\right| - |X|\cdot\frac{\left|W^{(s)}\right|}{2rq} > 0$, where in the last inequality we use the inequality $\left|W^{(s)}\right| > 0$ which is implied by property (i). ■

Proof of Lemma 3.4. We denote $H = K_{k-1}^{+q}(2q)$, and let

$$\gamma = \gamma_{3.5}\left(H,\tfrac{\varepsilon}{2k}\right),\ C = \max\left\{2k^2q^2\gamma^{-1}, 8(k-1)^2q, 4(k-1)q\varepsilon^{-1}, C_{3.5}(H,\tfrac{\varepsilon}{2k})\right\}.$$

Because $G = (V_1\cup\ldots\cup V_\ell, E)$ is a balanced ℓ-partite graph on n vertices, we must have

$$|V_1| = |V_2| = \ldots = |V_\ell| = \frac{n}{\ell} := m. \tag{2}$$

In the first step, we shall use Proposition 3.5 to show that G contains an almost spanning $(k-1)$-colourable subgraph. Indeed, by the choice of C we see that $n \geq \ell \geq C \geq C_{3.5}(H,\frac{\varepsilon}{2k})$. Moreover, since G contains no copy of H whose vertices lie in different parts of G, the number of copies of H in G is at most

$$\binom{v(H)}{2}\ell m^2 n^{v(H)-2} < \tfrac{2k^2q^2}{\ell}\cdot(\ell m)^2 n^{v(H)-2} \leq \gamma n^{v(H)},$$

since $n=\ell m$ and $\ell \geq C \geq 2k^2q^2\gamma^{-1}$. Also, by the density condition

$$e(G) \geq \binom{\ell}{2}\tfrac{k-2}{k-1}m^2 \overset{(2)}{\geq} \left(\tfrac{k-2}{k-1}-\tfrac{1}{\ell}\right)\binom{n}{2} \geq \left(\tfrac{k-2}{k-1}-\gamma\right)\binom{n}{2},$$

assuming $\ell \geq C \geq 2k^2q^2\gamma^{-1}$. Therefore, we can derive from Proposition 3.5 that G contains a $(k-1)$-colourable subgraph F' with

$$v(F') \geq (1-\tfrac{\varepsilon}{2k})n \text{ and } \delta(F') \geq \left(\tfrac{k-2}{k-1}-\tfrac{\varepsilon}{2k}\right)n. \tag{3}$$

If $W^{(1)},\dots,W^{(k-1)}$ are vertex classes of F', then (3) implies that

$$\left(\tfrac{1}{k-1}-\tfrac{\varepsilon}{2}\right)n \leq \left|W^{(s)}\right| \leq \left(\tfrac{1}{k-1}+\tfrac{\varepsilon}{2k}\right)n \quad \text{for } s \leq k-1. \tag{4}$$

In the second step, we shall prove that the induced subgraph $G[V(F')]$ of G does not contain a large monochromatic matching whose vertices are in different parts of G. Indeed, for $s \leq k-1$, let $\mathcal{M}_{(s)}$ denote a maximum matching in $G[W^{(s)}]$ whose vertices are in different parts of G, and let K be a subset of $[\ell]$ containing all indices i such that $\bigcup_{s\leq k-1}\mathcal{M}_{(s)}$ has a vertex in V_i. The size of K will be bounded from above in terms of k and q.

Claim 3.7. *$|K| < 2(k-1)q$.*

Proof. We prove the claim by contradiction. Suppose that for some $s \le k-1$, $\mathcal{M}_{(s)}$ contains a matching of size q, say $\{x_1x_2, \dots, x_{2q-1}x_{2q}\}$, . We wish to show that the following two properties holds:

(i) For $t \le k-1$ and $i \le \ell$, $W^{(t)} = W_1^{(t)} \dot\cup \dots \dot\cup W_\ell^{(t)}$ and $\left|W_i^{(t)}\right| < \frac{1}{4(k-1)q} \cdot \left|W^{(t)}\right|$;

(ii) For $t \le k-1$ and $v \in V(F') \setminus W^{(t)}$, $\deg_{F'}(v, W^{(t)}) > \left(1 - \frac{1}{4(k-1)q}\right) \cdot \left|W^{(t)}\right|$.

Property (i) follows from the estimate

$$\left|W_i^{(t)}\right| \le |V_i| = \frac{n}{\ell} < \frac{1}{4(k-1)q} \cdot \left(\frac{1}{k-1} - \frac{\varepsilon}{2}\right) n \overset{(4)}{<} \frac{1}{4(k-1)q} \cdot \left|W^{(t)}\right|$$

for $\ell \ge C \ge 8(k-1)^2 q$ and $\varepsilon < \frac{1}{8k^2q}$. To prove (ii), assume that $v \in W^{(s)}$ for some $s \ne t$. Because $W^{(s)}$ is an independent set in F', one has $\left|W^{(t)}\right| - d_{F'}(v, W^{(t)}) \le v(F') - \left|W^{(s)}\right| - \deg_{F'}(v)$. Hence by appealing to (3) and (4), we get

$$\begin{aligned}\left|W^{(t)}\right| - d_{F'}(v, W^{(t)}) &\le n - \left(\frac{1}{k-1} - \frac{\varepsilon}{2}\right) n - \left(\frac{k-2}{k-1} - \frac{\varepsilon}{2k}\right) n \\ &\le \varepsilon n < \frac{1}{4(k-1)q} \cdot \left(\frac{1}{k-1} - \frac{\varepsilon}{2}\right) n \le \frac{1}{4(k-1)q} \cdot \left|W^{(t)}\right|\end{aligned}$$

for $\varepsilon < \frac{1}{8k^2q}$. This finishes our verification of (i) and (ii).

Finally, properties (i) and (ii) ensure that we can apply Lemma 3.6 with $r_{3.6} = k-1$ and $q_{3.6} = 2q$ to $G[W^{(1)}, \dots, W^{(r)}]$ to find a copy of $K_{k-1}(2q)$ whose sth vertex class is $\{x_1, \dots, x_{2q}\}$ and vertices lie in different parts of G. Since $\{x_1, x_2, \dots, x_{2q-1}x_{2q}\}$ is a matching in G, the graph G contains a desired copy of H, which contradicts our hypothesis. ■

To finish the proof, we shall show that G contains an induced subgraph F with the desired properties. For this purpose, we let $X^{(s)} = W^{(s)} \setminus \bigcup_{i \in K} V_i$ for $s \le k-1$. The maximality of $\mathcal{M}_{(s)}$ implies that $X^{(s)}$ is an independent set in G. So the induced subgraph $F = G[X^{(1)} \cup \dots \cup X^{(k-1)}]$ is $(k-1)$-colourable. What is left is to prove that F has the desired properties. Since $\varepsilon < \frac{1}{8k^2q}$ and $\ell \ge C \ge 4(k-1)q\varepsilon^{-1}$, we find that

$$v(F) \ge v(F') - \left|\bigcup_{i \in K} V_i\right| \overset{(3),\text{Claim 3.7}}{\ge} \left(1 - \tfrac{\varepsilon}{2k}\right) n - 2(k-1)q \cdot \frac{n}{\ell} > (1-\varepsilon)n,$$

$$\delta(F) \ge \delta(F') - \left|\bigcup_{i \in K} V_i\right| \overset{(3),\text{Claim 3.7}}{\ge} \left(\tfrac{k-2}{k-1} - \tfrac{\varepsilon}{2k}\right) n - 2(k-1)q \cdot \frac{n}{\ell} > \left(\tfrac{k-2}{k-1} - \tfrac{\varepsilon}{2}\right) n.$$

Moreover, by (4) we see that $\left|X^{(s)}\right| \le \left|W^{(s)}\right| \le \left(\frac{1}{k-1} + \frac{\varepsilon}{2k}\right) n$ for $s \le k-1$, and hence $\left(\frac{1}{k-1} - \frac{\varepsilon}{2}\right) n \le \left|X^{(s)}\right| \le \left(\frac{1}{k-1} + \frac{\varepsilon}{2k}\right) n$ for $s \le k-1$. Therefore, for $s \le k-1$ and $v \in \bigcup_{t \ne s} X^{(t)}$, there are at most $n - \left|X^{(s)}\right| - d_F(v) \le n - \left(\frac{1}{k-1} - \frac{\varepsilon}{2}\right) n - \left(\frac{k-2}{k-1} - \frac{\varepsilon}{2}\right) n = \varepsilon n$ missing edges in F between v and $X^{(s)}$. This completes our proof of Lemma 3.4. ■

We also need the following elementary lemma. It is probably well-known, but we could not find a reference. For completeness we include its proof in Section 5.

Lemma 3.8. *Given integers $r \geq 1$ and $q \geq 2$ and a real number $d \in (0,1)$, there exist an integer $D = D(r,q,d)$ and a positive $\rho = \rho(r,q,d)$ so that the following holds. Suppose that G is an $(r+1)$-colourable graph with vertex classes $U, W_{(1)}, \ldots, W_{(r)}$. If $|U| \geq D$ and $\deg(u, W_{(s)}) \geq d\left|W_{(s)}\right|$ for all $u \in U$ and $s \leq r$, then there is a subset $A \in \binom{U}{q}$ with $\left|N(A) \cap W_{(s)}\right| \geq \rho\left|W_{(s)}\right|$ for $s \leq r$.*

To find an infracolourable structure in G we shall make use of a consequence of Lemmas 3.6 and 3.8.

Lemma 3.9. *Given integers $k \geq 3$ and $q \geq 1$ and a real number $\eta \in (0,1)$, there exist integers $C = C(k,q,\eta)$ and $D = D(k,q,\eta)$ and a positive $\delta = \delta(k,q,\eta)$ such that the following holds for $\ell \geq C$ and $\varepsilon \in (0,\delta)$. Suppose that $G = (V_1 \cup \ldots \cup V_\ell, E)$ is a balanced ℓ-partite graph containing no copy of $K_k(2q)$ in G whose vertices are in different parts of G. Assume $(X_i^{(s)})_{s \leq k-1, i \leq \ell}$ are vertex sets satisfying:*

(i) *For $i \leq \ell$, $X_i^{(1)}, \ldots, X_i^{(k-1)}$ are disjoint subsets of V_i,*

(ii) *For $i \leq \ell$ and $s \leq k-1$, $\left|X_i^{(s)}\right| = \left(\frac{1}{k-1} \pm \varepsilon\right)|V_i|$,*

(iii) *For every $s \leq k-1$ and $v \in \bigcup_{i \leq \ell, t \neq s} X_i^{(t)}$, $\deg(v, \bigcup_{i \leq \ell} X_i^{(s)}) \geq \left|\bigcup_{i \leq \ell} X_i^{(s)}\right| - \varepsilon \cdot v(G)$.*

Let I be the subset of $[\ell]$ consisting of all indices $i \in [\ell]$ such that V_i contains a vertex v with $\deg(v, \bigcup_{j \leq \ell} X_j^{(s)}) \geq \eta \cdot v(G)$ for $s \leq k-1$. Then $|I| \leq D$.

Proof. Let $D = D_{3.8}\left(k-1, 2q, \frac{k\eta}{4}\right)$, $C = \max\left\{4kD, 2\eta^{-1}D, \frac{9(k-1)kq}{\rho}\right\}$ and $\delta = \min\left\{\frac{1}{4k}, \frac{\rho}{8(k-1)kq}\right\}$, where $\rho = \rho_{3.8}\left(k-1, 2q, \frac{k\eta}{4}\right)$. We shall prove the lemma by contradiction. Assume that $|I| \geq D$. Let J be an arbitrary subset of I of size D. By the definition of I, for each index $j \in J$ we can find a vertex $v_j \in V_j$ such that $\deg(v_j, \bigcup_{i \leq \ell} X_i^{(s)}) \geq \eta \cdot v(G)$ for $s \leq k-1$. Let $U = \{v_j : j \in J\}$.

For simplicity of notation, let $X^{(s)} := \bigcup_{i \leq \ell} X_i^{(s)}$ and $W^{(s)} := \bigcup_{i \in [\ell] \setminus J} X_i^{(s)}$ for $s \leq k-1$. Then, property (i) implies that $W^{(1)}, \ldots, W^{(k-1)}$ are disjoint subsets of $V(G)$. By (i) and (ii), we find that

$$\left|W^{(s)}\right| \geq \left(\frac{1}{k-1} - \varepsilon - \frac{D}{\ell}\right) \cdot v(G) \geq \frac{v(G)}{2k} \tag{5}$$

for $\varepsilon \leq \delta \leq \frac{1}{4k}$ and $\ell \geq C \geq 4kD$. Also, (i) and (ii) force $\left|W^{(s)}\right| \leq \left(\frac{1}{k-1} + \varepsilon\right) v(G) \leq \frac{2v(G)}{k}$, since $\varepsilon \leq \delta \leq \frac{1}{4k}$. Combining these two inequalities, we conclude that

$$\deg(v, W^{(s)}) \geq \deg(v, X^{(s)}) - \left|\bigcup_{j \in J} V_j\right| \geq \eta \cdot v(G) - D \cdot \frac{v(G)}{\ell} \geq \frac{\eta}{2} \cdot v(G) \geq \frac{k\eta}{4} \cdot \left|W^{(s)}\right|$$

for $v \in U$ and $s \leq k-1$, as $\ell \geq 2\eta^{-1}D$. Furthermore, $|U| = D = D_{3.8}\left(k-1, 2q, \frac{k\eta}{4}\right)$, by the definition of D. By applying Lemma 3.8 to $G[U, W^{(1)}, \ldots, W^{(k-1)}]$ with $r_{3.8} = k-1$, $q_{3.8} = 2q$ and $d_{3.8} = \frac{k\eta}{4}$, we thus obtain a subset $A \in \binom{U}{2q}$ with

$$\left|N(A) \cap W^{(s)}\right| \geq \rho\left|W^{(s)}\right| \quad \text{for } s \leq k-1. \tag{6}$$

In the rest of the proof we shall use Lemma 3.6 to show that $G[N(A) \cap W^{(1)}, \ldots, N(A) \cap W^{(k-1)}]$ contains a copy of $K_{k-1}(2q)$ whose vertices are in different parts of G. Since this copy lies in $N(A)$, together with vertices of A it forms a copy of $K_k(2q)$ whose vertices belong to different parts of G, contradicting the assumption. It remains to verify the assumptions of Lemma 3.6. Indeed, for $s \le k-1$, $N(A) \cap W^{(s)}$ does admit the partition

$$N(A) \cap W^{(s)} = \dot{\bigcup}_{j \notin J} \big(N(A) \cap X_j^{(s)}\big). \tag{7}$$

Moreover, since $N(A) \cap W^{(s)} \subseteq X^{(s)}$ for $s \le k-1$, we must have, for $s \le k-1$ and $v \in \bigcup_{t \neq s} \big(N(A) \cap W^{(t)}\big)$,

$$\begin{aligned}\left|N(A) \cap W^{(s)}\right| - \deg(v, N(A) \cap W^{(s)}) &\le \left|\bigcup_{i \le \ell} X_i^{(s)}\right| - \deg(v, \bigcup_{i \le \ell} X_i^{(s)}) \\ &\overset{(iii)}{\le} \varepsilon \cdot v(G) \le \tfrac{1}{4(k-1)q} \cdot \rho \cdot \frac{v(G)}{2k} \overset{(5),(6)}{\le} \tfrac{1}{4(k-1)q} \cdot \left|N(A) \cap W^{(s)}\right|,\end{aligned}$$

assuming $\varepsilon \le \delta \le \frac{\rho}{8(k-1)kq}$. It can be rewritten as

$$\deg(v, N(A) \cap W^{(s)}) \ge \left(1 - \tfrac{1}{4(k-1)q}\right) \left|N(A) \cap W^{(s)}\right| \text{ for } s \le k-1 \text{ and } v \notin \bigcup_{t \neq s} (N(A) \cap W^{(t)}. \tag{8}$$

Also, for every $j \notin J$ and $s \le k-1$, we have

$$\left|N(A) \cap X_j^{(s)}\right| \le |V_j| = \tfrac{v(G)}{\ell} < \tfrac{1}{4(k-1)q} \cdot \rho \cdot \frac{v(G)}{2k} \overset{(5),(6)}{\le} \tfrac{1}{4(k-1)q} \cdot \left|N(A) \cap W^{(s)}\right| \tag{9}$$

because $\ell \ge C \ge \frac{9(k-1)kq}{\rho}$. The inequalities (7), (8) and (9) show that we can apply Lemma 3.6 to $G[N(A) \cap W^{(1)}, \dots, N(A) \cap W^{(k-1)}]$ with $r_{3.6} = k-1$ and $q_{3.6} = 2q$. ■

We also require another consequence of Lemma 3.6, stated below.

Lemma 3.10. *Given integers $k \ge 3$ and $q \ge 1$ and a real number $\eta \in \left(\frac{2q-1}{2(k-1)q}, 1\right)$, there exist an integer $C = C(k, q, \eta)$ and a positive $\delta = \delta(k, q, \eta)$ such that the following holds for every integer $\ell \ge C$ and every $\varepsilon \in (0, \delta)$. Let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph containing no copy of $K_{k-1}^{+q}(2q)$ whose vertices are in different parts of G. Assume $(X_i^{(s)}, Y_i^{(s)})_{s \le k-1, i \le \ell}$ are pairs of vertex sets satisfying:*

(i) *For $i \le \ell$ and $s \le k-1$, $Y_i^{(1)}, \dots, Y_i^{(k-1)}$ are disjoint subsets of V_i and $X_i^{(s)} \subseteq Y_i^{(s)}$,*

(ii) *For $i \le \ell$ and $s \le k-1$, $\left|X_i^{(s)}\right| = \left(\frac{1}{k-1} \pm \varepsilon\right) |V_i|$,*

(iii) *For $s \le k-1$ and $v \in \bigcup_{i \le \ell, t \neq s} X_i^{(t)}$, $\deg(v, \bigcup_{i \le \ell} X_i^{(s)}) \ge \left|\bigcup_{i \le \ell} X_i^{(s)}\right| - \varepsilon \cdot v(G)$.*

For $i \le \ell$ and $s \le k-1$, let $B_i^{(s)}$ denote a subset of $Y_i^{(s)}$ consisting of all vertices v with $\deg(v, \bigcup_{j \le \ell} X_j^{(t)}) < \eta \cdot v(G)$ for some $t \neq s$. For $s \le k-1$, write $\mathcal{M}_{(s)}$ for a maximal matching in the induced subgraph $G\big[\bigcup_{i \le \ell} Y_i^{(s)} \setminus B_i^{(s)}\big]$ of G whose vertices are in different parts of G, and set $J = \{j \in [\ell] : V_j$ contains some vertex in $\bigcup_{s \le k-1} \mathcal{M}_{(s)}\}$. Then, $|J| < 2(k-1)q$.

Proof. Choose

$$C = \frac{4(k-2)}{\eta'} \text{ and } \delta = \min\left\{\frac{q\eta'}{2q-1}, \frac{\eta'}{4(k-2)}\right\}, \text{ where } \eta' = \eta - \frac{2q-1}{2(k-1)q}.$$

Notice that $\eta' > 0$ as $\eta \in \left(\frac{2q-1}{2(k-1)q}, 1\right)$. We prove the statement by contradiction. Suppose that $\mathcal{M}_{(s)}$ contains a matching $\{x_1x_2, \ldots, x_{2q-1}x_{2q}\}$ of size q for some $s \leq k-1$. Let $X^{(t)}$ denote the vertex set $\bigcup_i X_i^{(s)}$ for $s \leq k-1$. For $t \neq s$, define $W_{(t)} = \bigcup_i \big(N(x_1, \ldots, x_{2q}) \cap X_i^{(t)}\big)$. Then property (i) implies that $W^{(1)}, \ldots, W^{(k-1)}$ are disjoint subsets of $V(G)$. We shall apply Lemma 3.6 to find a copy of $K_{k-2}(2q)$ in $G\big[W_{(1)}, \ldots, \widehat{W_{(s)}}, \ldots, W_{(k-1)}\big]$ whose vertices are in different parts of G (here $\widehat{W_{(s)}}$ stands for the empty set). Since this copy lies in $N(x_1, \ldots, x_{2q})$ and since $\{x_1x_2, \ldots, x_{2q-1}x_{2q}\}$ is a matching, G contains a copy of $K_{k-1}^{+q}(2q)$ whose vertices belong to different parts of G, which is impossible. The remaining task is thus to verify the assumptions of Lemma 3.6. Indeed, from the definition of $W_{(t)}$ we see that, for $t \neq s$,

$$W_{(t)} = \dot{\bigcup_i} \big(N(x_1, \ldots, x_{2q}) \cap X_i^{(t)}\big). \tag{10}$$

By the definition of $\mathcal{M}_{(s)}$, we have $\deg(x, X^{(t)}) \geq \eta \cdot v(G)$ for $x \in \{x_1, \ldots, x_{2q}\}$ and $t \neq s$. Hence

$$\begin{aligned} \left|W_{(t)}\right| = \left|N(x_1, \ldots, x_{2q}) \cap X^{(t)}\right| &\geq 2q\eta \cdot v(G) - (2q-1)\left|X^{(t)}\right| \\ &\overset{(ii)}{\geq} 2q\eta \cdot v(G) - (2q-1)\left(\tfrac{1}{k-1} + \varepsilon\right) v(G) \geq q\eta' \cdot v(G) \end{aligned} \tag{11}$$

for $\varepsilon \leq \delta \leq \frac{q\eta'}{2q-1}$. Together with the assumption $\ell \geq C = \frac{4(k-2)}{\eta'}$, this inequality implies that, for $i \leq \ell$ and $t \neq s$,

$$\left|N(x_1, \ldots, x_{2q}) \cap X_i^{(t)}\right| \leq |V_i| = \frac{v(G)}{\ell} \leq \tfrac{q\eta'}{4(k-2)q} \cdot v(G) \leq \tfrac{1}{4(k-2)q} \cdot \left|W_{(t)}\right|. \tag{12}$$

On the other hand, we can derive from property (iii) that, for $v \in \bigcup_{i \leq \ell, p \notin \{s,t\}} X_i^{(p)}$,

$$\left|W_{(t)}\right| - \deg(v, W_{(t)}) \leq \varepsilon \cdot v(G) \leq \tfrac{q\eta'}{4(k-2)q} \cdot v(G) \overset{(11)}{\leq} \tfrac{1}{4(k-2)q} \cdot \left|W_{(t)}\right|, \tag{13}$$

assuming $\varepsilon \leq \delta \leq \frac{\eta'}{4(k-2)}$. It follows from (10), (12) and (13) that we can apply Lemma 3.6 to $G\big[W_{(1)}, \ldots, \widehat{W_{(s)}}, \ldots, W_{(k-1)}\big]$ with $r_{3.6} = k-2$ and $q_{3.6} = 2q$. ■

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2. Let $k = \chi(H)$. If $k = 2$, then H is a matching. The density condition implies that there is at least one edge between any two parts of G. Hence G contains a matching of size $\frac{\ell}{2} \geq e(H)$ whose vertices are in different parts of G. So from now on we can focus on the case when $k \geq 3$. Moreover, as discussed in Remark 3.3, we can suppose that $H = K_{k-1}^{+q}(2q)$ for some positive integer q. To prove Theorem 3.2, we assume to the contrary that G does not contain a copy of H whose vertices are in different parts of G. Without loss of generality we can suppose that each part of G has exactly m vertices, where m is a sufficiently large integer. Otherwise, multiply each vertex in each part V_i by a factor of $\frac{m}{|V_i|}$, which has no effect on the densities, and creates no copy of H whose vertices lie in different parts of G.

Choose $\ell = \max\{C_{3.4}(k,q,\varepsilon), 1/\varepsilon\}$, where $\varepsilon > 0$ is sufficiently small (to be specified later). Let $\ell_1 = \frac{\ell}{2(k-1)!}$, $\ell_2 = \ell_1 - (k-1)$, $\ell_3 = \frac{\ell_2}{(k-1)!}$ and $\ell_4 = \ell_3 - 2(k-1)q - D$, where $D = D_{3.9}\left(k, q, \frac{1}{(6q+10)(k-1)(k-1)!}\right)$. Note that the parameters ℓ and ℓ_i both grow as $\Omega(1/\varepsilon)$.

Our goal is to find an infracolourable struture in G. In the first step, we apply Lemma 3.4 to G with $k_{3.4} = k$, $q_{3.4} = q$ and $\varepsilon_{3.4} = \varepsilon < \frac{1}{8k^2q}$ to obtain an induced $(k-1)$-colourable subgraph F of G whose vertex classes $X^{(1)}, \dots, X^{(k-1)}$ satisfy

$$\left|X^{(s)}\right| = \left(\tfrac{1}{k-1} \pm \varepsilon\right) n \text{ for } s \leq k-1, \tag{14}$$

$$\deg(v, X^{(s)}) \geq \left|X^{(s)}\right| - \varepsilon n \text{ for } s \leq k-1 \text{ and } v \in \bigcup_{t \neq s} X^{(t)}. \tag{15}$$

Let $T = V(G) \setminus V(F)$. The inequality (14) implies that $|T| \leq k\varepsilon n$. This forces $|T_i| \leq 2k\varepsilon m$ for at least half of indices $i \leq \ell$. Since $\ell_1 = \frac{\ell}{2(k-1)!}$, by the pigeon hole principle we can relabel the V_i and the $X^{(s)}$ such that $\left|X_i^{(1)}\right| \geq \left|X_i^{(2)}\right| \geq \ldots \geq \left|X_i^{(k-1)}\right|$ and

$$|T_i| \leq 2k\varepsilon m \text{ for } i \leq \ell_1. \tag{16}$$

Hence we can apply Lemma 2.4 with $\varepsilon_{2.4} = 2k\varepsilon < \frac{1}{4}$ to find a subset $I_0 \in \binom{\mathbb{N}}{k-1}$ such that $\left|X_i^{(s)}\right| = \left(\frac{1}{k-1} \pm k\sqrt{2k\varepsilon}\right) m$ for $s \leq k-1$ and $i \in [\ell_1] \setminus I_0$. By reordering parts if necessary, we may assume that

$$\left|X_i^{(s)}\right| = \left(\tfrac{1}{k-1} \pm k\sqrt{2k\varepsilon}\right) m \quad \text{for } s \leq k-1 \text{ and } i \leq \ell_2. \tag{17}$$

For $i \leq \ell_2$ we shall partition V_i into $k-1$ subsets $Y_i^{(1)}, \dots, Y_i^{(k-1)}$ as follows. A vertex $v \in V_i$ is assigned to $Y_i^{(s)}$ if $\deg\left(v, \bigcup_{j \leq \ell_2} X_j^{(s)}\right) = \min_{t \leq k-1} \deg\left(v, \bigcup_{j \leq \ell_2} X_j^{(t)}\right)$; if there are more than one such index s, arbitrarily choose one of them.

Claim 3.11. *$X_i^{(s)} \subseteq Y_i^{(s)} \subseteq X_i^{(s)} \dot\cup T_i$ and $\left|Y_i^{(s)}\right| = \left(\frac{1}{k-1} \pm 2k\sqrt{2k\varepsilon}\right) m$ for $s \leq k-1$ and $i \leq \ell_2$.*

Proof. Let v be an arbitrary vertex of $X_i^{(s)}$. Since $X^{(s)}$ is an independent set of G, v has no neighbours in $\bigcup_{j \leq \ell_2} X_j^{(s)}$. It thus follows from the definition of $Y_i^{(s)}$ that $v \in Y_i^{(s)}$, and so $X_i^{(s)}$ is a subset of $Y_i^{(s)}$. Combining with the fact that $V_i = \left(\dot\bigcup_s X_i^{(s)}\right) \dot\cup T_i = \dot\bigcup_s Y_i^{(s)}$, we conclude that $Y_i^{(s)} \subseteq X_i^{(s)} \dot\cup T_i$ for $i \leq \ell_2$ and $s \leq k-1$.

As $X_i^{(s)}$ is a subset of $Y_i^{(s)}$, (17) tells us that $\left|Y_i^{(s)}\right| \geq \left|X_i^{(s)}\right| \geq \left(\frac{1}{k-1} - k\sqrt{2k\varepsilon}\right) m$ for $i \leq \ell_2$ and $s \leq k-1$. Using (16) and (17), we get

$$\left|Y_i^{(s)}\right| \leq \left|X_i^{(s)}\right| + |T_i| \leq \left(\tfrac{1}{k-1} + k\sqrt{2k\varepsilon} + 2k\varepsilon\right) m \leq \left(\tfrac{1}{k-1} + 2k\sqrt{2k\varepsilon}\right) m$$

for $i \leq \ell_2$ and $s \leq k-1$, where the first inequality holds since $Y_i^{(s)}$ is a subset of $X_i^{(s)} \cup T_i$. ■

Let $I = \left\{i \in [\ell_2] : \exists\, v_i \in V_i \text{ with } \deg(v_i, X_1^{(s)} \cup \ldots \cup X_{\ell_2}^{(s)}) \geq \frac{1}{(6q+10)(k-1)!} \cdot \frac{\ell_2 m}{k-1} \text{ for } s \leq k-1\right\}$. We shall show that I has bounded size.

Claim 3.12. $|I| \le D$.

Proof. We require ε to be small enough so that $\max\{k\sqrt{2k\varepsilon}, k^k\varepsilon\} < \delta_{3.9}\left(k, q, \frac{1}{(6q+10)(k-1)(k-1)!}\right)$, and $\ell_2 \ge C_{3.9}\left(k, q, \frac{1}{(6q+10)(k-1)(k-1)!}\right)$. By (17), $\left|X_i^{(s)}\right| = \left(\frac{1}{k-1} \pm k\sqrt{2k\varepsilon}\right) m$ for $s \le k-1$ and $i \le \ell_2$. Moreover, for $s \le k-1$ and $v \in \bigcup_{i\le \ell_2, t\neq s} X_i^{(t)}$, we have

$$\begin{aligned}\deg(v, \bigcup_{i\le\ell_2} X_i^{(s)}) &\ge \left|\bigcup_{i\le\ell_2} X_i^{(s)}\right| + \deg(v, X^{(s)}) - \left|X^{(s)}\right| \\ &\overset{(15)}{\ge} \left|\bigcup_{i\le\ell_2} X_i^{(s)}\right| - \varepsilon n \ge \left|\bigcup_{i\le\ell_2} X_i^{(s)}\right| - k^k\varepsilon\ell_2 m.\end{aligned}$$

Therefore, we can apply Lemma 3.9 to $G[V_1 \cup \ldots \cup V_{\ell_2}]$ with input $k_{3.9} = k$, $q_{3.9} = q$ and $\eta_{3.9} = \frac{1}{(6q+10)(k-1)(k-1)!}$ to conclude that $|I| \le D_{3.9}\left(k, q, \frac{1}{(6q+10)(k-1)(k-1)!}\right) = D$. ■

As $\ell_3 = \frac{\ell_2}{(k-1)!}$, by reordering the V_i and $Y^{(s)}$ if necessary we can ensure

$$V_i = \dot{\bigcup_s} Y_i^{(s)} \text{ and } \left|Y_i^{(1)}\right| \ge \left|Y_i^{(2)}\right| \ge \ldots \ge \left|Y_i^{(k-1)}\right| \quad \text{for } i \le \ell_3. \tag{18}$$

For $i \le \ell_3$ and $s \le k-1$, let $B_i^{(s)}$ be the set of all vertices $v \in Y_i^{(s)}$ with the property that $\deg(v, X_1^{(t)} \cup \ldots \cup X_{\ell_3}^{(t)}) < \frac{2q}{2q+1} \cdot \frac{\ell_3 m}{k-1}$ for some $t \neq s$. For $s \le k-1$, let $\mathcal{M}_{(s)}$ denote a maximal matching in $G\big[\bigcup_{i\le\ell_3} Y_i^{(s)} \setminus B_i^{(s)}\big]$ whose vertices are in different parts of G, and write J for the collection of all indices $j \in [\ell_3]$ so that $\bigcup_{s\le k-1} \mathcal{M}_{(s)}$ contains some vertex in V_j.

Claim 3.13. $|J| < 2(k-1)q$.

Proof. We shall apply Lemma 3.10 to $G[V_1 \cup \ldots V_{\ell_3}]$ with $k_{3.10} = k$, $q_{3.10} = q$ and $\eta_{3.10} = \frac{2q}{(k-1)(2q+1)}$ to get $|J| < 2(k-1)q$. Note that $\left|X_i^{(s)}\right| = \left(\frac{1}{k-1} \pm k\sqrt{2k\varepsilon}\right) m$ for $s \le k-1$ and $i \le \ell_3$, by (17). Furthermore, for $s \le k-1$ and $v \in \bigcup_{i\le\ell_3, t\neq s} X_i^{(t)}$, we have

$$\deg(v, \bigcup_{i\le\ell_3} X_i^{(s)}) \overset{(15)}{\ge} \left|\bigcup_{i\le\ell_2} X_i^{(s)}\right| - \varepsilon n \ge \left|\bigcup_{i\le\ell_2} X_i^{(s)}\right| - k^{2k}\varepsilon\ell_3 m.$$

Finally, we can choose ε sufficiently small so that $\max\{k\sqrt{2k\varepsilon}, k^{2k}\varepsilon\} < \delta_{3.10}\left(k, q, \frac{2q}{(k-1)(2q+1)}\right)$ and $\ell_3 \ge C_{3.10}\left(k, q, \frac{2q}{(k-1)(2q+1)}\right)$. ■

From Claims 3.12 and 3.13 we can assume (relabelling parts once more if necessary) that $\{1, \ldots, \ell_3\} \setminus (I \cup J) = \{1, \ldots, \ell_4\}$. For $i \le \ell_4$ and $s \le k-1$, let $D_i^{(s)}$ be the set consisting of all vertices $v \in Y_i^{(s)}$ such that $\deg(v, Y_1^{(t)} \cup \ldots \cup Y_{\ell_4}^{(t)}) < \frac{2q+1}{2q+2} \cdot \frac{\ell_4 m}{k-1}$ for some $t \neq s$.

Claim 3.14. *The ℓ_4-partite graph $G[V_1 \cup \ldots \cup V_{\ell_4}]$ together with pairs $(D_i^{(s)}, Y_i^{(s)})_{s\le k-1, i\le\ell_4}$ of vertex sets form an $(\frac{1}{6q+9}, k, \ell_4)$-infracolourable structure.*

Proof. We have to verify the following three properties:

(i) For $i \le \ell_4$, $V_i = \dot{\bigcup}_{s\le k-1} Y_i^{(s)}$ and $\left|Y_i^{(1)}\right| \ge \left|Y_i^{(2)}\right| \ge \ldots \ge \left|Y_i^{(k-1)}\right|$;

(ii) For $i \le \ell_4$ and $s \le k-1$, $D_i^{(s)} \subseteq Y_i^{(s)}$ and $\bigcup_{i\le\ell_4} Y_i^{(s)} \setminus D_i^{(s)}$ is an independent set;

(iii) For $s \le k-1$, every vertex $v \in \bigcup_{i\le\ell_4} D_i^{(s)}$ has at most $\frac{1}{6q+9}\cdot\frac{\ell_4 m}{k-1}$ neighbours in $\bigcup_{i\le\ell_4} Y_i^{(s)}$ and at least $\frac{1}{2q+3}\cdot\frac{\ell_4 m}{k-1}$ non-neighbours in $\bigcup_{i\le\ell_4} V_i \setminus Y_i^{(s)}$.

Property (i) follows directly from (18). For (ii), we observe that $B_i^{(s)} \subseteq D_i^{(s)}$ for $i \le \ell_4$ and $s \le k-1$. We then deduce property (ii) from the maximality of $\mathcal{M}_{(s)}$. For (iii), we consider an arbitrary vertex $v \in \bigcup_{i\le\ell_4} D_i^{(s)}$. Assume to the contrary that $\deg(v, \bigcup_{i\le\ell_4} Y_i^{(s)}) > \frac{1}{6q+9}\cdot\frac{\ell_4 m}{k-1}$. Then, by Claim 3.11, we obtain

$$\deg(v, \bigcup_{i\le\ell_4} X_i^{(s)}) \ge \deg(v, \bigcup_{i\le\ell_4} Y_i^{(s)}) - \left|\bigcup_{i\le\ell_4} T_i\right| \overset{(16)}{\ge} \tfrac{1}{6q+9}\cdot\tfrac{\ell_4 m}{k-1} - 2k\varepsilon\ell_4 m > \tfrac{1}{(6q+10)(k-1)!}\cdot\tfrac{\ell_2 m}{k-1}$$

for ε sufficiently small. On the other hand, by (ii), we must have $v \in \bigcup_{i\le\ell_4} D_i^{(s)} \subseteq \bigcup_{i\le\ell_4} Y_i^{(s)}$, and so $\deg(v, \bigcup_{i\le\ell_2} X_i^{(t)}) \ge \deg(v, \bigcup_{i\le\ell_2} X_i^{(s)})$ for all $t \le k-1$. Therefore,

$$\deg(v, \bigcup_{i\le\ell_2} X_i^{(t)}) \ge \deg(v, \bigcup_{i\le\ell_2} X_i^{(s)}) \ge \deg(v, \bigcup_{i\le\ell_4} X_i^{(s)}) > \tfrac{1}{(6q+10)(k-1)!}\cdot\tfrac{\ell_2 m}{k-1}$$

for $t \le k-1$, as $v \in \bigcup_{i\le\ell_4} Y_i^{(s)}$. This contradicts the fact that $\{1,\ldots,\ell_4\}\cap I = \emptyset$. Finally, by the definition of $\bigcup_{i\le\ell_4} D_i^{(s)}$, there exists $t \ne s$ such that $\deg(v, \bigcup_{i\le\ell_4} Y_i^{(t)}) < \frac{2q+1}{2q+2}\cdot\frac{\ell_4 m}{k-1}$. Consequently, the number of non-neighbours of v in in $\bigcup_{i\le\ell_4} Y_i^{(t)}$ is at least

$$\left|\bigcup_{i\le\ell_4} Y_i^{(t)}\right| - \tfrac{2q+1}{2q+2}\cdot\tfrac{\ell_4 m}{k-1} \overset{\text{Claim 3.11}}{\ge} \left(\tfrac{1}{k-1} - 2k\sqrt{2k\varepsilon}\right)\ell_4 m - \tfrac{2q+1}{2q+2}\cdot\tfrac{\ell_4 m}{k-1} > \tfrac{1}{2q+3}\cdot\tfrac{\ell_4 m}{k-1},$$

assuming ε is sufficiently small. ■

Claim 3.14 tells us that $G[V_1\cup\ldots\cup V_{\ell_4}]$ is the base graph of an $(\frac{1}{6q+9}, k, \ell_4)$-infracolourable structure. By appealing to Lemma 2.3, we can find two indices $1 \le i < j \le \ell_4$ with $d(V_i, V_j) \le \frac{k-2}{k-1}$, contradicting the assumption that $d(V_i, V_j) > \frac{k-2}{k-1}$. This completes our proof of Theorem 3.2. ■

4 Proof of Theorem 1.4

In this section we shall prove a stronger version of Theorem 1.4.

Theorem 4.1. *Let k and ℓ be integers with $k \ge 3$ and $\ell \ge e^{2/c}$, where c is a real number with $0 < c \le k^{-(k+6)k}/2$. Suppose that $G = (V_1\cup\ldots\cup V_\ell, E)$ be a balanced ℓ-partite graph on n vertices such that*

$$d(V_i, V_j) \ge \tfrac{k-2}{k-1} \quad \text{for } i \ne j.$$

Then, G either contains a copy of $K_{k-1}^{+}\big(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lfloor n^{1-2\sqrt{c}}\rfloor\big)$ or is isomorphic to a graph in $\mathcal{G}_\ell^k$.

The idea of the proof is similar to that of Theorem 3.2. We assume that G does not contain a copy of $K^+_{k-1}(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lfloor n^{1-2\sqrt{c}}\rfloor)$. We wish to show that G is isomorphic to a graph in the family $\mathcal{G}^k_\ell$. For this purpose, we apply the stability lemma (Lemma 4.2) to find an induced $(k-1)$-colourable subgraph of G which almost spans $V(G)$. We then use the embedding lemma (Lemma 4.4) showing that G contains a large infracolourable structure. To conclude the proof, we shall use a bootstrapping argument (Lemma 4.8) which allows leveraging a weak structure result into a strong structure result.

In the proof of Theorem 4.1 we shall need the following stability lemma.

Lemma 4.2. *Let k and ℓ be integers with $k \geq 3$ and $\ell \geq e^{2/c}$, where c is a real number with $0 < c \leq k^{-(k+6)k}/2$. Let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph such that $d(V_i, V_j) \geq \frac{k-2}{k-1}$ for $i \neq j$. If G does not contain a copy of $K^+_{k-1}(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lfloor n^{1-2\sqrt{c}}\rfloor)$, then G has an induced $(k-1)$-colourable subgraph F whose vertex classes $X^{(1)}, \ldots, X^{(k-1)}$ satisfy the following properties with $\varepsilon = 4\ell^{-1/2}$*

(i) *For $s \leq k-1$, $\left|X^{(s)}\right| = \left(\frac{1}{k-1} \pm k\varepsilon\right) n$;*

(ii) *For $s \leq k-1$ and $v \in \bigcup_{t\neq s} X^{(t)}$, $\deg(v, X^{(s)}) \geq \left|X^{(s)}\right| - k\varepsilon n$.*

To prove the above statement we need a stability lemma of Nikiforov [17, Theorem 3].

Lemma 4.3. *Let $k \geq 3$ be an integer, and let c and δ be positive real numbers with $c < k^{-(k+6)k}/2$ and $\delta < \frac{1}{8k^8}$. Suppose that G is a graph of order $n \geq e^{2/c}$ with $e(G) \geq \left(\frac{k-2}{k-1} - \delta\right)\binom{n}{2}$. If G has no copy of $K^+_{k-1}(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lfloor n^{1-2\sqrt{c}}\rfloor)$, then G contains an induced $(k-1)$-colourable subgraph F of order $v(F) \geq (1 - 2\sqrt{\delta})n$ and minimum degree $\delta(F) \geq \left(\frac{k-2}{k-1} - 4\sqrt{\delta}\right) n$.*

Proof of Lemma 4.2. By the assumption, $|V_1| = \ldots = |V_\ell| = \frac{n}{\ell} := m$. Together with the density condition, we conclude that $e(G) \geq \binom{\ell}{2}\frac{k-2}{k-1}m^2 \geq \left(\frac{k-2}{k-1} - \frac{1}{\ell}\right)\frac{(\ell m)^2}{2} = \left(\frac{k-2}{k-1} - \frac{1}{\ell}\right)\frac{n^2}{2}$. Notice that $c \leq k^{-(k+6)k}$, $\frac{1}{\ell} < \frac{1}{8k^8}$ and $n \geq e^{2/c}$. Thus, by applying Lemma 4.2 to G with $\delta_{4.2} = \frac{1}{\ell}$ we obtain an $(k-1)$-colourable induced subgraph $F = G[X^{(1)} \cup \ldots \cup X^{(k-1)}]$ of G with $v(F) > (1-\varepsilon)n$ and $\delta(F) \geq \left(\frac{k-2}{k-1} - \varepsilon\right) n$. Since $\delta(F) \geq \left(\frac{k-2}{k-1} - \varepsilon\right) n$ and since $X^{(s)}$ is an independent set, we must have

$$\left|X^{(s)}\right| \leq n - \delta(F) \leq \left(\frac{1}{k-1} + \varepsilon\right) n$$

for $s \leq k-1$. This implies that

$$\left|X^{(s)}\right| \geq v(F) - (k-2)\left(\frac{1}{k-1} + \varepsilon\right) n \geq (1-\varepsilon)n - (k-2)\left(\frac{1}{k-1} + \varepsilon\right) n = \left(\frac{1}{k-1} - (k-1)\varepsilon\right) n$$

for $s \leq k-1$. Therefore, for $s \leq k-1$ and $v \in \bigcup_{t\neq s} X^{(t)}$, the number of non-neighbours of v in $X^{(s)}$ is at most

$$n - \left|X^{(s)}\right| - d_F(v) \leq n - \left(\frac{1}{k-1} - (k-1)\varepsilon\right) n - \left(\frac{k-2}{k-1} - \varepsilon\right) n = k\varepsilon n,$$

as desired. ■

The next ingredient we need is an embedding result.

Lemma 4.4. *Let $r \geq 2$ be an integer, and let G be an r-colourable graph with vertex classes $W_{(1)}, \ldots, W_{(r)}$ of the same size h. Suppose that $\deg(w, W_{(s)}) \geq \big(1 - \frac{1}{r^2}\big)h$ for $s \leq r$ and $w \in \bigcup_{t \neq s} W_{(t)}$. Then*

(1) *G contains at least $\frac{1}{2}h^r$ copies of K_r,*
(2) *For every $\alpha \in (0, \frac{1}{4})$ and $s \leq r$, G contains a copy of $K_r(\lfloor \alpha^r \ln h \rfloor, \ldots, \lfloor \alpha^r \ln h \rfloor, \lfloor h^{1-\alpha^{r-1}} \rfloor)$ whose sth vertex class is a subset of $W_{(s)}$.*

The proof of the above lemma requires a simple result of Nikiforov [17, Lemma 5].

Lemma 4.5. *Let $r \geq 2$ be an integer, and let α be a real number in $(0, \frac{1}{4})$. Suppose that $B[U, W]$ is a bipartite graph with $|U| = p$ and $|W| = q$. If $p \geq 4\lfloor \alpha^r \ln q \rfloor$ and $e(B[U, W]) \geq \frac{1}{2}pq$, then $B[U, W]$ contains the complete bipartite graph $K(a, b)$ with $a = \lfloor \alpha^r \ln q \rfloor$ and $b = \lfloor q^{1-\alpha^{r-1}} \rfloor$.*

Proof of Lemma 4.4. (1) Let $w_s \in W_{(s)}$ for $s = 1, \ldots, r$. Observe that $\{w_1, \ldots, w_r\}$ forms a clique of G if and only if $w_s \in N(w_1, \ldots, w_{s-1}) \cap W_{(s)}$ for $s = 2, \ldots, r$. In addition, $\big|N(w_1, \ldots, w_{s-1}) \cap W_{(s)}\big| \geq h - (s-1) \cdot \frac{h}{r^2}$. Thus, we can bound the number of copies of K_r in G from below by

$$h^r \cdot \prod_{s=1}^{r} \big(1 - \tfrac{s-1}{r^2}\big) \geq h^r \cdot \left(1 - \sum_{s=1}^{r} \tfrac{s-1}{r^2}\right) = \tfrac{r+1}{2r} \cdot h^r > \tfrac{1}{2}h^r.$$

(2) We proceed by induction on r. The base case $r = 2$ follows from the first assertion and Lemma 4.5. For the induction step, assume that $r > 2$. The induction hypothesis implies that $G[W_{(1)} \cup \ldots \cup W_{(r-1)}]$ contains a copy of $K_{r-1}(m)$ with $m = \lfloor \alpha^{r-1} \ln h \rfloor$. Let U denote a set of m disjoint copies of K_{r-1} in $K_{r-1}(m)$. Define a bipartite graph $B[U, W_{(r)}]$ with vertex classes U and $W_{(r)}$, joining $R \in U$ to $w \in W_{(r)}$ if $R \cup \{w\}$ is a clique. We see that $|U| = m$ and $\big|W_{(r)}\big| = h$. Since $0 < \alpha < 1/4$, we have $m = \lfloor \alpha^{r-1} \ln h \rfloor \geq \lfloor 4\alpha^r \ln h \rfloor \geq 4\lfloor \alpha^r \ln h \rfloor$. Furthermore, every vertex of U has at least $h - r \cdot \frac{h}{r^2} \geq h/2$ neighbours in $W_{(r)}$. Hence $e(B[U, W_{(r)}]) \geq mh/2$. The assertion then follows from the base case $r = 2$. ■

In order to find a large infracolourable structure in G we shall use the following consequence of Lemma 4.4.

Lemma 4.6. *Let $k \geq 3$ and $\ell \geq 2$ be integers, and let ε and α be positive real numbers with $\varepsilon < 10^{-2}k^{-k}$ and $\alpha < \frac{1}{4}$. Suppose that $G = (V_1 \cup \ldots \cup V_\ell, E)$ is a balanced ℓ-partite graph containing no copy of $K_{k-1}^{+}\big(\lfloor \alpha^{k-1} \ln(p) \rfloor, \ldots, \lfloor \alpha^{k-1} \ln(p) \rfloor, \lfloor p^{1-\alpha^{k-2}} \rfloor\big)$, where $p = \frac{1}{16(k-1)(k-1)!} \cdot v(G)$. Assume that $(X_i^{(s)})_{s \leq k-1, i \leq \ell}$ are vertex sets so that*

(i) *For $i \leq \ell$, $X_i^{(1)}, \ldots, X_i^{(k-1)}$ are disjoint subsets of V_i;*
(ii) *For $s \leq k-1$ and $v \in \bigcup_{i \leq \ell, t \neq s} X_i^{(t)}$, $\deg(v, \bigcup_{i \leq \ell} X_i^{(s)}) \geq \Big|\bigcup_{i \leq \ell} X_i^{(s)}\Big| - \varepsilon \cdot v(G)$.*

Then, there are no vertices $v \in V(G)$ such that $\deg(v, \bigcup_{i \leq \ell} X_i^{(s)}) \geq p$ for all $s \leq k-1$.

Proof. Suppose for the contradiction that there is $v \in V(G)$ with $\deg(v, \bigcup_i X_i^{(s)}) \geq p$ for all $s \leq k-1$. Then, for $s \leq k-1$ there exists a subset

$$W_{(s)} \subseteq N(v) \cap \left(\bigcup_i X_i^{(s)}\right) \text{ with } \big|W_{(s)}\big| = p. \tag{19}$$

By property (i), $W_{(1)},\ldots,W_{(k-1)}$ are disjoint subsets of $V(G)$. On the other hand, property (ii) shows that for all $s\le k-1$ and $v\in\bigcup_{t\ne s}W_{(t)}$ one has

$$\begin{aligned}\deg\bigl(v,W_{(s)}\bigr)&\ge\bigl|W_{(s)}\bigr|-\varepsilon\cdot v(G)\\&\ge\bigl|W_{(s)}\bigr|-\frac{1}{(k-1)^2}\cdot\frac{1}{16(k-1)(k-1)!}\cdot v(G)=\left(1-\frac{1}{(k-1)^2}\right)\cdot\bigl|W_{(s)}\bigr|,\end{aligned}\tag{20}$$

as $\varepsilon<10^{-2}k^{-k}$. Finally, it follows from (19) and (20) that we can apply Lemma 4.4(2) to the graph $G[W_{(1)},\ldots,W_{(k-1)}]$ with $r_{4.4}=k-1$, $h_{4.4}=p$ and $\alpha_{4.4}=\alpha$ to find a copy of $K_{k-1}\bigl(\lfloor\alpha^{k-1}\ln(p)\rfloor,\ldots,\lfloor\alpha^{k-1}\ln(p)\rfloor,\lfloor p^{1-\alpha^{k-2}}\rfloor\bigr)$. Since $W_{(1)}\cup\ldots\cup W_{(k-1)}$ lies in the neighbour of v, G contains a copy of $K^{+}_{k-1}\bigl(\lfloor\alpha^{k-1}\ln(p)\rfloor,\ldots,\lfloor\alpha^{k-1}\ln(p)\rfloor,\lfloor p^{1-\alpha^{k-2}}\rfloor\bigr)$, which contradicts our assumption. ■

To find a large infracolourable structure in G we also require the following consequence of Lemma 4.4.

Lemma 4.7. *Let $k\ge 3$ and $\ell\ge 2$ be integers, and let ε and α be positive real numbers with $\varepsilon<\frac{1}{12k^3}$ and $\alpha<\frac{1}{4}$. Let $G=(V_1\cup\ldots\cup V_\ell,E)$ be a balanced ℓ-partite graph containing no copy of $K^{+}_{k-1}\bigl(\lfloor\alpha^{k-1}\ln(p)\rfloor,\ldots,\lfloor\alpha^{k-1}\ln(p)\rfloor,\lfloor p^{1-\alpha^{k-2}}\rfloor\bigr)$, where $p=\frac{1}{4(k-1)}\cdot v(G)$. Suppose $(X_i^{(s)},Y_i^{(s)})_{s\le k-1,i\le\ell}$ are pairs of vertex sets which satisfy*

(i) *For every $i\le\ell$ and $s\le k-1$, $Y_i^{(1)},\ldots,Y_i^{(k-1)}$ are disjoint subsets of V_i and $X_i^{(s)}\subseteq Y_i^{(s)}$;*

(ii) *For $i\le\ell$ and $s\le k-1$, $\left|X_i^{(s)}\right|=\left(\frac{1}{k-1}\pm\varepsilon\right)|V_i|$;*

(iii) *For $s\le k-1$ and $v\in\bigcup_{i\le\ell,t\ne s}X_i^{(t)}$, $\deg(v,\bigcup_{i\le\ell}X_i^{(s)})\ge\left|\bigcup_{i\le\ell}X_i^{(s)}\right|-\varepsilon\cdot v(G)$.*

For $i\le\ell$ and $s\le k-1$, let $B_i^{(s)}$ stands for a subset of $Y_i^{(s)}$ consisting of all vertices v with $\deg(v,\bigcup_{j\le\ell}X_j^{(t)})<\frac{2}{3(k-1)}\cdot v(G)$ for some $t\ne s$. Then, for $s\le k-1$, $\bigcup_{i\le\ell}Y_i^{(s)}\setminus B_i^{(s)}$ is an independent set of G.

Proof. We prove by contradiction. Suppose that there exists an edge $\{x,y\}\in E(G)$ with $x,y\in\bigcup_i Y_i^{(s)}\setminus B_i^{(s)}$. Let $t\ne s$. By the definition of $\bigcup_i B_i^{(s)}$, both $\deg(x,\bigcup_i X_i^{(t)})$ and $\deg(y,\bigcup_i X_i^{(t)})$ are at least $\frac{2}{3(k-1)}\cdot v(G)$. Hence

$$\begin{aligned}\left|N(x,y)\cap\bigcup_i X_i^{(t)}\right|&\ge\deg(x,\bigcup_i X_i^{(t)})+\deg(y,\bigcup_i X_i^{(t)})-\left|\bigcup_i X_i^{(t)}\right|\\&\overset{(ii)}{\ge}\frac{4}{3(k-1)}\cdot v(G)-\left(\frac{1}{k-1}+\varepsilon\right)\cdot v(G)\ge\frac{1}{4(k-1)}\cdot v(G),\end{aligned}$$

as $\varepsilon<\frac{1}{12k^3}$. It means that there is a subset

$$W_{(t)}\subseteq N(x,y)\cap\bigcup_{i\le\ell_3}X_i^{(t)}\text{ with }\bigl|W_{(t)}\bigr|=\tfrac{1}{4(k-1)}\cdot v(G).$$

On the other hand, it follows from property (ii) that $\left|\bigcup_i X_i^{(s)}\right|\ge\left(\frac{1}{k-1}-\varepsilon\right)v(G)>\frac{1}{4(k-1)}\cdot v(G)$ for $0<\varepsilon<\frac{1}{12k^3}$, and so there exists a subset

$$W_{(s)}\subseteq\bigcup_i X_i^{(s)}\text{ with }\bigl|W_{(s)}\bigr|=\tfrac{1}{4(k-1)}\cdot v(G).$$

Analysis similar to that in the proof of Lemma 4.6 shows that $G[W_{(1)},\dots,W_{(k-1)}]$ must contain a copy of $K_{k-1}\big(\lfloor\alpha^{k-1}\ln(p)\rfloor,\dots,\lfloor\alpha^{k-1}\ln(p)\rfloor,\lfloor p^{1-\alpha^{k-2}}\rfloor\big)$ whose sth vertex class is of size $\lfloor\alpha^{k-1}\ln(p)\rfloor$. Adding back vertices x and y to this class one gets a supgraph of the graph $K^{+}_{k-1}\big(\lfloor\alpha^{k-1}\ln(p)\rfloor,\dots,\lfloor\alpha^{k-1}\ln(p)\rfloor,\lfloor p^{1-\alpha^{k-2}}\rfloor\big)$, contradicting the hypothesis. ■

The last component of the proof is a bootstrapping argument which allows us to leverage a weak structure result into a strong structure result. Roughly speaking, it says that if G contains an $\tilde{\ell}$-partite subgraph which is in $\mathcal{G}^k_{\tilde{\ell}}$, then G must belong to $\mathcal{G}^k_{\ell}$.

Lemma 4.8. *Let $k\geq 3$ be an integer, and let $G=(V_1\cup\ldots\cup V_\ell,E)$ be an ℓ-partite graph with $|V_1|=\ldots=|V_\ell|=m$ and $d(V_i,V_j)\geq\frac{k-2}{k-1}$ for all $i\neq j$. Suppose that there exist an integer $\tilde{\ell}$ and disjoint subsets $Y_i^{(1)},\dots,Y_i^{(k-1)}$ of V_i for $1\leq i\leq\tilde{\ell}$ so that $\left|Y_i^{(s)}\right|=\frac{m}{k-1}$ and $d(Y_i^{(s)},Y_j^{(t)})=1$ for all $i\neq j$ and $s\neq t$. If G does not contain a copy of $K^{+}_{k-1}\big(\frac{\tilde{\ell}m}{32k^2}\big)$, then G is isomorphic to a graph in the family $\mathcal{G}^k_{\ell}$.*

Proof. We wish to show that G is isomorphic to a graph in $\mathcal{G}^k_{\ell}$. According to Lemma 2.1, it suffices to prove G is $(k-1)$-colourable. By the assumption, we have

$$\left|Y_i^{(s)}\right|=\tfrac{m}{k-1},\ d(Y_i^{(s)},Y_j^{(t)})=1 \text{ for } s\neq t \text{ and } 1\leq i<j\leq\tilde{\ell}. \tag{21}$$

We shall show that for $v\in V(G)\setminus\big(V_1\cup\ldots\cup V_{\tilde{\ell}}\big)$ there does not exist $s\leq k-1$ with

$$\deg\big(v,Y_1^{(s)}\cup\ldots\cup Y_{\tilde{\ell}}^{(s)}\big)\geq 1,\ \deg\big(v,Y_1^{(t)}\cup\ldots\cup Y_{\tilde{\ell}}^{(t)}\big)\geq\frac{\tilde{\ell}m}{2k} \text{ for all } t\neq s. \tag{22}$$

We prove by contradiction. Suppose that (22) holds. We can pick an index $i_0\in\{1,2,\dots,\tilde{\ell}\}$ with $N(v)\cap Y_{i_0}^{(s)}\neq\emptyset$ whose existence is guaranteed by (22). We then arbitrarily add other indices to get a subset $I_{(s)}\subset\{1,\dots,\tilde{\ell}\}$ of size $\frac{\tilde{\ell}}{8k}$. It follows from (21) and (22) that for each $t\neq s$, there are at least $\frac{\tilde{\ell}}{4}$ indices $i\leq\tilde{\ell}$ with $\deg(v,Y_i^{(t)})\geq\frac{m}{4k}$. Hence we can find $k-1$ disjoint subsets $I_{(1)},\dots,I_{(k-1)}$ of size $\frac{\tilde{\ell}}{8k}$ of $\{1,\dots,\tilde{\ell}\}$ with the property that $\deg(v,Y_i^{(t)})\geq\frac{m}{4k}$ for all $t\neq s$ and $i\in I_{(t)}$. By (21), $G\big[\bigcup_{i\in I_{(1)}}Y_i^{(1)},\dots,\bigcup_{i\in I_{(k-1)}}Y_i^{(k-1)}\big]$ is a complete $(k-1)$-partite graph. In addition, we have $\left|N(v)\cap\bigcup_{i\in I_{(s)}}Y_i^{(s)}\right|\geq\left|N(v)\cap Y_{i_0}^{(s)}\right|>0$ and

$$\left|N(v)\cap\bigcup_{i\in I_{(t)}}Y_i^{(t)}\right|=\sum_{i\in I_{(t)}}\deg(v,Y_i^{(t)})\geq\left|I_{(t)}\right|\cdot\frac{m}{4k}=\frac{\tilde{\ell}m}{32k^2}\quad\text{for } t\neq s.$$

Therefore, by adding v to the sth part of $G\big[\bigcup_{i\in I_{(1)}}Y_i^{(1)},\dots,\bigcup_{i\in I_{(k-1)}}Y_i^{(k-1)}\big]$ we get a supergraph of $K^{+}_{k-1}\big(\frac{\tilde{\ell}m}{32k^2}\big)$ in G, contradicting our assumption.

We can infer from (22) that $\deg(v,V_1\cup\ldots\cup V_{\tilde{\ell}})\leq\frac{k-2}{k-1}\cdot\tilde{\ell}m$ for all $v\in V(G)\setminus\big(V_1\cup\ldots\cup V_{\tilde{\ell}}\big)$. By the density condition, equality must hold. Again (22) shows that for each $v\in V(G)\setminus\big(V_1\cup\ldots\cup V_{\tilde{\ell}}\big)$,

$$N(v)\cap\big(V_1\cup\ldots\cup V_{\tilde{\ell}}\big)=\bigcup_{i\leq\tilde{\ell}}V_i\setminus Y_i^{(s)}\quad\text{for some } s\leq k-1. \tag{23}$$

If $v \in V_i$ for some $i > \tilde{\ell}$, then we assign v to $Z_i^{(s)}$. For $i \leq \tilde{\ell}$ we let $Z_i^{(s)} = Y_i^{(s)}$ for $s \leq k-1$. If we denote $Z^{(s)} = \dot\bigcup_i Z_i^{(s)}$ for $s \leq k-1$, then $V = \dot\bigcup_s Z^{(s)}$. To prove G is $(k-1)$-colourable, it is enough to show that $Z^{(1)}, \ldots, Z^{(k-1)}$ are independent sets. Suppose to the contrary that for some $s \leq k-1$, $Z^{(s)}$ contains an edge $\{u,v\}$ with $u \in Z_{i_1}^{(s)}$ and $v \in Z_{i_2}^{(s)}$. We can easily find $k-1$ disjoint subsets $J_{(1)}, \ldots, J_{(k-1)}$ of size $\frac{\tilde{\ell}}{2(k-1)}$ of $[\tilde{\ell}] \setminus \{i_1, i_2\}$. Let $W^{(s)} = \{u,v\} \cup \left(\bigcup_{i \in J_{(s)}} Y_i^{(s)}\right)$ and $W^{(t)} = \bigcup_{i \in J_{(t)}} Y_i^{(t)}$ for $t \neq s$. It follows from (21) and (23) that $G[W^{(1)}, \ldots, W^{(k-1)}]$ is a complete $(k-1)$-colourable graph with $\left|W^{(t)}\right| \geq \frac{\tilde{\ell}}{2(k-1)} \cdot \frac{m}{k-1} > \frac{\tilde{\ell} m}{32k^2}$ for $t \leq k-1$. Combining this with the assumption that $\{u,v\} \in E(G)$, we conclude that G contains a copy of $K_{k-1}^{+}\left(\frac{\tilde{\ell} m}{32k^2}\right)$, a contradiction. ∎

We now have all the necessary tools to prove Theorem 4.1.

Proof of Theorem 4.1. For convenience, we write $H = K_{k-1}^{+}\left(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lfloor n^{1-2\sqrt{c}} \rfloor\right)$ and $H^{-} = K_{k-1}\left(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lfloor n^{1-2\sqrt{c}} \rfloor\right)$. Suppose G has no copy of H. We wish to show that G is isomorphic to a graph in $\mathcal{G}_{\ell}^{k}$. Since G is a balanced ℓ-partite graph on n vertices, each partition set of G has size $n/\ell := m$. Let $\varepsilon = 4\ell^{-1/2}$, $\ell_1 = \frac{\ell}{2(k-1)!} - (k-1)$, $\ell_2 = \frac{\ell_2}{(k-1)!}$ and $\ell_3 = \ell_2 - 1$.

By Lemma 4.2, G must contain an induced $(k-1)$-colourable subgraph F whose vertex classes $X^{(1)}, \ldots, X^{(k-1)}$ satisfy

$$\left|X^{(s)}\right| = \left(\tfrac{1}{k-1} \pm k\varepsilon\right) n \text{ for } s \leq k-1, \tag{24}$$

$$\deg(v, X^{(s)}) \geq \left|X^{(s)}\right| - k\varepsilon n \text{ for } s \leq k-1 \text{ and } v \in \bigcup_{t \neq s} X^{(t)}. \tag{25}$$

Let $T = V(G) \setminus V(F)$. As in the proof of Theorem 3.2, by relabelling parts we can assume that

$$|T_i| \leq 2k^2 \varepsilon m, \text{ and } \left|X_i^{(s)}\right| = \left(\tfrac{1}{k-1} \pm 2k^2\sqrt{\varepsilon}\right) m \quad \text{for } i \leq \ell_1 \text{ and } s \leq k-1. \tag{26}$$

For $i \leq \ell_1$ we shall partition V_i into $k-1$ subsets as follows. A vertex $v \in V_i$ is assigned to $Y_i^{(s)}$ if $\deg\left(v, \bigcup_{j \leq \ell_1} X_j^{(s)}\right) = \min_{t \leq k-1} \deg\left(v, \bigcup_{j \leq \ell_1} X_j^{(t)}\right)$; if there are more than one such index s, arbitrarily pick one of them.

Claim 4.9. $X_i^{(s)} \subseteq Y_i^{(s)} \subseteq X_i^{(s)} \dot\cup T_i$ *for* $i \leq \ell_1$ *and* $s \leq k-1$.

Proof. Because $X^{(s)}$ is an independent set in G, every vertex in $X_i^{(s)}$ has no neighbours in $\bigcup_{j \leq \ell_1} X_j^{(s)}$, and so $X_i^{(s)}$ is a subset of $Y_i^{(s)}$. Since $V_i = \left(\dot\bigcup_s X_i^{(s)}\right) \dot\cup T_i = \dot\bigcup_s Y_i^{(s)}$ and $X_i^{(s)} \subseteq Y_i^{(s)}$ for $i \leq \ell_1$ and $s \leq k-1$, the inclusion relation $Y_i^{(s)} \subseteq X_i^{(s)} \dot\cup T_i$ holds for $i \leq \ell_1$ and $s \leq k-1$. ∎

We proceed by showing that $\bigcup_{i \leq \ell_1} V_i$ does not contain a vertex which has relatively large degree to $\bigcup_{i \leq \ell_1} Y_i^{(s)}$ for all $s \leq k-1$.

Claim 4.10. *There are no vertices* $v \in \bigcup_{i \leq \ell_1} V_i$ *with* $\deg(v, \bigcup_{i \leq \ell_1} Y_i^{(s)}) \geq \frac{1}{15(k-1)(k-1)!} \cdot \ell_1 m$ *for all* $s \leq k-1$.

Proof. We can derive from (25) that, for $s \leq k-1$ and $v \in \bigcup_{i \leq \ell_1, t \neq s} X_i^{(t)}$,

$$\begin{aligned} \deg(v, \bigcup_{i \leq \ell_1} X_i^{(s)}) &\geq \left| \bigcup_{i \leq \ell_1} X_i^{(s)} \right| + \deg(v, X^{(s)}) - \left| X^{(s)} \right| \\ &\geq \left| \bigcup_{i \leq \ell_1} X_i^{(s)} \right| - k\varepsilon n \geq \left| \bigcup_{i \leq \ell_1} X_i^{(s)} \right| - k^k \varepsilon \cdot \ell_1 m. \end{aligned}$$

Applying Lemma 4.6 to $G[V_1 \cup \ldots \cup V_{\ell_1}]$ with $k_{4.6} = k$, $\varepsilon_{4.6} = k^k \varepsilon$ and $\alpha_{4.6} = (2c)^{1/(k-1)}$, we conclude that either $G[V_1 \cup \ldots \cup V_{\ell_1}]$ contains a copy of $K_{k-1}^+\big(\lfloor \alpha^{k-1} \ln(p) \rfloor, \ldots, \lfloor \alpha^{k-1} \ln(p) \rfloor, \lfloor p^{1-\alpha^{k-2}} \rfloor\big)$ or there are no vertices $v \in V_1 \cup \ldots \cup V_{\ell_1}$ with $\deg(v, X_1^{(s)} \cup \ldots \cup X_\ell^{(s)}) \geq p$ for all s, where $p = \frac{1}{16(k-1)(k-1)!} \cdot \ell_1 m$. Since $\alpha^{k-1} \ln(p) > c\ln(n)$, $p^{1-\alpha^{k-2}} > n^{1-2\sqrt{c}}$ and since G has no copy of $K_{k-1}^+\big(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lfloor n^{1-2\sqrt{c}} \rfloor\big)$, the former case is ruled out. The later case implies our statement. ∎

Since $\ell_2 = \frac{\ell_1}{(k-1)!}$, by reordering parts if necessary we can assume that

$$\left| Y_i^{(1)} \right| \geq \left| Y_i^{(2)} \right| \geq \ldots \geq \left| Y_i^{(k-1)} \right| \text{ for } i \leq \ell_2. \tag{27}$$

For $i \leq \ell_2$ and $s \leq k-1$, let us denote

$$D_i^{(s)} = \left\{ v \in Y_i^{(s)} : \deg(v, Y_1^{(t)} \cup \ldots \cup Y_{\ell_2}^{(t)}) < \tfrac{3}{4(k-1)} \cdot \ell_2 m \text{ for some } t \neq s \right\}.$$

Claim 4.11. *The vertex set $\bigcup_{i \leq \ell_2} Y_i^{(s)} \setminus D_i^{(s)}$ is an independent set of G for $s \leq k-1$.*

Proof. For $i \leq \ell_2$ and $s \leq k-1$, let $B_i^{(s)}$ be the vertex set consisting of all vertices $v \in Y_i^{(s)}$ such that $\deg(v, \bigcup_{i \leq \ell_2} X_i^{(t)}) < \frac{2}{3(k-1)} \cdot \ell_2 m$ for some $t \neq s$. Note that, for $s \leq k-1$ and $v \in \bigcup_{i \leq \ell_2, t \neq s} X_i^{(s)}$, one has

$$\begin{aligned} \deg(v, \bigcup_{i \leq \ell_2} X_i^{(s)}) &\geq \left| \bigcup_{i \leq \ell_2} X_i^{(s)} \right| + \deg(v, X^{(s)}) - \left| X^{(s)} \right| \\ &\overset{(25)}{\geq} \left| \bigcup_{i \leq \ell_2} X_i^{(s)} \right| - k\varepsilon n \geq \left| \bigcup_{i \leq \ell_2} X_i^{(s)} \right| - k^{2k} \varepsilon \cdot \ell_2 m. \end{aligned}$$

This estimate together with Claim 4.9 and (26) show that we can apply Lemma 4.7 to $G[V_1 \cup \ldots \cup V_{\ell_2}]$ with $k_{4.7} = k$, $\varepsilon_{4.7} = \max\{2k^2\sqrt{\varepsilon}, k^{2k}\varepsilon\}$ and $\alpha_{4.7} = (2c)^{1/(k-1)} := \alpha$ to conclude that either $G[V_1 \cup \ldots \cup V_{\ell_2}]$ contains $K_{k-1}^+\big(\lfloor \alpha^{k-1} \ln(p) \rfloor, \ldots, \lfloor \alpha^{k-1} \ln(p) \rfloor, \lfloor p^{1-\alpha^{k-2}} \rfloor\big)$ or $\bigcup_{i \leq \ell_2} Y_i^{(s)} \setminus B_i^{(s)}$ is an independent set of G for $s \leq k-1$, where $p = \frac{1}{4(k-1)} \cdot \ell_2 m$. Since $\alpha^{k-1} \ln(p) > c\ln(n)$, $p^{1-\alpha^{k-2}} > n^{1-2\sqrt{c}}$ and since G has no copy of $K_{k-1}^+\big(\lfloor c \ln n \rfloor, \ldots, \lfloor c \ln n \rfloor, \lfloor n^{1-2\sqrt{c}} \rfloor\big)$, the former case is ruled out. We can see that the later case implies our statement. ∎

Now we can find a large infracolourable structure in G, and then use Lemma 4.8 to show that G is isomorphic to a graph in $\mathcal{G}_\ell^k$.

Claim 4.12. *G is isomorphic to a graph in the family $\mathcal{G}_\ell^k$.*

Proof. Analogously to the proof of Claim 3.14, we can infer from Claims 4.10 and 4.11, (26) and (27) that $G[V_1 \cup \ldots \cup V_{\ell_2}]$ together with pairs $(D_i^{(s)}, Y_i^{(s)})_{s \le k-1, i \le \ell_2}$ form a $(\frac{1}{15}, k, \ell_2)$-infracolourable structure. By Lemma 2.3 this implies that $e(G[V_1 \cup \ldots \cup V_{\ell_2}]) \le \binom{\ell_2}{2}\frac{k-2}{k-1}m^2$ and hence the equality must occur by the density condition. Appealing to Lemma 2.3 once again, we see that there exists $i_0 \in \{0, 1, \ldots, \ell_2\}$ with $\left|Y_i^{(s)}\right| = \frac{m}{k-1}$ for all s and all $i \in [\ell_2] \setminus \{i_0\}$, and $d(Y_i^{(s)}, Y_j^{(t)}) = 1$ for all $s \ne t$ and $1 \le i < j \le \ell_2$. Hence we can apply Lemma 4.8 with $\tilde{\ell} = \ell_2 - 1$ to conclude that either G contains a copy of $K_{k-1}^+\big(\frac{(\ell_2-1)m}{32k^2}\big)$ or G is isomorphic to a graph in $\mathcal{G}_\ell^k$. The former can not happen since G has no copy of $K_{k-1}^+\big(\lfloor c\ln n\rfloor, \ldots, \lfloor c\ln n\rfloor, \lfloor n^{1-2\sqrt{c}}\rfloor\big)$ and since $\frac{(\ell_2-1)m}{32k^2} > \max\{n^{1-2\sqrt{c}}, c\ln n\}$. So G must isomorphic to a graph in the family $\mathcal{G}_\ell^k$. ■

This concludes our proof of Theorem 3.2. ■

5 Missing proofs

5.1 Proof of Theorem 1.5

In this section we sketch a proof of Theorem 1.5. We follow essentially the proof of Theorem 4.1. We make the following alterations. Instead of Lemma 4.3 we use a stability result due to Bollobás and Nikiforov [3, Theorem 9].

Lemma 5.1. *Let $k \ge 2$ be an integer, and let δ be a positive with $\delta < \frac{1}{16k^8}$. Suppose that G is a graph with $n > k^8$ vertices and $e(G) \ge \left(\frac{k-2}{k-1} - \delta\right)\binom{n}{2}$ edges. Then, either G contains a family of $k^{-(k+5)}n^{k-2}$ copies of K_k sharing a common edge, or G contains an induced $(k-1)$-colourable subgraph F of size $v(F) \ge (1 - 2\sqrt{\delta})n$ and minimum degree $\delta(F) \ge \left(\frac{k-2}{k-1} - 4\sqrt{\delta}\right)n$.*

We replace Lemma 4.4 by the following embedding result.

Lemma 5.2. *Let $r \ge 2$ be an integer, and let G be an r-colourable graph with classes $W_{(1)}, \ldots, W_{(r)}$ of the same size h. Suppose that $\deg(v, W_{(s)}) \ge \big(1 - \frac{1}{r^2}\big)h$ for $s \le r$ and $v \in \bigcup_{t \ne s} W_{(t)}$. Then for every pair (s,t) with $s \ne t$, there is an edge between $W_{(s)}$ and $W_{(t)}$ which is contained in $\frac{1}{2}h^{r-2}$ copies of K_r.*

Proof. According to Lemma 4.4, G contains at least $\frac{1}{2}h^r$ copies of K_r. Hence there exists an edge between $W_{(s)}$ and $W_{(t)}$ which is shared by at least $h^r/(2h^2) = \frac{1}{2}h^{r-2}$ copies of K_r. ■

The remainder of the proof is similar to that of Theorem 4.1.

5.2 Proofs of Proposition 3.5 and Lemma 3.8

To prove Proposition 3.5 we shall require the Erdős-Simonovits stability theorem (Erdős [8] and Simonovits [22, Theorem 8], and the graph removal lemma (Ruzsa and Szemerédi [21]).

Theorem 5.3 (**Stability theorem).** *For every graph H and every $\varepsilon > 0$, there exist positive constants $\delta = \delta(H, \varepsilon)$ and $C = C(H, \varepsilon)$ so that the following holds for every integer $n \geq C$. Every n-vertex H-free graph with at least $\left(\frac{\chi(H)-2}{\chi(H)-1} - \delta\right)\binom{n}{2}$ edges contains a $(\chi(H)-1)$-colourable subgraph of order at least $(1-\varepsilon)n$ and minimum degree at least $\left(\frac{\chi(H)-2}{\chi(H)-1} - \varepsilon\right)n$.*

Theorem 5.4 (**Graph removal lemma).** *For every graph H and every $\delta > 0$, there exists a positive constant $\gamma = \gamma(H, \delta)$ such that every graph on n vertices with at most $\gamma n^{v(H)}$ copies of H can be made H-free by removing from it at most $\delta\binom{n}{2}$ edges.*

Now we can deduce Proposition 3.5 from Theorems 5.3 and 5.4 as follows.

Proof of Proposition 3.5. Let $\delta = \delta_{5.3}(H, \varepsilon)/2$, $\gamma = \min\{\gamma_{5.4}(H, \delta), \delta\}$ and $C = C_{5.3}(H, \varepsilon)$. Since G contains at most $\gamma n^{v(H)}$ copies of H, Theorem 5.4 shows that G contains an H-free subgraph G' with $e(G') \geq e(G) - \delta\binom{n}{2}$. Hence

$$e(G') \geq \left(\frac{\chi(H)-2}{\chi(H)-1} - \gamma - \delta\right)\binom{n}{2} \geq \left(\frac{\chi(H)-2}{\chi(H)-1} - \delta_{5.3}(H, \varepsilon)\right)\binom{n}{2}.$$

Moreover, $v(G') = n \geq C = C_{5.3}(H, \varepsilon)$. Therefore, one can apply Theorem 5.3 to obtain a $(\chi(H)-1)$-colourable subgraph G'' of G' with $v(G'') \geq (1-\varepsilon)n$ and $\delta(G'') \geq \left(\frac{\chi(H)-2}{\chi(H)-1} - \varepsilon\right)n$. ■

Proof of Lemma 3.8. Choose $D = qd^{-r}$ and $\rho = e^{-q}d^{rq}$. Let S be the set of tuples $(w_1, \ldots, w_r, A)$ where $w_s \in W_{(s)}$ for all s, and $A \in \binom{N(w_1,\ldots,w_r)}{q}$. We find that

$$|S| = \sum_{A \in \binom{U}{q}} \prod_{s \leq r} \left|N(A) \cap W_{(s)}\right| = \sum_{(w_1,\ldots,w_r)} \binom{|N(w_1, \ldots, w_r)|}{q}. \tag{28}$$

Moreover, our assumption implies that

$$\sum_{(w_1,\ldots,w_r)} |N(w_1, \ldots, w_r)| = \sum_{u \in U} \prod_{s \leq r} \deg(u, W_{(s)}) \geq d^r\, |U| \cdot \prod_{s \leq r} \left|W_{(s)}\right|. \tag{29}$$

Note that the function

$$\binom{x}{q} = \begin{cases} x(x-1)\cdots(x-q+1)/q! & \text{if } x \geq q-1, \\ 0 & \text{if } x < q-1. \end{cases}$$

is convex. Thus, we can first apply Jensen's inequality to the right hand side of (28) and then use the inequality (29) to obtain $|S| \geq \binom{d^r|U|}{q} \prod\limits_{s \leq r} \left|W_{(s)}\right|$. We infer from this and the first identity in (28) that there is a subset $A \in \binom{U}{q}$ with

$$\prod_{s \leq r} \left|N(A) \cap W_{(s)}\right| \geq \frac{\binom{d^r|U|}{q}}{\binom{|U|}{q}} \cdot \prod_{s \leq r} \left|W_{(s)}\right| \geq e^{-q}d^{rq} \cdot \prod_{s \leq r} \left|W_{(s)}\right| = \rho \cdot \prod_{s \leq r} \left|W_{(s)}\right|,$$

where the second inequality holds since $\binom{|U|}{q} \leq \left(\frac{e|U|}{q}\right)^q$, and $\binom{d^r|U|}{q} \geq \left(\frac{d^r|U|}{q}\right)^q$ for $|U| \geq D = qd^{-r} \geq q$. Hence $\left|N(A) \cap W_{(s)}\right| \geq \rho\left|W_{(s)}\right|$ for $s \leq r$. ■

6 Concluding remarks

Bollobás [1, Corollary 3.5.4] showed that every n-vertex graph with $\lfloor \frac{n^2}{4} \rfloor + 1$ edges contains cycles of lengths from 3 up to $\lfloor \frac{n+3}{2} \rfloor$, and thus strengthened the Mantel theorem. Using techniques developed in this paper we can prove the following multipartite version of this result; we omit the details.

Theorem 6.1. *Let $\ell \geq 10^{20}$, and let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph on n vertices such that*

$$d(V_i, V_j) \geq \tfrac{1}{2} \quad \text{for } i \neq j.$$

Then, G either contains a cycle of length h for each integer h with $3 \leq h \leq (\frac{1}{2} - \frac{2}{\sqrt{\ell}})n$ or is isomorphic to a graph in $\mathcal{G}_\ell^3$.

The balanced ℓ-partite graph obtained by taking the disjoint union of $K_\ell\big(\lfloor \frac{n}{2\ell} \rfloor - 1\big)$ and $K_\ell\big(\lceil \frac{n}{2\ell} \rceil + 1\big)$ has edge densities between parts strictly greater than $\frac{1}{2}$. However, every cycle of this graph has length at most $\frac{1}{2}n + 2\ell = (\frac{1}{2} + o(1))n$ provided $\ell = o(n)$. Therefore, the bound $(\frac{1}{2} - \frac{2}{\sqrt{\ell}})n$ in the above result is asymptotically best possible.

A book in a graph is a collection of triangles sharing a common edge. The size of a book is the number of triangles. Let $b(G)$ be the size of the largest book in a graph G. Generalising Mantel's theorem, Erdős [6] showed that every n-vertex graph G with $\lfloor \frac{n^2}{4} \rfloor + 1$ edges satisfies $b(G) \geq \frac{n}{6} - O(1)$. The optimal bound $b(G) \geq \lfloor \frac{n}{6} \rfloor$ was obtained independently by Edwards in an unpublished manuscript [5], and by Khadžiivanov and Nikiforov in [14]. We wonder whether a similar result holds for balanced multipartite graphs.

Conjecture 6.2. *For every $\varepsilon > 0$, there is a constant $C = C(\varepsilon)$ such that the following holds for $\ell > C$. Let $G = (V_1 \cup \ldots \cup V_\ell, E)$ be a balanced ℓ-partite graph on n vertices such that*

$$d(V_i, V_j) > \tfrac{1}{2} \quad \text{for every } i \neq j.$$

Then, $b(G) > \left(\frac{1}{6} - \varepsilon\right) n$.

According to Theorem 1.5, the above conjecture is true for $\varepsilon \geq \frac{1}{6} - 3^{-18}$.

Assume H is not an almost colour-critical graph. Theorem 1.3(1) tells us that $d_\ell(H) \geq \frac{\chi(H)-2}{\chi(H)-1} + \frac{1}{(\chi(H)-1)^2(\ell-1)^2}$ for every $\ell \geq v(H)$. Furthermore, this estimate is tight for $H = K_{1,2}$, as shown in Remark 3.1. It would be very interesting to have a characterisation of the equality case.

Bondy, Shen, Thomassé and Thomassen [4] determined the value of $d_\ell(K_k)$ in the case when $\ell = k = 3$, while Pfender [19] obtained result in the case when ℓ is large enough in terms of k. The value of $d_\ell(K_k)$ is not known in the remaining cases. Nevertheless, when $\ell = k \geq 4$, Pfender [20] proposed the following conjecture (see [16, Section 5] for more details).

Conjecture 6.3. *The critical edge density $d_k = d_k(K_k)$ satisfies the following recurrence formula:*

$$d_2 = 0, \quad d_k^2(1 - d_{k-1}) + d_k - 1 = 0 \text{ for } k \geq 3.$$

Finally, we emphasise that there are other interesting multipartite versions of the Turán theorem. For instance, Bollobás, Erdős and Szemerédi [2] introduced the function $\delta_r(n)$ which is the smallest integer so that every r-partite graph with parts of size n and minimum degree $\delta_r(n) + 1$ contains a copy of K_r. The exact values of $\delta_r(n)$ was determined completely by Haxell and Szabó [13] (for odd r), and Szabó and Tardos [23] (for even r) via topological methods.

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References

[1] B. Bollobás, Extremal graph theory, *Academic Press*, 1978.

[2] B. Bollobás, P. Erdős, and E. Szemerédi, On complete subgraphs of r-chromatic graphs, *Discrete Math.* **13** (1975), 97–107.

[3] B. Bollobás and V. Nikiforov, Joints in graphs, *Discrete Math.* **308** (2008), 9–19.

[4] A. Bondy, J. Shen, S. Thomassé, and C. Thomassen, Density conditions for triangles in multipartite graphs, *Combinatorica* **26** (2006), 121–131.

[5] C. Edwards, A lower bound for the largest number of triangles with a common edge, *Unpublished manuscript*, 1977.

[6] P. Erdős, On a theorem of Rademacher–Turán, *Illinois J. Math.* **6** (1962), 122–127.

[7] P. Erdős, On the structure of linear graphs, *Israel J. Math.* **1** (1963), 156–160.

[8] P. Erdős, Some recent results on extremal problems in graph theory, *Theory of Graphs* (Internat. Sympos., Rome, 1966), pp. 117–123, Gordon and Breach, New York, 1967.

[9] P. Erdős, On the number of complete subgraphs and circuits contained in graphs, *Časopis Pěst. Mat.* **94** (1969), 290–296.

[10] P. Erdős and M. Simonovits, A limit theorem in graph theory, *Studia Sci. Math. Hung.*, **1** (1966) 51–57.

[11] P. Erdős and V. Sós, On Ramsey–Turán type theorems for hypergraphs, *Combinatorica* **2** (1982), 289–295.

[12] P. Erdős and A. Stone, On the structure of linear graphs, *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091.

[13] P. Haxell and T. Szabó, Odd independent transversals are odd, *Combin. Probab. Comput.* **15** (2006), 193–211.

[14] N. Khadžiivanov and V. Nikiforov, Solution of a problem of P. Erdős about the maximum number of triangles with a common edge in a graph, *C. R. Acad. Bulgare Sci.* **32** (1979), 1315–1318.

[15] W. Mantel, Problem 28, *Wiskundige Opgaven* **10** (1907), 60–61.

[16] Z. Nagy, A multipartite version of the Turán problem - density conditions and eigenvalues, *Elec. J. Combin.* **18** (2011), 1–15.

[17] V. Nikiforov, Turán's theorem inverted, *Discrete Math.* **310** (2010), 125–131.

[18] V. Nikiforov, Some new results in extremal graph theory, *Surveys in Combinatorics* (ed. R. Chapman), LMS Lecture Note Series 392, Cambridge University Press, pp. 141–183, 2011.

[19] F. Pfender, Complete subgraphs in multipartite graphs, *Combinatorica* **32** (2012), 483–495.

[20] F. Pfender, Personal communication.

[21] I. Ruzsa and E. Szemerédi, Triple systems with no six points carrying three triangles, *Combinatorics*, Proc. Fifth Hungarian Colloq., Keszthely, 1976.

[22] M. Simonovits, A method for solving extremal problems in graph theory, *Theory of Graphs*, Proc. Coll. Tihany, 279–319, 1966.

[23] T. Szabó and G. Tardos, Extremal problems for transversals in graphs with bounded degree, *Combinatorica* **26** (2006), 333–351.

[24] P. Turán, On an extremal problem in graph theory, *Mat. Fiz. Lapok.* **48** (1941), 436–452.