Decomposition of Graphs into (k, r)-Fans and Single Edges *

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Abstract

Let $\phi(n,H)$ be the largest integer such that, for all graphs G on n vertices, the edge set E(G) can be partitioned into at most $\phi(n,H)$ parts, of which every part either is a single edge or forms a graph isomorphic to H. Pikhurko and Sousa conjectured that $\phi(n,H)=\exp(n,H)$ for $\chi(H)\geqslant 3$ and all sufficiently large n, where $\exp(n,H)$ denotes the maximum number of edges of graphs on n vertices that does not contain H as a subgraph. A (k,r)-fan is a graph on (r-1)k+1 vertices consisting of k cliques of order r which intersect in exactly one common vertex. In this paper, we verify Pikhurko and Sousa's conjecture for (k,r)-fans. The result also generalizes a result of Liu and Sousa.

1 Introduction

All graphs considered in this paper are simple and finite. Given a graph G = (V, E) and a vertex $x \in V(G)$, the number of neighbors of x in G, denoted by $\deg_G(x)$, is

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called the degree of x in G. The number of edges of G is denoted by e(G). A matching in G is a subgraph of G such that each of its vertices has degree 1. The matching number of G is the maximum number of edges in a matching of G, denoted by $\nu(G)$. As usual, we use $\delta(G), \Delta(G)$ and $\chi(G)$ to denote the minimum degree, maximum degree, and chromatic number of G, respectively. For a graph G and G, G and G if G in the number of edges G instead of G induced by G in the subgraph of G induced by G integers G in the index G in the subgraph of G induced by G integers G in the index G in the index G in the subgraph of G induced by G integers G in the integers G in the integers G in the integers G integers G in the integers G in the integers G in the integers G integers G in the integers G in th

Let K_r denote the complete graph of order r and let $T_{n,r}$ denote the complete balanced r-partite graph of order n, also called the $Tur\'an\ graph$ in literature. For $k \geq 2$ and $r \geq 3$, a (k,r)-fan, denoted by $F_{k,r}$, is the graph on (r-1)k+1 vertices consisting of k K_r 's which intersect in exactly one common vertex, called the center of it. For some fixed graph H, let $\operatorname{ex}(n,H)$ be the maximum number of edges of graphs on n vertices that does not contain H as a subgraph. A graph G is called an extremal graph for H if G has n vertices with $e(G) = \operatorname{ex}(n,H)$ and does not contain H as a subgraph. Given two graphs G and H, an H-decomposition of G is a partition of edges of G such that every part is a single edge or forms a graph isomorphic to H. Let $\phi(G,H)$ be the smallest number of parts in an H-decomposition of G. Clearly, if H is non-empty, then

$$\phi(G, H) = e(G) - p_H(G)(e(H) - 1),$$

where $p_H(G)$ is the maximum number of edge-disjoint copies of H in G. Define

$$\phi(n, H) = \max\{\phi(G, H) : G \text{ is a graph on } n \text{ vertices}\}.$$

This function, motivated by the problem of representing graphs by set intersections, was first studied by Erdös, Goodman and Pósa [5], they proved that $\phi(n, K_3) = \exp(n, K_3)$. The result was generalized by Bollobás [2], he proved that $\phi(n, K_r) = \exp(n, K_r)$, for all $n \ge r \ge 3$. More generally, Pikhurko and Sousa [8] proposed the following conjecture.

Conjecture 1 ([8]). For any graph H with $\chi(H) \ge 3$, there is an $n_0 = n_0(H)$ such that $\phi(n, H) = \exp(n, H)$ for all $n \ge n_0$.

In [8], Pikhurko and Sousa also proved that $\phi(n, H) = \exp(n, H) + o(n^2)$. Recently, the error term improved to be $O(n^{2-\alpha})$ for some $\alpha > 0$ by Allen, Böttcher, and Person [1]. Sousa verified the conjecture for some families of edge-critical graphs, namely, clique-extensions of order $r \geq 4$ $(n \geq r)$ [11] and the cycles of length 5 $(n \geq 6)$ [9] and 7 $(n \geq 10)$ [10]. In [7], Özkahya and Person verified the conjecture for all edge-critical graphs with chromatic number $r \geq 3$. Here, a graph H is called edge-critical, if there is an edge $e \in E(H)$, such that $\chi(H) > \chi(H - e)$. For non-edge-critical graphs, Liu and Sousa [6] verified the conjecture for (k, 3)-fans, there result is as the following.

Theorem 2 ([6]). For $k \ge 1$, there exists $n_0 = n_0(k,r)$ such that $\phi(n, F_{k,3}) = \exp(n, F_{k,3})$ for all $n \ge n_0$. Moreover, the only graphs attaining $\exp(n, F_{k,3})$ are the extremal graphs for $F_{k,3}$.

In this paper, we verify Conjecture 1 for (k, r)-fans for $k \geq 2$ and $r \geq 3$ and hence generalizes Theorem 2. Our main result is the following.

Theorem 3. For $k \ge 2$ and $r \ge 3$, there exists $n_1 = n_1(k,r)$ such that $\phi(n, F_{k,r}) = \exp(n, F_{k,r})$ for all $n \ge n_0$. Moreover, the only graphs attaining $\exp(n, F_{k,r})$ are the extremal graphs for $F_{k,r}$.

The remaining of the paper is arranged as follows. Section 2 gives some lemmas. The proof of Theorem 3 is given in Section 3.

2 Lemmas

The extremal graphs for $F_{k,r}$ was determined by Chen, Gould, Pfender and Wei [4].

Lemma 4 (Theorem 2 in [4]). For every $k \ge 1$ and $r \ge 2$ and every $n \ge 16k^3r^8$, $ex(n, F_{k,r}) = ex(n, K_r) + g(k)$, where

$$g(k) = \begin{cases} k^2 - k & \text{if } k \text{ is odd,} \\ k^2 - \frac{3}{2}k & \text{if } k \text{ is even.} \end{cases}$$

And one of its extremal graphs, denoted by $G_{n,k,r}$, constructed as follows. If k is odd, $G_{n,k,r}$ is a Turán graph $T_{n,r-1}$ with two vertex disjoint copies of K_k embedding in one partite set. If k is even, $G_{n,k,r}$ is a Turán graph $T_{n,r-1}$ with a graph on 2k-1 vertices, $k^2 - \frac{3}{2}k$ edges, and maximum degree k-1 embedded in one partite set.

Lemma 5. Let G be an extremal graph on n vertices for $F_{k,r}$. Then $\delta(G) \geq \left| \frac{r-2}{r-1} n \right|$.

Proof. Suppose to the contrary that there is a vertex $v \in V(G)$ with $\deg_G(v) < \left| \frac{r-2}{r-1} n \right| = \delta(T_{n,r-1})$. Let G' = G - v. Then

$$e(G') \ge e(G) - \deg_G(v) \ge \exp(n, F_{k,r}) - \delta(T_{n,r-1}) + 1 = \exp(n-1, F_{k,r}) + 1$$

since $\operatorname{ex}(n, F_{k,r}) - \operatorname{ex}(n-1, F_{k,r}) = \delta(T_{n,r-1})$. By Lemma 4, G' (and hence G) contains a copy of $F_{k,r}$ as its subgraph, a contradiction.

Lemma 6. Let n_0 be an integer and let G be a graph on $n \ge n_0 + \binom{n_0}{2}$ vertices with $\phi(G, F_{k,r}) = \exp(n, F_{k,r}) + j$ for some integer j > 0. Then G contains a subgraph G' on $n' > n_0$ vertices such that $\delta(G') \ge \delta(T_{n-i,r-1})$ and $\phi(G', F_{k,r}) \ge \exp(n-i, F_{k,r}) + j + i$.

Proof. If $\delta(G) \geq \lfloor \frac{n}{2} \rfloor$, then G is the desired graph and we have nothing to do. So assume that $\delta(G) < \lfloor \frac{n}{2} \rfloor$. Let $v \in V(G)$ with $\deg_G(v) < \lfloor \frac{r-2}{r-1} n \rfloor$ and set $G_1 = G - v$. Then $\phi(G_1, F_{k,r}) \geqslant \phi(G, F_{k,r}) - \deg_G(v) \geqslant \exp(n, F_{k,r}) + j - \delta(T_{n,r-1}) + 1 = \exp(n - 1, F_{k,r}) + j + 1$, since $\exp(n, F_{k,r}) - \exp(n - 1, F_{k,r}) = \delta(T_{n,r-1})$ by Lemma 4. We may continue this procedure until we get a graph G' on n - i vertices with $\delta(G') \geqslant \lfloor \frac{r-2}{r-1}(n-i) \rfloor$ for some $i < n-n_0$, or until $i = n-n_0$. But the latter case can not occur since G' is a graph on n_0 vertices but $e(G') \geqslant \exp(n_0, F_{k,r}) + j + i \geqslant n - n_0 > \binom{n_0}{2}$, which is impossible.

The following stability lemma due to Özkahya and Person [7] is very important.

Lemma 7 ([7]). Let H be a graph with $\chi(H) = r \geqslant 3$ and $H \neq K_r$. Then, for every $\gamma > 0$ there exists $\delta > 0$ and $n_0 = n_0(H, \gamma) \in \mathbb{N}$ such that for every graph G on $n \geqslant n_0$ vertices with $\phi(G, H) \geqslant \operatorname{ex}(n, H) - \delta n^2$, then there exists a partition of $V(G) = V_1 \dot{\cup} ... \dot{\cup} V_{r-1}$ such that $\sum_{i=1}^{r-1} e(V_i) < \gamma n^2$.

Lemma 8 ([3]). For any graph G with maximum degree $\Delta \geqslant 1$ and matching number $\nu \geqslant 1$, then $e(G) \leqslant f(\nu, \Delta)$. Here, $f(\nu, \Delta) = \nu \Delta + \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\nu}{\lceil \Delta/2 \rceil} \right\rfloor$.

3 Proof of Theorem 3

The lower bound $\phi(n, H) \ge \exp(n, H)$ is trivial by the definition of $\phi(n, H)$ and $\exp(n, H)$. To prove Theorem 3, it is sufficient to prove that $\phi(n, F_{k,r}) \le \exp(n, F_{k,r})$ and the equality holds only for extremal graphs for $F_{k,r}$. Our proof is motivated by

the one in [6]. In outline, for every graph G with $\phi(G, F_{k,r}) \ge \operatorname{ex}(n, F_{k,r})$ and G is not an extremal graph for $F_{k,r}$, we will find sufficiently many edge-disjoint copies of $F_{k,r}$ in G which would imply that $\phi(G, F_{k,r}) = e(G) - p_{F_{k,r}}(G)(e(F_{k,r}) - 1) < \operatorname{ex}(n, F_{k,r})$, a contradiction. In other words, we will prove that

$$p_{F_{k,r}}(G) > \frac{e(G) - \exp(n, F_{k,r})}{e(F_{k,r}) - 1}.$$

For convenience, we set some constants as follows.

$$\begin{split} \gamma &= (40kr^4)^{-2}, \\ n_0 &= n_0(F_{k,r},\gamma) \text{ (which is determined by } F_{k,r} \text{ and } \gamma \text{ by applying Lemma 7)}, \\ \alpha &= \sqrt{1 - \frac{r-1}{r-2}\gamma}, \\ m_1 &= m_1(k,r) = \frac{(e(F_{k,r}) - 1)(r-1)k(k-1) - kg(k)}{e(F_{k,r}) - 1 - k}, \\ m_2 &= m_2(k,r) = \frac{(e(F_{k,r}) - 1)(r-1)k(k-1) - kg(k)}{(e(F_{k,r}) - 1)/2 - k}, \\ n_1 &= n_1(k,r) = 1 + \max\{\left\lceil \frac{r-1}{\alpha} \right\rceil, n_0 + \binom{n_0}{2}, 16(k+1)^3 r^8 + 6(k+1)^2 r^3 m_1\}. \end{split}$$

Now suppose that G is a graph on $n > n_1$ vertices, with $\phi(G, F_{k,r}) \geqslant \operatorname{ex}(n, F_{k,r})$ and G is not an extremal graph for $F_{k,r}$. Then $e(G) > \operatorname{ex}(n, F_{k,r})$. By Lemma 6, we may assume that $\delta(G) \geq \lfloor \frac{r-2}{r-1}n \rfloor$. Note that $\chi(F_{k,r}) = r$. Let $V_1, ..., V_{r-1}$ be a partition of V(G) such that $\sum_{1 \leq i < j \leq r-1} e(V_i, V_j)$ is maximized. Let $m = \sum_{i=1}^{r-1} e(V_i)$. By Lemma 7 and the choice of the partition of V(G), we have $m < \gamma n^2$. By Lemma 4, observe that

$$m = e(G) - \sum_{1 \le i < j \le r-1} e(V_i, V_j) > \exp(n, F_{k,r}) - e(T_{n,r-1}) = g(k),$$

and that

$$e(G) = m + \sum_{1 \le i < j \le r-1} e(V_i, V_j) \le m + e(T_{r-1,n}) = \exp(n, F_{k,r}) + m - g(k).$$

So to prove Theorem 3, it suffices to show that

$$p_{F_{k,r}}(G) > \frac{m - g(k)}{e(F_{k,r}) - 1} \ (\ge \frac{e(G) - \exp(n, F_{k,r})}{e(F_{k,r}) - 1}).$$

The following claim asserts that the partition $V(G) = V_1 \dot{\cup} V_2 \dot{\cup} \cdots \dot{\cup} V_{r-1}$ is very close to balance.

Claim 1. For every $i \in [1, r-1]$, $||V_i| - \frac{n}{r-1}| \leq 2\sqrt{\gamma}n$.

Proof. Let $a = \max\{\left||V_j| - \frac{n}{r-1}\right|, j \in [1, r-1].$ Wlog, suppose that $|V_1| - \frac{n}{r-1} = a$, then

$$\begin{split} e(G) &= \sum_{1\leqslant i < j\leqslant r-1} e(V_i,V_j) + m \\ &\leqslant \sum_{1\leqslant i < j\leqslant r-1} |V_i||V_j| + m \\ &= |V_1|(n-|V_1|) + \sum_{2\leqslant i < j\leqslant r-1} |V_i||V_j| + m \\ &= |V_1|(n-|V_1|) + \frac{1}{2} \left[\left(\sum_{j=2}^{r-1} |V_j| \right)^2 - \sum_{j=2}^{r-1} |V_j|^2 \right] + m \\ &\leqslant |V_1|(n-|V_1|) + \frac{1}{2}(n-|V_1|)^2 - \frac{1}{2(r-2)}(n-|V_1|)^2 + m \\ &\leqslant -\frac{r-1}{2(r-2)}a^2 + \frac{r-2}{2(r-1)}n^2 + \gamma n^2. \end{split}$$

The fifth inequality holds by Jensen's inequality and the last inequality holds since $|V_1| = a + \frac{n}{r-1}$, and $m < \gamma n^2$.

While on the other hand,

$$e(G) > ex(n, F_{k,r}) > e(T_{n,r-1})$$

$$\geqslant {r-1 \choose 2} \left(\frac{n}{r-1} - 1\right)^2$$

$$\geqslant {r-1 \choose 2} \left(\frac{\alpha n}{r-1}\right)^2$$

$$= \frac{r-2}{2(r-1)} n^2 \alpha^2.$$

Therefore,

$$\frac{r-1}{2(r-2)}a^2 < \left[\frac{r-2}{2(r-1)}(1-\alpha^2) + \gamma\right]n^2 < 2\gamma n^2,$$

which implies that $a < \sqrt{\frac{4(r-2)}{r-1}\gamma}n < 2\sqrt{\gamma}n$.

Recall that our purpose is to find more than $(m-g(k))/(e(F_{k,r})-1)$ edge-disjoint copies of $F_{k,r}$. We continue the proof by considering two different cases.

Case 1. $m > m_1$.

Set $t_1 = \frac{n}{16kr^3}$ and

$$t_2 = \frac{n}{r-1} - 2(r-3)\sqrt{\gamma}n - 2t_1 - 2.$$

For $v \in V_i$ $(i \in [1, r-1])$, we say that v is bad if $\deg_{G[V_i]}(v) > t_1$, otherwise v is said to be good. For each bad vertex $v \in V_i$, choose $k\lceil \deg_{G[V_i]}(v)/2k \rceil$ edges which connect v to good vertices in V_i . We keep these edges and delete other edges in $G[V_i]$ at v. We repeat this procedure to every bad vertex in G and denote the resulting graph by G_0 . This is possible since the number of bad vertices in G is at most

$$\frac{2m}{t_1} < \frac{2\gamma n^2}{t_1} = 32kr^3\gamma n < \frac{t_1}{2}.$$

Next we will find copies of $F_{k,r}$ in G_0 . Each time we find a copy of $F_{k,r}$, we delete the edges of $F_{k,r}$, and let G_s denote the graph obtained from G_0 after deleting the edges of s copies of $F_{k,r}$. For a vertex $v \in V_i$ in G_s , we say that v is active (in G_s) if $e_{G_s}(v, V_j) > t_2$ for every $j \neq i$, otherwise v is said to be inactive.

We have the following two claims about G_0 .

Claim 2.

$$\sum_{i=1}^{r-1} e_{G_0}(V_i) \geqslant \frac{m}{2}.$$

Proof. For every i, let $U_i \subset V_i$ be the set of good vertices in V_i . Then

$$e_{G_0}(V_i) = e_{G_0}(U_i) + \sum_{\substack{v \in V_i \\ v \text{ is bad}}} \deg_{G_0[V_i]}(v)$$

$$= e_G(U_i) + \sum_{\substack{v \in V_i \\ v \text{ is bad}}} k \left[\frac{1}{2k} \deg_{G[V_i]}(v) \right]$$

$$\geqslant \frac{1}{2} e_G(U_i) + \frac{1}{2} \sum_{\substack{v \in V_i \\ v \text{ is bad}}} \deg_{G[V_i]}(v)$$

$$\geqslant \frac{1}{2} e_G(V_i).$$

Claim 3. All of good vertices are active in G_0 .

Proof. Let $v \in V_i$ be a good vertex. Then for every $j \neq i$, by Claim 1 and Lemma 6,

$$e_{G_0}(v, V_j) \geq \delta(G) - \deg_{G_0[V_i]}(v) - (r - 3)(\frac{n}{r - 1} + 2\sqrt{\gamma}n)$$

$$\geq \delta(T_{n,r-1}) - \deg_{G[V_i]}(v) - (r - 3)(\frac{n}{r - 1} + 2\sqrt{\gamma}n)$$

$$\geq \frac{n}{r - 1} - 2(r - 3)\sqrt{\gamma}n - t_1 - 1$$

$$\geq t_2 + t_1.$$

The following two steps are the procedures to find edge-disjoint copies of $F_{k,r}$.

Step 1. Suppose that we have gotten G_s for some integer $s \ge 0$. If there exists a vertex $u \in V_i$ with $\deg_{G_s[V_i]}(u) \ge k$ (bad vertices are considered first, followed by good vertices), let $v_i^1, ..., v_i^k$ be k neighbors of u in V_i , then for every $j \ne i$ we find k good and active vertices $v_j^1, ..., v_j^k \in V_j$ such that for every $\ell \in [1, k], G_s[u, v_1^\ell, ..., v_{r-1}^\ell]$ is a copy of K_r , and thus $G_s[\{u\} \cup \{v_j^\ell : \ell \in [1, k], j \in [1, r-1]\}]$ contains a copy of $F_{k,r}$ centered at u. Let G_{s+1} be the updated new graph obtained from G_s by deleting the edges of this $F_{k,r}$.

Step 2. After Step 1 is completed, denote the remaining graph by G_a . Then $\Delta(G_a[V_i]) < k$ for each $i \in [1, r-1]$, and $\deg_{G_a[V_i]}(u) = 0$ for all bad vertices $u \in V_i$ since $\deg_{G_0[V_i]}(u)$ is a multiple of k. Suppose that we have get G_s for some $s \geqslant a$. If there is a matching of order k in $G_s[V_i]$ for some $i \in [1, r-1]$, for example, let $v_1^1w_1^1, ..., v_1^kw_1^k \in V_1$ be such a matching. We find good and active vertices $u \in V_2$ and $v_1^1, ..., v_j^k$ in V_j for every $j \in [3, r-1]$ such that $G_s[v_1^\ell, w_1^\ell, u, v_3^\ell, \cdots, v_{r-1}^\ell]$ is a copy of K_r for each $\ell \in [1, k]$ and so $G_s[\{u\} \cup \{v_1^\ell, w_1^\ell, v_j^\ell : j \in [3, r-1], \ell \in [1, k]\}]$ contains a copy of $F_{k,r}$ centered at u. Let G_{s+1} be the updated new graph obtained from G_s by deleting the edges of $F_{k,r}$. When Step 2 is completed, denote the remaining graph by G_b .

Note that after Step 1 and 2 are finished, we have found at least

$$\left[\frac{1}{k} \left(\sum_{i} e_{G_0}(V_i) - \sum_{i} e_{G_b}(V_i) \right) \right]$$

edge-disjoint $F_{k,r}$ from G since each copy of $F_{k,r}$ using exactly k edges from $\bigcup_{i=1}^{r-1} E(G[V_i])$. Since $\Delta(G_b[V_i]) \leq k-1$ and $\nu(G_b[V_i]) \leq k-1$ for $i=1,2,\cdots,r-1$, by Lemma 8,

$$e_{G_b}(V_i) \leqslant f(k-1,k-1) \leqslant k(k-1),$$

which implies that $\sum_{i} e_{G_b}(V_i) \leq (r-1)k(k-1)$. Let p be the number of removed edge-disjoint $F_{k,r}$ from G.

If $m > m_2(k,r)$ (> m_1), then, since $\sum_{i=1}^{r-1} e_{G_0}(V_i) \ge \frac{m}{2}$, we have

$$p_{F_{k,r}}(G) \ge p \geqslant \frac{1}{k} \left[\frac{m}{2} - (r-1)k(k-1) \right] > \frac{m-g(k)}{e(F_{k,r})-1}.$$

If $m_1 < m \le m_2(k, r)$, then for every vertex $u \in V_i$, $\deg_{G[V_i]}(u) \le m \le m_2(k, r) < t_1$ and hence u is good. That is G has no bad vertices and so $G_0 = G$. Therefore,

$$p_{F_{k,r}}(G) \ge p \geqslant \frac{1}{k} \left[m - (r-1)k(k-1) \right] > \frac{m - g(k)}{e(F_{k,r}) - 1}.$$

Therefore, to complete the proof of Case 1, it remains to prove the following claim.

Claim 4. Step 1 and 2 can be successfully iterated.

To proof the above claim, we first estimate the number of good and inactive vertices in each iteration of Step 1 or 2.

Claim 5. Let $G_s \subset G_0$ be a subgraph at some point of the iteration in Step 1 or Step 2. Then the number of good inactive vertices in G_s is at most $8kr^5\gamma n$.

Proof. Since in each iteration of Step 1 or Step 2, the number of removed edges with both endpoints in V_i is exactly $k, s \leq m/k < \gamma n^2/k$. So the total number of deleted edges from G_0 is at most $e(F_{k,r}) \cdot s < \frac{r(r-1)}{2} \gamma n^2$. By Claim 3, for every good vertex $u \in V_i$, u is active in G_0 and $e_{G_0}(v, V_j) \geq t_2 + t_1$. Thus the number of good vertices that are inactive in G_s is at most $\frac{e(F_{k,r}) \cdot s}{t_1} < 8kr^5 \gamma n$.

Proof of Claim 4: Let G_s be the graph obtained at some point of the iteration in Step 1. For any fixed $x \in V_i$ and every $j \neq i$. If x is bad and is involved in a previous iterate, then x is the center of a copy of $F_{k,r}$ and hence the number of removed edges that x sends to V_j is k. Since x involves at most $\frac{\deg_{G_0[V_i]}(x)}{k}$ iterates,

$$e_{G_s}(x, V_j) \ge e_{G_0}(x, V_j) - k \cdot \frac{\deg_{G_0[V_i]}(x)}{k}$$

$$= e_G(x, V_j) - k \left\lceil \frac{\deg_{G[V_i]}(x)}{2k} \right\rceil$$

$$\ge \frac{e_G(x, V_j)}{2} - k$$

$$\ge \frac{n}{4(r-1)} - \frac{r-3}{2} \sqrt{\gamma} n - \frac{1}{4} - k$$

$$\ge \frac{t_2}{4} + \frac{t_1}{2} + \frac{1}{4} - k,$$

the third inequality holds since

$$e_G(x, V_j) \ge \deg_{G[V_i]}(x) \tag{1}$$

((1) holds because of the maximality of $\sum_{1 \leq i < j \leq r-1} e(V_i, V_j)$); the forth inequality holds since

$$e_{G}(x, V_{j}) = \deg_{G}(x) - \deg_{G[V_{i}]}(x) - \sum_{\ell \neq i, j} e_{G}(x, V_{\ell})$$

$$\geq \delta(G) - \deg_{G[V_{i}]}(x) - (r - 3)(\frac{n}{r - 1} + 2\sqrt{\gamma}n)$$
(2)

and ((1)) and (2) implies that

$$e_G(x, V_j) \geqslant \frac{1}{2} [\delta(G) - (r-3)(\frac{n}{r-1} + 2\sqrt{\gamma}n)]$$

 $\geq \frac{n}{2(r-1)} - (r-3)\sqrt{\gamma}n - \frac{1}{2}.$

If x is good and active in G_s , then $e_{G_s}(x, V_j) \geq t_2$. Now suppose x is good but inactive in G_s . Then x becomes inactive in a previous iterate. If x is involved in a succeeding iterate, then x is chosen to be the center of a copy of $F_{k,r}$. So the number of edges that x sends to V_j is k and x is involved in at most $\frac{\deg_{G_0[V_i]}(x)}{k}$ succeeding iterates. Hence, by inequality (2) and $\deg_{G[V_i]}(x) \leq t_1$,

$$e_{G_s}(x, V_j) \geqslant t_2 - k - \deg_{G_0[V_i]}(x)$$

$$\geq t_2 - t_1 - k$$

$$(\geq \frac{t_2}{4} + \frac{t_1}{2} + \frac{1}{4} - k \text{ when } n \text{ is sufficiently large}).$$

Wlog, let u be in V_1 and $v_1^1,...,v_1^k \in V_1$ are k (good) neighbors of u. Suppose that for some $\ell \in [1,r-2]$, we have found good and active vertices $v_i^1,...,v_i^k \in V_i$ $(1 \le i \le l)$ such that for every $j \in [1,k]$, $G_s[u,v_1^j,...,v_\ell^j]$ is a copy of $K_{\ell+1}$. Then, for every $j \in [1,k]$, by Claim 5, the number of good and active common neighbors of

 $u, v_1^j, ..., v_{\ell}^j$ in $V_{\ell+1}$ of G_s , denoted by $L_{\ell+1}(u, v_1^j, ..., v_{\ell}^j)$, is

$$|L_{\ell+1}(u, v_1^j, ..., v_{\ell}^j)| \geq e_{G_s}(u, V_{\ell+1}) + \sum_{i=1}^{\ell} e_{G_s}(v_i^j, V_{\ell+1}) - \ell |V_{\ell+1}|$$

$$-32kr^3 \gamma n - 8kr^5 \gamma n$$

$$\geq \frac{t_2}{4} + \frac{t_1}{2} + \frac{1}{4} - k + \ell (t_2 - t_1 - k)$$

$$-\ell (\frac{n}{r-1} + 2\sqrt{\gamma}n) - 32kr^3 \gamma n - 8kr^5 \gamma n$$

$$\geq \frac{n}{4(r-1)} - \frac{13n}{40kr^2} - (k+2)r - k - \frac{1}{4}$$

$$\geq \frac{n}{4r} \text{ (when } n \text{ is sufficiently large)}.$$

Therefore, Step 1 can be performed successfully to find a copy of $F_{k,r}$ centered at u. Now, let G_s be the graph obtained at some point of the iteration in Step 2 for some $s \ge a$. Let $x \in V_i$ be a good vertex and $j \ne i$. Then, after x becomes inactive in previous iterates, the total number of removed edges that x sends to V_j are at most $\deg_{G_a[V_i]}(x)$. Hence

$$e_{G_s}(x, V_j) \geqslant t_2 - 2k - \deg_{G_a[V_i]}(x)$$

 $\geqslant t_2 - 2k - \deg_{G[V_i]}(x)$
 $\geqslant t_2 - 2k - t_1.$

Wlog, suppose that $v_1^1w_1^1, ..., v_1^kw_1^k$ is a matching in V_1 . Then $v_1^1, w_1^1, ..., v_1^k, w_1^k$ are good. Let $X = \{v_1^1, w_1^1, ..., v_1^k, w_1^k\}$. By CLaim 5, the number of common active good neighbors of X in V_2 of G_s , denoted by $L_2(X)$, is at least

$$|L_{2}(X)| = \sum_{x \in X} e_{G_{s}}(x, V_{2}) - (2k - 1)|V_{2}| - 32kr^{3}\gamma n - 8kr^{5}\gamma n$$

$$\geqslant 2k(t_{2} - 2k - t_{1}) - (2k - 1)(\frac{n}{r - 1} + 2\sqrt{\gamma}n) - 32kr^{3}\gamma n - 8kr^{5}\gamma n$$

$$= \frac{n}{r - 1} - 2[2k(r - 2) - 1]\sqrt{\gamma}n - 6kt_{1} - 4k(k + 1) - 32kr^{3}\gamma n - 8kr^{5}\gamma n$$

$$\geqslant \frac{n}{r - 1} - \frac{n}{2r^{2}} - 4k(k + 1)$$

$$\geq \frac{n}{r} \text{ (when } n \text{ is sufficiently large)}.$$

Choose such a common neighbor u of X in V_2 .

Now, suppose that we have found active and good vertices $v_i^1, ..., v_i^k$, for $i \in [3, \ell]$ $(2 \le \ell \le r-2)$ such that for every $j \in [1, k]$, $G_s[v_1^j, w_1^j, u, v_3^j, ..., v_\ell^j]$ is a copy of $K_{\ell+1}$.

Let $Y_j = \{v_1^j, w_1^j, u, v_3^j, ..., v_\ell^j\}$ for $j \in [1, k]$ and denote the the common active and good neighbors of Y_j in $V_{\ell+1}$ of G_s by $L_{\ell+1}(Y_j)$. Then the same reason as before,

$$|L_{\ell+1}(Y_j)| \geq \sum_{x \in Y_j} e_{G_s}(x, V_{\ell+1}) - \ell \cdot |V_{\ell+1}| - 32kr^3 \gamma n - 8kr^5 \gamma n$$

$$\geqslant (\ell+1)(t_2 - 2k - t_1) - \ell(\frac{n}{r-1} + 2\sqrt{\gamma}n) - 32kr^3 \gamma n - 8kr^5 \gamma n$$

$$= \frac{n}{r-1} - 2[(r-2)(\ell+1) - 1]\sqrt{\gamma}n - 3(\ell+1)t_1 - 32kr^3 \gamma n$$

$$-8kr^5 \gamma n - 2(\ell+1)(k+1)$$

$$\geqslant \frac{n}{r-1} - 2((r-2)(r-1) - 1)\sqrt{\gamma}n - 3(r-1)t_1 - 32kr^3 \gamma n$$

$$-8kr^5 \gamma n - 2(r-1)(k+1)$$

$$\geqslant \frac{n}{r-1} - \frac{3n}{8kr^2} - 2(r-1)(k+1)$$

$$\geq \frac{n}{r} \text{ (when } n \text{ is sufficiently large)}.$$

Hence k different active and good common neighbors $v_{\ell+1}^1, \dots, v_{\ell+1}^k$ with $v_{\ell+1}^j \in L_{\ell+1}(Y_j)$ for $j \in [1, k]$ do exist and therefore Step 2 can be successfully iterated.

Case 2. $m \leqslant m_1$.

For every $i \in [1, r-1]$, let $B_i \subset V_i$ be the set of isolated vertices of $G[V_i]$ and $A_i = V_i \setminus B_i$. Then $|A_i| \leq 2m \leq 2m_1$. Since $|V_i| \geqslant \frac{n}{r-1} - 2\sqrt{\gamma}n > 2m_1 \geq |A_i|$, we have $B_i \neq \emptyset$. Note that, for $u \in B_i$, $\deg_{G[V_i]}(u) = 0$. By $\lfloor \frac{r-2}{r-1}n \rfloor \leq \deg_G(u) \leq n - |V_i|$, we have $|V_i| \leq \lceil \frac{n}{r-1} \rceil$. Together with $|V_1| + ... + |V_{r-1}| = n$, we have the following claim.

Claim 6. For every $i \in [1, r-1]$, $\left\lceil \frac{n}{r-1} \right\rceil - (r-2) \leqslant |V_i| \leqslant \left\lceil \frac{n}{r-1} \right\rceil$. Particularly, if (r-1)|n, then $|V_i| = \frac{n}{r-1}$ for each $i \in [1, r-1]$.

Since $e(G) \ge \phi(G, F_{k,r}) \ge \exp(n, F_{k,r})$, there exists some integer $s \ge 0$, such that

$$s(e(F_{k,r}) - 1) \le e(G) - \exp(n, F_{k,r}) < (s+1)(e(F_{k,r}) - 1).$$

Note that $e(G) - \exp(n, F_{k,r}) \leq m - g(k) \leq m_1 - g(k)$. Hence $s \leq \frac{m_1 - g(k)}{e(F_{k,r}) - 1}$. Furthermore, we have a simple and useful upper bound for s as follows.

Claim 7. $s < \frac{(r-2)(k-1)}{2} + 1$. Particularly, $s < e(F_{k,r})$.

Proof. For convenience, write $e = e(F_{k,r}) = \frac{kr(r-1)}{2}$ and g = g(k). Since $s \leq \frac{m_1-g}{e-1}$, it's sufficient to prove $1 + \frac{(r-2)(k-1)}{2} > \frac{m_1-g}{e-1}$. Note that $m_1 = \frac{(e-1)(r-1)k(k-1)-kg}{e-k-1}$. So

$$\frac{(r-2)(k-1)}{2} + 1 > \frac{m_1 - g}{e - 1}$$

$$\Leftrightarrow \frac{(r-2)(k-1)}{2} + 1 > \frac{(r-1)k(k-1) - g}{e - k - 1}$$

$$\Leftrightarrow \frac{(r-2)(k-1)}{2} + 1 > \frac{(r-1)k(k-1) - k(k-3/2)}{e - k - 1} \text{ (since } g \ge k(k-3/2))$$

$$\Leftrightarrow \frac{(r-2)(k-1)}{2} + 1 > \frac{(r-2)k(k-1) + k/2}{e - k - 1}$$

$$\Leftrightarrow [(r-2)(k-1) + 2](e - k - 1) - 2(r-2)k(k-1) - k > 0$$

$$\Leftrightarrow (r-2)(k-1)(e - 3k - 1) + 2e - 3k - 2 > 0$$

Note that $e = \frac{kr(r-1)}{2} \ge 3k$ $(r \ge 3)$ and the equality holds if and only if r = 3. It is an easy task to check that the above inequality always holds for $k \ge 1$.

Clearly,
$$\frac{kr(r-1)}{2} > \frac{(r-2)(k-1)}{2} + 1$$
 when $r \ge 3$ and $k \ge 1$. So $s < e(F_{k,r})$.

If we can find s+1 edge-disjoint copies of $F_{k,r}$ in G, then we have

$$\phi(G, F_{k,r}) \le e(G) - (s+1)(e(F_{k,r}) - 1) < \exp(n, F_{k,r}),$$

a contradiction with the assumption that $\phi(G, F_{k,r}) \ge \exp(n, F_{k,r})$. So to complete the proof of Case 2 (and the proof of Theorem 3), it remains to prove the following claim.

Claim 8. G contains s + 1 edge-disjoint copies of $F_{k,r}$.

Before we prove Claim 8, we need an auxiliary claim.

Claim 9. For every $i \in [1, r-1]$, and any subset $A^i \subset A_i$, if $A^i \neq \emptyset$ and $|A^i| \leq (k+1)(s+1)$, then for every $j \neq i$, there exists a subset $B^j \subset B_j$ with $|B^j| = (k+1)(s+1)$ such that G contains a complete (r-1)-partite subgraph of G with partitions A^i , B^j $(j \in [1, r-1]$ and $j \neq i)$.

Proof. Wlog, let $A^1 \subset A_1$ and $|A^1| = (k+1)(s+1)$. Suppose that we have found $B^2, ..., B^{\ell}, B^j \subset B_j$ with $|B^j| = (k+1)(s+1), j \in [2, \ell], \ell \in [1, r-2]$, such that G contains a complete ℓ -partite subgraph with partitions A^1 and $B^2, ..., B^{\ell}$. Note that

 $|V_i| \leq \lceil \frac{n}{r-1} \rceil$ $(i \in [1, r-1])$ by Claim 6. Hence for any $u \in A^1$ and $v \in B^j$,

$$e_{G}(u, V_{\ell+1}) \geq \delta(G) - (r-3)(\frac{n}{r-1} + 1) - |A_{1}|$$

$$\geq \frac{n}{r-1} - (r-2) - 2m_{1},$$

$$e_{G}(v, V_{\ell+1}) \geq \delta(G) - (r-3)(\frac{n}{r-1} + 1)$$

$$\geq \frac{n}{r-1} - (r-2) - 2m_{1}.$$

Let $Z^{\ell} = A^1 \cup B^2 \cup ... \cup B^{\ell}$ and let $L_{\ell+1}(Z^{\ell})$ be the set of common neighbors of Z^{ℓ} in $B_{\ell+1}$. Then

$$|L_{\ell+1}(Z^{\ell})| \geq \sum_{v \in Z^{\ell}} e_G(v, V_{\ell+1}) - [(k+1)(s+1)\ell - 1]|V_{\ell+1}| - |A_{\ell+1}|$$

$$\geqslant (k+1)(s+1)\ell \left[\frac{n}{r-1} - (r-2) - 2m_1\right]$$

$$-[(k+1)(s+1)\ell - 1]\left(\frac{n}{r-1} + 1\right) - 2m_1$$

$$= \frac{n}{r-1} - (2m_1 + r - 1)(k+1)(s+1)\ell - 2m_1 + 1$$

$$\geqslant \frac{n}{r} \text{ (when } n \text{ is sufficiently large)}.$$

Hence we always can choose $B^{\ell+1} \subset B_{\ell+1}$ for $\ell \in [1, r-2]$ and so the result follows. \square

Proof of Claim 8: By Claim 7, $s < e(F_{k,r})$. By $e(G) \ge ex(n, F_{k,r}) + s(e(F_{k,r}) - 1)$, we have

$$e(G) - (s-1)e(F_{k,r}) \ge ex(n, F_{k,r}) + e(F_{k,r}) - s > ex(n, F_{k,r}).$$

So, G contains s edge-disjoint copies of $F_{k,r}$. Let G' be a subgraph of G by removing the edges of s copies of $F_{k,r}$. If there is a vertex u in G' with $\deg_{G'}(u) \leq \left\lfloor \frac{r-2}{r-1}n \right\rfloor - s$, then

$$\begin{split} e(G'-u) &= e(G') - \deg_{G'}(u) \\ &= e(G) - s \cdot e(F_{k,r}) - \deg_{G'}(u) \\ &\geqslant &\exp(n, F_{k,r}) - \left\lfloor \frac{r-2}{r-1} n \right\rfloor \\ &= &\exp(n-1, F_{k,r}). \end{split}$$

If the equality does not hold or the equality holds but we can show that G' - u is not an extremal graph for $F_{k,r}$, then G' - u contains a copy of $F_{k,r}$ and we are done.

In the following, we show that how to remove the edges of s copies of $F_{k,r}$ in G to get our desired subgraph G' and vertex u according to three cases.

- (I). If there are s vertices $u_1, ..., u_s \in A_1 \cup ... \cup A_{r-1}$ such that $\deg_{G[A_i]}(u_j) \geqslant k+s-1$ for $u_j \in A_i$. Let $q = \max\{|\{u_1, ..., u_s\} \cap A_i| : i \in [1, r-1]\}$. Then $s \geqslant q \geqslant \left\lceil \frac{s}{r-1} \right\rceil$. Wlog, assume that $u_1, ..., u_q \in A_1$. For each $j \in [1, q]$, choose k neighbors of u_j in $A_1 \setminus \{u_1, ..., u_q\}$, say $v_{j1}, ..., v_{jk}$, this is possible since there are at least $k+s-1-(q-1)=k+s-q\geqslant k$ such neighbors. We further assume that $v_{j,1} \neq v_{\ell,1}$ for $1 \leqslant j < \ell \leqslant q$. Let $A_j^1 = \{u_j, v_{j1}, ..., v_{jk}\}$ for $j \in [1, q]$. Then $|\cup_{j=1}^q A_j^1| \leq q(k+1)$. By Claim 9, we can find a common neighbor $u \in B_{i_0}$ of $\cup_{j=1}^q A_j^1$ for some $i_0 \neq 1$, here i_0 will be determined later. For each $j \in [1, q]$, since $|A_j^1| = k+1$, by Claim 9, we can choose $B_j^i \subset B_i$ for $i \in [2, r-1]$ such that
 - (a) $|B_i^i| = k$,
 - (b) $B_i^{i_0} \cap B_\ell^{i_0} = \{u\} \text{ for } j, \ell \in [1, q] \text{ and } j \neq \ell,$
 - (c) $B_j^i \cap B_\ell^i = \emptyset$ for $i \ge 2$, $i \ne i_0$ and $j \ne \ell$,
 - (d) G contains a complete (r-1)-partite graph with partitions $A_j^1, B_j^2, ..., B_j^{r-1}$.

Assume $B_j^i = \{u_{j1}^i, ..., u_{jk}^i\}$ and $u_{j1}^{i_0} = u$ for every $j \in [1, q]$ and $i \in [2, r - 1]$. Then for every $j \in [1, q]$ and every $\ell \in [1, k]$, $G_{j\ell} = G[u_j, v_{j\ell}, u_{j\ell}^2, ..., u_{j\ell}^{r-1}]$ is a copy of K_r and thus $G_j = G_{j1} \cup ... \cup G_{jk}$ contains a copy of $F_{k,r}$ centered at u_j . Furthermore, $G_1, ..., G_q$ are q edge-disjoint copies of $F_{k,r}$ with a common vertex u. Let G' be the subgraph obtained from G by deleting the edges of $G_1, G_2, ..., G_q$ and any other s-q copies of $F_{k,r}$ in G.

(I.1). If $q \geqslant \left\lceil \frac{s}{r-1} \right\rceil + 1$, then

$$\deg_{G'}(u) \leqslant n - |V_{i_0}| - (r - 1)q \leqslant \left| \frac{r - 2}{r - 1} n \right| - s - 1 < \left| \frac{r - 2}{r - 1} n \right| - s,$$

the second inequality holds since $|V_{i_0}| \ge \lceil \frac{n}{r-1} \rceil - (r-2)$ by Claim 6. Hence u is the desired vertex and we are done.

(I.2).
$$q = \left\lceil \frac{s}{r-1} \right\rceil$$
, let $s \equiv t \mod (r-1)$, $(t \in [0, r-2])$.

If t=0 or $t\geqslant 2$, then at least two subsets of A_1,\ldots,A_{r-1} satisfying that $|\{u_1,\ldots,u_s\}\cap A_i|=q$ by the choice of q. By Claim 6, at least one of $V_i,\ i\in[1,r-1]$ with $|V_i|=\left\lceil\frac{n}{r-1}\right\rceil$. Hence we can choose V_1 and $i_0\neq 1$ such that $|V_1|\leq \left\lceil\frac{n}{r-1}\right\rceil$ and $|V_{i_0}|=\left\lceil\frac{n}{r-1}\right\rceil$. So,

$$\deg_{G'}(u) \leqslant n - \left\lceil \frac{n}{r-1} \right\rceil - (r-1)q \leqslant \left\lfloor \frac{r-2}{r-1}n \right\rfloor - s.$$

Here we need to show that G'-u is not an extremal graph for $F_{k,r}$. Let H=G'-u. Consider $u_{12}^{i_0} \in V(H)$. Then $\deg_H(u_{12}^{i_0}) \leq n - \left\lceil \frac{n}{r-1} \right\rceil - (r-1) < \left\lfloor \frac{r-2}{r-1} (n-1) \right\rfloor$. By Lemma 5, H is not an extremal graph on n-1 vertices for $F_{k,r}$ and hence we are done.

If t=1, then $q=\frac{s-1}{r-1}+1$. Choose i_0 such that $|V_{i_0}|=\max_{\ell\in[2,r-1]}|V_\ell|$. Then $|V_{i_0}|\geqslant \left\lfloor\frac{n}{r-1}\right\rfloor$, otherwise, by Claim 6, $|V_1|+\ldots+|V_{r-1}|\leqslant \left\lceil\frac{n}{r-1}\right\rceil+(r-2)(\left\lfloor\frac{n}{r-1}\right\rfloor-1)< n$. Hence, by Claim 6,

$$\deg_{G'}(u) \leqslant n - \left\lfloor \frac{n}{r-1} \right\rfloor - (r-1)q = \left\lfloor \frac{r-2}{r-1}n \right\rfloor - s - (r-1) < \left\lfloor \frac{r-2}{r-1}n \right\rfloor - s.$$

So we are done in this case.

(II). Suppose that G contains a copy of $F_{qk,r}$ with center $u \in A_1 \cup ... \cup A_{r-1}$ and $q \geqslant 2$. We choose u so that q is maximum. If $q \geqslant s+1$ then we are done since $F_{qk,r}$ contains s+1 edge-disjoint copies of $F_{k,r}$. So $q \leqslant s$. Then we must have $\deg_G(u) < n - \left(\left\lceil \frac{n}{r-1} \right\rceil - (r-2)\right) + (q+1)k$, otherwise, $\deg_{G[A_i]}(u) \ge (q+1)k$ for some $i \in [1, r-1]$ with $u \in A_i$, then Claim 9 guarantees that there exists a copy of $F_{(q+1)k,r}$ centered at u in G, contradicting the choice of q. Let G' be the graph obtained from G by deleting the edges of the copy of $F_{qk,r}$ and the edges of any further s-q copies of $F_{k,r}$. We have

$$\deg_{G'}(u) \leq n - \left(\left\lceil \frac{n}{r-1} \right\rceil - (r-2) \right) + (q+1)k - 1 - (r-1)qk$$

$$= \left\lfloor \frac{r-2}{r-1}n \right\rfloor + (r-3) - ((r-2)q-1)k$$

$$\leq \left\lfloor \frac{r-2}{r-1}n \right\rfloor + (r-3) - (2r-5)k \text{ (since } q \geq 2)$$

$$= \left\lfloor \frac{r-2}{r-1}n \right\rfloor - \left\lfloor \frac{(r-2)(k-1)}{2} + 1 \right\rfloor - \frac{(r-2)(k-1)}{2} - (r-3)k$$

$$< \left\lfloor \frac{r-2}{r-1}n \right\rfloor - s \text{ (since } s < \frac{(r-2)(k-1)}{2} + 1 \text{ by Claim 7)}.$$

Hence u is the desired vertex and we are done.

(III). Now, suppose that (I) and (II) do not hold. We obtain G' from G by deleting the edges of any s copies of $F_{k,r}$. If there is a copy centered at $u \in B_1 \cup ... \cup B_{r-1}$,

then, by Claim 6,

$$\begin{split} \deg_{G'}(u) &\leqslant n - \left(\left\lceil \frac{n}{r-1} \right\rceil - (r-2) \right) - (r-1)k \\ &= \left\lfloor \frac{r-2}{r-1} n \right\rfloor - (r-1)(k-1) - 1 \\ &< \left\lfloor \frac{r-2}{r-1} n \right\rfloor - s \text{ (since } s < \frac{(r-2)(k-1)}{2} + 1 \text{ by Claim 7).} \end{split}$$

Hence, all the centers of the s copies of $F_{k,r}$ lie in $A_1 \cup ... \cup A_{r-1}$. By (II), they must be pairwisely distinct. By (I), at most s-1 of them have degree at least k+s-1 in $G[A_i]$, $i \in [1,r-1]$. Hence, at least one of the centers, say $u \in A_i$ with $\deg_{G[A_i]}(u) \leq k+s-2$. By Claim 6 and $s < \frac{(r-2)(k-1)}{2} + 1$, we have

$$\deg_{G'}(u) \leqslant n - \left(\left\lceil \frac{n}{r-1} \right\rceil - (r-2) \right) + k + s - 2 - (r-1)k$$

$$= \left\lfloor \frac{r-2}{r-1} n \right\rfloor - (r-2)(k-1) + s - 2$$

$$< \left\lfloor \frac{r-2}{r-1} n \right\rfloor - s.$$

Hence u is a desired vertex in G' and we are done.

This completes the proof of Claim 8 and also completes the proof of Theorem 3. **Acknowledgement.** We would like to thank Professor Jie Ma for many helpful comments and suggestions.

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