

Exponentially Many 4-List-Colorings of Triangle-Free Graphs on Surfaces

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Abstract

Thomassen proved that every planar graph G on n vertices has at least $2^{n/9}$ distinct L -colorings if L is a 5-list-assignment for G and at least $2^{n/10000}$ distinct L -colorings if L is a 3-list-assignment for G and G has girth at least five. Postle and Thomas proved that if G is a graph on n vertices embedded on a surface Σ of genus g , then there exist constants $\epsilon, c_g > 0$ such that if G has an L -coloring, then G has at least $c_g 2^{\epsilon n}$ distinct L -colorings if L is a 5-list-assignment for G or if L is a 3-list-assignment for G and G has girth at least five. More generally, they proved that there exist constants $\epsilon, \alpha > 0$ such that if G is a graph on n vertices embedded in a surface Σ of fixed genus g , H is a proper subgraph of G , and ϕ is an L -coloring of H that extends to an L -coloring of G , then ϕ extends to at least $2^{\epsilon(n-\alpha(g+|V(H)|))}$ distinct L -colorings of G if L is a 5-list-assignment or if L is a 3-list-assignment and G has girth at least five. We prove the same result if G is triangle-free and L is a 4-list-assignment of G , where $\epsilon = \frac{1}{8}$, and $\alpha = 130$.

1 Introduction

Let G be a graph with n vertices, and let $L = (L(v) : v \in V(G))$ be a collection of lists which we call *available colors*. If each set $L(v)$ is non-empty, then we say that L is a *list-assignment* for G . If k is an integer and $|L(v)| \geq k$ for every $v \in V(G)$, then we say that L is a *k-list-assignment* for G . An *L-coloring* of G is a mapping ϕ with domain $V(G)$ such that $\phi(v) \in L(v)$ for every $v \in V(G)$ and $\phi(v) \neq \phi(u)$ for every pair of adjacent vertices $u, v \in V(G)$. We say that a graph G is *k-choosable*, or *k-list-colorable*, if G has an L -coloring for every k -list-assignment L . If $L(v) = \{1, \dots, k\}$ for every $v \in V(G)$, then we call an L -coloring of G a *k-coloring*, and we say G is *k-colorable* if G has a k -coloring.

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If G has an L -coloring, it is natural to ask how many L -colorings G has. In particular, we are interested in when the number of L -colorings of G is exponential in the number of vertices. The Four Color Theorem states that every planar graph has a 4-coloring. A plane graph obtained from the triangle by recursively adding vertices of degree three inside facial triangles has only one 4-coloring up to permutation of the colors. So in general planar graphs do not have exponentially many 4-colorings. However, if ϕ is a k -coloring of G , then we may assume there is some $X \subseteq V(G)$ with $|X| \geq |V(G)|/k$ such that for all $v \in X$, $\phi(v) = 1$. It follows that G has at least $2^{|V(G)|/k} (k+1)$ -colorings, because for each subset of X , we can obtain a unique $(k+1)$ -coloring of G from ϕ by coloring it with the color $k+1$. Hence, planar graphs have exponentially many 5-colorings. In [2], Birkhoff and Lewis obtained an optimal bound on the number of 5-colorings of planar graphs, which is tight for the graph described above.

Theorem 1.1. [2] *Every planar graph on $n \geq 3$ vertices has at least $60 \cdot 2^{n-3}$ distinct 5-colorings*

In [8], Thomassen proved a similar result for graphs on surfaces.

Theorem 1.2. [8] *For every surface Σ there is some constant $c > 0$ such that every 5-colorable graph on n vertices embedded in Σ has at least $c \cdot 2^n$ distinct 5-colorings.*

In [8, Theorem 2.1], Thomassen gave a shorter proof using Euler's formula that for every fixed surface Σ , if a graph G embedded in Σ is 5-colorable, then it has exponentially many 5-colorings. The argument also applies to 4-colorings of triangle-free graphs and 3-colorings of graphs of girth at least five. We are interested in finding similar results for list-coloring.

In [6], Thomassen gave his classic proof that every planar graph is 5-choosable. Later, Thomassen proved that in fact every planar graph has exponentially many 5-list-colorings.

Theorem 1.3. [9] *If G is a planar graph on n vertices and L is a 5-list-assignment for G , then G has at least $2^{n/9}$ distinct L -colorings.*

In [7], Thomassen proved that every planar graph of girth at least five is 3-choosable. Later, he proved that in fact every planar graph of girth at least 5 has exponentially many 3-list-colorings.

Theorem 1.4. [10] *If G is a planar graph on n vertices of girth at least 5 and L is a 3-list-assignment for G , then G has at least $2^{n/10000}$ distinct L -colorings.*

An important proof technique is to extend a coloring of a subgraph to the entire graph. This can be viewed as list-coloring where the precolored vertices have lists of size one. The following theorem of Postle and Thomas [5, 4] utilizes this technique and extends Theorems 1.3 and 1.4 to graphs on surfaces.

Theorem 1.5. [5, 4] *There exist constants $\epsilon, \alpha > 0$ such that the following holds. Let G be a graph on n vertices embedded in a fixed surface Σ of genus g , and let H be a proper subgraph of G . If L is a 5-list-assignment for G , or L is a 3-list-assignment for G and G has girth at least five, and if ϕ is an L -coloring of H that extends to an L -coloring of G , then ϕ extends to at least $2^{\epsilon(n-\alpha(g+|V(H)|))}$ distinct L -colorings of G .*

A classical theorem of Grötzsch states that every triangle-free planar graph is 3-colorable. Hence, every triangle-free planar graph has exponentially many 4-colorings. Thomassen conjectured in [10] that in fact every triangle-free planar graph has exponentially many 3-colorings. The best progress towards this conjecture is the following result due to Asadi et al..

Theorem 1.6. [1] *Every triangle-free planar graph on n vertices has at least $2^{\sqrt{n/212}}$ distinct 3-colorings.*

Theorem 1.6 can not be extended to list-coloring, since there exist triangle-free planar graphs that are not 3-choosable. However, it is an easy consequence of Euler's formula that every triangle-free planar graph is 4-choosable. Thus, it is natural to ask if a result analagous to Theorem 1.5 holds for 4-list-coloring triangle-free graphs on surfaces. The following is our main theorem.

Theorem 1.7. *Let G be a triangle-free graph on n vertices embedded in a fixed surface Σ of genus g , and let L be a 4-list-assignment for G . If $H \subsetneq G$, and ϕ is an L -coloring of H that extends to G , then ϕ extends to $2^{(n-130(g+|V(H)|))/8}$ distinct L -colorings of G .*

In order to prove Theorem 1.7, we prove a stronger result for which we need the following definition.

Definition 1.8. Let $\epsilon, \alpha \geq 0$. Let G be a graph embedded in a surface Σ of Euler genus g , let H be a proper subgraph of G , and let L be a list-assignment for G . We say that (G, H) is (ϵ, α) -*exponentially-critical* with respect to L if for every proper subgraph G' of G such that $H \subseteq G'$, there exists an L -coloring ϕ of H such that there exists $2^{\epsilon(|V(G')|-\alpha(g+|V(H)|))}$ distinct L -colorings of G' extending ϕ , but there do not exist $2^{\epsilon(|V(G)|-\alpha(g+|V(H)|))}$ distinct L -colorings of G extending ϕ .

We prove the following theorem, which implies Theorem 1.7.

Theorem 1.9. *Suppose (G, H) is (ϵ, α) -exponentially-critical and G is triangle-free. For all $\alpha \geq 0$, if $0 \leq \epsilon \leq \frac{1}{8}$, then $|V(G)| \leq 50(|V(H)| - \frac{13}{5}) + 130g$.*

Proof of Theorem 1.7 assuming Theorem 1.9. Let (G, H) be a minimal counterexample. Then there exists an L -coloring ϕ of H that extends to G that does not extend to $2^{(V(G)|-130(g+|V(H)|))/8}$ distinct L -colorings of G . By the minimality of G , G is (ϵ, α) -exponentially-critical, where $\epsilon = \frac{1}{8}$ and $\alpha = 130$. Hence, by Theorem 1.9, $|V(G)| \leq 50(|V(H)| - \frac{13}{5}) + 130g$. Therefore ϕ does not extend to an L -coloring of G , a contradiction. \square

We prove Theorem 1.9 using the method of reducible configurations and discharging. In this paper, if G is a graph and $H \subsetneq G$, then a *reducible configuration* of (G, H) is a nonempty subgraph Q of $G - V(H)$ such that for every 4-list-assignment L of G , every L -coloring of $G - V(Q)$ extends to at least two distinct L -coloring of G . In Section 2, we prove that certain reducible configurations do not occur in (ϵ, α) -exponentially-critical graphs. In Section 3, we prove Theorem 1.9 using discharging.

Finally, we remark that a version of Theorem 1.9 can be proved if $\epsilon \leq \frac{1}{7}$, at the expense of a worse bound on $|V(G)|$ and a more complicated discharging argument.

2 Reducible Configurations

We first prove that small reducible configurations do not occur in (ϵ, α) -exponentially-critical graphs.

Proposition 2.1. *If (G, H) is (ϵ, α) -exponentially-critical with respect to some 4-list-assignment L , then (G, H) does not contain any reducible configurations of size at most $\frac{1}{\epsilon}$.*

Proof. Suppose that $Q \subseteq G - V(H)$ is a reducible configuration. We want to show $|V(Q)| > \frac{1}{\epsilon}$. Since (G, H) is (ϵ, α) -exponentially-critical, there exists an L -coloring ϕ of H such that there exists $2^{\epsilon(|V(G)| - |V(Q)| - \alpha(g + |V(H)|))}$ distinct L -colorings of $G - V(Q)$ extending ϕ , but there do not exist $2^{\epsilon(|V(G)| - \alpha(g + |V(H)|))}$ distinct L -colorings of G extending ϕ . Since Q is a reducible configuration, every L -coloring of $G - V(Q)$ extending ϕ has at least two extensions to an L -coloring of G . Hence, G has at least $2^{\epsilon(|V(G)| - |V(Q)| - \alpha(g + |V(H)|)) + 1} = 2^{\epsilon(|V(G)| - \alpha(g + |V(H)|)) + 1 - \epsilon|V(Q)|}$ distinct L -colorings extending ϕ . Therefore $|V(Q)| > \frac{1}{\epsilon}$, as desired. \square

We now present our first reducible configuration.

Lemma 2.2. *A 4-cycle $C \subseteq G - V(H)$ is a reducible configuration if for all $v \in V(C)$, v has degree at most four in G .*

Proof. Let L be some 4-list-assignment for G , and let ϕ be an L -coloring of $G - V(C)$. Note that there are two distinct list-colorings of a 4-cycle when every vertex has at least two available colors. Hence, there are at least two distinct L -colorings of G extending ϕ , as desired. \square

For our next reducible configuration, we need the following definitions.

Definition 2.3. If P is a path, and $v \in V(P)$ is not an end of P , then we say v is an *internal vertex* of P . If P' is also a path, we say P and P' are *internally disjoint* if they share no internal vertices.

Definition 2.4. We say a path $P \subseteq G$ is a *stamen* in (G, H) if there exists an end $u \in V(G) \setminus V(H)$ of P such that the degree of u is precisely three in G , and in addition, every internal vertex of P has degree four and is not in H . If $v \neq u$ is an end of P , then we say P is a *v-stamen*.

If $v \in V(G)$, let $d(v)$ denote the degree of v in G .

Definition 2.5. We say $G' \subseteq G - V(H)$ is a *poppy* of (G, H) if there is some $v \in V(G')$ such that G' is the union of v and at least $d(v) - 2$ internally disjoint v -stamens.



Figure 1: A v -stamen and a poppy

We next prove that a poppy is a reducible configuration, but first we need the following definition and a classical theorem of Erdős, Rubin, and Taylor [3].

Definition 2.6. We say G is *degree-choosable* if for every list-assignment L such that for all $v \in V(G)$, $|L(v)| \geq d(v)$, G has an L -coloring.

Theorem 2.7. [3] *A connected graph G is not degree-choosable if and only if every block of G is a clique or an odd cycle. Furthermore, if G does not have an L -coloring for some L with $|L(v)| \geq d(v)$, then for all $v \in V(G)$, $|L(v)| = d(v)$.*

Lemma 2.8. *If Q is a poppy of (G, H) , then Q is a reducible configuration.*

Proof. Let Q be a poppy of (G, H) . Let L be some 4-list-assignment of G , and let ϕ be an L -coloring of $G - V(Q)$. Say Q is the union of v and v -stamens P_1, \dots, P_k , where $k \geq d(v) - 2$. Let L' be a list-assignment for Q , where for every $u \in V(Q)$, $L'(u) = L(u) \setminus \{\phi(u') : uu' \in E(G), u' \in V(G) \setminus V(Q - v)\}$. Let $\phi_1 = \phi_2 = \phi$, and let $\phi_1(v) \neq \phi_2(v) \in L'(v)$.

Note that every connected component of $Q - v$ contains a vertex u of degree three in G , so $|L'(u)| = d_{Q-v}(u) + 1$. Therefore by Theorem 2.7, every connected component of $Q - v$ is L' -colorable. Hence, ϕ_1 and ϕ_2 extend to distinct L -colorings of G , so Q is a reducible configuration, as desired. \square

If $v \in V(G)$ has degree at most two, then v itself is a poppy. Hence, Lemma 2.8 implies the following.

Corollary 2.9. *If $v \in V(G)$ has degree at most two, then v is a reducible configuration.*

If $v \in V(G)$ has degree three, then a v -stamen in (G, H) is a poppy. Hence, Lemma 2.8 implies the following.

Corollary 2.10. *If $v \in V(G) \setminus V(H)$ has degree three, then a v -stamen is a reducible configuration of (G, H) .*

3 Discharging

Before proving Theorem 1.9, we need some definitions. In the following definitions, G is a graph and $H \subsetneq G$.

Definition 3.1. We say $v \in V(G)$ is a k -vertex if $d(v) = k$, a k^+ -vertex if $d(v) \geq k$, and a k^- -vertex if $d(v) \leq k$. If G is embedded in a surface, we define a k -face, a k^+ -face, and a k^- -face similarly.

Definition 3.2. We say $v \in V(G)$ is a *major vertex* of (G, H) if v is a 5^+ -vertex, or if $v \in V(H)$.

Definition 3.3. If every vertex of a stamen P of G is incident with a face f , then we say P is *incident with f* .

Definition 3.4. If G is 2-cell-embedded in some surface Σ and f is a face of G , then the boundary of f in Σ is the union of the vertices and edges of a closed walk in G , which we call the *boundary walk* of f .

If G is embedded in a surface, we let $F(G)$ denote the set of faces of G . If G is 2-cell-embedded and $f \in F(G)$, we let $|f|$ denote the length of the boundary walk of f . We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9. Suppose G is a triangle-free graph embedded in a surface Σ of Euler genus g , $H \subsetneq G$, and (G, H) is (ϵ, α) -exponentially-critical with respect to some 4-list-assignment L , where $0 \leq \epsilon \leq \frac{1}{8}$. Let G_1, \dots, G_m be the components of G , and let $H_i = G_i \cap H$. To prove Theorem 1.9, it suffices to show that for all $i = 1, \dots, m$, $|V(G_i)| \leq 50(|V(H_i)| - \frac{13}{5}) + 130g_i$ when $V(H_i) \subsetneq V(G_i)$ and g_i is the genus of G_i .

By Proposition 2.1, (G, H) has no reducible configurations of size at most $\frac{1}{\epsilon}$. Note that a reducible configuration of (G_i, H_i) is a reducible configuration of (G, H) . Thus, for all $i = 1, \dots, m$, (G_i, H_i) has no reducible configurations of size at most $\frac{1}{\epsilon}$. Hence, it suffices to show $|V(G)| \leq 50(|V(H)| - \frac{13}{5}) + 130g$, where G is a connected triangle-free graph embedded in a surface Σ of Euler genus g , $H \subsetneq G$, and (G, H) contains no reducible configurations of size at most $\frac{1}{\epsilon}$. We may assume G is 2-cell-embedded in Σ , or else we embed G in a surface of smaller genus.

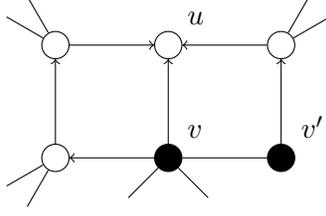


Figure 2: An Example of Rule 1

For $v \in V(G) \setminus V(H)$, let $ch(v) = d(v) - 4$, and for $v \in V(H)$, let $ch(v) = d(v) + 3\gamma - 1$ for some fixed constant $\gamma > 0$ to be determined later. For every $f \in F(G)$, let $ch(f) = |f| - 4$. By Euler's formula,

$$\sum_{v \in V(G)} ch(v) + \sum_{f \in F(G)} ch(f) = (3 + 3\gamma)|V(H)| + 4(2g - 2).$$

Redistribute the charges according to the following rules, and let ch_* denote the final charge.

1. Let v be a major vertex, and let $u \in V(G) \setminus V(H)$ be a 3-vertex at distance at most two from v . For every v -stamen P in G with an end at u such that there exists a 4-face f with P incident with f , let v send charge $\frac{1}{3} + \gamma$ to u .
2. Let v be a major vertex, and let $u \in V(G) \setminus V(H)$ be a 4-vertex at distance at most two from v . For each 4-face incident to both u and v , let v send charge $\frac{3\gamma}{4}$ to u .
3. If f is a 5^+ -face incident to a 3-vertex $u \in V(G) \setminus V(H)$, let f send charge $\frac{1}{3} + \gamma$ to u for every instance of u in the boundary walk of f .
4. If f is a 5^+ -face incident to a 4-vertex $u \in V(G) \setminus V(H)$, let f send charge $\frac{3\gamma}{4}$ to u for every instance of u in the boundary walk of f .

Figure 2 illustrates an instance of Rule 1. Major vertices are represented as black circles, and non-major vertices are represented as white circles. There are two v -stamens and one v' -stamen with ends at u (shown as directed paths), and each is incident with a 4-face. Hence, v sends charge at least $\frac{2}{3} + 2\gamma$ to u and v' sends charge at least $\frac{1}{3} + \gamma$ to u under Rule 1.

Claim 3.5. *If $u \in V(G) \setminus V(H)$ has degree at most four, $ch_*(u) \geq 3\gamma$.*

Proof. First suppose u is a 4-vertex. Note that u sends no charge under Rules 1-4. By Lemma 2.2, every 4-face f incident to u contains a major vertex v_f . Therefore, if u is adjacent to k 4-faces, u receives at least $\frac{3k\gamma}{4}$ charge under Rule 2. By Rule 4, u receives $\frac{3(4-k)\gamma}{4}$ charge from 5^+ -faces. Hence, u receives at least 3γ charge, as desired.

Therefore we may assume u is a 3-vertex. Note that u sends no charge under Rules 1-4. By Lemma 2.2, every 4-face f incident to u contains a major vertex. Hence, for every 4-face f incident to u , there are two internally disjoint stamens P_1 and P_2 with an end at u and an end at a major vertex such that every vertex in P_1 and P_2 is incident to f . Note that a stamen is incident with at most two 4-faces.

Therefore, if u is adjacent to k 4-faces, u receives at least $\frac{k(1+3\gamma)}{3}$ charge under Rule 1. By Rule 3, u receives $\frac{(3-k)(1+3\gamma)}{3}$ charge from 5^+ -faces. Hence, u receives at least $1+3\gamma$ charge, as desired. \square

Claim 3.6. *If $v \in V(G) \setminus V(H)$ has degree at least seven and $\gamma \leq \frac{2}{13}$, $ch_*(v) \geq \frac{2}{3} - \frac{91\gamma}{4}$.*

Proof. Let P_1 and P_2 be distinct v -stamens that are each incident with a 4-face. Suppose $vv' \in E(P_1) \cap E(P_2)$. Then $E(P_1) \cap E(P_2) = \{vv'\}$, and $P_1 \triangle P_2$ is a u -stamen of length at most five, where u is an end of P_1 , contradicting Corollary 2.10. Hence, P_1 and P_2 are internally disjoint. Therefore v sends charge at most $d(v)(\frac{1}{3} + \gamma)$ to 3-vertices under Rule 1. Note that v sends at most $d(v)\frac{9\gamma}{4}$ charge to 4-vertices under Rule 2. Therefore v sends charge at most $d(v)\left(\frac{1}{3} + \gamma + \frac{9\gamma}{4}\right)$. Since $\gamma \leq \frac{2}{13}$,

$$ch_*(v) \geq d(v) - 4 - d(v)\left(\frac{1}{3} + \gamma + \frac{9\gamma}{4}\right) = d(v)\left(\frac{2}{3} - \frac{13\gamma}{4}\right) - 4 \geq \frac{2}{3} - \frac{91\gamma}{4},$$

as desired. \square

Claim 3.7. *If $v \in V(G) \setminus V(H)$ has degree six, then $ch_*(v) \geq \frac{2}{3} - \frac{35\gamma}{2}$.*

Proof. Suppose v sends charge at most $\frac{4}{3} + 4\gamma$ to 3-vertices under Rule 1. Note that v sends at most $d(v)\frac{9\gamma}{4} = \frac{27\gamma}{2}$ charge to 4-vertices under Rule 2. Hence,

$$ch_*(v) \geq 2 - \left(\frac{4}{3} + 4\gamma\right) - \frac{54\gamma}{4} = \frac{2}{3} - \frac{35\gamma}{2},$$

as desired.

Therefore we may assume that v sends greater than $\frac{4}{3} + 4\gamma$ charge to 3-vertices. Then by Rule 1, there exist at least five v -stamens of G P_1, \dots, P_5 , where $u_i \neq v$ is an end of P_i , and each P_i is incident with a 4-face, f_i . Since $\epsilon \leq \frac{1}{5}$, by Corollary 2.10, the P_i are pairwise internally disjoint. Let $Q = \cup_{i=1}^5 P_i$. We choose P_1, \dots, P_5 such that $(|V(P_1)|, \dots, |V(P_5)|)$ is lexicographically minimum over all v -stamens of G , and subject to that, $|V(Q)|$ is minimum. Note that Q is a poppy of G . Since $\epsilon \leq \frac{1}{8}$, by Lemma 2.8, $|V(Q)| > 8$. Note that for all $i = 1, \dots, 5$, $2 \leq |V(P_i)| \leq 4$. Furthermore, if $|V(P_i)| = 4$, then v is adjacent to u_i , so there exists $j < i$ such that $u_j = u_i$ and $|V(P_j)| = 2$.

First we claim that $|V(P_2)| > 2$. Suppose not. Then $|V(P_1)| = |V(P_2)| = 2$. If $|V(P_3)| = 3$, then since $v \in V(P_i)$ for all i , $|V(Q)| \leq 8$, a contradiction. Therefore for

$i = 3, 4, 5$, $|V(P_i)| = 4$. Since $|V(Q)|$ is minimum, u_3 is either u_1 or u_2 . Hence, $|V(Q)| \leq 8$, a contradiction. Therefore $|V(P_2)| > 2$, as claimed.

We claim that $|V(P_1)| > 2$. Suppose not. Since $v \in V(P_i)$ for all i and $|V(Q)| > 8$, $|V(P_4)| = 4$. Since $|V(Q)|$ is minimum, $u_4 = u_1$. Since $|V(Q)| \leq 8$, $|V(P_3)| = 4$. Since $|V(P_2)| > 2$, $u_3 = u_1$. Since $|V(Q)| \leq 8$, $|V(P_2)| = 4$. Hence, $u_2 = u_1$, contradicting that u_1 has degree three. Therefore $|V(P_1)| > 2$, as claimed.

Thus $|V(P_i)| > 2$ for all $i = 1, \dots, 5$. But then $|V(P_i)| \neq 4$ for all i . Hence, $|V(P_i)| = 3$ for all $i = 1, \dots, 5$. Since $|V(Q)| > 8$ and $|V(Q)|$ is minimum, u_1, \dots, u_5 are distinct. For each $i = 1, \dots, 5$, let $w_i \in V(P_i) \setminus \{v, u_i\}$. If there exists i, j such that $i \neq j$ and w_i is adjacent to u_j , then $u_i w u_j$ is a u_i -stamen, contradicting Corollary 2.10. Therefore w_1, \dots, w_5 are distinct, and since the u_1, \dots, u_5 are distinct, f_1, \dots, f_5 are distinct. But each w_i is incident with at least two 4-faces that are incident to v . Since v is incident with at most six 4-faces, there exists some face f incident to v such that for all $i = 1, \dots, 5$, $f \neq f_i$ and w_i is incident with f . Therefore for some $i \neq j$, $w_i = w_j$, a contradiction. This completes the proof. \square

Claim 3.8. *If $v \in V(G) \setminus V(H)$ has degree five, then $ch_*(v) \geq \frac{1}{3} - \frac{53\gamma}{4}$.*

Proof. Suppose v sends charge at most $\frac{2}{3} + 2\gamma$ to 3-vertices under Rule 1. Note that v sends at most $d(v)\frac{9\gamma}{4} = \frac{45\gamma}{4}$ charge to 4-vertices under Rule 2. Hence,

$$ch_*(v) \geq 1 - \left(\frac{2}{3} + 2\gamma\right) - \frac{45\gamma}{4} = \frac{1}{3} - \frac{53\gamma}{4},$$

as desired.

Therefore we may assume that v sends greater than $\frac{2}{3} + 2\gamma$ charge to 3-vertices. Then by Rule 1, there exist v -stamens P_1, P_2 , and P_3 , where $u_i \neq v$ is an end of P_i , and each P_i is incident with a 4-face, f_i . Since $\epsilon \leq \frac{1}{5}$, by Corollary 2.10, the P_i are pairwise internally disjoint.

We choose P_1, P_2 , and P_3 such that $(|V(P_1)|, |V(P_2)|, |V(P_3)|)$ is lexicographically minimum over all v -stamens of G . Let $Q = \cup_{i=1}^3 P_i$. Note that Q is a poppy of G . Since $\epsilon \leq \frac{1}{8}$, by Lemma 2.8, $|V(Q)| > 8$. Note that for all $i = 1, 2, 3$, $2 \leq |V(P_i)| \leq 4$. Furthermore, if $|V(P_i)| = 4$, then v is adjacent to u_i , so there exists $j < i$ such that $u_j = u_i$ and $|V(P_i)| = 2$. Since $v \in V(P_i)$ for all i and $|V(Q)| > 8$, $|V(P_1)| + |V(P_2)| + |V(P_3)| > 10$. Since $|V(P_2)|, |V(P_3)| \leq 4$, $|V(P_1)| > 2$. Hence, $|V(P_i)| = 3$ for all $i = 1, 2, 3$. Then $|V(Q)| \leq 7$, a contradiction. This completes the proof. \square

Claim 3.9. *If $v \in V(H)$ and $\gamma \leq \frac{2}{13}$, then $ch_*(v) \geq \min\{3\gamma, \frac{1}{3} - \frac{7\gamma}{2}\}$.*

Proof. If v is a 1-vertex, then since G is simple, v is not incident to a 4-face unless G is the path of length three. Since H is a proper subgraph of G , there is a vertex of degree at most two in $V(G) \setminus V(H)$, contradicting Corollary 2.9. Therefore G is not the path of

length three, so v is not incident to a 4-face. Hence, v sends no charge under Rules 1-4, so $ch_*(v) \geq 3\gamma$, as desired.

Therefore we may assume $d(v) \geq 2$. Since $\epsilon \leq \frac{1}{5}$, by Corollary 2.10, if P_1 and P_2 are distinct v -stamens that are each incident with a 4-face, then P_1 and P_2 are internally disjoint. Therefore v sends charge at most $d(v)(\frac{1}{3} + \gamma)$ to 3-vertices under Rule 1. Note also that v sends charge at most $d(v)\frac{9\gamma}{4}$ to 4-vertices under Rule 2. Therefore,

$$ch_*(v) \geq d(v) + 3\gamma - 1 - d(v) \left(\frac{1}{3} + \gamma + \frac{9\gamma}{4} \right) = d(v) \left(\frac{2}{3} - \frac{13\gamma}{4} \right) + 3\gamma - 1 \geq \frac{1}{3} - \frac{7\gamma}{2},$$

as desired. \square

Claim 3.10. *If $f \in F(G)$ and $\gamma \leq \frac{1}{15}$, then $ch_*(f) \geq 0$.*

Proof. Let $f \in F(G)$. If $|f| = 4$, then f sends no charge under Rules 1-4. Therefore $ch_*(f) \geq 0$, as desired.

Suppose $|f| \geq 8$. Under Rule 3, f sends charge at most $|f|(\frac{1}{3} + \gamma)$ to 3-vertices. Under Rule 4, f sends charge at most $|f|\frac{3\gamma}{4}$ to 4-vertices. Since $\gamma \leq \frac{1}{15}$, f sends charge at most

$$|f| \left(\frac{1}{3} + \gamma + \frac{3\gamma}{4} \right) \leq \frac{27|f|}{60} < \frac{1}{2}|f|.$$

Hence, $ch_*(f) \geq |f| - 4 - \frac{|f|}{2} = \frac{|f|}{2} - 4 \geq 0$, as desired.

Suppose $5 < |f| < 8$. By Corollary 2.10, since $\epsilon \leq \frac{1}{2}$, G does not contain adjacent 3-vertices. Therefore f is incident to at most $\lfloor \frac{|f|}{2} \rfloor$ 3-vertices. Since G is triangle-free and $|f| < 8$, each 3-vertex appears at most once in the boundary walk of f . Hence, f sends charge at most $\frac{|f|}{2}(\frac{1}{3} + \gamma)$ to 3-vertices under Rule 3. Under Rule 4, f sends charge at most $|f|\frac{3\gamma}{4}$ to 4-vertices. Therefore f sends charge at most

$$\frac{|f|}{2} \left(\frac{1}{3} + \gamma \right) + |f|\frac{3\gamma}{4} = |f| \left(\frac{1}{6} + \frac{\gamma}{2} + \frac{3\gamma}{4} \right) = |f| \left(\frac{2 + 15\gamma}{12} \right).$$

Since $\gamma \leq \frac{1}{15}$, f sends at most $\frac{|f|}{4}$ charge. Hence, $ch_*(f) \geq |f| - 4 - \frac{|f|}{4} = \frac{3|f|}{4} - 4 \geq 0$, as desired.

Suppose $|f| = 5$. Since G is triangle-free, each vertex appears at most once in the boundary walk of f . If f is not incident to any 3-vertices, then f sends charge at most $5(\frac{3\gamma}{4}) \leq \frac{1}{4}$ under Rules 3 and 4, so $ch_*(f) \geq 0$, as desired. If f is incident to precisely one 3-vertex, then f sends charge at most $\frac{1}{3} + \gamma + 4(\frac{3\gamma}{4}) = \frac{1}{3} + 4\gamma \leq \frac{3}{5}$ under Rules 3 and 4, as desired. If f is incident to precisely two 3-vertices, then f sends charge at most $\frac{2}{3} + 2\gamma + 3(\frac{3\gamma}{4}) = \frac{2}{3} + \frac{17\gamma}{4} \leq \frac{57}{60}$ under Rules 3 and 4, as desired. Since $\epsilon \leq \frac{1}{2}$, G does not contain adjacent 3-vertices by Corollary 2.10. Hence, f is incident to at most two 3-vertices, so the proof is complete. \square

By Claims 3.5, 3.6, 3.7, 3.8, and 3.9, if $\gamma \leq \frac{1}{15}$, then for all $v \in V(G)$, $ch_*(v) \geq \min\{3\gamma, \frac{2}{3} - \frac{91\gamma}{4}, \frac{1}{3} - \frac{53\gamma}{4}\}$. So if $\gamma = \frac{4}{195}$, then $ch_*(v) \geq \frac{4}{65}$ for all $v \in V(G)$, and by Claim 3.10, for all $f \in F(G)$, $ch_*(f) \geq 0$. Therefore

$$\frac{4}{65}|V(G)| \leq \sum_{v \in V(G)} ch_*(v) + \sum_{f \in F(G)} ch_*(f) = \left(\frac{199}{65}\right)|V(H)| + 4(2g - 2).$$

Hence,

$$|V(G)| \leq \frac{199}{4}|V(H)| + 65(2g - 2) \leq 50 \left(|V(H)| - \frac{13}{5}\right) + 130g,$$

as desired. □

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