The Ramsey number of loose cycles versus cliques

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Abstract

Recently Kostochka, Mubayi and Verstraëte [6] initiated the study of the Ramsey numbers of uniform loose cycles versus cliques. In particular they proved that $R(C_3^r, K_n^r) = \tilde{\theta}(n^{3/2})$ for all fixed $r \ge 3$. For the case of loose cycles of length five they proved that $R(C_5^r, K_n^r) =$ $\Omega((n/\log n)^{5/4})$ and conjectured that $R(C_5^r, K_n^r) = O(n^{5/4})$ for all fixed $r \ge 3$. Our main result is that $R(C_5^3, K_n^3) = O(n^{4/3})$ and more generally for any fixed $l \ge 3$ that $R(C_l^3, K_n^3) = O(n^{1+1/\lfloor (l+1)/2 \rfloor})$.

We also explain why for every fixed $l \geq 5$, $r \geq 4$, $R(C_l^r, K_n^r) = O(n^{1+1/\lfloor l/2 \rfloor})$ if l is odd, which improves upon the result of Collier-Cartaino, Graber and Jiang [3] who proved that for every fixed $r \geq 3$, $l \geq 4$, we have $R(C_l^r, K_n^r) = O(n^{1+1/\lfloor l/2 \rfloor - 1})$.

1 Introduction

A loose cycle of length l is a hypergraph made of l edges e_1, e_2, \ldots, e_l such that, for any i, j, if $j = i + 1 \pmod{n}$ or $j = i - 1 \pmod{n}$ then $|e_i \cap e_j| = 1$ and otherwise $e_i \cap e_j = \emptyset$. For brevity, we shall denote by C_l such a hypergraph. An r-uniform loose cycle of length l is a loose cycle of length l whose edges all have size r. We shall denote by C_l^r such a hypergraph. This is one of the possible generalizations of graph cycles, and indeed corresponds to a cycle in the graph sense when r = 2. An r-uniform clique of order n is an r-uniform hypergraph on n vertices where all the sets of r vertices form an edge. We shall denote by K_n^r such a hypergraph.

The Ramsey number of an r-uniform loose cycle of length l versus an runiform clique of order n, denoted by $R(C_l^r, K_n^r)$, is the least integer m such that whenever the edges of K_m^r are coloured red and blue then K_m^r either contains a red C_l^r (that is, a copy of C_l^r all of whose edges are coloured red) or a blue K_n^r (that is, a copy of K_n^r all of whose edges are coloured blue). Determining the order of magnitude of $R(C_l^2, K_n^2)$ is a classical problem in

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graph theory. The best lower bound on $R(C_l^2, K_n^2)$ is due to Bohman and Keevash [1]. They proved that $R(C_l^2, K_n^2) = \Omega(n^{1+1/(l-2)}/\log(n))$. For l even, the best upper bound is due to Caro, Li, Rousseau and Zhang [2]; they proved that $R(C_l^2, K_n^2) = O((n/\log(n))^{1+1/(l/2-1)})$. For l odd, the best upper bound is due independently to Li and Zang [7] and to Sudakov [10]. They proved that $R(C_l^2, K_n^2) = O(n^{1+1/\lfloor l/2 \rfloor}/\log(n)^{1/\lfloor l/2 \rfloor})$. In the light of this, in subsequent discussion we shall refer to $n^{1+1/(\lfloor (l-1)/2 \rfloor)}$ as "the graph bound".

Recently Kostochka, Mubayi and Verstraëte initiated the study of $R(C_l^r, K_n^r)$ for $r \geq 3$. In [6] they proved the following theorems.

Theorem 1.1 (Kostochka, Mubayi and Verstraëte [6]). There exist constants $a, b_r > 0$ such that

$$a \frac{n^{3/2}}{(\log n)^{3/4}} \le R(C_3^3, K_n^3) \le b_3 n^{3/2},$$

and for $r \geq 4$,

$$\frac{n^{3/2}}{(\log n)^{3/4+o(1)}} \le R(C_3^r, K_n^r) \le b_r n^{3/2}$$

For loose cycles of length five, they proved the following.

Theorem 1.2 (Kostochka, Mubayi and Verstraëte [6]). There exist constants $c_r > 0$ such that

$$R(C_5^r, K_n^r) \ge c_r \left(\frac{n}{\log n}\right)^{5/4}$$

They also provided a more general lower bound of the form $R(C_l^r, K_n^r) = \Omega(n^{1+1/(3l-1)})$ for any fixed r and l. They made the following conjecture.

Conjecture 1.3 (Kostochka, Mubayi and Verstraëte [6]). For any fixed $r \geq 3$ we have

$$R(C_5^r, K_n^r) = O(n^{5/4}).$$

Collier-Cartaino, Graber and Jiang [3] proved the following.

Theorem 1.4 (Collier-Cartaino, Graber and Jiang [3]). For any fixed $r \ge 3$ and $l \ge 4$ there exist constants $b_{r,l}$ such that

$$R(C_l^r, K_n^r) \le b_{r,l} n^{1+1/(\lfloor l/2 \rfloor - 1)}$$

For l even, they are able to improve this bound by a polylogarithmic factor. We notice that this proves that the graph bound holds for $R(C_l^r, K_n^r)$ when l is even but falls short when l is odd. We also notice that for r = 3 and l = 5 this says that $R(C_5^3, K_n^3) = O(n^2)$.

Our main result is the following.

Theorem 1.5. There exists a constant $c_{3,5}$ such that $R(C_5^3, K_n^3) \leq c_{3,5}n^{4/3}$ and more generally for any fixed $l \geq 3$ there exists a constant $c_{3,l}$ such that $R(C_l^3, K_n^3) \leq c_{3,l}n^{1+1/\lfloor (l+1)/2 \rfloor}$.

For l = 3 notice that this follows from Theorem 1.1. We believe that this result is interesting for two reasons. First, it brings the bound of $O(n^2)$ on $R(C_5^3, K_n^3)$ due to Collier-Cartaino, Graber and Jiang down to $O(n^{4/3})$, a bound much closer to Conjecture 1.3. Secondly, the bound we obtain beats the best known upper bound on $R(C_l^2, K_n^2)$ i.e., the graph bound, for each $l \geq 3$ by an order of magnitude. In fact, we would expect that one should be able to prove that for every fixed $r \geq 3$, $l \geq 3$, there exist $\epsilon_{r,l} > 0$ and $c_{r,l} > 0$ such that $R(C_l^r, K_n^r) \leq c_{r,l}n^{1+1/\lfloor (l-1)/2 \rfloor - \epsilon_{r,l}}$. Thus Theorem 1.1 settles this question for l = 3 and $r \geq 3$, while Theorem 1.5 settles it for r = 3 and $l \geq 4$. However for $r \geq 4$ and $l \geq 4$ the methods we use do not seem to generalize in a straightforward way (See Section 8 for more details.) The proof of Theorem 1.5 relies on generalizing various ideas found in [3], [6], [7] and [10] together with some new ones of our own. Let us now state our second result.

Theorem 1.6. For any fixed $r \geq 3$, $l \geq 5$, there exists $c_{r,l} > 0$ such that $R(C_l^r, K_n^r) \leq c_{r,l} n^{1+1/\lfloor l/2 \rfloor}$ when l is odd.

The main point of Theorem 1.6 is that it essentially proves that the graph bound also holds for $R(C_l^r, K_n^r)$ when l is odd, thus completing the result of Theorem 1.4. We made no attempt at improving this by a polylogarithmic factor as we do not believe the exponent to be correct. We shall only sketch the proof of Theorem 1.6 as most of the ideas necessary to prove it will already have been developed to prove Theorem 1.5. This sketch also serves to highlight how we can beat the graph bound if r = 3.

2 Notation and Tools

In this section we review some basic notation and definitions related to hypergraphs. We also state a few results which we will need in order to prove Theorem 1.5.

For $a, b \in \mathbb{N}$, [b] denotes the set $\{1, 2, \dots, b\}$ and [a, b] denotes the set $\{a, a + 1, \dots, b\}$.

A hypergraph H = (A, B) is is a pair of finite sets A, B such that B is a set of subsets of A. The elements of A will be referred to as the *vertices* of H and those of B as the *edges* of H. For a given hypergraph H = (A, B)we let V(H) denote the set of vertices (that is, A) and let E(H) denote the set of edges (that is, B). Often when it is clear that say u, v are vertices of V, we shall write uv to mean the edge $\{u, v\}$. A hypergraph H is said to be r-uniform if all the elements of B have the same size r. We shall also call an r-uniform hypergraph an r-graph, for short. Notice that when r = 2 we get the classical definition of a loopless graph. If a, b are two integers with a < b then an [a, b]-graph means a hypergraph whose edges each have size lying in [a, b].

Given $v \in V(H)$ the degree of v in H, denoted by d(v), is the number of edges of H containing v. The average degree of H, denoted by d, is the quantity $(\sum_{v \in V(H)} d(v))/n$.

Given a hypergraph H and a subset X of V(H), a subhypergraph of H is a hypergraph with vertex $X \subseteq V(H)$ and set of edges a subset of E(H) made of edges contained in X. Given $X \subseteq V(H)$, the *induced* subhypergraph of H on vertex set X, denoted by H[X], is the hypergraph $(X, \{e \in E(H) : e \subseteq X\})$. A hypergraph H not containing (an isomorphic copy of) a hypergraph F as a subhypergraph is said to be F-free.

Given two hypergraph H_1 and H_2 we shall let $H_1 + H_2$ denote the hypergraph with vertex set $V(H_1) \cup V(H_2)$ and set of edges $E(H_1) \cup E(H_2)$.

A hypergraph H is said to be *simple* if for all $e, f \in E(H)$, if $e \neq f$ then $|e \cap f| \leq 1$. Notice that loose cycles as defined in Section 1 are simple hypergraphs, whereas r-uniform cliques for $r \geq 3$ and $n \geq r + 1$ are not.

A path of length l is a hypergraph P such that $E(P) = \{e_1, e_2, \ldots, e_l\}$ and for all $i \neq j$, if i < j then $e_i \cap e_j = \emptyset$ unless $i \leq l-1$ and j = i+1, in which case we require $e_i \cap e_j \neq \emptyset$. We also require all the edges of the path to have size at least 2. If P is a simple path of length l and $a \in e_1 \setminus e_2$ and $b \in e_l \setminus e_{l-1}$ then we say that P joins a to b.

A hypergraph H is said to be k-vertex-colourable (or k-colourable for brevity) if there exists a map $c: V(H) \longrightarrow [k]$ such that for any $e \in E(H)$ there exist $a, b \in e$ such that $c(a) \neq c(b)$. This condition on c makes it a proper colouring: we remark this because later we shall also refer to any map $f: V(H) \longrightarrow [k]$ as a colouring. Suppose V(H) is totally ordered by some ordering $\langle A$ path $P = \{e_1, e_2, \ldots, e_l\}$ of H is said to be increasing if $a \leq b$ whenever $a \in e_i$ and $b \in e_j$ for some i, j with i < j. Pluhár [8] proved the following.

Proposition 2.1 (Pluhár [8]). A hypergraph H is k-colourable if and only if there exists a total ordering < of V(H) for which there is no increasing simple path of length k in H.

Thus, in particular, if H contains no simple path of length k then H is k-colourable.

A set $X \subseteq V(H)$ is said to be *independent* in H if no edge of H is contained in X. The independence number of H, denoted by $\alpha(H)$, is the maximal size of an independent subset of V(H). An easy observation is that if H is k-colourable then $\alpha(H) \geq |V(H)|/k$. The following well-known result of Spencer [9] gives another way of bounding the independence number of a hypergraph. It is an analogue of Turán's Theorem [11] for $r \geq 3$. **Proposition 2.2** (Spencer [9]). Let *H* be an *r*-uniform hypergraph with average degree *d*. Then $\alpha(H) \geq 0.5n/d^{\frac{1}{r-1}}$.

The reason why we are interested in bounding the independence number of a hypergraph is that bounding $R(C_l^3, K_n^3)$ from above is clearly equivalent to bounding the independence number of C_l^3 -free 3-graphs from below.

3 Light and Heavy Pairs

Throughout this section and the next ones, l denotes a fixed integer which is at least 3.

In this section we introduce the concept of light and heavy pairs of vertices in a 3-graph H. These first appeared in the proof of the upper bound of Theorem 1.1 in [6]. We generalize the ideas found there as they will be equally useful when looking at C_l^3 -free 3-graphs.

Definition 3.1. Let H be a 3-graph. A pair ab of vertices of H is called light in H if it is contained in less than 2l-2 edges of H. It is called heavy in H otherwise. We let $P_{light}(H)$ be the set of all the pairs which are light in H and $P_{heavy}(H)$ the set of all the pairs which are heavy in H.

Definition 3.2. Let H be a 3-graph. An edge e of H is called heavy if it contains a pair which is heavy in H. Otherwise, it is called light. We let $E_{heavy}(H)$ be the set of heavy edges of H and $E_{light}(H)$ the set of light ones.

Definition 3.3. Let H be a 3-graph. The reduced hypergraph of H, denoted by H^* , is the hypergraph with vertex set V(H) and set of edges $E_{light}(H) \cup P_{heavy}(H)$.

Observe that H^* is a [2,3]-graph.

Lemma 3.4. Let H be a 3-graph. If $H + H^*$ contains a C_l then H contains a C_l^3 .

Proof. Let C be a C_l in $H + H^*$. Let A be the set of edges of C which belong to E(H) and let B be the set of edges of H which belong to $P_{\text{heavy}}(H)$. If $B = \emptyset$ then C is already a C_l^3 in H and there is nothing to prove. If $B \neq \emptyset$ then let $uv \in B$. We have $|V(C) \setminus \{u, v\}| = l + |A| - 2 \le 2l - 3$. Therefore since uv is contained in 2l - 2 edges of H at least one of them, call it e, doesn't meet $V(C) \setminus \{u, v\}$. Then if we remove uv from C and add e we obtain a C_l in $H + H^*$ with one less edge lying in $P_{\text{heavy}}(H)$. Repeat this operation until $B = \emptyset$.

Lemma 3.5. Let H be a C_l^3 -free 3-graph. Then there exist l-2 hypergraphs $H_1, H_2, \ldots, H_{l-2}$ with vertex set V(H) such that $E(H) = \bigcup_{i=1}^{l-2} E(H_i)$ and for each $i \in [l-2]$, each edge of H_i contains a light pair in H_i .

Proof. Suppose to the contrary that there exists a C_l^3 -free 3-graph H for which the conclusion of the lemma does not hold. Let $J_0 = H$ and let G_0 be the 2-graph with vertex set V(H) and set of edges $P_{\text{heavy}}(H)$. For $i \in [l-2]$, let H_i be the hypergraph with vertex set V(H) and edges those elements of $E(J_{i-1})$ containing an element of $P_{\text{light}}(J_{i-1})$. Let J_i be the hypergraph with vertex set V(H) and edges those elements of $E(J_{i-1})$ such that the three pairs of vertices that they contain all belong to $E(G_{i-1})$. Finally, let G_i be the 2-graph with vertex set V(H) and set of edges $P_{\text{heavy}}(H_i)$.

We shall now prove by induction on k the claim that for $k \in [0, l - 3]$, G_{l-3-k} contains a C_{k+3}^2 . First consider the base case k = 0. By our assumption on H, $E(H) \neq \bigcup_{i=1}^{l-2} E(H_i)$ and therefore $E(J_{l-2}) \neq \emptyset$ since $E(H) = \bigcup_{i=1}^{l-2} E(H_i) \cup E(J_{l-2})$. Let $x_1 x_2 x_3 \in E(J_{l-2})$. By definition of J_{l-2} , the pairs $x_1 x_2$, $x_2 x_3$ and $x_3 x_1$ are heavy in J_{l-3} so that G_{l-3} contains the C_3^2 formed by these three pairs. Suppose now that the induction hypothesis holds for some $k \in [0, l-4]$ and let us prove it holds for k+1. Let $C = x_1 x_2 \dots x_{k+3}$ be a C_{k+3}^2 in G_{l-3-k} . Since the pair $x_{k+3} x_1$ is heavy in J_{l-3-k} , it is contained in at least 2l - 2 edges of J_{l-3-k} . One of these therefore does not meet any vertex of C other than x_{k+3} and x_1 since $|C| \leq l-1$; let $x_{k+3} x_1 y$ be such an edge. The pairs $x_{k+3} y$ and $y x_1$ are then heavy in $J_{l-3-(k+1)}$ by definition of J_{l-3-k} , are also heavy in $J_{l-3-(k+1)}$. Thus $x_1 x_2 \dots x_{k+3} y$ is a C_{k+4}^2 in $G_{l-3-(k+1)}$, as required. This finishes the proof of the claim.

So there exists a C_l^2 in $P_{\text{heavy}}(H)$. By Lemma 3.4 there exists a C_l^3 in H, which is a contradiction.

Let us remark that the quantity l-2 in Lemma 3.5 is by no means tight, but since we do not care about the constant implicit in Theorem 1.5, this shall be enough for our purposes. Indeed, in the rest of this paper we shall not seek to optimize the constants appearing in the various lemmas. The next lemma will play a very important role in our argument.

Lemma 3.6. Let H be a 3-graph. Then $E_{light}(H)$ is the union of at most 6l - 11 simple hypergraphs.

Proof. Let G be the 2-graph with vertex set $E_{\text{light}}(H)$ and where e is joined to f by an edge if $|e \cap f| = 2$. Then this graph has maximum degree at most 6l - 12 (for if $e \in E_{\text{light}}(H)$ has degree at least 6l - 11 in this hypergraph then one of the 3 pairs of vertices contained in e is contained in at least 2l - 3edges of H other than e and hence is a heavy pair in H, a contradiction). Therefore there exists a proper vertex colouring of G on 6l - 11 colours; now clearly each colour class of this colouring forms a simple hypergraph with vertex set V(H).

The idea of Lemma 3.6 essentially appears in the proof of Lemma 7.7 of

[3]. Without delving into the details, let us point out that the difference here is that the authors of [3] were only seeking a large simple subhypergraph of some carefully chosen C_l -free hypergraph, whereas in our case the fact that each edge of $E_{\text{light}}(H)$ is contained in one of the simple hypergraphs given by Lemma 3.6 is vital.

4 Extenders

In a C_l^2 -free 2-graph, the neighbourhood of a vertex v contains no path of length l-2, hence is (l-2)-colourable by Proposition 2.1, and so contains an independent set of size at least $|\Gamma(v)|/(l-2)$. Furthermore, there is always a vertex whose neighbourhood is at least as large as the average degree dof the graph. Thus an elementary argument to find a large independent set in a C_l^2 -free 2-graph is, provided d is large, to find it as a subset of a neighbourhood of a vertex of large degree and, if d is small, to apply Turán's Theorem [11] which guarantees the existence of an independent set of size at least n/(1+d) where n is the number of vertices of the 2-graph in question. For 3-graphs, the situation is a bit more complicated. Extenders, which are introduced in Definition 4.1 below, will play the same role as the neighbourhood of a vertex in the argument we just gave. The various lemmas of this section are aimed at proving that extenders satisfy all the properties that are required to make the argument work, and we shall use some ideas from [6].

Definition 4.1. A pair (X, Y) of disjoint subsets of V(H) is called an extender if

- 1. For any $u, v \in X$ with $u \neq v$ and any set $S \subseteq V(H) \setminus (\{u, v\} \cup Y)$ of size at most 2l-5 there exists a simple path of length two in H joining u to v and which contains no element of S;
- 2. $|Y| \le 2|X|$.

The size of the extender (X, Y) is defined to be |X|.

Thus an extender is a generalization of the neighbourhood of a vertex in a 2-graph in the sense that, if v is a vertex of a 2-graph G, then by letting $X = \Gamma(v)$ and $Y = \{v\}$ we see that (X, Y) certainly satisfies the requirement of being an extender. The difference for a 3-graph is that Y will typically have size much larger than one and also that proving that large extenders exist is not as straighforward as in the 2-graph case. The following is an easy corollary to Lemma 3.4.

Lemma 4.2. Let H be a C_l^3 -free 3-graph and let (X, Y) be an extender in H. Then for any $u, v \in X$ with $u \neq v$, there does not exist a simple path of length l-2 in $(H+H^*)[V(H)\backslash Y]$ joining u to v.

Proof. If there were such a path P, then if we let $S = V(P) \setminus \{u, v\}$, clearly $|S| \leq 2l - 5$ and so there exists a simple path of length two joining u to v and containing no other vertex of P; thus it forms a C_l in $H + H^*$. But then by Lemma 3.4 H contains a C_l^3 , a contradiction.

The next lemma proves that in a C_l^3 -free 3-graph H there exists an extender of size at least a constant times the average degree of H.

Lemma 4.3. Let H be a C_l^3 -free 3-graph of average degree d. Then there exists an extender (X, Y) such that $|X| \ge d/(24l^2)$.

Proof. Let $H_1, H_2, \ldots, H_{l-2}$ be hypergraphs such that $E(H) = \bigcup_{i=1}^{l-2} E(H_i)$ and for each *i*, each edge of H_i contains a pair that is light in H_i . Such a collection of hypergraphs exists by Lemma 3.6. By the pigeonhole principle there exists $i \in [l-2]$ such that $|E(H_i)| \ge |E(H)|/(l-2)$. We may assume without loss of generality that i = 1. For j = 1, 2, 3 let H_{1j} be the hypergraph with vertex set V(H) and consisting of those edges of H_1 containing precisely *j* light pairs in H_1 . We consider two different cases.

Case 1: $|E(H_{11})| \geq |E(H_1)|/2$. Each edge of H_{11} gives two pairs (x, e) of a vertex v and an edge e of H_1 such that v is contained in the (unique) light pair of H_1 contained in e. Therefore there exists a vertex v contained in the light pair of at least $2|E(H_{11})|/n$ edges of H_{11} . But $2|E(H_{11})|/n \geq$ $|E(H_1)|/n \geq E(H)/(n(l-2)) = d/(3(l-2))$. List these edges as vx_iy_i : $i = 1, \ldots, m, m \geq d/(3(l-2))$, so that the pairs vx_i are light and the pairs vy_i are heavy in H_1 . In particular, no y_i can occur as x_j for any j. As the pairs vx_i are light, at least m/(2l-3) of the x_i 's are pairwise distinct; so we let X be a set of m/(2l-3) pairwise distinct x_i 's and let Y be $\{v\} \cup \{y_i : x_i \in X\}$. We claim that the pair (X, Y) is an extender. It is clear that $|Y| \leq 2|X|$, so let us check that the first condition holds. Let $x_i, x_j \in X$. Let $S \subseteq V(H) \setminus (\{x_i, x_j\} \cup Y), |S| \leq 2l-5$. If $y_i \neq y_j$ then $\{vx_iy_i, vx_jy_j\}$ forms a required path of length two. If $y_i = y_j$ then since x_jy_i is heavy in H there exists $z \in V(H) \setminus (S \cup \{v, x_i\})$ such that $y_ix_jz \in E(H)$. Then $\{vx_iy_i, y_ix_jz\}$ forms the required path of length two.

Case 2: $|E(H_{12}) \cup E(H_{13})| \ge |E(H_1)|/2$. Each edge of $E(H_{12}) \cup E(H_{13})$ defines at least one pair (v, e) of a vertex v and an edge e of H_1 such that v belongs to two light pairs of e. Thus there exists a vertex v contained in two light pairs of at least $|E(H_1)|/(2n) \ge d/(6(l-2))$ edges of H_1 . List these edges as vx_iy_i , $1 \le i \le m; m \ge d/(6(l-2))$, so that the pairs vx_i and vy_i are light for all i. The fact that these pairs are light implies that we may find at least m/(4l-7) pairs x_iy_i which are pairwise disjoint; without loss of generality the pairs x_iy_i are pairwise disjoint for $i \in [1, \lceil m/(4l-7) \rceil]$. We let $X = \{x_i : i \in \lceil m/(4l-7) \rceil\}$ and we let $Y = \{v\} \cup \{y_i : i \in [\lceil m/(4l-7) \rceil]\}$. It is clear that $|Y| \le 2|X|$ and for any $i \ne j$ and any $S \subseteq V(H) \setminus (\{x_i, x_j\} \cup Y), \{vx_iy_i, vx_jy_j\}$ is a path of length two not meeting S which joins x_i to x_j .

We are now ready to prove Lemma 4.4, which is the main result of this section. It says that if (X, Y) is an extender, then X contains a large independent set in H^* . Observe that being independent in H^* is stronger than being independent in H, i.e. that any set independent in H^* is also independent in H. This is because any edge of H contains an edge of H^* . The reason why we wish to find an independent set in H^* rather than merely H is that in the proof of Theorem 1.5 we will seek an independent set in H which is made of subsets of several disjoint extenders (X_1, Y_1) , $(X_2, Y_2), \ldots$ (for a suitable definition of "disjoint"), and so we shall need not only that each subset of each extender is independent in H but also that there are no edges *between* these subsets. This is where the extra information given by the lemma will be useful.

Lemma 4.4. Let H be a C_l^3 -free 3-graph and let (X, Y) be an extender in H. Then X contains a subset Z which

- 1. is independent in H^* ;
- 2. has size at least |X|/(l-2).

Proof of Lemma 4.4. By Lemma 4.2 $H^*[X]$ contains no simple path of length l-2 and hence by Proposition 2.1 is (l-2)-colourable. Hence X contains a set Z which is independent in H^* and has size at least |X|/(l-2).

5 Neighbourhoods of Extenders

In a C_l^2 -free 2-graph, the i^{th} neighbourhood of a vertex v (that is, the vertices at distance precisely i from v) is also (l-2)-colourable provided $i \leq \lfloor (l-1)/2 \rfloor$ (see Erdös, Faudree, Rousseau and Schelp [4]). The arguments of [7] and [10] use this in order to bound $R(C_l^2, K_n^2)$. Likewise, the i^{th} neighbourhood of an extender, if carefully defined, will be useful in the proof of Theorem 1.5.

Definition 5.1. Let H be a 3-graph and let (X, Y) be an extender in H. Let $S \subseteq V(H), X \cup Y \subseteq S$. For $i \in \mathbb{N}$ and $v \in S \setminus (X \cup Y)$, the distance between v and (X, Y) within S, denoted by $d_S(v, (X, Y))$, is the minimal length of a simple path P in $H^*[S \setminus Y]$ joining v to a vertex x of X.

The *i*th neighbourhoud of (X, Y) within S, denoted by $\Gamma_{S,i}(X, Y)$, is the set $\{v \in V(H) \setminus Y : d_S(v, (X, Y)) = i\}$. We also let $\Gamma_{S,0}(X, Y) = X$. Finally, we define $\Gamma_{S,\leq i}(X, Y)$ to be $\bigcup_{j \in [0,i]} \Gamma_{S,j}(X, Y)$.

So the i^{th} neighbourhood of (X, Y) within S, for $i \ge 1$, consists of those vertices of S which do not lie in X and which can be joined to a vertex of

X by a path of $H^*[S]$ of length *i* which does not meet Y, but by no such path of length less than *i*. Notice that a key element of this definition is that we are looking at paths in H^* , not H. Our motivation for introducing neighbourhoods of extenders is the following lemma, which plays the same role for neighbourhoods of extenders as Lemma 4.4 does for extenders. Again the stronger statement that $\Gamma_{S,i}(X,Y)$ contains a large independent set in H^* rather than H will be necessary in the proof of Theorem 1.5.

Lemma 5.2. Let H be a 3-uniform, C_l^3 -free hypergraph. Let (X, Y) be an extender in H. Let $S \subseteq V(H)$, $X \cup Y \subseteq S$. Let $i \in [m-1]$ where $m = \lfloor (l-1)/2 \rfloor$ and $m \geq 2$. Then $\Gamma_{S,i}(X,Y)$ contains a set Z which

- 1. is independent in H^* ;
- 2. is such that $|Z| \ge |\Gamma_{S,i}(X,Y)|/b_l$ where $b_l = (6l 10)^{m-1} \cdot (2m 1)^{2m-1} \cdot (l-2)^{3^{2m-1}}$.

Let us point out that the parameter S in the definition of the i^{th} neighbourhood of an extender plays no important role in this lemma. It will only be required later on in the proof of Theorem 1.5. What the proof of Lemma 5.2 actually shows is that amongst n vertices each joined to X by a path in H^* of length i not meeting Y, we may find n/b_l vertices which form an independent set in H^* , where b_l is a constant whose value does not matter to us.

Proof of Lemma 5.2. For the sake of clarity, the proof will contain several subclaims.

By Lemma 3.6, $E_{\text{light}}(H)$ can be partitioned into 6l - 11 simple hypergraphs which we denote by $H_1, H_2, \ldots, H_{6l-11}$. Furthermore let $H_0 = P_{\text{heavy}}(H)$. Thus H^* is a [2,3]-graph whose set of edges is partitioned by the simple hypergraphs $H_0, H_1, H_2, \ldots, H_{6l-11}$.

For each $v \in \Gamma_{S,i}(X,Y)$ let P_v be a simple path of length i in $H^*[S \setminus Y]$ which joins v to some vertex x_v of X. For the rest of the proof, the choice of P_v and x_v is fixed for each $v \in \Gamma_{S,i}(X,Y)$. We also fix an enumeration of the edges of P as $E(P_v) = \{f_1^v, f_2^v, \ldots, f_i^v\}$ with $x_v \in f_1^v$ and $v \in f_i^v$ and $f_k^v \cap f_{k+1}^v \neq \emptyset$ for all $k \in [i-1]$, as well as an enumeration of the vertices of P as $V(P) = \{x_1^v, x_2^v, \ldots, x_{|V(P)|}^v\}$ so that $x_1^v = x_v$ and $x_{|V(P)|}^v = v$, and if $x_r^v \in f_j^v, x_s^v \in f_k^v$ with j < k then $r \leq s$. It is easy to see that both of the enumerations we just described exist and are unique, because the path P_v is simple and its edges have size no more than 3.

The type of P_v is the tuple $(t_k)_{k=1}^i$ which is such that for each $k \in [i]$, $f_k^v \in H_{t_k}$. Clearly for any v, P_v has one of $(6l-10)^i$ possible types. Therefore we have our first claim.

Claim 5.3. There exists a subset Z_1 of $\Gamma_{S,i}(X,Y)$ of size at least $|\Gamma_{S,i}(X,Y)|/(6l-10)^i$ such that all the paths P_v for $v \in Z_1$ are of the same type.

Since all the paths P_v for $v \in Z_1$ have the same type, it is clear that they contain the same number of vertices, and we denote by p this quantity, so $p \leq 2i + 1$. Let $c: V(H) \longrightarrow [2i + 1]$ be a (not necessarily proper) colouring of the vertices of V(H). We say that a path P_v for $v \in Z_1$ is rainbow with respect to c if $c(x_k^v) = k$ for all $k \in [p]$. Our second claim is the following.

Claim 5.4. There exists a colouring $c: V(H) \longrightarrow [2i+1]$ and a subset Z_2 of Z_1 of size at least $|Z_1|/(2i+1)^{2i+1}$ such that P_v is rainbow with respect to c for all $v \in Z_2$.

Proof of Claim 5.4. Let c be the colouring obtained by attributing to each vertex of H one of the 2i + 1 possible colours uniformly at random and independently of other vertices. Then for any $v \in Z_1$ the probability that P_v is rainbow with respect to c is precisely $1/(2i+1)^p \ge 1/(2i+1)^{2i+1}$. Thus the expected number of vertices v such that P_v is rainbow with respect to c is at least $|Z_1|/(2i+1)^{2i+1}$ and so there exists a choice of c and of Z_2 such that the claim holds.

In order to prove that $\Gamma_{S,i}(X,Y)$ contains a large independent set in H^* , we wish to apply Proposition 2.1, in the same way as we did in the proof of lemma 4.4 above. Ideally we would like to say that " $\Gamma_{S,i}(X,Y)$ cannot contain a simple path P of length l-2 since otherwise some subpath of P, call it P', would join two vertices a, b of $\Gamma_{S,i}(X,Y)$ such that there is a subpath P'_a of P_a and a subpath P'_b of P_b , such that $P' + P'_a + P'_b$ forms a C_l , a contradiction". There are several difficulties which prevent us from doing this directly, however (What happens if P'_a and P'_b meet each other more than once? What happens if P'_a or P'_b meets P' more than once? How do we ensure that $P'_a + P'_b + P'$ has length l?) The fact that the paths P_v for $v \in \mathbb{Z}_2$ are rainbow goes a long way towards resolving these problems. Indeed, notice that if $u, v \in Z_2$ then $V(P_u) \cap V(P_v) = \{x_i^u : x_i^u = x_i^v, i \in [p]\}$ since P_v and P_u are rainbow. This would guarantee in the discussion above, for example, that P_a and P_b intersect P only once each. But this is not enough, as the main problem we are faced with is to guarantee that $P'_a + P'_b + P'$ has length l. This is why we introduce the class of an edge of $H^*[Z_2]$ in what follows.

Order the vertices of H arbitrarily and let < be the chosen total ordering. Given $e \in E(H^*[Z_2])$, let a be the smallest vertex under < which is contained in e and let b be the largest one. The class of e is the tuple $(m_k)_{k=1}^p$ whose coordinates take values in $\{0, 1, 2\}$ such that $m_k = 0$ if $x_k^a < x_k^b$, $m_k = 1$ if $x_k^a = x_k^b$ and $m_k = 2$ if $x_k^a > x_k^b$. So there are $3^p \leq 3^{2i+1}$ possible classes for an element of $E(H^*[Z_2])$, and for each possible class t of an element of $E(H^*[Z_2])$ we let J_t be the hypergraph with vertex set Z_2 and whose edges are all the elements of $E(H^*[Z_2])$ of type t. Our third claim is the following.

Claim 5.5. For each element t of $\{0, 1, 2\}^p$, J_t is (l-2)-colourable.

Proof of Claim 5.5. By Proposition 2.1 it is enough to show that J_t does not contain an increasing simple path of length l-2 with respect to <. Suppose, to the contrary, that it did contain such a path P. Let a be the smallest vertex of P and b the largest one (with respect to <). Write $E(P) = \{e_1, e_2, \ldots, e_{l-2}\}$ with $a \in e_1$, $b \in e_{l-2}$ and $e_j \cap e_{j+1} \neq \emptyset$ for each $j \in [l-3]$. For each $j \in [l-3]$ let u_j be the vertex of P belonging to $e_j \cap e_{j+1}$. Furthermore let $u_0 = a$, $u_{l-2} = b$.

Recall that by definition of J_t , each of $e_1, e_2, \ldots, e_{l-2}$ has class t. Suppose that P_{u_0} and P_{u_1} do not intersect. Then, since e_1 has type t, we have $m_k = 0$ or $m_k = 2$ for any $k \in [i]$. Let $k \in [i]$. If $m_k = 0$ then we know, since each of $e_1, e_2, \ldots, e_{l-2}$ has class t, that $x_k^{u_0} < x_k^{u_1} < \cdots < x_k^{u_{l-2}}$. Likewise, if $m_k = 2$ we then know that $x_k^{u_0} > x_k^{u_1} > \cdots > x_k^{u_{l-2}}$. So in either case, for any $k \in [p]$, the $x_k^{u_j}$'s, $j \in [0, l-2]$, are pairwise distinct. But, since the paths $P_{u_0}, P_{u_1}, \ldots, P_{u_{l-2}}$ are rainbow with respect to c, this implies that $P_{u_0} \cap P_{u_j} = \emptyset$ for all $j \in [l-2]$. Thus in particular $P_{u_0} \cap P_{u_{l-2i-2}} = \emptyset$. Hence $P_{u_0} + \{e_1, e_2, \ldots, e_{l-2i-2}\} + P_{u_{l-2i-2}}$ is a simple path of length l-2 in H^* not meeting Y and joining two vertices of X, a contradiction to Lemma 4.2.

Thus we may assume that P_{u_0} and P_{u_1} do intersect in some vertex. This means that some entry of t is equal to one; let q be largest such that $t_q = 1$, and let *h* be largest such that $x_q^{u_0} \in f_h^{u_0}$. Let $A = \{f_h^{u_0}, f_{h+1}^{u_0}, \dots, f_i^{u_0}\}, B = \{f_h^{u_{l-2i+2h-2}}, f_{h+1}^{u_{l-2i+2h-2}}, \dots, f_i^{u_{l-2i+2h-2}}\}$ and $D = \{e_1, e_2, \dots, e_{l-2i+2h-2}\}$. We shall now prove that C := A + D + B is a C_l in $H + H^*$, which is a contradiction by Lemma 3.4 and thus finishes the proof of the claim. Let $W = V(A) \cap V(B)$. To prove that C is a C_l it is enough to prove that $W = \{x_q^{u_0}\}$, since the rainbow property of P_{u_0} and $P_{u_{l-2i+2h-2}}$ implies that A and B only intersect D in u_0 and $u_{l-2i+2h-2}$ respectively. Since $t_q = 1$ we have $x_q^{u_0} = x_q^{u_1}, x_q^{u_1} = x_q^{u_2}, \ldots, x_q^{u_{l-2i+2h-3}} = x_q^{u_{l-2i+2h-2}}$ and so $x_q^{u_0} = x_q^{u_{l-2i+2h-2}}$. Thus $x_q^{u_0} \in W$. Suppose now that there exists $u \in W$, $u \neq x_q^{u_0}$. So $u = x_w^{u_0}$ for some $w \neq q$. If w > q then as $t_w \neq 1$ we either $u \neq x_q^{w_0}$. So $u = x_w^{v_0}$ for some $w \neq q$. If $w \neq q$ for $u \neq x_w^{u_0}$ have $x_w^{u_0} < x_w^{u_1} < \ldots < x_w^{u_{l-2i+2h-2}}$ or $x_w^{u_0} > x_w^{u_1} > \ldots > x_w^{u_{l-2i+2h-2}}$ and so either way $x_w^{u_0} \neq x_w^{l-2i+2h-2}$. But then as the paths P_{u_0} and $P_{u_{l-2i+2h-2}}$ are rainbow, $x_w^{u_0} \notin W$, a contradiction. Thus w < q. Then $w \in f_h^{u_0}$ since w < q, $w \in A$ and $x_q^{u_0} \in f_h^{u_0}$. Also $w \in f_h^{u_{l-2i+2h-2}}$ by the rainbow property of P_{u_0} and $P_{u_{l-2i+2h-2}}$. Hence $\{x_q^{u_0}, x_w^{u_0}\} \subseteq f_h^{u_0} \cap f_h^{u_{l-2i+2h-2}}$ and so $|f_h^{u_0} \cap f_h^{u_{l-2i+2h-2}}| \ge 2$. Notice also that $f_h^{u_0} \ne f_h^{u_{l-2i+2h-2}}$ (if not then by definition of a this can only because if $x^{u_0} \in \mathcal{L}^{u_0}$ by definition of q this can only happen if $x_q^{u_0} \in f_{h+1}^{u_0}$ and this contradicts the definition of h). But $f_h^{u_0}$ and $f_h^{u_{l-2i+2h-2}}$ belong to the same simple hypergraph H_j (for some j) since P_{u_0} and $P_{u_{l-2i+2h-2}}$ are of the same type, and this is a contradiction. The claim is proved.

Since J_t is (l-2)-colourable for any class t, we see that $H^*[Z_2]$ is $(l-2)^{3^{2i+1}}$ -colourable, since any edge of $H^*[Z_2]$ belongs to J_t for some t and there are at most 3^{2i+1} values of t. We therefore have the following.

Claim 5.6. Z_2 contains a set Z of size at least $|Z_2|/(l-2)^{3^{2i+1}}$ which is independent in H^* .

Lemma 5.2 is now proved.

6 Proof of Theorem 1.5

Before proving Theorem 1.5 in earnest, let us briefly explain how one might find a large independent set in a C_l^2 -free 2-graph. Let $m = \lfloor (l-1)/2 \rfloor$. As mentioned at the beginning of Section 4, a simple bound on the independence number of a C_l^2 -free 2-graph G can be found by considering the average degree d of G. However, when $l \geq 5$, we can do better. Indeed in a "typical" 2-graph G of average degree d, we expect the m^{th} neighbourhood of a vertex to have size about d^m . This is why, if d is large, it might be a better idea to seek an independent set in $\Gamma_m(v)$ rather than $\Gamma(v)$ (since $\Gamma_m(v)$ is (l-2)colourable), and when d is small to still apply Turán's theorem as explained in Section 4. This in itself is not a valid argument, obviously, since it is not the case that $|\Gamma_m(v)| \geq d^m$ (for some v) in every graph. But, if the m^{th} neighbourhood of a vertex is bounded in G, then it is a better idea to look at the ith neighbourhood of a vertex for $1 \le i \le m-1$. In fact, both [7] and [10] either find a large independent set which is the union of large subsets of i^{th} neighbourhoods of vertices (where some care is needed to make sure that there is no edge between these sets), or find an induced subgraph of G of small average degree, where one can then apply Turán's Theorem to find a large independent set. We adopt the same strategy, and are able to improve upon the graph bound because we are able to put ourselves in the position of applying Proposition 2.2 with r = 3 rather than Turán's Theorem. The main difficulty for hypergraphs is to introduce useful definitions of what is meant by a "neighbourhood" and this was the subject of the previous sections.

Let H be a C_l^3 -free 3-graph on n vertices. The statement of Theorem 1.5 is equivalent to proving that the independence number of H is at least a constant times $n^{(m+1)/(m+2)}$ where $m = \lfloor (l-1)/2 \rfloor$, and this is what we shall prove. Let us notice that if l = 3 or l = 4, i.e. m = 1, then the theorem can be proved as follows: either H contains an extender of size at least $n^{2/3}$ hence contains an independent set of size at least $n^{2/3}/(l-2)$ by Lemma 4.4 or has average degree no more than $24l^2n^{2/3}$ by Lemma 4.3 and hence contains an independent set of size at least $0.5n/(24l^2n^{2/3})^{1/2} = n^{2/3}/(4\sqrt{6}l)$ by Proposition 2.2 with r = 3. Thus henceforth we shall assume $l \geq 5$ and so $m \geq 2$.

Notice that for every extender (X, Y) in H such that $|X| \ge n^{2/(m+2)}$ and every $S \subseteq V(H)$ containing $X \cup Y$, we may assume that there exists an integer $i \in [0, m-2]$ such that $|\Gamma_{S,i+1}(X,Y)| \le n^{1/(m+2)} |\Gamma_{S,i}(X,Y)|$ for

otherwise we have $|\Gamma_{S,m-1}(X,Y)| \ge n^{(m+1)/(m+2)}$ and so by Lemma 5.2 we can find an independent set of size at least $n^{(m+1)/(m+2)}/b_l$ in H.

Consider the following procedure producing a sequence $((X_k, Y_k))_{k \in [t]}$ of extenders of H, a sequence $(S_k)_{k \in [t+1]}$ of subsets of H, and a sequence $(i_k)_{k \in [t]}$ of elements of [0, m-2].

- Initially, we let $S_1 = V(H)$. If there is no extender (X, Y) in H with $|X| \ge n^{2/(m+2)}$ then we STOP. Otherwise we let (X_1, Y_1) be an extender in H with $|X_1| \ge n^{2/(m+2)}$, and we let i_1 be the least $i \in [0, m-2]$ such that $|\Gamma_{S_1,i+1}(X_1, Y_1)| \le n^{1/(m+2)}|\Gamma_{S_1,i}(X_1, Y_1)|$.
- Having obtained sequences $((X_j, Y_j))_{j \in [k]}$, $(S_j)_{j \in [k]}$ and $(i_j)_{j \in [k]}$, we let

$$S_{k+1} = V(H) \setminus \left(\bigcup_{j \in [k]} \left(Y_j \cup \Gamma_{S_j, \leq i_j+1}(X_j, Y_j) \right) \right).$$

If there is no extender (X, Y) in $H[S_{k+1}]$ with $|X| \ge n^{2/(m+2)}$ then we STOP. Otherwise, we let (X_{k+1}, Y_{k+1}) be such an extender. This is clearly an extender in H, as $H[S_{k+1}]$ is a subhypergraph of H. We let i_{k+1} be the least $i \in [0, m-2]$ such that $|\Gamma_{S_{k+1},i+1}(X_{k+1}, Y_{k+1})| \le n^{1/(m+2)}|\Gamma_{S_{k+1},i}(X_{k+1}, Y_{k+1})|$.

Clearly, this procedure must terminate. The sequence produced has the following important property.

For every $k_1, k_2 \in [t]$ with $k_1 < k_2$,

$$\left(Y_{k_1} \cup \Gamma_{S_{k_1}, \leq i_{k_1}+1}(X_{k_1}, Y_{k_1})\right) \cap \Gamma_{S_{k_2}, i_{k_2}}(X_{k_2}, Y_{k_2}) = \emptyset.$$
(1)

Suppose first that $|S_{t+1}| \ge n/2$. Then $H[S_{t+1}]$ contains no extender of size at least $n^{2/(m+2)}$. By Lemma 4.3 this implies that the average degree of $H[S_{t+1}]$ is no more than $24l^2n^{2/(m+2)}$. Hence by Proposition $2.2 \ \alpha(H[S_{t+1}]) \ge 0.5(n/2)/(24l^2n^{2/(m+2)})^{1/2} = n^{(m+1)/(m+2)}/(8\sqrt{6}l)$. But clearly $\alpha(H) \ge \alpha(H[S_{t+1}])$ and so we are done.

So we may assume that that $|S_{t+1}| \leq n/2$. We shall find a large set which is independent in H^* (rather than H). As mentioned above in Section 4 such a set is also independent in H. The reason why we look for an independent set in H^* rather than H is that neighbourhoods of extenders are defined in terms of paths in H^* rather than H. Let

$$T = \bigcup_{k \in [t]} \left(Y_k \cup \Gamma_{S_k, \leq i_k + 1}(X_k, Y_k) \right).$$

Since $S_{t+1} \leq n/2$ we have $|T| \geq n/2$. For each $k \in [t]$, by the definition of i_k we have $|\Gamma_{S_k,i_k}(X_k,Y_k)| \geq |\Gamma_{S_k,i}(X_k,Y_k)|$ for any $i \leq i_k$ and $|\Gamma_{S_k,i_k}(X_k,Y_k)| \geq |\Gamma_{S_k,i_k+1}(X_k,Y_k)|/n^{1/(m+2)}$. Thus,

$$|\Gamma_{S_k, i_k}(X_k, Y_k)| \ge |Y_k \cup \Gamma_{S_k, \le i_k+1}(X_k, Y_k)| / ((m+2)n^{1/(m+2)})$$

(Recall that by definition of an extender, $|Y_k| \leq 2|X_k|$.) By Lemma 5.2 (Lemma 4.4 when $i_k = 0$), for every $k \in [t]$, $\Gamma_{S_k,i_k}(X_k,Y_k)$ contains a set Z_k of size at least $|\Gamma_{S_k,i_k}(X_k,Y_k)|/b_l$ which is independent in H^* . Let $Z = \bigcup_{k \in [t]} Z_k$, so that $|Z| \geq |T|/(b_l(m+2)n^{1/(m+2)}) \geq n^{(m+1)/(m+2)}/(2b_l(m+2))$. We shall find $W \subseteq Z$ of size at least $|Z|/2^{2m-1}$ which is independent in H^* , and this shall finish the proof of Theorem 1.5.

Let $v \in Z$. Then $v \in Z_k$ for some $k \in [t]$. Suppose $i_k \geq 1$. Then as $v \in \Gamma_{S_k,i_k}(X_k, Y_k)$, there exists a simple path P_v in $H^*[S_k]$ which joins v to an element of X_k , which is disjoint from Y_k , and which has length i_k . As in the proof of Lemma 5.2, we select one such path P_v for each $v \in W$ and fix our choice for the rest of the proof. If $i_k = 0$ we let $P_v = \{v\}$. Let $c : V(H) \longrightarrow \{$ blue, red $\}$ be a (not necessarily proper) 2-colouring of the vertices of H. We say that P_v is well-coloured by c if the colour given to v is red and the colour of any other vertex of P_v is blue. If c is a colouring obtained by randomly and uniformly assigning the colour blue or red to each vertex of H, independently of other vertices, then the probability that P_v is well-coloured is clearly $(1/2)^{|V(P_v)|}$, which is at least $(1/2)^{2i_k+1}$. Therefore by a mere expectation argument there exists a colouring c and $W \subseteq Z$ with $|W| \geq |Z|/2^{2m-1}$ such that for each $v \in W$, P_v is well-coloured.

Let us check that W is independent in H^* . Indeed suppose to the contrary that it isn't. Let $e \in H^*[W]$. As each Z_k is independent in H^* , e meets at least two distinct Z_k 's. Let k_1 be minimal such that $e \cap Z_{k_1} \neq \emptyset$. Let $k_2 > k_1$ be such that $e \cap Z_{k_2} \neq \emptyset$. Let $a \in e \cap Z_{k_1}$ and let $b \in e \cap Z_{k_2}$. By definition of k_1 and the fact that $S_{k_1} \supseteq S_{k_1+1} \supseteq S_{k_1+2} \supseteq \cdots$, we have $e \subseteq S_{k_1}$, in other words $e \in E(H^*[S_{k_1}])$. By (1) we have that $e \cap Y_{k_1} = \emptyset$ (Indeed, (1) implies that that all the elements of e not lying in $\Gamma_{S_{k_1},i_{k_1}}(X_{k_1},Y_{k_1})$ do not lie in Y_{k_1} .) Thus in fact $e \in E(H^*[S_{k_1} \setminus Y_{k_1}])$.

We now consider two different cases: $i_{k_1} = 0$ and $i_{k_1} \ge 1$. Consider first the case $i_{k_1} = 0$. Since e joins b to $a \in X_{k_1}$ and $e \in E(H^*[S_{k_1} \setminus Y_{k_1}])$, we have $b \in \Gamma_{S_{k_1}, \le 1}(X_{k_1}, Y_{k_1})$. This is a contradiction to (1) given that $b \in \Gamma_{S_{k_2}, k_2}(X_{k_2}, Y_{k_2})$. Hence we may assume that $i_{k_1} \ge 1$. In this case, as P_a is well-coloured by c, its sole red vertex is a, and since $e \subseteq W$, all its vertices are coloured red. Hence $P_a \cap e = \{a\}$, so that the path $P_a + e$ is simple. But then $P_a + e$, being a simple path in $H^*[S_{k_1} \setminus Y_{k_1}]$ of length $i_{k_1} + 1$ joining b to an element of X_{k_1} , shows that $b \in \Gamma_{S_{k_1}, \le i_{k_1}+1}(X_{k_1}, Y_{k_1})$. This is a contradiction to (1) and finishes the proof of Theorem 1.5.

7 Proof of Theorem 1.6

Here we shall give a sketch of the proof of Theorem 1.6. Let r, l be fixed integers with $r \ge 2$ and l odd, $l \ge 5$. Let H be a C_l^r -free r-graph on n vertices.

The first step of the proof is to show that there exists a C_l -free [2, r]-

graph H' such that V(H') = V(H), $\alpha(H) \geq \alpha(H')$ and E(H') can be partitioned into a constant number of simple hypergraphs. We shall use the same reduction ideas as in Lemma 7.5 and Lemma 7.6 of [3], but we go one step further by partitioning E(H') into simple hypergraphs. A sunflower Swith core C is a collection S of sets such that $e, f \in S$ and $e \neq f$ implies $e \cap f = C$. If |S| = p and |C| = a then we say that S is an (a, p)-sunflower. The Sunflower Lemma is the following statement.

Proposition 7.1 (P. Erdös, R. Rado [5]). Let \mathcal{F} be a collection of sets of size at most r. If $|\mathcal{F}| \geq r!(p-1)^r$ then \mathcal{F} contains a sunflower with p members.

To find H', one can now proceed as follows. Suppose H contains an (a, rl)-sunflower S for $a \geq 2$. Let C be the core of S. Remove all the edges of H containing C and then add C to H. It can be checked that this does not increase the independence number of H or create a C_l . Repeat this procedure until no (a, rl)-sunflower exists in H with $a \geq 2$. Call the resulting hypergraph H'.

Clearly no pair of vertices of H' is contained in $r!(rl-1)^r$ edges of H' else by Proposition 7.1 H' would contain an (a, rl)-sunflower with $a \ge 2$. This implies that the graph with vertex set E(H') where e and f are adjacent if $|e \cap f| \ge 2$, has maximal degree less than $\binom{r}{2}r!(rl-1)^r$, and hence by the same argument as in Lemma 3.6 we see that E(H') can be partitioned into $\binom{r}{2}r!(rl-1)^r$ simple hypergraphs, which we denote by H'_1, H'_2, \ldots, H'_p , where $p = \binom{r}{2}r!(rl-1)^r$.

Since $\alpha(H) \geq \alpha(H')$, it is enough to find in H' an independent set of size at least a constant times $n^{m/(m+1)}$ where $m = \lfloor l/2 \rfloor$. We shall proceed as in the proof of Theorem 1.5.

Extenders are defined in the same way as before, except that now we require $|Y| \leq (r-1)|X|$. In a simple [2, r]-graph there is always an extender (X, Y) of size at least d/r where d is the average degree of the [2, r]-graph in question (consider the neighbourhood of a vertex of maximal degree in the hypergraph). Hence in any subhypergraph of H' there is an extender of size at least a constant times the average degree of that subhypergraph, because its edges can be partitioned into a constant number of simple hypergraphs.

The i^{th} neighbourhood of an extender within a set S is defined as before except that we regard $(H')^*$ as being equal to H' (Because "H' is already reduced".) Thus as before, for an extender (X, Y), if $0 \le i \le m - 1$ then $\Gamma_{S,i}(X, Y)$ contains a large independent set in H'. To see why, it suffices to follow the argument of Lemma 5.2 where the hypergraphs $H_0, H_1, \ldots,$ H_{6l-11} are replaced by H'_1, H'_2, \ldots, H'_p and H^* by H'. Here we modify the definition of the type of a path P_v slightly: for P_u and P_v to be of the same type, we require as before that corresponding edges along P_u and P_v belong to the same simple hypergraph but we now also require that they have the same size. This only affects the bound obtained on |Z| in Lemma 5.2 by a constant factor.

Finally we proceed as in Section 6, but with different parameters. Namely, an extender (X, Y) is added to the sequence if $|X| \ge n^{1/(m+1)}$, and i_{k+1} is the least $i \in [0, m-2]$ such that $|\Gamma_{S_{k+1},i+1}(X_{k+1}, Y_{k+1})| \le n^{1/(m+1)}|\Gamma_{S_{k+1},i}(X_{k+1}, Y_{k+1})|$ (As before, it is easy to see that we may assume without loss of generality that i_{k+1} exists.) If $|S_{t+1}| \ge n/2$, then $H'[S_{t+1}]$ is a [2, r]-graph of average degree no more than a constant times $n^{1/(m+1)}$. By applying Proposition 2.2 with r = 2 to the 2-graph with vertex set S_{t+1} and set of edges $\{\{u, v\} \subseteq S_{t+1} : \exists e \in H'[S_{t+1}] \text{ s.t. } \{u, v\} \subseteq e\}$ we find an independent set of size at least a constant times $n^{m/(m+1)}$ in H'. If $|S_{t+1}| \le n/2$ then we follow the rest of the proof of Theorem 1.5 (where $(H')^* = H')$; no Z_k can contain an edge of H' by Lemma 5.2 (and Lemma 4.4) and the existence of an edge of H' containing vertices from two distinct Z_k 's again eventually implies a violation of (1).

8 Further Remarks

We mentioned in the introduction that we believe that one should be able, for any fixed $r \ge 3$ and $l \ge 3$, to beat the graph bound for $R(C_l^r, K_n^r)$ by an order of magnitude. However, the obvious generalization of the methods we use fails for $r \ge 4$ and $l \ge 4$. An example of a 4-uniform hypergraph where our attempts fail is the following. Let V(H) = [2n] and let E(H) = $\{\{i, i+n, j, j+n\} : i, j \in [n], i \ne j\}$. It is clear that this hypergraph contains no loose cycle since any two of its edges meet in 0 or 2 vertices. Naturally, this hypergraph does contain a very large independent set, but there are no useful extenders in this hypergraph, so there is no obvious argument which can make use of Proposition 2.2. It might still be possible to improve upon the graph bound for some specific values of r and l, but as no straighforward generalization seems possible we did not cover this here. The first natural case to consider is $R(C_5^4, K_n^4)$, where we do not know how to beat the graph bound of $O(n^{3/2})$.

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