COVERING 2-CONNECTED 3-REGULAR GRAPHS WITH DISJOINT PATHS

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Abstract. A path cover of a graph is a set of disjoint paths so that every vertex in the graph is contained in one of the paths. The path cover number $p(G)$ of graph G is the cardinality of a path cover with the minimum number of paths. Reed in 1996 conjectured that a 2-connected 3-regular graph has path cover number at most $\lceil n/10 \rceil$. In this paper, we confirm this conjecture.

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1. Introduction

A *path cover* of a graph is a set of disjoint paths that contain all the vertices of the graph. The *path cover number* of graph G, written as $p(G)$, is the cardinality of a path cover with the minimum number of paths.

Ore [11] initiated the study of path covers. A graph has path cover number 1 precisely when it has a Hamiltonian path. It is well-known that if the minimum degree of an n-vertex graph is at least $n/2$ then the graph is Hamiltonian. Because of its natural connection with hamiltonian graphs, people were interested in the sufficient conditions for a graph to have path cover number at most $k \geq 2$, see, for example, [3, 8]. In more recent years, path covers have been used to study other graph parameters, such as domination numbers [12, 5, 6], $L(2,1)$-labelling [2], independence number [3], and graphic-TSP [1], just to name a few.

Every n-vertex graph have a path cover of order at most n, and one would imagine that a graph with more edges will require fewer paths to cover. However, an n-vertex graph with minimum degree t could have path cover number as high as $n-2t$, for example $K_{t,n-t}$. Thus, we are more interested in path cover of regular graphs. Jackson [4] showed that 2-connected k-regular graphs with at most $3k+1$ vertices have a hamiltonian path (actually they have a hamiltonian cycle except the Petersen graph), thus the path cover number is 1. Magnant and Martin [7] studied path cover numbers of k-regular graphs for $k \geq 3$, and they showed that for $k \leq 5$, a k-regular graph has path cover number at most $n/(k+1)$, which they conjectured to be true for $k > 5$. Note that if every component of a graph G is a clique of $k+1$ vertices, then $p(G) = n/(k+1)$, thus the bound is sharp for general graphs. As they pointed out, it is more difficult to find the path cover numbers of connected regular graphs.

The following example gives a general lower bound for the path cover numbers of connected k-regular graphs. Take $K_{2,k-1}$ and replace every vertex of degree 2 with K_{k+1}^- (a $k+1$-clique minus an edge), and call this graph H, in which two vertices have degree $k-1$ and the rest have degree k. Now let G be the k-regular graph with n vertices formed from $\frac{n}{k^2+1}$ pairwise disjoint H by adding $\frac{n}{k^2+1}$ edges to link them in a ring. It is not hard to see that the path cover number of G is at least $\frac{n(k-3)}{k^2+1}$ for $k \geq 5$. Therefore for $k \geq 13$, one cannot find a path cover with fewer than $n/(k+4)$

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paths in connected k-regular graphs (note that the examples are actually 2-connected). Some more examples from [9, 10] also show that $n/(k+4)$ paths are necessary.

Intuitively, one may need more paths to cover the vertices when there are fewer edges in the graphs. This initiated the study of path covers for connected 3-regular graphs. Reed [12] showed that *a connected* 3*-regular graph with* n *vertices has path cover number at most* $\lceil n/9 \rceil$, and also gave examples that need $\lceil n/9 \rceil$ paths. He conjectured [12] that it suffices to use at most $\lceil n/10 \rceil$ paths to cover 2-connected 3-regular graphs. In this article, we confirm this conjecture.

Theorem 1.1. *Every* 2*-connected* 3*-regular graph with* $n \geq 10$ *vertices has path cover number at most* $n/10$.

It follows that every 2-connected 3-regular graph with at most 20 vertices contains a hamiltonian path. Reed [12] gave the following example to show that one cannot improve $\lceil n/10 \rceil$ in general: let $C = u_1v_1u_2v_2 \dots u_kv_k$ be a cycle of $2k$ vertices, let H be the the graph obtained from the Petersen graph by removing an edge, say uv, and let G be the graph obtained by replacing edge u_iv_i for $1 \leq i \leq k$ with H so that $u = u_i$ and $v = v_i$. He claimed that the path cover number of G is $n/10$, based on the observation that one needs a path to cover each H. However, we can use one path to cover two consecutive copies of H, thus only need $n/20$ paths to cover $V(G)$. Here we give infinitely many 2-connected 3-regular n-vertex graphs whose path cover numbers are at least $n/14$.

Theorem 1.2. *There are infinitely many* 2*-connected* 3*-regular* n*-vertex graphs whose path cover numbers are at least* $n/14$.

Proof. Let G be an arbitrary 2-connected 3-regular graph, and let H be the graph obtained from G by replacing each edge of G with a K_4^- (that is, delete the edge, and connect two endpoints of the edge to the two degree-2 vertices on K_4^-, respectively). Then $n(H) = n(G) + 4 \cdot \frac{3n(G)}{2} = 7n(G)$. We now show that $p(H) \geq n(G)/2 = n(H)/14$.

Let $\mathcal{P}$ be a path cover of H. Let $e = uv$ be the edge between $u \in V(G)$ and v in some K_4^-. Then either uv is on a path of $\mathcal{P}$, or v is on some path in $\mathcal{P}$ that contains all vertices of the K_4^-. In the latter case, we may reroute the path so that v is an endpoint, thus extend the path to include the edge vu. Therefore, we may obtain a path decomposition $\mathcal{P}'$ of G (a set of edge-disjoint paths $\mathcal{P}'$ containing all the edges of G) with $|\mathcal{P}'| = |\mathcal{P}|$. Each path in $\mathcal{P}'$ contains a vertex in G as either an internal point or an endpoint, and only when it is an endpoint, the parity of its degree changes when we remove the edges on the path. But each path can only change the degree parities of at most two vertices in G. As G has $n(G)$ vertices whose degree parity need to be changed, there are at least $n(G)/2$ paths in $\mathcal{P}'$. Thus, $\mathcal{P}$ contains at least $n(H)/14$ paths. $\square$

It is an interesting question to determine the sharp bounds for the path cover numbers of 2-connected 3-regular graphs in terms of the orders of the graphs.

We will often use the following notation for a path and its segments. A *k-path* is a path of k vertices. For a k-path P, if $G[V(P)]$ contains a spanning cycle, we call it a *cyclic k-path* or a *k-cycle*, otherwise *non-cyclic*. A vertex on a non-cyclic path P is called *weighty* if it is adjacent to an endpoint of P by an edge not on P. If a path P contains vertex x, then we sometimes write P as P_v, and let v^-, v^+ be the vertices (neighbors) next to v on P, respectively. If the endpoints of P_v are x and y, then we also write P_v as xPy, or even as xPv^-vv^+Py. We will use uPv to denote the segment on P from u to v. If v is an endpoint of P_v, we sometime use P_vv to denote the path P_v with endpoint v. For other notation, we refer to West [13].

2. Sketch of the proof of Theorem 1.1

The idea of the proof of the theorem is quite simple. We consider a specially chosen minimal path cover $\mathcal{P}$, and assign a weight of 10 to each path in the cover initially. We then redistribute the weights among the paths and show that the final weight on each path is at most its order. It follows that the total weight is $10|\mathcal{P}|$ on one hand, and at most n on the other hand, therefore $|\mathcal{P}| \le n/10$. The difficulty lies on the choice of minimal path cover and on the way to redistribute the weights. Below we give some insights on how we make the choices.

In the minimal path covers we can show that none of the paths are single vertices or contain a spanning cycle. We may think that the weights of the paths are all on the endpoints, 5 for each. Let x be an endpoint of a path P, and xu be an edge not on $E(P)$. Clearly, u is not an endpoint of another path, or we will combine the paths into one to get a cover with fewer paths. We transfer a weight of 2 from x to u. The vertex u is called *weighty* if u is on P and *heavy* otherwise. Then on each path, the four edges incident with the endpoints will send out a weight of 8 and only a weight of 1 remains on each endpoint.

Note that heavy vertices are not next to each other on the paths, or we can rearrange the paths to get a cover with fewer paths. Therefore, if there are no consecutive vertices on paths that are either heavy or weighty, and the number of vertices on a path is not odd, then the final weight on each path is at most the number of vertices on the path, as desired. Therefore the problematic cases are the existence of consecutive weighty and heavy vertices, or the number of vertices is odd and every other vertex on a path is weighty or heavy. They force us to identify more vertices to transfer weights from one path to another, and suggest such vertices to be the ones incident with a vertex whose neighbors are weighty and heavy on the path. It turns out that we only need to use such vertices, namely *P(seudo)E(ndpoint)-vertices*, to transfer a weight of 1. PE-vertices share a lot of common features with the endpoints. For example, a light vertex cannot be next to a heavy vertex on a path, where a light vertex is the neighbor of a PE-vertex not on the path of the PE-vertex. Light vertices make the proof more complicated.

There are still bad situations that a path may have too much weight. For example, a 3-path with a heavy middle vertex, or a 5-path $P = xu_1u_2u_3y$ such that $xu_2, xu_3 \in E(G)$, or a 6-path $P = xu_1u_2u_3u_4y$ such that $xu_3, x'u_1, x'u_4 \in E(G)$ where x' is an endpoint of another path, or a 7-path $P = xu_1u_2u_3u_4u_5y$ such that $xu_4, x'u_1, x'u_5 \in E(G)$ where x' is an endpoint of another path. Fortunately, we can carefully define the optimal path covers to avoid all those situations.

Each pair of consecutive heavy/weighty vertices on a path contains a neighbor of the endpoints, so there are at most four such pairs on each path. To show that each path has no more weight than its number of vertices, we show that in each of the bad cases, the path has enough neutral vertices (vertices do not receive weights) and/or PE-vertices.

We define optimal path covers and study their properties in Section 3. The special vertices (heavy, light, PE-vertices) and their properties are studied in Section 4. Then in Section 5, we prove the main lemma that the total weight on each path does not exceed its order and finish the proof of the theorem.

3. Optimal path covers and their basic properties

Let G be a minimum counterexample to Theorem 1.1. Among all path covers of G, choose $\mathcal{P}$ to be an optimal path cover subject to the following:

(i) the number of paths is minimized.

(ii) subject to (i), the number of 1-paths is minimized.
(iii) subject to (i)-(ii), the number of 3-paths and cyclic paths is minimized.
(iv) subject to (i)-(iii), the number of bad endpoints is minimized, where an endpoint $x' \in P' \in \mathcal{P}$ is bad if (v1) x' is adjacent to $u_1, u_4 \in P$ and $xu_3 \in E(G)$, or (v2) x' is adjacent to $u_1, u_5 \in P$ and $xu_4 \in E(G)$, where $P = xu_1u_2u_3u_4 \dots u_ky \in \mathcal{P} - \{P'\}$.
(v) subject to (i)-(iv), the number of annoying endpoints is minimized, where an endpoint $x' \in P' = x'u'_1 \dots u'_l y'$ is annoying if $x'u'_{s+1}, x'u_i, xu'_{s-1}, u'_s u_{i+1} \in E(G)$ with $P = xu_1u_2 \dots u_k y \in \mathcal{P} - \{P'\}$ and $2 \le s \le l-1$.
(vi) subject to (i)-(v), the number of weighty vertices is minimized.
(vii) subject to (i)-(vi), for each non-cyclic path P, the number of vertices on P between the endpoints and their corresponding furthest neighbors on P is maximized.

We shall call a path cover satisfying the first t conditions above as $\mathcal{P}_t$. Thus $\mathcal{P}$ is $\mathcal{P}_7$, and $\mathcal{P}_0$ is just a path cover to G. Clearly, $\mathcal{P}_{i+1} \subseteq \mathcal{P}_i$, so $\mathcal{P}_{i+1}$ has all the properties that $\mathcal{P}_i$ has.

A *net* is a triangle whose three neighbors not on the triangle are distinct. The following was observed in [12].

Lemma 3.1. *The graph G contains no net.*

Proof. For otherwise, let $u_1u_2u_3$ be a triangle with $u_iu'_i \in E(G)$ such that u'_i's are distinct. Then we contract the triangle to a single vertex u and get a graph G'. Now G' has a path cover with at most $|V(G')|/10$ paths, but then we can get a path cover of G by replacing u with a path containing u_1, u_2, u_3. □

Lemma 3.2. *The following are true about $\mathcal{P}_1$:*

(1) Endpoints of different paths in $\mathcal{P}_1$ are not adjacent. In particular, there is no edge between cyclic paths or between a cyclic path and an endpoint of a non-cyclic path.
(2) every cyclic path has at least two neighbors not on the path.

Proof. (1) is true because our cover used the minimum number of paths. (2) is true because G has no cut-vertices. □

The following lemma from [7] says that a path cover subject to (i) and (ii) contains no 1-paths. We give an alternative proof here, whose idea will be used to prove more results about path covers.

Lemma 3.3 ([7]). *The path cover $\mathcal{P}_2$ contains no 1-paths.*

Proof. Suppose that $P \in \mathcal{P}_2$ consists of vertex v. By Lemma 3.2, v is not adjacent to an endpoint of another path. We also note that v is not adjacent to an interior vertex on a path P' of order at least 4, for otherwise, one can easily decompose $P \cup P'$ into two paths, each of order at least 2. Therefore v must be adjacent to the midpoints of 3-paths. Furthermore, if v is adjacent to the vertex $w \in P' = xwy$, then we may rearrange paths to form the paths xwv and y or vwy and x. This implies x and y must also be adjacent only to the midpoints of 3-paths.

Let T be the set of 1-paths and 3-paths that are involved in the above rearrangement process. We consider an auxiliary digraph D whose vertices are the paths in T, and there is a directed edge from $P_1 \in T$ to $P_2 \in T$ if and only if an endpoint of P_1 is adjacent to the midpoint of P_2. Clearly, each vertex in D has in-degree at most 1 and out-degree at least 3, which is impossible. Therefore, $\mathcal{P}$ contains no 1-paths. □

Lemma 3.4. *The path cover $\mathcal{P}_3$ contains no 1-paths, 3-paths, or cyclic paths.*

Proof. We call a path *bad* if it is cyclic or has order 1 or 3. By Lemma 3.3, we may assume that each bad path in $\mathcal{P}_3$ is a non-cyclic 3-path (note that a cyclic 3-path must be a net) or a cyclic path with order at least 4.

Let P be a bad path in $\mathcal{P}_3$ and $x \in P$ be a potential endpoint of P, which is an endpoint if P is non-cyclic, or any vertex on P if P is cyclic. Suppose that $xw \in E(G)$ with $w \in Q \in \mathcal{P}_3 - \{P\}$. Then Q is not cyclic, and $Q - w$ splits into two paths Q_1 and Q_2. In fact, the path P' obtained by concatenating P, w, Q_i cannot be cyclic (as then $P + Q$ would have a Hamilton path contradicting minimality) or have length less than 4. Thus both Q_1 and Q_2 must be bad by the minimality of the cover. Furthermore, neither Q_1 nor Q_2 is a cyclic 3-path, or we would have a net or a path cover with fewer paths. We shall call w a *special vertex* on Q, and Q_1, Q_2 *bad components* on Q.

Now, for $i \in \{1, 2\}$, provided Q_i has order more than one, replacing P, Q in $\mathcal{P}$ with Q_i and $P' = P + w + Q_{3-i}$ gives a new minimal path cover. We can repeat our argument using Q_i in the place of P and any other non-bad path of the new cover other than P' in the place of Q.

We build a directed graph whose vertices are the paths in $\mathcal{P}$, and a family $\mathcal{F}$ of subpaths of these paths as follows.

(A) The set $\mathcal{F}_0$ consists of bad paths in $\mathcal{P}_3$; and we add a directed edge from $P \in \mathcal{F}_0$ to $Q \in \mathcal{F}_0$ if a potential endpoint of P is adjacent to a special vertex on Q (note that this can only happen if Q is a non-cyclic 3-path);

(B) If an endpoint of a Hamilton path on the vertex set of a bad path $P \in \mathcal{F}_0$ is adjacent to w on a non-bad path $Q \in \mathcal{P}$, we add to $\mathcal{F}_1$ all the bad components of $Q - w$ which do not have order 1, and add to our digraph an edge from P to Q;

(C) For $i \geq 1$, if an endpoint of a Hamilton path on the vertex set of some bad path $P \in \mathcal{F}_i$ is adjacent to a special vertex w of some non-bad path $R \in \mathcal{P}_3$, we add to $\mathcal{F}_{i+1}$ all the components of $R - w$ which do not have order 1, and we add to the digraph the edge from the path in $\mathcal{P}_3$ that contains P as a bad component described in (B) to R. Note that multi-edges are allowed, but we only allow one directed edge implied by the middle vertex of each 5-path.

We let $\mathcal{F}$ be the union of the $\mathcal{F}_i$. By definition, the in-degree of a path equals to the number of special vertices on it. Note that a cyclic bad path or component does not contain special vertices. It follows that if a non-bad path P contains two special vertices w_1 and w_2, then the bad component in $P - w_1$ that contains w_2 must be a 3-path, and the bad component in $P - w_2$ that contains w_1 must also be a 3-path, so P must be a non-cyclic 5-path. Therefore, the in-degrees of 5-paths are at most 2 and all other paths are at most 1. Note that there may be isolated vertices in the digraph.

Now we count the out-degrees. The out-degree of a path P equals to the number of edges that connect one endpoint of a bad component of order more than 1 and a special vertex not on P. Let Q_1, Q_2 be the two bad components of a path $P \in \mathcal{P}_3$ in the digraph.

If Q_1 and Q_2 both have order 1, then P is a bad 3-path. By (A), P has out-degree 4. So let Q_1 have order more than 1. Note that Q_1 has at least two edges out of Q_1 (as G is 2-connected), one of which is not adjacent to the special vertex on P.

If Q_1 and Q_2 are both cyclic or have order 1, then there can be no edge between them, as if $Q' = Q_1 \cup Q_2$ has a Hamiltonian cycle, we can rearrange $P \cup Q$ into one path, contradicting the minimality of the cover, and otherwise $P' = P + w$ and Q' are both non-cyclic and we contradict the minimality of the number of bad paths in the cover. So P has out-degree at least 1 (actually 2 if both Q_1, Q_2 are cyclic).

Now, if Q_1 has order three and Q_2 is cyclic or has order 1, then (a) the endpoint of Q_1 which is an endpoint of Q cannot be adjacent to any vertex on Q_2 or we could find a Hamilton path on $P \cup Q$ contradicting the minimality of the cover, and (b) the other endpoint x' of Q_1 can be adjacent to none of the vertices on Q_2 or we could find a Hamiltonian path P' on $P \cup Q_2 + w + x'$, which together with $Q = Q_1 - x'$ contradicts the minimality of the cover. Similar arguments show that if Q_1 and Q_2 both have order three then there are no edges joining their endpoints. So P has out-degree at least 3.

Since the out-degrees of the paths are as large as their in-degrees, and the bad paths have higher out-degrees than their in-degrees, such a digraph does not exist, a contradiction. □

From now on, we assume that $\mathcal{P}_3$ consists of non-cyclic paths with order other than 1 and 3.

Lemma 3.5. *There are no bad endpoints described in (iv) in $\mathcal{P}_4$.*

Proof. Suppose otherwise. Consider an endpoint $x' \in P' = x'u'_1u'_2 \ldots u'_t y'$ in (v1). We replace P, P' with $P'x'u_1x$ and u_2Py. We lose x', and do not create cyclic paths or we would contradict the minimality of the cover. We do not gain a new bad endpoint described in (v1) because (a) x is not adjacent to u'_1 or we could rearrange $P' + P$ into one path $P'u'_1xu_3u_2u_1x'u_4Py$, and (b) u_2 is not adjacent to u_5 or we could rearrange $P' + P$ into one path $P'x'u_4u_3xu_1u_2u_5Py$. We also do not gain a new bad endpoint described in (v2) because (a) x cannot be, or $u_2u_6, xu_7 \in E(G)$, which allows us to reroute P, P' into one path $P'x'u_1u_2u_6u_5u_4u_3xu_7Py$, and (b) u_2 cannot be, or $u_2u'_3, xu'_2 \in E(G)$, which allows us to reroute P, P' into one path $P'u'_3u_2u_1x'u'_1u'_2xu_3u_4Py$.

Consider an endpoint $x' \in P'$ in (v2). If u_5Py is a 3-path, then we replace P, P' with u_6y and $xPu_5x'P'$. We lose x', and do not create cyclic paths, but clearly do not gain a new bad endpoint, as u_1 is now adjacent to a vertex on the path, and u_6 has at most one neighbor on other paths. If u_5Py is not a 3-path, then we replace P, P' with u_5Py and $u_2u_3u_4xu_1x'P'$. We lose x', and do not create cyclic paths. We do not gain new bad endpoints, since u_2, u_5 cannot be as they have at most one neighbor on other paths, and no other vertex can be adjacent to x' as it is already adjacent to u_5, u_1, u'_1, and no vertex from other path can be a bad endpoint (to u_5Py) as u_5 has only one neighbor on the path. □

Lemma 3.6. *There are no annoying endpoints described in (v) in $\mathcal{P}_5$.*

Proof. Let $x' \in P' = x'u'_1 \ldots u'_l y' \in \mathcal{P}_5$ be an annoying endpoint. Then for $P = xu_1 \ldots u_k y \in \mathcal{P}_5 - \{P'\}$, $x'u_i, x'u'_{s+1}, xu'_{s-1}, u'_s u_{i+1} \in E(G)$. Note that P, P' can be decomposed into $yPu_{i+1}u'_sP'y'$ and cyclic path $x'P'u'_{s-1}xPu_i$. So y, y', and endpoint of the paths in $\mathcal{P}_5 - \{P, P'\}$ cannot have neighbors on the cyclic path.

Case 1. $s = 3$ and $u'_1u_{i+2} \in E(G)$. We replace P, P' with $P_1 = x'u'_1$ and $P'_1 = yPxu'_2P'y'$. Since P_1 is a 2-path and $xy, xy' \notin E(G)$, P_1, P'_1 are not 1-paths, 3-paths, or cyclic paths, and none of the endpoints (x', u'_1, y, y') becomes bad or annoying. But we have fewer annoying endpoints, a contradiction.

Case 2. $s > 3$, or $s = 3$ but $u'_1u_{i+2} \notin E(G)$. We replace P, P' with $P_2 = u_iPxu'_{s-1}P'x'u'_{s+1}P'y'$ and $P'_2 = u'_su_{i+1}u_{i+2}Py$. Note that none of P_2, P'_2 can be 1-paths, 3-paths, or cyclic paths. Since u_i has at most one neighbor on paths other than P_2, u_i is not bad. Since u'_{s-1} is nto next to u_i (the endpoint of P_2), u'_s is not bad. Since u'_s has only one neighbor on P'_2, it is not annoying. Note that u_i is annoying only if $u'_1u_{i+2} \in E(G)$ and $s = 3$ (so that $u'_{s-1} = u'_2$), so u_i is not annoying. Since y, y' have no neighbors in u_iP_2x', they cannot become new bad or annoying endpoints. Therefore, we have fewer annoying endpoints, a contradiction. □

Lemma 3.7. *Let $P = xu_1u_2\dots u_ky \in \mathcal{P}_7$ be a non-cyclic path so that $xu_i, xu_j \in E(G)$ with $1 < i < j \le k$. Then $j \ne i+1$, and the neighbors of u_{i-1} and u_{j-1} are on P. Furthermore, if y has no neighbors on xPu_j, then the neighbors of u_{j-1} and u_{i-1} must be on xPu_j.*

Proof. If $j = i+1$ and $i \ne 2$, then xu_iu_{i+1} is a net, a contradiction to Lemma 3.1. If $j = i+1 = 3$, then u_1 will be a better endpoint than x subject to (vi) (with fewer weighty vertices) or (vii) (weighty neighbors are further away from the endpoints). Let $j > i+1$. If u_{i-1} (or u_{j-1}) has a neighbor outside of P, then we can reroute P so that u_{i-1} (or u_{j-1}) is an endpoint, which would give fewer weighty vertices, a contradiction to the optimality of $\mathcal{P}$. If y has no neighbors on xPu_j, and u_{j-1} (or u_{i-1}) has a neighbor u_t with $t > j$, but then u_{j-1} (or u_{i-1}) is a better endpoint than x subject to (vii) (and we do not change the number of vertices between y and its furthest neighbor). □

4. Properties of heavy, light, and PE-vertices

In this section, we study the properties of some special vertices on the paths in $\mathcal{P}$.

Definition 4.1. *Let u be an endpoint of a path $P \in \mathcal{P}_4$ and $uv \in E(G) - E(P)$. Then v is called a heavy vertex if $v \notin V(P)$ (and a weighty vertex is $v \in V(P)$).*

Definition 4.2. *Let uv be an edge between $u = u_i \in P = xu_1\dots u_ky \in \mathcal{P}_4$ and $v \in P_v \in \mathcal{P}_4 - \{P\}$. Then u is called a PE-vertex (aka, pseudo-endpoint) and v is called a light vertex if one of the following is true*

(1a) $xu_{i+1}, yu_{i-1} \in E(G)$; or
(1b) $xu_{i+1} \in E(G)$, and u_{i-1} is heavy; or $yu_{i-1} \in E(G)$ and u_{i+1} is heavy; or
(1c) both u_{i-1} and u_{i+1} are heavy.

A vertex is neutral *if it is not heavy or light or weighty.*

Note that a PE-vertex is also a neutral vertex. Also note that if u is a PE-vertex defined in (1a) and (1b), then P_u can be rerouted so that u (and x or y) is an endpoint of the path.

Lemma 4.3. *Let $u \in P \in \mathcal{P}, v \in P_v \in \mathcal{P} - \{P\}$ with $uv \in E(G)$. If $P = xPy$ can be rerouted so that u is an endpoint, then v cannot be an endpoint or a PE-vertex, unless $xv^-, yv^+ \in E(G)$. Consequently, if u and y are the endpoints, then $ux \in E(G)$ or u is neutral.*

Proof. If P_v can also be routed so that v is an endpoint, then P, P_v can be combined into one path, a contradiction. So v cannot be an endpoint or a PE-vertex defined as in (1a) or (1b).

Let v be a PE-vertex defined as in (1c). Let $P_v = x_vP_vv^-vv^+P_vy_v$. Then v^- and v^+ are heavy. We assume that $v^-x_s, v^+x_t \in E(G)$, where x_s, x_t are endpoints of $P_s, P_t \in \mathcal{P} - \{P_v\}$, respectively. If one of P_s and P_t, say P_s, is not P, then we can decompose P, P_s, P_v into two paths: $P_sx_sv^-P_vx_v$ and y_vP_vvuP, a contradiction. So $P_s = P_t = P$. If y (and by symmetry, x) has only one neighbor on P, then y must be the other endpoint when u is an endpoint of P, thus $yv^+ \notin E(G)$ (and similarly, $yv^- \notin E(G)$), or P, P_v can be combined into one path $y_vP_vv^+yPuvP_vx_v$. It follows that $xv^-, xv^+ \in E(G)$, and thus x has only one neighbor on P, a contradiction. So both x and y have at least two neighbors on P. Then we must have $xv^-, yv^+ \in E(G)$.

When u and y are the endpoints, v cannot be an endpoint or a PE-vertex, so u is not heavy or light and $uy \notin E(G)$. Then u is neutral or weighty, and when it is weighty, we have $xu \in E(G)$. □

Corollary 4.4. *Let $P = xu_1\dots u_ky \in \mathcal{P}$ and $1 \le i < j \le k$. Then*

- *if $xu_j \in E(G)$, then u_{j-1} is neutral;*
- *if $xu_{i+1} \in E(G)$ and $u_iu_j \in E(G)$, then u_{j-1} is neutral or $xu_{j-1} \in E(G)$;*
- *if $xu_{j+1} \in E(G)$ and $u_iu_j \in E(G)$, then u_{i+1} is neutral or $xu_{i+1} \in E(G)$;*
- *if $xu_{j-1} \in E(G)$ and $u_iu_j \in E(G)$, then u_{i-1} and u_{i+1} are neutral or adjacent to x;*
- *if $xu_{i+1}, xu_{j+1}, u_iu_j \in E(G)$, then u_1, u_{j-1} are neutral.*

Corollary 4.5. *Let $P = xu_1 \dots u_ky \in \mathcal{P}$. If xPu_i is cyclic and u_{i+1} is heavy or light, then a vertex $u \in xPu_{i-1}$ is adjacent to y or u_j with $yu_{j-1} \in E(G)$ only when u_{i+1} is light and is adjacent to $v_s \in P' = x'P'v_{s-1}v_sv_{s+1}P'y'$ such that $xv_{s-1}, yv_{s+1} \in E(G)$.*

Proof. Under the condition, P can be rerouted so that u_{i+1} is an endpoint. So the statement follows from Lemma 4.3. □

Lemma 4.6. *PE-vertices form an independent set. Consequently, no light vertex is a PE-vertex.*

Proof. Let $u, u' \in E(G)$ be PE-vertices such that $uu' \in E(G)$ with $u \in P_u = xPu^-uu^+Py$ and $u' \in P_{u'} = x'Pu'^-u'u'^+Py' \in \mathcal{P} - \{P\}$. By Lemma 4.3, we may assume that u, u' are PE-vertices defined as in (1c), thus assume that u^-, u^+, u'^-, u'^+ are adjacent to endpoints $s \in P_s, t \in P_t, s' \in P_{s'}, t' \in P_{t'}$, respectively, where $P_s, P_t \neq P_u$ and $P_{s'}, P_{t'} \neq P_{u'}$.

First assume that none of the pairs $(P_s, P_{s'}), (P_s, P_{t'}), (P_t, P_{s'}), (P_t, P_{t'})$ contains two different paths. Then $P_s = P_{s'} = P_{t'} = P_t$. Without loss of generality, we may assume that s, s' are the endpoints of P_s. Then $P_u, P_{u'}$ and P_s can be combined into two paths: $xP_uu^-sP_ss'u'^-P_{u'}x'$ and $yP_uuu'P_{u'}y'$, a contradiction. Therefore, without loss of generality, we assume that $P_s \neq P_{s'}$.

If $P_s \neq P_{u'}$ and $P_{s'} \neq P_u$, then we reach a contradiction by combining $P_u, P_s, P_{u'}, P_{s'}$ into three paths: $P_su^-P_ux, P_{s'}u'^-P_{u'}x'$ and $yP_uuu'P_{u'}y'$. Thus, we may assume that $P_s = P_{u'}$ and let $s = x'$.

If $P_{s'} \neq P_u$, we can decompose $P_{s'}, P_u, P_{u'}$ into fewer paths: $xP_uu^-x'P_{u'}u'^-s'P_{s'}$ and $yPuu'P_{u'}y'$, again a contradiction. Therefore, we may assume that $P_{s'} = P_u$. By symmetry, we also know that $P_t = P_{u'}$ and $P_{t'} = P_u$.

Let $xu'^- \in E(G)$. If $x'u^+ \in E(G)$ (or by symmetry $y'u^- \in E(G)$), then we reach a contradiction by combing P_u and $P_{u'}$ into one path $yP_uu^+x'P_{u'}u'^-xP_uuu'P_{u'}y'$. Thus, we let $x'u^-, y'u^+ \in E(G)$. But we again can combine the two paths into one path $yP_uu^+y'P_{u'}u'uP_uxu'^-P_{u'}x'$. □

Lemma 4.7. *Let $P = xu_1u_2 \dots u_ky \in \mathcal{P}_4$. Assume that for some $1 < s < i < t < k$, the subgraphs induced by $V(xPu_i)$ and $V(u_{i+1}Py)$ contain spanning paths so that u_s and u_t are the endpoints, respectively. If u_s, u_t are heavy or light, then*

(a) u_s and u_t are both light; or

(b) u_s, u_t are heavy and adjacent to a same endpoint of $P' \in \mathcal{P}_4 - \{P\}$; or

(c) u_s is heavy and u_t is light (or by symmetry u_t is heavy and u_s is light) with $x_wu_s, vu_t \in E(G)$, where x_w is an endpoint of $P_w \in \mathcal{P}_4 - \{P\}$ and $v \in P_v = x_vP_vv^-vv^+P_vy_v$, such that

  (c1) *$P_w = P_v$, and $x_vu_s, x_vv^+ \in E(G)$ and v^- is adjacent to x or y, or*

  (c2) *$P_w \neq P_v$, and v^-, v^+ are adjacent to x, y or x_w.*

Consequently, let xPu_i be cyclic, then

  (1) if u_{i+1} is heavy, then xPu_i contains at most one heavy or light vertex; and

  (2) if u_{i+1} is light, then xPu_i contains at most one heavy vertex.

Proof. Let P_1, P_2 be the spanning paths on $V(xPu_i)$ and $V(u_{i+1}Py)$ so that u_s, x' and u_t, y' are endpoints, respectively. We may assume that at least one of u_s, u_t (say u_s) is heavy, or we have (a). Let $x_w u_s \in E(G)$ from the endpoint $x_w \in P_w \in \mathcal{P}_4 - \{P\}$ and $vu_t \in E(G)$ from $v \in P_v = x_v P_v v^- v v^+ P_v y_v \in \mathcal{P}_4 - \{P\}$.

Assume first that P_v can be rerouted such that v is an endpoints. If v is heavy, then we must have (b), or P, P_w, P_v can be replaced with paths $P_w wuP_1$ and $P_v vu_{i+1}P_2$ to obtain a better path cover. So let v be light, and by symmetry let $x_v v^+ \in E(G)$ and v^- be heavy and adjacent to an endpoint $z \in P_z \in \mathcal{P} - \{P_v\}$. Then $x_v = x_w$, or we replace P, P_v, P_w with $y_v P v^+ x_v P_v v u_t P_2 y'$ and $y_w P_w x_w u_s P_1 x'$ (when $x_w \neq y_v$) or $x' P_1 u_s P_1 y_v P_v v^+ x_v P_v v u_t P_2 y'$ (when $x_w = y_v$). Now z must be x or y, as in (c1), or we could get a cover with fewer paths: $P_1' = P_z z v^- P_v x_v u_s P_1 x'$ and $P_2' = y_v P_v v u_t P_2 y'$.

Now assume that both v^- and v^+ are heavy. We may assume that v^- is adjacent to an endpoint $z \in P_z \in \mathcal{P} - \{P_v\}$. If $z \notin \{x, y, x_w\}$, then we can replace P, P_w, P_z, P_v with paths $P_w wuP_1, P_2 u_{i+1} v P_v y_v$ and $P_z z v^- P_v x_v$, a contradiction. So v^-, and by symmetry v^+, is adjacent to x, y or x_w, as in (c2). □

Lemma 4.8. *The subgraph induced by the set of heavy and light vertices contains no edges.*

Proof. By Lemma 4.6, two vertices that are heavy or light are adjacent only if they are consecutive vertices on a path in $\mathcal{P}$. Let u_i, u_{i+1} be two vertices on P that are heavy or light. We may assume that u_{i+1} is adjacent to x_v or the vertex v on $P_v = x_v P_v v^- v v^+ P_v y_v \in \mathcal{P} - \{P\}$. As xPu_i and $u_{i+1}Py$ contain spanning trees such that u_i (and x) and u_{i+1} (and y) are endpoints, respectively, by Lemma 4.7, the following are the possible cases:

Case 1. both u_i and u_{i+1} are heavy. Then they are adjacent to the same endpoint $x_v \in P_v$. In this case, $x_v u_i u_{i+1}$ is a net, which cannot occur by Lemma 3.1.

Case 2. u_i is heavy and u_{i+1} is light. Then by Lemma 4.7, we consider the following cases.

Case 2.1. $x_v u_i, xv^-, vu_{i+1}, x_v v^+ \in E(G)$, or $x_v u_i, yv^-, vu_{i+1}, x_v v^+ \in E(G)$. In the former case, x_v is an annoying endpoint, which by Lemma 3.6 cannot exist, and in the latter case, we can combine P, P_v into one path: $xPu_i x_v P_v v^- y P u_{i+1} v P_v y_v$.

Case 2.2. $vu_{i+1}, xv^-, x_w u_i \in E(G)$, where x_w is an endpoint of $P_w \in \mathcal{P} - \{P, P_v\}$, and v^+ is adjacent to y or x_w. Then P, P_v, P_w can be combined into two paths: $P_w x_w u_i P x v^- P_v x_v$ and $y_v P_v v u_{i+1} P y$, a contradiction.

Case 3. both u_i and u_{i+1} are light. Then $wu_i, vu_{i+1} \in E(G)$ for PE-vertices $w \in P_w = x_w w_1 \dots w_{s-1} w w_{s+1} \dots w_l y_w \in \mathcal{P} - \{P\}$ and $v \in P_v = x_v v_1 \dots v_{t-1} v v_{t+1} \dots v_m y_v \in \mathcal{P} - \{P\}$.

Case 3.1 w and v are PE-vertices defined as in (1a) or (1b). Then P_w can be rerouted so that w and y_w are endpoints, and P_v can be rerouted so that v and y_v are endpoints.

- If $P_w \neq P_v$, then P, P_v, P_w can be combined into paths $P_w w u_i P x$ and $P_v v u_{i+1} P y$, a contradiction.
- If $P_w = P_v$, then x_w or y_w cannot be adjacent to two weighty vertices, by Lemma 3.7, so we may assume that $x_w w_{s+1} \in E(G)$ and $v = w_j$ so that $y_w w_{j-1} \in E(G)$. Clearly, $j < s$, or P, P_w can be combined into one path $xPu_i w P_w x_w w_{s+1} P_w w_{j-1} y_w P_w w_j u_{i+1} P y$. By definition, w_{j+1} is heavy is adjacent to an endpoint $z \in P_z \neq P_w$, so P, P_w, P_z can be decomposed into paths $P_z z w_{j+1} P_w w_s u_i P x$ and $y P u_{i+1} w_j P_w x_w w_{s+1} P_w y_w$, a contradiction.

Case 3.2 w is a PE-vertex defined as in (1a) or (1b), and v is a PE-vertex defined as in (1c). Let $x_w w_{s+1} \in E(G)$. By definition, v_{t-1} and v_{t+1} are heavy, then at least one of them, say v_{t-1}, is not adjacent to x_w. Let v_{t-1} be adjacent to the endpoint $z \in P_z \neq P_v$. When $P_z = P$, we assume that $z = y$.

- If $P_v \neq P_w$, then P, P_w, P_v, P_z can be decomposed into fewer paths:

$$P_z z v_{t-1} P_v x_v,\ y_v P_v v u_{i+1} P y,\ P_w w u_i P x.$$

- If $P_v = P_w$ and $v = w_j$ with $j > s$, then P, P_w, P_z can also be decomposed into fewer paths: $P_z z w_{j-1} P_w w_{s+1} x_w P_w w u_i P x, \quad y_w P_w w_j u_{i+1} P y$.
- If $P_v = P_w$ and $v = w_j$ with $j < s$, then P, P_w, P_z can be decomposed into fewer paths: $P_z z w_{j-1} P_w x_w w_{s+1} P_w y_w, \quad x P u_i w P_w w_j u_{i+1} P y$.

There is a contradiction in each of the cases.

Case 3.3. Both w and v are PE-vertices defined as in (1c).

Let $z_1 w_{s-1}, z_2 w_{s+1}, z_3 v_{t-1}, z_4 v_{t+1} \in E(G)$ such that z_i is an endpoint of $P_{z_i} \in \mathcal{P}$, respectively.

Let $P_w \neq P_v$. As each endpoint is adjacent to at most two heavy vertices, we may assume that $P_{z_1} \neq P_{z_4}$ or $P_{z_1} = P_{z_4}$ and z_1, z_4 are the endpoints. Then $P, P_w, P_v, P_{z_1}, P_{z_4}$ can be decomposed into fewer paths: $P_{z_1} z_1 w_{s-1} P_w x_w, \quad P_{z_4} z_4 v_{t+1} P_v y_v, \quad y_w P_w w u_i P x, \quad x_v P_v v u_{i+1} P y$.

Let $P_w = P_v$. Assume that $v = w_t$ for some $t > s$. Note that $P_{z_1} = P_{z_4}$ and $z_1 = z_4$, or P, P_w, z_1, z_4 can be decomposed into fewer paths: $x P u_i w_s P_w w_t u_{i+1} P y, P_{z_1} z_1 w_{s-1} P_w x_w$ and $P_{z_4} z_4 w_{t+1} P_w y_w$. We may also assume that $P_{z_2} = P_{z_4}$ (and similarly, $P_{z_3} = P_{z_4}$), or P, P_w, P_{z_2}, P_{z_4} can be combined into fewer paths: $x P u_i w P_w x_w, P_{z_2} w_{s+1} P_w v u_{i+1} P y, P_{z_4} z_4 w_{t+1} P_w y_w$. Therefore $P_{z_1} = P_{z_2} = P_{z_3} = P_{z_4}$, and z_1 and z_2 are the endpoints. But then P, P_w and P_{z_1} can be combined into two paths: $x P u_i w_s P_w w_{t-1} z_2 P_{z_2} z_1 w_{s-1} P_w x_w$ and $y P u_{i+1} w_t P_w y_w$, a contradiction. $\square$

Lemma 4.9. *Let $P = x u_1 u_2 \dots u_k y \in \mathcal{P}_4$ and $u_i u_j \in E(G)$ for some i, j with $j \neq i-1, i+1$.*

(a) Let u_{i-1} and u_{i+1} be heavy. Then u_{j-1} is not weighty, and u_{j-1} is heavy only if u_{i-1} and u_{j-1} are adjacent to the same endpoint of $P' \in \mathcal{P} - \{P\}$; Similarly, u_{j+1} is heavy only u_{i+1} and u_{j+1} are adjacent to the same endpoint of $P'' \in \mathcal{P} - \{P\}$. Furthermore, if both u_{j-1} and u_{j+1} are heavy, then $P' = P''$.

(b) Let $x u_{i+1} \in E(G)$ and u_{i-1} be heavy or $y u_{i-1} \in E(G)$. Then u_{j-1} is neutral when $j > i$, and u_{j+1} is neutral when $j < i$.

Proof. (a) As u_{i-1} and u_{i+1} are heavy, we may assume that they are adjacent to endpoints x', x'' of $P', P'' \in \mathcal{P} - \{P\}$, respectively.

First note that u_{j-1} is not weighty. If $x u_{j-1} \in E(G)$, then P, P' can be combined into one path $P' x' u_{i-1} P x u_{j-1} P u_i u_j P y$, a contradiction. If $y u_{j-1} \in E(G)$, then P, P'' can be combined into one path $P'' x'' u_{i+1} P u_{j-1} y P u_j u_i P x$, a contradiction again.

Note that u_{i-1} and u_{j-1} are the endpoints of the spanning paths $x P u_{i-1}$ and $u_{j-1} P u_i u_j P y$, respectively. By Lemma 4.7, if u_{j-1} is not neutral, then it is adjacent to x', as claimed, or it is light as in (c1) or (c2) in Lemma 4.7.

If it is the case as (c1), then u_{j-1} is adjacent to a PE-vertex $v \in P'$ such that $x' v^+ \in E(G)$. Now P', P'', P can be combined into fewer paths: $P'' u_{i+1} P u_{j-1} v P' x' v^+ P' y'$ and $x P u_i u_j P y$. So we may assume that it is the case as (c2). Then u_{j-1} is adjacent to a PE-vertex $v \in P_v = x_v P_v v^- v v^+ P_v y_v \in \mathcal{P} - \{P, P'\}$ such that v^-, v^+ are adjacent to x, y or x'. We may assume that v^- is adjacent to x or y. Then P', P_v, P can be combined into two paths in either case: in the former case $x_v P_v v^- P u_{i-1} x' P'$ and $y_v P_v v u_{j-1} P u_i u_j P y$, and in the latter case, $P' x' u_{i-1} P x$ and $x_v P_v v^- y P u_j u_i P u_{j-1} v P_v y_v$, a contradiction.

For the furthermore part, if both u_{j-1} and u_{j+1} are heavy and adjacent to different paths, say P' and P'', then we can replace P, P', P'' with $P' x' u_{j-1} P u_{i+1} x'' P''$ and $x P u_i u_j P y$, a contradiction.

(b) When $j > i$, P can be rerouted so that u_{j-1}, y (or u_{j+1}, y when $j < i$) are endpoints, it follows from Lemma 4.3 that u_{j-1} is neutral or $xu_{j-1} \in E(G)$; but in the latter case, P can be rerouted so that u_{i-1} and y are endpoints, thus by the same lemma, u_{i-1} cannot be heavy or adjacent to y, a contradiction. $\square$

Lemma 4.10. *Let $P = xu_1u_2 \dots u_ky \in \mathcal{P}$. If $xu_3 \in E(G)$ and u_4 is heavy, then u_1, u_2 are neutral.*

Proof. By Lemma 3.7, $u_2x \notin E(G)$, and then by Lemma 4.3, u_2 is neutral, and $u_1y \notin E(G)$. Assume that u_1 is not neutral. Then u_1 must be light or heavy. If u_1 is heavy, then by Lemma 4.7, u_1 and u_4 are adjacent to the same endpoint of a path, thus we have a bad endpoint, a contradiction to Lemma 3.5. So u_1 must be light.

Let u_1 be adjacent to the PE-vertex $v \in P_v = x_vP_vv^-vv^+Py_v \in \mathcal{P} - \{P\}$. Let u_4 be adjacent to an endpoint $x_w \in P_w \in \mathcal{P} - \{P\}$. The following cases (c1) and (c2) from Lemma 4.7 must be true.

(c1). $P_w = P_v$, and $x_vu_4, x_vv^+ \in E(G)$ and v^- is adjacent to x or y. If $v^-x \in E(G)$, then P, P_v can be combined into one path $yPu_4x_vP_vv^-xu_3u_2u_1vP_vy_v$, a contradiction. So $yv^- \in E(G)$.

Note that P, P_v can be replaced by $y_vP_vvu_1u_2u_3x$ (or $y_vP_vvu_1xu_3u_2$) and cyclic path $x_vPv^-yPu_4$, so x, u_2, y_v have no neighbors on $x_vP_vv^-$ and u_4Py, or we can combine P, P_v into one path.

If x has no neighbors on P_v and u_4Py is not a 3-path, let $P_1 = y_vPv^+x_vP_vvu_1u_2u_3x$ and $P_2 = u_4Py$; If x has no neighbors on P_v and u_4Py is a 3-path, let $P_1 = y_vPx_vu_4Px$ and $P_2 = u_5y$; If x has a neighbor on P_v, let $P_1 = xu_3u_4Py$ and $P_2 = y_vP_vv^+x_vP_vvu_1u_2$. Note that each of x, x_v is adjacent to at least one weighty vertex in P, P_v, respectively. We replace P, P_v with P_1, P_2 in the corresponding cases, and claim that there are fewer weighty vertices in the new cover. In the first two cases, x has one weighty neighbor on P_1 and u_4 or u_5 has none in P_2, and in the last case, x has no weighty neighbors on P_1, and u_2 has at most one weighty neighbor on P_2. Clearly, we do not add bad paths (1-, 3- or cyclic paths) to the cover. To obtain a contradiction, we show below that we do not create new bad or annoying endpoints.

Only x in the last case could be a bad endpoint described in (iv), and when it is, we must have $u_2g, xg^- \in E(G)$, where $P_v = x_vP_vg^-gv^-vv^+P_vy_v$; but in this case, we can combine P, P_v into one path $y_vP_vvu_1xu_3u_2gv^-yPu_4x_vP_vg^-$, a contradiction.

We also claim that no new annoying endpoints are added. In the first case, u_4 cannot be, since it has only one neighbors on P_2, and if x is one, then u_2 must be adjacent to a vertex in u_5Py, which is impossible. In the second case, no vertex is an annoying endpoint as P_2 is a 2-path. In the last case, x cannot be as it has only one neighbor on P_1, and if u_2 is one, then u_4 should be light and be adjacent to a PE-vertex, but $u_4x_v \in E(G)$ and x_v is not a PE-vertex in P_2.

(c2). $P_w \neq P_v$, and v^-, v^+ are adjacent to x_w, x or y. If $xv^- \in E(G)$, then P, P_v, P_w can be combined into two paths: $P_wx_wu_4Py$ and $x_vP_vv^-xu_3u_2u_1vP_vy_v$, a contradiction. So we may assume that $xv^-, xv^+ \notin E(G)$. It follows, by symmetry, that $yv^+ \in E(G)$. But again, P, P_v, P_w can be combined into two paths: $P_wx_wu_4Pyv^+P_vy_v$ and $x_vP_vvu_1u_2u_3x$, a contradiction. $\square$

5. Weights on paths

We give an initial weight of 10 to each path in $\mathcal{P}$. By Lemma 3.4, all paths in $\mathcal{P}$ are non-cyclic paths with order more than 1. So we may think that each endpoint of the paths in $\mathcal{P}$ gets an initial weight of 5. Here is the rule to transfer weights between (vertices on) paths:

> **Rule to transfer weights:** Each endpoint sends a weight of 2 to the adjacent weighty or heavy vertex, and each PE-vertex transfers 1 to the adjacent light vertex.

For convenience, we let $w(P)$ be the final weight on a path or a segment P. For a path $P \in \mathcal{P}$, let $s_1(P), s_2(P), s_3(P)$ and $n_o(P)$ be the number of weighty and heavy vertices, light vertices, neutral vertices, and PE-vertices, respectively. Then

$$w(P) = 2 + 2s_1(P) + s_2(P) - n_o(P). \tag{1}$$

Definition 5.1. *Let $P = xu_1u_2 \dots u_ky \in \mathcal{P}$. For each $1 \le i \le j \le k-1$,*

- *a neutral vertex u_i is* free *if there are neither heavy nor weighty vertices on P between u_i and an endpoint (x or y);*
- *u_iPu_j is a* heavy segment *if both u_i and u_j are heavy or weighty and there is no neutral vertices on it; (so a single heavy or weighty vertex is also a heavy segment)*
- *u_iPu_j is a* neutral segment *if u_i, u_j are non-free neutral and there is no heavy or weighty vertices on it. (so a single neutral vertex is also a neutral segment)*

Note that a light vertex may be on a heavy or neutral segment. By Corollary 4.4 and Lemma 4.8, there are at least one neutral vertices between any two heavy vertices, so a heavy segment with more than one vertices must contain at least one weighty vertex, and contain at most three vertices, and when it contains three vertices, it must be a light vertex adjacent to two weighty vertices. Let a *heavy pair* be a pair of vertices in a heavy segment that are both heavy or weighty. So every heavy segment contains either 0 or 1 heavy pair. Let $n_h(P)$ be the number of heavy pairs on $P \in \mathcal{P}$. So $n_h(P) \le 4$ for each $P \in \mathcal{P}$.

Similarly, a pair of neutral vertices in a neutral segment that are consecutive or separated by a light vertex is called a *neutral pair*. So a neutral segment with s neutral vertices contains $s-1$ *neutral pairs*. A heavy segment (and similarly, a neutral segment) is *maximal* if it is not contained in a larger heavy segment (neutral segment).

Lemma 5.2. *For a path $P \in \mathcal{P}$ with $w(P) > |V(P)|$,*

$$n_h(P) \ge n_o(P) + n_q(P) + n_r(P),$$

where $n_h(P), n_o(P), n_q(P), n_r(P)$ are the numbers of heavy pairs, PE-vertices, neutral pairs, and free neutral vertices on P, respectively.

Proof. As above, assume that P contains a maximal heavy segments, then P contains $a-1$ maximal neutral segments. It follows that $s_1(P) = n_h(P) + a$ and $s_3(P) = n_q(P) + a - 1 + n_r(P)$. By (1),

$$\begin{aligned} w(P) &= 2 + s_1(P) + s_2(P) + (n_h(P) + s_3(P) - n_q(P) + 1 - n_r(P)) - n_o(P) \\ &= |V(P)| + n_h(P) + 1 - (n_q(P) + n_r(P) + n_o(P)). \end{aligned}$$

It follows that if $w(P) > |V(P)|$ then $n_h(P) \ge n_q(P) + n_r(P) + n_o(P)$, as claimed. $\square$

For convenience, we call the number $n_o(P) + n_q(P) + n_r(P)$ the *good number of P*, and in particular, it is called the *good number of the segment* if P is a segment of a path.

Lemma 5.3. *If for some $i > 1$, $xu_i \in E(G)$ and u_{i+1} is heavy (so $\{u_i, u_{i+1}\}$ is a heavy pair), then the good number of xPu_i is at least 1, with equality if and only if $i = 4$ and u_2 is heavy.*

Proof. First of all, xPu_{i-1} contains no neighbors of y by Corollary 4.5, and contains at most one heavy or light vertex by Lemma 4.7. We may assume that the good number of xPu_i is at most 1.

Clearly, $i > 2$, or xu_1u_2 is a net. Also, $i \ne 3$, or by Lemma 4.10, both u_1, u_2 are neutral (and free), so the good number of xPu_i is 2. So $i \ge 4$, and xPu_{i-1} contains at least one heavy, weighty, or light vertices, or there are at least two free neutral vertices on xPu_i.

First assume that xPu_{i-1} contains no weighty vertices. Then it must contain exactly one heavy or light vertex, by Lemma 4.7. If it contains a light vertex, then all neutral vertices on xPu_{i-1} are free, thus the good number is at least $i-1-1 \geq 2$. Therefore, it contains a heavy vertex. If u_1 is heavy, then $u_2, \ldots, u_{i-1}$ are all neutral, so it contains $i-3$ distinct neutral pairs, namely, $\{u_2, u_3\}, \{u_3, u_4\}, \ldots, \{u_{i-2}, u_{i-1}\}$; If u_2 is heavy, then u_1 is free and $\{u_3, u_4\}, \ldots, \{u_{i-2}, u_{i-1}\}$ are $i-4$ distinct neutral pairs; if u_1, u_2 are not heavy, then u_1, u_2 are free neutral vertices. So in either case, the good number is at least $i-3$. Then $i = 4$ and u_1 or u_2 is heavy. By Lemma 3.5, u_1 cannot be heavy, so u_2 is heavy.

Now assume that xPu_{i-1} contains a weighty vertex, say u_j for some $1 < j < i$. Then $xu_j \in E(G)$. If $j = 2$, then $u_1u_3 \in E(G)$, or we will have a net. Note that $u_4 \neq u_i$ (otherwise u_4u_5 is a cut-edge), u_4 cannot be neutral (otherwise, $\{u_3, u_4\}$ is a neutral pair and u_1 is free, so the good number of xPu_i is at least 2), and u_4 cannot be light (otherwise, u_5 must be neutral by Corollary 4.3, thus $\{u_3, u_5\}$ is a neutral pair and u_1 is free, so the good number of xPu_i is at least 2). So u_4 is heavy. Now u_5 is neutral and u_6 must be u_i, or $\{u_5, u_6\}$ is a neutral pair. But then by Lemma 3.7, u_5 must be adjacent to some vertex on xPu_i other than u_4, u_6, which is impossible.

So $j \geq 3$, and one vertex on xPu_{j-1} must be heavy or light (otherwise there are at least two free neutral vertices). As xPu_{j-1} contains no weighty vertices, the above argument shows that xPu_{j-1} has good number at least 2, unless $j = 4$ and u_2 is heavy. In the bad case, $i = j+2 = 6$ (for otherwise $\{u_{j+1}, u_{j+2}\}$ is a neutral pair), and by Lemma 3.7, u_1, u_3, u_5 must be adjacent only to vertices on xPu_6, which is impossible. $\square$

Lemma 5.4. *Let $P = xu_1 \ldots u_k y \in \mathcal{P}$. If for some $1 < i < k$, $xu_i, yu_{i+2} \in E(G)$ and u_{i+1} is light, then $w(P) \leq |V(P)|$.*

Proof. Assume that $w(P) > |V(P)|$. Then by Lemma 5.2, $n_h(P) \geq n_o(P) + n_q(P) + n_r(P)$.

By Lemma 4.7, xPu_i and yPu_{i+2} contain at most one heavy vertex altogether. By Corollary 4.5, xPu_i contains no neighbors of y and $u_{i+1}Py$ contains no neighbor of x. We may assume that yPu_{i+2} contains no heavy vertices. As $i+2 \neq k-1$ (otherwise $u_{i+2}u_k y$ is a net), yPu_{i+2} contains at least one free neutral vertex, so the good number of yPu_{i+2} is at least one.

If xPu_i contains no heavy vertices, then similarly the good number of xPu_i is also at least one, so the good number of P is at least 2, but P has only one heavy pair, a contradiction. Thus, we may assume that xPu_i contains exactly one heavy vertex.

If xPu_i contain no heavy pairs, then its good number must be zero. It follows that u_1 is heavy (otherwise $n_r \geq 1$), $xu_3 \in E(G)$ (otherwise $n_q \geq 1$), and $i \geq 3$. As u_2 cannot be a PE-vertex (otherwise $n_o + n_r + n_q \geq 2$ but $n_h = 1$), $u_2u_a \in E(G)$ for some $a \neq 1, 3$. By Lemma 3.7, $i > a$. By Lemma 4.9 (b), u_{a-1} is neutral, so $n_q \geq 1$ thus $n_q + n_r \geq 2$, a contradiction.

Thus, we may assume that xPu_i contains a heavy pair, that is, $xu_j \in E(G)$ and u_{j+1} is heavy for some $2 \leq j < i-1$. By Lemma 5.3, the good number of xPu_j is at least 1. Thus it must be exactly 1, $j = 4$ and u_2 is heavy. But then xPu_i contains two heavy vertices, a contradiction. $\square$

Lemma 5.5. *For each $P \in \mathcal{P}$, $w(P) \leq |V(P)|$.*

Proof. Let $P = xu_1u_2 \ldots u_k y \in \mathcal{P}$. By Lemma 3.5, P is non-cyclic and is not a 1-path or a 3-path. Assume that $w(P) > |V(P)|$. Let n_h, n_o, n_q, n_r, as defined in the Lemma 5.2, be the numbers of heavy pairs, PE-vertices, neutral pairs, and free neutral vertices on P, respectively. By Lemma 5.2,

$$n_o + n_q + n_r \leq n_h \leq 4. \tag{2}$$

By Lemma 5.4 and Lemma 4.8, we may assume that each heavy pair consists of two consecutive vertices on P, at least one of which is a weighty vertex.

Case 1. P contains a heavy pair consisting of two weighty vertices. Assume that $xu_i, yu_{i+1} \in E(G)$ for some $1 < i < k-1$. Note that by Lemma 3.7, $xu_{i-1}, yu_{i+2} \notin E(G)$.

Case 1.1. P contains at least two heavy pairs.

First assume that P contains another heavy pair consisting of two weighty vertices, say $\{u_j, u_{j+1}\}$ with $xu_j, yu_{j+1} \in E(G)$ for some $j > i$. In this case, there are no other heavy pairs. So $n_h = 2$. As P can be rerouted so that u_1 and u_k are the endpoints, u_1, u_k must be neutral by Corollary 4.3, thus free, so $n_r = 2$ and $n_o = n_q = 0$. Then u_{j-1} is not in a neutral pair. It follows that u_{j-2} is heavy or $u_{j-2} = u_{i+1}$, and in either case, $u_{j-1}u_s \in E(G)$ for some $s \neq j-2, j$, or u_{j-1} is a PE-vertex (a contradiction to $n_o = 0$). By Lemma 4.9, u_{s+1} (if $s < j$) or u_{s-1} (if $s > j$) is neutral, so we have a neutral pair, a contradiction to $n_q = 0$.

Now we assume that $\{u_i, u_{i+1}\}$ is the only heavy pair containing two weighty vertices. Note that if a heavy pair is $\{u_s, u_{s+1}\}$ such that $xu_s \in E(G)$ and u_{s+1} is heavy, then by Corollary 4.5, $s < i$. Similarly for such a heavy pair involving y. Also note that $n_h \leq 3$.

Consider the case $n_h = 3$. Let $\{u_s, u_{s+1}\}$ and $\{u_t, u_{t-1}\}$ be the other two heavy pairs such that $s < i, t > i+1$, $xu_s, yu_t \in E(G)$, and u_{s+1}, u_{t-1} are heavy. By Lemma 4.7, P contains no other heavy or light vertices. Now by Lemma 5.3, each of xPu_i and $u_{t+1}Py$ has the good number at least two, thus the good number of P is at least 4, a contradiction.

Now let $n_h = 2$ and $\{u_s, u_{s+1}\}$ be the only other heavy pair such that $xu_s \in E(G)$ and u_{s+1} is heavy. By Lemma 5.3, xPu_s has the good number at least 2 or $s = 4$ such that u_2 is heavy. In the latter case, $u_{i+1}Py$ contains no heavy or light vertices by Lemma 4.7, so has at least two free neutral vertices, thus the good number of P is at least 3, a contradiction. For the former case, the good number of $u_{i+1}Py$ is 0. It follows that u_k must be heavy, and by Lemma 4.7, there are no other heavy or light vertices on P other than u_{s+1} and u_k. Then $s = 3$ and $i = 6$. Now u_{i-1} must be adjacent u_t for some $t \neq i-2, i$. Clearly, $t \neq 1, 2, i+2$. Then $t > i+1$, and by Corollary 4.4, u_{t-1} is neutral, so $\{u_{t-1}, u_t\}$ is a neutral pair, a contradiction to the fact that the good number of $u_{i+1}Py$ is 0.

Case 1.2. $\{u_i, u_{i+1}\}$ is the only heavy pair on P. Then the good number of P is at most one. By symmetry, we may assume that the good number of xPu_i is at most one and the good number of $u_{i+1}Py$ is 0. It follows that u_k cannot be neutral (otherwise it is free).

Let $xu_j \in E(G)$ for some $j > i$. Then P can be rerouted so that u_1 is an endpoint. By Lemma 4.3, u_1 is neutral (and free). So $n_r \geq 1$. It follows that $n_r = 1$ and $n_o = n_q = 0$. By Lemma 3.7, $u_{j-1}u_s \in E(G)$ for some $s \neq j-2, j$. By Corollary 4.4, u_{s+1} (if $s < j$) or u_{s-1} (if $s > j$) is neutral, thus $n_q \geq 1$, a contradiction. It follows that $u_{i+1}Py$ contains no neighbor of x, and by symmetry, xPu_i contains no neighbour of y.

Now that u_k cannot be neutral or weighty, it must be light or heavy. But if it is light, then u_{k-1} cannot be a neighbour of y (otherwise yu_ku_{k-1} is a net), or heavy by Lemma 4.8, so u_{k-1} is neutral, thus the good number of $u_{i+1}Py$ is not 0, a contradiction. So we assume that u_k is heavy.

Observe that u_{k-1} is neutral. We may assume that u_{k-2} is not neutral, or we have a neutral pair $\{u_{k-2}, u_{k-1}\}$ on $u_{i+1}Py$, a contradiction. So u_{k-2} is heavy, light or $yu_{k-2} \in E(G)$.

Note that u_{k-2} cannot be heavy. Suppose otherwise. By Lemma 4.7, xPu_i contains no heavy or light vertices. So u_1, u_2 are free neutral vertices. It follows that $n_r \geq 2$, a contradiction.

We also have $u_{k-2}y \notin E(G)$. Suppose otherwise. Then $u_{k-1}u_s \in E(G)$ for some $i+1 < s < k-2$, or u_{k-1} is a PE-vertex. By Lemma 4.9, u_{s+1} is neutral, thus $\{u_s, u_{s+1}\}$ is a neutral pair on $u_{i+1}Py$, a contradiction.

So let u_{k-2} be light. By Lemma 4.8, u_{k-3} is not heavy or light. But u_{k-3} is not a neighbour of x or y, so u_{k-3} must be neutral. Now $\{u_{k-3}, u_{k-1}\}$ is a neutral pair, a contradiction to the assumption that the good number of $u_{i+1}Py$ is 0.

Case 2. P contains no heavy pairs consisting of only weighty vertices.

Case 2.1. $n_h = 0$. It follows that $n_o = n_q = n_r = 0$. Recall that $k > 1$, by Lemma 3.4.

Assume first that P contains no weighty vertices. Then u_1, u_k are heavy, and the vertices on u_1Pu_k are alternatively heavy and neutral. Let $v \neq u_1, u_3$ be the third neighbor of u_2. Then $v = u_t \in P$ for some $t > 3$, or u_2 is a PE-vertex (this contradicts to the fact that $n_o = 0$). But u_{t-1} and u_{t+1} are heavy. So by Lemma 4.9 (a), u_1, u_{t-1} are adjacent to one endpoint of a path $Q \neq P$ and u_3, u_{t+1} are adjacent to the other endpoint of Q. Note that $t \neq 4$, or $u_2u_3u_4$ is a net. Now u_4 is neutral and must be adjacent to $v' \notin P$ or u_s for some $s > 5$. In the former case, u_{i+3} is a PE-vertex, and in the latter case, u_{s-1} or u_{s+1} cannot be heavy by Lemma 4.9 (a), a contradiction.

Now let $xu_i \in E(G)$ for some $i > 1$, and we may assume that $xu_j \notin E(G)$ for $1 < j < i$. First let $i > 2$. Then u_{i-2} cannot be neutral (or $\{u_{i-2}, u_{i-1}\}$ is a neutral pair) or light (otherwise, u_{i-1} is free if $i = 3$, or u_{i-3} must be neutral thus $\{u_{i-3}, u_{i-1}\}$ is a neutral pair). So u_{i-2} is heavy or $yu_{i-2} \in E(G)$. Now u_{i-1} must be adjacent to u_t for some $t \neq i-3, i-1$, or it is a PE-vertex. By Lemma 4.9 (b), u_{t-1} (if $t > i-1$) or u_{t+1} (if $t < i-3$) is neutral, thus u_t is in a neutral pair, a contradiction to $n_q = 0$. Now let $i = 2$. Then $u_1u_3 \in E(G)$, or xu_1u_2 is a net. So $xu_j \in E(G)$ for some $j > 3$, or u_1 is a better endpoint of P than x. Clearly $j \neq 4$, or u_4u_5 is a cut-edge. So u_{j-2} must be heavy or adjacent to y. Now a similar argument as above shows $n_q > 0$, a contradiction.

Case 2.2. $n_h = 1$. Let $\{u_i, u_{i+1}\}$ be the only heavy pair with $xu_i \in E(G)$ for some $1 < i < k$ and u_{i+1} is heavy. As the good number of xPu_i is at most 1, by Lemma 5.3, $i = 4$ and u_2 is heavy. So u_1 is free, and $n_q = n_o = 0$.

Now, u_3u_t for some $t \neq 2, 4$, or u_3 is a PE-vertex, a contradiction to $n_o = 0$. Clearly, $t \neq 1$, so $t \geq 6$. By Lemma 4.3, u_{t-1} is neutral or $xu_{t-1} \in E(G)$. In the former case, we have a neutral pair, thus $n_q > 0$, a contradiction. As y has no neighbors on $xu_1u_2u_3$, Lemma 3.7 implies that the latter case cannot happen.

Case 2.3. $n_h = 2$. Let $\{u_i, u_{i+1}\}$ and $\{u_{t-1}, u_t\}$ be the two heavy pairs for some $i < t-1$.

First assume that $xu_i, xu_{t-1} \in E(G)$ and u_{i+1}, u_t are both heavy. By Lemma 4.7, u_{i+1}, u_t are adjacent to the same endpoint $x' \in P' \in \mathcal{P}$, and they are the only heavy/light vertices on xPu_t. As $i \geq 3$ (otherwise there is a net) and $u_1, u_2, \ldots, u_{i-1}$ are free neutral vertices, $n_r \geq 2$. It follows that $n_r = 2$ and $i = 3$, and $n_o = n_q = 0$. By Lemma 3.7, u_2 can only be adjacent to vertices on P, thus $u_2u_a \in E(G)$ for some $a \geq 4$, and by Lemma 4.9, u_{a-1} is neutral, so $n_q \geq 1$, a contradiction.

Note that if $xu_{t-1}, yu_{i+1} \in E(G)$ and u_t, u_i are heavy, then P can be rerouted so that u_i and u_t are endpoints, a contradiction to Lemma 4.3. Thus, we assume that $1 < i < t < k$, $xu_i, yu_t \in E(G)$ and u_{i+1}, u_{t-1} are heavy, and y has no neighbors on xPu_i and x has no neighbors on u_tPy. Let u_{i+1} be adjacent to an endpoint $x' \in P' \in \mathcal{P} - \{P\}$.

By Lemma 5.3, xPu_{i-1} and $u_{t+1}Py$ both have the good number at least 1, thus $i = 4, t = k-3$ and u_2 and u_{k-1} are heavy. So there are two free neutral vertices, and $n_q = n_o = 0$. Note that $i+1 < t-1$. For otherwise, by Lemma 4.7, xPu_i and u_tPy contain at most one heavy/light vertex altogether, a contradiction.

Now, u_3u_t for some $t \neq 2,4$, or u_3 is a PE-vertex. Clearly, $t \neq 1$, so $t \geq 6$. By Lemma 4.3, u_{t-1} is neutral or $xu_{t-1} \in E(G)$. In the former case, we have a neutral pair, a contradiction. As y has no neighbors on $xu_1u_2u_3$, Lemma 3.7 implies that the latter case cannot happen.

Case 2.4. $n_h \in \{3,4\}$. As in Case 2.3, we may assume that the neighbors of x are u_1, u_i, u_j and the neighbor of y are u_s, u_t, u_k with $1 < i < j < s \leq t < k$ such that $u_{i+1}, u_{j+1}, u_{s-1}, u_{t-1}$ are all heavy.

By Lemma 4.7, u_{i+1}, u_{j+1} are adjacent to the same endpoint of a path in $\mathcal{P}$, and there are no other heavy or light vertices on xPu_j. Note that $i \geq 3$ (otherwise xu_1u_2 is a net), so xPu_i contains at least $i-1 \geq 2$ free neutral vertices, i.e., $u_1, u_2, \ldots, u_{i-1}$. Similarly, u_sPy contains at least $k-t \geq 2$ free neutral vertices when $n_h = 4$ (that is, $s < t$). Note that when $n_h = 3$ (that is, $s = t$), the good number of u_sPy is 1, thus by Lemma 5.3, $s = k-3$, u_{k-1} is heavy and u_k is free. So P contains n_h free neutral vertices. It follows that $n_q = n_o = 0$, and $i = 3$ and $j = 6$. By Lemma 3.7, u_5 must be adjacent to u_1 or u_2, but then P, P' can be combined into one path: $P'x'u_4u_5u_1u_2u_3xu_6Py$ if $u_5u_1 \in E(G)$, and $P'x'u_4u_3xu_1u_2u_5Py$ if $u_5u_2 \in E(G)$, a contradiction. □

Proof of Theorem 1.1: Consider an optimal path cover $\mathcal{P}$ of G, and assign a weight of 10 to each path in $\mathcal{P}$. By Lemma 5.5, the total weight is $10|\mathcal{P}| = \sum_{P\in\mathcal{P}} w(P) \leq \sum_{P\in\mathcal{P}} |V(P)| = n$. So $|\mathcal{P}| \leq n/10$, that is, G has a path cover with at most $n/10$ paths. □

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