# Tight lower bounds on the matching number in a graph with given maximum degree 

Michael A. Henning ${ }^{1, *}$ and Anders Yeo ${ }^{1,2}$<br>${ }^{1}$ Department of Pure and Applied Mathematics University of Johannesburg<br>Auckland Park, 2006 South Africa<br>Email: mahenning@uj.ac.za<br>${ }^{2}$ Engineering Systems and Design<br>Singapore University of Technology and Design<br>8 Somapah Road, 487372, Singapore<br>Email: andersyeo@gmail.com


#### Abstract

Let $k \geq 3$. We prove the following three bounds for the matching number, $\alpha^{\prime}(G)$, of a graph, $G$, of order $n$ size $m$ and maximum degree at most $k$. - If $k$ is odd, then $\alpha^{\prime}(G) \geq\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{k-1}{k\left(k^{2}-3\right)}$. - If $k$ is even, then $\alpha^{\prime}(G) \geq \frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k}$. - If $k$ is even, then $\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\frac{k+2}{k^{2}+k+2}$.

In this paper we actually prove a slight strengthening of the above for which the bounds are tight for essentially all densities of graphs.

The above three bounds are in fact powerful enough to give a complete description of the set $L_{k}$ of pairs $(\gamma, \beta)$ of real numbers with the following property. There exists a constant $K$ such that $\alpha^{\prime}(G) \geq \gamma n+\beta m-K$ for every connected graph $G$ with maximum degree at most $k$, where $n$ and $m$ denote the number of vertices and the number of edges, respectively, in $G$. We show that $L_{k}$ is a convex set. Further, if $k$ is odd, then $L_{k}$ is the intersection of two closed half-spaces, and there is exactly one extreme point of $L_{k}$, while if $k$ is even, then $L_{k}$ is the intersection of three closed half-spaces, and there are precisely two extreme points of $L_{k}$.


Keywords: Matching number, maximum degree, convex set
AMS subject classification: 05C65

[^0]
## 1 Introduction

Two edges in a graph $G$ are independent if they are not adjacent in $G$. A set of pairwise independent edges of $G$ is called a matching in $G$, while a matching of maximum cardinality is a maximum matching. The number of edges in a maximum matching of $G$ is called the matching number of $G$ which we denote by $\alpha^{\prime}(G)$. Matchings in graphs are extensively studied in the literature (see, for example, the classical book on matchings my Lovász and Plummer [11], and the excellent survey articles by Plummer [14] and Pulleyblank [15]).

For $k \geq 3$, let $L_{k}$ be the set of all pairs $(\gamma, \beta)$ of real numbers for which there exists a constant $K$ such that

$$
\alpha^{\prime}(G) \geq \gamma n+\beta m-K
$$

holds for every connected graph $G$ with maximum degree at most $k$, where $n$ and $m$ denote the number of vertices and the number of edges, respectively, in $G$. Our main result is to give a complete description of the set $L_{k}$. For this purpose, let $\ell_{1}, \ell_{2}, \ell_{3}$ and $\ell_{4}$ be the following four closed half-spaces over the reals $\gamma$ and $\beta$ :

$$
\begin{array}{lll}
\ell_{1}: \beta \leq-\gamma & + & \frac{1}{k} \\
\ell_{2}: \beta \leq-\left(\frac{2}{k}\right) \gamma & + & +\frac{k^{3}-k^{2}-2}{k^{2}\left(k^{2}-3\right)} \\
\ell_{3}: \beta \leq-\left(\frac{2}{k}\right) \gamma & & +\frac{k^{2}+4}{k\left(k^{2}+k+2\right)} \\
\ell_{4}: \beta \leq & -\left(\frac{2 k^{2}}{k^{3}-k+2}\right) \gamma & +\frac{k^{2}-k+2}{k^{3}-k+2}
\end{array}
$$

We are now in a position to state our main result.

Theorem A For $k \geq 3$, the set $L_{k}$ is a convex set. Further, the following holds.
(a) If $k \geq 3$ is odd, then $L_{k}$ is the intersection of the two closed half-spaces $\ell_{1}$ and $\ell_{2}$, and there is exactly one extreme point of $L_{k}$, namely

$$
\left(\frac{k-1}{k\left(k^{2}-3\right)}, \frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right)
$$

(b) If $k \geq 4$ is even, then $L_{k}$ is the intersection of the three closed half-spaces $\ell_{1}, \ell_{3}$ and $\ell_{4}$, and there are precisely two extreme points of $L_{k}$, namely

$$
\left(\frac{1}{k(k+1)}, \frac{1}{k+1}\right) \quad \text { and } \quad\left(-\frac{k-2}{k^{2}+k+2}, \frac{k+2}{k^{2}+k+2}\right)
$$

Theorem A is illustrated in Figure 1 for small values of $k$, namely $k \in\{3,4,5,6\}$, where the convex set $L_{k}$ corresponds to the grey area in the pictures.

In order to prove Theorem A, we shall prove the following two key results on the matching number.


$k=4$

$k=5$


$$
k=6
$$

Figure 1: The convex set $L_{k}$ for small $k$

Theorem B If $k \geq 3$ is an odd integer and $G$ is a connected graph of order $n$, size $m$, and with maximum degree $\Delta(G) \leq k$, then

$$
\alpha^{\prime}(G) \geq\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{k-1}{k\left(k^{2}-3\right)} .
$$

Further, this lower bound is achieved for infinitely many trees, and for infinitely many $k$ regular graphs.

In fact, Theorem B is tight for essentially all possible densities of connected graphs with maximum degree $k$.

Theorem C If $k \geq 4$ is an even integer and $G$ is a connected graph of order $n$, size $m$ and maximum degree $\Delta(G) \leq k$, then the following holds.

$$
\begin{aligned}
& \text { (a) } \alpha^{\prime}(G) \geq \frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k} \\
& \text { (b) } \alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\frac{k+2}{k^{2}+k+2}
\end{aligned}
$$

We will later see how to slightly improve the bounds in Theorem C, such that they are also achieved for infinitely many trees and for infinitely many $k$-regular graphs and essentially for all possible densities in between.

## 2 Known Matching Results

We shall need the following theorem of Berge [1] about the matching number of a graph, which is sometimes referred to as the Tutte-Berge formulation for the matching number.

Theorem 1 (Tutte-Berge Formula) For every graph $G$,

$$
\alpha^{\prime}(G)=\min _{X \subseteq V(G)} \frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X)) .
$$

An elegant proof of Theorem 1 was given by West [16]. We remark that as a consequence of the Tutte-Berge Formula, it is well-known that if $X$ is a proper subset of vertices of $G$ such that $(|V(G)|+|X|-\mathrm{oc}(G-X)) / 2$ is minimum, then every odd component $C$ of $G$ contains an almost perfect matching; that is, $\alpha^{\prime}(C)=(|V(C)|-1) / 2$.

The following results from [7] establishes a tight lower bound on the matching number of a regular graph.

Theorem 2 ([7]) For $k \geq 2$ even, if $G$ is a connected $k$-regular graph of order $n$, then

$$
\alpha^{\prime}(G) \geq \min \left\{\left(\frac{k^{2}+4}{k^{2}+k+2}\right) \times \frac{n}{2}, \frac{n-1}{2}\right\},
$$

and this bound is tight.

Theorem 3 ([7]) For $k \geq 3$ odd, if $G$ is a connected $k$-regular graph of order n, then

$$
\alpha^{\prime}(G) \geq \frac{\left(k^{3}-k^{2}-2\right) n-2 k+2}{2\left(k^{3}-3 k\right)}
$$

and this bound is tight.

For small values of $k \geq 3$, the results of Theorem 2 and 3 are summarized in Table 1 .

| $G$ is a connected $k$-regular graph |  |  |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k$ | 3 | 4 | 5 | 6 | 7 | 8 |
| $\alpha^{\prime}(G) \geq$ | $\frac{4 n-1}{9}$ | $\min \left\{\frac{5 n}{11}, \frac{n-1}{2}\right\}$ | $\frac{49 n-4}{110}$ | $\min \left\{\frac{5 n}{11}, \frac{n-1}{2}\right\}$ | $\frac{73 n-3}{161}$ | $\min \left\{\frac{17 n}{37}, \frac{n-1}{2}\right\}$ |

Table 1. Tight lower bounds on the matching number of a $k$-regular, connected graph.

## 3 Main Results

Our main result establishes tight lower bounds on the matching number of a graph in terms of its maximum degree, order, size, and number of components. Our first result is the following result, a proof of which is given in Section 4 .

Theorem 4 Let $k \geq 3$ be an integer and let $G$ be a graph with $c$ components and of order $n$ and size $m$ and maximum degree $\Delta(G) \leq k$. If no component of $G$ is $k$-regular, then

$$
\alpha^{\prime}(G) \geq \begin{cases}\left(\frac{1}{k(k+1)}\right)(n-c)+\left(\frac{1}{k+1}\right) m & \text { if } k \text { is even } \\ \left(\frac{k-1}{k\left(k^{2}-3\right)}\right)(n-c)+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m & \text { if } k \text { is odd. }\end{cases}
$$

For small values of $k \geq 3$, the results of Theorem 4 are summarized in Table 2.

| $k$ | Exact bound |  | Approximate equivalent to |
| :---: | ---: | :---: | :---: |
| 3 | $9 \alpha^{\prime}(G)$ | $\geq n+2 m-c$ | $\alpha^{\prime}(G) \geq 0.11111 \cdot n+0.22222 \cdot m-0.11111 \cdot c$ |
| 4 | $20 \alpha^{\prime}(G)$ | $\geq n+4 m-c$ | $\alpha^{\prime}(G) \geq 0.05000 \cdot n+0.20000 \cdot m-0.05000 \cdot c$ |
| 5 | $55 \alpha^{\prime}(G)$ | $\geq 2 n+9 m-2 c$ | $\alpha^{\prime}(G) \geq 0.03636 \cdot n+0.16364 \cdot m-0.03636 \cdot c$ |
| 6 | $42 \alpha^{\prime}(G)$ | $\geq n+6 m-c$ | $\alpha^{\prime}(G) \geq 0.02381 \cdot n+0.14286 \cdot m-0.02381 \cdot c$ |
| 7 | $161 \alpha^{\prime}(G)$ | $\geq 3 n+20 m-3 c$ | $\alpha^{\prime}(G) \geq 0.01863 \cdot n+0.12422 \cdot m-0.01863 \cdot c$ |
| 8 | $72 \alpha^{\prime}(G)$ | $\geq n+8 m-c$ | $\alpha^{\prime}(G) \geq 0.01389 \cdot n+0.11111 \cdot m-0.01389 \cdot c$ |
| 9 | $351 \alpha^{\prime}(G)$ | $\geq 4 n+35 m-4 c$ | $\alpha^{\prime}(G) \geq 0.01140 \cdot n+0.09972 \cdot m-0.01140 \cdot c$ |
| 10 | $110 \alpha^{\prime}(G)$ | $\geq n+10 m-c$ | $\alpha^{\prime}(G) \geq 0.00909 \cdot n+0.09091 \cdot m-0.00909 \cdot c$ |
| 11 | $649 \alpha^{\prime}(G)$ | $\geq 5 n+54 m-5 c$ | $\alpha^{\prime}(G) \geq 0.00770 \cdot n+0.08320 \cdot m-0.00770 \cdot c$ |

Table 2. Tight lower bounds on the matching number of a graph with maximum degree $k$ and with no $k$-regular component.

The following result presents another lower bound on the matching number when $k \geq 2$ is even. A proof of Theorem 5 is given in Section 5 ,

Theorem 5 Let $k \geq 2$ be an even integer and let $G$ be any graph of order $n$ and size $m$ and maximum degree $\Delta(G) \leq k$. If no component of $G$ is $k$-regular, then

$$
\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n .
$$

If $G$ is a connected graph of order $n$ with maximum degree at most 2 , then $G$ is a either a path or a cycle, implying that $\alpha^{\prime}(G)=\left\lceil\frac{n-1}{2}\right\rceil$. Hence, it is only of interest to focus on connected graphs with maximum degree at most $k$, where $k \geq 3$. As a consequence of Theorem 2 and Theorem 4, we have the following result when $k \geq 4$ is even. A proof of Corollary 1 is given in Section 6

Corollary 1 If $k \geq 4$ is an even integer and $G$ is a connected graph of order $n$, size $m$ and maximum degree $\Delta(G) \leq k$, then

$$
\alpha^{\prime}(G) \geq \frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)},
$$

unless the following holds.
(a) $G$ is $k$-regular and $n=k+1$, in which case $\alpha^{\prime}(G)=\frac{n-1}{2}=\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k}$.
(b) $G$ is $k$-regular and $n=k+3$, in which case $\alpha^{\prime}(G)=\frac{n-1}{2}=\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{3}{k(k+1)}$.

As a consequence of Theorem 3 and Theorem [4 we have the following result when $k \geq 3$ is odd. A proof of Corollary 2 is given in Section 7

Corollary 2 If $k \geq 3$ is an odd integer and $G$ is a connected graph of order $n$, size $m$, and with maximum degree $\Delta(G) \leq k$, then

$$
\alpha^{\prime}(G) \geq\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{k-1}{k\left(k^{2}-3\right)} .
$$

As a consequence of Theorem 2 and Theorem 囵, we have the following result when $k \geq 4$ is even. A proof of Corollary 3 is given in Section 8 ,

Corollary 3 If $k \geq 4$ is an even integer and $G$ is a graph of order $n$, size $m$ and maximum degree $\Delta(G) \leq k$, then

$$
\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n
$$

unless the following holds.
(a) $G$ is $k$-regular and $n=k+1$, in which case $\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\frac{k+2}{k^{2}+k+2}$.
(b) $G$ is $k$-regular and $n=k+3$, in which case $\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\frac{4}{k^{2}+k+2}$.
(c) $G$ is 4 -regular and $n=9$, in which case $\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\frac{2}{k^{2}+k+2}$.

Theorem B and Theorem C follow from Corollaries 1, 2, and 3,
Let $n_{i}$ denotes the number of vertices of degree $i$ in a graph $G$. We remark that substituting $k=3, n=n_{1}+n_{2}+n_{3}$, and $m=\frac{1}{2}\left(n_{1}+2 n_{2}+3 n_{3}\right)$ into the lower bound in the statement of Corollary 2 yields the following result of Haxwell and Scott [5].

Corollary 4 ([5]) If $G$ is a graph with maximum degree $\Delta(G) \leq 3$, then

$$
\alpha^{\prime}(G) \geq \frac{4}{9} n_{3}+\frac{1}{3} n_{2}+\frac{2}{9} n_{1}-\frac{1}{9} c .
$$

### 3.1 Motivation

Our aim in this paper, is for all $k \geq 3$, to give a complete description of the set $L_{k}$ of pairs $(\gamma, \beta)$ of real numbers for which there exists a constant $K$ such that $\alpha^{\prime}(G) \geq \gamma n+\beta m-K$ holds every connected graph $G$ with maximum degree at most $k$, where $n$ and $m$ denote the number of vertices and the number of edges, respectively, in $G$. Similar work was done by Chvátal and McDiarmid [3] for the transversal number of a $k$-uniform hypergraph, for $k \geq 2$, in terms of its order and size. In their case, the resulting convex set has infinitely many extreme points. In our case, we show that in contrast to the Chvátal-McDiarmid result, our convex set has exactly one extreme point when $k$ is odd and exactly two extreme points when $k$ is even.

Various lower bounds on the matching number for regular graphs have appeared in the literature. For example, Biedl et. al [2] proved that if $G$ is a cubic graph, then $\alpha^{\prime}(G) \geq$ $(4 n-1) / 9$. This result was generalized to regular graphs of higher degree by Henning and Yeo [7] (see also, O and West [12]). O and West [13] established lower bounds on the matching number with given edge-connectivity in regular graphs. Cioabă, Gregory, and Haemers 44 studied matchings in regular graphs from eigenvalues. Lower bounds on the matching number for general graphs and bipartite graphs were obtained by Jahanbekam and West 9 .

Lower bounds on the matching number in subcubic graphs (graphs with maximum degree at most 3) were studied by, among others, Henning, Löwenstein, and Rautenbach 6]. Recently, Haxell and Scott [5] gave a complete description of the set of triples $(\alpha, \beta, \gamma)$ of real numbers for which there exists a constant $K$ such that $\alpha^{\prime}(G) \geq \alpha n_{3}+\beta n_{2}+\gamma n_{1}-K$ for every connected subcubic graph $G$, where $n_{i}$ denotes the number of vertices of degree $i$ for each $i \in[3]$. Here, the resulting convex set is shown to be a 3 -dimensional polytope determined by the the intersection of the six half-spaces.

In this paper, we establish a tight lower bound on the matching number of a graph with given maximum degree in terms of its order and size.

For graph theory and terminology, we generally follow 8]. In particular, we denote the degree of a vertex $v$ in the graph $G$ by $d_{G}(v)$. The maximum degree among the vertices of $G$ is denoted by $\Delta(G)$. For a subset $S$ of vertices of a graph $G$, we let $G[S]$ denote the subgraph induced by $S$. The number of odd components of a graph $G$ we denote by oc $(G)$. We use the standard notation $[k]=\{1,2, \ldots, k\}$.

## 4 Proof of Theorem (4)

Let

$$
\varepsilon_{k}=\left\{\begin{array}{cl}
\frac{2}{k(k+1)} & \text { for } k \geq 2 \text { even } \\
\frac{2 k-2}{k\left(k^{2}-3\right)} & \text { for } k \geq 3 \text { odd }
\end{array}\right.
$$

For $k \geq 2$, let

$$
a_{k}=\frac{\varepsilon_{k}}{2} \quad \text { and } \quad b_{k}=\frac{2-k \varepsilon_{k}}{2 k} .
$$

We note that for $k \geq 2$ even,

$$
a_{k}=\frac{1}{k(k+1)} \quad \text { and } \quad b_{k}=\frac{1}{k+1},
$$

and for $k \geq 3$ odd,

$$
a_{k}=\frac{k-1}{k\left(k^{2}-3\right)} \quad \text { and } \quad b_{k}=\frac{k^{2}-k-2}{k\left(k^{2}-3\right)} .
$$

Further, in both cases,

$$
a_{k}+b_{k}=\frac{1}{k},
$$

and so $k a_{k}+k b_{k}=1$. Theorem 4 can now be restated as follows.
Theorem 4 Let $k \geq 4$ be an integer and let $G$ be a graph with $c$ components and of order $n$ and size $m$ and maximum degree $\Delta(G) \leq k$. If no component of $G$ is $k$-regular, then $\alpha^{\prime}(G) \geq a_{k} n+b_{k} m-a_{k} c$.

Proof of Theorem 4, Let $c\left(G^{*}\right)$ denote the number of components of a graph $G^{*}$. Define the following five values of a graph $G^{*}$ and vertex set $X^{*} \subseteq V\left(G^{*}\right)$.
(1): $\beta_{1}\left(G^{*}, X^{*}\right)$ is the number of edges in $G^{*}\left[X^{*}\right]$.
(2): $\beta_{2}\left(G^{*}, X^{*}\right)$ is the number of even components in $G^{*}-X^{*}$.
(3): $\beta_{3}\left(G^{*}, X^{*}\right)$ is the number of vertices in $X^{*}$ with degree less than $k$.
(4): $\beta_{4}\left(G^{*}, X^{*}\right)$ is the number of components in $G^{*}-X^{*}$ that do not have exactly one edge to $X^{*}$.
(5): $\beta_{5}\left(G^{*}, X^{*}\right)$ is the number of odd components in $G^{*}-X^{*}$ with order between 1 and $k+1$.

For the sake of contradiction suppose that the theorem is false and that $G$ is a counter example to the theorem. That is, $G$ has maximum degree at most $k$ and no component of $G$
is $k$-regular and $\alpha^{\prime}(G)<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G)$. By the Tutte-Berge formula in Theorem 1 we may assume that $G$ and $X$ are chosen such that the following holds.

$$
\begin{equation*}
\frac{1}{2}(n+|X|-\mathrm{oc}(G-X))<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G) \tag{1}
\end{equation*}
$$

Furthermore we may assume that $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$ is lexicographically minimum of all $G$ and $X$ satisfying the above. We proceed further with the following series of claims.

Claim A $\beta_{1}(G, X)=0$.

Proof. Suppose, to the contrary, that $\beta_{1}(G, X) \geq 1$, and let $x_{1} x_{2} \in E(G)$ be arbitrary where $x_{1}, x_{2} \in X$. Delete the edge $x_{1} x_{2}$ and add a new vertex $u$ and the edges $x_{1} u$ and $u x_{2}$. Let $G^{\prime}$ be the resulting graph. We note that $c\left(G^{\prime}\right)=c(G),\left|V\left(G^{\prime}\right)\right|=|V(G)|+1$, $\left|E\left(G^{\prime}\right)\right|=|E(G)|+1$ and oc $\left(G^{\prime}-X\right)=\mathrm{oc}(G-X)+1$. Also, no component in $G^{\prime}$ is $k$-regular (as if any component in $G^{\prime}$ is $k$-regular, then the corresponding component in $G$ would also be $k$-regular). The following now holds.

$$
\begin{aligned}
\frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right) & =\frac{1}{2}((|V(G)|+1)+|X|-(\operatorname{oc}(G-X)+1)) \\
& =\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X)) \\
& <a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G) \\
& <a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right) .
\end{aligned}
$$

As $\beta_{1}\left(G^{\prime}, X\right)<\beta_{1}(G, X)$ this contradicts the lexicographical minimality of $\left(\beta_{1}(G, X)\right.$, $\left.\beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$. (ㅁ)

Claim B $\beta_{2}(G, X)=0$.

Proof. Suppose, to the contrary, that $\beta_{2}(G, X) \geq 1$. Let $C$ be an even component $C$ in $G-X$, and let $u$ be a leaf in some spanning tree of $C$. In this case $C-\{u\}$ is connected. Let $G^{\prime}=G-u$ and note that $\left|V\left(G^{\prime}\right)\right|=|V(G)|-1,\left|E\left(G^{\prime}\right)\right| \geq|E(G)|-k, c\left(G^{\prime}\right) \leq c(G)+k-1$ (as we delete at most $k-1$ edges not in $C$ incident to $u$ ) and oc $\left(G^{\prime}-X\right)=o \mathrm{oc}(G-X)+1$. The following now holds (as we have shown that $k a_{k}+k b_{k}=1$ ).

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\mathrm{oc}\left(G^{\prime}-X\right)\right)=\frac{1}{2}((|V(G)|-1)+|X|-(\mathrm{oc}(G-X)+1)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\mathrm{oc}(G-X))-1 \\
& \quad<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G)-1 \\
& \quad \leq a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|+1\right)+b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|+k\right)-a_{k} \cdot\left(c\left(G^{\prime}\right)-k+1\right)-1 \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)+\left(k a_{k}+k b_{k}-1\right) \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right) .
\end{aligned}
$$

Any component in $G^{\prime}$ is either a component in $G$ or contains a vertex of degree at most $k-1$ (adjacent to $u$ in $G$ ), which implies that no component of $G^{\prime}$ is $k$-regular. Furthermore as $\beta_{1}\left(G^{\prime}, X\right)=\beta_{1}(G, X)=0$ and $\beta_{2}\left(G^{\prime}, X\right)<\beta_{2}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$. (ם)

Claim C $\beta_{3}(G, X)=0$.

Proof. Suppose, to the contrary, that $\beta_{3}(G, X) \geq 1$. Let $x$ be a vertex in $X$ with $d_{G}(x)<$ $k$. Let $G^{\prime}$ be obtained by adding $s=k-d_{G}(x)$ new vertices $u_{1}, u_{2}, \ldots, u_{s}$ and the edges $u_{1} x, u_{2} x, \ldots, u_{s} x$ to $G$. Note that $\left|V\left(G^{\prime}\right)\right|=|V(G)|+s$ and $\left|E\left(G^{\prime}\right)\right|=|E(G)|+s$ and $c\left(G^{\prime}\right)=c(G)$ and oc $\left(G^{\prime}-X\right)=\mathrm{oc}(G-X)+s$. The following now holds.

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right)=\frac{1}{2}((|V(G)|+s)+|X|-(\mathrm{oc}(G-X)+s)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\mathrm{oc}(G-X)) \\
& \quad<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G) \\
& \quad \leq a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|-s\right)+b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|-s\right)-a_{k} \cdot c\left(G^{\prime}\right) \\
& \quad<a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right) .
\end{aligned}
$$

Any component in $G^{\prime}$ is either a component in $G$ or contains vertices of degree $1<k-1$ (adjacent to $x$ in $G$ ), which implies that no component of $G^{\prime \prime}$ is $k$-regular. As $\beta_{i}\left(G^{\prime}, X\right)=$ $\beta_{i}(G, X)$ for $i \in[2]$ and $\beta_{3}\left(G^{\prime}, X\right)<\beta_{3}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$.(口)

Claim D $\beta_{4}(G, X)=0$.

Proof. Suppose, to the contrary, that $\beta_{4}(G, X) \geq 1$. Let $C$ be an odd component in $G-X$ with $s$ edges to $X$, where $s \neq 1$. Let $q=|V(C)|$.

Suppose that $s=0$ and let $G^{\prime}=G-C$. Suppose further that $q \leq k$. In this case, we note that $|E(C)| \leq\binom{ q}{2}$, which implies the following (as $k a_{k}+k b_{k}=1$ ).

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\mathrm{oc}\left(G^{\prime}-X\right)\right) \leq \frac{1}{2}((|V(G)|-q)+|X|-(\mathrm{oc}(G-X)-1)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\mathrm{oc}(G-X))+\frac{1-q}{2} \\
& \quad<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G)+\frac{1-q}{2} \\
& \quad \leq a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|+q\right)+b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|+\frac{q(q-1)}{2}\right)-a_{k} \cdot\left(c\left(G^{\prime}\right)+1\right)+\frac{1-q}{2} \\
& =a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\frac{q-1}{2}\left(1-2 a_{k}-b_{k} q\right) \\
& \leq a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\frac{q-1}{2}\left(1-k a_{k}-k b_{k}\right) \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right) .
\end{aligned}
$$

As $\beta_{i}\left(G^{\prime}, X\right)=\beta_{i}(G, X)$ for $i \in[3]$ and $\beta_{4}\left(G^{\prime}, X\right)<\beta_{4}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$. Therefore, $q \geq k+1$.

Suppose that $k$ is even. Recall that $a_{k}=1 /(k(k+1))$ and $b_{k}=1 /(k+1)$. In this case, as $\Delta(G) \leq k$ and $C$ is not $k$-regular we have $|E(C)| \leq \frac{1}{2} k q-1$. This implies the following, as $q \geq k+1$.

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right)=\frac{1}{2}((|V(G)|-q)+|X|-(\operatorname{oc}(G-X)-1)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X))+\frac{1-q}{2} \\
& \quad<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G)+\frac{1-q}{2} \\
& \quad \leq a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|+q\right)+b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|+\frac{k q}{2}-1\right)-a_{k} \cdot\left(c\left(G^{\prime}\right)+1\right)+\frac{1-q}{2} \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\left(\frac{q-1}{2}-a_{k}(q-1)-b_{k} \frac{k q}{2}+b_{k}\right) \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\left(\frac{q-1}{2}-\frac{q-1}{k(k+1)}-\frac{k q}{2(k+1)}+\frac{1}{k+1}\right) \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\frac{q}{2(k+1)}\left((k+1)-\frac{2}{k}-k\right)+\frac{1}{2}-\frac{1}{k(k+1)}-\frac{1}{k+1} \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\frac{q}{2(k+1)}\left(1-\frac{2}{k}\right)+\frac{k(k+1)-2-2 k}{2 k(k+1)} \\
& \quad \leq a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\frac{1}{2}\left(\frac{k-2}{k}\right)+\frac{k^{2}-k-2}{2 k(k+1)} \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)+\frac{-(k-2)(k+1)+k^{2}-k-2}{2 k(k+1)} \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right) .
\end{aligned}
$$

As $\beta_{i}\left(G^{\prime}, X\right)=\beta_{i}(G, X)$ for $i \in[3]$ and $\beta_{4}\left(G^{\prime}, X\right)<\beta_{4}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$. Therefore, $k$ is odd, and so $a_{k}=(k-1) /\left(k\left(k^{2}-3\right)\right)$ and $b_{k}=\left(k^{2}-k-2\right) /\left(k\left(k^{2}-3\right)\right)$. In this case, as $\Delta(G) \leq k$ and $C$ is not $k$-regular we have $|E(C)| \leq \frac{1}{2}(k q-1)$. Further, since $C$ is an odd component, we note that $q$ is odd and $q \geq k+2$. This implies the following.

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right)=\frac{1}{2}((|V(G)|-q)+|X|-(\operatorname{oc}(G-X)-1)) \\
&=\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X))+\frac{1-q}{2} \\
&<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G)+\frac{1-q}{2} \\
& \leq a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|+q\right)+b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|+\frac{1}{2}(k q-1)\right)-a_{k} \cdot\left(c\left(G^{\prime}\right)+1\right)+\frac{1-q}{2} \\
&= a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\left(\frac{q-1}{2}-(q-1) a_{k}-\frac{1}{2}(k q-1) b_{k}\right) \\
&= a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\left(\frac{q-1}{2}-\frac{(k-1)(q-1)}{k\left(k^{2}-3\right)}-\frac{(k q-1)\left(k^{2}-k-2\right)}{2 k\left(k^{2}-3\right)}\right) \\
&=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)- \\
& \quad \frac{q}{2 k\left(k^{2}-3\right)}\left(k\left(k^{2}-3\right)-2(k-1)-k\left(k^{2}-k-2\right)\right)+ \\
& \quad \frac{1}{2 k\left(k^{2}-3\right)}\left(k\left(k^{2}-3\right)-2(k-1)-\left(k^{2}-k-2\right)\right) \\
&= a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\frac{q\left(k^{2}-3 k+2\right)}{2 k\left(k^{2}-3\right)}+\frac{k^{3}-k^{2}-4 k+4}{2 k\left(k^{2}-3\right)} \\
& \leq a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\frac{(k+2)\left(k^{2}-3 k+2\right)}{2 k\left(k^{2}-3\right)}+\frac{k^{3}-k^{2}-4 k+4}{2 k\left(k^{2}-3\right)} \\
&=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right) .
\end{aligned}
$$

As $\beta_{i}\left(G^{\prime}, X\right)=\beta_{i}(G, X)$ for $i \in[3]$ and $\beta_{4}\left(G^{\prime}, X\right)<\beta_{4}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$. Therefore, $s \geq 2$.

Let $x_{1} u_{1}, x_{2} u_{2}, \ldots, x_{s} u_{s}$ be distinct edges from $x_{i} \in X$ to $u_{i} \in V(C)$ for $i \in[s]$. Let $G^{\prime}$ be obtained from $G$ by adding $s-1$ new vertices $w_{2}, w_{3}, \ldots, w_{s}$, deleting the edges $x_{2} u_{2}, x_{3} u_{3}, \ldots, x_{s} u_{s}$ and adding the edges $x_{2} w_{2}, x_{3} w_{3}, \ldots, x_{s} w_{s}$. It is not difficult to see that no component of $G^{\prime}$ is $k$-regular. Note that $\left|V\left(G^{\prime}\right)\right|=|V(G)|+s-1$ and $\left|E\left(G^{\prime}\right)\right|=|E(G)|$ and $c\left(G^{\prime}\right) \leq c(G)+s-1$ and $\operatorname{oc}\left(G^{\prime}-X\right)=\mathrm{oc}(G-X)+s-1$. The following now holds.

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right)=\frac{1}{2}((|V(G)|+s-1)+|X|-(\operatorname{oc}(G-X)+s-1)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X)) \\
& \quad<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G) \\
& \quad \leq a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|-s+1\right)+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left(c\left(G^{\prime}\right)-s+1\right) \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)
\end{aligned}
$$

As $\beta_{i}\left(G^{\prime}, X\right)=\beta_{i}(G, X)$ for $i \in[3]$ and $\beta_{4}\left(G^{\prime}, X\right)<\beta_{4}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$. (ם)

Claim $\mathbf{E} \beta_{5}(G, X)=0$.

Proof. Suppose, to the contrary, that $\beta_{5}(G, X) \geq 1$. Let $C$ be an odd component $C$ in $G-X$ with $1<|V(C)|<k+1$. As $\beta_{4}(G, X)=0$ there is exactly one edge from $C$ to $X$. Let $x c$ be the edge with $x \in X$ and $c \in C$. Let $r=|V(C)|-1$.

Let $G^{\prime}=G-(V(C) \backslash\{c\})$. That is, $G^{\prime}$ is obtained from $G$ by removing all vertices of $C$, except $c$, from $G$. Note that $\left|V\left(G^{\prime}\right)\right|=|V(G)|-r$ and $\left|E\left(G^{\prime}\right)\right|=|E(G)|-|E(C)|$ and $c\left(G^{\prime}\right)=c(G)$ and oc $\left(G^{\prime}-X\right)=\operatorname{oc}(G-X)$. It is not difficult to see that no component of $G^{\prime}$ is $k$-regular. The following now holds.

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\mathrm{oc}\left(G^{\prime}-X\right)\right)=\frac{1}{2}((|V(G)|-r)+|X|-(\mathrm{oc}(G-X))) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\mathrm{oc}(G-X))-\frac{r}{2} \\
& \quad<a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G)-\frac{r}{2} \\
& \quad=a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|+r\right)+b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|+|E(C)|\right)-a_{k} \cdot c\left(G^{\prime}\right)-\frac{r}{2} \\
& \quad=a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)-\left(\frac{r}{2}-a_{k} r-b_{k}|E(C)|\right)
\end{aligned}
$$

We will now evaluate $\frac{r}{2}-a_{k} r-b_{k}|E(C)|$. Note that $|E(C)| \leq \frac{r(r+1)}{2}$ as $|V(C)|=r+1$ and $r \leq k-1$. If $k$ is even, then $a_{k}=1 /(k(k+1))$ and $b_{k}=1 /(k+1)$, which implies the following (as $0<r<k$ and $k \geq 2$ ).

$$
\begin{aligned}
\frac{r}{2}-a_{k} r-b_{k}|E(C)| & \geq \frac{r}{2}-\frac{1}{k(k+1)} r-\frac{1}{k+1} \times \frac{r(r+1)}{2} \\
& =\frac{r}{2 k(k+1)}(k(k+1)-2-k(r+1)) \\
& =\frac{r}{2 k(k+1)}(k(k-r)-2) \\
& \geq 0 .
\end{aligned}
$$

If $k$ is odd, then $a_{k}=(k-1) /\left(k\left(k^{2}-3\right)\right)$ and $b_{k}=\left(k^{2}-k-2\right) /\left(k\left(k^{2}-3\right)\right)$, which implies the following (as $0<r<k$ and $k \geq 3$ ).

$$
\begin{aligned}
\frac{r}{2}-a_{k} r-b_{k}|E(C)| & \geq \frac{r}{2}-\frac{(k-1) r}{k\left(k^{2}-3\right)}-\frac{r(r+1)\left(k^{2}-k-2\right)}{2 k\left(k^{2}-3\right)} \\
& =\frac{r}{2 k\left(k^{2}-3\right)}\left(k\left(k^{2}-3\right)-2(k-1)-\left(k^{2}-k-2\right)(r+1)\right) \\
& \geq \frac{r}{2 k\left(k^{2}-3\right)}\left(k^{3}-5 k+2-\left(k^{2}-k-2\right) k\right) \\
& \geq \frac{r}{2 k\left(k^{2}-3\right)}\left(k^{2}-3 k+2\right) \\
& \geq \frac{r(k-1)(k-2)}{2 k\left(k^{2}-3\right)} \\
& >0 .
\end{aligned}
$$

In both cases, $\frac{r}{2}-a_{k} r-b_{k}|E(C)| \geq 0$. This implies the following.

$$
\frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right)<a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot c\left(G^{\prime}\right)
$$

As $\beta_{i}\left(G^{\prime}, X\right)=\beta_{i}(G, X)$ for $i \in[4]$ and $\beta_{5}\left(G^{\prime}, X\right)<\beta_{5}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\beta_{1}(G, X), \beta_{2}(G, X), \ldots, \beta_{5}(G, X)\right)$. (口)

We now return to the proof of Theorem 4. Let $\mathcal{C}$ be the set of all components of $G$, and so $|\mathcal{C}(G)|=c(G)$.

Claim $\mathbf{F}$ If $C \in \mathcal{C}$, then

$$
\frac{1}{2}\left(|V(C)|+\left|X_{C}\right|-\operatorname{oc}\left(C-X_{C}\right)\right) \geq a_{k} \cdot|V(C)|+b_{k} \cdot|E(C)|-a_{k}
$$

Proof. Let $C \in \mathcal{C}$, and let $X_{C}=X \cap V(C)$. By Claim A, $\beta_{1}(G, X)=0$, and so there is no edge in $G\left[X_{C}\right]$. By Claim D, $\beta_{4}(G, X)=0$, and so every component in $C-X_{C}$ has exactly one edge to $X_{C}$. Since $C$ is connected, we therefore note that $\left|X_{C}\right|=1$. Let $X_{c}=\{x\}$. By Claim C, $\beta_{3}(G, X)=0$, and so $d_{G}(x)=d_{C}(x)=k$. Let $n_{i}$ denote that number of components in $C-X_{C}$ of order $i$, and so

$$
\mathrm{oc}\left(C-X_{C}\right)=k=\sum_{i=1}^{\infty} n_{i} .
$$

By Claim $\mathrm{B}, \beta_{2}(G, X)=0$ and by Claim $\mathrm{E}, \beta_{5}(G, X)=0$, implying that all $n_{i}$ are zero except possibly if $i=1$ or $i \geq k+1$ and $i$ is odd. We note that the expression $\frac{1}{2}\left(|V(C)|+\left|X_{C}\right|-\mathrm{oc}\left(C-X_{C}\right)\right)$ can be written in terms of $k$ and $\varepsilon_{k}$ as follows.

$$
\begin{aligned}
\frac{1}{2} & \left(|V(C)|+\left|X_{C}\right|-\operatorname{oc}\left(C-X_{C}\right)\right) \\
& =\frac{1}{2}\left(\left|X_{C}\right|+\left(\sum_{i=1}^{\infty} i \cdot n_{i}\right)+\left|X_{C}\right|-\operatorname{oc}\left(G-X_{G}\right)\right) \\
& =\frac{1}{2}\left(2+\left(\sum_{i=1}^{\infty} i \cdot n_{i}\right)-\left(1-\varepsilon_{k}\right) \operatorname{oc}\left(G-X_{G}\right)-\varepsilon_{k} \operatorname{oc}\left(G-X_{G}\right)\right) \\
& =\frac{1}{2}\left(2+\left(\sum_{i=1}^{\infty} i \cdot n_{i}\right)-\left(1-\varepsilon_{k}\right)\left(\sum_{i=1}^{\infty} n_{i}\right)-\varepsilon_{k} k\right) \\
& =1-\left(\frac{\varepsilon_{k}}{2}\right) k+\sum_{i=1}^{\infty}\left(i-1+\varepsilon_{k}\right) \frac{n_{i}}{2} \\
& =1-\left(\frac{\varepsilon_{k}}{2}\right) k+\left(\frac{\varepsilon_{k}}{2}\right) n_{1}+\sum_{i=k+1}^{\infty}\left(i-1+\varepsilon_{k}\right) \frac{n_{i}}{2} .
\end{aligned}
$$

Recall that

$$
a_{k}=\frac{\varepsilon_{k}}{2} \quad \text { and } \quad b_{k}=\frac{2-k \varepsilon_{k}}{2 k} .
$$

Hence, the expression $\frac{1}{2}\left(|V(C)|+\left|X_{C}\right|-\mathrm{oc}\left(C-X_{C}\right)\right)$ can be written as follows.

$$
\begin{equation*}
\frac{1}{2}\left(|V(C)|+\left|X_{C}\right|-\mathrm{oc}\left(C-X_{C}\right)\right)=k b_{k}+a_{k} n_{1}+\sum_{i=k+1}^{\infty}\left(i-1+\varepsilon_{k}\right) \frac{n_{i}}{2} \tag{2}
\end{equation*}
$$

We consider two cases, depending on the parity of $k$.
Case 1. $k$ is even. For all components, $C^{*}$, in $C-X_{C}$ we note that $C^{*}$ is not $k$-regular. Further, if $C^{*}$ has order $r$, then since $k$ is even, $\left|E\left(C^{*}\right)\right| \leq(r k-2) / 2$. Thus,

$$
\begin{align*}
a_{k} & \cdot|V(C)|+b_{k} \cdot|E(C)|-a_{k} \\
& \leq a_{k}\left(\left|X_{C}\right|+\sum_{i=1}^{\infty} i \cdot n_{i}\right)+b_{k}\left(k+\sum_{i=k+1}^{\infty} \frac{k i-2}{2}\right)-a_{k} \\
& =a_{k}\left(1+n_{1}+\sum_{i=k+1}^{\infty} i \cdot n_{i}\right)+b_{k}\left(k+\sum_{i=k+1}^{\infty} \frac{k i-2}{2}\right)-a_{k} \\
& =k b_{k}+a_{k} n_{1}+\sum_{i=k+1}^{\infty}\left(a_{i} \cdot i+b_{k}\left(\frac{k i-2}{2}\right)\right) n_{i} . \tag{3}
\end{align*}
$$

By Equation (2) and Inequality (3), we note that the desired result follows if the following is true for all $i \geq k+1$.

$$
\begin{aligned}
& \frac{1}{2}\left(i-1+\varepsilon_{k}\right) \geq a_{k} \cdot i+b_{k}\left(\frac{k i-2}{2}\right) \\
& \text { ॥ } \\
& \frac{1}{2}\left(i-1+\varepsilon_{k}\right) \geq\left(\frac{\varepsilon_{k}}{2}\right) i+\left(\frac{1}{k}-\frac{\varepsilon_{k}}{2}\right)\left(\frac{k i-2}{2}\right) \\
& \Uparrow \\
& 2 k\left(i-1+\varepsilon_{k}\right) \geq 2 k \varepsilon_{k} i+\left(2-k \varepsilon_{k}\right)(k i-2) \\
& \Uparrow \\
& 2 k i-2 k+2 k \varepsilon_{k} \geq 2 k \varepsilon_{k} i+2 k i-i k^{2} \varepsilon_{k}-4+2 k \varepsilon_{k} \\
& \text { I } \\
& i k \varepsilon_{k}(k-2) \geq 2 k-4 \\
& \Uparrow \\
& (k+1) k \varepsilon_{k}(k-2) \geq 2(k-2) \\
& \varepsilon_{k} \geq \frac{2}{k(k+1)}
\end{aligned}
$$

The above clearly holds since in this case when $k$ is even, $\varepsilon_{k}=2 /(k(k+1)$.
Case 2. $k \geq 3$ is odd. In this case, we note that all $n_{i}$ are zero except possibly if $i=1$ or $i \geq k+2$ and $i$ is odd. In particular, we note that in Equation (2) the term $n_{k+1}=0$. For all components, $C^{*}$, in $C-X_{C}$ we note that $C^{*}$ is not $k$-regular. Further, if $C^{*}$ has order $r$, then since $k$ is odd, $\left|E\left(C^{*}\right)\right| \leq(r k-1) / 2$. Thus,

$$
\begin{align*}
a_{k} & \cdot|V(C)|+b_{k} \cdot|E(C)|-a_{k} \\
& \leq a_{k}\left(\left|X_{C}\right|+\sum_{i=1}^{\infty} i \cdot n_{i}\right)+b_{k}\left(k+\sum_{i=k+2}^{\infty}\left(\frac{k i-1}{2}\right) n_{i}\right)-a_{k} \\
& =a_{k}\left(1+n_{1}+\sum_{i=k+2}^{\infty} i \cdot n_{i}\right)+b_{k}\left(k+\sum_{i=k+2}^{\infty}\left(\frac{k i-1}{2}\right) n_{i}\right)-a_{k} \\
& =k b_{k}+a_{k} n_{1}+\sum_{i=k+2}^{\infty}\left(a_{k} \cdot i+b_{k}\left(\frac{k i-1}{2}\right)\right) n_{i} . \tag{4}
\end{align*}
$$

By Equation (2) and Inequality (4), we note that the desired result follows if the following is true for all $i \geq k+2$.

$$
\begin{array}{rlrl} 
& & \frac{1}{2}\left(i-1+\varepsilon_{k}\right) & \geq a_{k} \cdot i+b_{k}\left(\frac{k i-1}{2}\right) \\
\Uparrow & \frac{1}{2}\left(i-1+\varepsilon_{k}\right) & \geq \frac{\varepsilon_{k}}{2} i+\left(\frac{1}{k}-\frac{\varepsilon_{k}}{2}\right)\left(\frac{k i-1}{2}\right) \\
\Uparrow & 2 k\left(i-1+\varepsilon_{k}\right) & \geq 2 k \varepsilon_{k} i+\left(2-k \varepsilon_{k}\right)(k i-1) \\
\Uparrow & 2 k i-2 k+2 k \varepsilon_{k} & \geq 2 k \varepsilon_{k} i+2 k i-i k^{2} \varepsilon_{k}-2+k \varepsilon_{k} \\
\Uparrow & i k \varepsilon_{k}(k-2) & \geq 2 k-k \varepsilon_{k}-2 \\
\Uparrow & & \\
\Uparrow & (k+2) k \varepsilon_{k}(k-2) & \geq 2 k-k \varepsilon_{k}-2 \\
\Uparrow & k \varepsilon_{k}\left(k^{2}-4+1\right) & \geq 2 k-2 \\
\Uparrow & \varepsilon_{k} & \geq \frac{2 k-2}{k\left(k^{2}-3\right)}
\end{array}
$$

The above clearly holds in this case when $k$ is odd, $\varepsilon_{k}=(2 k-2) /\left(k\left(k^{2}-3\right)\right)$. This completes the proof of Claim F. (ㅁ)

Applying Claim F to each component $C$ in $\mathcal{C}$, the following holds.

$$
\begin{aligned}
\frac{1}{2}(n+|X|-\mathrm{oc}(G-X)) & =\sum_{C \in \mathcal{C}} \frac{1}{2}\left(|V(C)|+\left|X_{C}\right|-\mathrm{oc}\left(C-X_{C}\right)\right) \\
& \geq \sum_{C \in \mathcal{C}}\left(a_{k} \cdot|V(C)|+b_{k} \cdot|E(C)|-a_{k}\right) \\
& =a_{k} \cdot|V(G)|+b_{k} \cdot|E(G)|-a_{k} \cdot c(G),
\end{aligned}
$$

which contradicts Inequality (1), thereby proving the theorem.

## 5 Proof of Theorem 5

For all even $k \geq 4$, let

$$
a_{k}=\frac{k-2}{k^{2}+k+2} \quad \text { and } \quad b_{k}=\frac{k+2}{k^{2}+k+2} .
$$

We note that for $k \geq 2$ even,

$$
k b_{k}-a_{k}=\frac{k(k+2)-(k-2)}{k^{2}+k+2}=1 .
$$

Theorem 5 can now be restated as follows.
Theorem 5 Let $k \geq 2$ be an even integer and let $G$ be any graph of order $n$ and size $m$ and maximum degree $\Delta(G) \leq k$. If no component of $G$ is $k$-regular, then $\alpha^{\prime}(G) \geq b_{k} \cdot m-a_{k} \cdot n$.

Proof of Theorem 5. Let $K_{k+1}-e$ denote the complete graph on $k+1$ vertices after removing one edge. Define the following five values of a graph $G^{*}$ and vertex set $X^{*} \subseteq V\left(G^{*}\right)$.
(1): $\xi_{1}\left(G^{*}, X^{*}\right)$ is the number of edges in $G^{*}\left[X^{*}\right]$.
(2): $\xi_{2}\left(G^{*}, X^{*}\right)$ is the number of even components in $G^{*}-X^{*}$.
(3): $\xi_{3}\left(G^{*}, X^{*}\right)$ is the number of vertices in $X^{*}$ with degree less than $k$.
(4): $\xi_{4}\left(G^{*}, X^{*}\right)$ is the number of components in $G^{*}-X^{*}$ that do not have exactly one edge to $X^{*}$.
(5): $\xi_{5}\left(G^{*}, X^{*}\right)$ is the number of odd components in $G^{*}-X^{*}$ not isomorphic to $K_{k+1}-e$.

For the sake of contradiction suppose that the theorem is false and that $G$ is a counter example to the theorem. That is, $G$ has maximum degree at most $k$ and no component of $G$ is $k$-regular and $\alpha^{\prime}(G)<b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)|$. By Theorem 1 (the Tutte-Berge formula) we may assume that $G$ and $X$ are chosen such that the following holds.

$$
\begin{equation*}
\frac{1}{2}(n+|X|-\mathrm{oc}(G-X))<b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)| . \tag{5}
\end{equation*}
$$

Furthermore we may assume that $\left(\xi_{1}(G, X), \xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$ is lexicographically minimum of all $G$ and $X$ satisfying the above. Note that if $G$ is not connected, then one of the components of $G$ is also a counter example to the theorem and this component either has the same value of $\left(\xi_{1}(G, X), \xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$ or smaller. We may therefore assume that $G$ is connected, for otherwise we consider the before mentioned component of $G$. We proceed further with the following series of claims.

Claim I $\xi_{1}(G, X)=0$.

Proof. Suppose, to the contrary, that $\xi_{1}(G, X) \geq 1$, and let $x_{1} x_{2} \in E(G)$ be arbitrary where $x_{1}, x_{2} \in X$. Delete the edge $x_{1} x_{2}$ and add a new vertex $u$ and the edges $x_{1} u$ and $u x_{2}$. Let $G^{\prime}$ be the resulting graph. We note that $\left|V\left(G^{\prime}\right)\right|=|V(G)|+1,\left|E\left(G^{\prime}\right)\right|=|E(G)|+1$ and $\mathrm{oc}\left(G^{\prime}-X\right)=\mathrm{oc}(G-X)+1$. Also, no component in $G^{\prime}$ is $k$-regular (as if any component in $G^{\prime}$ is $k$-regular, then the corresponding component in $G$ would also be $k$-regular). The following now holds, as $b_{k}>a_{k}$.

$$
\begin{aligned}
\frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right) & =\frac{1}{2}((|V(G)|+1)+|X|-(\mathrm{oc}(G-X)+1)) \\
& =\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X)) \\
& <b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)| \\
& <b_{k}(|E(G)|+1)-a_{k}(|V(G)|+1) \\
& =b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right| .
\end{aligned}
$$

As $\xi_{1}\left(G^{\prime}, X\right)<\xi_{1}(G, X)$ this contradicts the lexicographical minimality of ( $\xi_{1}(G, X)$, $\left.\xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$.(ロ)

Claim II $\xi_{2}(G, X)=0$.

Proof. Suppose, to the contrary, that $\xi_{2}(G, X) \geq 1$. Let $C$ be an even component $C$ in $G-X$, and let $u$ be a leaf in some spanning tree of $C$. In this case $C-\{u\}$ is connected. Let $G^{\prime}=G-u$ and note that $\left|V\left(G^{\prime}\right)\right|=|V(G)|-1,\left|E\left(G^{\prime}\right)\right| \geq|E(G)|-k$ and oc $\left(G^{\prime}-X\right)=\mathrm{oc}(G-X)+1$. The following now holds (as we have shown that $k b_{k}-a_{k}=1$ ).

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right)=\frac{1}{2}((|V(G)|-1)+|X|-(\operatorname{oc}(G-X)+1)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X))-1 \\
& \quad<b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)|-1 \\
& \quad \leq b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|+k\right)-a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|+1\right)-1 \\
& \quad=b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right| .
\end{aligned}
$$

Any component in $G^{\prime}$ is either a component in $G$ or contains a vertex of degree at most $k-1$ (adjacent to $u$ in $G$ ), which implies that no component of $G^{\prime}$ is $k$-regular. Furthermore as $\xi_{1}\left(G^{\prime}, X\right)=\xi_{1}(G, X)=0$ and $\xi_{2}\left(G^{\prime}, X\right)<\xi_{2}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\xi_{1}(G, X), \xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$. (ם)

Claim III $\xi_{3}(G, X)=0$.

Proof. Suppose, to the contrary, that $\xi_{3}(G, X) \geq 1$. Let $x$ be a vertex in $X$ with $d_{G}(x)<$ $k$. Let $G^{\prime}$ be obtained by adding $s=k-d_{G}(x)$ new vertices $u_{1}, u_{2}, \ldots, u_{s}$ and the edges $u_{1} x, u_{2} x, \ldots, u_{s} x$ to $G$. Note that $\left|V\left(G^{\prime}\right)\right|=|V(G)|+s$ and $\left|E\left(G^{\prime}\right)\right|=|E(G)|+s$ and $\operatorname{oc}\left(G^{\prime}-X\right)=\operatorname{oc}(G-X)+s$. The following now holds, as $b_{k}>a_{k}$.

$$
\begin{aligned}
\frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right) & =\frac{1}{2}((|V(G)|+s)+|X|-(\mathrm{oc}(G-X)+s)) \\
& =\frac{1}{2}(|V(G)|+|X|-\mathrm{oc}(G-X)) \\
& <b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)| \\
& =b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|-s\right)-a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|-s\right) \\
& =b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|-s\left(b_{k}-a_{k}\right) \\
& <b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|
\end{aligned}
$$

Any component in $G^{\prime}$ is either a component in $G$ or contains vertices of degree $1<k$ (adjacent to $x$ in $G$ ), which implies that no component of $G^{\prime}$ is $k$-regular. As $\xi_{i}\left(G^{\prime}, X\right)=$ $\xi_{i}(G, X)$ for $i \in[2]$ and $\xi_{3}\left(G^{\prime}, X\right)<\xi_{3}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\xi_{1}(G, X), \xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$. (ロ)

Claim IV $\xi_{4}(G, X)=0$.

Proof. Suppose, to the contrary, that $\xi_{4}(G, X) \geq 1$. Let $C$ be an odd component in $G-X$ with $s$ edges to $X$, where $s \neq 1$. Let $q=|V(C)|$.

Suppose that $s=0$, which as $G$ is connected (which was proved before Claim I) implies that $|X|=0$. By Claim II we note that $q$ is odd. By Equation (5) the following holds.

$$
\begin{equation*}
\frac{q-1}{2}=\frac{1}{2}(n+|X|-\mathrm{oc}(G-X))<b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)| \tag{6}
\end{equation*}
$$

Suppose that $q \leq k$. In this case, we note that $|E(C)| \leq\binom{ q}{2}$, which implies the following.

$$
\frac{q-1}{2}<b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)| \leq b_{k} \cdot \frac{q(q-1)}{2}-a_{k} \cdot q=q\left(\left(\frac{q-1}{2}\right) b_{k}-a_{k}\right)
$$

If $q=1$, then $b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)| \leq-a_{k}<0=(q-1) / 2$, a contradiction. By Claim II we note that $q$ is odd, and so $q \geq 3$, which implies that $b_{k} \frac{(q-1)}{2}-a_{k}>0$. Therefore the following holds.

$$
\begin{aligned}
\frac{q-1}{2} & <q\left(b_{k} \frac{(q-1)}{2}-a_{k}\right) \\
& \leq k\left(b_{k} \frac{(q-1)}{2}-a_{k}\right) \\
& =k\left(\frac{k+2}{k^{2}+k+2} \times \frac{(q-1)}{2}-\frac{k-2}{k^{2}+k+2}\right) \\
& =\frac{q-1}{2}+\left(\frac{q-1}{2}\right)\left(\frac{k(k+2)}{k^{2}+k+2}-1\right)-\frac{k(k-2)}{k^{2}+k+2} \\
& =\frac{q-1}{2}+\left(\frac{q-1}{2}\right)\left(\frac{k-2}{k^{2}+k+2}\right)-k\left(\frac{k-2}{k^{2}+k+2}\right) \\
& =\frac{q-1}{2}+\left(\frac{k-2}{k^{2}+k+2}\right)\left(\frac{q-1}{2}-k\right) \\
& <\frac{q-1}{2} .
\end{aligned}
$$

This is clearly a contradiction, which implies that, $q \geq k+1$. Therefore, since $G$ is not $k$-regular, $|E(G)| \leq \frac{k q}{2}-1$ and

$$
\begin{aligned}
\frac{q-1}{2} & <b_{k}\left(\frac{q k-2}{2}\right)-a_{k} \cdot q \\
& =\frac{q}{2}\left(k b_{k}-a_{k}\right)-\frac{a_{k} q}{2}-b_{k} \\
& =\frac{q}{2}-\frac{a_{k} q}{2}-b_{k} \\
& \leq \frac{q}{2}-\frac{a_{k}(k+1)}{2}-b_{k} \\
& =\frac{q}{2}-\frac{(k-2)(k+1)}{2\left(k^{2}+k+2\right)}-\frac{k+2}{k^{2}+k+2} \\
& =\frac{q}{2}-\frac{k^{2}+k+2}{2\left(k^{2}+k+2\right)} \\
& =\frac{q-1}{2}
\end{aligned}
$$

This is a contradiction which implies that $s \geq 2$. Recall that $C$ is an odd component in $G-X$ with $s$ edges to $X$. Let $H_{k+1}$ denote $K_{k+1}-e$ and let the two vertices of $H_{k+1}$ with degree $k-1$ be called link vertices. Let $x_{1} u_{1}, x_{2} u_{2}, \ldots, x_{s} u_{s}$ be distinct edges from $x_{i} \in X$ to $u_{i} \in V(C)$ for $i \in[s]$. Let $G^{\prime}$ be obtained from $G$ by adding $s-1$ vertex disjoint copies, $G_{1}, G_{2}, \ldots, G_{s-1}$, of $H_{k+1}$, and deleting the edges $x_{1} u_{1}, x_{2} u_{2}, \ldots, x_{s-1} u_{s-1}$ and adding an edge from $x_{i}$ to a link vertex of $G_{i}$ for all $i \in[s-1]$. Since no component of $G$ is $k$-regular, it is not difficult to see that no component of $G^{\prime}$ is $k$-regular. Note that the following holds.

- $\left|V\left(G^{\prime}\right)\right|=|V(G)|+(s-1)(k+1)$
- $\left|E\left(G^{\prime}\right)\right|=|E(G)|+(s-1)\left(\frac{k(k+1)}{2}-1\right)$
- oc $\left(G^{\prime}-X\right)=\mathrm{oc}(G-X)+s-1$

This implies the following.

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right) \\
& \quad=\frac{1}{2}((|V(G)|+(s-1)(k+1))+|X|-(\operatorname{oc}(G-X)+s-1)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X))+\frac{(s-1)(k+1)}{2}-\frac{s-1}{2} \\
& \quad<b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)|+\frac{k(s-1)}{2} \\
& \quad=b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|-(s-1)\left(\frac{k^{2}+k-2}{2}\right)\right)-a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|-(s-1)(k+1)\right)+\frac{k(s-1)}{2} \\
& \quad=b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+(s-1)\left(\frac{k}{2}-b_{k}\left(\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1)\right) .
\end{aligned}
$$

As $\xi_{i}\left(G^{\prime}, X\right)=\xi_{i}(G, X)$ for $i \in[3]$ and $\xi_{4}\left(G^{\prime}, X\right)<\xi_{4}(G, X)$ we would obtain a contradiction to the lexicographical minimality of $\left(\xi_{1}(G, X), \xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$ if the following holds.

$$
\begin{align*}
& 0 \geq \frac{k}{2}-b_{k}\left(\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1) \\
& \Downarrow \\
& \frac{k}{2} \leq \frac{k+2}{k^{2}+k+2} \times \frac{k^{2}+k-2}{2}-\left(\frac{k-2}{k^{2}+k+2}\right)(k+1) \\
& \Uparrow \\
& k\left(k^{2}+k+2\right) \leq(k+2)\left(k^{2}+k-2\right)-2(k-2)(k+1) \\
& \downarrow \\
& k^{3}+k^{2}+2 k \leq k^{3}+3 k^{2}-4-\left(2 k^{2}-2 k-4\right) \\
& k^{3}+k^{2}+2 k \leq k^{3}+k^{2}+2 k
\end{align*}
$$

As the last statement is clearly true, Claim IV is proved. (ם)

Claim V $\xi_{5}(G, X)=0$.

Proof. Suppose, to the contrary, that $\xi_{5}(G, X) \geq 1$. Let $C$ be a component in $G-X$ which in not isomorphic to $K_{k+1}-e$. By Claim II and Claim IV we note that $C$ is odd and has exactly one edge to $X$. Let $q=|V(C)|$ and let $G^{\prime}$ be obtained from $G$ by deleting $C$ and adding a copy of $K_{k+1}-e$ and adding an edge from a degree $k-1$ vertex in $K_{k+1}-e$ to the vertex of $X$ that was adjacent to a vertex of $C$ in $G$. The following now holds.

- $\left|V\left(G^{\prime}\right)\right|=|V(G)|+k+1-q$
- $\left|E\left(G^{\prime}\right)\right|=|E(G)|+\frac{k(k+1)}{2}-1-|E(C)|$
- oc $\left(G^{\prime}-X\right)=$ oc $(G-X)$

Since no component of $G$ is $k$-regular, it is not difficult to see that no component of $G^{\prime}$ is $k$-regular. The following now holds.

$$
\begin{aligned}
& \frac{1}{2}\left.\left(\left|V\left(G^{\prime}\right)\right|+|X|-\mathrm{oc}\left(G^{\prime}-X\right)\right)\right) \\
& \quad=\frac{1}{2}((|V(G)|+k+1-q)+|X|-(\mathrm{oc}(G-X)) \\
& \quad=\frac{1}{2}(|V(G)|+|X|-\operatorname{oc}(G-X))+\frac{k+1-q}{2} \\
& \quad<b_{k} \cdot|E(G)|-a_{k} \cdot|V(G)|+\frac{k+1-q}{2} \\
& \quad=b_{k} \cdot\left(\left|E\left(G^{\prime}\right)\right|-\frac{k(k+1)}{2}+1+|E(C)|\right)-a_{k} \cdot\left(\left|V\left(G^{\prime}\right)\right|-(k+1-q)\right)+\frac{k+1-q}{2} \\
& \quad=b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+\frac{k+1-q}{2}+b_{k}\left(|E(C)|-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q) .
\end{aligned}
$$

We will now consider the cases when $q \leq k$ and $q \geq k+1$ seperately. First consider the case when $q \leq k$. In this case, $|E(C)| \leq q(q-1) / 2$ and the above implies the following.

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\operatorname{oc}\left(G^{\prime}-X\right)\right) \\
& \quad<b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+\frac{k+1-q}{2}+b_{k}\left(|E(C)|-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q) \\
& \quad \leq b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+\frac{k+1-q}{2}+b_{k}\left(\frac{q(q-1)}{2}-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q)
\end{aligned}
$$

As $\xi_{i}\left(G^{\prime}, X\right)=\xi_{i}(G, X)$ for $i \in[4]$ and $\xi_{5}\left(G^{\prime}, X\right)<\xi_{5}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\xi_{1}(G, X), \xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$ if the following holds

$$
\Uparrow \begin{aligned}
\frac{k+1-q}{2}+b_{k}\left(\frac{q(q-1)}{2}-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q) & \leq 0 \\
-q+b_{k} q(q-1)-2 a_{k} q & \leq b_{k}\left(k^{2}+k-2\right)-(k+1)-2 a_{k}(k+1)
\end{aligned}
$$

Recall that $1 \leq q \leq k$. Differentiating the left-hand-side twice with respect to $q$ we note that the largest value of $-q+b_{k} q(q-1)-2 a_{k} q$ in the interval $[1, q]$ is obtained at the end points of the interval, when $q=1$ or $q=k$. When $q=1$,

$$
-q+b_{k} q(q-1)-2 a_{k} q=-1-2 a_{k}
$$

and when $q=k$, we get

$$
\begin{aligned}
-q+b_{k} q(q-1)-2 a_{k} q & =-k+b_{k} k(k-1)-2 a_{k} k \\
& =-1+(k-1)\left(-1+b_{k} k-a_{k}\right)-a_{k}(k+1) \\
& =-1-a_{k}(k+1) \\
& <-1-2 a_{k} .
\end{aligned}
$$

Therefore the following holds.

$$
\begin{align*}
& \frac{k+1-q}{2}+b_{k}\left(\frac{q(q-1)}{2}-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q) \leq 0 \\
& \text { ॥ } \\
& -q+b_{k} q(q-1)-2 a_{k} q \leq b_{k}\left(k^{2}+k-2\right)-(k+1)-2 a_{k}(k+1) \\
& \text { 介 } \\
& -1-2 a_{k} \leq b_{k}\left(k^{2}+k-2\right)-(k+1)-2 a_{k}(k+1) \\
& \text { I } \\
& 0 \leq b_{k}\left(k^{2}+k\right)-2 b_{k}-(k+1)-2 a_{k}(k+1)+1+2 a_{k} \\
& 0 \leq(k+1)\left(k b_{k}-a_{k}-1\right)-2 b_{k}-a_{k}(k+1)+1+2 a_{k} \\
& 0 \leq-2 b_{k}-(k-1) a_{k}+1 \\
& 0 \leq \frac{-2(k+2)}{k^{2}+k+2}-\frac{(k-1)(k-2)}{k^{2}+k+2}+\frac{k^{2}+k+2}{k^{2}+k+2} \\
& 0 \leq \frac{(-2 k-4)-\left(k^{2}-3 k+2\right)+\left(k^{2}+k+2\right)}{k^{2}+k+2} \\
& 0 \leq \frac{2 k-4}{k^{2}+k+2}
\end{align*}
$$

As the last statement is true we have completed the proof for the case when $q \leq k$ ．Now consider the case when $q \geq k+1$ ．In this case $|E(C)| \leq \frac{q k}{2}-1$（as $C$ is not $k$－regular）and the following holds．

$$
\begin{aligned}
& \frac{1}{2}\left(\left|V\left(G^{\prime}\right)\right|+|X|-\mathrm{oc}\left(G^{\prime}-X\right)\right) \\
& \quad<b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+\frac{k+1-q}{2}+b_{k}\left(|E(C)|-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q) \\
& \quad \leq b_{k} \cdot\left|E\left(G^{\prime}\right)\right|-a_{k} \cdot\left|V\left(G^{\prime}\right)\right|+\frac{k+1-q}{2}+b_{k}\left(\frac{q k-2}{2}-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q) .
\end{aligned}
$$

As $\xi_{i}\left(G^{\prime}, X\right)=\xi_{i}(G, X)$ for $i \in[4]$ and $\xi_{5}\left(G^{\prime}, X\right)<\xi_{5}(G, X)$ we obtain a contradiction to the lexicographical minimality of $\left(\xi_{1}(G, X), \xi_{2}(G, X), \ldots, \xi_{5}(G, X)\right)$ if the following holds

$$
\begin{aligned}
& \frac{k+1-q}{2}+b_{k}\left(\frac{q k-2}{2}-\frac{k^{2}+k-2}{2}\right)+a_{k}(k+1-q) \leq 0 \\
& \Uparrow \\
& q\left(-1+k b_{k}-2 a_{k}\right) \leq b_{k}\left(k^{2}+k-2+2\right)-(k+1)-2 a_{k}(k+1) \\
& \Uparrow \\
& q\left(-1+1-a_{k}\right) \leq b_{k} k(k+1)-(k+1)-2 a_{k}(k+1) \\
& \text { 介 } \\
& (k+1)\left(-a_{k}\right) \leq(k+1)\left(b_{k} k-1-2 a_{k}\right) \\
& -a_{k} \leq\left(b_{k} k-a_{k}\right)-1-a_{k} \\
& \text { I } \\
& -a_{k} \leq-a_{k}
\end{aligned}
$$

The last statement is clearly true which completes the proof of Claim V．（ם）

We now return to the proof of Theorem 5, As mentioned before the statement of Claim I, we may assume that $G$ is connected. By Claim $\mathrm{I}, \xi_{1}(G, X)=0$, and so there is no edge in $G[X]$. By Claim IV, $\xi_{4}(G, X)=0$, and so every component in $C-X$ has exactly one edge to $X$. Since $C$ is connected, we therefore note that $|X|=1$. Let $X=\{x\}$. By Claim III, $\xi_{3}(G, X)=0$, and so $d_{G}(x)=k$. By Claim V, every component in $C-X$ is isomorphic to $K_{k+1}-e$, which implies the following.

- $|V(G)|=k(k+1)+1=k^{2}+k+1$.
- $|E(G)|=k\left(\frac{k(k+1)}{2}-1\right)+d(x)=k\left(\frac{k^{2}+k-2}{2}\right)+k$.
- $\alpha^{\prime}(G)=k\left(\frac{k}{2}\right)+1=\frac{k^{2}+2}{2}$.

Therefore, the following holds.

$$
\begin{aligned}
b_{k}|E(G)|-a_{k}|V(G)| & =\frac{k+2}{k^{2}+k+2} \times k \times\left(\frac{k^{2}+k-2}{2}+1\right)-\frac{k-2}{k^{2}+k+2} \times\left(k^{2}+k+1\right) \\
& =\frac{1}{2\left(k^{2}+k+2\right)}\left(k(k+2)\left(k^{2}+k\right)-2(k-2)\left(k^{2}+k+2\right)\right) \\
& =\frac{\left(k^{4}+3 k^{3}+2 k^{2}\right)-\left(2 k^{3}-2 k^{2}-2 k-4\right)}{2\left(k^{2}+k+2\right)} \\
& =\frac{k^{4}+k^{3}+4 k^{2}+2 k+4}{2\left(k^{2}+k+2\right)} \\
& =\frac{\left(k^{2}+2\right)\left(k^{2}+k+2\right)}{2\left(k^{2}+k+2\right)} \\
& =\frac{k^{2}+2}{2} \\
& =\alpha^{\prime}(G) .
\end{aligned}
$$

This contradicts the fact that $G$ was a counter example to the theorem.

## 6 Proof of Corollary 1

Recall the statement of Corollary (1.
Corollary 1 If $k \geq 4$ is an even integer and $G$ is a connected graph of order $n$, size $m$ and maximum degree $\Delta(G) \leq k$, then

$$
\alpha^{\prime}(G) \geq \frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)},
$$

unless the following holds.
(a) $G$ is $k$-regular and $n=k+1$, in which case $\alpha^{\prime}(G)=\frac{n-1}{2}=\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k}$.
(b) $G$ is $k$-regular and $n=k+3$, in which case $\alpha^{\prime}(G)=\frac{n-1}{2}=\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{3}{k(k+1)}$.

Proof of Corollary 1, If $G$ is not $k$-regular, then the result follows from Theorem 4, so assume that $G$ is $k$-regular. By the $k$-regularity of $G$ we have $n k=2 m$, which together with the observation that

$$
\frac{k^{2}+2}{k^{2}+k} \leq \frac{k^{2}+4}{k^{2}+k+2} .
$$

when $k \geq 4$, implies the following.

$$
\begin{aligned}
\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)} & =\frac{n}{k(k+1)}+\frac{n k}{2(k+1)}-\frac{1}{k(k+1)} \\
& =\frac{n}{2}\left(\frac{2}{k(k+1)}+\frac{k}{k+1}\right)-\frac{1}{k(k+1)} \\
& <\frac{n}{2}\left(\frac{2}{k(k+1)}+\frac{k^{2}}{k(k+1)}\right) \\
& =\frac{n}{2} \times \frac{k^{2}+2}{k(k+1)} \\
& \leq \frac{n}{2} \times \frac{k^{2}+4}{k^{2}+k+2} .
\end{aligned}
$$

By Theorem 2 we therefore have the following.

$$
\alpha^{\prime}(G) \geq \min \left\{\frac{k^{2}+4}{k^{2}+k+2} \times \frac{n}{2}, \frac{n-1}{2}\right\} \geq \min \left\{\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)}, \frac{n-1}{2}\right\} .
$$

As $\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)}<\frac{n}{2}$ by the above, this proves the theorem when $n$ is even. So let $n$ be odd. We will now determine when the following holds.

$$
\frac{n}{k(k+1)}+\frac{n k}{2(k+1)}-\frac{1}{k(k+1)}>\frac{n-1}{2} .
$$

Define $r$ such that $n=k+r$ and note that the above is equivalent to the following.

$$
\begin{aligned}
0 & <\frac{n}{k(k+1)}+\frac{n k}{2(k+1)}-\frac{1}{k(k+1)}-\frac{n-1}{2} \\
& =n\left(\frac{2+k^{2}-k(k+1)}{2 k(k+1)}\right)-\frac{1}{k(k+1)}+\frac{1}{2} \\
& =(k+r)\left(\frac{2-k}{2 k(k+1)}\right)+\frac{-2+k(k+1)}{2 k(k+1)} \\
& =\frac{-k^{2}+2 k+2 r-k r}{2 k(k+1)}+\frac{k^{2}+k-2}{2 k(k+1)} \\
& =\frac{3 k+2 r-k r-2}{2 k(k+1)} .
\end{aligned}
$$

This is equivalent to $3 k-2-r(k-2)>0$, which is equivalent to the following (as $k \geq 4$ ).

$$
r<\frac{3 k-2}{k-2}
$$

When $k \geq 4$ this is equivalent to $r<5$ (as $r$ is odd) and therefore the following holds (as we already handled the case when $n$ was even).

$$
\begin{aligned}
\alpha^{\prime}(G) & \geq \min \left\{\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)}, \frac{n-1}{2}\right\} \\
& =\left\{\begin{array}{cc}
\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)} & \text { if } n \notin\{k+1, k+3\} \\
\frac{n-1}{2} & \text { if } n \in\{k+1, k+3\} .
\end{array}\right.
\end{aligned}
$$

We will therefore determine $\theta$ such that the following holds.

$$
\begin{aligned}
\frac{n-1}{2} & =\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{\theta}{k(k+1)} \\
\Uparrow & \\
2 \theta & =2 n+2 k m-(n-1)(k(k+1)) \\
& =2 n+n k^{2}-n\left(k^{2}+k\right)+\left(k^{2}+k\right) \\
& =n(2-k)+k^{2}+k \\
& =(k+r)(2-k)+k^{2}+k \\
& =3 k+2 r-r k
\end{aligned}
$$

So, if $r=3$, then $\theta=3$ and if $r=1$, then $\theta=k+1$. This completes the proof of Corollary 1 ,
The lower bound in Corollary 1 is tight when $G$ is $k$-regular and $n \in\{k+1, k+3\}$. We will furthermore show that they are tight for infinite classes of trees and infinite classes of graphs with any given average degree between 2 and $k-\frac{k-2}{k^{2}}$, implying that the corollaries are tight for almost all possible densities. Note that for small values of $k$ the value $k-\frac{k-2}{k^{2}}$ is the following.

| $k$ | 4 | 6 | 8 | 10 | 12 | 14 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $k-\frac{k-2}{k^{2}}$ | 3.875 | $5.888 \ldots$ | $7.906 \ldots$ | 9.92 | $11.93 \ldots$ | $13.938 \ldots$ |

Table 3. The value $k-\frac{k-2}{k^{2}}$ for small values of $k \geq 4$ with $k$ even.

We will illustrate how to obtain the above. Let $k \geq 4$ be even and let $r \geq 1$ be arbitrary and let $\ell=r(k-1)+1$. Let $X_{1}, X_{2}, \ldots, X_{\ell}$ be a number of vertex disjoint graphs such that each $X_{i}$ where $i \in[\ell]$ is either a single vertex or it is a $K_{k+1}$ where an arbitrary edge has been deleted. Let $Y=\left\{y_{1}, y_{2}, \ldots, y_{r}\right\}$ and build the graph $G_{k, r}$ as follows. Let $G_{k, r}$ be obtained from the disjoint union of the graphs $X_{1}, X_{2}, \ldots, X_{\ell}$ by adding to it the vertices in $Y$ and furthermore, for every $i \in[r]$, adding an edge from $y_{i}$ to a vertex in each graph $X_{(i-1)(k-1)+1}$, $X_{(i-1)(k-1)+2}, X_{(i-1)(k-1)+3}, \ldots, X_{(i-1)(k-1)+k}$ in such a way that no vertex degree becomes more than $k$. Let $\mathcal{G}_{k, r}$ be the family of all such graph $G_{k, r}$.

When $k=4$ and $r=2$, an example of a graph $G$ in the family $\mathcal{G}_{k, r}$ is illustrated in Figure 2, where $G$ has order $n=21$, size $m=35$ and matching number $\alpha^{\prime}(G)=8$.


Figure 2: A graph $G$ in the family $\mathcal{G}_{4,2}$

Proposition 1 For $k \geq 4$ an even integer and $r \geq 1$ arbitrary, if $G \in \mathcal{G}_{k, r}$ has order $n$ and size $m$, then

$$
\alpha^{\prime}(G)=\frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k}
$$

Proof. Assume that in $X_{1}, X_{2}, \ldots, X_{\ell}$ we have $\ell_{1}$ single vertices and $\ell_{2}$ copies of $K_{k+1}$ 's minus an edge. Note that $\ell=\ell_{1}+\ell_{2}$ and $n=r+\ell_{1}+\ell_{2}(k+1)$. Furthermore we have $m=r k+\ell_{2}(k(k+1) / 2-1)$ and $\alpha^{\prime}(G)=r+\ell_{2}(k / 2)$. Therefore the following holds.

$$
\begin{aligned}
& \frac{n}{k(k+1)}+\frac{m}{k+1}-\frac{1}{k(k+1)} \\
& =\frac{r+\ell_{1}+\ell_{2}(k+1)}{k(k+1)}+\frac{2 r k+\ell_{2}\left(k^{2}+k-2\right)}{2(k+1)}-\frac{1}{k(k+1)} \\
& =\frac{r+\left(\ell-\ell_{2}\right)+\ell_{2} k+\ell_{2}}{k(k+1)}+\frac{2 r k^{2}+\ell_{2} k\left(k^{2}+k-2\right)}{2 k(k+1)}-\frac{2}{2 k(k+1)} \\
& =\frac{2 r+2 \ell+2 \ell_{2} k+2 r k^{2}+\ell_{2} k\left(k^{2}+k-2\right)-2}{2 k(k+1)} \\
& =\frac{2 r+2(r(k-1)+1)+2 r k^{2}+\ell_{2} k\left(k^{2}+k\right)-2}{2 k(k+1)} \\
& =\frac{2 r+2 r k-2 r+2+2 r k^{2}+\ell_{2} k^{2}(k+1)-2}{2 k(k+1)} \\
& =\frac{2 r\left(k^{2}+k\right)+\ell_{2} k^{2}(k+1)}{2 k(k+1)} \\
& =r+\frac{1}{2} k \ell_{2} \\
& =\alpha^{\prime}(G) .
\end{aligned}
$$

By Proposition the lower bound on the matching number in Corollary 1 is tight for every graph in the family $\mathcal{G}_{k, r}$. We remark that if all $X_{1}, X_{2}, \ldots, X_{\ell}$ used to construct a graph in the family $\mathcal{G}_{k, r}$ are single vertices, then clearly we have a tree and as $r \geq 1$ was arbitrary we obtain an infinite class of trees where the corollaries are tight. If all $X_{1}, X_{2}, \ldots, X_{\ell}$ are copies of $K_{k+1}$ minus an edge, then we denote the resulting subfamily of graphs of $\mathcal{G}_{k, r}$ by $\mathcal{G}_{k, r}^{\prime}$. Hence, each graph in the family $\mathcal{G}_{k, r}^{\prime}$ achieves the lower bound in Corollary (1) We remark, further, that if we build a graph in the family $\mathcal{G}_{k, r}$ using only single vertices for the copies of $X_{i}$ for each $i \in[\ell]$, then the resulting graph $G$ is a tree. Hence, as an immediate consequence of Proposition 1 and the above observations, we have the following result.

Proposition 2 The lower bound in Corollary $\mathbb{1}$ is achieved for both trees and for the class of graphs in the family $\mathcal{G}_{k, r}^{\prime}$.

We note that the average degree of a graph in the family $\mathcal{G}_{k, r}^{\prime}$ is the following, as $n=$ $\ell(k+1)+r$ and there are $\ell-(r-1)$ vertices of degree $k-1$.

$$
\begin{aligned}
\frac{1}{n} \sum_{v \in V(G)} d(v) & =\frac{1}{n}(n k-(\ell-(r-1))) \\
& =k-\frac{1}{n}(\ell-r+1) \\
& =k-\frac{r(k-1)+1-r+1}{(r(k-1)+1)(k+1)+r} \\
& =k-\frac{r(k-2)+2}{r k^{2}+k+1}
\end{aligned}
$$

So when $r$ is large the average degree can get arbitrarily close to $k-\frac{k-2}{k^{2}}$. Clearly, as the average degree of a tree is less than 2 , the average degree can be made arbitrarily close to any $\beta$ where $2 \leq \beta<k-\frac{k-2}{k^{2}}$ by picking the correct proportion of single vertices in $X_{1}, X_{2}, \ldots, X_{\ell}$.

## 7 Proof of Corollary 2

Recall the statement of Corollary 2.
Corollary 2 If $k \geq 3$ is an odd integer and $G$ is a connected graph of order $n$, size $m$, and with maximum degree $\Delta(G) \leq k$, then

$$
\alpha^{\prime}(G) \geq\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{k-1}{k\left(k^{2}-3\right)} .
$$

Proof of Corollary 2, If $G$ is not $k$-regular, then the result follows from Theorem 4, so assume that $G$ is $k$-regular. By the $k$-regularity of $G$ we have $n k=2 m$, which implies the following by Theorem 3,

$$
\begin{aligned}
& \left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{k-1}{k\left(k^{2}-3\right)} \\
& \quad=\quad\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k\left(k^{2}-k-2\right)}{2 k\left(k^{2}-3\right)}\right) n-\frac{k-1}{k\left(k^{2}-3\right)} \\
& \quad=\quad \frac{\left(k^{3}-k^{2}-2\right) n-2 k+2}{2 k\left(k^{2}-3\right)} \\
& \stackrel{(T h m}{\leq} 3) \\
& \leq
\end{aligned}
$$

We show that the lower bound on the matching number in Corollary 2 is tight for infinite classes of trees and other infinite classes (including the class of $k$-regular graphs) of connected graphs with maximum degree at most $k$.

For $k \geq 3$ odd, let $H_{k+2}$ be the graph of (odd) order $k+2$ obtained from $K_{k+2}$ by removing the edges of an almost perfect matching; that is, the complement $\overline{H_{k+2}}$ of $H_{k+2}$ is isomorphic to $P_{3} \cup\left(\frac{k-1}{2}\right) P_{2}$. We note that every vertex in $H_{k+2}$ has degree $k$, except for exactly one vertex, which has degree $k-1$. We call the vertex of degree $k-1$ in $H_{k+2}$ the link vertex of $H_{k+2}$. We note that $H_{k+2}$ has size $m\left(H_{k+2}\right)=\frac{1}{2}\left(k^{2}+2 k-1\right)$.

For $k \geq 3$ odd and $r \geq 1$ arbitrary, let $T_{k, r}$ be a tree with maximum degree at most $k$ and with partite sets $V_{1}$ and $V_{2}$, where $\left|V_{2}\right|=r$. Let $H_{k, r}$ be obtained from $T_{k, r}$ as follows: For every vertex $x$ in $V_{2}$ with $d_{T_{k, r}}(x)<k$, add $k-d_{T_{k, r}}(x)$ copies of the subgraph $H_{k+2}$ to $T_{k, r}$ and in each added copy of $H_{k+2}$, join the link vertex of $H_{k+2}$ to $x$. We note that every vertex in the resulting graph $H_{k, r}$ has degree $k$, except possibly for vertices in the set $V_{1}$ whose degrees belong to the set $\{1,2, \ldots, k\}$. Let $\mathcal{H}_{k, r}$ be the family of all such graph $H_{k, r}$.

When $k=3$ and $r=4$, an example of a graph $G$ in the family $\mathcal{H}_{k, r}$ is illustrated in Figure 3, where $G$ has order $n=29$, size $m=40$ and matching number $\alpha^{\prime}(G)=12$.


Figure 3: A graph $G$ in the family $\mathcal{H}_{3,4}$

Proposition 3 For $k \geq 3$ an odd integer and $r \geq 1$ arbitrary, if $G \in \mathcal{H}_{k, r}$ has order $n$ and size $m$, then

$$
\alpha^{\prime}(G)=\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{k-1}{k\left(k^{2}-3\right)} .
$$

Proof. Let $G \cong H_{k, r} \in \mathcal{H}_{k, r}$ have order $n$ and size $m$. Suppose that $\ell$ copies of the graph $H_{k+2}$ were added when constructing the graph $G$. Thus,

$$
\begin{aligned}
\ell & =k\left|V_{2}\right|-\sum_{x \in V_{2}} d_{T_{k, r}}(x) \\
& =k\left|V_{2}\right|-m\left(T_{k, r}\right) \\
& =k\left|V_{2}\right|-\left(\left|V_{1}\right|+\left|V_{2}\right|-1\right) \\
& =(k-1)\left|V_{2}\right|-\left|V_{1}\right|+1 .
\end{aligned}
$$

The graph $G$ has order

$$
\begin{aligned}
n=n(G) & =\left|V_{1}\right|+\left|V_{2}\right|+\ell(k+2) \\
& =\left|V_{1}\right|+\left|V_{2}\right|+\left((k-1)\left|V_{2}\right|-\left|V_{1}\right|+1\right)(k+2) \\
& =\left(k^{2}+k-1\right)\left|V_{2}\right|-(k+1)\left|V_{1}\right|+(k+2) .
\end{aligned}
$$

and size

$$
\begin{aligned}
m=m(G) & =k\left|V_{2}\right|+\frac{1}{2} \ell\left(k^{2}+2 k-1\right) \\
& =k\left|V_{2}\right|+\frac{1}{2}\left((k-1)\left|V_{2}\right|-\left|V_{1}\right|+1\right)\left(k^{2}+2 k-1\right) \\
& =\frac{1}{2}\left(k^{3}+k^{2}-k+1\right)\left|V_{2}\right|-\frac{1}{2}\left(k^{2}+2 k-1\right)\left|V_{1}\right|+\frac{1}{2}\left(k^{2}+2 k-1\right) .
\end{aligned}
$$

Furthermore by deleting the vertices $V_{2}$ from $G$ we obtain $\ell+\left|V_{1}\right|=(k-1)\left|V_{2}\right|+1$ odd components. Therefore, by Theorem [1,

$$
\begin{aligned}
2 \alpha^{\prime}(G) & \leq|V(G)|+\left|V_{2}\right|-\operatorname{oc}\left(G-V_{2}\right) \\
& =\left(\left(k^{2}+k-1\right)\left|V_{2}\right|-(k+1)\left|V_{1}\right|+(k+2)\right)+\left|V_{2}\right|-\left((k-1)\left|V_{2}\right|+1\right) \\
& =\left(k^{2}+1\right)\left|V_{2}\right|-(k+1)\left|V_{1}\right|+(k+1)
\end{aligned}
$$

However, the lower bound of Corollary 2 shows that

$$
\begin{aligned}
& 2 \alpha^{\prime}(G) \\
\geq & 2\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+2\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{2(k-1)}{k\left(k^{2}-3\right)} \\
= & 2\left(\frac{k-1}{k\left(k^{2}-3\right)}\right)\left(\left(k^{2}+k-1\right)\left|V_{2}\right|-(k+1)\left|V_{1}\right|+(k+2)\right) \\
& +\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right)\left(\left(k^{3}+k^{2}-k+1\right)\left|V_{2}\right|-\left(k^{2}+2 k-1\right)\left|V_{1}\right|+\left(k^{2}+2 k-1\right)\right) \\
& \quad-\frac{2(k-1)}{k\left(k^{2}-3\right)} \\
= & \left(\frac{k^{5}-2 k^{3}-3 k}{k\left(k^{2}-3\right)}\right)\left|V_{2}\right|-\left(\frac{k^{4}+k^{3}-3 k^{2}-3 k}{k\left(k^{2}-3\right)}\right)\left|V_{1}\right|+\left(\frac{k^{4}+k^{3}-3 k^{2}-3 k}{k\left(k^{2}-3\right)}\right) \\
= & \left(k^{2}+1\right)\left|V_{2}\right|-(k+1)\left|V_{1}\right|+(k+1) .
\end{aligned}
$$

Consequently, we must have equality throughout the above inequality chains. In particular,

$$
\alpha^{\prime}(G)=\left(\frac{k-1}{k\left(k^{2}-3\right)}\right) n+\left(\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}\right) m-\frac{k-1}{k\left(k^{2}-3\right)} .
$$

This completes the proof of the proposition.
We remark that if the tree $T_{k, r}$ used to construct the graph $H_{k, r}$ is chosen so that every vertex in $V_{2}$ has degree $k$, then $H_{k, r}$ is a tree. If, however, tree $T_{k, r}$ used to construct the graph $H_{k, r}$ is chosen so that every vertex in $V_{1}$ has degree $k$, then $H_{k, r}$ is a $k$-regular graph. Hence, as an immediate consequence of Proposition 3, we have the following result.

Proposition 4 The lower bound in Corollary 图 is achieved for an infinite class of both trees and $k$-regular graphs.

## 8 Proof of Corollary 3

Recall the statement of Corollary 3.
Corollary [3 If $k \geq 4$ is an even integer and $G$ is a graph of order $n$, size $m$ and maximum degree $\Delta(G) \leq k$, then

$$
\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n
$$

unless the following holds.
(a) $G$ is $k$-regular and $n=k+1$, in which case $\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2} n\right)-\frac{k+2}{k^{2}+k+2}$.
(b) $G$ is $k$-regular and $n=k+3$, in which case $\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\frac{4}{k^{2}+k+2}$.
(c) $G$ is 4 -regular and $n=9$, in which case $\alpha^{\prime}(G) \geq\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\frac{2}{k^{2}+k+2}$.

Proof of Corollary 3, If $G$ is not $k$-regular, then the result follows from Theorem [5, so assume that $G$ is $k$-regular. By the $k$-regularity of $G$ we have $n k=2 m$, which implies the following

$$
\begin{aligned}
\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n & =\frac{k+2}{k^{2}+k+2} \times \frac{n k}{2}-\left(\frac{2 k-4}{2\left(k^{2}+k+2\right)}\right) n \\
& =\frac{n}{2} \times \frac{k(k+2)-(2 k-4)}{k^{2}+k+2} \\
& =\frac{n}{2} \times \frac{k^{2}+4}{k^{2}+k+2}
\end{aligned}
$$

Applying Theorem 2 to the $k$-regular graph $G$, the matching number of $G$ is bounded below as follows.

$$
\alpha^{\prime}(G) \geq \min \left\{\frac{k^{2}+4}{k^{2}+k+2} \times \frac{n}{2}, \frac{n-1}{2}\right\}
$$

Therefore we have shown that the corollary holds in all cases, except when

$$
\frac{k^{2}+4}{k^{2}+k+2} \times \frac{n}{2}>\alpha^{\prime}(G) \geq \frac{n-1}{2}
$$

So assume that this exceptional case occurs. Since $k \geq 4$, we note that in this case

$$
\frac{n-1}{2} \leq \alpha^{\prime}(G)<\frac{k^{2}+4}{k^{2}+k+2} \times \frac{n}{2}<1 \times \frac{n}{2}=\frac{n}{2},
$$

implying that $\alpha^{\prime}(G)=(n-1) / 2$, and so $n \geq k+1$ is odd. Thus, $n=k+i$ for some $i \geq 1$ odd. This implies the following.

$$
\begin{aligned}
b_{k}|E(G)|-a_{k}|V(G)|-\alpha^{\prime}(G) & =\left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n-\alpha^{\prime}(G) \\
& =\frac{k^{2}+4}{k^{2}+k+2} \times \frac{n}{2}-\frac{n-1}{2} \\
& =\frac{k^{2}+4}{k^{2}+k+2} \times \frac{k+i}{2}-\frac{(k+i-1)\left(k^{2}+k+2\right)}{2\left(k^{2}+k+2\right)} \\
& =\frac{1}{2\left(k^{2}+k+2\right)}\left((k+i)\left(k^{2}+4\right)-(k+i-1)\left(k^{2}+k+2\right)\right) \\
& =\frac{1}{2\left(k^{2}+k+2\right)}\left((k+i)(2-k)+k^{2}+k+2\right) \\
& =\frac{1}{2\left(k^{2}+k+2\right)}(3 k+2-i(k-2))
\end{aligned}
$$

or, equivalently,

$$
\begin{equation*}
\alpha^{\prime}(G)=b_{k}|E(G)|-a_{k}|V(G)|-\frac{3 k+2-i(k-2)}{2\left(k^{2}+k+2\right)} . \tag{7}
\end{equation*}
$$

This proves the case when $n=k+1$, as when $i=1$ the right-hand side of Equation (7) is equivalent to the lower bound on $\alpha^{\prime}(G)$ in the statement of Corollary 3(a). We also get the case when $n=k+3$, as when $i=3$ we again note that the the right-hand side of Equation (7) is equivalent to the lower bound on $\alpha^{\prime}(G)$ in the statement of Corollary 3(b). When $n=k+5$, we have $i=5$ and in this case $3 k+2-i(k-2)=12-2 k$. When $k=4$, the right-hand side of Equation (7) is equivalent to the lower bound on $\alpha^{\prime}(G)$ in the statement of Corollary 3 (c). When $k \geq 6$, we note that $12-2 k \leq 0$, implying that

$$
\alpha^{\prime}(G) \geq b_{k}|E(G)|-a_{k}|V(G)|=\frac{k^{2}+4}{k^{2}+k+2} \times \frac{n}{2},
$$

a contradiction to our exceptional case. When $n \geq k+7$, then $i \geq 7$ and $3 k+2-i(k-2) \leq$ $16-4 k \leq 0$, implying that $\alpha^{\prime}(G) \geq b_{k}|E(G)|-a_{k}|V(G)|$, again a contradiction to our exceptional case.

Following the notation in the proof of Proposition (1) for $k \geq 4$ an even integer and $r \geq 1$ arbitrary, let $G$ be a graph in the family $\mathcal{G}_{k, r}^{\prime}$ of order $n$ and size $m$. We recall that contains $\ell$ copies of $K_{k+1}-e$ with $\ell=r(k-1)+1$. Note that in this case, $n=r+\ell(k+1)$, $m=r k+\frac{1}{2} \ell\left(k^{2}+k-2\right)$ and $\alpha^{\prime}(G)=r+\frac{1}{2} \ell k$. Therefore the following holds.

$$
\begin{aligned}
& \left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n \\
& =\left(\frac{k+2}{k^{2}+k+2}\right)\left(r k+\frac{\ell\left(k^{2}+k-2\right)}{2}\right)-\left(\frac{k-2}{k^{2}+k+2}\right)(r+\ell(k+1)) \\
& =\frac{1}{2\left(k^{2}+k+2\right)}(r k(k+2)-r(k+2)) \\
& +\frac{\ell}{2\left(k^{2}+k+2\right)}\left((k+2)\left(k^{2}+k-2\right)-2(k-2)(k+1)\right) \\
& =\frac{2 r\left(k^{2}+k+2\right)}{2\left(k^{2}+k+2\right)}+\frac{\ell\left(k^{3}+k^{2}+2 k\right)}{2\left(k^{2}+k+2\right)} \\
& =r+\frac{1}{2} \ell k \\
& =\alpha^{\prime}(G) \text {. }
\end{aligned}
$$

Thus, the lower bound in Corollary 3 is tight for the class of graphs in the family $\mathcal{G}_{k, r}^{\prime}$. We show next that Corollary 3 is tight for an infinite family of $k$-regular graphs. For $k \geq 4$ an even integer and $r \geq 1$ arbitrary, let $F_{k, r}$ be a graph of order $n$ and size $m$ obtained of the disjoint union of $k r / 2$ copies of $K_{k+1}-e$ by adding a set $X$ of $r$ new vertices, and adding edges between $X$ and the $k r$ link vertices of degree $k-1$ in the copies of $K_{k+1}-e$ in such a way that $F_{k, r}$ is a connected, $k$-regular graph. We note that $n=r+\frac{1}{2} k r(k+1)$,
$m=r k+\frac{1}{2} k r \times \frac{1}{2}\left(k^{2}+k-2\right)=r k+\frac{1}{4} k\left(k^{2}+k-2\right) r$ and $\alpha^{\prime}(G)=r+\frac{1}{2} k r \times \frac{1}{2} k=r+\frac{1}{4} k^{2} r$. Therefore the following holds.

$$
\begin{aligned}
& \left(\frac{k+2}{k^{2}+k+2}\right) m-\left(\frac{k-2}{k^{2}+k+2}\right) n \\
& \quad=\left(\frac{r(k+2)}{k^{2}+k+2}\right)\left(k+\frac{k\left(k^{2}+k-2\right)}{4}\right)-\left(\frac{r(k-2)}{k^{2}+k+2}\right)\left(1+\frac{k(k+1)}{2}\right) \\
& \quad=\left(\frac{r(k+2)}{k^{2}+k+2}\right)\left(\frac{k\left(k^{2}+k+2\right)}{4}\right)-\left(\frac{r(k-2)}{k^{2}+k+2}\right)\left(\frac{k^{2}+k+2}{2}\right) \\
& \quad=\frac{1}{4} r k(k+2)-\frac{1}{2} r(k-2) \\
& \quad=r+\frac{1}{4} k^{2} r \\
& \quad=\alpha^{\prime}(G)
\end{aligned}
$$

Thus, Corollary 3 is tight for an infinite family of $k$-regular graphs. We state these results formally as follows.

Proposition 5 The lower bound in Corollary 3 is achieved for both $k$-regular graphs and for the class of graphs in the family $\mathcal{G}_{k, r}^{\prime}$.

## 9 The convex set $L_{k}$

Let $k \geq 3$ be an integer. Let $\mathcal{G}_{k}$ denote the class of connected graphs with maximum degree at most $k$. For every pair $(a, b)$ of real numbers $a$ and $b$ we define the concept of $k$-good, $k$-bad and $k$-tight as follows.

- $(a, b)$ is called $k$-good if there exists a constant $T_{a, b}$ such that

$$
\alpha^{\prime}(G) \geq a|V(G)|+b|E(G)|-T_{a, b}
$$

holds for all $G \in \mathcal{G}_{k}$.

- $(a, b)$ is called $k$-bad if it is not $k$-good.
- $(a, b)$ is called $k$-tight if it is $k$-good and there exists a constant $S_{a, b}$ such that

$$
\alpha^{\prime}(G) \leq a|V(G)|+b|E(G)|-S_{a, b}
$$

holds for infinitely many graphs $G \in \mathcal{G}_{k}$.

If we say that $(a, b)$ is $k$-tight for a certain subset of $\mathcal{G}_{k}$ (for example, the class of trees or $k$-regular graphs), then we mean that there are infinitely many graphs from this class that satisfy $\alpha^{\prime}(G) \leq a|V(G)|+b|E(G)|-S_{a, b}$ for some constant $S_{a, b}$.

Suppose that $(a, b)$ is $k$-good and $\varepsilon \geq 0$. Then, there exists a constant $T_{a, b}$ such that $\alpha^{\prime}(G) \geq a|V(G)|+b|E(G)|-T_{a, b} \geq a|V(G)|+(b-\varepsilon)|E(G)|-T_{a, b}$, implying that ( $a, b-\varepsilon$ ) is $k$-good where $T_{a, b-\varepsilon}=T_{a, b}$. We state this formally as follows.

Observation 1 If $(a, b)$ is $k$-good and $\varepsilon \geq 0$, then $(a, b-\varepsilon)$ is $k$-good.
Lemma 6 If $(a, b)$ is $k$-good and $\varepsilon \geq 0$, then both $(a+\varepsilon, b-\varepsilon)$ and $(a-\varepsilon \cdot k, b+2 \varepsilon)$ are $k$-good. Furthermore the following holds.
(a) If $(a, b)$ is $k$-tight for trees, then $(a+\varepsilon, b-\varepsilon)$ is $k$-tight.
(b) If $(a, b)$ is $k$-tight for $k$-regular graphs, then $(a-\varepsilon \cdot k, b+2 \varepsilon)$ is $k$-tight.

Proof. Let $G \in \mathcal{G}_{k}$ have order $n$ and size $m$. Since $G$ is connected, we note that $m \geq n-1$. Since ( $a, b$ ) is $k$-good, this implies that there exists a constant $T_{a, b}$ such that the following also holds for $\varepsilon \geq 0$.

$$
\begin{aligned}
\alpha^{\prime}(G) & \geq a \cdot n+b \cdot m-T_{a, b} \\
& \geq a \cdot n+b \cdot m-T_{a, b}+\varepsilon(n-1-m) \\
& =(a+\varepsilon) n+(b-\varepsilon) m-\left(T_{a, b}+\varepsilon\right) .
\end{aligned}
$$

So letting $T_{a+\varepsilon, b-\varepsilon}=T_{a, b}+\varepsilon$, the pair $(a+\varepsilon, b-\varepsilon)$ is $k$-good. If $(a, b)$ is $k$-tight for trees, then there exists a constant $S_{a, b}$ such that for infinitely many trees, $G^{\prime}$, in $\mathcal{G}_{k}$ we have $\alpha^{\prime}\left(G^{\prime}\right) \leq a\left|V\left(G^{\prime}\right)\right|+b\left|E\left(G^{\prime}\right)\right|-S_{a, b}$. Let $G^{\prime}$ has order $n^{\prime}$ and size $m^{\prime}$. Then, $m^{\prime}=n^{\prime}-1$ and, analogously as before, the following holds.

$$
\alpha^{\prime}\left(G^{\prime}\right) \leq a \cdot n^{\prime}+b \cdot m^{\prime}-S_{a, b}=(a+\varepsilon) n^{\prime}+(b-\varepsilon) m^{\prime}-\left(S_{a, b}+\varepsilon\right) .
$$

So letting $S_{a+\varepsilon, b-\varepsilon}=S_{a, b}+\varepsilon$, the pair $(a+\varepsilon, b-\varepsilon)$ is $k$-tight in this case. Recall that $G \in \mathcal{G}_{k}$ has order $n$ and size $m$. As $G$ has maximum degree at most $k$, we have $n k \geq$ $\sum_{v \in V(G)} d_{G}(v)=2 m$, which implies that the following also holds for all $\varepsilon \geq 0$.

$$
\begin{aligned}
\alpha^{\prime}(G) & \geq a \cdot n+b \cdot m-T_{a, b} \\
& \geq a \cdot n+b \cdot m-T_{a, b}+\varepsilon(2 m-n k) \\
& =(a-\varepsilon \cdot k) n+(b+2 \varepsilon) m-T_{a, b} .
\end{aligned}
$$

So letting $T_{a-\varepsilon \cdot k, b+2 \varepsilon}=T_{a, b}$, we note that $(a-\varepsilon \cdot k, b+2 \varepsilon)$ is $k$-good. If $(a, b)$ is $k$ tight for $k$-regular graphs, then for infinitely many $k$-regular graphs, $G^{\prime}$, in $\mathcal{G}_{k}$ we have $\alpha^{\prime}\left(G^{\prime}\right) \leq a\left|V\left(G^{\prime}\right)\right|+b\left|E\left(G^{\prime}\right)\right|-S_{a, b}$. Let $G^{\prime}$ has order $n^{\prime}$ and size $m^{\prime}$. Then, $n^{\prime} k=2 m^{\prime}$ and, analogously as before, the following holds.

$$
\begin{aligned}
\alpha^{\prime}(G) & \leq a \cdot n^{\prime}+b \cdot m^{\prime}-S_{a, b} \\
& =a \cdot n^{\prime}+b \cdot m^{\prime}-S_{a, b}+\varepsilon\left(2 m^{\prime}-n^{\prime} k\right) \\
& =(a-\varepsilon \cdot k) n^{\prime}+(b+2 \varepsilon) m^{\prime}-S_{a, b} .
\end{aligned}
$$

So letting $S_{a-\varepsilon \cdot k, b+2 \varepsilon}=S_{a, b}$, the pair $(a-\varepsilon \cdot k, b+2 \varepsilon)$ is $k$-tight in this case.

Lemma 7 If $(a, b)$ is $k$-tight, then $(a+\varepsilon, b)$ and $(a, b+\varepsilon)$ are both $k$-bad for all $\varepsilon>0$.

Proof. Assume that $(a, b)$ is $k$-tight and, for the sake of contradiction, suppose that $(a+\varepsilon, b)$ is $k$-good for some $\varepsilon>0$. That is, there exists a constant $T_{a+\varepsilon, b}$ such that

$$
\alpha^{\prime}(G) \geq(a+\varepsilon) \cdot|V(G)|+b \cdot|E(G)|-T_{a+\varepsilon, b}
$$

holds for all $G \in \mathcal{G}_{k}$. Since $(a, b)$ is $k$-tight, there exists a constant $S_{a, b}$ such that

$$
\alpha^{\prime}(G) \leq a \cdot|V(G)|+b \cdot|E(G)|-S_{a, b}=(a+\epsilon) \cdot|V(G)|+b \cdot|E(G)|-S_{a, b}-\varepsilon|V(G)|
$$

holds for infinitely many $G \in \mathcal{G}_{k}$. However as there are infinitely many such graphs, we can choose such a graph $G$ (of sufficiently large order) such that $S_{a, b}+\varepsilon|V(G)|>T_{a+\varepsilon, b}$. For this graph $G$, we have $\alpha^{\prime}(G)<(a+\varepsilon)|V(G)|+b|E(G)|-T_{a+\varepsilon, b}$, a contradiction. Therefore, $(a+\varepsilon, b)$ is $k$-bad.

The fact that $(a, b+\varepsilon)$ is $k$-bad can be proved analogously.

Lemma 8 If $\left(a_{1}, b_{1}\right)$ and $\left(a_{2}, b_{2}\right)$ are both $k$-good, then $\left(\varepsilon a_{1}+(1-\varepsilon) a_{2}, \varepsilon b_{1}+(1-\varepsilon) b_{2}\right)$ is also $k$-good for all $0 \leq \varepsilon \leq 1$.

Furthermore if $\left(a_{1}, b_{1}\right)$ and $\left(a_{2}, b_{2}\right)$ are both $k$-tight for the same infinite class $\mathcal{G}^{\prime} \subseteq \mathcal{G}_{k}$, then $\left(\varepsilon a_{1}+(1-\varepsilon) a_{2}, \varepsilon b_{1}+(1-\varepsilon) b_{2}\right)$ is also $k$-tight.

Proof. Since $\left(a_{1}, b_{1}\right)$ and $\left(a_{2}, b_{2}\right)$ are both $k$-good, there exists constants $T_{a_{1}, b_{1}}$ and $T_{a_{2}, b_{2}}$ such that

$$
\alpha^{\prime}(G) \geq a_{1}|V(G)|+b_{1}|E(G)|-T_{a_{1}, b_{1}} \quad \text { and } \quad \alpha^{\prime}(G) \geq a_{2}|V(G)|+b_{2}|E(G)|-T_{a_{2}, b_{2}}
$$

hold for all $G \in \mathcal{G}_{k}$. Let $0 \leq \varepsilon \leq 1$. Multiplying the first equation by $\varepsilon$ and the second by ( $1-\varepsilon$ ), and then adding the equations together shows that

$$
\alpha^{\prime}(G) \geq\left(\varepsilon a_{1}+(1-\varepsilon) a_{2}\right)|V(G)|+\left(\varepsilon b_{1}+(1-\varepsilon) b_{2}\right)|E(G)|-\varepsilon T_{a_{1}, b_{1}}-(1-\varepsilon) T_{a_{2}, b_{2}}
$$

holds for all $G \in \mathcal{G}_{k}$. This proves that $\left(\varepsilon a_{1}+(1-\varepsilon) a_{2}, \varepsilon b_{1}+(1-\varepsilon) b_{2}\right)$ is $k$-good, with $T_{\varepsilon a_{1}+(1-\varepsilon) a_{2}, \varepsilon b_{1}+(1-\varepsilon) b_{2}}=\varepsilon T_{a_{1}, b_{1}}+(1-\varepsilon) T_{a_{2}, b_{2}}$.

Assume that $\left(a_{1}, b_{1}\right)$ and $\left(a_{2}, b_{2}\right)$ are both $k$-tight for the same infinite class $\mathcal{G}^{\prime} \subseteq \mathcal{G}_{k}$. Thus there exists constants $S_{a_{1}, b_{1}}$ and $S_{a_{2}, b_{2}}$ such that

$$
\alpha^{\prime}(G) \leq a_{1}|V(G)|+b_{1}|E(G)|-S_{a_{1}, b_{1}} \quad \text { and } \quad \alpha^{\prime}(G) \leq a_{2}|V(G)|+b_{2}|E(G)|-S_{a_{2}, b_{2}}
$$

hold for all $G \in \mathcal{G}^{\prime}$. Again, multiplying the first equation by $\varepsilon$ and the second by $(1-\varepsilon)$ and adding the equations together shows that

$$
\alpha^{\prime}(G) \leq\left(\varepsilon a_{1}+(1-\varepsilon) a_{2}\right)|V(G)|+\left(\varepsilon b_{1}+(1-\varepsilon) b_{2}\right)|E(G)|-\varepsilon S_{a_{1}, b_{1}}-(1-\varepsilon) S_{a_{2}, b_{2}} .
$$

holds for all $G \in \mathcal{G}^{\prime}$. Therefore, $\left(\varepsilon a_{1}+(1-\varepsilon) a_{2}, \varepsilon b_{1}+(1-\varepsilon) b_{2}\right)$ is $k$-tight letting

$$
S_{\varepsilon a_{1}+(1-\varepsilon) a_{2}, \varepsilon b_{1}+(1-\varepsilon) b_{2}}=\varepsilon S_{a_{1}, b_{1}}+(1-\varepsilon) S_{a_{2}, b_{2}} .
$$

## $9.1 k$ odd

Theorem 9 Let $k \geq 3$ be odd and let $a^{*}=\frac{k-1}{k\left(k^{2}-3\right)}$ and $b^{*}=\frac{k^{2}-k-2}{k\left(k^{2}-3\right)}$. For any pair $(a, b)$, the following holds.
(a) If $a \leq a^{*}$, then $(a, b)$ is $k$-good if and only if $b \leq b^{*}+2\left(\frac{a^{*}-a}{k}\right)$.
(b) If $a>a^{*}$, then $(a, b)$ is $k$-good if and only if $b \leq b^{*}+a^{*}-a$.

Proof. By Corollary 2, the pair $\left(a^{*}, b^{*}\right)$ is $k$-good with

$$
T_{a^{*}, b^{*}}=\frac{k-1}{k\left(k^{2}-3\right)} .
$$

By Proposition 4, the lower bound in Corollary 2 is achieved for an infinite class of both trees and $k$-regular graphs, implying that $\left(a^{*}, b^{*}\right)$ is $k$-tight for both trees and $k$-regular graphs.

Suppose $a \leq a^{*}$, and let $\varepsilon=\left(a^{*}-a\right) / k$. By Lemma 6, we note that $\left(a^{*}-\varepsilon \cdot k, b^{*}+2 \varepsilon\right)$ is $k$-good. Further, since $\left(a^{*}, b^{*}\right)$ is $k$-tight for $k$-regular graphs, by Lemma 6(b), we note that $\left(a^{*}-\varepsilon \cdot k, b^{*}+2 \varepsilon\right)$ is $k$-tight. Since $\varepsilon=\left(a^{*}-a\right) / k$, this is equivalent to $\left(a, b^{*}+2\left(\frac{a^{*}-a}{k}\right)\right)$ being $k$-tight. If $b \leq b^{*}+2\left(\frac{a^{*}-a}{k}\right)$, then by Observation 1, the pair $(a, b)$ is $k$-good. If $b>b^{*}+2\left(\frac{a^{*}-a}{k}\right)$, then by Lemma 7 and our earlier observation that $\left(a, b^{*}+2\left(\frac{a^{*}-a}{k}\right)\right)$ is $k$-tight, the pair ( $a, b$ ) is $k$-bad. This completes the case when $a \leq a^{*}$.

Suppose next that $a>a^{*}$, and let $\varepsilon=a-a^{*}$. By Lemma6, we note that ( $a^{*}+\varepsilon, b^{*}-\varepsilon$ ) is $k$-good. Further since $\left(a^{*}, b^{*}\right)$ is $k$-tight for trees, we note by Lemma 6(a) that ( $a^{*}+\varepsilon, b^{*}-\varepsilon$ ) is $k$-tight. Since $\varepsilon=\left(a^{*}-a\right) / k$, this is equivalent to $\left(a, b^{*}+a^{*}-a\right)$ being $k$-tight. If $b \leq b^{*}+a^{*}-a$, then by Observation 1, the pair $(a, b)$ is $k$-good. If $b>b^{*}+a^{*}-a$, then by Lemma 7 and our earlier observation that $\left(a, b^{*}+a^{*}-a\right)$ is $k$-tight, the pair $(a, b)$ is $k$-bad.

We remark that the equation in Theorem 9(a) corresponds to the half-plane $\ell_{2}$ described in the introductory section, noting that

$$
b \leq b^{*}+2\left(\frac{a^{*}-a}{k}\right)=-\left(\frac{2}{k}\right) a+\frac{k^{3}-k^{2}-2}{k^{2}\left(k^{2}-3\right)} .
$$

The equation in Theorem $9(b)$ corresponds to the half-plane $\ell_{1}$ described in the introductory section, noting that

$$
b \leq b^{*}+a^{*}-a=-a+\frac{1}{k} .
$$

Theorem 9 is illustrated in Figure 1 when $k=3$ and $k=5$. The grey area corresponds to all $k$-good pairs $(a, b)$ while the non-grey area corresponds to the $k$-bad pairs.

## $9.2 k$ even

Theorem 10 Let $k \geq 4$ be even and let $a_{1}^{*}=\frac{1}{k(k+1)}$ and $b_{1}^{*}=\frac{1}{k+1}$ and $a_{2}^{*}=-\frac{k-2}{k^{2}+k+2}$ and $b_{2}^{*}=\frac{k+2}{k^{2}+k+2}$. For any pair ( $a, b$ ), the following holds.
(a) If $a \leq a_{2}^{*}$, then $(a, b)$ is $k$-good if and only if $b \leq b_{2}^{*}+2\left(\frac{a^{*}-a}{k}\right)$.
(b) If $a>a_{1}^{*}$, then $(a, b)$ is $k$-good if and only if $b \leq b_{1}^{*}+a_{1}^{*}-a$.
(c) If $a_{2}^{*}<a \leq a_{1}^{*}$, then $(a, b)$ is $k$-good if and only if $b \leq b_{2}^{*}+\frac{\left(b_{1}^{*}-b_{2}^{*}\right)\left(a-a_{2}^{*}\right)}{a_{1}^{*}-a_{2}^{*}}$.

Proof. By Corollary 1, the pair $\left(a_{1}^{*}, b_{1}^{*}\right)$ is $k$-good with $T_{a_{1}^{*}, b_{1}^{*}}=-1 / k$, while by Corollary 3, the pair $\left(a_{2}^{*}, b_{2}^{*}\right)$ is $k$-good with

$$
T_{a_{2}^{*}, b_{2}^{*}}=-\frac{k+2}{k^{2}+k+2} .
$$

By Proposition 2, the lower bound in Corollary 1 is achieved for both trees and for the class of graphs in the family $\mathcal{G}_{k, r}^{\prime}$, implying that $\left(a_{1}^{*}, b_{1}^{*}\right)$ is $k$-tight for both trees and graphs in the family $\mathcal{G}_{k, r}^{\prime}$. By Proposition [5, the lower bound in Corollary 3 is achieved for the class of graphs in the family $\mathcal{G}_{k, r}^{\prime}$ and for the class of $k$-regular graphs, implying that $\left(a_{2}^{*}, b_{2}^{*}\right)$ is $k$-tight for these classes of graphs.

Suppose that $a \leq a_{2}^{*}$, and let $\varepsilon=\left(a_{2}^{*}-a\right) / k$. By Lemma 6, we note that $\left(a_{2}^{*}-\varepsilon \cdot k, b_{2}^{*}+2 \varepsilon\right)$ is $k$-good. Further, since $\left(a_{2}^{*}, b_{2}^{*}\right)$ is $k$-tight for $k$-regular graphs, by Lemma 6 (b), we note that $\left(a_{2}^{*}-\varepsilon \cdot k, b_{2}^{*}+2 \varepsilon\right)$ is $k$-tight. Since $\varepsilon=\left(a_{2}^{*}-a\right) / k$, this is equivalent to $\left(a, b_{2}^{*}+2\left(\frac{a_{2}^{*}-a}{k}\right)\right)$ being $k$-tight. If $b \leq b_{2}^{*}+2\left(\frac{a^{*}-a}{k}\right)$, then by Observation 1, the pair $(a, b)$ is $k$-good. If $b>b_{2}^{*}+2\left(\frac{a^{*}-a}{k}\right)$, then by Lemma 7 and our earlier observation that $\left(a, b_{2}^{*}+2\left(\frac{a_{2}^{*}-a}{k}\right)\right)$ is $k$-tight, the pair $(a, b)$ is $k$-bad. This completes the case when $a \leq a_{2}^{*}$.

Suppose that $a>a_{1}^{*}$, and let $\varepsilon=a_{1}-a^{*}$. By Lemma 6, we note that ( $a_{1}^{*}+\varepsilon, b_{1}^{*}-\varepsilon$ ) is $k$-good. Further since $\left(a_{1}^{*}, b_{1}^{*}\right)$ is $k$-tight for trees, we note by Lemma 6(a) that ( $a_{1}^{*}+\varepsilon, b_{1}^{*}-\varepsilon$ ) is $k$-tight. Since $\varepsilon=\left(a^{*}-a\right) / k$, this is equivalent to $\left(a, b_{1}^{*}+a_{1}^{*}-a\right)$ being $k$-tight. If $b \leq b_{1}^{*}+a_{1}^{*}-a$, then by Observation 1, the pair $(a, b)$ is $k$-good. If $b>b_{1}^{*}+a_{1}^{*}-a$, then by Lemma 7 and our earlier observation that $\left(a, b_{1}^{*}+a_{1}^{*}-a\right)$ is $k$-tight, the pair $(a, b)$ is $k$-bad.

Finally, suppose that $a_{2}^{*}<a \leq a_{1}^{*}$ and let $\varepsilon=\left(a-a_{2}^{*}\right) /\left(a_{1}^{*}-a_{2}^{*}\right)$. By Lemma 8 , we note that $\left(\varepsilon a_{1}^{*}+(1-\varepsilon) a_{2}^{*}, \varepsilon b_{1}^{*}+(1-\varepsilon) b_{2}^{*}\right)$ is $k$-good. Furthermore, since $\left(a_{1}^{*}, b_{1}^{*}\right)$ and $\left(a_{2}^{*}, b_{2}^{*}\right)$ are both $k$-tight for graphs in the family $\mathcal{G}_{k, r}^{\prime}$, we note that $\left(\varepsilon a_{1}^{*}+(1-\varepsilon) a_{2}^{*}, \varepsilon b_{1}^{*}+(1-\varepsilon) b_{2}^{*}\right)$ is $k$ tight. Since $\varepsilon=\left(a-a_{2}^{*}\right) /\left(a_{1}^{*}-a_{2}^{*}\right)$, this is equivalent to $\left(a, b_{2}^{*}+\frac{\left(b_{1}^{*}-b_{2}^{*}\right)\left(a-a_{2}^{*}\right)}{a_{1}^{*}-a_{2}^{*}}\right)$ being $k$-tight. If $b \leq b_{2}^{*}+\frac{\left(b_{1}^{*}-b_{2}^{*}\right)\left(a-a_{2}^{*}\right)}{a_{1}^{*}-a_{2}^{*}}$, then by Observation 1, the pair $(a, b)$ is $k$-good. If $b>b_{2}^{*}+\frac{\left(b_{1}^{*}-b_{2}^{*}\right)\left(a-a_{2}^{*}\right)}{a_{1}^{*}-a_{2}^{*}}$, then by Lemma 7 and our earlier observation that $\left(a, b_{2}^{*}+\frac{\left(b_{1}^{*}-b_{2}^{*}\right)\left(a-a_{2}^{*}\right)}{a_{1}^{*}-a_{2}^{*}}\right)$ is $k$-tight, the pair $(a, b)$ is $k$-bad.

We remark that the equation in Theorem 10(a) corresponds to the half-plane $\ell_{3}$ described in the introductory section, noting that

$$
b \leq b_{2}^{*}+2\left(\frac{a_{2}^{*}-a}{k}\right)=-\left(\frac{2}{k}\right) a+\frac{k^{2}+4}{k\left(k^{2}+k+2\right)} .
$$

The equation in Theorem (10) corresponds to the half-plane $\ell_{1}$ described in the introductory section, noting that

$$
b \leq b_{1}^{*}+a_{1}^{*}-a=-a+\frac{1}{k} .
$$

The equation in Theorem 10(c) corresponds to the half-plane $\ell_{4}$ described in the introductory section, noting that

$$
b \leq b_{2}^{*}+\frac{\left(b_{1}^{*}-b_{2}^{*}\right)\left(a-a_{2}^{*}\right)}{a_{1}^{*}-a_{2}^{*}}=-\left(\frac{2 k^{2}}{k^{3}-k+2}\right) a+\frac{k^{2}-k+2}{k^{3}-k+2} .
$$

Theorem 10 is illustrated in Figure 1 when $k=4$ and $k=6$. The grey area corresponds to all $k$-good pairs $(a, b)$ while the non-grey area corresponds to the $k$-bad pairs.

## References

[1] C. Berge, C. R. Acad. Sci. Paris Ser. I Math. 247, (1958) 258-259 and Graphs and Hypergraphs (Chap. 8, Theorem 12), North-Holland, Amsterdam, 1973.
[2] T. Biedl, E. D. Demaine, C. A. Duncan, R. Fleischer and S. G. Kobourov, Tight bounds on maximal and maximum matchings. Discrete Math. 285 (2004), 7-15.
[3] V. Chvátal and C. McDiarmid, Small transversals in hypergraphs. Combinatorica 12 (1992), 19-26.
[4] S. M. Cioabă, D. A. Gregory, and W. H. Haemers, Matchings in regular graphs from eigenvalues. J. Combinatorial Theory Ser. B 99 (2009), 287-297.
[5] P. E. Haxell and A. D. Scott, On lower bounds for the matching number of subcubic graphs. Manuscript, June 2014. http://arxiv.org/abs/1406.7227
[6] M. A. Henning, C. Löwenstein, and D. Rautenbach, Independent sets and matchings in subcubic graphs. Discrete Math. 312 (2012), 1900-1910.
[7] M. A. Henning and A. Yeo, Tight lower bounds on the size of a matching in a regular graph. Graphs Combin. 23 (2007), 647-657.
[8] M. A. Henning and A. Yeo, Total domination in graphs (Springer Monographs in Mathematics). ISBN-13: 978-1461465249 (2013).
[9] S. Jahanbekam and D. B. West, New lower bounds on matching numbers of general and bipartite graphs. Congr. Numer. 218 (2013), 57-59.
[10] P. Katerinis, Maximum matching in a regular graph of specified connectivity and bounded order. J. Graph Theory 11 (1987), 53-58.
[11] L. Lovász and M. D. Plummer, Matching Theory, North-Holland Mathematics Studies, vol. 121, Ann. Discrete Math., vol. 29, North-Holland, 1986.
[12] Suil O and D. B. West, Balloons, cut-edges, matchings, and total domination in regular graphs of odd degree. J. Graph Theory 64 (2010), 116-131.
[13] Suil O and D. B. West, Matching and edge-connectivity in regular graphs. European J. Comb. 32 (2011), 324-329.
[14] M. Plummer, Factors and Factorization. 403-430. Handbook of Graph Theory ed. J. L. Gross and J. Yellen. CRC Press, 2003, ISBN: 1-58488-092-2.
[15] W. R. Pulleyblank, Matchings and Extension. 179-232. Handbook of Combinatorics ed. R. L. Graham, M. Grötschel, L. Lovász. Elsevier Science B.V. 1995, ISBN 0-444-82346-8.
[16] D. B. West, A short proof of the Berge-Tutte Formula and the Gallai-Edmonds Structure Theorem. European J. Comb. 32 (2011), 674-676.


[^0]:    *Research supported in part by the South African National Research Foundation and the University of Johannesburg

