

# $\chi$ -bounds, operations and chords

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## Abstract

A *long unichord* in a graph is an edge that is the unique chord of some cycle of length at least 5. A graph is *long-unichord-free* if it does not contain any long-unichord. We prove a structure theorem for long-unichord-free graph. We give an  $O(n^4m)$ -time algorithm to recognize them. We show that any long-unichord-free graph  $G$  can be colored with at most  $O(\omega^3)$  colors, where  $\omega$  is the maximum number of pairwise adjacent vertices in  $G$ .

**Key Words:** amalgam,  $\chi$ -bounded, chords.

**AMS classification:** 05C75, 05C15, 05C85.

## 1 Introduction

In this article, all graphs are finite and simple. We denote by  $\chi(G)$  the *chromatic number* of a graph  $G$ , that is the minimum number of colors needed to give a color to each vertex in such a way that any two adjacent vertices receive different colors. We denote by  $\omega(G)$  the maximum size of a set of pairwise adjacent vertices (that we call a *clique*). It is clear that for every graph,  $\chi(G) \geq \omega(G)$ , while the converse inequality is false in general (the smallest example is the chordless cycle on five vertices).

Let  $f$  be any function from  $\mathbb{R}$  to  $\mathbb{R}$ . A graph  $G$  is  *$\chi$ -bounded by  $f$*  if every induced subgraph  $H$  of  $G$  satisfies  $\chi(H) \leq f(\omega(H))$ . This notion first

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appeared in an article of Gyárfás [17]. A class of graphs is  $\chi$ -*bounded* if for some function  $f$ , every graph of the class is  $\chi$ -bounded by  $f$ . It is well known that the class of all graphs is not  $\chi$ -bounded, this follows from the existence of graphs with  $\omega = 2$  and arbitrarily large chromatic number, see for instance [26].

Graphs that are  $\chi$ -bounded by the identity function are known as *perfect graphs*. They were the object of much research (see [24] for a survey), and the notion of  $\chi$ -boundedness was invented to try to have some insight on them. In his seminal paper, Gyárfás [17] made many conjectures, and some of them were claiming that excluding chordless cycles with various constraints on their length should lead to  $\chi$ -bounded classes. Recently, much progress has been reported toward these conjectures, see for instance [9].

In this paper, we focus on excluding cycles with constraints on their chords. A *unichord* in a graph is an edge that is the unique chord of some cycle (note that the cycle has length at least 4 because of the chord). A graph is *unichord-free* if it does not contain any unichord. A *long unichord* in a graph is an edge that is the unique chord of some cycle of length at least 5. The *house* is the graph on five vertices  $a, b, c, d, e$  with the following edges:  $ab, bc, cd, da, ea, eb$  (so the house is the smallest graph that contains a long unichord). A *house\** is any graph obtained from the house by repeatedly subdividing edges. A graph is *house\*-free* if it does not contain any house\* as an induced subgraph.

It is straightforward to check that long-unichord-free graphs form a generalisation of unichord-free graphs and of house\*-free graphs. They also form a generalisation of the classical class of chordal graphs (a graph is *chordal* if it contains no chordless cycle of length at least 4). A classical result states that chordal graphs are perfect (equivalently, they are  $\chi$ -bounded by the identity function). In [25], it is proved that unichord-free graphs are  $\chi$ -bounded by the function  $f(x) = \max(3, x)$ , and in [19], it is proved that house\*-free graphs are  $\chi$ -bounded by some exponential function. We generalise these theorems and we provide a better bound for the last one by showing that long-unichord-free graphs are  $\chi$ -bounded by a polynomial function of degree 3, namely  $f_3$ , to be defined later:

**Theorem 6.4** *Long-unichord-free graphs are  $\chi$ -bounded by  $f_3$  (in particular, by a polynomial of degree 3).*

Our proof relies on a decomposition theorem that is easily obtained from [25] and [10] (again, we postpone the definitions).

**Theorem 5.4** *Let  $G$  be a connected long-unichord-free graph. Then either:*

- $G$  is an induced subgraph of the Petersen graph;
- $G$  is an induced subgraph of the Heawood graph;
- $G$  is chordal;
- $G$  is bipartite and one side of the bipartition is made of vertices of degree at most 2;
- $G$  has a universal vertex;
- $G$  has a cutvertex;
- $G$  has an amalgam;
- $G$  has proper 2-cutset.

It must be stressed that applying decomposition theorems to prove  $\chi$ -boundedness is not at all straightforward. There are several papers dealing with the following question: when a prescribed operation is applied repeatedly to some graphs from a class  $\chi$ -bounded by  $f$ , is the larger class that is obtained  $\chi$ -bounded by a possibly different function  $g$ ? This has been answered positively for several operations, most notably for one that we use in our decomposition theorem, the so called amalgam operation (to be defined in the next section), see [19]. But the theorem from [19] is not enough for our purpose because it leads to an exponential  $\chi$ -bounding function. Here, to obtain  $\chi$ -boundedness we prove a stronger property for the sake of induction, roughly we find graphs with a special structure that intersect all inclusion-wise maximal cliques of the graph to be colored, and we apply this procedure inductively on what remains (where the maximum clique is smaller).

Our decomposition theorem turns out to be a complete structural description of long-unichord-free graphs: it tells how all long-unichord-free graphs can be constructed from simple pieces. As a byproduct of this description we obtain the following:

**Theorem 7.6** *Deciding whether an input graph  $G$  has a long-unichord can be performed in time  $O(n^4m^2)$  (where  $n = |V(G)|$  and  $m = |E(G)|$ ).*

This answers an open question mentioned in [25] (where a similar algorithm is given for unichord-free graphs). It should be pointed out that in [25], a problem of the very same flavour is proved to be NP-complete: deciding whether a graph contains a cycle  $C$  with a unique chord  $uv$  such

that  $u$  and  $v$  are at distance at least 4 along the cycle. This shows that naive attempts to obtain our recognition algorithm are likely to fail.

## Outline of the paper

In Section 2, we define all the decompositions and operations on graphs that we need, and we survey several results about how they preserve perfection and  $\chi$ -boundedness.

In Section 3, we study a technique to prove that an operation on graphs (namely, the substitution operation, everything is defined in the next section) preserves  $\chi$ -boundedness. This technique is analogous to the one used in the proof of the replication lemma of Lovász (see [24]). It consists in identifying a particular subgraph that intersects all maximal cliques of a graph, and in showing that the existence of such a subgraph is preserved by the operation. Our technique yields short proofs of known results and may provide good bounds in some situations. For instance we prove that the closure of 3-colourable graphs under substitution is  $\chi$ -bounded by a quadratic function, a seemingly new result. Note that the results from Section 3 are not used in the rest of the article. They illustrate our method and are of independent interest.

In Section 4, we apply a similar technique to a larger set of operations (namely, we consider 1-joins, amalgams and proper 2-cutsets). The price to pay for that is that the classes of graphs where the technique can be applied are even more restricted. But fortunately, it does not vanish to nothing as shown afterward.

In Section 5, we prove the structure theorem for long-unichord-free graphs.

In Section 6, we apply the results of the previous sections to prove that long-unichord-free graphs are  $\chi$ -bounded.

In Section 7 we provide a polynomial time algorithm to recognize long-unichord-free graphs, based on the decomposition theorem.

Section 8 is devoted to open questions.

## 2 Operations and properties preserved by them

We now define several classical decompositions for graphs, that are all partitions of the vertex-set with some structural constraints. For each of them, we explain how it enables us to obtain smaller graphs called *blocks of decomposition*, and how the decomposition can be reversed into an operation that allows building a graph from smaller pieces.

A vertex  $x$  in a graph  $G$  is *complete* to  $A \subseteq V(G) \setminus \{x\}$  if for all  $y \in A$ ,  $xy \in E(G)$ . We also say that  $x$  is *A-complete*. A set  $A \subseteq V(G)$  is *complete* to a set  $B \subseteq V(G)$  disjoint from  $A$  if every vertex of  $A$  is *B-complete*.

A vertex  $x$  in a graph  $G$  is *anticomplete* to  $A \subseteq V(G) \setminus \{x\}$  if for all  $y \in A$ ,  $xy \notin E(G)$ . We also say that  $x$  is *A-anticomplete*. A set  $A \subseteq V(G)$  is *anticomplete* to a set  $B \subseteq V(G)$  disjoint from  $A$  if every vertex of  $A$  is *B-anticomplete*.

### Gluing along a clique

A (possibly empty) clique  $K$  of a graph  $G$  is a clique cutset of  $G$  if there exists a partition  $(X_1, K, X_2)$  of  $V(G)$  such that  $X_1, X_2 \neq \emptyset$  and there are no edges of  $G$  between  $X_1$  and  $X_2$ . We then say that  $(X_1, K, X_2)$  is a *split* for this clique cutset, and that  $G_1 = G[X_1 \cup K]$  and  $G_2 = G[K \cup X_2]$  are the *blocks of decomposition* of  $G$  with respect to this split.

Note that  $G = G_1 \cup G_2$ , and we say that  $G$  is obtained from  $G_1$  and  $G_2$  by *gluing along a clique*. This operation can be performed for any pair of graphs  $G_1, G_2$  such that  $G_1 \cap G_2$  is a clique. If  $K = \emptyset$ , this operation can also be referred to as *disjoint union*. When  $|K| = 1$ , the operation can be referred to as *gluing along a vertex*, and the unique vertex in  $K$  is called a *cutvertex*.

### Substitutions

A set  $X$  of vertices of a graph  $G$  is a *homogeneous set* if  $|X| \geq 2$ ,  $X \subsetneq V(G)$ , and every vertex of  $V(G) \setminus X$  is either complete or anticomplete to  $X$ . We then denote by  $G/X$  the graph obtained from  $G$  by deleting  $X$  and adding a vertex  $v$  adjacent to all  $X$ -complete vertices of  $G$ . The graphs  $G[X]$  and  $G/X$  are the *blocks of decomposition of  $G$*  with respect to the homogeneous set  $X$ .

When  $G$  is a graph on at least two vertices,  $v$  is a vertex of  $G$  and  $H$  is a graph on at least two vertices vertex-disjoint from  $G$ , then the graph  $G'$  obtained from  $G$  by deleting  $v$ , adding  $H$  and all possible edges between vertices of  $H$  and the neighbors of  $v$  in  $G$  is called the graph obtained from  $G$  by *substituting  $H$  for  $v$* . We also say that  $G'$  is obtained from  $G$  and  $H$  by a *substitution*. Observe that  $V(H)$  is a homogeneous set of  $G'$ .

### 1-join composition

A *1-join* of a graph  $G$  is a partition of  $V(G)$  into sets  $X_1$  and  $X_2$  such that there exist sets  $A_1, A_2$  satisfying:

- $\emptyset \neq A_1 \subseteq X_1, \emptyset \neq A_2 \subseteq X_2$ ;
- $|X_1| \geq 2$  and  $|X_2| \geq 2$ ;
- there are all possible edges between  $A_1$  and  $A_2$ ;
- there are no other edges between  $X_1$  and  $X_2$ .

We say that  $(X_1, X_2, A_1, A_2)$  is a *split* of this 1-join. For  $i = 1, 2$ , the *block of decomposition*  $G_i$  with respect to this split is the graph obtained from  $G[X_i]$  by adding a vertex  $u_{3-i}$  complete to  $A_i$ .

The operation that is the reverse of the 1-join decomposition is defined as follows. Start with two vertex-disjoint graphs  $G_1$  and  $G_2$  on at least 3 vertices. For some vertex  $u_2$  (resp.  $u_1$ ) of  $G_1$  (resp.  $G_2$ ) such that  $N_{G_1}(u_2)$  (resp.  $N_{G_2}(u_1)$ ) is non-empty,  $G$  is obtained from the disjoint union of  $G_1 \setminus \{u_2\}$  and  $G_2 \setminus \{u_1\}$  by adding all possible edges between  $N_{G_1}(u_2)$  and  $N_{G_2}(u_1)$ . We say that  $G$  is obtained from  $G_1$  and  $G_2$  by a *1-join composition*.

### Amalgam composition

An *amalgam* of a graph  $G$  is a partition  $(K, X_1, X_2)$  of  $V(G)$  such that  $K$  is a (possibly empty) clique,  $(X_1, X_2)$  is a 1-join of  $G \setminus K$  with a split  $(X_1, X_2, A_1, A_2)$  and  $K$  is complete to  $A_1 \cup A_2$  (possibly, vertices of  $K$  have neighbors in  $V(G) \setminus (A_1 \cup A_2)$ ).

We say that  $(X_1, X_2, A_1, A_2, K)$  is a *split* of the amalgam defined above. For  $i = 1, 2$ , the *block of decomposition*  $G_i$  with respect to this split is the graph obtained from  $G[X_i \cup K]$  by adding a vertex  $u_{3-i}$  complete to  $A_i \cup K$ .

The operation that is the reverse of the amalgam decomposition is defined as follows. Start with two graphs  $G_1$  and  $G_2$  whose intersection forms a clique  $K$  with  $|K| \leq |V(G_1)| - 3, |V(G_2)| - 3$  and such that for  $i = 1, 2$  there is a vertex  $u_{3-i}$  in  $V(G_i) \setminus K$  whose neighborhood is  $K \cup A_i$  where  $A_i$  is non-empty, disjoint from  $K$  and  $K$ -complete. Let  $G$  be obtained from the union of  $G_1 \setminus \{u_2\}$  and  $G_2 \setminus \{u_1\}$  by adding all edges between  $A_1$  and  $A_2$ . We say that  $G$  is obtained from  $G_1$  and  $G_2$  by an *amalgam composition*.

The amalgam is obviously a generalisation of the 1-join. If  $A_1$  or  $A_2$  were allowed to be empty, it could be also be seen as a generalisation of the clique cutset, but it is not (we keep this distinction that might seem artificial, for historical reasons and compatibility of definitions with previous papers).

If  $X_1 = A_1$  then  $X_1$  is a homogeneous set of  $G$ . Yet, formally the amalgam is not a generalisation of the homogeneous set, because a homogeneous set  $X$  such that  $|X| = |V(G)| - 1$  (which is allowed) does not imply the

presence of an amalgam. Setting  $K = \emptyset$ ,  $X_1 = X$ , and  $X_2 = V(G) \setminus X$  does not work because then  $|X_2| = 1$ . However, it works for all homogeneous sets  $X$  such that  $|X| \leq |V(G)| - 2$ . This remark leads us to consider the following trivial decomposition and lemma.

### Adding a universal vertex

A *universal vertex* in a graph  $G$  is a vertex  $v$  complete to  $V(G) \setminus \{v\}$ . Note that  $V(G) \setminus \{v\}$  is then a homogeneous set of  $G$  (that does not yield a 1-join or an amalgam as noted above). From the discussion above, the following is trivial.

**Lemma 2.1** *If  $G$  has a homogenous set, then either  $G$  has a 1-join (and therefore an amalgam) or  $G$  has a universal vertex.*

The operation that is the reverse of “having a universal vertex” is simply *adding a universal vertex*, which means adding a vertex  $v$  to a graph  $G$ , and all possible edges between  $v$  and  $V(G)$ .

### Proper 2-cutset composition

A *proper 2-cutset* of a connected graph  $G$  is a pair of non-adjacent vertices  $a, b$ , such that  $V(G)$  can be partitioned into non-empty sets  $X_1$ ,  $X_2$  and  $\{a, b\}$  so that:  $|X_1| \geq 2$ ,  $|X_2| \geq 2$ ; there are no edges between  $X_1$  and  $X_2$ ; and both  $G[X_1 \cup \{a, b\}]$  and  $G[X_2 \cup \{a, b\}]$  contain a path from  $a$  to  $b$ . We say that  $(X_1, X_2, a, b)$  is a *split* of this proper 2-cutset.

For  $i = 1, 2$ , the *block of decomposition*  $G_i$  with respect to this split is the graph obtained from  $G[X_i \cup \{a, b\}]$  by adding a vertex  $x_{3-i}$  complete to  $\{a, b\}$ .

The operation that is the reverse of the proper 2-cutset is defined as follows. Start with two graphs  $G_1$  and  $G_2$  whose intersection is a pair of vertices  $a, b$  non-adjacent in both  $G_1$  and  $G_2$ , and such that  $a, b$  have a common neighbor  $x_2$  in  $G_1$ , and a common neighbor  $x_1$  in  $G_2$ . Suppose furthermore that for  $i = 1, 2$ , there exists a path from  $a$  to  $b$  in  $G_i \setminus x_{3-i}$ . Let  $G$  be the union of  $G_1 \setminus \{x_2\}$  and  $G_2 \setminus \{x_1\}$ . We say that  $G$  is obtained from  $G_1$  and  $G_2$  by a *proper 2-cutset composition*.

### Heredity of decompositions

The next two lemmas are very easy to prove and we give them without proofs.

**Lemma 2.2** *Suppose that  $G$  is obtained from  $G_1$  by substituting  $G_2$  for  $v$ . If  $G'$  is an induced subgraph of  $G$ , then either  $G'$  is isomorphic to an induced subgraph of  $G_1$ , or  $G'$  is an induced subgraph of  $G_2$ , or  $G'$  is obtained from an induced subgraph of  $G_1$  by substituting an induced subgraph of  $G_2$  for  $v$ .*

**Lemma 2.3** *Suppose that  $G$  is obtained from  $G_1$  and  $G_2$  by one of the operations from  $S = \{\text{gluing along a clique, substitution, 1-join composition, amalgam composition, gluing along a proper 2-cutset}\}$ .*

*If  $G'$  is an induced subgraph of  $G$ , then either  $G'$  is isomorphic to an induced subgraph of  $G_1$ , or  $G'$  is isomorphic to an induced subgraph of  $G_2$ , or  $G'$  is obtained from an induced subgraph of  $G_1$  and an induced subgraph of  $G_2$  by an operation from  $S$ .*

### Properties preserved by the operations

Theorem 2.4 below was proved by Gallai [16] (gluing along a clique), Lovász [18] (substitutions), Cunningham [13] (1-join), Burlet and Fonlupt [5] (amalgams), Cornuéjols and Cunningham [12] (proper 2-cutset).

Note that in [12], an operation more general than gluing along a proper 2-cutset is considered (the so-called *2-join*, not worth defining here). Note also that with our definitions, it could be that for a graph  $G$  with a proper 2-cutset, the blocks of decompositions are not perfect. This happens for instance with the chordless cycle  $v_1 \dots v_6 v_1$  and the proper 2-cutset  $v_1, v_4$ . The blocks of decomposition are then both isomorphic to a cycle of length 5, a notoriously non-perfect graph. But the converse works smoothly: if two graphs are perfect, then a perfect graph is obtained by gluing them along a proper 2-cutset. A proof of this is implicit in [12]. Another simple way to check this is to note that when a vertex  $v$  has degree 2 and non-adjacent neighbors  $a, b$  in a perfect graph  $G$ , then all paths from  $a$  to  $b$  in  $G$  have even length (otherwise,  $G$  contains an odd chordless cycle of length at least 5). Such a pair  $a, b$  is what is called an *even pair*, and it is proved in [15] that there exists an optimal coloring of  $G$  such that  $a$  and  $b$  have the same color. The perfection of a graph obtained from two perfect graphs by gluing  $G_1$  and  $G_2$  along a proper 2-cutset  $\{a, b\}$  is then easy to prove by a direct coloring argument: use colorings of  $G_1$  and  $G_2$  that both give the same color to  $a$  and  $b$ .

**Theorem 2.4** *Perfect graphs are closed under the following operations: gluing along a clique, substitution, 1-join composition, amalgam composition, gluing along a proper 2-cutset.*



We now turn our attention to the preservation of  $\chi$ -boundedness under the operations, but there is an important technicality. A class of graphs is *hereditary* if it is closed under taking induced subgraphs. The *closure* of a class  $\mathcal{B}$  of graphs under a set of graph operations is the class  $\mathcal{C}$  obtained from the graphs of  $\mathcal{B}$  by performing the operations repeatedly and in any order. A set of operations *preserves  $\chi$ -boundedness* if the closure of any hereditary  $\chi$ -bounded class under the set of operations is a  $\chi$ -bounded class. Of course, the function that bounds  $\chi$  needs not be the same in  $\mathcal{B}$  and  $\mathcal{C}$ , and in most cases, it is not. This leads to a potential problem: it may happen that an operation  $O_1$  preserves  $\chi$ -boundedness, that another operation  $O_2$  also preserves  $\chi$ -boundedness, but that the set of operations  $\{O_1, O_2\}$  does not preserve  $\chi$ -boundedness. This is explained in [7], where an actual (but slightly artificial) example of this phenomenon is provided.

It is very easy to prove that gluing along a clique preserves  $\chi$ -boundedness. In [7], it is proved that substitution preserves  $\chi$ -boundedness. In [14], it is proved that 1-join composition preserves  $\chi$ -boundedness. But it is not at all easy to prove for instance that the set of operations {1-join composition, gluing along a clique} preserves  $\chi$ -boundedness or that amalgam composition preserves  $\chi$ -boundedness. However, these are true statements, and corollaries of the next theorem from [19].

**Theorem 2.5 (Penev)** *If a class of graphs is  $\chi$ -bounded, then its closure under the following set of operations is  $\chi$ -bounded: {substitution, amalgam composition, gluing along a clique}.*

Also gluing along a proper 2-cutset preserves  $\chi$ -boundedness as shown in [7].

**Theorem 2.6 (Chudnovsky, Penev, Scott and Trotignon)** *If a class of graphs is  $\chi$ -bounded, then its closure under the operation of gluing along a proper 2-cutset is  $\chi$ -bounded.*

Note that in Theorem 2.5, if the class we start with is  $\chi$ -bounded by a function  $f$ , then the closure is  $\chi$ -bounded by an exponential in  $f$  (something close to  $g(x) = (xf(x))^x$ ). In Theorem 2.6, the situation is much better, and the resulting function is linear in the function  $f$  we start with. The function was even improved by Penev, Thomassé and Trotignon, see [20]. Note that in [7, 20] an operation more general than the proper 2-cutset is considered (namely, the operation of gluing along a 2-cutset, not worth defining here).

### 3 A property closed under substitutions

Say that a graph  $G$  has Property  $P_0$  if it has no edges (such a graph is called an *independent graph*). We now define inductively a Property  $P_k$  for all  $k \geq 1$  as follows: a graph  $G$  has Property  $P_k$  if for every induced subgraph  $G'$  of  $G$  there exists an induced subgraph  $H$  of  $G'$  that has Property  $P_{k-1}$  and that intersects every maximal clique of  $G'$ . From the definition, it is clear that Property  $P_k$  is hereditary (if a graph has it, then so are all its induced subgraphs).

Graphs with Property  $P_1$  are exactly the graphs  $G$  such that for every induced subgraph  $H$  of  $G$  there exists a stable set of  $H$  that intersects every maximal clique of  $H$  (where a *stable set* in a graph is a set of vertices that induces an independent graph). Graphs satisfying Property  $P_1$  are known as *strongly perfect graphs*, see [21] for a survey about them. They form a (proper) subclass of perfect graphs. To the best of our knowledge, for  $k \geq 2$ , graphs with Property  $P_k$  were not studied so far.

The following provides examples of graphs with Property  $P_k$ .

**Lemma 3.1** *For all  $k \geq 1$ , graphs with chromatic number at most  $k$  have Property  $P_{k-1}$ .*

PROOF — We proceed by induction on  $k$ . For  $k = 1$ , the result is obvious. Suppose it holds for some fixed  $k \geq 1$ . Let  $G$  be a graph with chromatic number at most  $k + 1$ , and  $G'$  an induced subgraph of  $G$ . In  $G'$ , there exists an induced subgraph  $H$  of chromatic number at most  $k$  that intersects all maximal cliques of  $G'$ : consider for instance the union of the first  $k$  (possibly empty) colour classes in a colouring of  $G'$  with  $k + 1$  colours. By the induction hypothesis,  $H$  has Property  $P_{k-1}$ . This proves that  $G$  has Property  $P_k$ .  $\square$

The following is similar to the Lovász's replication lemma, stating that perfect graphs are closed under substitutions.

**Lemma 3.2** *For all  $k \geq 0$ , Property  $P_k$  is closed under substitution.*

PROOF — We proceed by induction on  $k$ . If  $k = 0$ , we have to prove that substituting an independent graph for a vertex  $v$  of an independent graph yields an independent graph, which is obvious. So, suppose  $k \geq 1$  and suppose Property  $P_{k-1}$  is closed under substitution.

Suppose that  $G$  is a graph obtained from  $G_1$  by substituting  $G_2$  for  $v \in V(G_1)$  and  $G_1$  and  $G_2$  have Property  $P_k$ . We will prove that  $G$  contains an induced subgraph with Property  $P_{k-1}$  that intersect all maximal cliques

of  $G$ . By Lemma 2.2, the same proof can be done for induced subgraphs  $G'$  of  $G$ .

A maximal clique in  $G$  is either a maximal clique of  $G_1$  that does not contain  $v$  (we say that such a maximal clique has *type non- $v$* ), or is equal to  $K_1 \cup K_2$  where  $K_1 \cup \{v\}$  is a maximal clique of  $G_1[\{v\} \cup N(v)]$  and  $K_2$  is a maximal clique of  $G_2$  (we say that such a maximal clique has *type  $v$* ).

For  $i = 1, 2$ , because  $G_i$  has Property  $P_k$ , there exists an induced subgraph  $H_i$  of  $G_i$  that has Property  $P_{k-1}$  and that intersects every maximal clique of  $G_i$ . There are now two cases.

If  $v \in V(H_1)$ , then let  $H$  be the graph obtained from  $H_1$  by substituting  $H_2$  for  $v$ . By the induction hypothesis,  $H$  has Property  $P_{k-1}$ . Let  $K$  be a maximal clique in  $G$ . If  $K$  is of type  $v$ , then  $K \cap V(G_2)$  is a maximal clique of  $G_2$ , and it is intersected by  $V(H_2)$ , so it is intersected by  $H$ . If  $K$  is of type non- $v$ , then  $K$  is a maximum clique of  $G_1$ , so it must intersect  $H_1$ , and not in  $v$ , so it intersects  $H$ . We proved that  $H$  intersects all maximal cliques of  $G$ .

If  $v \notin V(H_1)$ , then we set  $H = H_1$ . Let  $K$  be a maximal clique in  $G$ . If  $K$  is of type  $v$ , then  $(K \cap V(G_1)) \cup \{v\}$  is a maximal clique of  $G_1$ , and it is intersected by  $V(H_1) = V(H)$ . If  $K$  is of type non- $v$ , then  $K$  is a maximum clique of  $G_1$ , so it must intersect  $V(H_1) = V(H)$ . We proved again that  $H$  intersects all maximal cliques of  $G$ .  $\square$

Since substitution is one of the simplest operation that preserves perfection, it is worth asking whether Property  $P_k$  is closed under gluing along a clique (another simple operation that preserves perfection and  $\chi$ -boundedness). It turns out that it is not the case for  $k = 1$  (examples are provided in [3]). Here we give another example showing that  $P_2$  is not closed under gluing along a clique.

To check this, it is convenient to rephrase Property  $P_2$ : for every induced subgraph  $G'$ , there is a strongly perfect graph  $H$  that is an induced subgraph of  $G'$  and that intersects all maximal cliques of  $G'$ . Chordal graphs are shown to be strongly perfect in [3]. On Figure 1, three graphs are represented. Graph  $G_1$  is obtained from a copy of  $C_5$  and a copy of  $K_5$  by adding a matching. In  $G_2$ , there are five copies of  $K_5$ , say  $H_1, \dots, H_5$ , and there are all possible edges between  $H_i$  and  $H_{i+1}$  for all  $i = 1, \dots, 5$  (taken modulo 5). Four of the copies have a  $C_5$  matched to them.

It is easy to check that  $G_1$  and  $G_2$  both have Property  $P_2$  (in fact, they have the stronger property that a chordal graph intersects all maximal cliques). For instance, in  $G_1$ , by picking a vertex in the  $K_5$  and by taking all its non-neighbors, we obtain a chordal graph  $H$  that intersects all maximal

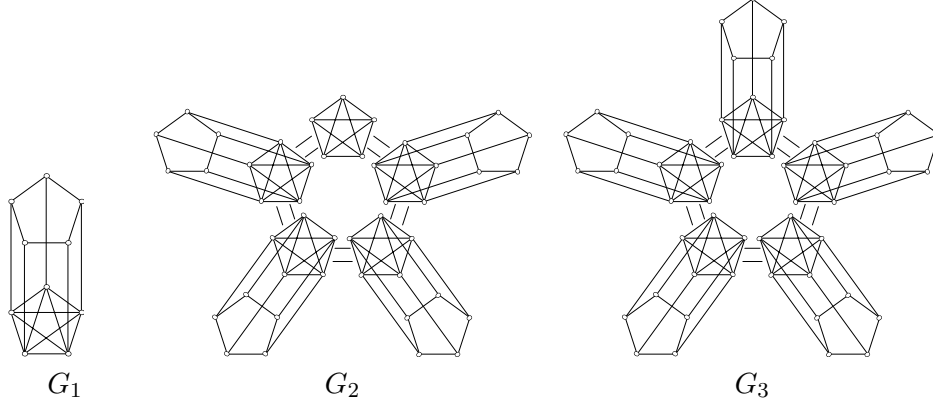


Figure 1: Three graphs

cliques of  $G_1$ . In  $G_2$ , we can take four copies of the chordal graph used for  $G_1$  that exist in the matched  $K_5$ 's. Note that in  $G_2$ , no vertex of the top clique needs to be taken in chordal graph that intersects all maximal cliques.

However,  $G_3$ , that is obtained by gluing  $G_1$  and  $G_2$  along a  $K_5$ , does not have Property  $P_2$ . To see this, note that every matching edge is a maximal clique. Also,  $H$  cannot contain a vertex in each of the  $K_5$ 's (because this would form a  $C_5$ , that is not strongly perfect), so at least one copy of  $K_5$  does not intersect  $H$ . The  $C_5$  matched to this copy therefore has to be all in  $H$ , a contradiction to the strong perfection of  $H$ .

We now explain how Property  $P_k$  is related to  $\chi$ -boundedness. We define  $f_0(0) = 0$  and  $f_0(x) = 1$  for all integers  $x \geq 1$ . For all integers  $k \geq 1$  and  $x \geq 0$ , we set

$$f_k(x) = \sum_{i=0}^x f_{k-1}(i).$$

By an easy induction, for all integer  $k \geq 0$ ,  $f_k(0) = 0$ ,  $f_k(1) = 1$  and  $f_1$  is the identity function. Also, it is easy to check that  $f_k(x) \leq x^k$  for all integers  $x, k \geq 0$ . Hence  $f_k$  is a polynomial of degree  $k$  (with the convention that  $0^0 = 0$ ).

**Lemma 3.3** *Graphs with Property  $P_k$  are  $\chi$ -bounded by the function  $f_k$  (and therefore by a polynomial of degree  $k$ ).*

PROOF — For  $k = 0$ , this is trivial. Let us prove it by induction on  $k$  for  $k \geq 1$ . Let  $G$  be a graph that has Property  $P_k$  and set  $\omega = \omega(G)$ . By

Property  $P_k$ ,  $G$  contains an induced subgraph  $H_\omega$  that has Property  $P_{k-1}$  and intersects all maximal cliques of  $G$ , so that  $\omega(G \setminus H_\omega) = \omega(G) - 1$ . In  $G \setminus H_\omega$ , there exists also an induced subgraph  $H_{\omega-1}$  that has Property  $P_{k-1}$  and intersects all maximal cliques of  $G \setminus H_\omega$ , and continuing like that, we prove that  $G$  can be vertex-wise partitioned into  $\omega(G)$  induced subgraphs  $H_1, H_2, \dots, H_\omega$ , such that for all  $j = 1, \dots, \omega$ ,  $H_j$  has Property  $P_{k-1}$  and  $\omega(H_j) = j$ . By the induction hypothesis, we have

$$\chi(G) \leq \sum_{i=1}^{\omega} \chi(H_i) \leq \sum_{i=0}^{\omega} f_{k-1}(i) = f_k(\omega(G)).$$

The same proof can be made for all induced subgraphs of  $G$ .  $\square$

**Theorem 3.4** *The closure by substitutions of the class of  $k$ -colourable graphs is a class of graph that is  $\chi$ -bounded by  $f_{k-1}$  (in particular, by a polynomial of degree  $k - 1$ ).*

PROOF — Every graph in the class has Property  $P_{k-1}$ , either by Lemma 3.1 or by Lemma 3.2. So, by Lemma 3.3, it is  $\chi$ -bounded by  $f_{k-1}$ .  $\square$

As observed by Penev, for large values of  $k$ , a stronger result was implicitly proved in [7].

**Theorem 3.5 (Chudnovsky, Penev, Scott and Trotignon)** *If a class of graphs is  $\chi$ -bounded by  $f(x) = x^A$ , then the closure of the class under substitution is  $\chi$ -bounded by  $g(x) = x^{3A+11}$ .*

Since  $k$ -colourable graphs are  $\chi$ -bounded by  $f(x) = x^{\log_2 k}$ , we know by Theorem 3.5 that the closure of  $k$ -colorable graphs under substitution forms a class  $\chi$ -bounded by  $g(x) = x^{11+3\log_2 k}$ . So, when  $k$  is large,  $g(x)$  is smaller than  $x^k$ , but for small values, Theorem 3.4 provides the best bound known so far. For instance, the fact that the closure of 3-colorable graphs under substitution is  $\chi$ -bounded by a quadratic function is seemingly a new theorem.

## 4 A property closed under amalgam and proper 2-cutset

In the rest of the paper, we adopt the unusual convention that *no vertex of a graph is complete to the empty set*. We call a constraint for a graph  $G$

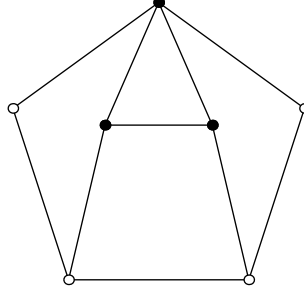


Figure 2: A graph that does not have Property  $Q_k$  for any  $k$

any pair  $(K^{\text{in}}, K^+)$  such that  $K^{\text{in}}$  and  $K^+$  are disjoint sets and  $K^{\text{in}} \cup K^+$  is a clique of  $G$ . Note that  $K^{\text{in}}$  and  $K^+$  are therefore disjoint possibly empty cliques of  $G$ . When  $(K^{\text{in}}, K^+)$  is a constraint for a graph  $G$ , a *splitter* for  $(G, K^{\text{in}}, K^+)$  is an induced subgraph  $H$  of  $G$  that satisfies the following.

- $H$  intersects all maximal cliques of  $G$  (except possibly  $K^+$  when  $K^+$  is a maximal clique of  $G$ );
- $H$  contains all vertices of  $K^{\text{in}}$  and  $H$  contains no  $K^{\text{in}}$ -complete vertex (when  $K^{\text{in}} = \emptyset$ , this constraint can be forgotten);
- $H$  contains no vertex of  $K^+$ .

We now define inductively a Property  $Q_k$  for all  $k \geq 1$ . A graph has *Property*  $Q_1$  if it is perfect. For  $k \geq 1$ , a graph has *Property*  $Q_{k+1}$  if for every induced subgraph  $G'$  and every constraint  $(K^{\text{in}}, K^+)$  for  $G'$ , there is splitter  $H$  for  $(G', K^{\text{in}}, K^+)$  with the additional property that  $G[V(H) \cup K^+]$  has Property  $Q_k$ . Such a splitter is called a *k-splitter*.

It is obvious that if a graph  $G$  has Property  $Q_k$  then every induced subgraph of  $G$  has Property  $Q_k$ . On Figure 2, we show a graph  $G$  that does not have Property  $Q_k$  for any  $k$ . To see this, suppose for a contradiction that  $G$  has Property  $Q_k$  for some  $k \geq 1$ , and consider the minimum such  $k$ . Since  $G$  contains a  $C_5$  and is therefore not perfect, we have  $k \geq 2$ . Define  $K^+$  as the set of black vertices on the figure. It is straightforward that the only splitter for  $(G, \emptyset, K^+)$  is  $H = G \setminus K^+$ , so  $G[V(H) \cup K^+] = G$  must have Property  $Q_{k-1}$ , a contradiction to the minimality of  $k$ .

The next lemma gives the taste of our main theorem on Property  $Q_k$  (and is a particular case of it, but we prefer proving it separately). It is very easy, but it seems impossible to prove it formally without an induction.

**Lemma 4.1** *Property  $Q_k$  is closed under disjoint union.*

PROOF — We prove the lemma by induction on  $k$ . If  $k = 1$ , the result follows directly from Theorem 2.4 (because taking the disjoint union means gluing along an empty clique). So, suppose  $k \geq 1$  and let  $G$  be the disjoint union of two graphs  $G_1$  and  $G_2$  that have Property  $Q_{k+1}$ . Let  $(K^{\text{in}}, K^+)$  be a constraint for  $G$ . Up to symmetry, we may assume that  $K^{\text{in}} \cup K^+ \subseteq V(G_1)$ , and consider a  $k$ -splitter  $H_1$  for  $(G_1, K^{\text{in}}, K^+)$ . In  $G_2$ , we consider a  $k$ -splitter for  $(H_2, \emptyset, \emptyset)$ . It is straightforward to check that  $H_1 \cup H_2$  is a splitter for  $(G, K^{\text{in}}, K^+)$ , and by the induction hypothesis, it is a  $k$ -splitter.  $\square$

**Lemma 4.2** *For every graph  $G$  and every constraint  $(K^{\text{in}}, K^+)$  for  $G$  there exist a splitter for  $(G, K^{\text{in}}, K^+)$ .*

PROOF — Define  $H$  as the graph induced by all vertices of  $G \setminus K^+$  that are not complete to  $K^{\text{in}}$  (in particular, if  $K^{\text{in}} = \emptyset$  then  $H = G \setminus K^+$ ). We claim that  $H$  is a splitter for  $(G, K^{\text{in}}, K^+)$ . It intersects all maximal cliques of  $G$  (except possibly  $K^+$  when  $K^+$  is a maximal clique) because any vertex in  $K^{\text{in}}$  is complete to  $V(G) \setminus V(H)$  (and when  $K^{\text{in}} = \emptyset$  and  $K^+$  is not maximal, there must be a vertex complete to  $K^+$  in  $H$ ). Obviously,  $H$  contains all vertices of  $K^{\text{in}}$ , no  $K^{\text{in}}$ -complete vertex, and no vertex of  $K^+$ .  $\square$

**Lemma 4.3** *For all  $k \geq 1$ , any graph with Property  $Q_k$  has Property  $Q_{k+1}$ .*

PROOF — Let  $(K^{\text{in}}, K^+)$  be a constraint for a graph  $G$  with Property  $Q_k$ . Lemma 4.2 provides a splitter for  $(G, K^{\text{in}}, K^+)$ . This splitter has Property  $Q_k$  because so does  $G$ . The same proof can be done for all induced subgraphs of  $G$ , so every induced subgraph of  $G$  has a  $k$ -splitter.  $\square$

**Lemma 4.4** *Let  $(K^{\text{in}}, K^+)$  be a constraint for a non-bipartite triangle-free connected graph  $G$ . Then there exists a splitter  $H$  for  $(G, K^{\text{in}}, K^+)$  such that  $|V(H) \cup K^+| < |V(G)|$ .*

PROOF — Since  $G$  is triangle-free,  $|K^{\text{in}} \cup K^+| \leq 2$ . Also, since  $G$  is non-bipartite and triangle-free,  $|V(G)| \geq 5$ . We claim that  $G$  contains a vertex  $v$  such that:

- $v \notin K^{\text{in}} \cup K^+$ ;
- $v$  has no neighbor in  $K^+$ ;

- $v$  is not adjacent to any  $K^{\text{in}}$ -complete vertex.

To prove the claim, we break into cases according to the sizes of  $K^{\text{in}}$  and  $K^+$ .

If  $|K^{\text{in}}| = 0$  and  $|K^+| = 0$ , then any vertex  $v$  satisfies the constraint.

If  $|K^{\text{in}}| = 0$  and  $|K^+| = 1$ , then any vertex not in  $K^+$  and non-adjacent to the unique vertex in  $K^+$  satisfies the constraint (and there is such a vertex, for otherwise  $G$  is bipartite).

If  $|K^{\text{in}}| = 1$  and  $|K^+| = 0$ , then any  $K^{\text{in}}$ -complete vertex  $v$  satisfies the constraint (and there is such a vertex since  $G$  is connected).

If  $|K^{\text{in}}| = 1$  and  $|K^+| = 1$ , then any  $K^{\text{in}}$ -complete vertex not in  $K^+$  satisfies the constraints. Since  $G$  is connected, we may therefore assume that  $K^{\text{in}}$  is made of a vertex  $x$  whose only neighbor is the vertex  $y$  from  $K^+$ . If all vertices of  $G \setminus K^+$  are adjacent to  $y$ , then since  $G$  is triangle free,  $N(y)$  is a stable set, and  $G$  is bipartite, a contradiction. It follows that  $v$  can be chosen among the non-neighbors of  $y$ .

If  $|K^{\text{in}}| = 2$  and  $|K^+| = 0$ , then no vertex in  $G$  is  $K^{\text{in}}$ -complete since  $G$  is triangle-free, so any vertex  $v$  not in  $K^{\text{in}}$  satisfies the constraint (and there exists such a vertex since  $G$  is not bipartite).

If  $|K^{\text{in}}| = 0$  and  $|K^+| = 2$ , then any vertex not in  $K^+$  and with no neighbor in  $K^+$  satisfies the constraint, so suppose that no such vertex exists. It follows that  $V(G) = \{x, y\} \cup N(x) \cup N(y)$  where  $K^+ = \{x, y\}$ . Since  $G$  is triangle-free, we see that  $x \cup N(y)$  and  $y \cup N(x)$  are stable sets, so  $G$  is bipartite, a contradiction.

This proves the claim. Now define  $X$  as the set of  $K^{\text{in}}$ -complete vertices. Since  $G$  is triangle-free, we see that  $X \cup K^+ \cup \{v\}$  is a stable set of  $G$  (except when  $|K^+| = 2$ ). It follows that  $H = G \setminus (X \cup K^+ \cup \{v\})$  is a splitter for  $(G, K^{\text{in}}, K^+)$ . And because of  $v$ , we have  $|V(H) \cup K^+| < |V(G)|$ .  $\square$

**Lemma 4.5** *Every triangle-free graph on  $n \geq 4$  vertices has Property  $Q_{n-3}$ .*

PROOF — We prove the property by induction on  $n$ . It is well known that all graphs on at most 4 vertices are perfect (they are all chordal except the cycle of length 4 that is bipartite). So by Lemma 4.3, the property is true for  $n = 4$ . Suppose it holds for  $n \geq 4$ , and consider a graph  $G$  on  $n + 1$  vertices and a constraint  $(K^{\text{in}}, K^+)$  for  $G$ . By Lemma 4.3, we may assume that  $G$  is not perfect (and in particular not bipartite). It is enough to find an  $(n - 3)$ -splitter for  $(G, K^{\text{in}}, K^+)$ , because the proper induced subgraphs of  $G$  have Property  $Q_{n-3}$  by the induction hypothesis. Lemma 4.4 provides



a splitter  $H$  such that  $|V(H) \cup K^+| < |V(G)|$ , so the induction hypothesis shows that this splitter has Property  $Q_{n-3}$ .  $\square$

**Theorem 4.6** *Property  $Q_k$  is closed under the following four operations: gluing along a clique, amalgam (and therefore 1-join composition), substitution and proper 2-cutset composition.*

PROOF — We proceed by induction on  $k$ . For  $k = 1$ , the result follows directly from Theorem 2.4. Suppose it holds for some fixed  $k \geq 1$ . We consider a graph  $G$  obtained from two graphs  $G_1$  and  $G_2$  with Property  $Q_{k+1}$  by one of the operations, and we show that for every constraint  $(K^{\text{in}}, K^+)$ , there exists a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ . Each time, the splitter is obtained by combining  $k$ -splitters of  $G_1$  and  $G_2$  with well chosen constraints (they exist by assumption), and the combination has Property  $Q_k$  by the induction hypothesis. Note that by Lemma 2.3, the same proof can be done for induced subgraphs of  $G$ , so we do not need to consider induced subgraphs of  $G$ . Let us consider the operations one by one.

### Gluing along a clique

We suppose that  $(X_1, K, X_2)$  is a split for a clique cutset of  $G$ , so  $G$  is obtained from  $G_1 = G[X_1 \cup K]$  and  $G_2 = G[X_2 \cup K]$  by gluing along  $K$ . We suppose that  $G_1$  and  $G_2$  have Property  $Q_{k+1}$ .

Up to symmetry, we may assume that  $K^{\text{in}} \cup K^+ \subseteq X_1 \cup K$ . Set  $K_1^{\text{in}} = K^{\text{in}}, K_1^+ = K^+$  and let  $H_1$  be a splitter for  $(G_1, K_1^{\text{in}}, K_1^+)$ . There are two cases.

**Case 1:**  $V(H_1) \cap K \neq \emptyset$ .

Set  $K_2^{\text{in}} = V(H_1) \cap K$  and  $K_2^+ = K^+ \cap K$ . Note that  $K_2^+$  is not a maximal clique of  $G_2$  since  $K_2^{\text{in}} \neq \emptyset$ . Let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ . Let  $H = H_1 \cup H_2$ . Since  $H_2$  contains no  $K_2^{\text{in}}$ -complete vertex, we have  $V(H_2) \cap K = K_2^{\text{in}}$ . We now check that  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

First,  $H$  contains all vertices of  $K^{\text{in}}$  and  $H$  contains no vertex of  $K^+$ . Vertices of  $H_1$  are not complete to  $K_1^{\text{in}} = K^{\text{in}}$ . Also a vertex  $v \in V(H_2)$  is not complete to  $K^{\text{in}}$ , for otherwise,  $v \in V(H_2) \setminus K$  (because as noted already  $V(H_2) \cap K = K_2^{\text{in}}$ ). It follows that  $K^{\text{in}} \subseteq K$ . But then,  $K^{\text{in}} \subseteq K_2^{\text{in}}$ , so  $v$  is  $K_2^{\text{in}}$ -complete, a contradiction. Hence,  $H$  contains no  $K^{\text{in}}$ -complete vertex.

Moreover,  $H$  intersects all maximal cliques of  $G$  (except  $K^+$  when  $K^+$  is a maximal clique of  $G$  and therefore of  $G_1$ ), because all such cliques are either in  $G_1$  or in  $G_2$ .

Since  $G_1[V(H_1) \cup K^+]$  and  $G_2[V(H_2) \cup K_2^+]$  have Property  $Q_k$  and  $G[V(H) \cup K^+]$  is obtained from these two graphs by gluing along  $K_2^{\text{in}} \cup K_2^+$ , we know by the induction hypothesis that  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

**Case 2:**  $V(H_1) \cap K = \emptyset$ .

Note that  $K^{\text{in}} \subseteq X_1$ . Set  $K_2^{\text{in}} = \emptyset$  and  $K_2^+ = K$  and let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ . Let  $H = H_1 \cup H_2$ . So  $H$  contains all vertices of  $K^{\text{in}}$ , no vertex of  $K^+$  and no  $K^{\text{in}}$ -complete vertex.

Observe that  $G[V(H) \cup K^+]$  is obtained from  $G_1[V(H_1) \cup K^+]$  and  $G_2[V(H_2) \cup (K^+ \cap K)]$  by gluing along  $K^+ \cap K$ . And  $G_2[V(H_2) \cup (K^+ \cap K)]$  has Property  $Q_k$  because it is an induced subgraph of  $G_2[V(H_2) \cup K_2^+]$ . So by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ . It remains to prove that  $H$  intersects all maximal cliques of  $G$  (except  $K^+$  when  $K^+$  is a maximal clique of  $G$ ).

Let  $K'$  be any maximal clique of  $G$ . Since  $K$  is a clique cutset of  $G$ ,  $K'$  is a maximal clique of  $G_1$  or  $G_2$ . If  $K' \subseteq V(G_1)$ , then  $K'$  is intersected by  $H_1$  (and therefore  $H$ ) unless  $K' = K_1^+ = K^+$ . If  $K' \subseteq V(G_2)$ , then  $K'$  is intersected by  $H_2$ , unless  $K' = K_2^+ = K$ . In this last case,  $K$  is a maximal clique of  $G$  (and therefore  $G_1$ ), and since it is not intersected by  $H_1$  (because  $V(H_1) \cap K = \emptyset$ ), it must be that  $K' = K_1^+ = K^+$ . In all cases,  $H$  intersects  $K'$  except when  $K' = K^+$ .

## Amalgam

We suppose that  $(X_1, X_2, A_1, A_2, K)$  is a split for an amalgam of  $G$ . For  $i = 1, 2$ , the *block of decomposition*  $G_i$  with respect to this split is the graph obtained from  $G[X_i \cup K]$  by adding a vertex  $u_{3-i}$  complete to  $A_i \cup K$ , so  $G$  is obtained from  $G_1$  and  $G_2$  by an amalgam composition. We suppose that  $G_1$  and  $G_2$  have Property  $Q_{k+1}$ .

Let  $(K^{\text{in}}, K^+)$  be a constraint for  $G$ .

**Case 1**  $X_1 \cup K$  does not contain  $K^{\text{in}} \cup K^+$  and  $X_2 \cup K$  does not contain  $K^{\text{in}} \cup K^+$ . Since  $K^{\text{in}} \cup K^+$  is a clique,  $K^{\text{in}} \cup K^+$  belongs to  $A_1 \cup A_2 \cup K$ ,  $(K^{\text{in}} \cup K^+) \cap A_1 \neq \emptyset$ ,  $(K^{\text{in}} \cup K^+) \cap A_2 \neq \emptyset$ . There are three subcases.

**Case 1a:**  $K^{\text{in}} \cap K \neq \emptyset$ .

Set  $K_1^{\text{in}} = K^{\text{in}} \cap (K \cup A_1)$  and  $K_1^+ = (K^+ \cap (K \cup A_1)) \cup \{u_2\}$ . Note that  $K_1^+$  is not a maximal clique of  $G_1$  since  $K^{\text{in}} \cap K \neq \emptyset$ . Let  $H_1$  be a splitter for  $(G_1, K_1^{\text{in}}, K_1^+)$ .

Set  $K_2^{\text{in}} = K^{\text{in}} \cap (K \cup A_2)$  and  $K_2^+ = (K^+ \cap (K \cup A_2)) \cup \{u_1\}$ . Note that  $K_2^+$  is not a maximal clique of  $G_2$  since  $K^{\text{in}} \cap K \neq \emptyset$ . Let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ .

Let  $H = H_1 \cup H_2$ . We now check that  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

It is obvious that  $H$  contains all vertices of  $K^{\text{in}}$ . If a vertex  $v \in V(H_1)$  is complete to  $K^{\text{in}}$ , then  $v$  is complete to  $K_1^{\text{in}}$ , a contradiction. This implies that  $H_1$  contains no  $K^{\text{in}}$ -complete vertex, no vertex of  $K^+$ . Similarly,  $H_2$  contains no  $K^{\text{in}}$ -complete vertex, no vertex of  $K^+$ . Hence,  $H$  contains no  $K^{\text{in}}$ -complete vertex, no vertex of  $K^+$ .

$H$  contains  $K^{\text{in}}$  and  $K^{\text{in}} \cap K \neq \emptyset$  so  $H$  intersects all maximal cliques of  $A_1 \cup A_2 \cup K$ . Therefore,  $H$  intersects all maximal cliques of  $G$ .

$G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[V(H_1) \cup K_1^+]$  and  $G_2[V(H_2) \cup K_2^+]$  (both graphs have Property  $Q_k$ ) by an amalgam composition, so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

**Case 1b:**  $K^{\text{in}} \cap K = \emptyset$ ,  $K^{\text{in}} \neq \emptyset$ .

Up to symmetry, we may assume that  $K^{\text{in}} \cap A_1 \neq \emptyset$ .

Set  $K_1^{\text{in}} = K^{\text{in}} \cap A_1$  and  $K_1^+ = (K^+ \cap (K \cup A_1)) \cup \{u_2\}$ . Note that  $K_1^+$  is not a maximal clique of  $G_1$  since  $K^{\text{in}} \cap A_1 \neq \emptyset$ . Let  $H_1$  be a splitter for  $(G_1, K_1^{\text{in}}, K_1^+)$ .

Set  $K_2^{\text{in}} = (K^{\text{in}} \cap A_2) \cup \{u_1\}$  and  $K_2^+ = (K^+ \cap (K \cup A_2))$ . Note that  $K_2^+$  is not a maximal clique of  $G_2$  since  $u_2 \notin K_2^+$ . Let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ .

Let  $H = G[V(H_1) \cap (V(H_2) \setminus \{u_1\})]$ . We now check that  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

As in the preceding case,  $H$  contains all vertices of  $K^{\text{in}}$ , no  $K^{\text{in}}$ -complete vertex, no vertex of  $K^+$ .

Since  $H_1$  intersects all maximal cliques of  $G_1$  then  $P_1 = V(H_1) \cap A_1 \neq \emptyset$  and  $P_1$  must intersect all maximal cliques of  $A_1$  (if not this clique combined with  $u_2$  and  $K$  would be a maximal clique of  $G_1$  that  $H_1$  does not intersect, a contradiction). This implies that  $P_1$  intersects all maximal cliques of  $A_1 \cup A_2 \cup K$ . Hence,  $H$  intersects all maximal cliques of  $G$ .

Because  $(K^{\text{in}} \cup K^+) \cap A_2 \neq \emptyset$ ,  $(V(H_2) \cup K_2^+) \cap A_2 \neq \emptyset$ . Hence,  $G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[V(H_1) \cup K_1^+]$  and  $G_2[V(H_2) \cup K_2^+]$  (both graphs have Property  $Q_k$ ) by an amalgam composition, so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

**Case 1c:**  $K^{\text{in}} = \emptyset$ .

If  $(A_1 \cup A_2 \cup K) \setminus K^+ \neq \emptyset$ , then, we choose  $v \in (A_1 \cup A_2 \cup K) \setminus K^+$ , set  $K^{\text{in}} = \{v\}$  and by the proof of case 1 and case 2 we obtain a  $k$ -splitter  $H$  for  $(G, K^{\text{in}}, K^+)$ . So, suppose  $K^+ = A_1 \cup A_2 \cup K$ . This means  $K^+$  is a maximal clique of  $G$ .

Set  $K_1^{\text{in}} = \emptyset$  and  $K_1^+ = (K^+ \cap (K \cup A_1)) \cup \{u_2\}$ . Let  $H_1$  be a splitter for  $(G_1, K_1^{\text{in}}, K_1^+)$ .

Set  $K_2^{\text{in}} = \emptyset$  and  $K_2^+ = (K^+ \cap (K \cup A_2)) \cup \{u_1\}$ . Let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ .

Let  $H = H_1 \cup H_2$ . We now check that  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

It is obvious that  $H$  contains no vertex of  $K^+$  and  $H$  intersects all maximal cliques of  $G$  except  $K^+$ .

$G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[V(H_1) \cup K_1^+]$  and  $G_2[V(H_2) \cup K_2^+]$  (both graphs have Property  $Q_k$ ) by an amalgam composition, so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

**Case 2** We are not in Case 1, so up to symmetry, we may assume that  $K^{\text{in}} \cup K^+ \subseteq X_1 \cup K$ . Set  $K_1^{\text{in}} = K^{\text{in}}, K_1^+ = K^+$  and let  $H_1$  be a splitter for  $(G_1, K_1^{\text{in}}, K_1^+)$ . There are three cases.

**Case 2a:**  $V(H_1) \cap K \neq \emptyset$ .

Set  $K_2^{\text{in}} = V(H_1) \cap K$  and  $K_2^+ = K^+ \cap K$ . Note that  $K_2^+$  is not a maximal clique of  $G_2$  since  $K_2^{\text{in}} \neq \emptyset$ . Let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ . Since  $H_2$  contains no  $K_2^{\text{in}}$ -complete vertex, we have  $V(H_2) \cap K = K_2^{\text{in}}, V(H_2) \cap A_2 = \emptyset$  and  $u_1 \notin V(H_2)$ .

Let  $H = G[(V(H_1) \setminus \{u_2\}) \cup V(H_2)]$ . We now check that  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

It is obvious that  $H$  contains all vertices of  $K^{\text{in}}$  and  $H$  contains no vertex of  $K^+$ . Vertices of  $H_1$  are not complete to  $K_1^{\text{in}} = K^{\text{in}}$ . Also a vertex  $v \in V(H_2)$  is not complete to  $K^{\text{in}}$ , for otherwise,  $v \in V(H_2) \setminus K$  (because as noted already  $V(H_2) \cap K = K_2^{\text{in}}$ ). It follows that  $K^{\text{in}} \subseteq K$ . But then,  $V(H_1) \cap K = K_1^{\text{in}} = K^{\text{in}}$  since  $H_1$  contains no  $K_1^{\text{in}}$ -complete vertex. This implies  $K^{\text{in}} = K_2^{\text{in}}$ , so  $v$  is  $K_2^{\text{in}}$ -complete, a contradiction. Hence,  $H$  contains no  $K^{\text{in}}$ -complete vertex.

$H$  contains  $V(H_1) \cap K \neq \emptyset$  so  $H$  intersects all maximal cliques of  $A_1 \cup A_2 \cup K$ . Hence,  $H$  intersects all maximal cliques of  $G$ .

Observe that  $G[V(H) \cup K^+]$  is obtained from  $G_1[(V(H_1) \setminus \{u_2\}) \cup K_1^+]$  and  $G_2[V(H_2) \cup K_2^+]$  by gluing along  $K_2^+ \cap K_2^{\text{in}}$ . And  $G_1[(V(H_1) \setminus \{u_2\}) \cup K_1^+]$  has Property  $Q_k$  because it is an induced subgraph of  $G_1[V(H_1) \cup K_1^+]$ . So by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

**Case 2b:**  $V(H_1) \cap K = \emptyset$  and  $u_2 \notin V(H_1)$ .

$H_1$  contains all vertices of  $K^{\text{in}}$  so  $K^{\text{in}} \cap K = \emptyset$ . Set  $K_2^{\text{in}} = \{u_1\}$  and  $K_2^+ = K^+ \cap K$ . Note that  $K_2^+$  is not a maximal clique of  $G_2$  since  $A_2 \neq \emptyset$ . Let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ . Since  $H_2$  contains no  $K_2^{\text{in}}$ -complete vertex, we have  $V(H_2) \cap K = \emptyset, V(H_2) \cap A_2 = \emptyset$ .

Let  $H = G[V(H_1) \cup (V(H_2) \setminus \{u_1\})]$ . We now check that  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

It is obvious that  $H$  contains all vertices of  $K^{\text{in}}$  and  $H$  contains no vertex of  $K^+$ .  $H$  contains no  $K^{\text{in}}$ -complete vertex since vertices of  $H_1$  are not complete to  $K_1^{\text{in}} = K^{\text{in}}$  and  $V(H_2)$  is anticomplete to  $K^{\text{in}}$  (because as noted already  $V(H_2) \cap K = \emptyset, V(H_2) \cap A_2 = \emptyset$  and  $K^{\text{in}} \cap K = \emptyset$ ).

Since  $H_1$  intersects all maximal cliques of  $G_1$  then  $P_1 = V(H_1) \cap A_1 \neq \emptyset$  and  $P_1$  must intersect all maximal cliques of  $A_1$  (if not this clique combined with  $u_2$  and  $K$  would be a maximal clique of  $G_1$  that  $H_1$  does not intersect, a contradiction). This implies that  $P_1$  intersects all maximal cliques of  $A_1 \cup A_2 \cup K$ . Hence,  $H$  intersects all maximal cliques of  $G$ .

Observe that  $G[V(H) \cup K^+]$  is obtained from  $G_1[V(H_1) \cup K_1^+]$  and  $G_2[(V(H_2) \setminus \{u_1\}) \cup K_2^+]$  by gluing along  $K_2^+$  (possibly empty). And  $G_2[(V(H_2) \setminus \{u_1\}) \cup K_2^+]$  has Property  $Q_k$  because it is an induced subgraph of  $G_2[V(H_2) \cup K_2^+]$ . So by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

**Case 2c**  $V(H_1) \cap K = \emptyset$  and  $u_2 \in V(H_1)$ .

$H_1$  contains all vertices of  $K^{\text{in}}$  so  $K^{\text{in}} \cap K = \emptyset$ . Also  $K^{\text{in}} \not\subseteq A_1$ , for otherwise  $u_2 \in V(H_1)$  is complete to  $K^{\text{in}}$ .

Choose any vertex  $v_2 \in A_2$ , set  $K_2^{\text{in}} = \{v_2\}$  and  $K_2^+ = (K^+ \cap K) \cup \{u_1\}$ . Note that  $K_2^+$  is not a maximal clique of  $G_2$  since  $A_2 \neq \emptyset$ . Let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ . Since  $H_2$  contains no  $K_2^{\text{in}}$ -complete vertex, we have  $V(H_2) \cap K = \emptyset$ .

Let  $H = G[(V(H_1) \setminus \{u_2\}) \cup V(H_2)]$ . We now check that  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

It is obvious that  $H$  contains all vertices of  $K^{\text{in}}$  and  $H$  contains no vertex of  $K^+$ . Vertices of  $H_1$  are not complete to  $K_1^{\text{in}} = K^{\text{in}}$ . Also a vertex  $v \in V(H_2)$  is not complete to  $K^{\text{in}}$ , for otherwise,  $v \in A_2$  since  $K^{\text{in}} \cap K = \emptyset$ . But then  $v$  is adjacent to a vertex of  $X_1 \setminus A_1$  (because as noted already  $K^{\text{in}} \not\subseteq A_1$ ), a contradiction. Hence,  $H$  contains no  $K^{\text{in}}$ -complete vertex.

Since  $H_2$  intersects all maximal cliques of  $G_2$  then  $P_2 = V(H_2) \cap A_2 \neq \emptyset$  and  $P_2$  must intersect all maximal cliques of  $A_2$  (if not this clique combined with  $u_1$  and  $K$  would be a maximal clique of  $G_2$  that  $H_2$  does not intersect, a contradiction). This implies that  $P_2$  intersects all maximal cliques of  $A_1 \cup A_2 \cup K$ . Hence,  $H$  intersects all maximal cliques of  $G$ .

Consider  $V(H_1) \cap A_1 = \emptyset$ ,  $G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[(V(H_1) \setminus \{u_2\}) \cup K_1^+]$  and  $G_2[V(H_2) \cup (K^+ \cap K)]$  (both graphs have Property  $Q_k$ ) by gluing along a clique  $K^+ \cap K$  (possibly empty), so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ . In the case  $V(H_1) \cap A_1 \neq \emptyset$ ,  $G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[V(H_1) \cup K_1^+]$  and  $G_2[V(H_2) \cup K_2^+]$  (both graphs have Property  $Q_k$ ) by an amalgam composition, so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

We are done when  $K^{\text{in}} \cup K^+ \subseteq X_1 \cup K$ .

Hence,  $G$  has Property  $Q_{k+1}$ .

### Substitution

Since the amalgam is already treated, by Lemma 2.1, it is enough to prove that Property  $Q_k$  is closed under adding a universal vertex. Let  $G$  be a graph obtained from a graph  $G'$  by adding a universal vertex  $v$ . We suppose that  $G'$  has Property  $Q_{k+1}$ . Let  $(K^{\text{in}}, K^+)$  be a constraint for  $G$ . There are two cases.

**Case 1:**  $v \in K^{\text{in}}$ .

Let  $H = G[K^{\text{in}}]$ . So  $H$  contains all vertices of  $K^{\text{in}}$ , no vertex of  $K^+$  and no  $K^{\text{in}}$ -complete vertex. Every maximal clique of  $G$  contains  $v$  since  $v$  is a universal vertex, this means  $H$  intersects all maximal cliques of  $G$ .  $G[V(H) \cup K^+]$  is a clique, so it has Property  $Q_1$ . Therefore,  $G[V(H) \cup K^+]$  has Property  $Q_k$ . Hence,  $H$  is a  $k$ -splitter for  $(G, K^{\text{in}}, K^+)$ .

**Case 2:**  $v \notin K^{\text{in}}$ .

Let  $H$  be a splitter for  $(G', K^{\text{in}}, K^+ \setminus \{v\})$ .

We have  $v \notin H$  so in  $G$ ,  $H$  contains all vertices of  $K^{\text{in}}$ , no vertex of  $K^+$ , no  $K^{\text{in}}$ -complete vertex.

Assume  $H$  does not intersect a maximal clique  $P$  ( $P \neq K^+$ ) of  $G$ . It follows that  $H$  does not intersect a maximal clique  $P \setminus \{v\}$  of  $G'$  since  $v$  is a universal vertex of  $G$ , a contradiction.

Hence,  $H$  intersects all maximal cliques of  $G$  (except  $K^+$  when  $K^+$  is a maximal clique).

If  $v \notin K^+$  then  $G[V(H) \cup K^+]$  has Property  $Q_k$  since  $G[V(H) \cup K^+] = G'[V(H) \cup K^+]$ . If  $v \in K^+$  then  $G[V(H) \cup K^+]$  is obtained from graph  $G'[V(H) \cup (K^+ \setminus \{v\})]$  by adding a universal vertex  $v$ . So by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

Hence,  $G$  has Property  $Q_{k+1}$ .

### Proper 2-cutset

We suppose that  $(X_1, X_2, a, b)$  is a split for a proper 2-cutset. For  $i = 1, 2$ , the block of decomposition  $G_i$  with respect to this split is the graph obtained from  $G[X_i \cup \{a, b\}]$  by adding a vertex  $x_{3-i}$  complete to  $\{a, b\}$ , so  $G$  is obtained from  $G_1$  and  $G_2$  by a proper 2-cutset composition. We suppose that  $G_1$  and  $G_2$  have Property  $Q_{k+1}$ .

Let  $(K^{\text{in}}, K^+)$  be a constraint for  $G$ . Up to symmetry, we may assume that  $K^{\text{in}} \cup K^+ \subseteq X_1 \cup \{a, b\}$ . Set  $K_1^{\text{in}} = K^{\text{in}}, K_1^+ = K^+$  and let  $H_1$  be a splitter for  $(G_1, K_1^{\text{in}}, K_1^+)$ . There are two cases.

**Case 1:**  $K^+ \cap \{a, b\} \neq \emptyset$ .

Because  $K^+$  is a clique, we have  $K^+ \cap \{a, b\} \neq \{a, b\}$ . Without loss

of generality, we can assume  $K^+ \cap \{a, b\} = \{a\}$ . Note that  $b \notin K^{\text{in}}$  since  $K^+ \cup K^{\text{in}}$  is a clique of  $G$ . It follows  $K^{\text{in}} \subseteq X_1$ .

The set  $H_1$  intersects maximal clique  $\{a, x_2\}$  and  $a \notin V(H_1)$  since  $H_1$  does not contain vertices of  $K_1^+$ . It follows  $x_2 \in V(H_1)$ .

- If  $b \in V(H_1)$  then set  $K_2^{\text{in}} = \emptyset, K_2^+ = \{a, x_1\}$  and let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ .

Let  $H = G[(V(H_1) \setminus \{x_2\}) \cup V(H_2)]$  so  $H$  intersects all maximal cliques of  $G$  (except possibly  $K^+$  when  $K^+$  is a maximal clique) because all such cliques are either in  $G_1$  or in  $G_2$ .

$H$  contains all vertices of  $K^{\text{in}}$  and no vertex of  $K^+$ . Also  $b$  is not complete to  $K^{\text{in}}$  since  $b \in V(H_1)$  and  $H_2 \setminus \{b\}$  is anticomplete to  $K^{\text{in}}$ . This implies that  $H$  contains no  $K^{\text{in}}$ -complete vertex.

$G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[V(H_1) \cup K_1^+]$  and  $G_2[V(H_2) \cup K_2^+]$  (both graphs have Property  $Q_k$ ) by a proper 2-cutset composition so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

- If  $b \notin V(H_1)$  then set  $K_2^{\text{in}} = \{x_1\}, K_2^+ = \{a\}$  and let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ .

Let  $H = G[(V(H_1) \setminus \{x_2\}) \cup (V(H_2) \setminus \{x_1\})]$  so  $H$  intersects all maximal cliques of  $G$  (except possibly  $K^+$  when  $K^+$  is a maximal clique) because all such cliques are either in  $G_1$  or in  $G_2$ .

The graph  $H$  contains all vertices of  $K^{\text{in}}$  and no vertex of  $K^+$ . Also  $H_2$  contains no  $K_2^{\text{in}}$ -complete vertex so  $H_2$  does not contain  $a$  or  $b$ . It follows  $H_2$  is anticomplete to  $K^{\text{in}}$ . Hence,  $H$  contains no  $K^{\text{in}}$ -complete vertex.

$G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[(V(H_1) \setminus \{x_2\}) \cup K_1^+]$  and  $G_2[(V(H_2) \setminus \{x_1\}) \cup K_2^+]$  (both graphs have Property  $Q_k$ ) by a gluing at clique  $\{a\}$ , so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

**Case 2:**  $K^+ \cap \{a, b\} = \emptyset$ .

- If  $V(H_1) \cap \{a, b\} = \emptyset$  then  $K^{\text{in}}$  does not contain  $a$  or  $b$ . Set  $K_2^{\text{in}} = \{x_1\}, K_2^+ = \emptyset$  and let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ .

Let  $H = G[(V(H_1) \setminus \{x_2\}) \cup (V(H_2) \setminus \{x_1\})]$  so  $H$  intersects all maximal cliques of  $G$  (except possibly  $K^+$  when  $K^+$  is a maximal clique) because all such cliques are either in  $G_1$  or in  $G_2$ .

The graph  $H$  contains all vertices of  $K^{\text{in}}$  and no vertex of  $K^+$ . Also  $H_2$  contains no  $K_2^{\text{in}}$ -complete vertex so  $H_2$  does not contain  $a$  or  $b$ . It follows  $H_2$  is anticomplete to  $K^{\text{in}}$ . Hence,  $H$  contains no  $K^{\text{in}}$ -complete vertex.

The graph  $G[V(H) \cup K^+]$  is obtained from two disjoint graphs  $G_1[(V(H_1) \setminus \{x_2\}) \cup K_1^+]$  and  $G_2[V(H_2) \setminus \{x_1\}]$  (both graphs have Property  $Q_k$ ) so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

- If  $V(H_1) \cap \{a, b\} = \{a, b\}$  and  $\{a, b\} \cap K^{\text{in}} = \emptyset$  then set  $K_2^{\text{in}} = \{a\}$ . If  $\{a, b\} \cap K^{\text{in}} \neq \emptyset$  then set  $K_2^{\text{in}} = \{a, b\} \cap K^{\text{in}}$  and  $K^{\text{in}}$  is a clique so up to symmetry, we may assume that  $K_2^{\text{in}} = \{a\}$ .

Set  $K_2^+ = \emptyset$  and let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ . Because  $H_2$  does not contain  $\{a\}$ -complete vertices,  $x_1 \notin V(H_2)$ , so  $H_2$  contains  $b$  since  $H_2$  intersects all maximal cliques.

Let  $H = G[(V(H_1) \setminus \{x_2\}) \cup (V(H_2) \setminus \{a\})]$  so  $H$  intersects all maximal cliques of  $G$  (except possibly  $K^+$  when  $K^+$  is a maximal clique) because all such cliques are either in  $G_1$  or in  $G_2$ .

The graph  $H$  contains all vertices of  $K^{\text{in}}$  and no vertex of  $K^+$ . Also  $H_2$  contains no  $K^{\text{in}}$ -complete vertex since  $H_2$  does not contain  $\{a\}$ -complete vertex. Hence,  $H$  contains no  $K^{\text{in}}$ -complete vertex.

The graph  $G[V(H) \cup K^+]$  is obtained from graphs  $G_1[(V(H_1) \setminus \{x_2\}) \cup K_1^+]$  and  $G_2[V(H_2) \setminus \{a\}]$  (both graphs have Property  $Q_k$ ) by gluing along a clique  $\{b\}$ , so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .

- If  $V(H_1) \cap \{a, b\} \neq \emptyset$  and  $V(H_1) \cap \{a, b\} \neq \{a, b\}$  then without loss of generality, we can assume that  $V(H_1) \cap \{a, b\} = \{a\}$ . Because  $H_1$  intersects the maximal clique  $\{x_2, b\}$ ,  $x_2 \in V(H_1)$ . Hence,  $K^{\text{in}} \neq \{a\}$  (since if  $K^{\text{in}} = \{a\}$ , then  $x_2 \notin V(H_1)$ , a contradiction).

Set  $K_2^{\text{in}} = \{x_1\}$ ,  $K_2^+ = \{a\}$  and let  $H_2$  be a splitter for  $(G_2, K_2^{\text{in}}, K_2^+)$ . Because  $H_2$  does not contain  $\{x_1\}$ -complete vertex,  $H_2$  does not contain  $a$  or  $b$ .

Let  $H = G[(V(H_1) \setminus \{x_2\}) \cup (V(H_2) \setminus \{x_1\})]$  so  $H$  intersects all maximal cliques of  $G$  (except possibly  $K^+$  when  $K^+$  is a maximal clique) because all such cliques are either in  $G_1$  or in  $G_2$ .

The graph  $H$  contains all vertices of  $K^{\text{in}}$  and no vertex of  $K^+$ . Also  $H$  contains no  $K^{\text{in}}$ -complete vertex since  $K^{\text{in}} \neq \{a\}$ .



The graph  $G[V(H) \cup K^+]$  is obtained from two graphs  $G_1[(V(H_1) \setminus \{x_2\}) \cup K_1^+]$  and  $G_2[(V(H_2) \setminus \{x_1\}) \cup \{a\}]$  (both graphs have Property  $Q_k$ ) by gluing at clique  $\{a\}$ , so by the induction hypothesis,  $G[V(H) \cup K^+]$  has Property  $Q_k$ .  $\square$

Recall that the function  $f_k$  was defined in the previous section.

**Lemma 4.7** *For all  $k \geq 1$ , graphs with Property  $Q_k$  are  $\chi$ -bounded by the function  $f_k$ .*

PROOF — The proof is similar to the proof of Lemma 3.3. In the induction step, when we consider a graph with Property  $Q_{k+1}$ , we use the constraint  $(\emptyset, \emptyset)$  to find an induced subgraph with Property  $Q_k$  that intersects all maximal cliques.  $\square$

We can now prove the following theorem that is a seemingly new and non-trivial result. Note that  $\chi$ -boundedness of the class under consideration can easily be obtained by Theorem 2.5, but this approach would only provide an exponential function, while we provide a polynomial.

**Theorem 4.8** *The closure by the set of operations  $S = \{\text{gluing along a clique, substitutions, 1-join composition, amalgam compositions, gluing along a proper 2-cutset}\}$  of the class of graphs of triangle-free graphs of order at most  $k + 3$  is a class of graph that is  $\chi$ -bounded by  $f_k$  (in particular, by a polynomial of degree  $k$ ).*

PROOF — Every graph in the class has Property  $Q_k$ , either by Lemma 4.5 or by Theorem 4.6. So, by Lemma 4.7, it is  $\chi$ -bounded by  $f_k$ .  $\square$

## 5 Structure of long-unichord-free graphs

Recall that a *long-unichord* in a graph is an edge that is the unique chord of some cycle of length at least 5. A graph is *long-unichord-free* if it does not contain any long-unichord. In this section, we prove a decomposition theorem for long-unichord-free graphs. We obtain its proof somehow for free, by combining two known theorems.

The theorem below is proven in [10]. The original statement is slightly more precise, but this one is enough for our purpose. A *cap* in a graph is a cycle of length at least 5, with a unique chord  $ab$  such that  $a$  and  $b$  are at distance two along the cycle.

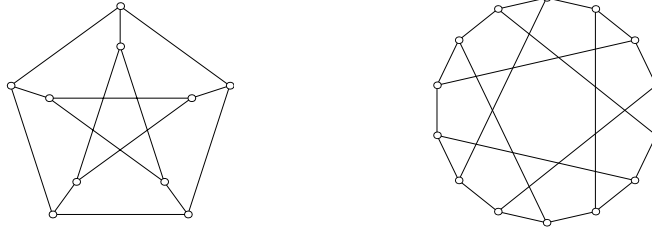


Figure 3: The Petersen and the Heawood graph

**Theorem 5.1 (Conforti, Cornuéjols, Kapoor and Vušković)** *If  $G$  is a connected cap-free graph, then either:*

- $G$  is chordal;
- $G$  is triangle-free;
- $G$  has a universal vertex;
- $G$  has a cutvertex;
- $G$  has an amalgam.

Here is useful corollary.

**Theorem 5.2** *If  $G$  is long-unichord-free, then either:*

- $G$  is chordal;
- $G$  is unichord-free;
- $G$  has a universal vertex;
- $G$  has a cutvertex;
- $G$  has an amalgam.

PROOF — Since a cap has a long unichord,  $G$  is cap-free. If  $G$  contains a triangle, one of the outcome follows from Theorem 5.1. And if  $G$  is triangle-free, then every unichord of  $G$  is a long-unichord. So  $G$  is unichord-free.  $\square$

The Petersen and the Heawood graphs are the graphs represented on Figure 3. The following theorem is proved in [25].

**Theorem 5.3 (Trotignon and Vušković)** *If  $G$  is a connected unichord-free graph, then either:*

- $G$  is a clique;
- $G$  is an induced subgraph of the Petersen graph;
- $G$  is an induced subgraph of the Heawood graph;
- $G$  is bipartite and one side of the bipartition is made of vertices of degree at most 2;
- $G$  has a cutvertex;
- $G$  has a 1-join (and therefore an amalgam);
- $G$  has a proper 2-cutset.

Our main decomposition theorem is the following.

**Theorem 5.4** *Let  $G$  be a connected long-unichord-free graph. Then either:*

- $G$  is an induced subgraph of the Petersen graph;
- $G$  is an induced subgraph of the Heawood graph;
- $G$  is chordal;
- $G$  is bipartite and one side of the bipartition is made of vertices of degree at most 2;
- $G$  has a universal vertex;
- $G$  has a cutvertex;
- $G$  has an amalgam;
- $G$  has proper 2-cutset.

PROOF — We apply Theorem 5.2. So either  $G$  satisfies one of the outcomes, or  $G$  is unichord free. In this last case, the result follows from Theorem 5.3.  $\square$

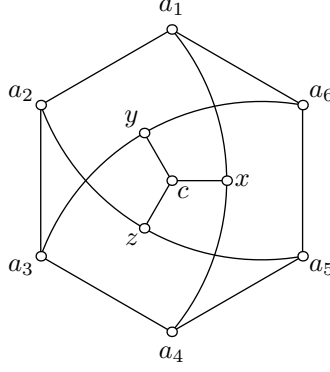


Figure 4: The Petersen graph

## 6 $\chi$ -bounding long-unichord-free graphs

Our main purpose is to prove that all long-unichord-free graphs are  $\chi$ -bounded. This is a direct consequence of Theorems 5.2, 2.5 and the fact proved in [25] that unichord-free graphs are  $\chi$ -bounded. But this approach would only provide an exponential bound (because of Theorem 2.5). Here, by using Property  $Q_k$ , we prove that the class is  $\chi$ -bounded by a polynomial.

By Theorem 4.6 and 5.4, to prove that long-unichord-free graphs have Property  $Q_k$ , it is enough to prove that the basic graphs from Theorem 5.4 have Property  $Q_k$ . It turns out that most of these basic graphs are perfect : chordal graphs and bipartite graphs are perfect, and the Heawood graph is bipartite. So, all these graphs have Property  $Q_1$ , and Property  $Q_k$  for all  $k \geq 1$  by Lemma 4.3. So, the only problem is the Petersen graph, but it has Property  $Q_7$  by Lemma 4.5. Hence, we have a short proof that long-unichord-free graphs have all Property  $Q_7$ .

We now prove several lemmas needed to show that in fact, long-unichord-free graphs have Property  $Q_3$ . The only problem is to handle the Petersen graph. We rely on the labeling of the Petersen graph represented on Figure 4. We first observe that  $Q_3$  is best possible: by setting  $K^{\text{in}} = \{c\}$  and  $K^+ = \{x\}$ , it can be checked that the Petersen graph does not have Property  $Q_2$ . Indeed, the splitter  $H$  cannot contain  $y$  and  $z$  that are  $K^{\text{in}}$ -complete, so it would have to contain  $a_5, a_6$  and also  $a_1$  and  $a_4$  (because it does not contain  $x$ ), so that  $H \cup \{x\}$  contains a  $C_5$ .

**Lemma 6.1** *The graph induced by the Petersen graph on  $\{a_1, a_2, a_3, a_4, a_5, a_6, x, c\}$  has Property  $Q_2$*

PROOF — All the odd cycles of the graph under consideration go through  $a_1$ ,  $x$  and  $a_4$ , so that whatever  $K^{\text{in}}$  and  $K^+$ , one of these three vertices is neither in  $K^+$  nor in the splitter obtained by taking all other vertices. It follows that a splitter  $H$  such that  $V(H) \cup K^+$  induces a perfect graph exists.  $\square$

**Lemma 6.2** *The Petersen graph has Property  $Q_3$ .*

PROOF — We rely on the representation of the Petersen graph  $G$  given on Figure 4. It is well known that all pairs of adjacent vertices in the Petersen graph are equivalent, that is for  $xy, x'y' \in E(G)$ , there is an automorphism  $\tau$  of  $G$  such that  $\tau(x) = x'$  and  $\tau(y) = y'$ . This property is referred to as *the symmetry of  $G$* . Let  $G'$  be an induced subgraph of  $G$ , and  $C = (K^{\text{in}}, K^+)$  be a constraint for  $G'$ .

If  $|K^{\text{in}}| = 2$  (so  $|K^+| = 0$ ), then because of the symmetry of  $G$ , we may assume that  $K^{\text{in}} = \{a_1, a_2\}$  and  $V(G') \cap \{a_1, \dots, a_6, c\}$  induces a 1-splitter for  $(G', K^{\text{in}}, K^+)$ .

If  $|K^{\text{in}}| = 1$ , then we may assume because of the symmetry of  $G$  that  $K^{\text{in}} = \{c\}$  and  $K^+ = \{x\}$  or  $K^+ = \emptyset$ . In both cases,  $V(G') \cap \{a_1, \dots, a_6, c\}$  induces a 2-splitter for  $(G', K^{\text{in}}, K^+)$ . Note that  $G[a_1, \dots, a_6, c, x]$  has Property  $Q_2$  by Lemma 6.1.

We may therefore assume that  $K^{\text{in}} = \emptyset$ . Because of the symmetry of  $G$ , we may assume that  $K^+ \subseteq \{x, a_1\}$ . We now observe as above that  $V(G') \cap \{a_1, \dots, a_6, c, x\} \setminus K^+$  induces a 1-splitter for  $(G', K^{\text{in}}, K^+)$  (it has Property  $Q_2$  as above).  $\square$

**Lemma 6.3** *Every long-unichord-free graph has Property  $Q_3$ .*

PROOF — As noted at the beginning of the section, a part from the Petersen graph, all basic graphs in Theorem 5.4 are perfect (and have therefore Property  $Q_1$ , and Property  $Q_3$  by Lemma 4.3). Also the Petersen graph has Property  $Q_3$  by Lemma 6.2. The result now follows from Theorems 5.4 and 4.6.  $\square$

**Theorem 6.4** *Long-unichord-free graphs are  $\chi$ -bounded by  $f_3$  (in particular, by a polynomial of degree 3).*

PROOF — Follows directly from Lemmas 6.3 and 4.7.  $\square$

## 7 Recognizing long-unichord-free graphs

In this section, we describe a polytime algorithm that decides whether a graph contains a long-unichord. The next lemma is straightforward to check (while a formal proof would be very long), so we prefer letting it without proof.

**Lemma 7.1** *Suppose that  $(X_1, X_2, A_1, A_2, K)$  is a split of an amalgam of a graph  $G$  and let  $H$  be a hole in  $G$ . Then, one of the following occurs:*

- (i)  $V(H) \subseteq X_1$ ;
- (ii)  $V(H) \subseteq X_2$ ;
- (iii)  $H = a_1 b_1 a_2 b_2 a_1$  where  $a_1, a_2 \in A_1$  and  $b_1, b_2 \in A_2$ ;
- (iv)  $H = b a_1 p_1 \dots p_k a_2 b$  where  $k \geq 1$ ,  $b \in A_2 \cup K$ ,  $a_1, a_2 \in A_1$ ,  $p_1, \dots, p_k \in X_1 \setminus A_1$ ;
- (v)  $H = a b_1 p_1 \dots p_k b_2 a$  where  $k \geq 1$ ,  $a \in A_1 \cup K$ ,  $b_1, b_2 \in A_2$ ,  $p_1, \dots, p_k \in X_2 \setminus A_2$ ;
- (vi)  $H = c p_1 \dots p_k c$  where  $k \geq 3$ ,  $c \in K$ ,  $p_1 \dots p_k \in X_1$ ;
- (vii)  $H = c p_1 \dots p_k c$  where  $k \geq 3$ ,  $c \in K$ ,  $p_1 \dots p_k \in X_2$ ;
- (viii)  $H = c_1 c_2 p_1 \dots p_k c_1 c_2$  where  $k \geq 2$ ,  $c_1, c_2 \in K$ ,  $p_1 \dots p_k \in X_1 \setminus A_1$ ;
- (ix)  $H = c_1 c_2 p_1 \dots p_k c_1 c_2$  where  $k \geq 2$ ,  $c_1, c_2 \in K$ ,  $p_1 \dots p_k \in X_2 \setminus A_2$ .

And we call them hole type 1, 2, 3, 4, 5, 6, 7, 8, 9 respectively.

A *decomposition* for a graph is either an amalgam or a cutvertex, and blocks of decomposition of these are defined in Section 2.

**Lemma 7.2** *Let  $G$  be a graph that admits a decomposition. Then  $G$  is a long-unichord-free graph if and only if the blocks of decomposition  $G_1$  and  $G_2$  are long-unichord-free.*

PROOF — If the decomposition under consideration is a cutvertex, the result is clear. So, suppose it is an amalgam. Because  $G_1$  and  $G_2$  are induced subgraph of  $G$ , if  $G$  is long-unichord-free then both  $G_1$  and  $G_2$  are long-unichord-free. Conversely, suppose that  $G_1$  and  $G_2$  are long-unichord-free and assume  $C$  is the cycle with unique chord with length at least 5 of  $G$ .

Suppose first that the unique chord divides  $C$  into two holes. We say that the type of  $C$  is  $XY$ , according to the types  $X$  and  $Y$  of the two holes with respect to the amalgam. Based on Lemma 7.1, and the property that these two holes share only one edges, there are 13 possible types for  $C$ : 11, 14, 16, 18, 22, 25, 27, 29, 46, 48, 59, 57, 89. If  $C$  belongs to type 89 then any vertex of  $A_2$  (this vertex exists since  $A_2 \neq \emptyset$ ) with hole type 8 induces a cycle with a long unichord in  $G_1$ , a contradiction. If  $C$  belongs to the remaining types, either  $G_1$  or  $G_2$  contains  $C$ , contradiction.

Hence, we may assume that the unique chord divides  $C$  into a hole and a triangle ( $C$  is a cap). This also means that  $C$  consists of a hole  $H$  plus a vertex  $x$  that is adjacent to two adjacent vertices of this hole. If  $H$  is of type 3, then  $x$  does not exist. If  $H$  is of type 1, 2, 4, 5, 6, 7, every choice of  $x$  makes  $C$  belong to  $G_1$  and  $G_2$ . If  $H$  is of type 8 (or 9), this hole with a vertex of  $A_2$  (or  $A_1$ ) induces a cycle with a long-unichord in  $G_1$ , a contradiction.

This proves the lemma.  $\square$

We now describe our algorithm (similar to the algorithm to recognize cap-free graph from [10]). A graph is *basic* if it is chordal or unichord-free. A *decomposition tree*  $T_G$  of a graph  $G$  is defined as follows :

- The root of  $T_G$  is  $G$ .
- If some node  $H$  of  $T_G$  is not basic and has a universal vertex, then its unique child is  $H \setminus X$  where  $X$  is the set of all universal vertices of  $H$ .
- If some node of  $T_G$  is not basic, has no universal vertex and has a decomposition, then its children are its blocks of decomposition.
- All nodes not handled in the previous cases are leaves of  $T_G$  (to be more specific: basic nodes and nodes that are not basic, without universal vertices and decomposition).

Note that the definition is not fully deterministic since different decompositions can be present in a graph. In this case, one of the decomposition should be used (so a graph may have different decomposition trees). Note that every graph has a decomposition tree, and that every decomposition tree of a graph is finite (because the children of a given node are smaller than the node).

For every graph  $G$  we define

$$f(G) = \max(E(\overline{G}), 1) .$$

**Lemma 7.3** *Suppose that  $G$  is a non-basic graph, with no universal vertex and with an amalgam or a cutvertex. Let  $G_1$  and  $G_2$  be the blocks of decomposition of  $G$  with respect to this amalgam or cutvertex. Then  $f(G_1) + f(G_2) \leq f(G)$ .*

PROOF — If for  $i = 1, 2$  we have  $f(G_i) = |E(\overline{G_i})|$ , then every pair  $\{u, v\}$  such that  $uv \notin G_i$  can be associated injectively to a similar pair in  $G$ , so the inequality holds. Hence, we may assume that  $f(G_1) = 1$ .

If we have  $f(G_2) = |E(\overline{G_2})|$ , then every pair  $\{u, v\}$  such that  $uv \notin G_2$  can be associated injectively to a similar pair in  $G$ . Since  $G$  has no universal vertex, some vertex  $v \in V(G_1) \cap V(G)$  has a non-neighbor in  $G$  and provides an extra non-adjacent pair in  $G$ , so that the inequality holds. Hence, we may assume that  $f(G_2) = 1$ .

So, we just have to check that  $f(G) \geq 2$ . This is the case because  $G$  is not basic, so it is not chordal and contains a chordless cycle of length at least 4 (that provides at least two non-adjacent pairs).  $\square$

The next lemma is implicitly proved in [11] (as Corollary 2.16), but the machinery there is much heavier and relies on many definitions, so we prefer to give our own simple proof. Note that in [12], it is claimed without proof that any graph with an amalgam should have an amalgam such that at least one block of decomposition has no amalgam. Such a result would imply the existence of a decomposition tree of linear size, but unfortunately, in [11], a counter-example to the claim is provided.

**Lemma 7.4** *Any decomposition tree of a graph  $G$  has at most  $O(n^2)$  nodes.*

PROOF — If a graph  $H$  has a universal vertex  $v$ , then  $f(H) = f(H \setminus v)$ . Hence, by Lemma 7.3 the number of leaves of  $T_G$  is at most  $f(G) \leq O(n^2)$ . Since removing the set of universal vertices can be done at most once to any node of the decomposition tree, we obtain the bound  $O(n^2)$ .  $\square$

**Lemma 7.5** *A graph is long-unichord-free if and only if all the leaves of its decomposition tree are basic.*

PROOF — Follows directly from Lemma 7.2 and Theorem 5.2.  $\square$

**Theorem 7.6** *Deciding whether an input graph  $G$  has a long-unichord can be performed in time  $O(n^4 m^2)$  (where  $n = |V(G)|$  and  $m = |E(G)|$ ).*



PROOF — The first step is to build a decomposition tree for  $G$ . Deciding whether a graph is basic can be performed in time  $O(nm)$  (see [25] for unichord-free graphs and [22] for chordal graphs). Finding a universal vertex, a cutvertex or an amalgam can be performed in time  $O(n^2m)$  (see [12] for the amalgam and the other claims are trivial). So, the tree can be constructed. Once the tree is given, the algorithm checks whether all leaves are basics, and by Lemma 7.5, this decides whether the graph is long-unichord-free.

**Complexity analysis:** the most expensive step is to find an amalgam in time  $O(n^2m)$ , and it is performed at most  $O(n^2)$  times by Lemma 7.4.  $\square$

## 8 Open questions

As observed by Esperet (personal communication), no counter-example to the following statement is known: every hereditary  $\chi$ -bounded class is  $\chi$ -bounded by some polynomial. This would have several consequences. For instance, consider the following well known conjecture:

**Conjecture 8.1 (Erdős and Hajnal, see [6])** *For every hereditary class  $C$  of graphs, except the class of all graphs, there exist a constant  $c$  such that every graph  $G$  in  $C$  contains a clique or a stable set on at least  $|V(G)|^c$  vertices.*

The conjecture is true for any class  $C$  that is  $\chi$ -bounded by some polynomial (because, if for some  $d$ ,  $\chi(G) \leq \omega(G)^d$ , then  $\alpha(G)\omega(G)^d \geq |V(G)|$ ). Let us see several results and open problems related to the question of polynomial  $\chi$ -bounds.

### “Big” $\chi$ -bounding functions

First, let us recall a known observation, that is seemingly unpublished. For every integers  $s, t$ , define the Ramsey number  $R(s, t)$  as the smallest integer  $n$  such that every graph on  $n$  vertices contains a stable set of size  $s$  or clique of size  $t$ . By celebrated theorems of Bohman and Keevash [4], and Ajtai, Komlós and Szemerédi [1], for every fixed interger  $s$ , there exists constants  $c_s, c'_s$  such that:

$$c_s t^{s/2} \leq R(s, t) \leq c'_s t^s$$

The inequalities that we give are not as good than the ones in papers, but are enough for our purpose. Now, for every integer  $s$ , consider the class of graphs that do not contain a stable set of size  $s$  and denote it by  $C_s$ . This class is clearly hereditary. By the definition of Ramsey numbers, a graph  $G$  in  $C_s$  has less than  $R(s, \omega(G) + 1)$  vertices, and therefore chromatic number less than  $R(s, \omega(G) + 1)$ . It follows that

$$\chi(G) \leq c'_s(\omega(G) + 1)^s$$

Hence,  $C_s$  is  $\chi$ -bounded by some polynomial. But the interesting point about  $C_s$  is that every graph  $G$  in  $C_s$  has chromatic number at least  $|V(G)|/(s-1)$  (because the maximum stable set in  $G$  has size at most  $s-1$ ). Hence, if for every integer  $\omega$  we choose in  $C_s$  a graph  $G_\omega$  on  $R(s, \omega + 1) - 1$  vertices, we have :

$$\chi(G_\omega) \geq \frac{|V(G)|}{(s-1)} \geq \frac{R(s, \omega + 1) - 1}{s-1} \geq \frac{c_s}{s-1} \omega^{s/2}$$

It follows that there cannot exist an integer  $d$  such that every hereditary class is  $\chi$ -bounded by a polynomial of degree  $d$ . To our knowledge, this example is the best attempt so far to construct a class with a “big”  $\chi$ -bounding function.

### Constraints on $\chi$ -bounding functions

More generally, what are the functions that can be the minimal  $\chi$ -bounding function of some hereditary class of graph? We suppose that the class contains complete graphs of all sizes (otherwise, that class is  $\chi$ -bounded by a constant, and any discussion about how big can be the  $\chi$ -bounding function is pointless). Let  $f$  be such a function. Let us see evidences that there are restrictions on  $f$ . Clearly,  $f(0) = 0$  and  $f(1) = 1$  and for all integers  $x$ ,  $f(x) \geq x$ . By classical constructions of triangle-free graph with high chromatic number, we can see that  $f(2)$  can be any integer higher than 1.

Suppose now that  $f(2) = 2$ . Can  $f(3)$  be any integer ? It is not the case. Since  $f(2) = 2$ , we know that any graph  $G$  in the class contains no odd hole. Therefore, by a theorem of Chudnovsky, Robertson, Seymour and Thomas [8], stating that any odd-hole-free graph with no  $K_4$  is 4-colourable, we know that  $f(3) \leq 4$ . More generally, odd-hole-free graphs are  $\chi$ -bounded by a theorem of Scott and Seymour [23]. So, in fact, for all integers  $x$ ,  $f(x) \leq g(x)$  where  $g(x)$  is the function proven to  $\chi$ -bound odd-hole-free graphs (unfortunately, this function  $g$  is bigger than an exponential).

This seems to be the only general statements that can be made about  $\chi$ -bounding functions in general. In particular, the following is still open : suppose  $f(2) = 3$ . Can  $f(3)$  be any integer ?

### Polynomial $\chi$ -bounding functions

For each particular  $\chi$ -bounded class, one might want try to prove a polynomial bound. The remarks above suggest that the most important class to think of should be odd-hole-free graphs. Also,  $P_k$ -free graphs should be of interest. Because they form the simplest case where the so-called method of extending a path developped by Gyárfás can be applied (see [17]). And this method seems to be the most succesfull to provide proofs of  $\chi$ -boundedness, see [23, 9] for instance. However, the method notoriously produces exponential bounds.

The following is still open for all integers  $k$  greater than 4 : is the class of  $P_k$ -free graphs  $\chi$ -bounded by a polynomial ?

### Polynomial $\chi$ -bounds and decomposition

It seems that the most successful attempts to prove polynomial  $\chi$ -bounds make use of decomposition theorems. It is therofore interesting to provide proofs that operations preserve the property of being  $\chi$ -bounded by a polynomial. This is known only for gluing along a clique (trivial), substitutions (see [7]) and gluing along a fixed number of vertices (see [20]). Is it true for 1-join compositions and amalgams ?

## References

- [1] M. Ajtai, J. Komlós, and E. Szemerédi. A note on Ramsey numbers. *Journal of Combinatorial Theory, Series A*, 29:354–360, 1980.
- [2] C. Berge and V. Chvátal, editors. *Topics on Perfect Graphs*, volume 21 of *Annals of Discrete Mathematics*. North Holland, Amsterdam, 1984.
- [3] C. Berge and P. Duchet. Strongly perfect graphs. In Berge and Chvátal [2], pages 57–61.
- [4] T. Bohman and P. Keevash. The early evolution of the  $h$ -free process. *Inventiones mathematicae*, 181:291–336, 2010.
- [5] M. Burlet and J. Fonlupt. A polynomial time algorithm to recognize a Meyniel graph. In Berge and Chvátal [2], pages 225–252.

- [6] M. Chudnovsky. The Erdős-Hajnal conjecture — a survey. *Journal of Graph Theory*, 75:178–190, 2014.
- [7] M. Chudnovsky, I. Penev, A.D. Scott, and N. Trotignon. Substitution and  $\chi$ -boundedness. *Journal of Combinatorial Theory, Series B*, 103(5):567–586, 2013.
- [8] M. Chudnovsky, N. Robertson, P.D. Seymour, and R. Thomas.  $k_4$ -free graphs with no odd holes. *Journal of Combinatorial Theory, Series B*, 100(3):313–331, 2010.
- [9] M. Chudnovsky, A. Scott, and P.D. Seymour. Induced subgraphs of graphs with large chromatic number. II. Three steps towards Gyárfás’ conjectures. *Journal of Combinatorial Theory, Series B*, 118:109–128, 2016.
- [10] M. Conforti, G. Cornuéjols, A. Kapoor, and K. Vušković. Even and odd holes in cap-free graphs. *Journal of Graph Theory*, 30:289–308, 1999.
- [11] M. Conforti, B. Gerards, and K. Pashkovich. Stable sets and graphs with no even holes. *Mathematical Programming*, 153(1):13–39, 2015.
- [12] G. Cornuéjols and W.H. Cunningham. Composition for perfect graphs. *Discrete Mathematics*, 55:245–254, 1985.
- [13] W.H. Cunningham. Decomposition of directed graphs. *SIAM Journal on Algebraic and Discrete Methods*, 3:214–228, 1982.
- [14] Zdenek Dvorak and Daniel Král’. Classes of graphs with small rank decompositions are  $\chi$ -bounded. *European Journal of Combinatorics*, 33(4):679–683, 2012.
- [15] J. Fonlupt and J.P. Uhry. Transformations which preserve perfectness and  $h$ -perfectness of graphs. In A. Bachem, M. Grötschel, and B. Korte, editors, *Bonn Workshop on Combinatorial Optimization*, pages 83–85. North-Holland, 1982. *Annals of Discrete Mathematics*, 16.
- [16] T. Gallai. Graphen mit triangulierbaren ungeraden Vielecken. *A Magyar Tudományos Akadémia — Matematikai Kutató Intézetének Közleményei*, 7:3–36, 1962.
- [17] A. Gyárfás. Problems from the world surrounding perfect graphs. *Zastowania Matematyki Applicationes Mathematicae*, 19:413–441, 1987.

- [18] L. Lovász. Normal hypergraphs and the perfect graph conjecture. *Discrete Mathematics*, 2:253–267, 1972.
- [19] I. Penev. Amalgams and  $\chi$ -boundedness. Manuscript available at <http://perso.ens-lyon.fr/irena.penev/>, 2014.
- [20] I. Penev, S. Thomassé, and N. Trotignon. Isolating highly connected induced subgraphs. arXiv:1406.1671, 2014.
- [21] G. Ravindra. Some classes of strongly perfect graphs. *Discrete Mathematics*, 206(1–3):197–203, 1999.
- [22] D.J. Rose, R.E. Tarjan, and G.S. Lueker. Algorithmic aspects of vertex elimination on graphs. *SIAM Journal on Computing*, 5:266–283, 1976.
- [23] A. Scott and P.D. Seymour. Induced subgraphs of graphs with large chromatic number. I. Odd holes. *Journal of Combinatorial Theory, Series B*, 121:68–84, 2016.
- [24] N. Trotignon. Perfect graphs: a survey. arXiv:1301.5149, 2013.
- [25] N. Trotignon and K. Vušković. A structure theorem for graphs with no cycle with a unique chord and its consequences. *Journal of Graph Theory*, 63(1):31–67, 2010.
- [26] A.A. Zykov. On some properties of linear complexes. *Matematicheskii Sbornik*, 24(2):163–188, 1949. In Russian.