# Edge Clique Cover of Claw-free Graphs 

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#### Abstract

The smallest number of cliques, covering all edges of a graph $G$, is called the (edge) clique cover number of $G$ and is denoted by $\operatorname{cc}(G)$. It is an easy observation that if $G$ is a line graph on $n$ vertices, then $\operatorname{cc}(G) \leq n$. G. Chen et al. [Discrete Math. 219 (2000), no. 1-3, 17-26; MR1761707] extended this observation to all quasi-line graphs and questioned if the same assertion holds for all claw-free graphs. In this paper, using the celebrated structure theorem of claw-free graphs due to Chudnovsky and Seymour, we give an affirmative answer to this question for all claw-free graphs with independence number at least three. In particular, we prove that if $G$ is a connected claw-free graph on $n$ vertices with three pairwise nonadjacent vertices, then $\operatorname{cc}(G) \leq n$ and the equality holds if and only if $G$ is either the graph of icosahedron, or the complement of a graph on 10 vertices called "twister" or the $p^{\text {th }}$ power of the cycle $C_{n}$, for some positive integer $p \leq\lfloor(n-1) / 3\rfloor$.


Keywords: edge clique covering, edge clique cover number, claw-free graphs, triangle-free graphs.

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## 1 Introduction

Throughout the paper all graphs are finite without parallel edges and loops, unless it is explicitly mentioned. By a clique of $G$ we mean a subset of pairwise adjacent vertices. The size of the largest clique of $G$ is called the clique number of $G$ and is denoted by $\omega(G)$. Also, a subset of pairwise nonadjacent vertices is called a stable set of $G$ and the size of the largest stable set of $G$ is called the independence number of $G$ and is denoted by $\alpha(G)$. A clique of size three and a stable set of size three are called a triangle and a triad, respectively. The graph $K_{1,3}$ (a star graph with three edges) is called a claw. We also use the term "a claw in $G$ " for a subset of vertices $X \subseteq V(G)$ where the induced subgraph of $G$ on $X$ is isomorphic to $K_{1,3}$. Given a graph $H$, a graph $G$ is called $H$-free, if $G$ does not have the graph $H$ as an induced subgraph. Thus, claw-free graphs are $K_{1,3}$-free graphs. Also, trianglefree graphs and triad-free graphs are graphs with no triangle and no triad, respectively. A graph $G$ is called a line graph if $G$ is the line graph of a graph $H$ (i.e. the vertices of $G$ correspond to the edges of $H$ and two vertices are adjacent in $G$ if their corresponding edges in $H$ have a common endpoint). Also, a graph is called a quasi-line graph if the neighbourhood of every vertex can be expressed as the union of two cliques. Line graphs, quasi-line graphs and triad-free graphs are examples of claw-free graphs. In a series of elegant papers [3-7], Seymour and Chudnovsky gave a complete description that explicitly explains the structure of all claw-free graphs. Since then, their structure theorem has been used to prove several conjectures for claw-free graphs. In this paper, we are going to apply this machinery to investigate the edge clique covering of claw-free graphs.
Given a graph $G$, let us say that an edge of $G$ is covered by a clique $C$ of $G$ if both of its endpoints belong to $C$. An (edge) clique covering for $G$ is a collection $\mathscr{C}$ of cliques of $G$ such that every edge of $G$ is covered by a clique in $\mathscr{C}$. The (edge) clique cover number (or the intersection number, see [11]) of $G$, denoted by $\operatorname{cc}(G)$, is the least number $k$ such that $G$ admits a clique covering of size $k$. The clique cover number has been studied widely in the literature (see e.g. [9,11,16,17]). In [2], Chen et al. proved that the clique cover number of a quasi-line graph on $n$ vertices is at most $n$. Claw-free graphs being generalization of quasi-line graphs, they questioned if the same result holds for all claw-free graphs. In order to answer the question of Chen et al., it is enough to answer it for connected claw-free graphs. Regarding this question, in [14], it is proved that if $G$ is a connected claw-free graph on $n$ vertices with the minimum degree $\delta$, then $\mathrm{cc}(G) \leq n+c \delta^{4 / 3} \log ^{1 / 3} \delta$, for some constant $c$. Moreover, assuming $\Delta$ as the maximum degree of $G$, it is proved there that $G$ admits a clique covering $\mathscr{C}$ such that each vertex is in at most $c \Delta / \log \Delta$ cliques of $\mathscr{C}$, where $c$ is a constant (though they did not use the structure theorem of claw-free graphs). In this paper, we apply the structure theorem to give an affirmative answer to the question of Chen et al. for all claw-free graphs $G$ with $\alpha(G) \geq 3$. We also characterize the equality cases, i.e. all such graphs whose clique cover number is equal to their number of vertices. Let us describe the main results of this paper in more detail.
First, we need a couple of definitions. For a graph $G$ and a nonnegative integer $p$, the $p^{\text {th }}$ power of $G$, denoted by $G^{p}$, is defined as the graph on the same vertices such that two distinct vertices $u, v$ are adjacent in $G^{p}$ if and only if there is a path in $G$ between $u$ and $v$ with at most $p$ edges (thus, $G^{1}=G$ ). For instance, when $n \geq 2 p+1$, the $p^{t h}$ power of the cycle $C_{n}$ (also called a circulant graph), denoted by $C_{n}^{p}$, is the graph with vertices $v_{0}, v_{1}, \ldots, v_{n-1}$, such that $v_{i}$ is adjacent to $v_{j}$ if and only if either $i-j$ or $j-i$ modulo $n$ is in $\{1, \ldots, p\}$. All graphs $C_{n}^{p}$ are clearly claw-free (they are in fact quasi-line graphs). The graph of icosahedron and the complement of a twister (depicted in Figure 1) are two more examples of claw-free graphs. The main result of this paper is as follows.

Theorem 1.1. Let $G$ be a connected claw-free graph on $n$ vertices with $\alpha(G) \geq 3$. Then, $\operatorname{cc}(G) \leq n$ and equality holds if and only if $G$ is either the graph of icosahedron, or the complement of a twister, or the $p^{t h}$ power of the cycle $C_{n}$, for some positive integer $p \leq\lfloor(n-1) / 3\rfloor$.

According to the above theorem, in order to answer the question of Chen et al. completely, it suffices to prove the upper bound $n$ for the clique cover number of $n$-vertex triad-free graphs (i.e. graphs $G$ with $\alpha(G) \leq 2$ ). Since there is no structure theorem for triangle-free graphs (and so triad-free graphs), dealing with the question for these graphs seems to require a completely different approach. In [13], it


Figure 1: The graph of icosahedron $G_{0}$ (left) and the complement of a twister (right).
is proved that for every triad-free graph $G$ on $n$ vertices, $\operatorname{cc}(G) \leq 2(1-o(1)) n$. Combining this result with Theorem 1.1 gives the following corollary.

Corollary 1.2. For every connected claw-free graph $G$ on $n$ vertices, we have $\operatorname{cc}(G) \leq 2(1-o(1)) n$.
However, improving the upper bound to $n$ for all triad-free graphs remains open.
Conjecture 1.3. If $G$ is a graph on $n$ vertices with $\alpha(G) \leq 2$, then $\operatorname{cc}(G) \leq n$.
In this paper, we also prove the upper bound $n+1$ for the clique cover number of a special class of triad-free graphs. Pursuing [8], a graph $G$ is said to be tame if $G$ is an induced subgraph of a connected claw-free graph $H$ such that $H$ has a triad. For instance, every triad-free graph whose vertex set is the union of three cliques is tame. To see this, let $G$ be a triad-free graph, where $V(G)$ is the union of three cliques $A, B$ and $C$. Now, define $H$ to be the graph obtained from $G$ by adding three new vertices $a, b, c$, such that $\{a, b, c\}$ is a triad, $a$ is adjacent to every vertex in $A \cup B, b$ is adjacent to every vertex in $B \cup C$ and $c$ is adjacent to every vertex in $C \cup A$. Then, $H$ is a claw-free graph with a triad and $G$ is an induced subgraph of $H$. Thus, $G$ is tame. In fact, we prove the following theorem.

Theorem 1.4. Let $G$ be a connected graph on $n$ vertices which is tame. If $V(G)$ is not the union of three cliques, then $\operatorname{cc}(G) \leq n$ and otherwise, $\operatorname{cc}(G) \leq n+1$.

Finally, it is worth noting that investigating the clique cover number of $K_{1, t}$-free graphs $(t \geq 4)$, as a generalization of claw-free graphs, seems to be an interesting and challenging problem. For a graph $G$, let us define the local independence number of $G$, denoted by $\alpha_{l}(G)$, as the maximum number $t$ such that $G$ has a stable set of size $t$ within the neighbourhood of a vertex. Thus, claw-free graphs are exactly the graphs with $\alpha_{l}(G) \leq 2$. Towards the generalization of Theorem 1.1, the following question is naturally raised in the same line of thought, relating the clique cover number and the local independence number.

Question 1.5. Is it true that for every graph $G$ on $n$ vertices, $2 \operatorname{cc}(G) \leq n \alpha_{l}(G)$ ?
Moreover, a relaxed version of the above question can be asked about the independence number as follows (note that this is a generalization of Conjecture 1.3).

Question 1.6. Is it true that for every graph $G$ on $n$ vertices, $2 \operatorname{cc}(G) \leq n \alpha(G)$ ?
In order to prove Theorem 1.1, we deploy the structure theorem in [7]. It basically asserts that for every connected claw-free graph $G$, either $V(G)$ is the union of three cliques, or $G$ is some kind of "generalized line graph" which admits a certain structure called non-trivial strip-structure, or $G$ is
in one of the three basic classes, namely, graphs from the icosahedron, fuzzy long circular interval graphs and fuzzy antiprismatic graphs (see Section 2 for the definitions and the exact statement of the theorem). We will apply this theorem and more intricate structure of the three-cliqued claw-free graphs as well as the antiprismatic graphs to prove Theorem 1.1. The most difficult and lengthiest part of the proof (like most of the other applications of the structure theorem) is dealing with antiprismatic graphs.
The organisation of forthcoming sections is as follows. In Section 2, we recall from [7] the necessary definitions and the structure theorem. In Section 3, we discuss the claw-free graphs whose clique cover number is equal to the number of their vertices. In Section 4, we give a sketch of the proof of the main results (Theorems 1.1 and 1.4). In Section 5, we study the clique covering of claw-free graphs which admit a non-trivial strip-structure. Section 6 is devoted to the clique covering of the fuzzy long circular interval graphs. In Section 7, using the structure of three-cliqued claw-free graphs from [7], we investigate their clique cover number. Finally, in Sections 8 and 9 , we go through the clique covering of the antiprismatic graphs.

### 1.1 Notation and terminology

Here, we collect some notation and terminology that is used throughout the paper. For two graphs $G, H$, we write $G=H$ if $G$ is isomorphic to $H$. In all definitions, we often omit the subscript $G$ (if exists) whenever there is no ambiguity. The graph $G$ is called non-null if $V(G)$ is nonempty. The complement of $G$, denoted by $\bar{G}$, is a graph on the same vertices such that two distinct vertices are adjacent in $\bar{G}$ if and only if they are nonadjacent in $G$. For a vertex $u \in V(G)$, let $N_{G}(u)$ stand for the set of all neighbours of $u$ in $G$ and define $N_{G}[u]=N_{G}(u) \cup\{u\}$ as the closed neighbourhood of $u$ in $G$. For a graph $G$ and a subset of vertices $X \subseteq V(G)$, the induced subgraph of $G$ on $X$ is denoted by $G[X]$. For two sets $X, Y \subseteq V(G), E_{G}(X, Y)$ stands for the set of edges in $G$ whose one end is in $X$ and another is in $Y$. Also, $E_{G}(X)$ is a shorthand for $E_{G}(X, X)$. For a vertex $u \in V(G)$ and a set $X \subseteq V(G)$, define $N_{G}(u, X)=N_{G}(u) \cap X$ as the set of all neighbours of $u$ in $X$, and $N_{G}[u, X]=N_{G}(u, X) \cup\{u\}$. For two sets $X, Y \subseteq V(G)$, define $\mathscr{N}_{G}[X ; Y]=\left\{N_{G}[x, Y]: x \in X\right\}$. If $F$ is a set of unordered pairs of $V(G)$, then the graph $G \backslash F$ (resp. $G+F$ ) is obtained from $G$ by deleting (resp. adding) all edges in $F \cap E(G)$ (resp. $F \cap E(\bar{G})$ ). Also, when $F$ is singleton, we often drop the brackets. Let $\mathscr{C}$ be a collection of cliques of $G$ and $C_{1}, \ldots, C_{k} \in \mathscr{C}$. If $C=C_{1} \cup \ldots \cup C_{k}$ is a clique of $G$, then by merging $\left(C_{1}, \ldots, C_{k}\right)$ in $\mathscr{C}$ we mean removing the cliques $C_{1}, \ldots, C_{k}$ from $\mathscr{C}$ and adding the clique $C$.
Furthermore, we follow some definitions from [3,7]. A vertex $u \in V(G)$ is called simplicial if $N_{G}(u)$ is a clique of $G$. For a vertex $u \in V(G)$ and a set $X \subseteq V(G) \backslash\{u\}$, we say that $u$ is complete to $X$ if $u$ is adjacent to every vertex in $X$ and that $u$ is anticomplete to $X$ if $u$ has no neighbour in $X$. For two disjoint sets $X, Y \subseteq V(G)$, we say that $X$ is complete (resp. anticomplete) to $Y$ if every vertex in $X$ is complete (resp. anticomplete) to $Y$. Also, we say that $X, Y$ are matched, if $|X|=|Y|$ and every vertex in $X$ has a unique neighbour in $Y$ and vice versa. Also, we say that $X, Y$ are anti-matched, if $X, Y$ are matched in $\bar{G}$. For a claw-free graph $G$, the set of all vertices of $G$ which are in at least one triad of $G$ is called the core of $G$ and is denoted by $W(G)$. The set $V(G) \backslash W(G)$ is called the non-core of $G$ and is denoted by $\tilde{W}(G)$.

## 2 Global structure of claw-free graphs

In this section, we recall from $[7,8]$ the exact statement of the theorem that explicitly describes the structure of claw-free graphs. It will be later applied to prove the main results stated in Section 1. For it, we have to recall a couple of definitions (for more details, see [3-7]). Given a graph $G$, a set $F$ of unordered pairs of $V(G)$ is called a valid set for $G$, if every vertex of $G$ belongs to at most one member of $F$. For a graph $G$ and a valid set $F$, the graph $G^{\prime}$ is called a thickening of $(G, F)$, if its vertex set $V\left(G^{\prime}\right)$ is the disjoint union of some sets $\left(X_{v}\right)_{v \in V(G)}$ such that
(T1) for each $v \in V(G), X_{v} \subseteq V\left(G^{\prime}\right)$ is a clique of $G^{\prime}$,
(T2) if $u, v \in V(G)$ are adjacent in $G$ and $\{u, v\} \notin F$, then $X_{u}$ is complete to $X_{v}$ in $G^{\prime}$,
(T3) if $u, v \in V(G)$ are nonadjacent in $G$ and $\{u, v\} \notin F$, then $X_{u}$ is anticomplete to $X_{v}$ in $G^{\prime}$,
(T4) if $\{u, v\} \in F$, then $X_{u}$ is neither complete nor anticomplete to $X_{v}$ in $G^{\prime}$.
Here are three classes of claw-free graphs which are needed for the statement of the structure theorem.

- $\mathcal{T}_{1}$ : Graphs from the icosahedron. The graph of icosahedron is the unique planar five regular graph $G_{0}$ with twelve vertices. In particular, $G_{0}$ has vertices $v_{0}, v_{1}, \ldots, v_{11}$, where for $1 \leq i \leq 10$, $v_{i}$ is adjacent to $v_{i+1}, v_{i+2}$ (reading subscripts modulo 10) and $v_{0}$ is adjacent to $v_{1}, v_{3}, v_{5}, v_{7}, v_{9}$ and $v_{11}$ is adjacent to $v_{2}, v_{4}, v_{6}, v_{8}, v_{10}$ (see Figure 1). Let $G_{1}$ be obtained from $G_{0}$ by deleting $v_{11}$ and let $G_{2}$ be obtained from $G_{1}$ by deleting $v_{10}$. Furthermore, let $F^{\prime}=\left\{\left\{v_{1}, v_{4}\right\},\left\{v_{6}, v_{9}\right\}\right\}$. The class $\mathcal{T}_{1}$ is the set of all graphs $G$ where $G$ is a thickening of either $\left(G_{0}, \emptyset\right)$, or $\left(G_{1}, \emptyset\right)$, or $\left(G_{2}, F\right)$, for some $F \subseteq F^{\prime}$.
- $\mathcal{T}_{2}$ : Fuzzy long circular interval graphs. Let $\Sigma$ be a circle and let $\mathcal{I}=\left\{I_{1}, \ldots, I_{k}\right\}$ be a collection of subsets of $\Sigma$, such that each $I_{i}$ is homeomorphic to the interval $[0,1]$, no two of $I_{1}, \ldots, I_{k}$ share an endpoint, and no three of them have union $\Sigma$. Let $H$ be the graph whose vertex set is a finite subset $V \subseteq \Sigma$ and distinct vertices $u, v \in V$ are adjacent precisely if $u, v \in I_{i}$ for some $i$. The graph $H$ is called a long circular interval graph. The powers of cycles defined in Section 1 are examples of long circular interval graphs. Furthermore, let $F^{\prime}$ be the set of all pairs $\{u, v\}$ such that $u, v \in V$ are distinct endpoints of $I_{i}$ for some $i$ and there exists no $j \neq i$ for which $u, v \in I_{j}$. Also, let $F \subseteq F^{\prime}$. Then, for some such $H$ and $F$, any thickening $G$ of $(F, H)$ is called a fuzzy long circular interval graph. The class of all fuzzy long circular interval graphs is denoted by $\mathcal{T}_{2}$.
- $\mathcal{T}_{3}$ : Fuzzy antiprismatic graphs. A graph $H$ is called antiprismatic if for every triad $\tau$ and every vertex $v \in V(H) \backslash \tau, v$ has exactly two neighbours in $\tau$. Let $u, v$ be two vertices of an antiprismatic graph $H$. We say that the pair $\{u, v\}$ is changeable if $u$ is nonadjacent to $v$, and $G+u v$ is also antiprismatic. Let $H$ be an antiprismatic graph and let $F$ be a valid set of changeable pairs of $H$. Then, for some such $H$ and $F$, every thickening of pair $(H, F)$ is called a fuzzy antiprismatic graph. The class of all fuzzy antiprismatic graphs is denoted by $\mathcal{T}_{3}$.

Now, we recall the structure theorem of claw-free graphs from [8]. See Section 5 for the definition of a non-trivial strip-structure.

Theorem 2.1. [7, 8] Let $G$ be a connected claw-free graph. Then either

- G admits a non-trivial strip-structure, or
- $G$ is in $\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \mathcal{T}_{3}$, or
- $V(G)$ is the union of three cliques of $G$.

Finally, we conclude this section with two useful lemmas showing how one can extend a clique covering of a graph to a clique covering of its thickening.

Lemma 2.2. Let $H$ be a graph and $F$ be a valid set for $H$. Also, let $G$ be a thickening of $(H, F)$. Assume that no isolated vertex of $H \backslash F$ belongs to a member of $F$ and $H \backslash F$ admits a clique covering of size at most $|V(H)|-t$, for some number $t$. Then $G$ admits a clique covering of size at most $|V(G)|-t$.

Proof. Let $\left(X_{v}\right)_{v \in V(H)}$ be as in the definition of thickening. Suppose that $\mathscr{C}$ is a clique covering for $H \backslash F$ of size at most $|V(H)|-t$. For each $C \in \mathscr{C}$, define $X_{C}=\cup_{u \in C} X_{u}$ which is a clique of $G$. Note that if $F=\emptyset$, then $\left\{X_{C}: C \in \mathscr{C}\right\}$ is a clique covering for $G$ of size at most $|V(G)|-t$. Now, let $F=\left\{\left\{u_{1}, v_{1}\right\}, \ldots,\left\{u_{\ell}, v_{\ell}\right\}\right\}$ and w.l.o.g. assume that for every $i \in\{1, \ldots, \ell\},\left|X_{u_{i}}\right| \leq\left|X_{v_{i}}\right|$. Note
that by (T4), $\left|X_{v_{i}}\right| \geq 2$. Also, let $I$ be the set of all isolated vertices of $H \backslash F$. Now, the collection of cliques

$$
\left(\bigcup_{i=1}^{l} \mathscr{N}_{H}\left[X_{u_{i}} ; X_{v_{i}}\right]\right) \cup\left\{X_{v}: v \in I,\left|X_{v}\right| \geq 2\right\} \cup\left\{X_{C}: C \in \mathscr{C}\right\},
$$

is a clique covering for $G$ of size at most

$$
\begin{aligned}
\sum_{i=1}^{\ell}\left|X_{u_{i}}\right|+\sum_{v \in I}\left(\left|X_{v}\right|-1\right)+|V(H)|- & t \leq \sum_{i=1}^{\ell}\left(\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|\right)-2 \ell+\sum_{v \in I}\left|X_{v}\right|-|I|+|V(H)|-t \\
& \leq|V(G)|-t .
\end{aligned}
$$

The following lemma is the counterpart of Lemma 2.2 for the thickening of an antiprismatic graph.
Lemma 2.3. Let $H$ be an antiprismatic graph and $F$ be a valid set of changeable pairs of $H$. Also, let $G$ be a thickening of $(H, F)$ containing a triad. Assume that $H$ admits a clique covering of size at $\operatorname{most}|V(H)|-t$, for some number $t \leq 1$. Then, $G$ admits a clique covering of size at most $|V(G)|-t$.

Proof. First, note that the members of $F$ are non-edges of $H$. Thus, $H \backslash F=H$ and since $G$ contains a triad, $H$ contains a triad, too. If no isolated vertex of $H$ belongs to a member of $F$, then by Lemma 2.2, we are done. Now, assume that $F$ contains a pair $\left\{u_{1}, v_{1}\right\}$ such that $u_{1}$ is an isolated vertex of $H$. Since $\left\{u_{1}, v_{1}\right\}$ is a changeable pair of $H, V(H) \backslash\left\{u_{1}, v_{1}\right\}$ is a clique. Also, since $H$ is antiprismatic and contains a triad, $v_{1}$ has exactly one non-neighbour, say $v_{1}^{\prime}$, in $V(H) \backslash\left\{u_{1}, v_{1}\right\}$. Thus, since $F$ is valid, we have $F=\left\{\left\{u_{1}, v_{1}\right\}\right\}$. Assume that $\{a, b\}=\left\{u_{1}, v_{1}\right\}$ such that $\left|X_{a}\right| \leq\left|X_{b}\right|$ and so $\left|X_{b}\right| \geq 2$. Then, consider the collection $\mathscr{N}\left[X_{a} ; X_{b}\right] \cup\left\{X_{u_{1}}, \cup_{v \in V(H) \backslash\left\{u_{1}, v_{1}^{\prime}\right\}} X_{v}\right\}$ of cliques of $G$ and if $\left|\cup_{v \in V(H) \backslash\left\{u_{1}, v_{1}\right\}} X_{v}\right| \geq 2$, then add the clique $\cup_{v \in V(H) \backslash\left\{u_{1}, v_{1}\right\}} X_{v}$ to obtain a clique covering for $G$ of size at most $\left|X_{a}\right|+2+\left|\cup_{v \in V(H) \backslash\left\{u_{1}, v_{1}\right\}} X_{v}\right|-1 \leq|V(G)|-\left|X_{b}\right|+1 \leq|V(G)|-t$.

## 3 Equality cases

In this section, we investigate the claw-free graphs whose clique cover number is equal to the number of their vertices. Theorem 1.1 asserts that these graphs are merely the graph of icosahedron, the complement of a twister, and some powers of cycles. First, we need the following simple lemma which gives a lower bound for the clique cover number of a graph.

Lemma 3.1. Let $G$ be a graph and $\left(\chi_{v}: v \in V(G)\right)$ be some positive integers such that for every vertex $v \in V(G)$, the chromatic number of $\bar{G}\left[N_{G}(v)\right]$ is at least $\chi_{v}$. Then, $\operatorname{cc}(G) \geq\left\lceil\left(\sum_{v \in V(G)} \chi_{v}\right) / \omega(G)\right\rceil$, where $\omega(G)$ denotes the clique number of $G$.

Proof. Let $\mathscr{C}$ be a clique covering for $G$ of $\operatorname{size} \operatorname{cc}(G)$. For every vertex $v \in V(G)$, let $r_{v}$ be the number of cliques in $\mathscr{C}$ containing $v$. Then, $N_{G}(v)$ can be partitioned into at most $r_{v}$ cliques of $G$ and so $r_{v} \geq \chi_{v}$, for all $v \in V(G)$. Hence,

$$
\sum_{v \in V(G)} \chi_{v} \leq \sum_{v \in V(G)} r_{v}=\sum_{C \in \mathscr{C}}|C| \leq|\mathscr{C}| \omega(G),
$$

as desired.
Now, we prove the assertion of Theorem 1.1 for the graphs in $\mathcal{T}_{1}$, in the following lemma.
Lemma 3.2. Let $G$ be a graph in $\mathcal{T}_{1}$ on $n$ vertices. Then, $\operatorname{cc}(G) \leq n$ and equality holds if and only if $G$ is isomorphic to the graph of icosahedron.

Proof. Let $G_{0}, G_{1}, G_{2}$ be as in the definition of $\mathcal{T}_{1}$ in Section 2, where $\left|V\left(G_{0}\right)\right|=12,\left|V\left(G_{1}\right)\right|=11$ and $\left|V\left(G_{2}\right)\right|=10\left(G_{0}\right.$ is depicted in Figure 1 and $G_{1}$ and $G_{2}$ are obtained from $G_{0}$ by deleting $v_{11}$ and $v_{10}, v_{11}$, respectively). First, note that the family of cliques

$$
\begin{aligned}
\mathscr{C}_{0}= & \left\{\left\{v_{0}, v_{1}, v_{9}\right\},\left\{v_{0}, v_{1}, v_{3}\right\},\left\{v_{2}, v_{3}, v_{4}\right\},\left\{v_{3}, v_{4}, v_{5}\right\},\left\{v_{0}, v_{5}, v_{7}\right\},\left\{v_{1}, v_{2}, v_{10}\right\},\right. \\
& \left.\left\{v_{8}, v_{9}, v_{10}\right\},\left\{v_{7}, v_{8}, v_{9}\right\},\left\{v_{4}, v_{6}, v_{11}\right\},\left\{v_{2}, v_{10}, v_{11}\right\},\left\{v_{6}, v_{8}, v_{11}\right\},\left\{v_{5}, v_{6}, v_{7}\right\}\right\},
\end{aligned}
$$

is a clique covering for $G_{0}$ and thus, $\operatorname{cc}\left(G_{0}\right) \leq 12$. On the other hand, note that $\omega\left(G_{0}\right)=3$ and for every vertex $v \in V\left(G_{0}\right), \overline{G_{0}}\left[N_{G_{0}}(v)\right]$ is a cycle on five vertices whose chromatic number is equal to three. Thus, by Lemma 3.1, $\operatorname{cc}\left(G_{0}\right) \geq(12 \times 3) / 3=12$ and so, $\operatorname{cc}\left(G_{0}\right)=\left|V\left(G_{0}\right)\right|=12$. Also, if $G \neq G_{0}$ is a thickening of $\left(G_{0}, \emptyset\right)$, then by replacing each clique $C \in \mathscr{C}_{0}$ with the clique $\cup_{v \in C} X_{v}$, one may obtain a clique covering for $G$ of size $12 \leq n-1$.
In order to obtain a clique covering $\mathscr{C}_{1}$ for $G_{1}$, in $\mathscr{C}_{0}$, replace the cliques $\left\{v_{4}, v_{6}, v_{11}\right\},\left\{v_{2}, v_{10}, v_{11}\right\}$, $\left\{v_{6}, v_{8}, v_{11}\right\}$ and $\left\{v_{5}, v_{6}, v_{7}\right\}$ with the cliques $\left\{v_{4}, v_{5}, v_{6}\right\}$ and $\left\{v_{6}, v_{7}, v_{8}\right\}$. Also, to obtain a clique covering for $G_{2}$, in $\mathscr{C}_{1}$, replace the cliques $\left\{v_{1}, v_{2}, v_{10}\right\}$ and $\left\{v_{8}, v_{9}, v_{10}\right\}$ with the clique $\left\{v_{1}, v_{2}\right\}$. Thus, $\operatorname{cc}\left(G_{1}\right) \leq 10$ and $\operatorname{cc}\left(G_{2}\right) \leq 9$. Let $F^{\prime}=\left\{\left\{v_{1}, v_{4}\right\},\left\{v_{6}, v_{9}\right\}\right\}$ and $F \subseteq F^{\prime}$. Since $v_{1} v_{4}$ and $v_{6} v_{9}$ are nonedges of $G_{2}$, by Lemma 2.2, if $G$ is a thickening of either ( $G_{1}, \emptyset$ ) or $\left(G_{2}, F\right)$, then $\operatorname{cc}(G) \leq n-1$.

The following lemma proves the assertion of Theorem 1.1 for the powers of cycles.
Lemma 3.3. Let $p, n$ be two positive integers such that $n \geq 2 p+1$. Then $\operatorname{cc}\left(C_{n}^{p}\right) \leq n$ and equality holds if and only if $n \geq 3 p+1$.

Proof. Let $V=\left\{v_{0}, \ldots, v_{n-1}\right\}$ be the vertex set of $C_{n}^{p}$ and read all subscripts modulo $n$. Define $S=\left\{\left\{v_{k}, v_{k+p}\right\}: 0 \leq k \leq n-1\right\}$ as a subset of $E\left(C_{n}^{p}\right)$ of size $n$. We claim that if $n \geq 3 p+1$, then no pair of edges in $S$ can be covered by a single clique of $G$. To see this, on the contrary and w.l.o.g. assume that the edges $\left\{v_{0}, v_{p}\right\},\left\{v_{k}, v_{k+p}\right\}$ are covered by a clique, for some $k \in\{1, \ldots, n-1\}$. Since $v_{k}$ is adjacent to $v_{0}$, either $1 \leq k \leq p$ or $n-p \leq k \leq n-1$. In the former case, since $p+1 \leq k+p \leq$ $2 p \leq n-p-1, v_{0}$ is not adjacent to $v_{k+p}$, and in the latter case, since $2 p+1 \leq n-p \leq k \leq n-1, v_{k}$ is not adjacent to $v_{p}$, a contradiction. This proves the claim, and thus for $n \geq 3 p+1, \operatorname{cc}(G) \geq n$. Also the family of cliques $\left\{\left\{v_{i}, v_{i+1}, \ldots, v_{i+p}\right\}: 0 \leq i \leq n-1\right\}$ is a clique covering for $G$ of size $n$. Thus, when $n \geq 3 p+1$, we have $\operatorname{cc}(G)=n$. Also, if $n=2 p+1$, then $G$ is isomorphic to the complete graph $K_{n}$ and thus, $\operatorname{cc}(G)=1 \leq n-2$. Now, assume that $2 p+2 \leq n \leq 3 p$. We provide a clique covering for $G$ of size $n-1$. Note that the cliques $C_{i}=\left\{v_{2 i}, v_{2 i+1}, \ldots, v_{2 i+p}\right\}, i \in\{0,1, \ldots,\lceil n / 2\rceil-1\}$, cover all edges of $G$ except the edges in $S^{\prime}=\left\{\left\{v_{2 j+1}, v_{2 j+1+p}\right\}: 0 \leq j \leq\lfloor n / 2\rfloor-1\right\}$. It can be easily seen that $K=\left\{v_{1}, v_{p+1}, v_{2 p+1}, v_{3 p+1}\right\}$ is a clique of $G$ covering two distinct edges $\left\{v_{1}, v_{p+1}\right\}$ and $\left\{v_{2 p+1}, v_{3 p+1}\right\}$ in $S^{\prime}$. Hence, $\left\{C_{i}: 0 \leq i \leq\lceil n / 2\rceil-1\right\} \cup\{K\} \cup S^{\prime} \backslash\left\{\left\{v_{1}, v_{p+1}\right\},\left\{v_{2 p+1}, v_{3 p+1}\right\}\right\}$ is a clique covering for $G$ of size $\lceil n / 2\rceil+\left|S^{\prime}\right|-1=n-1$. This proves Lemma 3.3.

Finally, the following lemma provides the appropriate clique covering for the complement of a twister (an antiprismatic graph depicted in Figure 1) and its thickening.

Lemma 3.4. If $G$ is the complement of a twister, then $\operatorname{cc}(G)=|V(G)|=10$. Also, if $G^{\prime}$ is a thickening of $(G, F)$, for a valid set $F$ of changeable pairs of $G$, and $G^{\prime} \neq G$, then $\operatorname{cc}\left(G^{\prime}\right) \leq\left|V\left(G^{\prime}\right)\right|-1$.

Proof. Let $V(G)=\left\{u_{1}, u_{2}, v_{1}, \ldots, v_{8}\right\}$ as in Figure 1. First, note that $\omega(G)=3$. For every vertex $u \in\left\{u_{1}, u_{2}\right\}, \bar{G}\left[N_{G}(u)\right]$ is a matching on four vertices and thus has the chromatic number equal to two. Also, for every vertex $v \in\left\{v_{1}, \ldots, v_{8}\right\}, \bar{G}\left[N_{G}(v)\right]$ is a cycle on five vertices and so has the chromatic number equal to three. Hence, by Lemma 3.1, $\operatorname{cc}(G) \geq\lceil(8 \times 3+2 \times 2) / 3\rceil=10$. Moreover, the following collection is a clique covering for $G$ and $\operatorname{so}, \operatorname{cc}(G)=10$.

$$
\begin{aligned}
\mathscr{C}=\left\{C_{1}\right. & =\left\{u_{2}, v_{1}, v_{7}\right\}, C_{2}=\left\{u_{2}, v_{3}, v_{5}\right\}, C_{3}=\left\{u_{1}, v_{4}, v_{6}\right\}, C_{4}=\left\{u_{1}, v_{2}, v_{8}\right\}, C_{5}=\left\{v_{1}, v_{3}, v_{6}\right\}, \\
& \left.C_{6}=\left\{v_{2}, v_{5}, v_{7}\right\}, C_{7}=\left\{v_{3}, v_{6}, v_{8}\right\}, C_{8}=\left\{v_{3}, v_{5}, v_{8}\right\}, C_{9}=\left\{v_{1}, v_{4}, v_{7}\right\}, C_{10}=\left\{v_{2}, v_{4}, v_{7}\right\}\right\} .
\end{aligned}
$$

Now, let $G^{\prime} \neq G$ be a thickening of $(G, F)$. If $F=\emptyset$, then it is clear that $\operatorname{cc}\left(G^{\prime}\right) \leq 10 \leq\left|V\left(G^{\prime}\right)\right|-1$. Thus, assume that $F \neq \emptyset$. It is easy to check that the only changeable pairs of $G$ are $v_{1} v_{5}, v_{2} v_{6}, v_{3} v_{7}$
and $v_{4} v_{8}$. Let $\left(X_{v}\right)_{v \in V(G)}$ be as in the definition of thickening. By symmetry, assume that $\left\{v_{1}, v_{5}\right\} \in F$ and $\left|X_{v_{1}}\right| \geq 2$. Then, the collection of cliques,

$$
\begin{aligned}
& \left\{\cup_{v \in C_{i}} X_{v}: 5 \leq i \leq 10\right\} \cup \mathscr{N}\left[X_{v_{5}} ; X_{u_{2}} \cup X_{v_{1}} \cup X_{v_{3}}\right] \cup \mathscr{N}\left[X_{v_{6}} ; X_{u_{1}} \cup X_{v_{2}} \cup X_{v_{4}}\right] \cup \\
& \mathscr{N}\left[X_{v_{7}} ; X_{u_{2}} \cup X_{v_{1}} \cup X_{v_{3}}\right] \cup \mathscr{N}\left[X_{v_{8}} ; X_{u_{1}} \cup X_{v_{2}} \cup X_{v_{4}}\right]
\end{aligned}
$$

is a clique covering for $G^{\prime}$ of size $\left|X_{v_{5}}\right|+\left|X_{v_{6}}\right|+\left|X_{v_{7}}\right|+\left|X_{v_{8}}\right|+6 \leq\left|V\left(G^{\prime}\right)\right|-1$, where the last inequality holds since $\left|X_{v_{1}}\right| \geq 2$.

## 4 Outline of the proofs

In this section, we give an outline of our approach towards the proofs of the main theorems of this paper, i.e. Theorems 1.1 and 1.4. The main tool is Theorem 2.1 which asserts that for every connected claw-free graph $G$ either $G$ admits a non-trivial strip-structure, or $V(G)$ is the union of three cliques, or $G$ is in $\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \mathcal{T}_{3}$. We prove our results for each of these classes separately. First, in the following theorem, we deal with those claw-free graphs which admit a non-trivial strip-structure. The proof of this theorem is given in Section 5.

Theorem 4.1. Let $G$ be a connected claw-free graph on $n$ vertices such that $V(G)$ is not the union of three cliques and $G$ is not in $\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \mathcal{T}_{3}$. Then, $\operatorname{cc}(G) \leq n-1$.

Then, in Section 6, we prove the following theorem regarding the clique covering of the graphs in $\mathcal{T}_{2}$.
Theorem 4.2. Let $H$ be a long circular interval graph and $G$ be a connected graph on $n$ vertices which is a thickening of $(H, F)$, for some $F$ as in the definition of $\mathcal{T}_{2}$. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $G=H$ and both $G$ and $H$ are isomorphic to the $p^{t h}$ power of the cycle $C_{n}$, for some positive integer $p \leq\lfloor(n-1) / 3\rfloor$.

The following two theorems, proved in Section 7, consider the clique covering of those claw-free graphs whose vertex set is the union of three cliques.

Theorem 4.3. Let $G$ be a claw-free graph on $n$ vertices such that $V(G)$ is the union of three cliques of $G$. Then, $\operatorname{cc}(G) \leq n+1$.

Theorem 4.4. Let $G$ be a claw-free graph on $n$ vertices which contains at least one triad and $V(G)$ is the union of three cliques of $G$. Then $\operatorname{cc}(G) \leq n$, and equality holds if and only if $n=3 p+3$ for some positive integer $p$ and $G$ is isomorphic to the $p^{\text {th }}$ power of the cycle $C_{n}$.

Dealing with the case of antiprismatic graphs is the most difficult part of the proof. In [3, 4], the study of antiprismatic graphs is divided into two different parts (depending on whether the triads of the graph admit a certain kind of orientation or not) called orientable and non-orientable antiprismatic graphs (to see the definitions of these graphs, see $[3,4]$ ). In Section 8, we go through the clique covering of orientable antiprismatic graphs and prove the following.

Theorem 4.5. Let $G$ be an orientable antiprismatic graph on $n$ vertices which contains at least one triad. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $n=3 p+3$, for some positive integer $p$ and $G$ is isomorphic to the $p^{t h}$ power of the cycle $C_{n}$.

In Section 9, we look into non-orientable antiprismatic graphs and prove the following.
Theorem 4.6. Let $G$ be a non-orientable antiprismatic graph on $n$ vertices. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to a twister.

The two preceding theorems enable us to prove the following which handles the case of graphs in $\mathcal{T}_{3}$.
Theorem 4.7. Let $G$ be a connected fuzzy antiprismatic graph on $n$ vertices which contains at least one triad. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if either $n=3 p+3$, for some positive integer $p$ and $G$ is isomorphic to the $p^{\text {th }}$ power of the cycle $C_{n}$, or $\bar{G}$ is isomorphic to a twister.

Proof. Let $H$ be an antiprismatic graph and $F$ be a valid set of changeable pairs of $H$, where $G$ is a thickening of $(H, F)$. Note that since the pairs in $F$ are non-edges of $H$ and $G$ contains a triad, $H$ contains a triad as well. On the other hand, since $H$ contains a triad, by Theorems 4.5 and 4.6, either $\operatorname{cc}(H) \leq|V(H)|-1$, or $H$ is isomorphic to the complement of a twister, or the graph $C_{3 p+3}^{p}$, for some positive integer $p$. In the former case, the result follows from Lemma 2.3. If $H$ is isomorphic to the complement of a twister, then Lemma 3.4 implies the assertion. Finally, assume that $H$ is isomorphic to the graph $C_{3 p+3}^{p}$. Let $V(H)=\left\{v_{0}, v_{1}, \ldots, v_{3 p+2}\right\}$, where adjacency is as in the definition (see Section 1). It is evident that $H$ contains exactly $p+1$ disjoint triads and $F$ is a subset of $\left\{\left\{v_{i}, v_{i+p+1}\right\}: 0 \leq i \leq 3 p+2\right\}$ (reading subscripts modulo $3 p+3$ ). If $G=H$, then by Lemma 3.3, we are done. Now, assume that $G \neq H$ and so $n>3 p+3$. Let $H^{\prime}$ be the graph obtained from $H$ by adding edges between all the pairs of vertices in $F$. Thus, $H^{\prime}$ is a long circular interval graph and $F$ can be considered as a set of pairs of distinct endpoints of some intervals. Therefore, $G$ is a thickening of $\left(H^{\prime}, F\right)$ and $G \neq H^{\prime}$. Hence, by Theorem 4.2, $\operatorname{cc}(G) \leq n-1$.

With all these theorems in hand, now we can prove Theorem 1.1.
Proof of Theorem 1.1. By Theorem 2.1, either $G$ admits a non-trivial strip-structure, or $V(G)$ is the union of three cliques, or $G \in \mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \mathcal{T}_{3}$. Hence, Theorem 1.1 follows immediately from Theorems 3.2, 4.1, 4.2, 4.4 and 4.7.

For the purpose of proving Theorem 1.4, we need the following lemma from [8].
Lemma 4.8. [8] Let $G$ be a claw-free graph and $X, Y$ be disjoint subsets of $V(G)$ with $X \neq \emptyset$. Assume that for every two nonadjacent vertices in $Y$, every vertex in $X$ is adjacent to exactly one of them. Then $Y$ is the union of two cliques.

Proof of Theorem 1.4. Let $G$ be a connected graph on $n$ vertices which is tame. If $G$ contains a triad, then Theorem 1.1 implies the desired bound. Also, if $V(G)$ is the union of three cliques, then the result follows from Theorem 4.3. Thus, assume that $G$ is triad-free and $V(G)$ is not the union of three cliques. In this case, we prove that $G$ is obtained from a fuzzy antiprismatic graph containing a triad by deleting a vertex. Let $H$ be a connected claw-free graph containing a triad, where $G$ is an induced subgraph of $H$. One can obtain a sequence of induced subgraphs of $H$, say $G_{0}, G_{1}, \ldots, G_{s}$, such that $G_{0}=G, G_{s}=H$ and $G_{i}$ is obtained from $G_{i-1}$ by adding a vertex $v_{i}$ such that $v_{i}$ is not anticomplete to $V\left(G_{i-1}\right)$. Suppose that $G_{t}$ is the first graph in the sequence which contains a triad. Let $X, Y$ be the sets of neighbours and non-neighbours of $v_{t}$ in $V\left(G_{t-1}\right)$, respectively.
(1) There exists two nonadjacent vertices in $Y$ and for every two nonadjacent vertices $y, y^{\prime} \in Y$ and every vertex $x \in X, x$ is adjacent to exactly one of $y$ and $y^{\prime}$. Moreover, $X$ is the disjoint union of two cliques $N_{H}(y, X)$ and $N_{H}\left(y^{\prime}, X\right)$.
Since $G_{t-1}$ is triad-free and $G_{t}$ contains a triad, there exist two nonadjacent vertices in $Y$. Now, if $x$ is adjacent to both $y$ and $y^{\prime}$, then $\left\{x, v_{t}, y, y^{\prime}\right\}$ is a claw in $H$, a contradiction. Also, if $x$ is nonadjacent to both $y$ and $y^{\prime}$, then $\left\{x, y, y^{\prime}\right\}$ is a triad in $G_{t-1}$, a contradiction. Therefore, $x$ is adjacent to exactly one of $y$ and $y^{\prime}$. Hence, since $G_{t-1}$ is triad-free, it turns out that $X$ is partitioned into two disjoint cliques $N_{H}(y, X)$ and $N_{H}\left(y^{\prime}, X\right)$. This proves (1).
By (1) and Lemma 4.8, $Y$ is the union of two cliques $Y_{1}$ and $Y_{2}$. Also, let $\tilde{Y} \subseteq Y$ be the set of all vertices in $Y$ which have no non-neighbour in $Y$. Now, note that $\overline{G_{t}}$ induces a bipartite graph on $Y \backslash \tilde{Y}$ with bipartition $\left(Y_{1} \backslash \tilde{Y}, Y_{2} \backslash \tilde{Y}\right)$ and let $G_{1}^{\prime}, \ldots, G_{k}^{\prime}$ be the connected components of $\overline{G_{t}}[Y \backslash \tilde{Y}]$. Also, let $A_{i}=V\left(G_{i}^{\prime}\right) \cap Y_{1}$ and $B_{i}=V\left(G_{i}^{\prime}\right) \cap Y_{2}, 1 \leq i \leq k$, where both are nonempty.
(2) For $i=1, \ldots, k$, every vertex in $X$ is either complete to $A_{i}$ and anticomplete to $B_{i}$, or vice versa.

If $a, a^{\prime} \in A_{i}$, then there exist vertices $u_{1}, \ldots, u_{2 l+1}$ in $V\left(G_{i}^{\prime}\right)$, such that $u_{1}=a, u_{2 l+1}=a^{\prime}$ and $u_{j}$ is nonadjacent to $u_{j+1}$ in $G_{t}$, for every $j \in\{1, \ldots, 2 l\}$. Thus, by (1), $N_{H}\left(u_{1}, X\right)=N_{H}\left(u_{3}, X\right)=\cdots=$ $N_{H}\left(u_{2 l+1}, X\right)$. Therefore, every vertex in $X$ is either complete or anticomplete to $A_{i}$ and the same holds for $B_{i}$ similarly. Now, since $A_{i}$ is not complete to $B_{i}$, (2) follows from (1).

Now, let $G^{\prime}$ be the graph obtained from $G_{t}$ by deleting all vertices in $Y \backslash \tilde{Y}$ and adding the new vertices $Y^{\prime}=\left\{a_{1}, b_{1}, \ldots, a_{k}, b_{k}\right\}$ such that both $\left\{a_{1}, \ldots, a_{k}\right\}$ and $\left\{b_{1}, \ldots, b_{k}\right\}$ are cliques in $G^{\prime}, a_{i}$ is adjacent to $b_{j}$ in $G^{\prime}$ if and only if $i \neq j, Y^{\prime}$ is complete to $Y$ and anticomplete to $v_{t}$ and every vertex vertex $x \in X$ is adjacent to the vertex $a_{i} \in Y^{\prime}$ (resp. $b_{i} \in Y^{\prime}$ ) if and only if $x$ is complete to $A_{i}$ (resp. $B_{i}$ ). Adjacency of the other vertices is the same as in $G_{t}$. Let $F$ be the set of all the pairs $\left\{a_{i}, b_{i}\right\}$ where $A_{i}$ is not anticomplete to $B_{i}$ in $G_{t}$. It is clear that $G_{t}$ is a thickening of ( $\left.G^{\prime}, F\right)$. Also, the only triads of $G^{\prime}$ are $\left\{v_{t}, a_{1}, b_{1}\right\}, \ldots,\left\{v_{t}, a_{k}, b_{k}\right\}$. Every vertex in $Y^{\prime} \cup \tilde{Y} \backslash\left\{a_{i}, b_{i}\right\}$ is adjacent to both $a_{i}$ and $b_{i}$ and nonadjacent to $v_{t}$ and every vertex in $X$ is adjacent to $v_{t}$ and, by (2), exactly one of $a_{i}$ and $b_{i}$. Hence, $G^{\prime}$ is an antiprismatic graph (in fact, $G^{\prime}$ is a non-2-substantial graph, see Section 8 for the definition) and $F$ is a valid set of changeable pairs of $G^{\prime}$. Therefore, $G_{t}$ is a fuzzy antiprismatic graph. Let $H^{\prime}$ be the induced subgraph of $H$ on $V(G) \cup\left\{v_{t}\right\}$. Since, $H^{\prime}$ is an induced subgraph of $G_{t}, H^{\prime}$ is also a fuzzy antiprismatic graph on $n+1$ vertices. If $V(G) \cap Y$ is a clique, then by (1), $V(G)$ is the union of three cliques, a contradiction. Hence, $V(G) \cap Y$ contains two nonadjacent vertices and so $H^{\prime}$ contains a triad. Hence, by Theorem 4.7, $\operatorname{cc}\left(H^{\prime}\right) \leq\left|V\left(H^{\prime}\right)\right|=n+1$ and equality holds, if and only if $H^{\prime}$ is isomorphic to either the graph $C_{3 p+3}^{p}$, for some positive integer $p$, or the complement of a twister. On the other hand, removing the vertex $v_{t}$ from $H^{\prime}$ yields the triad-free graph $G$. However, both $C_{3 p+3}^{p}$ and the complement of a twister contain two disjoint triads. Consequently, $H^{\prime}$ is isomorphic to neither $C_{3 p+3}^{p}$ nor the complement of a twister and thus, $\operatorname{cc}(G) \leq \operatorname{cc}\left(H^{\prime}\right) \leq\left|V\left(H^{\prime}\right)\right|-1=n$. This proves Theorem 1.4.

Remark 4.9. The proof of Theorem 1.4 is actually explaining the structure of tame graphs which is interesting in its own right. In fact, it is proved there that if $G$ is connected and tame, then either $G$ contains a triad, or $V(G)$ is the union of three cliques, or $G$ is obtained from a fuzzy antiprismatic graph whose all triads are on a vertex $v$, by deleting the vertex $v$.

## 5 Claw-free graphs with non-trivial strip-structure

In this section, we are going to prove Theorem 4.1. First, we recall some definition from [7]. Given a graph $G$, a non-trivial strip-structure of $G$ is a pair $(H, \eta)$, where $H$ is a graph with no isolated vertex (which is allowed to have loops and parallel edges) with $|E(H)| \geq 2$ and a function $\eta$ mapping each $f \in E(H)$ to a subset $\eta(f)$ of $V(G)$, and mapping each pair $(f, h)$ where $f \in E(H)$ and $h$ is an endpoint of $f$, to a subset $\eta(f, h)$ of $\eta(f)$, satisfying the following conditions.
(S1) The sets $\eta(f)(f \in E(H))$ are nonempty and partition $V(G)$.
(S2) For each $h \in V(H)$, the union of the sets $\eta(f, h)$ for all $f \in E(H)$ incident with $h$ is a clique of $G$.
(S3) For all distinct $f_{1}, f_{2} \in E(H)$, if $v_{1} \in \eta\left(f_{1}\right)$ and $v_{2} \in \eta\left(f_{2}\right)$ are adjacent, then there exists $h \in V(H)$ incident with $f_{1}, f_{2}$, such that $v_{i} \in \eta\left(f_{i}, h\right)$, for $i=1,2$.
(S4) For every $f \in E(H), h \in V(H)$ incident with $f$ and $v \in \eta(f, h), N_{G}(v, \eta(f) \backslash \eta(f, h))$ is a clique of $G$.
(S5) For every edge $f \in E(H)$ with distinct endpoints $h_{1}$ and $h_{2}$, either $\eta\left(f, h_{1}\right) \cap \eta\left(f, h_{2}\right)=\emptyset$ or $\eta\left(f, h_{1}\right)=\eta\left(f, h_{2}\right)=\eta(f)$.

Also, by a stripe, we mean a pair $(G, Z)$ where $G$ is a claw-free graph such that $|V(G)| \geq 2$, and $Z \subseteq V(G)$ is a stable set of simplicial vertices of $G$ such that every vertex of $G$ has at most one neighbour in $Z$. Moreover, for a non-trivial strip-structure $(H, \eta)$ of a graph $G$ and each edge $f \in E(H)$, let us denote the set of all vertices of $H$ incident with $f$ by $\bar{f}$. Also, define $J_{f}$ to be a graph obtained from $G[\eta(f)]$ by adding a new vertex $z_{h}$ for every $h \in \bar{f}$, such that $z_{h}$ is complete to $\eta(f, h)$ and anticomplete to $\eta(f) \backslash \eta(f, h)$. Then, assuming $Z_{f}=\left\{z_{h}: h \in \bar{f}\right\},\left(J_{f}, Z_{f}\right)$ is called the strip of $(H, \eta)$ at $f$. The following theorem is a restatement of the structure theorem (Theorem 2.1) which is proved in [7].

Theorem 5.1. [7] Let $G$ be a connected claw-free graph. If $V(G)$ is not the union of three cliques and $G$ is not in $\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \mathcal{T}_{3}$, then $G$ admits a non-trivial strip-structure such that for each strip $(J, Z)$, we have $1 \leq|Z| \leq 2$, and either

- $|V(J)|=3$ and $|Z|=2$, or
- $(J, Z)$ is a stripe.

In order to deal with the claw-free graphs with non-trivial strip-structure, first we have to provide appropriate clique coverings for stripes (or more generally, claw-free graphs with simplicial vertices), which is done in the following theorem. For a graph $G$, by $Z(G)$ we denote the set of all simplicial vertices of $G$. Also, a clique covering for $G$ is said to be a simplicial clique covering, if it contains the clique $N[z]$, for every $z \in Z(G)$.

Theorem 5.2. Let $G$ be a connected claw-free graph on $n$ vertices such that $Z(G) \neq \emptyset$. Then, $G$ admits a simplicial clique covering of size at most $n-|Z(G)|$, unless $G$ is isomorphic to the line graph of a tree on $n+1$ vertices, and in this case, $G$ admits a simplicial clique covering of size at most $n-|Z(G)|+1$.

Proof. We begin with setting up some notation and convention. Pick a vertex $z_{0} \in Z(G)$ and due to the connectedness of $G$, let $V_{0}, \ldots, V_{d}, d \geq 1$ be the partition of $V(G)$, where $V_{i}$ is the set of the vertices of $G$ at distance $i$ from $z_{0}, 0 \leq i \leq d$ (thus, $V_{0}=\left\{z_{0}\right\}$ ). For every $i \in\{1, \ldots, d\}$ and every nonempty set $X \subseteq V_{i}$, let $N^{-}(X)$ denote the set of vertices in $V_{i-1}$ with a neighbour in $X$ (which is clearly nonempty). It is clear that if $i \geq 2$, then $N^{-}(X) \cap Z(G)=\emptyset$. Also, for every $X \subseteq V(G)$, let $\mathcal{K}(X)$ denote the family of connected components of $G[X]$. Now, note that every connected component of $G[Z(G)]$ is a clique, and let $\mathcal{K}(Z(G))=\left\{Z_{0}, \ldots, Z_{k}\right\}$, where $k \geq 0$ and $z_{0} \in Z_{0}$. Also, for every $i \in\{0, \ldots, k\}, V(G) \backslash Z(G)$ can be partitioned into the sets $N_{i}$ and $V(G) \backslash\left(Z(G) \cup N_{i}\right)$, such that $Z_{i}$ is complete to $N_{i}$ and anticomplete to $V(G) \backslash\left(Z(G) \cup N_{i}\right)$. In fact, $N\left[z_{0}\right]=Z_{0} \cup N_{0}$ and for every $i \in\{1, \ldots, k\}$, there exists a unique $j \in\{2, \ldots, d\}$ such that $Z_{i} \subseteq V_{j}$ and $N_{i} \backslash N^{-}\left(Z_{i}\right) \subseteq V_{j}$.
Finally, for every $i \in\{0, \ldots, d\}$ and every vertex $v \in V_{i}$ (assuming $V_{-1}=V_{d+1}=\emptyset$ ), let $N^{\circ}(v)$ be the set of all neighbours of $v$ in $V_{i}$ which have no common neighbour with $v$ in $V_{i-1}$, and $N^{+}(v)$ be the set of all neighbours of $v$ in $V_{i+1}$. Also, let $C(v)=N^{\circ}(v) \cup N^{+}(v)$ and $C[v]=C(v) \cup\{v\}$. We observe that,
(1) The following hold.
(i) For every vertex $v \in V(G), C[v]$ is a clique of $G$. Also, $C\left[z_{0}\right]=V_{0} \cup V_{1}$ and for every $z \in$ $Z(G) \backslash\left\{z_{0}\right\}, C[z]=\{z\}$.
(ii) Let $i \in\{1, \ldots, d\}$ and $K \in \mathcal{K}\left(V_{i}\right)$. Then for every vertex $u \in N^{-}(K), C[u] \subseteq N^{-}(K) \cup K$.

The second and third assertion of (i) is obvious. To see the first assertion of (i), note that for every $v \in V(G) \backslash\left\{z_{0}\right\}$, if $C(v)$ contains two nonadjacent vertices $v_{1}$ and $v_{2}$, then $\left\{v, v^{-}, v_{1}, v_{2}\right\}$ is a claw for a vertex $v^{-} \in N^{-}(\{v\})$. This proves (i). Also, to see (ii), note that for every vertex $u \in N^{-}(K)$, since $C[u]$ is a clique, we have $N^{\circ}(u) \subseteq N^{-}(K)$ and $N^{+}(u) \subseteq K$, and thus, $C[u] \subseteq N^{-}(K) \cup K$. This proves (1).

We continue with a couple of definitions. Let us say that $G$ is irreducible, if
(R1) for every vertex $v \in V(G) \backslash Z(G), N^{+}(v) \neq \emptyset$, and
(R2) for every nonempty set $U \subseteq V(G) \backslash Z(G)$, if $\cup_{u \in U} C[u]$ is a clique of $G$, then $|U|=1$.
Otherwise, we say that $G$ is reducible. For every $i \in\{1, \ldots, d\}$, let us say that $V_{i}$ is a nest-hotel, if each member of $\mathcal{K}\left(V_{i}\right)$ is a clique, and for every clique $K \in \mathcal{K}\left(V_{i}\right)$, there exists a vertex $x_{K} \in V_{i-1}$ such that $N^{+}\left(x_{K}\right)=K$ and $V_{i-1} \backslash\left\{x_{K}\right\}$ is anticomplete to $K$.
(2) If $G$ is irreducible, then for every $i \in\{1, \ldots, d\}, V_{i}$ is a nest-hotel, and consequently, $G$ is isomorphic to the line graph of a tree on $n+1$ vertices.

The assertion is trivial for $i=1$. First, we prove the assertion for $i=d \geq 2$. Since $G$ is irreducible, by (R1), we have $V_{d} \subseteq Z(G)$ and thus, $\mathcal{K}\left(V_{d}\right) \subseteq \mathcal{K}(Z(G))$. Therefore, by (1)-(ii), for every clique $Z_{i} \in \mathcal{K}\left(V_{d}\right)$ and every $u \in N^{-}\left(Z_{i}\right), C[u] \subseteq N^{-}\left(Z_{i}\right) \cup Z_{i}$. Also, $N^{-}\left(Z_{i}\right) \cup Z_{i}$ is a clique and $N^{-}\left(Z_{i}\right) \subseteq$ $V(G) \backslash Z(G)$. Hence, since $G$ is irreducible, by (R2), $\left|N^{-}\left(Z_{i}\right)\right|=1$. Also, by (1)-(i), for every two distinct cliques $Z_{i}, Z_{j} \in \mathcal{K}\left(V_{d}\right)$, we have $N^{-}\left(Z_{i}\right) \cap N^{-}\left(Z_{j}\right)=\emptyset$. Hence, defining $x_{Z_{i}}$ as the single member of $N^{-}\left(Z_{i}\right)$, for every $Z_{i} \in \mathcal{K}\left(V_{d}\right)$, it turns out that $V_{d}$ is a nest-hotel.
Now, contrary to (2), let $i_{0} \geq 2$ be the maximal $i$ such that $V_{i}$ is not a nest-hotel. By the above argument, we have $2 \leq i_{0} \leq d-1$. First, assume that $V_{i_{0}}$ contains vertices $u, v, w$ such that $v$ is adjacent to $u, w$ and $u, w$ are nonadjacent. Then, $v \in V(G) \backslash Z(G)$ and by (R1), we have $N^{+}(v) \neq \emptyset$. Since $V_{i_{0}+1}$ is a nest-hotel, $N^{+}(u), N^{+}(v)$ and $N^{+}(w)$ are disjoint and thus, for every vertex $v^{+} \in N^{+}(v)$, $\left\{v, v^{+}, u, w\right\}$ is a claw, a contradiction. Therefore, $G\left[V_{i_{0}}\right]$ contains no induced path of length two, and thus each of its connected components is a clique. Now, by (1)-(ii), for every $K \in \mathcal{K}\left(V_{i_{0}}\right)$ and every $u \in N^{-}(K), N^{+}(u) \subseteq K$. We claim that $N^{+}(u)=K$. For if there exists $v \in K \backslash N^{+}(u)$, then for every $u^{+} \in N^{+}(u)$, we have $u^{+} \in V(G) \backslash Z(G)$, and thus by (R1), $N^{+}\left(u^{+}\right) \neq \emptyset$. Consequently, since $V_{i_{0}+1}$ is a nest-hotel, for every vertex $u^{++} \in N^{+}\left(u^{+}\right),\left\{u^{+}, u^{++}, u, v\right\}$ is a claw, a contradiction. This proves that $N^{+}(u)=K$, i.e. $N^{-}(K)$ is complete to $K$. Now, if for some $K \in \mathcal{K}\left(V_{i_{0}}\right), N^{-}(K)$ contains two nonadjacent vertices $u_{1}, u_{2}$, then since $N^{-}(K)$ is complete to $K$, we have $K \cap Z(G)=\emptyset$. Thus by (R1), for every vertex $v \in K, N^{+}(v) \neq \emptyset$, and so for every $v^{+} \in N^{+}(v),\left\{v, v^{+}, u_{1}, u_{2}\right\}$ is a claw, a contradiction. Therefore, $N^{-}(K)$ and so $N^{-}(K) \cup K$ is a clique. Now, by (1)-(ii), for every $u \in N^{-}(K)$, we have $C[u] \subseteq N^{-}(K) \cup K$, and since $N^{-}(K) \subseteq V(G) \backslash Z(G)$, by (R2), we have $\left|N^{-}(K)\right|=1$. Also, by (1)-(i), for every two distinct cliques $K, K^{\prime} \in \mathcal{K}\left(V_{i_{0}}\right)$, we have $N^{-}(K) \cap N^{-}\left(K^{\prime}\right)=\emptyset$. Now, defining $x_{K}$ as the single member of $N^{-}(K)$, it turns out that $V_{i_{0}}$ is a nest-hotel, a contradiction. This proves (2).

Now, we claim that $\{C[v]: v \in V(G)\}$ is a clique covering for $G$ of size $n$. To see this, note that for every edge $u v \in E(G)$, either $u \in V_{i}$ and $v \in V_{i-1}$, or $u, v \in V_{i}$, for some $i \in\{1, \ldots, d\}$. In the former case, $u v$ is covered by $C[v]$. In the latter case, if $u$ and $v$ have no common neighbour in $V_{i-1}$, then $u v$ is covered by both $C[u]$ and $C[v]$. Also, if $u$ and $v$ have a common neighbour in $V_{i-1}$, say $w$, then $u v$ is covered by $C[w]$. This proves the claim. Moreover, by (1)-(i), $C[z]=\{z\}$, for every $z \in Z(G) \backslash\left\{z_{0}\right\}$. Also, for every $i \in\{1, \ldots, k\}$ and every $u \in N^{-}\left(Z_{i}\right)$, since $Z_{i} \subseteq C[u]$, $C[u] \subseteq N_{i} \cup Z_{i}$. Now, let $V^{\prime}=\left(Z(G) \backslash\left\{z_{0}\right\}\right) \bigcup\left(N^{-}\left(Z_{i}\right): i \in\{1, \ldots, k\}\right)$ and $V^{\prime \prime}=V(G) \backslash V^{\prime}$. Therefore, the family of cliques $\mathscr{C}=\left\{C[v]: v \in V^{\prime \prime}\right\} \cup\left\{N_{i} \cup Z_{i}: i \in\{1, \ldots, k\}\right\}$ is a clique covering for $G$ of size $n-|Z(G)|-\sum_{i=1}^{k}\left|N^{-}\left(Z_{i}\right)\right|+k+1 \leq n-|Z(G)|+1$. Also, since $N_{G}[z]=N_{i} \cup Z_{i}$, for every $z \in Z_{i}, i \in\{1, \ldots, k\}, \mathscr{C}$ is a simplicial clique covering for $G$. In addition, for every vertex $v \in V(G)$, there exists a clique in $\mathscr{C}$ containing $C[v]$.
Now, suppose that $G$ is not isomorphic to the line graph of a tree on $n+1$ vertices. Then, by (2), $G$ is reducible. Contrary to (R1), assume that $N^{+}(v)=\emptyset$, for some vertex $v \in V(G) \backslash Z(G)$, say $v \in V_{i}$ for some $i \in\{1, \ldots, d\}$. Then evidently $v \in V^{\prime \prime}$ (i.e. $C[v] \in \mathscr{C}$ ) and $C(v)=N^{\circ}(v)$. We claim that $\mathscr{C} \backslash\{C[v]\}$ is a simplicial clique covering of size at most $n-|Z(G)|$. To see this, let $x y$ be an edge with $x, y \in C[v]$, where $y \in N^{\circ}(v)$. Now, if $x \in N^{\circ}(y)$, then $x y$ is covered by a clique in $\mathscr{C} \backslash\{C[v]\}$ containing $C[y]$. Otherwise, $x$ and $y$ has a common neighbour $w$ in $V_{i-1}$, and $x y$ is covered by a clique in $\mathscr{C} \backslash\{C[v]\}$ containing $C[w]$. This proves the claim.
Next, assume that (R1) holds and contrary to (R2), assume that there exist distinct vertices $u, v \in$ $V(G) \backslash Z(G)$ such that $K=C[u] \cup C[v]$ is a clique of $G$. If $u, v \in V^{\prime \prime}$, then $(\mathscr{C} \cup\{K\}) \backslash\{C[u], C[v]\}$ is a simplicial clique covering for $G$ of size at most $n-|Z(G)|$. Otherwise, if say $u \in N^{-}\left(Z_{i}\right)$, where $Z_{i} \subseteq V_{j}, i \in\{1, \ldots, k\}, j \in\{1, \ldots, d\}$, then $Z_{i} \subseteq C[u] \subseteq K$, and since $C[v] \subseteq K$, it turns out that $v \in N_{i}$. Now, if $v \notin N^{-}\left(Z_{i}\right)$, then since $N^{+}(v) \neq \emptyset$ (due to (R1)), we have $K \cap V_{j+1} \neq \emptyset$. This contradicts the fact that $K$ is a clique containing $u \in V_{j-1}$. Thus, $v \in N^{-}\left(Z_{i}\right)$, and so $\left|N^{-}\left(Z_{i}\right)\right| \geq 2$, which implies that $|\mathscr{C}| \leq n-|Z(G)|$. This completes the proof of Theorem 5.2.

Orlin in [18] (see also [15]) proved that for every graph $H \neq K_{3}$, the clique cover number of the line graph of $H$ is equal to the number of vertices of $H$ of degree at least two minus the number of its wings (triangles with two vertices of degree two). If $G$ is the line graph of a tree $T$ on $n+1$ vertices,
then $Z(G)$ is equal to the set of pendant edges of $T$ and thus, $\operatorname{cc}(G)=n+1-|Z(G)|$. This, together with Theorem 5.2, implies the following which is not needed for us, however, it seems to be of interest in its own right.

Corollary 5.3. Let $G$ be a connected claw-free graph on $n$ vertices such that $Z(G) \neq \emptyset$. Then $\operatorname{cc}(G) \leq n-|Z(G)|+1$ and equality holds if and only if $G$ is isomorphic to the line graph of a tree on $n+1$ vertices.

Now, we are ready to prove Theorem 4.1 which we restate here.
Theorem 5.4. Let $G$ be a connected claw-free graph on $n$ vertices, such that $V(G)$ is not the union of three cliques and $G$ is not in $\mathcal{T}_{1} \cup \mathcal{T}_{2} \cup \mathcal{T}_{3}$. Then, $\operatorname{cc}(G) \leq n-1$.

Proof. By Theorem 5.1, $G$ admits a non-trivial strip-structure such that for each strip $(J, Z)$, we have $1 \leq|Z| \leq 2$, and either $|V(J)|=3$ and $|Z|=2$, or $(J, Z)$ is a stripe. Let $(H, \eta)$ be such a strip-structure for $G$ with $|V(H)|=\nu$ and $|E(H)|=\epsilon \geq 2$. Also, let $\lambda \geq 0$ be the number of loops in $E(H)$. For every $h \in V(H)$, define $\eta(h)$ to be the union of the sets $\eta(f, h)$ for all $f \in E(H)$ incident with $h$ (i.e. $h \in \bar{f}$ ). By (S2), $\eta(h)$ is a clique of $G$. Also, for every $f \in E(H)$, let $\left(J_{f}, Z_{f}\right)$ be the strip of $(H, \eta)$ at $f$, where $1 \leq\left|Z_{f}\right| \leq 2$ and define $n_{f}=|\eta(f)|=\left|V\left(J_{f}\right)\right|-\left|Z_{f}\right|$.
Note that since $H$ has no isolated vertex, the connectedness of $G$ along with (S3) implies that $H$ is also connected and for every $f \in E(H)$, the set $Z_{f}$ intersects each connected component of $J_{f}$. In particular, if $\left|Z_{f}\right|=1$, then $J_{f}$ is connected and if $\left|Z_{f}\right|=2$, then either $J_{f}$ is connected or $J_{f}$ has exactly two connected components, each of which containing exactly one member of $Z_{f}$. With this observation, the following statement is directly deduced from Theorem 5.2.
(1) For every $f \in E(H)$, $J_{f}$ admits a simplicial clique covering $\mathscr{C}_{f}$ such that $\left|\mathscr{C}_{f}\right|=n_{f}+1$, when $J_{f}$ is a path on $n_{f}+2$ vertices and $\left|Z_{f}\right|=2$, and $\left|\mathscr{C}_{f}\right| \leq n_{f}$, otherwise. In particular, if $f$ is a loop, then $\left|\mathscr{C}_{f}\right| \leq n_{f}$.
Henceforth, for each $f \in E(H)$, let $\mathscr{C}_{f}$ be as in (1). Now, remove all vertices in $Z_{f}$ from the cliques in $\mathscr{C}_{f}$ to obtain a clique covering $\mathscr{C}_{f}^{\prime}$ for $G[\eta(f)]$. Thus, since $\mathscr{C}_{f}$ is a simplicial clique covering, we have $\eta(f, h) \in \mathscr{C}_{f}^{\prime}$, for every $h \in \bar{f}$. Now, consider the collection $\bigcup_{f \in E(H)} \mathscr{C}_{f}^{\prime}$ of cliques of $G$, remove the clique $\eta(f, h)$ for every $f \in E(H)$ and every $h \in \bar{f}$ and add the clique $\eta(h)$ for every $h \in V(H)$. Note that the resulting family, called $\mathscr{C}$, is a clique covering for $G$, because by (S3), for every distinct edges $f_{1}, f_{2} \in E(H)$, the edges in $E_{G}\left(\eta\left(f_{1}\right), \eta\left(f_{2}\right)\right)$ are covered by the cliques $\eta(h), h \in \bar{f}_{1} \cap \bar{f}_{2}$. Hence,
(2) We have,

$$
\operatorname{cc}(G) \leq|\mathscr{C}| \leq \nu+\sum_{f \in E(H)}\left(\left|\mathscr{C}_{f}\right|-|\bar{f}|\right)=\nu+\lambda-2 \epsilon+\sum_{f \in E(H)}\left|\mathscr{C}_{f}\right| \leq \nu-\epsilon+\sum_{f \in E(H)} n_{f},
$$

where the last inequality is by (1) (applying $\left|\mathscr{C}_{f}\right| \leq n_{f}$, if $f$ is a loop, and $\left|\mathscr{C}_{f}\right| \leq n_{f}+1$, otherwise).
If $\epsilon \geq \nu+1$, then we are done. On the other hand, since $H$ is connected, we have $\epsilon \geq \nu+\lambda-1$. Hence, it remains to discuss two possibilities that either $\epsilon=\nu-1$ and $\lambda=0$ or $\epsilon=\nu$ and $\lambda \leq 1$. First, consider the former case. In this case, $H$ is a tree and the right hand side of (2) is equal to $n+1$. Let $h$ be a leaf (a vertex of degree one in $V(H)$ ) and $f \in E(H)$ be the edge incident with $h$. If $J_{f}$ is not a path, then by (1), $\left|\mathscr{C}_{f}\right| \leq n_{f}$. Also, if $J_{f}$ is a path, then $|\eta(h)|=|\eta(f, h)|=1$, and thus, we can remove the singleton clique $\eta(h)$ from $\mathscr{C}$. Hence, for each leaf in $V(H)$, we can subtract the right hand side of (2) by one and since $H$ has at least two leaves, we are done.
Finally, consider the latter case $\epsilon=\nu$ and $\lambda \leq 1$. In this case, the right hand side of (2) is equal to $n$. If $H$ has a leaf, then by the above argument, we can subtract the right hand side of (2) by one and $\operatorname{cc}(G) \leq n-1$. Now, assume that $H$ has no leaf. If $\lambda=1$, then the graph obtained from $H$ by removing its loop is a tree and thus $H$ has a leaf. Therefore, $\lambda=0$ and $H$ is a unicyclic graph with no leaf and no loop, i.e. a cycle (we consider two parallel edges as a cycle). If for some $f \in E(H)$, either $J_{f}$ is not a path on $n_{f}+2$ vertices or $\left|Z_{f}\right|=1$, then $\left|\mathscr{C}_{f}\right| \leq n_{f}$ and by (2), cc $(G) \leq n-1$. Therefore, we can assume that for every $f \in E(H), J_{f}$ is a path and $Z_{f}$ is the set of its both endpoints. Consequently,
$G$ is obtained from $H$ by replacing each edge $f \in E(H)$ by a path on $n_{f}$ vertices. Hence, $G$ is a cycle. But then $G \in \mathcal{T}_{2}$, a contradiction. This proves Theorem 5.4.

## 6 Fuzzy long circular interval graphs

In this section, we are going to prove Theorem 4.2. Note that in a long circular interval graph, the vertices included in each interval form a clique and these cliques easily comprise a clique covering, although there are some difficulties in the case of thickening, where Lemma 2.2 cannot be applied. We restate Theorem 4.2 as follows.

Theorem 6.1. Let $H$ be a long circular interval graph and $G$ be a connected graph on $n$ vertices which is a thickening of $(H, F)$, for some $F$ as in the definition of $\mathcal{T}_{2}$. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $G=H$ and both $G$ and $H$ are isomorphic to the $p^{\text {th }}$ power of the cycle $C_{n}$, for some positive integer $p \leq\lfloor(n-1) / 3\rfloor$.
Proof. Assume that $|V(H)|=m$ and $\Sigma$ and $\mathcal{I}=\left\{I_{1}, \ldots, I_{k}\right\}$ are as in the definition. Throughout the proof, read all subscripts modulo $k$. For simplicity, we may assume that the number of intervals $k$ is minimal, in the sense that there is no long circular interval graph isomorphic to $H$ whose number of intervals is less than $k$. For every two distinct points $x, y \in \Sigma$, let $[x, y]$ denote the arc in $\Sigma$ joining $x$ to $y$ clockwise. Also, let $[x, y)$ be obtained from $[x, y]$ by removing $y$, and $(x, y]$ and $(x, y)$ are defined similarly. Considering a clockwise orientation for $\Sigma$, for every $i \in\{1, \ldots, k\}$, we denote the opening and closing points of $I_{i}$ by $o_{i}, c_{i} \in \Sigma$, respectively, such that $I_{i}=\left[o_{i}, c_{i}\right]$. Furthermore, assume that $o_{1}, o_{2}, \ldots, o_{k}$ are placed on $\Sigma$ exactly in this order (i.e. $\left(o_{i}, o_{i+1}\right) \cap\left\{o_{1}, o_{2}, \ldots, o_{k}\right\}=\emptyset$, for every $i \in\{1, \ldots, k\}$ ). First, note that no interval in $\mathcal{I}$ contains another interval in $\mathcal{I}$ (otherwise one might reduce the number of intervals $k$ by deleting the smaller interval without changing adjacency). Thus, the closing points $c_{1}, c_{2}, \ldots, c_{k}$ are placed on $\Sigma$ exactly in this order, and for each $i \in\{1, \ldots, k\}$, $c_{i} \in\left(o_{i}, c_{i+1}\right)$. We need the following.
(1) If for each $i \in\{1, \ldots, k\},\left|\left[o_{i}, o_{i+1}\right) \cap V(H)\right|=1$, then $H$ is isomorphic to the $p^{\text {th }}$ power of the cycle $C_{m}$, for some positive integer $p$.
For every $i \in\{1, \ldots, k\}$, assume that $\left|I_{i} \cap V(H)\right|=t_{i}$. We can observe that there exists a positive integer $t$ such that $t_{i}=t$ for all $i \in\{1, \ldots, k\}$, and so $H$ is isomorphic to $C_{m}^{t-1}$. For the contrary assume that there exists $i \in\{1, \ldots, k\}$ such that $t_{i}>t_{i+1}$. This implies that $I_{i+1} \cap V(H) \subseteq I_{i}$ and we can remove $I_{i+1}$ from $\mathcal{I}$ without changing adjacency, a contradiction with the minimality of $k$. This proves (1).
Note that w.l.o.g. we may assume that for every $i \in\{1, \ldots, k\}$, the set $\left[o_{i}, o_{i+1}\right) \cap V(H)$ is nonempty (otherwise one could delete the interval $I_{i}$ without changing adjacency). Hence,
(2) We have,

$$
k \leq \sum_{i=1}^{k}\left|\left[o_{i}, o_{i+1}\right) \cap V(H)\right|=m .
$$

Now, if there exists some $i \in\{1, \ldots, k\}$, for which $\left|\left[o_{i}, o_{i+1}\right) \cap V(H)\right| \geq 2$, then by (2), we have $k \leq m-1$, and thus the collection of cliques $\left\{I_{i} \cap V(H): 1 \leq i \leq k\right\}$ is a clique covering for $H$ of size at most $m-1$. Also, if for each $i \in\{1, \ldots, k\},\left|\left[o_{i}, o_{i+1}\right) \cap V(H)\right|=1$, then (1) implies that $H$ is isomorphic to $C_{m}^{p}$, for some positive integer $p$. This argument together with Lemma 3.3, implies that
(3) We have $\operatorname{cc}(H) \leq m$ and equality holds if and only if $H$ is isomorphic to $C_{m}^{p}$, for $p \leq\lfloor(m-1) / 3\rfloor$.

Now, let $G$ be a connected graph on $n$ vertices which is a thickening of $(H, F)$, for some $F \subseteq F^{\prime}$, where $F^{\prime}$ is the set of pairs $\{u, v\}$ such that $u, v \in V(H)$ are distinct endpoints of $I_{i}$ for some $i \in\{1, \ldots, k\}$. Also, let $\left(X_{v}\right)_{v \in V(H)}$ be the subsets of $V(G)$ as in the definition of thickening. If $G=H$, then by (3) we are done. Thus, suppose that $G \neq H$, which implies that $m \leq n-1$. For each interval $I_{i} \in \mathcal{I}$, let $C_{i}=\cup_{v \in I_{i} \cap V(H)} X_{v}$. If $F=\emptyset$, then trivially $\left\{C_{1}, \ldots, C_{k}\right\}$ is a clique covering for $G$ of size $k$, and thus by $(2), \operatorname{cc}(G) \leq n-1$. Now, assume that $F \neq \emptyset$, say $F=\left\{\left\{u_{1}, v_{1}\right\}, \ldots,\left\{u_{\ell}, v_{\ell}\right\}\right\}$, where for each
$i \in\{1, \ldots, \ell\}, u_{i}, v_{i} \in V(H)$ are the distinct endpoints of the interval $I_{p_{i}}, p_{i} \in\{1, \ldots, k\}$. Note that w.l.o.g. we may assume that $\left|X_{v_{1}}\right| \geq 2$. Also, since $G$ is connected, for each $i \in\{1, \ldots, \ell\}$, there exists $q_{i} \in\{1, \ldots, k\}$ such that either $u_{i}$ or $v_{i}$ belongs to the interior of $I_{q_{i}}$, and by symmetry, we assume that for each $i, u_{i}$ is in the interior of $I_{q_{i}}$. Also, for every $i \in\{1, \ldots, \ell\}$, let $Y_{i}=C_{p_{i}} \backslash X_{u_{i}}$.
In the sequel, we prove that $G$ admits a clique covering of size at most $n-1$ which completes the proof. First, suppose that there exists some $i \in\{1, \ldots, k\}$, for which $\left|\left[o_{i}, o_{i+1}\right) \cap V(H)\right| \geq$ 2. Thus, by (2), $k \leq m-1$. For every $i \in\{1, \ldots, \ell\}$, let $\mathscr{C}_{i}=\left\{Y_{i}\right\}$, when $\left|X_{v_{i}}\right| \geq 2$, and let $\mathscr{C}_{i}=\emptyset$, when $\left|X_{v_{i}}\right|=1$. Thus, $\left|\mathscr{C}_{i} \cup \mathscr{N}\left[X_{u_{i}} ; Y_{i}\right]\right| \leq\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|-1$. Now, the family of cliques $\left(\bigcup_{i=1}^{\ell}\left(\mathscr{C}_{i} \cup \mathscr{N}\left[X_{u_{i}} ; Y_{i}\right]\right)\right) \cup\left\{C_{i}: i \in\{1, \ldots, k\} \backslash\left\{j_{1}, \ldots, j_{\ell}\right\}\right\}$ is a clique covering for $G$ of size at most $\sum_{i=1}^{\ell}\left(\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|-1\right)+k-\ell \leq \sum_{i=1}^{\ell}\left(\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|\right)+m-1-2 \ell \leq n-1$, as required (note that the edges in $E\left(X_{u_{i}}\right)$ are covered since $u_{i}$ is in the interior of $\left.I_{q_{i}}\right)$.
Next, assume that $\left|\left[o_{i}, o_{i+1}\right) \cap V(H)\right|=1$ for all $i \in\{1, \ldots, k\}$ (and thus by (1), $H$ is isomorphic to the $p^{t h}$ power of the cycle $C_{m}$, for some positive integer $p$ ). Note that w.l.o.g. we may assume $u_{i}=o_{p_{i}}$ and $v_{i}=c_{p_{i}}$. For each $i \in\{1, \ldots, \ell\}$, if $p_{i}+1 \notin\left\{p_{1}, \ldots, p_{\ell}\right\}$, then the edges in $E_{G}\left(Y_{i}\right)$ are covered by the clique $C_{p_{i}+1}$. Otherwise, if $p_{i}+1=p_{j}$, for some $j \in\{1, \ldots, \ell\}$, then these edges are covered by the cliques in $\mathscr{N}\left[X_{u_{i}} ; Y_{i}\right] \cup \mathscr{N}\left[X_{u_{j}} ; Y_{j}\right]$. Hence, $\left(\bigcup_{i=1}^{\ell} \mathscr{N}\left[X_{u_{i}} ; Y_{i}\right]\right) \cup\left\{C_{i}: i \in\{1, \ldots, k\} \backslash\left\{p_{1}, \ldots, p_{\ell}\right\}\right\}$ is a clique covering for $H$ of size $\sum_{i=1}^{\ell}\left|X_{u_{i}}\right|+k-\ell \leq \sum_{i=1}^{\ell}\left(\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|\right)+m-2 \ell-1 \leq n-1$, where the first inequality is due to (2) and the fact that $\left|X_{v_{1}}\right| \geq 2$ (note that for each $i \in\{1, \ldots, \ell\}$, the edges in $E\left(X_{u_{i}}\right)$ are covered, since $u_{i}$ is included in the interior of $\left.I_{q_{i}}\right)$.

## 7 Three-cliqued claw-free graphs

The goal of this section is to prove Theorems 4.3 and 4.4. A three-cliqued graph $(G, A, B, C)$ consists of a graph $G$ and three cliques $A, B, C$ of $G$, pairwise disjoint and with the union $V(G)$. If $G$ is also claw-free, then $(G, A, B, C)$ is called a three-cliqued claw-free graph. First, as a warm-up, we prove Theorem 4.3, restated as follows. Then, using more subtle structure of these graphs from [7], we improve the upper bound $n+1$ to $n$ for the three-cliqued claw-free graphs containing a triad.

Theorem 7.1. Let $(G, A, B, C)$ be a three-cliqued claw-free graph on $n$ vertices. Then $\operatorname{cc}(G) \leq n+1$.
Proof. Let $C_{1}, C_{2}, C_{3}$ be three sets of vertices such that $\{A, B, C\}=\left\{C_{1}, C_{2}, C_{3}\right\}$ and $\left|C_{1}\right| \leq\left|C_{2}\right| \leq$ $\left|C_{3}\right|$. For each vertex $x \in V(G)$ and $i \in\{1,2,3\}$, let $N_{i}(x)=N\left(x, C_{i}\right)$ and $N_{i}[x]=N\left[x, C_{i}\right]$. Also, for every $i, j \in\{1,2,3\}$, define $\mathscr{N}_{j}^{i}=\mathscr{N}\left[C_{i} ; C_{j}\right]$. Evidently, $\mathscr{N}_{2}^{1} \cup \mathscr{N}_{3}^{1} \cup \mathscr{N}_{3}^{2} \cup\left\{C_{1}, C_{2}, C_{3}\right\}$ is a clique covering for $G$ of size $2\left|C_{1}\right|+\left|C_{2}\right|+3=n+\left|C_{1}\right|-\left|C_{3}\right|+3$. Thus, whenever $\left|C_{3}\right|-\left|C_{1}\right| \geq 2$, we are done. Now, assume that $\left|C_{3}\right| \leq\left|C_{1}\right|+1$. We consider two cases.
First, assume that there exists some $i \in\{1,2,3\}$, for which some edges in $E\left(C_{i}\right)$ are not covered by the cliques in $\cup_{j \in\{1,2,3\} \backslash\{i\}} \mathscr{N}_{i}^{j}$. Let $i_{0}$ be the smallest such $i$ and also let $\left\{i_{0}, j_{0}, k_{0}\right\}=\{1,2,3\}$, where $j_{0}<k_{0}$. Therefore, there exist two vertices $x, y \in C_{i_{0}}$ such that for every $j \in\{1,2,3\} \backslash\left\{i_{0}\right\}, N_{j}(x)$ and $N_{j}(y)$ are disjoint. Since $G$ is claw-free, $N_{j_{0}}(x)$ is complete to $N_{k_{0}}(x)$ and $N_{j_{0}}(y)$ is complete to $N_{k_{0}}(y)$. Now, in the clique covering $\mathscr{N}_{j_{0}}^{i_{0}} \cup \mathscr{N}_{k_{0}}^{i_{0}} \cup \mathscr{N}_{k_{0}}^{j_{0}} \cup\left\{C_{1}, C_{2}, C_{3}\right\}$, merge the pairs $\left(N_{j_{0}}[x], N_{k_{0}}[x]\right)$ and $\left(N_{j_{0}}[y], N_{k_{0}}[y]\right)$ to obtain the clique covering $\mathscr{C}$. If $i_{0} \leq 2$, then $|\mathscr{C}| \leq 2\left|C_{i_{0}}\right|+\left|C_{j_{0}}\right|-2+3 \leq$ $\left|C_{i_{0}}\right|+\left|C_{j_{0}}\right|+\left|C_{k_{0}}\right|+1=n+1$. Also, if $i_{0}=3$, then $j_{0}=1, k_{0}=2$ and, by minimality of $i_{0}$, the cliques in $\mathscr{N}_{2}^{1} \cup \mathscr{N}_{2}^{3}$ cover all edges in $E\left(C_{2}\right)$. Thus, removing the clique $C_{2}$ from $\mathscr{C}$ yields a clique covering for $G$ of size $2\left|C_{3}\right|+\left|C_{1}\right|-2+2 \leq\left|C_{3}\right|+\left|C_{1}\right|+1+\left|C_{2}\right|=n+1$.
Now, assume that for each $i \in\{1,2,3\}$, all edges in $E\left(C_{i}\right)$ are covered by the cliques in $\cup_{j \in\{1,2,3\} \backslash\{i\}} \mathscr{N}_{i}^{j}$. If all edges in $E\left(C_{2}\right)$ are covered by the cliques in $\mathscr{N}_{2}^{3}$ and all edges in $E\left(C_{1}\right)$ are covered by the cliques in $\mathscr{N}_{1}^{2}$, then the family $\mathscr{N}_{3}^{1} \cup \mathscr{N}_{1}^{2} \cup \mathscr{N}_{2}^{3} \cup\left\{C_{3}\right\}$ is a clique covering of size $n+1$. Hence, suppose that either some edges in $E\left(C_{2}\right)$ are not covered by the cliques in $\mathscr{N}_{2}^{3}$, or some edges in $E\left(C_{1}\right)$ are not covered by the cliques in $\mathscr{N}_{1}^{2}$, and w.l.o.g. assume that the former occurs. Thus, there exist two vertices $x, y \in C_{2}$ for which $N_{3}(x) \cap N_{3}(y)=\emptyset$. Since $G$ is claw-free, $N_{3}(x)$ is complete to $N_{1}(x) \backslash N_{1}(y)$ and $N_{3}(y)$ is complete to $N_{1}(y) \backslash N_{1}(x)$. Now, in the clique covering $\mathscr{N}_{3}^{1} \cup \mathscr{N}_{1}^{2} \cup \mathscr{N}_{3}^{2} \cup\left\{C_{1}, C_{2}\right\}$, replace the cliques $N_{1}[x], N_{3}[x], N_{1}[y]$ and $N_{3}[y]$ with the cliques $\left(N_{1}(x) \cap N_{1}(y)\right) \cup\{x, y\},\left(N_{1}[x] \cup N_{3}[x]\right) \backslash N_{1}(y)$
and $\left(N_{1}[y] \cup N_{3}[y]\right) \backslash N_{1}(x)$, thereby obtaining a clique covering for $G$ of size $\left|C_{1}\right|+2\left|C_{2}\right|+2-1 \leq$ $\left|C_{1}\right|+\left|C_{2}\right|+\left|C_{3}\right|+1=n+1$. This proves Theorem 7.1.

In order to improve the bound in Theorem 7.1 for three-cliqued claw-free graphs containing a triad, we need to know their more detailed structure. It is proved in [7] that every three-cliqued claw-free graph can be obtained from some special graphs using a certain construction called "worn hex-chain". Let us recall the definition of this construction from [7].
Let $k \geq 1$, and for every $1 \leq i \leq k$, let $\left(G_{i}, A_{i}, B_{i}, C_{i}\right)$ be a three-cliqued graph, where $V\left(G_{1}\right), \ldots, V\left(G_{k}\right)$ are all nonempty and pairwise disjoint. Let $A=\cup_{i=1}^{k} A_{i}, B=\cup_{i=1}^{k} B_{i}$ and $C=\cup_{i=1}^{k} C_{i}$, and let $G$ be the graph with vertex set $V(G)=\cup_{i=1}^{k} V\left(G_{i}\right)$ and with adjacency as follows,
(W1) for every $1 \leq i \leq k, G\left[V\left(G_{i}\right)\right]=G_{i}$,
(W2) for every $1 \leq i<j \leq k, A_{i}$ is complete to $V\left(G_{j}\right) \backslash B_{j} ; B_{i}$ is complete to $V\left(G_{j}\right) \backslash C_{j}$; and $C_{i}$ is complete to $V\left(G_{j}\right) \backslash A_{j}$, and
(W3) for every $1 \leq i<j \leq k$, if $u \in A_{i}$ and $v \in B_{j}$ are adjacent, then $u, v$ are both in no triads; and the same holds if $u \in B_{i}$ and $v \in C_{j}$, and if $u \in C_{i}$ and $v \in A_{j}$.

Thus, $A, B, C$ are cliques of $G$, and so $(G, A, B, C)$ is a three-cliqued graph. We call the sequence $\left(G_{i}, A_{i}, B_{i}, C_{i}\right)(i=1, \ldots, k)$ a worn hex-chain for $(G, A, B, C)$. Note that every triad of $G$ is a triad of one of $G_{1}, \ldots, G_{k}$, and if each term $G_{i}$ is claw-free, then so is $G$.
Let $(G, A, B, C)$ be a three-cliqued graph and $F$ be a valid set for $G$ such that for every pair $\{u, v\} \in F$, $u, v$ are not both in the same set $A, B$ or $C$. Then, every thickening $G^{\prime}$ of $(G, F)$ is also a threecliqued graph $\left(G^{\prime}, A^{\prime}, B^{\prime}, C^{\prime}\right)$, where $A^{\prime}=\cup_{v \in A} X_{v}$ and $B^{\prime}, C^{\prime}$ are defined similarly. If $(G, A, B, C)$ is a three-cliqued graph and $\{\hat{A}, \hat{B}, \hat{C}\}=\{A, B, C\}$, then $(G, \hat{A}, \hat{B}, \hat{C})$ is also a three-cliqued graph, called a permutation of $(G, A, B, C)$. The following theorem from [7] states that every three-cliqued claw-free graph admits a worn hex-chain whose all terms are obtained from graphs in five basic classes $\mathcal{T C}_{1}, \ldots, \mathcal{T C}_{5}($ defined in Appendix A).

Theorem 7.2. [7] Every three-cliqued claw-free graph admits a worn hex-chain into terms each of which is a thickening of a permutation of a member of one of $\mathcal{T C}_{1}, \ldots, \mathcal{T C}_{5}$ with respect to their corresponding sets $F$ (see Appendix $A$ for the definition of $\mathcal{T C}_{1}, \ldots, \mathcal{T} \mathcal{C}_{5}$ ).

We will apply Theorem 7.2 to prove Theorem 4.4. First, we need a couple of lemmas. We begin with a simple observation showing that one can change the ordering in a worn hex-chain.

Lemma 7.3. Assume that $G$ admits a worn hex-chain $\left(G_{i}, A_{i}, B_{i}, C_{i}\right)(i=1, \ldots, k)$ and let $i_{0}, j_{0} \in$ $\{1, \ldots, k\}$. Then,
(i) $G$ admits a worn hex-chain $\left(G_{i}^{\prime}, A_{i}^{\prime}, B_{i}^{\prime}, C_{i}^{\prime}\right)(i=1, \ldots, k)$ such that for every $i \in\{1, \ldots, k\}$, $\left(G_{i}^{\prime}, A_{i}^{\prime}, B_{i}^{\prime}, C_{i}^{\prime}\right)$ is a permutation of $\left(G_{i+i_{0}-j_{0}}, A_{i+i_{0}-j_{0}}, B_{i+i_{0}-j_{0}}, C_{i+i_{0}-j_{0}}\right)$ (reading subscripts modulo k). Also, $A_{j_{0}}^{\prime}=A_{i_{0}}, B_{j_{0}}^{\prime}=B_{i_{0}}$ and $C_{j_{0}}^{\prime}=C_{i_{0}}$.
(ii) if $\left(G_{i_{0}}, \hat{A}_{i_{0}}, \hat{B}_{i_{0}}, \hat{C}_{i_{0}}\right)$ is a permutation of $\left(G_{i_{0}}, A_{i_{0}}, B_{i_{0}}, C_{i_{0}}\right)$, then $G$ admits a worn hex-chain $\left(G_{i}^{\prime}, A_{i}^{\prime}, B_{i}^{\prime}, C_{i}^{\prime}\right)(i=1, \ldots, k)$ such that $G_{i_{0}}^{\prime}=G_{i_{0}}, A_{i_{0}}^{\prime}=\hat{A}_{i_{0}}, B_{i_{0}}^{\prime}=\hat{B}_{i_{0}}$ and $C_{i_{0}}^{\prime}=\hat{C}_{i_{0}}$.

Proof. (i) Let $k_{0}=i_{0}-j_{0}$ and for each $i \in\{1, \ldots, k\}$, if $1 \leq i+k_{0} \leq k$, then define $G_{i}^{\prime}=G_{i+k_{0}}$, $A_{i}^{\prime}=A_{i+k_{0}}, B_{i}^{\prime}=B_{i+k_{0}}$ and $C_{i}^{\prime}=C_{i+k_{0}}$, if $i+k_{0} \leq 0$, then define $G_{i}^{\prime}=G_{i+k_{0}+k}, A_{i}^{\prime}=C_{i+k_{0}+k}$, $B_{i}^{\prime}=A_{i+k_{0}+k}$ and $C_{i}^{\prime}=B_{i+k_{0}+k}$ and if $i+k_{0} \geq k+1$, then define $G_{i}^{\prime}=G_{i+k_{0}-k}, A_{i}^{\prime}=B_{i+k_{0}-k}$, $B_{i}^{\prime}=C_{i+k_{0}-k}$ and $C_{i}^{\prime}=A_{i+k_{0}-k}$. It is easy to check that $\left(G_{i}^{\prime}, A_{i}^{\prime}, B_{i}^{\prime}, C_{i}^{\prime}\right)(i=1, \ldots, k)$ is a worn hex-chain for $G$.
(ii) Suppose that $\hat{A}_{i_{0}}=A_{i_{0}}, \hat{B}_{i_{0}}=C_{i_{0}}$ and $\hat{C}_{i_{0}}=B_{i_{0}}$ (the other cases are similar). For every $i \in\{1, \ldots, k\}$, if $1 \leq 2 i_{0}-i \leq k$, then define $G_{i}^{\prime}=G_{2 i_{0}-i}, A_{i}^{\prime}=A_{2 i_{0}-i}, B_{i}^{\prime}=C_{2 i_{0}-i}$ and $C_{i}^{\prime}=B_{2 i_{0}-i}$, if $2 i_{0}-i \leq 0$, then define $G_{i}^{\prime}=G_{2 i_{0}-i+k}, A_{i}^{\prime}=C_{2 i_{0}-i+k}, B_{i}^{\prime}=B_{2 i_{0}-i+k}$ and $C_{i}^{\prime}=A_{2 i_{0}-i+k}$ and finally if $2 i_{0}-i \geq k+1$, then define $G_{i}^{\prime}=G_{2 i_{0}-i-k}, A_{i}^{\prime}=B_{2 i_{0}-i-k}, B_{i}^{\prime}=A_{2 i_{0}-i-k}$ and $C_{i}^{\prime}=C_{2 i_{0}-i-k}$. Again, it is easy to check that $\left(G_{i}^{\prime}, A_{i}^{\prime}, B_{i}^{\prime}, C_{i}^{\prime}\right)(i=1, \ldots, k)$ is a worn hex-chain for $G$.

In the following lemma, we elaborate how one can extend a (special) clique covering of one of the terms of a worn hex-chain to a clique covering of the whole graph. Let us say that a clique covering $\mathscr{C}$ of a three-cliqued graph $(G, A, B, C)$ is splitting if
(SP1) we have $A, B, C \in \mathscr{C}$, and
(SP2) for every $u \in A \cap \tilde{W}(G)$, there exists a clique $C_{u}$ in $\mathscr{C} \backslash\{A\}$ such that $C_{u} \cap A=\{u\}$. The same holds for every $u \in B \cap \tilde{W}(G)$ and every $u \in C \cap \tilde{W}(G)$, where $\tilde{W}(G)$ is the non-core of $G$, defined in Subsection 1.1.

Lemma 7.4. Let $G$ be a three-cliqued graph which admits a worn hex-chain $\left(G_{i}, A_{i}, B_{i}, C_{i}\right)(i=$ $1, \ldots, k)$. Assume that there exists $i_{0} \in\{1, \ldots, k\}$ such that $G_{i_{0}}$ has a splitting clique covering $\mathscr{C}$ of size at most $\left|V\left(G_{i_{0}}\right)\right|-t$, for some number $t$. Then $G$ admits a clique covering of size at most $|V(G)|-t$.

Proof. Due to Lemma 7.3(i), w.l.o.g. we may assume that $i_{0}=1$. Define $A=\cup_{i=1}^{k} A_{i}, B=\cup_{i=1}^{k} B_{i}$ and $C=\cup_{i=1}^{k} C_{i}$. By (W3), $A_{1} \cap W\left(G_{1}\right)$ is anticomplete to $B \backslash B_{1}, B_{1} \cap W\left(G_{1}\right)$ is anticomplete to $C \backslash C_{1}$ and $C_{1} \cap W\left(G_{1}\right)$ is anticomplete to $A \backslash A_{1}$. For every $u \in \tilde{W}\left(G_{1}\right)$, let $C_{u} \in \mathscr{C}$ be as in (SP2). For every $u \in A_{1} \cap \tilde{W}\left(G_{1}\right)$, define $\tilde{C}_{u}=N\left[u, B \backslash B_{1}\right] \cup C_{u}$ which is a clique of $G$. Similarly, for every $u \in B_{1} \cap \tilde{W}\left(G_{1}\right)$, define $\tilde{C}_{u}=N\left[u, C \backslash C_{1}\right] \cup C_{u}$ and for every $u \in C_{1} \cap \tilde{W}\left(G_{1}\right)$, define $\tilde{C}_{u}=N\left[u, A \backslash A_{1}\right] \cup C_{u}$ which are all cliques of $G$. Moreover, by (SP1), we have $A_{1}, B_{1}, C_{1} \in \mathscr{C}$. Now, let $\mathscr{C}^{\prime}$ be the collection of cliques of $G$ obtained from $\mathscr{C}$ by replacing the cliques $A_{1}, B_{1}$ and $C_{1}$ with the cliques $A, B$, and $C$, and also replacing the cliques $C_{u}, u \in \tilde{W}\left(G_{1}\right)$, with the cliques $\tilde{C}_{u}$, $u \in \tilde{W}\left(G_{1}\right)$. It can be easily seen that the collection $\mathscr{C}^{\prime} \cup \mathscr{N}\left[A \backslash A_{1} ; B\right] \cup \mathscr{N}\left[B \backslash B_{1} ; C\right] \cup \mathscr{N}\left[C \backslash C_{1} ; A\right]$ is a clique covering for $G$ of size at most $\left|V\left(G_{1}\right)\right|-t+|V(G)|-\left|V\left(G_{1}\right)\right|=\mid V(G)-t$. This proves Lemma 7.4.

In the following, as a modification of Lemma 2.2, we prove how one can derive a splitting clique covering for a thickening of a three-cliqued graph.

Lemma 7.5. Let $(H, A, B, C)$ be a three-cliqued graph and $F$ be a valid set for $H$ such that for every pair $\{u, v\} \in F, u, v$ are not both in the same set $A, B$ or $C$. Also, let $G$ be a thickening of $(H, F)$. Suppose that $\tilde{W}(H \backslash F)=\emptyset$ and $H \backslash F$ admits a splitting clique covering $\mathscr{C}$ of size at most $|V(H)|-t$, for some number $t$. Then $\left(G, A^{\prime}, B^{\prime}, C^{\prime}\right)$ admits a splitting clique covering of size at most $|V(G)|-t$, where $A^{\prime}=\cup_{v \in A} X_{v}$ and $B^{\prime}, C^{\prime}$ are defined similarly.

Proof. The proof is similar to the proof of Lemma 2.2. Let $\left(X_{v}\right)_{v \in V(H)}$ be as in the definition of thickening. For each clique $K \in \mathscr{C}$, define $X_{K}=\cup_{u \in K} X_{u}$ which is a clique of $G$ and let $\mathscr{C}^{\prime}=\left\{X_{K}\right.$ : $K \in \mathscr{C}\}$. By (SP1), we have $A, B, C \in \mathscr{C}$ and thus $A^{\prime}, B^{\prime}, C^{\prime} \in \mathscr{C}^{\prime}$. If $F=\emptyset$, then since $\tilde{W}(H \backslash F)=\emptyset$, we have $\tilde{W}(G)=\emptyset$ and thus $\mathscr{C}^{\prime}$ is a splitting clique covering for $G$ of size at most $|V(H)|-t \leq|V(G)|-t$. Now, assume that $F \neq \emptyset$, say $F=\left\{\left\{u_{1}, v_{1}\right\}, \ldots,\left\{u_{\ell}, v_{\ell}\right\}\right\}$. For every $i \in\{1, \ldots, \ell\}$, let $X_{u_{i}} \subseteq X_{u_{i}}$ (resp. $\tilde{X}_{v_{i}} \subseteq X_{v_{i}}$ ) be the set of vertices in $X_{u_{i}}$ (resp. $X_{v_{i}}$ ) which are complete to $X_{v_{i}}$ (resp. $X_{u_{i}}$ ). Note that by (T4) in the definition of thickening, $X_{u_{i}} \backslash \tilde{X}_{u_{i}}$ and $X_{v_{i}} \backslash \tilde{X}_{v_{i}}$ are both nonempty. Assume w.l.o.g. that for every $i \in\{1, \ldots, \ell\}, 1 \leq\left|X_{u_{i}} \backslash \tilde{X}_{u_{i}}\right| \leq\left|X_{v_{i}} \backslash \tilde{X}_{v_{i}}\right|$ and for each $x \in \tilde{X}_{u_{i}}$, define $C_{x}=X_{v_{i}} \cup\{x\}$ and for each $x \in \tilde{X}_{v_{i}}$, define $C_{x}=X_{u_{i}} \cup\{x\}$. Now, if $\left|X_{v_{i}} \backslash \tilde{X}_{v_{i}}\right|=1$, then let $\mathscr{C}_{i}=\left\{C_{x}: x \in \tilde{X}_{u_{i}} \cup \tilde{X}_{v_{i}}\right\}$, and if $\left|X_{v_{i}} \backslash \tilde{X}_{v_{i}}\right| \geq 2$, then let $\mathscr{C}_{i}=\left\{C_{x}: x \in \tilde{X}_{u_{i}} \cup \tilde{X}_{v_{i}}\right\} \cup \mathscr{N}\left[X_{u_{i}} \backslash \tilde{X}_{u_{i}} ; X_{v_{i}}\right]$. Note that $\left|\mathscr{C}_{i}\right| \leq\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|-2$ and the cliques in $\mathscr{C}_{i}$ cover all the edges in $E\left(X_{u_{i}}, X_{v_{i}}\right)$. Now, the collection of cliques $\mathscr{C}^{\prime \prime}=\mathscr{C}^{\prime} \cup\left(\cup_{i=1}^{\ell} \mathscr{C}_{i}\right)$ is a clique covering for $G$ of size at most

$$
|V(H)|-t+\sum_{i=1}^{\ell}\left(\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|-2\right) \leq|V(H)|-t-2 \ell+\sum_{i=1}^{\ell}\left(\left|X_{u_{i}}\right|+\left|X_{v_{i}}\right|\right) \leq|V(G)|-t .
$$

It remains to prove that $\mathscr{C}^{\prime \prime}$ is splitting. First, since $A^{\prime}, B^{\prime}, C^{\prime} \in \mathscr{C} \mathscr{C}^{\prime \prime}$, (SP1) is satisfied. In addition, since $\tilde{W}(H \backslash F)=\emptyset$, we have $\tilde{W}(G) \subseteq \bigcup_{i=1}^{\ell}\left(\tilde{X}_{u_{i}} \cup \tilde{X}_{v_{i}}\right)$, and since no pair in $F$ is included in the same set $A, B$ or $C$, for every $x \in \bigcup_{i=1}^{\ell}\left(\tilde{X}_{u_{i}} \cup \tilde{X}_{v_{i}}\right)$, the clique $C_{x}$ defined above fulfills the conditions in (SP2). This proves Lemma 7.5.

Now, we are ready to prove Theorem 4.4, which is restated below. It should be noted that in the proof, we presume the truth of Theorem 4.7. In fact, within the proof, we need appropriate clique coverings for three-cliqued antiprismatic graphs which will be given in Subsection 8.1.

Theorem 7.6. Let $G$ be a three-cliqued claw-free graph on $n$ vertices which contains at least one triad. Then $\operatorname{cc}(G) \leq n$, and equality holds if and only if $n=3 p+3$, for some positive integer $p$, and $G$ is isomorphic to the $p^{t h}$ power of the cycle $C_{n}$.

Proof. By Theorem 7.2, $G$ admits a worn hex-chain $\left(G_{i}, A_{i}, B_{i}, C_{i}\right)(i=1, \ldots, k)$, where for each $i \in\{1, \ldots, k\},\left(G_{i}, A_{i}, B_{i}, C_{i}\right)$ is a thickening of a permutation of a member of one of $\mathcal{T} \mathcal{C}_{1}, \ldots, \mathcal{T C}_{5}$. First, assume that for each $i, G_{i}$ is a thickening of a member of $\mathcal{T} \mathcal{C}_{4}$. Since every triad of $G$ is a triad of one of $G_{1}, \ldots, G_{k}, G$ is also a thickening of a member of $\mathcal{T} \mathcal{C}_{4}$, i.e. $G$ is a fuzzy antiprismatic graph. Now, since $G$ contains a triad, by Theorem 4.7, $\operatorname{cc}(G) \leq n$ and equality holds if and only if $G$ is isomorphic to $C_{3 p+3}^{p}$, for some positive integer $p$ (note that the complement of a twister is not threecliqued). Now, assume that there exists some $i_{0} \in\{1, \ldots, k\}$ such that $G_{i_{0}}$ is not a thickening of a member of $\mathcal{T} \mathcal{C}_{4}$, i.e. $G_{i_{0}}$ is a thickening of a permutation of a graph $G^{\prime} \in\left(\mathcal{T} \mathcal{C}_{1} \cup \mathcal{T} \mathcal{C}_{2} \cup \mathcal{T} \mathcal{C}_{3} \cup \mathcal{T} \mathcal{C}_{5}\right) \backslash \mathcal{T C} \mathcal{C}_{4}$, with respect to the corresponding valid set $F$. Due to Lemma $7.3(\mathrm{ii})$, we may assume that $G_{i_{0}}$ is a thickening of $\left(G^{\prime}, F\right)$.
Now, we claim that $G^{\prime} \backslash F$ admits a splitting clique covering of size at most $\left|V\left(G^{\prime}\right)\right|-1$ (note that by the definitions in Appendix A, we have $\tilde{W}\left(G^{\prime} \backslash F\right)=\emptyset$ and thus, it is enough to check (SP1) in the definition of splitting clique covering). This along with Lemma 7.5 implies that $G_{i_{0}}$ admits a splitting clique covering of size at most $\left|V\left(G_{i_{0}}\right)\right|-1$. Thus, by Lemma $7.4, G$ admits a clique covering of size at most $n-1$ and the proof is complete. It just remains to prove the claim. For this, we consider the following four cases.

- $G^{\prime} \in \mathcal{T C}_{1}$. Let $\left(G^{\prime}, A, B, C\right), H, F^{\prime}$ and $F$ be as in the definition of $\mathcal{T C}_{1}$. For every vertex $v \in V(H)$ of degree at least three, let $K_{v}$ be the set of all edges of $H$ incident with $v$ and note that $K_{v}$ is a clique of $G^{\prime} \backslash F$. These cliques together with all pairs in $F^{\prime} \backslash F$ comprise a clique covering $\mathscr{C}$ for $G^{\prime} \backslash F$. Also, $A=K_{v_{1}}, B=K_{v_{2}}$ and $C=K_{v_{3}}$ are all in $\mathscr{C}$. Thus, $\mathscr{C}$ satisfies (SP1). Moreover, note that if $H$ has $t$ vertices of degree at least three, then $|\mathscr{C}| \leq t+\left|F^{\prime}\right|$. On the other hand, $\left|F^{\prime}\right|$ is equal to the number of vertices of $H$ with degree two, and thus,

$$
\left|V\left(G^{\prime}\right)\right|=|E(H)|=\frac{1}{2} \sum_{u \in V(H)} \operatorname{deg}(u) \geq \frac{3}{2} t+\left|F^{\prime}\right| \geq \frac{1}{2} t+|\mathscr{C}|
$$

Therefore, since $t \geq 3$, we have $|\mathscr{C}| \leq\left|V\left(G^{\prime}\right)\right|-2$.

- $G^{\prime} \in \mathcal{T} \mathcal{C}_{2} \backslash \mathcal{T} \mathcal{C}_{4}$. Let $\left(G^{\prime}, A, B, C\right)$ be a graph in $\mathcal{T} \mathcal{C}_{2}$ with $\Sigma, \mathcal{I}=\left\{I_{1}, \ldots, I_{k}\right\}, L_{1}, L_{2}, L_{3}$ and the valid set $F$ as in the definition, where $A=V\left(G^{\prime}\right) \cap L_{1}, B=V\left(G^{\prime}\right) \cap L_{2}$ and $C=V\left(G^{\prime}\right) \cap L_{3}$. Then, $G^{\prime} \backslash F$ is also a long circular interval graph with some intervals, say $\mathcal{J}=\left\{J_{1}, \ldots, J_{l}\right\}$ (to obtain these intervals, it is enough to exchange each interval $I_{i} \in \mathcal{I}$ whose endpoints are in $F$ with two other intervals, each of which is obtained from $I_{i}$ by a slight moving of one of its endpoints). We may also assume that for every $i \in\{1,2,3\}, L_{i}$ is contained in an interval in $\mathcal{J}$. Moreover, since for every $\{u, v\} \in F, u, v$ do not belong to the same set $A, B$, or $C, G^{\prime} \backslash F$ is three-cliqued.
First, we observe that $G^{\prime} \backslash F$ is not isomorphic to $C_{m}^{p}$, for any positive integers $m, p$. For, on the contrary, if $m \geq 3 p+4$, then $G^{\prime} \backslash F$ is not three-cliqued. Otherwise, if $m \leq 3 p+3$, then $G^{\prime} \backslash F$ is in $\mathcal{T} \mathcal{C}_{4}$ and since $G^{\prime}$ is claw-free, we have $G^{\prime} \in \mathcal{T} \mathcal{C}_{4}$, a contradiction. Therefore, $G^{\prime} \backslash F$ is not isomorphic to $C_{m}^{p}$, for any positive integers $m, p$. This observation, besides an argument similar to the proof Theorem 6.1 (Statements (1) and (2)), allows us to assume that the number of intervals $l$ is at most $\left|V\left(G^{\prime}\right)\right|-1$ and so the collection of cliques $\mathscr{C}=\left\{J_{i} \cap V\left(G^{\prime}\right): 1 \leq i \leq l\right\}$ is a clique covering for $G^{\prime} \backslash F$ of size at most $\left|V\left(G^{\prime}\right)\right|-1$. Moreover, let $J_{j}$ be an interval in $\mathcal{J}$ containing $L_{1}$. Thus, $J_{j} \cap V\left(G^{\prime} \backslash F\right)$ contains $A$. Also, every vertex $u \in J_{j} \cap(B \cup C)$ is complete to either $A \cup B \backslash\{u\}$ or $A \cup C \backslash\{u\}$, and so $u$ is in no triad, a contradiction with $\tilde{W}\left(G^{\prime} \backslash F\right)=\emptyset$.

Thus, $J_{j} \cap(B \cup C)=\emptyset$ and $J_{j} \cap V\left(G^{\prime} \backslash F\right)=A$ belongs to $\mathscr{C}$. By a similar argument, $B, C \in \mathscr{C}$. Hence, $\mathscr{C}$ satisfies (SP1).

- $G^{\prime} \in \mathcal{T} \mathcal{C}_{3}$. Let $m \geq 2,\left(G^{\prime}, A, B, C\right), X$ and $F$ be as in the definition of $\mathcal{T} \mathcal{C}_{3}$ and $\left|V\left(G^{\prime}\right)\right|=3 m+$ $2-|X|$. For every edge $e=\left\{a_{i}, b_{i}\right\} \in E\left(G^{\prime} \backslash F\right)$, define $K_{e}=\left(C \cup\left\{a_{i}, b_{i}\right\}\right) \backslash\left\{c_{i}\right\}$, which is a clique of $G^{\prime} \backslash F$. Let $A^{\prime}$ (resp. $B^{\prime}$ ) be the set of vertices in $A \backslash\left\{a_{0}\right\}$ (resp. $B \backslash\left\{b_{0}\right\}$ ) which are anticomplete to $B$ (resp. $A$ ) in $G^{\prime} \backslash F$. Now, the family $\left\{K_{e}: e \in E_{G^{\prime} \backslash F}(A, B)\right\} \cup \mathscr{N}\left[A^{\prime} \cup B^{\prime} ; C\right] \cup\{A, B, C\}$ is a splitting clique covering for $G^{\prime} \backslash F$ of size at most

$$
\left|E_{G^{\prime} \backslash F}(A, B)\right|+\left|A^{\prime}\right|+\left|B^{\prime}\right|+3=|A|+|B|-\left|E_{G^{\prime} \backslash F}(A, B)\right|+1 \leq\left|V\left(G^{\prime}\right)\right|-1
$$

where the last inequality is due to the fact that $|C| \geq 2$.

- $G^{\prime} \in \mathcal{T} \mathcal{C}_{5}$. Let $\left(G^{\prime}, A, B, C\right), X$ and $F$ be as in the first construction in the definition of $\mathcal{T} \mathcal{C}_{5}$. First, note that $\mathscr{C}_{1}=\left\{A, B, C,\left\{v_{1}, v_{6}, v_{7}\right\},\left\{v_{2}, v_{3}, v_{4}\right\} \backslash X,\left\{v_{3}, v_{4}, v_{5}\right\} \backslash X\right\}$ is a splitting clique covering for $G^{\prime} \backslash F=G^{\prime}$ of size 6 . If $|X| \leq 1$, then $\left|V\left(G^{\prime}\right)\right| \geq 7$ and we are done. Otherwise, if $X=\left\{v_{3}, v_{4}\right\}$, then $\left|V\left(G^{\prime}\right)\right|=6$ and we can remove the cliques $\left\{v_{2}\right\}$ and $\left\{v_{5}\right\}$ from $\mathscr{C}_{1}$ to obtain a clique covering for $G^{\prime}$ of size 4 , as desired.
Next, let $\left(G^{\prime}, A, B, C\right), X$ and $F$ be as in the second construction in the definition of $\mathcal{T} \mathcal{C}_{5}$. Consider the collection of cliques $\left\{A, B, C,\left\{v_{1}, v_{8}, v_{9}\right\},\left\{v_{2}, v_{3}\right\} \backslash X,\left\{v_{6}, v_{7}\right\} \backslash X\right\}$ of size 6 and if $v_{2} v_{4} \in E\left(G^{\prime} \backslash F\right)$, then replace $\left\{v_{2}, v_{3}\right\} \backslash X$ with $\left\{v_{2}, v_{3}, v_{4}\right\} \backslash X$. Also, if $v_{5} v_{7} \in E\left(G^{\prime} \backslash F\right)$, then replace $\left\{v_{6}, v_{7}\right\} \backslash X$ with $\left\{v_{5}, v_{6}, v_{7}\right\} \backslash X$. The resulting family called $\mathscr{C}_{2}$ is a splitting clique covering for $G^{\prime} \backslash F$. If $|X| \leq 2$, then $\left|V\left(G^{\prime}\right)\right| \geq 7$ and we are done. If $|X|=3$, then either $\left\{v_{3}, v_{4}\right\} \subseteq X$ or $\left\{v_{5}, v_{6}\right\} \subseteq X$ and consequently either $\left\{v_{2}\right\}$ or $\left\{v_{7}\right\}$ is in $\mathscr{C}_{2}$, which can be removed. Finally, if $|X|=4$, i.e. $X=\left\{v_{3}, v_{4}, v_{5}, v_{6}\right\}$, then both $\left\{v_{2}\right\}$ and $\left\{v_{7}\right\}$ are in $\mathscr{C}_{2}$, which can be removed.


## 8 Orientable antiprismatic graphs

The main goal of this and next section is to prove the upper bound $n$ for the clique cover number of $n$-vertex antiprismatic graphs which contain a triad (see Theorem 4.7). This is the most cumbersome part of the proof of Theorem 1.1 which will occupy the whole rest of this paper. To warm-up, we first prove a weaker version of Theorem 4.7 which has a short independent proof.

Theorem 8.1. If $G$ is a connected fuzzy antiprismatic graph on $n$ vertices which contains at least one triad, then $\operatorname{cc}(G) \leq n+3(1+o(1)) \log n$.

Proof. Let $G$ be a thickening of $(H, F)$, where $H$ is an antiprismatic graph on $m$ vertices and $F$ is a valid set of changeable pairs of $H$. Since the pairs in $F$ are non-edges of $H$ and $G$ contains a triad, $H$ contains a triad, say $\tau=\{x, y, z\}$, as well. First, we claim that $\operatorname{cc}(H) \leq m+3(1+o(1)) \log m$. To see this, note that every vertex in $V(H) \backslash \tau$ is nonadjacent to exactly one of the vertices $x, y$ or $z$. Let $X, Y, Z$ be the set of non-neighbours of $x, y, z$ in $H$, respectively. Therefore, $V(H)$ is the disjoint union of $X \backslash \tau, Y \backslash \tau, Z \backslash \tau$ and $\tau$. Since $H$ is antiprismatic, we observe that,

- For every vertex $u$ in $V(H) \backslash X$, the set $N(u, X)$ is a clique of $H$. The same holds for $Y, Z$.
- The induced subgraph $\bar{H}[X]$ is the union of an induced matching and some isolated vertices. The same holds for $Y, Z$.
(The proofs are straightforward and left to the reader.) Gregory and Pullman in [12] proved that the edges of the complement of a perfect matching on $2 t$ vertices can be covered by at most $(1+o(1)) \log t$ cliques. Thus, all the edges in $E(X)$ can be covered by at most $(1+o(1)) \log m$ cliques and the same holds for $Y$ and $Z$. On the other hand, the collection of cliques $\mathscr{N}[X \backslash\{y, z\} ; Y] \cup \mathscr{N}[Y \backslash\{x, z\} ; Z] \cup$
$\mathscr{N}[Z \backslash\{x, y\} ; X]$ covers all the remaining edges. Hence, $H$ admits a clique covering of size at most $m+3(1+o(1)) \log m$. Now, by Lemma 2.3, we have $c c(G) \leq n+3(1+o(1)) \log m \leq n+3(1+o(1)) \log n$. This proves Theorem 8.1.

In this section, we focus particularly on a special class of antiprismatic graphs called orientable antiprismatic graphs (see Section 4) and prove Theorem 4.5, restated as follows.

Theorem 8.2. Let $G$ be an orientable antiprismatic graph on $n$ vertices which contains at least one triad. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $n=3 p+3$, for some positive integer $p$ and $G$ is isomorphic to the $p^{\text {th }}$ power of the cycle $C_{n}$.

In order to prove this theorem, we apply the structure theorem of orientable antiprismatic graphs from [3] as stated in the following. Note that the description of "prismatic" graphs (the graphs whose complement are antiprismatic) is given in [3], so we have reformulated the definitions and the statements in terms of the complements. For every positive integer $k$, an antiprismatic graph $G$ is called to be $k$-substantial if for every $S \subseteq V(G)$ of size at most $k-1$, there exists a triad $\tau$ in $G$ such that $\tau \cap S=\emptyset$.

Theorem 8.3. [3] Every 3-substantial orientable antiprismatic graph is either three-cliqued, or the complement of a cycle of triangles graph, or the complement of a ring of five, or the complement of a mantled $L\left(K_{3,3}\right)$ (see the definitions of these graphs in Appendix B).

In Subsection 8.1, we deal with three-cliqued antiprismatic graphs. In Subsection 8.2, using Theorem 8.3, we complete the proof for 3 -substantial graphs and finally in Subsection 8.3 we tackle the case of non-3-substantial graphs, thereby establishing Theorem 8.2.

### 8.1 Three-cliqued antiprismatic graphs

Firstly, we consider the case of three-cliqued antiprismatic graphs and prove the following.
Theorem 8.4. If $G$ is a three-cliqued antiprismatic graph on $n$ vertices which contains at least one triad, then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $n=3 p+3$, for some positive integer $p$ and $G$ is isomorphic to the $p^{\text {th }}$ power of the cycle $C_{n}$.

To prove Theorem 8.4, we apply the following structure theorem from [3] (reformulated in terms of the complements) as well as Lemma 7.4. Note that the line graph of $K_{3,3}$ is self-complementary.

Theorem 8.5. [3] Every three-cliqued antiprismatic graph admits a worn hex-chain whose all terms are either triad-free, or isomorphic to the line graph of $K_{3,3}$, or the complement of a canonicallycoloured path of triangles graph (see the definitions of these graphs in Appendix B).

Let $G$ be a graph whose vertex set is the union of three disjoint cliques $X_{1}, X_{2}, X_{3}$ such that $X_{2}$ contains a subset $\hat{X}_{2}$ with $\left|\hat{X}_{2}\right|=1$ which is anticomplete to $X_{1}, X_{3}$. Also, $1 \leq\left|X_{1}\right|=\left|X_{3}\right| \leq 2$, $X_{1}, X_{3}$ are anti-matched and every vertex in $X_{2} \backslash \hat{X}_{2}$ is either complete to $X_{1}$ and anticomplete to $X_{3}$, or anticomplete to $X_{1}$ and complete to $X_{3}$. Let us call the three-cliqued graph ( $G, X_{1}, X_{2}, X_{3}$ ) a tripod. A tripod is an example of the complement of a canonically-coloured path of triangles graph (with setting $m=1$ in the definition) which should be treated separately. In the following, we give a splitting clique covering for the complement of a canonically-coloured path of triangles graph which is not a tripod (see the definition of a splitting clique covering in Section 7).

Lemma 8.6. Let $G$ be the complement of a canonically-coloured path of triangles graph on $n$ vertices which is not a tripod. Then $G$ admits a splitting clique covering of size at most $n-1$.

Proof. Let $X_{1}, \ldots, X_{2 m+1}$ be as in the definition. Also, for every $i \in\{1, \ldots, m\}$, define $\tilde{X}_{2 i}=$ $X_{2 i} \backslash \hat{X}_{2 i}$ (for every $i \notin\{1, \ldots, 2 m+1\}$, all sets $X_{i}, \tilde{X}_{i}, L_{i}, M_{i}, R_{i}$ are supposed to be empty). Let $A=X_{1} \cup X_{4} \cup X_{7} \cup \ldots, B=X_{2} \cup X_{5} \cup X_{8} \cup \ldots, C=X_{3} \cup X_{6} \cup X_{9} \cup \ldots$, and for every $i \in\{-1, \ldots, m-1\}$, define

$$
C_{i}=\ldots \cup X_{2 i-8} \cup X_{2 i-5} \cup X_{2 i-2} \cup R_{2 i-1} \cup X_{2 i+2} \cup X_{2 i+5} \cup X_{2 i+8} \cup \ldots,
$$

(note that $C_{-1}=C, C_{0}=B$ and $C_{1}=A$ ). Moreover, for every $x \in R_{2 m-1}$, define

$$
C_{x}^{\prime}=N\left[x, L_{2 m+1}\right] \cup X_{2 m-2} \cup X_{2 m-5} \cup X_{2 m-8} \cup \ldots
$$

Note that, by (P2)-(2) and (P5), all above sets are cliques of $G$. Now, for every $i \in\{1, \ldots, m-1\}$ and every $x \in M_{2 i+1}$, define $C_{i, x}^{\prime \prime}$ as follows,

- if $\left|\hat{X}_{2 i}\right|=1$, then $C_{i, x}^{\prime \prime}=M_{2 i-1} \cup R_{2 i-1} \cup N\left[x, M_{2 i+3}\right] \cup L_{2 i+3}$,
- if $\left|\hat{X}_{2 i}\right|>1$, then $C_{i, x}^{\prime \prime}=N\left[x, M_{2 i-1} \cup X_{2 i}\right] \cup M_{2 i+3} \cup L_{2 i+3}$.

Note that the former is a clique due to (P2)-(2), (P6)-(1) and (P7)-(1). Also, note that if $\left|\hat{X}_{2 i}\right|>1$, then by (P1), $1<i<m$ and $\left|\hat{X}_{2 i-2}\right|=\left|\hat{X}_{2 i+2}\right|=1$ and thus by (P6)-(4) and (P7)-(2), $M_{2 i-1}, M_{2 i+1}$ are matched and both are anti-matched to $\hat{X}_{2 i}$. This along with (P2)-(2) and (P5) implies that the latter is also a clique. Also, by (P1) and (P6)-(4), for every $i \in\{1, \ldots, m-1\}, M_{2 i+1}$ is nonempty. Finally, for every $i \in\{1, \ldots, m\}$, define

$$
Y_{i}=\ldots \cup \tilde{X}_{2 i-8} \cup \tilde{X}_{2 i-2} \cup \tilde{X}_{2 i+2} \cup \tilde{X}_{2 i+8} \cup \ldots,
$$

and for every $i \in\{1, \ldots, m\}$ and every $x \in \tilde{X}_{2 i}$, define $\tilde{C}_{i, x}$ as follows,

- if $\left|\hat{X}_{2 i}\right|=1$, then $\tilde{C}_{i, x}=N\left[x, Y_{i} \cup M_{2 i-1} \cup R_{2 i-1} \cup M_{2 i+1} \cup L_{2 i+1}\right]$,
- if $\left|\hat{X}_{2 i}\right|>1$, then $\tilde{C}_{i, x}=N\left[x, Y_{i}\right]$,
which is a clique due to (P2)-(2) and (P5). Define the collection of cliques $\mathscr{C}$ as follows.

$$
\begin{aligned}
\mathscr{C}= & \left\{C_{i}:-1 \leq i \leq m-1\right\} \cup\left\{N\left[x, C_{i}\right]: 1 \leq i \leq m-1, x \in L_{2 i+1}\right\} \cup\left\{C_{x}^{\prime}: x \in R_{2 m-1}\right\} \\
& \cup\left\{C_{i, x}^{\prime \prime}: 1 \leq i \leq m-1, x \in M_{2 i+1}\right\} \cup\left\{\tilde{C}_{i, x}: 1 \leq i \leq m, x \in \tilde{X}_{2 i}\right\} .
\end{aligned}
$$

Note that by (P6)-(1) and (P7)-(1), $\left|R_{2 i-1}\right|=\left|L_{2 i+1}\right|$, for every $i \in\{1, \ldots, m\}$. This together with (P3) implies that,

$$
\begin{aligned}
|\mathscr{C}| & =m+1+\sum_{i=1}^{m-1}\left(\left|L_{2 i+1}\right|+\left|M_{2 i+1}\right|\right)+\left|R_{2 m-1}\right|+\sum_{i=1}^{m}\left|\tilde{X}_{2 i}\right| \\
& =m+1+\sum_{i=1}^{m}\left(\left|L_{2 i+1}\right|+\left|M_{2 i+1}\right|+\left|\tilde{X}_{2 i}\right|\right) \\
& =n+1-\sum_{i=1}^{m}\left(\left|R_{2 i-1}\right|+\left|\hat{X}_{2 i}\right|-1\right) .
\end{aligned}
$$

First, suppose that $m=1$. Then, $B, C \in \mathscr{C}$ and by (P4), $R_{1} \neq \emptyset$. Thus, adding the clique $A=X_{1}$ to $\mathscr{C}$ yields a splitting clique covering for $G$ of size $n+2-\left|R_{1}\right|$. If $\left|R_{1}\right| \geq 3$, then we are done. If $\left|R_{1}\right| \leq 2$, then since $G$ is not a tripod, we have $\left|R_{1}\right|=2$ and at least one of the two edges in $E\left(R_{1}, L_{3}\right)$ are covered by the cliques $\tilde{C}_{1, x}, x \in \tilde{X}_{2}$. Thus, we may remove one of the cliques $C_{x}^{\prime}, x \in R_{1}$, from $\mathscr{C}$ to obtain a splitting clique covering of size $n-1$.
Now, suppose that $m \geq 2$ and let $I=\left\{i: 1<i<m,\left|\hat{X}_{2 i}\right|>1, R_{2 i+1} \neq \emptyset\right\}$. We leave the reader to check that the only edges of $G$ which are not covered by $\mathscr{C}$ are the edges in $E\left(R_{2 i+1}, L_{2 i-1}\right)$, $i \in I$. If $\left|\hat{X}_{2 i}\right|>1$, then by (P5) and (P7)-(2), $R_{2 i+1}$ is complete to $L_{2 i-1}$. Therefore, the collection $\mathscr{C}^{\prime}=\mathscr{C} \cup\left\{R_{2 i+1} \cup L_{2 i-1}: i \in I\right\}$ is a clique covering for $G$. Also, $A, B, C \in \mathscr{C}$ and $\tilde{W}(G)=\cup_{i=1}^{m} \tilde{X}_{2 i}$. Thus, the cliques $\tilde{C}_{i, x}, x \in \tilde{X}_{2 i}$, satisfy (SP2) in the definition of splitting clique covering and $\mathscr{C}^{\prime}$ is a splitting clique covering. It remains to calculate the cardinality of $\mathscr{C}^{\prime}$.
If at least two of the sets $R_{2 i-1}, i \in\{1, \ldots, m\}$, are nonempty, then since $|I| \leq \sum_{i=1}^{m}\left(\left|\hat{X}_{2 i}\right|-1\right)$, we have $\left|\mathscr{C}^{\prime}\right| \leq n-1$. Now, assume that at most one of the sets $R_{2 i-1}, i \in\{1, \ldots, m\}$, is nonempty and thus $|I| \leq 1$. If $R_{1} \neq \emptyset$, then $I=R_{2 m-1}=\emptyset$ and thus by (P4), $\left|\hat{X}_{2 m-2}\right|>1$. Therefore, $\left|\mathscr{C}^{\prime}\right|=|\mathscr{C}| \leq n-1$. Finally, if $R_{1}=\emptyset$, then by (P4), $\left|\hat{X}_{4}\right|>1$. Now, if $m=2$, then $I=\emptyset$ and by (P4), we have $R_{3} \neq \emptyset$ and thus $\left|\mathscr{C}^{\prime}\right|=|\mathscr{C}| \leq n-1$. Also, if $m \geq 3$, then for every $x \in M_{3}, C_{1, x}^{\prime \prime}=N\left[x, M_{5}\right] \subseteq \cup_{y \in M_{5}} C_{2, y}^{\prime \prime}$ and we may remove the cliques $C_{1, x}^{\prime \prime}, x \in M_{3}$, from $\mathscr{C}^{\prime}$ to obtain a splitting clique covering of size at most $n-1$.

We also need the following lemma.
Lemma 8.7. Let $G$ be a three-cliqued graph on $n$ vertices which admits a worn hex-chain $\left(G_{i}, A_{i}, B_{i}, C_{i}\right)$ $(i=1, \ldots, k)$. If for some $i_{0} \in\{1, \ldots, k\}, G_{i_{0}}$ is triad-free and $G_{i_{0}+1}$ is a triad (reading $i_{0}+1$ modulo $k)$, then $\operatorname{cc}(G) \leq n-1$.

Proof. Due to Lemma 7.3(i), w.l.o.g. we may assume that $G_{1}$ is a triad and $G_{k}$ is triad-free. Let $A=\cup_{i=1}^{k} A_{i}, B=\cup_{i=1}^{k} B_{i}$ and $C=\cup_{i=1}^{k} C_{i}, A^{\prime}=\cup_{i=2}^{k-1} A_{i}, B^{\prime}=\cup_{i=2}^{k-1} B_{i}$ and $C^{\prime}=\cup_{i=2}^{k-1} C_{i}$. Note that $A_{1}$ is anticomplete to $B, B_{1}$ is anticomplete to $C$ and $C_{1}$ is anticomplete to $A$. Define $\mathscr{C}=$ $\mathscr{N}\left[A^{\prime} ; B\right] \cup \mathscr{N}\left[B^{\prime} ; C\right] \cup \mathscr{N}\left[C^{\prime} ; A\right]$. First, assume that there are two of the sets $A_{k}, B_{k}, C_{k}$, say $A_{k}$ and $B_{k}$, such that some vertex in one of them is complete to the other (note that it includes the case that at least one of $A_{k}, B_{k}, C_{k}$ is empty). If there is a vertex in $A_{k}$ complete to $B_{k}$, then the collection of cliques $\mathscr{C} \cup \mathscr{N}\left[A_{k} ; B\right] \cup \mathscr{N}\left[B_{k} ; C\right] \cup \mathscr{N}\left[C_{k} ; A\right] \cup\{A, C\}$ is a clique covering for $G$ of size $\left|A^{\prime}\right|+\left|B^{\prime}\right|+\left|C^{\prime}\right|+\left|A_{k}\right|+\left|B_{k}\right|+\left|C_{k}\right|+2=n-1$. Also, if there is a vertex in $B_{k}$ complete to $A_{k}$, then the collection $\mathscr{C} \cup \mathscr{N}\left[A_{k} ; A^{\prime} \cup A_{1} \cup C_{k}\right] \cup \mathscr{N}\left[B_{k} ; B^{\prime} \cup B_{1} \cup A_{k}\right] \cup \mathscr{N}\left[C_{k} ; C^{\prime} \cup C_{1} \cup B_{k}\right] \cup\left\{C^{\prime} \cup C_{1} \cup B_{k}, A^{\prime} \cup A_{1} \cup C_{k}\right\}$ is a clique covering for $G$ of size $n-1$. Therefore, we may assume that every vertex in one of the sets $A_{k}, B_{k}, C_{k}$ has a non-neighbour in each of two others. In addition, we claim that every vertex in one of the sets $A_{k}, B_{k}, C_{k}$ has a neighbour in each of two others. For if say there exists a vertex $x \in A_{k}$ which is anticomplete to $B_{k}$, then by the assumption, $x$ has a non-neighbour $y$ in $C_{k}$ and also $y$ has a non-neighbour $z$ in $B_{k}$, and so $\{x, y, z\}$ is a triad in $G_{k}$, a contradiction. This proves the claim. Moreover, assume that $\left|A_{k}\right| \leq\left|B_{k}\right| \leq\left|C_{k}\right|$ (the other cases are similar).
Now, if every two distinct vertices in $B_{k}$ have a common neighbour in $A_{k} \cup C_{k}$, then $\mathscr{C} \cup \mathscr{N}\left[A_{k} ; B\right] \cup$ $\mathscr{N}\left[C_{k} ; B_{k} \cup C_{1} \cup C^{\prime}\right] \cup \mathscr{N}\left[A_{k} ; C_{k} \cup A_{1} \cup A^{\prime}\right] \cup\{A, C\}$ is a clique covering for $G$ of size $|\mathscr{C}|+2\left|A_{k}\right|+\left|C_{k}\right|+2 \leq$ $|\mathscr{C}|+\left|A_{k}\right|+\left|B_{k}\right|+\left|C_{k}\right|+2=n-1$ (note that, by the above claim, the edges in $E(B)$ are covered by the cliques in $\left.\mathscr{N}\left[A_{k} ; B\right] \cup \mathscr{N}\left[C_{k} ; B_{k} \cup C_{1} \cup C^{\prime}\right]\right)$. Finally, if there exist two distinct vertices $u, v \in B_{k}$ with no common neighbour in $A_{k} \cup C_{k}$, then since $G_{k}$ is triad free, $N\left[u, A_{k} \cup C_{k}\right]$ and $N\left[v, A_{k} \cup C_{k}\right]$ are both cliques. Hence, the collection of cliques $\mathscr{C} \cup \mathscr{N}\left[B_{k} \backslash\{u, v\} ; A_{k}\right] \cup \mathscr{N}\left[B_{k} \backslash\{u, v\} ; C_{k}\right] \cup \mathscr{N}\left[A_{k} ; C_{k} \cup\right.$ $\left.A_{1} \cup A^{\prime}\right] \cup\left\{N\left[u, A_{k} \cup C_{k}\right], N\left[v, A_{k} \cup C_{k}\right], A_{k} \cup B_{1} \cup B^{\prime}, B_{k} \cup C_{1} \cup C^{\prime}, B, C\right\}$ is a clique covering for $G$ of size $|\mathscr{C}|+2\left(\left|B_{k}\right|-2\right)+\left|A_{k}\right|+6 \leq|\mathscr{C}|+\left|A_{k}\right|+\left|B_{k}\right|+\left|C_{k}\right|+2=n-1$ (note that the edges in $E(A)$ are covered by the cliques in $\left.\mathscr{N}\left[A_{k} ; C_{k} \cup A_{1} \cup A^{\prime}\right] \cup\left\{A_{k} \cup B_{1} \cup B^{\prime}\right\}\right)$. This proves Lemma 8.7.

Now, we are ready to prove Theorem 8.4.
Proof of Theorem 8.4. By Theorem 8.5, $G$ admits a worn hex-chain $\left(G_{i}, A_{i}, B_{i}, C_{i}\right)(i=1, \ldots, k)$, where each $G_{i}$ is either triad-free or isomorphic to the line graph of $K_{3,3}$, or the complement of a canonically-coloured path of triangles graph.
Firstly, suppose that there exists $i_{0} \in\{1, \ldots, k\}$ such that $G_{i_{0}}$ is isomorphic to the line graph of $K_{3,3}$ on the vertex set $\left\{a_{j}^{i}: 1 \leq i, j \leq 3\right\}$, where $a_{j}^{i}$ is adjacent to $a_{j^{\prime}}^{i^{\prime}}$ if and only if $i=i^{\prime}$ or $j=j^{\prime}$. Then, $A_{i_{0}}, B_{i_{0}}, C_{i_{0}}$ are three disjoint triangles of $G_{i_{0}}$ and $\left\{\left\{a_{1}^{i}, a_{2}^{i}, a_{3}^{i}\right\},\left\{a_{i}^{1}, a_{i}^{2}, a_{i}^{3}\right\}: 1 \leq i \leq 3\right\}$ is a splitting clique covering for $G_{i_{0}}$ of size six. Hence, Lemma 7.4 implies that $G$ admits a clique covering of size at most $n-3$, as desired.
Secondly, suppose that there exists $i_{0} \in\{1, \ldots, k\}$ such that $G_{i_{0}}$ is the complement of a canonicallycoloured path of triangles graph which is not a tripod. Then, by Lemma 8.6, $G_{i_{0}}$ admits a splitting clique covering of size at most $\left|V\left(G_{i_{0}}\right)\right|-1$, and thus by virtue of Lemma 7.4 , we obtain a clique covering for $G$ of size at most $n-1$.
Thirdly, assume that there exists $i_{0} \in\{1, \ldots, k\}$ such that $G_{i_{0}}$ is a tripod with the sets $X_{1}, X_{2}, \hat{X}_{2}, X_{3}$ as in the definition, where $X_{2} \backslash \hat{X}_{2} \neq \emptyset$. In the light of Lemma 7.3(i), assume that $i_{0}=k$. Let $\tilde{X}_{1}$ (resp. $\tilde{X}_{3}$ ) be the set of vertices in $X_{2} \backslash \hat{X}_{2}$ which are complete to $X_{1}$ (resp. $X_{3}$ ). Then, since $X_{2} \backslash \hat{X}_{2} \neq \emptyset$, either $\tilde{X}_{1}$ or $\tilde{X}_{3}$, say $\tilde{X}_{1}$, is nonempty and by Lemma 7.3 (ii), we may suppose that $A_{k}=X_{1}, B_{k}=X_{2}$ and $C_{k}=X_{3}$. Let $\left(H, A^{\prime}, B^{\prime}, C^{\prime}\right)$ be a three-cliqued graph which admits a worn hexchain $\left(H_{i}, A_{i}^{\prime}, B_{i}^{\prime}, C_{i}^{\prime}\right)(i=1, \ldots, k+2)$ such that for every $i \in\{1, \ldots, k-1\}, H_{i}=G_{i}, A_{i}^{\prime}=A_{i}, B_{i}^{\prime}=B_{i}$ and $C_{i}^{\prime}=C_{i}$. Also, $H_{k}$ and $H_{k+2}$ are both cliques and $H_{k+1}$ is a triad on the vertex set $\{a, b, c\}$, where $A_{k}^{\prime}=\emptyset, B_{k}^{\prime}=\tilde{X}_{1}, C_{k}^{\prime}=\emptyset, A_{k+1}^{\prime}=\{a\}, B_{k+1}^{\prime}=\{b\}, C_{k+1}^{\prime}=\{c\}, A_{k+2}^{\prime}=\emptyset, B_{k+2}^{\prime}=\tilde{X}_{3}$ and $C_{k+2}^{\prime}=\emptyset$. Since $H_{k}$ is triad-free and $H_{k+1}$ is a triad, by Lemma 8.7, $H$ admits a clique covering of size at most
$|V(H)|-1$. Now, if $\left|X_{1}\right|=\left|X_{3}\right|=1$, then $G$ is isomorphic to $H$ (by an isomorphism mapping the single vertices of $G$ in $X_{1}, \hat{X}_{2}, X_{3}$ to the vertices $a, b, c$ in $H$, respectively) and if $\left|X_{1}\right|=\left|X_{3}\right|=2$, then $G$ is a thickening of $(H, F)$, for $F=\{\{a, c\}\}$ (we leave the reader to check that $\{a, c\}$ is a changeable pair of $H$ ). Thus, by Lemma 2.3, $\operatorname{cc}(G) \leq n-1$.
Therefore, we may assume that for all $i \in\{1, \ldots, k\}, G_{i}$ is either triad-free or a tripod with $X_{2}=\hat{X}_{2}$. Hence, each $G_{i}$ is either triad-free, or a thickening of a triad. First, assume that there is some $i$ such that $G_{i}$ is triad-free. If all terms $G_{i}$ are triad-free, then so is $G$, a contradiction. Therefore, there is some $i_{0} \in\{1, \ldots, k\}$ such that $G_{i_{0}}$ is triad-free and $G_{i_{0}+1}$ is a thickening of a triad (reading $i_{0}+1$ modulo $k$ ). Hence, Lemmas 8.7 and 2.3 yield that $\operatorname{cc}(G) \leq n-1$. Finally, assume that each term $G_{i}$ is a thickening of a triad. Note that if all terms $G_{i}$ are triads, then $G$ is isomorphic to the graph $C_{3 k}^{k-1}$ (by the isomorphism mapping the single vertices of $G$ in $A_{1}, \ldots, A_{k}, B_{1}, \ldots, B_{k}, C_{1}, \ldots, C_{k}$ to the vertices $v_{0}, \ldots, v_{3 k-1}$ of $C_{3 k}^{k-1}$, respectively), and if each $G_{i}$ is a thickening of a triad, then it is easy to check that $G$ is a fuzzy long circular interval graph. Thus, in this case, by Theorem $6.1, \operatorname{cc}(G) \leq n$ and equality holds if and only if $G$ is isomorphic to $C_{n}^{p}$, for some positive integer $p \leq\lfloor(n-1) / 3\rfloor$. However, if $n>3 p+3$, then $C_{n}^{p}$ is not three-cliqued and if $n<3 p+3$, then $C_{n}^{p}$ is triad-free. Therefore, $\operatorname{cc}(G) \leq n$ and equality holds if and only if $n=3 p+3$ and $G$ is isomorphic to $C_{n}^{p}$. This proves Theorem 8.4.

### 8.2 Other 3-substantial antiprismatic graphs

According to Theorems 8.3 and 8.4, in order to complete the proof of Theorem 8.2 for 3 -substantial orientable antiprismatic graphs, we need to prove the assertion for the complement of a cycle of triangles graph, the complement of a mantled $L\left(K_{3,3}\right)$ and the complement of a ring of five (see Appendix B for the definitions).

Lemma 8.8. If $G$ is the complement of a cycle of triangles graph on $n$ vertices, then $\operatorname{cc}(G) \leq n-1$.
Proof. Let $m \geq 5$ and $m=3 k+2$ for some integer $k$ and also let $X_{1}, \ldots, X_{2 m}$ be as in the definition and define $\tilde{X}_{2 i}=X_{2 i} \backslash \hat{X}_{2 i}$. Throughout the proof, read subscripts modulo $2 m$. For every $i \in\{1, \ldots, m\}$, define

$$
C_{i}=L_{2 i-1} \cup X_{2 i} \cup X_{2 i+3} \cup X_{2 i+6} \cup \ldots \cup X_{2 i+6 k},
$$

which by (C2) and (C4) is a clique of $G$. Note that by (C1) and (C5)-(4), $M_{2 i+1}$ is nonempty for every $i \in\{1, \ldots, m\}$. Now, for every $i \in\{1, \ldots, m\}$ and every $x \in M_{2 i+1}$, define the clique $C_{i, x}^{\prime}$ as follows,

- if $\left|\hat{X}_{2 i}\right|=1$, then $C_{i, x}^{\prime}=M_{2 i-1} \cup R_{2 i-1} \cup N\left[x, M_{2 i+3}\right] \cup L_{2 i+3}$,
- if $\left|\hat{X}_{2 i}\right|>1$, then $C_{i, x}^{\prime}=N\left[x, M_{2 i-1} \cup X_{2 i}\right] \cup M_{2 i+3} \cup L_{2 i+3}$.

Finally, for every $i \in\{1, \ldots, m\}$, define

$$
Y_{i}=\tilde{X}_{2 i+2} \cup \tilde{X}_{2 i+8} \cup \ldots \cup \tilde{X}_{2 i+2+6 k},
$$

and for every $x \in \tilde{X}_{2 i}$, define $\tilde{C}_{i, x}$ as follows,

- if $\left|\hat{X}_{2 i}\right|=1$, then $\tilde{C}_{i, x}=N\left[x, M_{2 i-1} \cup R_{2 i-1} \cup M_{2 i+1} \cup L_{2 i+1} \cup Y_{i}\right]$,
- if $\left|\hat{X}_{2 i}\right|>1$, then $\tilde{C}_{i, x}=N\left[x, Y_{i}\right]$.

Now, define $\mathscr{C}=\left\{C_{i}: 1 \leq i \leq m\right\} \cup\left\{N\left[x, C_{i}\right]: 1 \leq i \leq m, x \in R_{2 i-3}\right\} \cup\left\{C_{i, x}^{\prime}: 1 \leq i \leq m, x \in\right.$ $\left.M_{2 i+1}\right\} \cup\left\{\tilde{C}_{i, x}: 1 \leq i \leq m, x \in \tilde{X}_{2 i}\right\}$. Thus,

$$
|\mathscr{C}|=m+\sum_{i=1}^{m}\left(\left|M_{2 i-1}\right|+\left|R_{2 i-1}\right|+\left|\tilde{X}_{2 i}\right|\right)=n-\sum_{i=1}^{m}\left(\left|L_{2 i-1}\right|+\left|\hat{X}_{2 i}\right|-1\right) .
$$

Also, let $I=\left\{i: 1 \leq i \leq m,\left|\hat{X}_{2 i}\right|>1, L_{2 i-1} \neq \emptyset\right\}$. It is easy to check that the only edges which are not covered by $\mathscr{C}$ are the edges in $E\left(L_{2 i-1}, R_{2 i+1}\right), i \in I$. Therefore, $\mathscr{C}^{\prime}=\mathscr{C} \cup\left\{L_{2 i-1} \cup R_{2 i+1}: i \in I\right\}$
is a clique covering for $G$. If either $L_{2 i-1}$ is nonempty or $\left|\hat{X}_{2 i}\right|>1$ for some $i$, then $\left|\mathscr{C}^{\prime}\right| \leq n-1$. Now, assume that $L_{2 i-1}=R_{2 i-1}=\emptyset$ and $\left|\hat{X}_{2 i}\right|=1$ for all $i \in\{1, \ldots, m\}$. In this case, remove the cliques $C_{i, x}^{\prime}$, for all $1 \leq i \leq m$ and $x \in M_{2 i+1}$ from $\mathscr{C}$ and add the following cliques

$$
\forall 1 \leq i \leq\left\lfloor\frac{m+1}{2}\right\rfloor, C_{i}^{\prime}=M_{4 i-3} \cup M_{4 i-1} \cup M_{4 i+1} \cup M_{4 i+3}
$$

Thus, in this case, $\operatorname{cc}(G) \leq m+\lfloor(m+1) / 2\rfloor+\sum_{i=1}^{m}\left|\tilde{X}_{2 i}\right|=\lfloor(m+1) / 2\rfloor+\sum_{i=1}^{m}\left|X_{2 i}\right| \leq n-1$.
The following two lemmas handle the last two cases of the 3-substantial antiprismatic graphs.
Lemma 8.9. If $G$ is the complement of a mantled $L\left(K_{3,3}\right)$ on $n$ vertices, then $\operatorname{cc}(G) \leq n-1$.
Proof. Let $a_{j}^{i}, V^{i}, V_{j}, i, j \in\{1,2,3\}$, be as in the definition and read all subscripts and superscripts modulo 3. For every $i, j \in\{1,2,3\}, i \neq j$, define

$$
A_{j}^{i}=V^{i} \cup\left\{a_{1}^{j}, a_{2}^{j}, a_{3}^{j}\right\}, \quad B_{j}^{i}=V_{i} \cup\left\{a_{j}^{1}, a_{j}^{2}, a_{j}^{3}\right\}
$$

First, assume that at least one of $V_{1}, V_{2}, V_{3}$ and at least one of $V^{1}, V^{2}, V^{3}$ are empty, say $V_{3}=V^{3}=\emptyset$. In this case, the family of cliques $\mathscr{N}\left[V^{1} ; V^{2} \cup V_{1}\right] \cup \mathscr{N}\left[V_{1} ; V_{2} \cup V^{2}\right] \cup \mathscr{N}\left[V^{2} ; V_{2}\right] \cup \mathscr{N}\left[V_{2} ; V^{1}\right] \cup\left\{A_{j}^{i}, B_{j}^{i}\right.$ : $1 \leq i \leq 2,1 \leq j \leq 3, j \neq i\}$ is a clique covering for $G$ of size $n-1$. Now, assume that either all $V^{1}, V^{2}, V^{3}$, or all $V_{1}, V_{2}, V_{3}$ are nonempty. Assume w.l.o.g. that the former case occurs. Since $G\left[V^{1} \cup V^{2} \cup V^{3}\right]$ contains no triad, there exists $i_{0} \in\{1,2,3\}$ such that every vertex in $V^{i_{0}+1}$ has a neighbour in $V^{i_{0}}$. For every $i \in\{1,2,3\}$, define $U^{i}=V^{i+1} \cup\left\{a_{i}^{i+2}, a_{i+1}^{i+2}, a_{i+2}^{i+2}\right\}$, if $V_{i+1}=\emptyset$, and define $U^{i}=V^{i+1} \cup V_{i+1} \cup\left\{a_{i}^{i+2}, a_{i+2}^{i+2}\right\}$, otherwise. Also, let $U_{i}=V_{i+1} \cup V^{i+1} \cup\left\{a_{i+2}^{i}, a_{i+2}^{i+2}\right\}$ and let

$$
\mathscr{C}=\bigcup_{i=1}^{3}\left(\left\{A_{i+1}^{i}, B_{i+1}^{i}\right\} \cup \mathscr{N}\left[V^{i} ; U^{i}\right] \cup \mathscr{N}\left[V_{i} ; U_{i}\right]\right)
$$

Note that the edges in $E\left(V^{i_{0}+1}, V_{i_{0}+1}\right)$ are covered by the cliques in $\mathscr{N}\left[V^{i_{0}} ; U^{i_{0}}\right]$. For every $i \in$ $\left\{i_{0}, i_{0}+2\right\}$, if $V_{i} \neq \emptyset$, then the edges in $E\left(V^{i}, V_{i}\right)$ are possibly not covered by the cliques in $\mathscr{C}$. Also, for every $i \in\{1,2,3\}$, if $V_{i-1}=\emptyset$ and $V_{i+1} \neq \emptyset$, then the edges in $E\left(V^{i},\left\{a_{i+1}^{i+2}\right\}\right)$ are not covered by the cliques in $\mathscr{C}$. In both of these cases, add the clique $V^{i} \cup V_{i} \cup\left\{a_{i+1}^{i+2}\right\}$ to $\mathscr{C}$. We leave the reader to check that this procedure adds at most two cliques to $\mathscr{C}$ and yields a clique covering for $G$ of size at $\operatorname{most}|\mathscr{C}|+2=n-1$.

Lemma 8.10. If $G$ is the complement of a ring of five on $n$ vertices, then $\operatorname{cc}(G) \leq n-2$.
Proof. Let $V_{i}, a_{i}, b_{i}, 1 \leq i \leq 5$, be as in the definition and consider the following cliques in $G$ (reading subscripts modulo 5 ).

$$
\begin{aligned}
& C_{0}=\left\{b_{1}, \ldots, b_{5}\right\}, \\
\forall i \in\{1, \ldots, 5\}, & C_{i}=V_{i} \cup V_{i+2} \cup\left\{a_{i}, b_{i+1}, a_{i+2}\right\}, \\
\forall i \in\{1,2,3\}, & C_{i}^{\prime}=V_{0} \cup V_{i} \cup V_{i+2} \cup\left\{a_{i}, a_{i+2}\right\}, \\
\forall i \in\{1, \ldots, 5\}, \forall x \in V_{i}, & C_{i, x}=N\left[x, V_{i+1}\right] \cup\left\{b_{i+2}, b_{i+3}\right\} .
\end{aligned}
$$

The collection of cliques $\mathscr{C}=\left\{C_{i}: 0 \leq i \leq 5\right\} \cup\left\{C_{i}^{\prime}: 1 \leq i \leq 3\right\} \cup\left\{C_{i, x}: 1 \leq i \leq 5, x \in V_{i}\right\}$ forms a clique covering for $G$. If $V_{0}$ is nonempty, then $|\mathscr{C}| \leq n-2$ and if $V_{0}$ is empty, then one can remove the cliques $C_{1}^{\prime}, C_{2}^{\prime}, C_{3}^{\prime}$ from $\mathscr{C}$ to obtain a clique covering of size $n-4$.

### 8.3 Non-3-substantial antiprismatic graphs

Finally, we encounter the case of non-3-substantial antiprismatic graphs, i.e. antiprismatic graphs whose all triads meet a fixed set of at most two vertices. They include several cases that should be dealt separately and this makes the proof lengthy. The following lemma would be fruitful which will be used in the proof of the next theorem as well as in Section 9.

Lemma 8.11. Assume that $G$ is an antiprismatic graph on $n$ vertices which is not three-cliqued. Also, let $a_{0}$ be a vertex of $G$ which is contained in a triad, let $A=N_{\bar{G}}\left(a_{0}\right)$ and $\tilde{A} \subseteq A$ be the set of all vertices in $A$ which have no non-neighbour in $A$. If for every $v \in A \backslash \tilde{A}, N_{G}(v, V(G) \backslash A)$ is a clique of $G$, then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to a twister.

Proof. Let $B=N_{G}\left(a_{0}\right)=V(G) \backslash\left(A \cup\left\{a_{0}\right\}\right)$. Since $G$ is antiprismatic and $a_{0}$ is in a triad, we may assume that $A \backslash \tilde{A}=X \cup Y$, where $X=\left\{x_{1}, \ldots, x_{k}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{k}\right\}$, such that the only non-edges whose both endpoints are in $A$ are $x_{i} y_{i}, i=1, \ldots, k$. Also, every vertex in $B$ is adjacent to exactly one of $x_{i}$ and $y_{i}$, for each $i \in\{1, \ldots, k\}$. Thus, by the assumption, for each $i \in\{1, \ldots, k\}$, $B$ is partitioned into two disjoint cliques $U_{i}=N\left(x_{i}, B\right)$ and $V_{i}=N\left(y_{i}, B\right)$. We may assume w.l.o.g. that $\left|V_{i}\right| \leq\left|U_{i}\right|$, for all $i \in\{1, \ldots, k\}$. First, we claim that
(1) We have $k \geq 2$. Also, for every $i \in\{1, \ldots, k\}, U_{i}$ is not complete to $V_{i}$ and every vertex in $U_{i}$ has a neighbour in $V_{i}$ and vice versa.
For if $k=1$, then $V(G)$ is the union of three cliques $A \backslash\left\{y_{1}\right\}, V_{1} \cup\left\{y_{1}\right\}$ and $U_{1} \cup\left\{a_{0}\right\}$, a contradiction. Also, if $U_{i}$ is complete to $V_{i}$ for some $i \in\{1, \ldots, k\}$, then $V(G)$ is the union of three cliques $X, A \backslash X$ and $U_{i} \cup V_{i} \cup\left\{a_{0}\right\}$, a contradiction. Finally, assume that there exists a vertex $u \in U_{i}$ with no neighbour in $V_{i}$. If $u$ is nonadjacent to some $x_{j} \in X$, then $u \in V_{j}$ and since $V_{j}$ is a clique, we have $V_{i} \cap V_{j}=\emptyset$ and thus $x_{j}$ is complete to $V_{i}$. Similarly, if $u$ is nonadjacent to some $y_{j} \in Y$, then $y_{j}$ is complete to $V_{i}$. Therefore, $V_{i}$ is complete to $N_{\bar{G}}(u, X \cup Y)$ and consequently $V$ is the union of three cliques $U_{i} \cup\left\{a_{0}\right\}$, $V_{i} \cup N_{\bar{G}}(u, X \cup Y)$ and $A \backslash N_{\bar{G}}(u, X \cup Y)$, a contradiction. This proves (1).

By (1), for each $i \in\{1, \ldots, k\}$, there exist $u_{i} \in U_{i}$ and $v_{i} \in V_{i}$, such that $u_{i}$ and $v_{i}$ are nonadjacent. Then, for each $i \in\{1, \ldots, k\}, X_{i}=N\left(u_{i}, X \cup Y\right)$ and $Y_{i}=N\left(v_{i}, X \cup Y\right)$ are two disjoint cliques partitioning $X \cup Y$ and it is evident that $\left|X_{i}\right|=\left|Y_{i}\right|=k$. Now, for every $i \in\{1, \ldots, k\}$, define

$$
\mathscr{C}_{i}^{1}=\mathscr{N}[B ; A] \cup \mathscr{N}\left[X_{i} ; Y_{i} \cup \tilde{A}\right] \cup \mathscr{N}\left[V_{i} ; U_{i} \cup\left\{a_{0}\right\}\right] \cup\left\{U_{i} \cup\left\{a_{0}\right\}, V_{i} \cup\left\{a_{0}\right\}\right\}
$$

and

$$
\mathscr{C}_{i}^{2}=\mathscr{N}[A \backslash \tilde{A} ; B] \cup \mathscr{N}\left[\tilde{A} ; U_{i}\right] \cup \mathscr{N}\left[\tilde{A} ; V_{i}\right] \cup \mathscr{N}[X ; Y \cup \tilde{A}] \cup \mathscr{N}\left[V_{i} ; U_{i} \cup\left\{a_{0}\right\}\right] \cup\{X \cup \tilde{A}\}
$$

First, note that due to (1), the edges in $E\left(\left\{a_{0}\right\}, U_{i} \cup V_{i}\right)$ are covered by the cliques in $\mathscr{N}\left[V_{i} ; U_{i} \cup\left\{a_{0}\right\}\right]$. Also, by (1), we have $k \geq 2$. Thus, for each $i \in\{1, \ldots k\}$, the edges in $E(A)$ are covered by the cliques in $\mathscr{N}[B ; A] \cup \mathscr{N}\left[X_{i} ; Y_{i} \cup \tilde{A}\right] \subseteq \mathscr{C}_{i}^{1}$, and so $\mathscr{C}_{i}^{1}$ is a clique covering for $G$ of size $\left|U_{i}\right|+2\left|V_{i}\right|+k+2$.
Moreover, note that for each $i \in\{1, \ldots, k\}$, the edges in $E\left(U_{i}\right)$ and $E\left(V_{i}\right)$ are covered by the cliques in $\mathscr{N}[A \backslash \tilde{A} ; B] \subseteq \mathscr{C}_{i}^{2}$. Also, if $k \geq 3$, then the edges in $E(Y)$ are covered by the cliques in $\mathscr{N}[X ; Y \cup$ $\tilde{A}] \subseteq \mathscr{C}_{i}^{2}$. Hence, if $k \geq 3$, then for every $i \in\{1, \ldots, k\}, \mathscr{C}_{i}^{2}$ is also a clique covering for $G$ of size $|A|+|\tilde{A}|+k+\left|V_{i}\right|+1$. Now, we prove the lemma in two cases of $k \geq 3$ and $k=2$.
First, assume that $k \geq 3$. If there exists some $i \in\{1, \ldots, k\}$, such that either $\left|V_{i}\right| \leq|\tilde{A}|+k-2$, or $|\tilde{A}|+k \leq\left|U_{i}\right|-1$, then either $\mathscr{C}_{i}^{1}$ or $\mathscr{C}_{i}^{2}$ is a clique covering for $G$ of size at most $n-1$. Thus, suppose that for every $i \in\{1, \ldots, k\}$,

$$
|\tilde{A}|+k-1 \leq\left|V_{i}\right| \leq\left|U_{i}\right| \leq|\tilde{A}|+k
$$

We provide a clique covering for $G$ of size at most $n-1$ in the following three possibilities.
(2) Assume that $k \geq 3$ and for every vertex $x_{i} \in X$ and $v \in A \backslash\left\{y_{i}\right\}$, $x_{i}$ and $v$ have a common neighbour in $B$. Then $\operatorname{cc}(G) \leq n-1$.
In this case, the edges whose one end is in $X$ and the other end is in $A$ are covered by the cliques in $\mathscr{N}[B ; A]$. Now, in the clique covering $\mathscr{C}_{1}^{1}$, replace the cliques in $\mathscr{N}\left[X_{1} ; Y_{1} \cup \tilde{A}\right]$ with the clique $Y \cup \tilde{A}$ to obtain a clique covering of size $\left|U_{1}\right|+2\left|V_{1}\right|+3 \leq\left|U_{1}\right|+\left|V_{1}\right|+|\tilde{A}|+2 k \leq n-1$. This proves (2).
(3) Assume that $k \geq 3$ and for every vertex $x_{i} \in X$ and $v \in X \cup Y \backslash\left\{y_{i}\right\}_{\tilde{A}}, x_{i}$ and $v$ have a common neighbour in $B$, however, there exists a vertex $x_{i_{0}} \in X$ and a vertex $v_{0} \in \tilde{A}$ such that $x_{i_{0}}$ and $v_{0}$ have no common neighbour in $B$. Then $\operatorname{cc}(G) \leq n-1$.

In this case, $N\left(v_{0}, B\right) \subseteq V_{i_{0}}$. Now, if $\left|U_{i_{0}}\right|=|\tilde{A}|+k$, then remove the clique $N\left[v_{0}, U_{i_{0}}\right]$ from $\mathscr{C}_{i_{0}}^{2}$ to obtain a clique covering of size $n-1$. Otherwise, if $\left|U_{i_{0}}\right|=\left|V_{i_{0}}\right|=|\tilde{A}|+k-1$, then in the collection $\mathscr{C}_{i_{0}}^{1}$, replace the cliques in $\mathscr{N}\left[X_{i_{0}} ; Y_{i_{0}} \cup \tilde{A}\right]$ with the cliques $X \cup \tilde{A}$ and $Y \cup \tilde{A}$ to obtain a clique covering of size $\left|U_{i_{0}}\right|+2\left|V_{i_{0}}\right|+4 \leq\left|U_{i_{0}}\right|+\left|V_{i_{0}}\right|+|\tilde{A}|+2 k \leq n-1$ (note that due to the assumption, the edges in $E(X, Y)$ are covered by the cliques in $\mathscr{N}[B ; A])$. This proves (3).
(4) Assume that $k \geq 3$ and there exist vertices $x_{i_{0}} \in X$ and $v_{0} \in X \cup Y \backslash\left\{y_{i_{0}}\right\}$ which have no common neighbour in $B$. Then, $\operatorname{cc}(G) \leq n-1$.
First, assume that $v_{0}=y_{j_{0}} \in Y$, for some $j_{0} \neq i_{0}$. Thus, $U_{i_{0}} \cap V_{j_{0}}=\emptyset$ and so $U_{i_{0}} \subseteq U_{j_{0}}$ and $V_{j_{0}} \subseteq V_{i_{0}}$. Therefore, since $\left|U_{j_{0}}\right|-\left|V_{j_{0}}\right| \leq 1$, we have $U_{i_{0}}=U_{j_{0}}$ and $V_{i_{0}}=V_{j_{0}}$. In this case, in $\mathscr{C}_{i_{0}}^{2}$, replace the cliques $N[x, B], x \in\left\{x_{i_{0}}, y_{i_{0}}, x_{j_{0}}, y_{j_{0}}\right\}$, with the cliques $U_{i_{0}} \cup\left\{x_{i_{0}}, x_{j_{0}}\right\}$ and $V_{i_{0}} \cup\left\{y_{i_{0}}, y_{j_{0}}\right\}$ to obtain a clique covering of size at most $n-1$. Now, assume that $v_{0}=x_{j_{0}} \in X$, for some $j_{0} \neq i_{0}$. Thus, $U_{i_{0}} \cap U_{j_{0}}=\emptyset$ and then $U_{j_{0}} \subseteq V_{i_{0}}$ and $U_{i_{0}} \subseteq V_{j_{0}}$. Since $\left|V_{i}\right| \leq\left|U_{i}\right|$ for all $i$, we have $U_{j_{0}}=V_{i_{0}}$ and $U_{i_{0}}=V_{j_{0}}$ and in $\mathscr{C}_{i_{0}}^{2}$, replacing the cliques $N[x, B], x \in\left\{x_{i_{0}}, y_{i_{0}}, x_{j_{0}}, y_{j_{0}}\right\}$ with the cliques $U_{i_{0}} \cup\left\{x_{i_{0}}, y_{j_{0}}\right\}$ and $V_{i_{0}} \cup\left\{y_{i_{0}}, x_{j_{0}}\right\}$ yields a clique covering of size at most $n-1$. This proves (4).
Hence, if $k \geq 3$, then by (2), (3) and (4), we are done. Now, assume that $k=2$. Note that for every $i \in\{1,2\}, \mathscr{C}_{i}^{1}$ is a clique covering for $G$. Also, adding the clique $Y$ to $\mathscr{C}_{i}^{2}$ yields a clique covering for $G$. Thus, if there exists some $i \in\{1,2\}$, such that either $\left|V_{i}\right| \leq|\tilde{A}|$, or $|\tilde{A}| \leq\left|U_{i}\right|-4$, then $\operatorname{cc}(G) \leq n-1$. Hence, suppose that for every $i \in\{1,2\}$,

$$
|\tilde{A}|+1 \leq\left|V_{i}\right| \leq\left|U_{i}\right| \leq|\tilde{A}|+3
$$

Now, let $B_{1}=U_{1} \cap V_{2}, B_{2}=U_{1} \cap U_{2}, B_{3}=V_{1} \cap U_{2}$ and $B_{4}=V_{1} \cap V_{2}$. Also, for every $i \in\{1,2,3,4\}$, let $\hat{B}_{i} \subseteq B_{i}$ be the set of vertices in $B_{i}$ which are anticomplete to $B_{i+2}$ (reading $i+2$ modulo 4 ). In the sequel, we need the following three facts.
(5) For every $i \in\{1,2\}, B_{i}$ is not complete to $B_{i+2}$ and the edges $x_{1} x_{2}, x_{1} y_{2}, y_{1} x_{2}$ and $y_{1} y_{2}$ are covered by the cliques in $\mathscr{N}[B ; A]$. Also, for every $v \in \tilde{A}$, if $N(v, B)$ is not a clique, then all edges incident with $v$ are covered by the cliques in $\mathscr{N}[B ; A]$.
To see the first claim, note that if say $B_{1}$ is complete to $B_{3}$, then $V(G)$ is the union of three cliques $B_{1} \cup B_{2} \cup B_{3} \cup\left\{a_{0}\right\}, B_{4} \cup Y$ and $\tilde{A} \cup X$, a contradiction. Therefore, for every $i \in\{1,2\}, B_{i}$ is not complete to $B_{i+2}$ and thus, all sets $B_{i}, 1 \leq i \leq 4$, are nonempty. Hence, the four edges $x_{1} x_{2}, x_{1} y_{2}, y_{1} x_{2}$ and $y_{1} y_{2}$ are covered by the cliques in $\mathscr{N}[B ; A]$. Now, assume that the vertex $v \in \tilde{A}$ is adjacent to vertices $b, b^{\prime} \in B$, where $b$ and $b^{\prime}$ are nonadjacent. Let $v^{\prime} \in A$ be a vertex distinct from $v$. Since $G$ is claw-free, $v^{\prime}$ is adjacent to at least one of $b, b^{\prime}$ and thus the edge $v v^{\prime}$ is covered by the clique $N[b, A]$ or $N\left[b^{\prime}, A\right]$. This proves (5).
(6) For every $v \in \tilde{A}$, either $v$ is anticomplete to $B$, or there is some $i \in\{1,2,3,4\}$ such that $v$ is complete to $\hat{B}_{i} \cup \hat{B}_{i+1}$ (reading $i+1$ modulo 4 ).
Assume that $v \in \tilde{A}$ and for every $i \in\{1,2,3,4\}, v$ is not complete to $\hat{B}_{i} \cup \hat{B}_{i+1}$. Thus, there exists some $i \in\{1,2\}$ such that $v$ has two non-neighbours in $\hat{B}_{i}$ and $\hat{B}_{i+2}$, say $b_{1} \in \hat{B}_{1}$ and $b_{3} \in \hat{B}_{3}$. Therefore, $v$ is anticomplete to $B_{2} \cup B_{4}$ (since if say $v$ has a neighbour $b$ in $B_{2} \cup B_{4}$, then $\left\{b, b_{1}, b_{3}, v\right\}$ would be a claw). Also, since by (5), $B_{2}$ is not complete to $B_{4}$, by a similar argument, $v$ is anticomplete to $B_{1} \cup B_{3}$ and so $v$ is anticomplete to $B$. This proves (6).
Now, let $\mathscr{C}^{\prime}$ be the collection of cliques obtained from $\mathscr{C}_{1}^{2}$ by removing all the cliques in $\mathscr{N}[A \backslash \tilde{A} ; B] \cup$ $\mathscr{N}[X ; Y \cup \tilde{A}]$ and adding the cliques $B_{1} \cup\left\{x_{1}, y_{2}\right\}, B_{2} \cup\left\{x_{1}, x_{2}\right\}, B_{3} \cup\left\{x_{2}, y_{1}\right\}, B_{4} \cup\left\{y_{1}, y_{2}\right\}, Y \cup \tilde{A}$, $U_{1}$ and $V_{1}$. Note that $\mathscr{C}^{\prime}$ is a clique covering for $G$ of size $2|\tilde{A}|+\left|V_{1}\right|+8$. Also, we have
(7) If there exists $v_{0} \in \tilde{A}$ which is complete to $\hat{B}_{1} \cup \hat{B}_{2}$, then $\mathscr{C}^{\prime} \backslash\left\{U_{1}\right\}$ is a clique covering for $G$.

Since $U_{1}=B_{1} \cup B_{2}$, it is enough to show that edges in $E\left(B_{1}, B_{2}\right)$ are covered by the cliques in $\mathscr{C}^{\prime} \backslash\left\{U_{1}\right\}$. To see this, note that the cliques in $\mathscr{N}\left[V_{1} ; U_{1} \cup\left\{a_{0}\right\}\right]$ cover all edges between $B_{1}$ and $B_{2}$ except the edges between $\hat{B}_{1}$ and $\hat{B}_{2}$, while these edges are covered by the clique $N\left[v_{0}, U_{1}\right]$. This proves (7).

Now, we complete the proof for $k=2$ through the following four possibilities.
(8) Assume that $k=2$ and for every vertex $v \in \tilde{A}, N(v, B)$ is not a clique of $G$. Then $\operatorname{cc}(G) \leq n-1$.

First, note that by (5), all edges whose both endpoints are in $A$ are covered by the cliques in $\mathscr{N}[B ; A]$. Hence, removing the cliques in $\mathscr{N}\left[X_{i} ; Y_{i} \cup \tilde{A}\right]$ from $\mathscr{C}_{i}^{1}$ leads to a clique covering for $G$ of size $\left|U_{i}\right|+$ $2\left|V_{i}\right|+2$. Therefore, if there exists some $i \in\{1,2\}$ such that $\left|V_{i}\right| \leq|\tilde{A}|+2$, then $\operatorname{cc}(G) \leq n-1$. Now, assume that for every $i \in\{1,2\},\left|U_{i}\right|=\left|V_{i}\right|=|\tilde{A}|+3$. Thus, $\left|B_{1}\right|=\left|B_{3}\right|$ and $\left|B_{2}\right|=\left|B_{4}\right|$. If $\tilde{A}$ is empty, then remove the cliques $X \cup \tilde{A}$ and $Y \cup \tilde{A}$ from $\mathscr{C}^{\prime}$ to get a clique covering of size $\left|V_{1}\right|+6=n-2$. Now, assume that $\tilde{A}$ is nonempty and let $v_{0} \in \tilde{A}$. Due to (6), assume w.l.o.g. that $v_{0}$ is complete to $\hat{B}_{1} \cup \hat{B}_{2}$. Thus, by (7), $\mathscr{C}^{\prime} \backslash\left\{U_{1}\right\}$ is a clique covering of size $2|\tilde{A}|+\left|V_{1}\right|+7=n-1$. This proves (8).
(9) Assume that $k=2$ and there are at least two vertices $v, v^{\prime} \in \tilde{A}$ such that $N(v, B)$ and $N\left(v^{\prime}, B\right)$ are both cliques of $G$. Then $\operatorname{cc}(G) \leq n-1$.

First, assume that $N(v, B)$ is empty. In this case, in the clique covering $\mathscr{C}_{i}^{2}$, one can replace the cliques $N\left[v, U_{i}\right], N\left[v, V_{i}\right], N\left[v^{\prime}, U_{i}\right]$ and $N\left[v^{\prime}, V_{i}\right]$ with the cliques $N\left[v^{\prime}, B\right]$ and $Y$ to obtain a clique covering of size $2|\tilde{A}|+\left|V_{i}\right|+5 \leq|\tilde{A}|+\left|U_{i}\right|+\left|V_{i}\right|+4=n-1$. Now, assume that $N(v, B)$ is nonempty. If there exists some $i \in\{1,2\}$ such that $\left|U_{i}\right| \geq|\tilde{A}|+2$, then adding the clique $N[v, B]$ to the above collection yields a clique covering of size at most $n-1$. Finally, assume that for every $i \in\{1,2\},\left|U_{i}\right|=\left|V_{i}\right|=|\tilde{A}|+1$. By (6), w.l.o.g. we may assume that $v$ is complete to $\hat{B}_{1} \cup \hat{B}_{2}$. Thus, by (7), the collection obtained from $\mathscr{C}^{\prime}$ by removing the clique $U_{1}$ and merging the pairs $\left(N\left[v, U_{1}\right], N\left[v, V_{1}\right]\right)$ and $\left(N\left[v^{\prime}, U_{1}\right], N\left[v^{\prime}, V_{1}\right]\right)$ is a clique covering for $G$ of size $2|\tilde{A}|+\left|V_{1}\right|+5=n-1$. This proves (9).
(10) Assume that $k=2$ and there exists a vertex $v_{0} \in \tilde{A}$ such that $N\left(v_{0}, B\right)$ is a nonempty clique of $G$ and for every vertex $v \in \tilde{A} \backslash\left\{v_{0}\right\}, N(v, B)$ is not a clique of $G$. Then $\operatorname{cc}(G) \leq n-1$.
Assume w.l.o.g. that $v_{0}$ has a neighbour in $B_{1}$. First, note that in the collection $\mathscr{C}_{i}^{2}$, merging the pair $\left(N\left[v_{0}, U_{i}\right], N\left[v_{0}, V_{i}\right]\right)$ and adding the clique $Y$ yield a clique covering for $G$ of size $2|\tilde{A}|+\left|V_{i}\right|+7$. Thus, if there exists some $i \in\{1,2\}$, such that $\left|U_{i}\right|=|\tilde{A}|+3$, then $\operatorname{cc}(G) \leq n-1$. Moreover, by (5), the cliques in $\mathscr{N}[B ; A]$ cover all the edges in $E(A) \backslash\left\{v_{0} x_{2}, v_{0} y_{1}\right\}$ (note that since $v_{0}$ has a neighbour in $B_{1}$, the edges $v_{0} x_{1}$ and $v_{0} y_{2}$ are covered). Thus, replacing the cliques in $\mathscr{N}\left[X_{i} ; Y_{i} \cup \tilde{A}\right]$ with the clique $\tilde{A} \cup\left\{x_{2}, y_{1}\right\}$ in $\mathscr{C}_{i}^{1}$ leads to a clique covering for $G$ of size $\left|U_{i}\right|+2\left|V_{i}\right|+3$. Thus, if there exists some $i \in\{1,2\}$, such that $\left|V_{i}\right|=|\tilde{A}|+1$, then $\operatorname{cc}(G) \leq n-1$. Consequently, assume that for every $i \in\{1,2\},\left|U_{i}\right|=\left|V_{i}\right|=|\tilde{A}|+2$. Thus, $\left|B_{1}\right|=\left|B_{3}\right|$ and $\left|B_{2}\right|=\left|B_{4}\right|$. By (6), w.l.o.g. we may assume that $v_{0}$ is complete to $\hat{B}_{1} \cup \hat{B}_{2}$. Thus, by (7), the collection obtained from $\mathscr{C}^{\prime}$ by removing $U_{1}$ and merging the pair $\left(N\left[v_{0}, U_{1}\right], N\left[v_{0}, V_{1}\right]\right)$ is a clique covering for $G$ of size $2|\tilde{A}|+\left|V_{1}\right|+6=n-1$. This proves (10).
(11) Assume that $k=2$ and there exists a vertex $v_{0} \in \tilde{A}$, such that $N\left(v_{0}, B\right)$ is empty and for every vertex $v \in \tilde{A} \backslash\left\{v_{0}\right\}, N(v, B)$ is not a clique of $G$. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to a twister.
Note that removing the cliques $N\left[v_{0}, U_{i}\right]$ and $N\left[v_{0}, V_{i}\right]$ from $\mathscr{C}_{i}^{2}$ and adding the clique $Y$ yield a clique covering for $G$ of size $2|\tilde{A}|+\left|V_{i}\right|+6$. Thus, if there exists some $i \in\{1,2\}$, such that $|\tilde{A}|+2 \leq\left|U_{i}\right|$, then $\operatorname{cc}(G) \leq n-1$. Hence, suppose that for every $i \in\{1,2\},\left|U_{i}\right|=\left|V_{i}\right|=|\tilde{A}|+1$. First, assume that $|\tilde{A}| \geq 2$ and let $v \in \tilde{A} \backslash\left\{v_{0}\right\}$. By (6), assume w.l.o.g. that $v$ is complete to $\hat{B}_{1} \cup \hat{B}_{2}$. Thus, by (7), the collection of cliques obtained from $\mathscr{C}^{\prime}$ by removing the cliques $U_{1}, N\left[v_{0}, U_{1}\right]$ and $N\left[v_{0}, V_{1}\right]$ is a clique covering for $G$ of size $2|\tilde{A}|+\left|V_{1}\right|+5=n-1$. Finally, assume that $|\tilde{A}|=1$. Then, $\left|B_{1}\right|=\left|B_{2}\right|=\left|B_{3}\right|=\left|B_{4}\right|=1$, say $B_{i}=\left\{b_{i}\right\}, i=1,2,3,4$, and so $|V(G)|=10$. Therefore, by (5), for every $i \in\{1,2\}, B_{i}$ is anticomplete to $B_{i+2}$. Hence, $G$ is isomorphic to the complement of a twister (by mapping the vertices $a_{0}, v_{0}, b_{1}, b_{2}, b_{3}, b_{4}, x_{1}, x_{2}, y_{1}, y_{2}$ of $G$ to the vertices $u_{1}, u_{2}, v_{4}, v_{6}, v_{8}, v_{2}, v_{1}, v_{3}, v_{5}, v_{7}$ of the complement of a twister in Figure 1, respectively). Thus, by Lemma 3.4, $\operatorname{cc}(G)=10$. This proves (11).
Finally, (8), (9), (10) and (11) imply the assertion for $k=2$. This proves Lemma 8.11.
Now, we are ready to complete the proof of Theorem 8.2 providing appropriate clique covering for
non-3-substantial antiprismatic graphs.
Theorem 8.12. Let $G$ be an antiprismatic graph on $n$ vertices which is neither 3 -substantial nor three-cliqued and contains a triad. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to a twister.

Proof. If $G$ is not 2-substantial, then there exists a vertex $u$ meeting all triads of $G$. Let $A$ be the set of non-neighbours of $u$ and let $\{u, v, w\}$ be a triad of $G$. Then, $N(v, V(G) \backslash A)$ is a clique (otherwise, $v$ is adjacent to $x, y \in V(G) \backslash A$, where $x, y$ are nonadjacent and then $\{x, y, w\}$ is a triad not containing $u$, a contradiction). Therefore, the result follows from Lemma 8.11. Now, assume that $G$ is 2 -substantial and not 3 -substantial. Then, there exist two vertices $u, v$ meeting all triads. We prove the theorem in the following two cases.
(1) If $u$ and $v$ are nonadjacent, then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to $a$ twister.

If $u$ and $v$ are not contained in a common triad, then the result follows from Lemma 8.11. Thus, assume that $u$ and $v$ are contained in a triad, say $T=\{u, v, w\}$. Let $A, B, C$ be the sets of non-neighbours of $u, v, w$ in $V(G) \backslash T$, respectively, which partition $V(G) \backslash T$. By the assumption, $C$ is a clique of $G$. If $C=\emptyset$, then $G$ is an extension of a member of $\mathcal{F}_{1}$ that will be handled in Lemma 9.8 (for the definition of $\mathcal{F}_{1}$ and the extension, see Appendix C and Subsection 9.1, respectively). So, suppose that $C$ is nonempty. Also, let $A=X \cup Y \cup A^{\prime}$ and $B=Z \cup W \cup B^{\prime}$, where $X=\left\{x_{1}, \ldots, x_{k}\right\}, Y=\left\{y_{1}, \ldots, y_{k}\right\}$, $Z=\left\{z_{1}, \ldots, z_{l}\right\}$ and $W=\left\{w_{1}, \ldots, w_{l}\right\}$ and the only non-edges in $A$ are $x_{i} y_{i}, 1 \leq i \leq k$, and the only non-edges in $B$ are $z_{i} w_{i}, 1 \leq i \leq l$. Also, since $G$ is 2 -substantial, $k, l \geq 1$. Assume w.l.o.g. that $x_{1}$ is complete to $Z$ and anticomplete to $W, y_{1}$ is complete to $W$ and anticomplete to $Z, z_{1}$ is complete to $X$ and anticomplete to $Y$ and $w_{1}$ is complete to $Y$ and anticomplete to $X$. Moreover, let $X^{\prime}=N\left(z_{1}, A^{\prime}\right)$, $Y^{\prime}=N\left(w_{1}, A^{\prime}\right), Z^{\prime}=N\left(x_{1}, B^{\prime}\right), W^{\prime}=N\left(y_{1}, B^{\prime}\right), Z^{\prime \prime}=N\left(x_{1}, C\right)$ and $W^{\prime \prime}=N\left(y_{1}, C\right)$. Thus, $A^{\prime}=X^{\prime} \cup Y^{\prime}, B^{\prime}=Z^{\prime} \cup W^{\prime}$ and $C=Z^{\prime \prime} \cup W^{\prime \prime}$. Since $x_{i}, y_{i}$ are contained in no triad than $\left\{x_{i}, y_{i}, u\right\}$, $N\left(x_{i}, B \cup C\right)$ and $N\left(y_{i}, B \cup C\right)$ are cliques of $G$. Thus, $Z \cup Z^{\prime} \cup Z^{\prime \prime}$ and $W \cup W^{\prime} \cup W^{\prime \prime}$ are cliques. Similarly, $X \cup X^{\prime} \cup Z^{\prime \prime}$ and $Y \cup Y^{\prime} \cup W^{\prime \prime}$ are cliques. Moreover, assume w.l.o.g. that $k \leq l$ and $Z^{\prime \prime}$ is nonempty. Now, we claim that $X$ is complete to $Z$. Since $Z^{\prime \prime}$ is nonempty, let $z \in Z^{\prime \prime}$. If $x_{i} \in X$ is adjacent to some $w_{j} \in W$, then $z$ is adjacent to $w_{j}$ (because $N\left(x_{i}, B \cup C\right)$ is a clique). Thus, $G$ induces a claw on $\left\{z, v, z_{j}, w_{j}\right\}$, a contradiction. Hence, $X$ is anticomplete to $W$ and thus complete to $Z$ and then $Y$ is anticomplete to $Z$ and complete to $W$. Let $\tilde{X} \subseteq X^{\prime}$ (resp. $\tilde{Y} \subseteq Y^{\prime}$ ) be the set of all vertices in $X^{\prime}\left(\right.$ resp. $\left.Y^{\prime}\right)$ which are complete to $Z$ (resp. $W$ ). Now, we claim that $X^{\prime} \backslash \tilde{X}$ and $Y^{\prime} \backslash \tilde{Y}$ are complete to $C$. For assume that $x \in X^{\prime} \backslash \tilde{X}$ has a non-neighbour $w^{\prime \prime} \in W^{\prime \prime}$. By the definition, $x$ also has a non-neighbour $z_{j} \in Z$. Thus, $\left\{x, w^{\prime \prime}, z_{j}\right\}$ is a triad disjoint from $\{u, v\}$, a contradiction. Hence, $X^{\prime} \backslash \tilde{X}$ (and similarly $Y^{\prime} \backslash \tilde{Y}$ ) are complete to $C$. Now, define

$$
\begin{aligned}
\mathscr{C}= & \mathscr{N}[C ; B \cup\{u\}] \cup \mathscr{N}\left[B^{\prime} ; A \cup\{w\}\right] \cup \mathscr{N}\left[\left(X^{\prime} \backslash \tilde{X}\right) \cup\left(Y^{\prime} \backslash \tilde{Y}\right) ; B \cup\{w\}\right] \\
& \cup \mathscr{N}\left[\tilde{X} ; W^{\prime \prime} \cup Y \cup Y^{\prime} \cup\{v\}\right] \cup \mathscr{N}\left[\tilde{Y} ; Z^{\prime \prime} \cup X \cup X^{\prime} \cup\{v\}\right] \cup \mathscr{N}[X ; Y] \cup \mathscr{N}\left[Z ; W \cup B^{\prime} \cup\{w\}\right] \\
& \cup\left\{C_{1}=X \cup X^{\prime} \cup\left(Y^{\prime} \backslash \tilde{Y}\right) \cup Z^{\prime \prime} \cup\{v\}, C_{2}=Y \cup Y^{\prime} \cup\left(X^{\prime} \backslash \tilde{X}\right) \cup W^{\prime \prime} \cup\{v\}\right\} \\
& \cup\left\{C_{3}=X \cup \tilde{X} \cup Z \cup\{w\}, C_{4}=Y \cup \tilde{Y} \cup W \cup\{w\}, C_{5}=C \cup\{v\}\right\} .
\end{aligned}
$$

If $W^{\prime \prime}=\emptyset$, then in $\mathscr{C}$, replace $C_{5}$ with the clique $W \cup B^{\prime} \cup\{u\}$. If $l \geq 2$, then it is easy to check that $\mathscr{C}$ is a clique covering for $G$ of size $|C|+\left|A^{\prime}\right|+\left|B^{\prime}\right|+k+l+5 \leq n-1$, as desired. Now, assume that $k=l=1$ and thus, $\tilde{X}=X^{\prime}$ and $\tilde{Y}=Y^{\prime}$. In this case, let $\mathscr{C}^{\prime}$ be the collection obtained from $\mathscr{C}$ by removing the degenerate clique in $\mathscr{N}[X ; Y]$. If $W^{\prime \prime}=\emptyset$, then $\mathscr{C}^{\prime}$ is a clique covering of size $n-1$ and we are done. If $W^{\prime \prime} \neq \emptyset$, then the only edges which are possibly not covered by $\mathscr{C}^{\prime}$ are the edges in $E\left(W, Z^{\prime}\right)$. Thus, if $Z^{\prime}$ is empty, then again we are done. Finally, assume that $W^{\prime \prime}$ and $Z^{\prime}$ (and by symmetry all sets $W^{\prime}, X^{\prime}, Y^{\prime}$ ) are nonempty. In this case, replace four cliques $C_{1}, C_{2}, C_{3}, C_{4}, C_{5}$ in $\mathscr{C}^{\prime}$ with five cliques $X \cup X^{\prime} \cup Z \cup Z^{\prime \prime}, Y \cup Y^{\prime} \cup W \cup W^{\prime \prime}, W \cup B^{\prime} \cup\{w\}, A^{\prime} \cup\{w\}$ and $C \cup\{v\}$ to form a clique covering of size $n-1$.
(2) If $u$ and $v$ are adjacent, then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to $a$ twister.

If any triad on $u$ does not intersect any triad on $v$, then the result follows from Lemma 8.11. Thus, assume that there exist two triads $T_{1}=\{u, w, x\}$ and $T_{2}=\{v, w, z\}$ and thus $x$ is complete to $\{z, v\}$ and $z$ is complete to $\{x, u\}$. Note that $V^{\prime}=V(G) \backslash\{u, v, w, z, x\}$ is partitioned into five cliques $A=N_{G}\left(u, V^{\prime}\right) \cap N_{\bar{G}}\left(v, V^{\prime}\right), B=N_{\bar{G}}\left(u, V^{\prime}\right) \cap N_{G}\left(v, V^{\prime}\right), C=N_{\bar{G}}\left(u, V^{\prime}\right) \cap N_{\bar{G}}\left(v, V^{\prime}\right), D=N_{\bar{G}}\left(w, V^{\prime}\right)$ and $E=N_{G}\left(u, V^{\prime}\right) \cap N_{G}\left(v, V^{\prime}\right) \backslash D$. Thus, $u$ is complete to $A \cup D \cup E, v$ is complete to $B \cup D \cup E, w$ is complete to $A \cup B \cup C \cup E, x$ is complete to $B \cup C \cup D$ and $z$ is complete to $A \cup C \cup D$. Let $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$ be the sets of vertices respectively in $A$ and $B$ which are complete to $C$. Also, let $\tilde{C} \subseteq C$ be the set of vertices in $C$ which are complete to $A \cup B$. It is evident that $\tilde{W}(G)=\tilde{A} \cup \tilde{B} \cup \tilde{C} \cup D \cup E$. We may assume w.l.o.g. that $\tilde{A} \neq A, \tilde{B} \neq B$ and $\tilde{C} \neq C$, otherwise if say $A=\tilde{A}$, then every triad of $G$ meets $\{u, w\}$ and $u, w$ are nonadjacent and by (1), we are done. Moreover, note that since every triad of $G$ meets $\{u, v\}$, both $N_{\bar{G}}\left(z, V^{\prime}\right)=B \cup E$ and $N_{\bar{G}}\left(x, V^{\prime}\right)=A \cup E$ are cliques. Thus, since $G$ is antiprismatic, $E$ is anticomplete to $C \backslash \tilde{C}$. Moreover, since $G$ is antiprismatic and $N_{\bar{G}}\left(u, V^{\prime}\right)=B \cup C$, every vertex in $B$ (resp. $C$ ) has at most one non-neighbour in $C$ (resp. $B$ ) and for every vertex $y$ in $V^{\prime} \backslash(B \cup C), N(y, B \cup C)$ is a clique. Similarly, every vertex in $A$ (resp. $C$ ) has at most one non-neighbour in $C$ (resp. $A$ ) and for every vertex $y$ in $V^{\prime} \backslash(A \cup C), N(y, A \cup C)$ is a clique. We prove (2) in two cases.

Firstly, assume that $|C \backslash \tilde{C}| \geq 2$. Also, assume w.l.o.g. that $|A \backslash \tilde{A}| \geq|B \backslash \tilde{B}|$. Thus, every vertex in $B$ has a non-neighbour in $A \backslash \tilde{A}$. Now, we claim that $E$ is complete to $D$. For contrary, suppose that $d \in D$ is not adjacent to $e \in E$. The facts that $E$ is anticomplete to $C \backslash \tilde{C}$ and the induced subgraph of $G$ on $C \cup D \cup E$ is triad-free yield that $d$ is complete to $C \backslash \tilde{C}$. Therefore, since $G$ is antiprismatic, $d$ is anticomplete to $B \backslash \tilde{B}$ and $A \backslash \tilde{A}$. Thus, there exists a triad in $A \cup B \cup D$, a contradiction. This shows that $E$ is complete to $D$. Now, let $D^{\prime} \subseteq D$ be the set of vertices in $D$ which are complete to $B$ and define

$$
\begin{aligned}
\mathscr{D}= & \mathscr{N}[A ; D \cup\{z, u\}] \cup \mathscr{N}[B ; A \cup C \cup\{w\}] \cup \mathscr{N}[C ; A \cup\{w\}] \\
& \cup \mathscr{N}[E ; B \cup C \cup\{w\}] \cup \mathscr{N}\left[D \backslash D^{\prime} ; B \cup C \cup\{x\}\right] \cup \mathscr{N}\left[D^{\prime} ; C \cup\{x, z\}\right] \\
& \cup\left\{C \cup\{x, z\}, B \cup D^{\prime} \cup\{x, v\}, D \cup E \cup\{u, v\}, A \cup E \cup\{w\}\right\} .
\end{aligned}
$$

Note that every vertex in $D \backslash D^{\prime}$ has a neighbour in $A$, otherwise there would be a triad in $A \cup B \cup D$. Therefore, the edges in $E(\{z\}, D)$ are covered by the cliques in $\mathscr{N}[A ; D \cup\{z, u\}] \cup \mathscr{N}\left[D^{\prime} ; C \cup\{x, z\}\right]$. This implies that $\mathscr{D}$ is a clique covering for $G$ of size $|A|+|B|+|C|+|D|+|E|+4=n-1$.
Secondly, assume that $|C \backslash \tilde{C}|=1$ and $C \backslash \tilde{C}=\left\{c_{0}\right\}$. Thus, $|B \backslash \tilde{B}|=|A \backslash \tilde{A}|=1$ and $A \backslash \tilde{A}$ is complete to $B \backslash \tilde{B}$. Assume w.l.o.g. that $|\tilde{B}| \leq|\tilde{A}|$. Also, let $D_{1} \subseteq D$ be the set of vertices in $D$ which are adjacent to $c_{0}$ and let $D_{2}=D \backslash D_{1}$. By above argument, $D_{2}$ is complete to $E \cup(B \backslash \tilde{B}) \cup(A \backslash \tilde{A})$ and $D_{1}$ is complete to $\tilde{A} \cup \tilde{B}$ (if say $d_{1} \in D_{1}$ is nonadjacent to $\tilde{a} \in \tilde{A}$, then $\left\{\tilde{a}, d_{1}\right\} \cup B \backslash \tilde{B}$ is a triad). Now, define

$$
\begin{aligned}
\mathscr{D}^{\prime}= & \mathscr{N}[A ; D \cup\{z, u\}] \cup \mathscr{N}[B ; A \cup C \cup\{w\}] \cup \mathscr{N}[C ; D \cup\{z, x\}] \\
& \cup \mathscr{N}[E ; B \cup C \cup\{w\}] \cup \mathscr{N}\left[D_{1} ; E \cup D_{2} \cup\{u, v\}\right] \cup \mathscr{N}\left[D_{2} ; B \cup C \cup\{x\}\right] \\
& \cup\left\{C \cup \tilde{A} \cup\{w\}, D_{2} \cup E \cup\{u, v\}, B \cup\{x, v\}, A \cup E \cup\{w\} .\right.
\end{aligned}
$$

It is easy to verify that the cliques in $\mathscr{D}^{\prime}$ cover all edges of $G$ except the edges in $E\left(D_{1}, \tilde{B}\right)$ (note that the edges in $E\left(D_{1} \cup\{z, x\}\right)$ are covered by the cliques in $\mathscr{N}[C ; D \cup\{z, x\}]$ and the edges in $E(A \backslash \tilde{A}, C)$ are covered by the cliques in $\mathscr{N}[B ; A \cup C \cup\{w\}])$. Thus, if $\tilde{B}=\emptyset$, then $\mathscr{D}^{\prime}$ is a clique covering for $G$ of size $n-1$. Now, assume that $\tilde{B} \neq \emptyset$. Note that $\tilde{C} \neq \emptyset$, because if $\tilde{C}=\emptyset$, then $V(G)$ is the union of three cliques $A \cup\{z, u\}, \tilde{B} \cup D_{1} \cup\left\{x, c_{0}\right\}$ and $D_{2} \cup E \cup(B \backslash \tilde{B}) \cup\{v\}$, a contradiction. Now, let $\mathscr{D}^{\prime \prime}$ be the collection obtained from $\mathscr{D}^{\prime}$ by replacing the cliques $N\left[c_{0}, D \cup\{z, x\}\right]=D_{1} \cup\left\{c_{0}, z, x\right\}$ and $C \cup \tilde{A} \cup\{w\}$ with the cliques $D_{1} \cup \tilde{B} \cup\left\{c_{0}, x\right\}$ and $C \cup \tilde{A} \cup\{z\}$. The facts that $\tilde{B}$ and thus $\tilde{A}$ are nonempty, $D_{1}$ is complete to $\tilde{A}$ and $D_{2}$ is complete to $A \backslash \tilde{A}$, imply that the edges in $E(\{z\}, D)$ are covered by the cliques in $\mathscr{N}[A ; D \cup\{z, u\}]$ and the edges in $E(\{w\}, C)$ are covered by the cliques in $\mathscr{N}[B ; A \cup C \cup\{w\}]$. Also, since $\tilde{C} \neq \emptyset$, the edge $z x$ is covered by the cliques in $\mathscr{N}[\tilde{C} ; D \cup\{z, x\}]$. Hence, $\mathscr{D}^{\prime \prime}$ is a clique covering for $G$ of size $n-1$.


Figure 2: The complement of a rotator.

Now, we are ready to prove of Theorem 8.2.
Proof of Theorem 8.2. Let $G$ be an orientable antiprismatic graph on $n$ vertices containing a triad. If $G$ is three-cliqued, then the result follows from Theorem 8.4. If $G$ is neither three-cliqued nor 3substantial, then by Theorem $8.12, \operatorname{cc}(G) \leq n-1$ (note that the complement of a twister is not an orientable antiprismatic graph, see [3]). Now, assume that $G$ is 3 -substantial and not three-cliqued. Thus, by Theorem 8.3, $G$ is either the complement of a cycle of triangles graph, or the complement of a ring of five, or the complement of a mantled $L\left(K_{3,3}\right)$. Hence, by Lemmas 8.8, 8.9 and 8.10, $\operatorname{cc}(G) \leq n-1$.

## 9 Non-orientable antiprismatic graphs

The main goal of this section is to prove Theorem 4.6, which we restate as follows.
Theorem 9.1. Let $G$ be a non-orientable antiprismatic graph on $n$ vertices. Then $\operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to a twister.

Chudnovsky and Seymour in [4] proved that every non-orientable antiprismatic graph can be obtained from graphs in some explicitly described classes, whose union is called the menagerie, by applying a certain operation, what we call "extension" in the sequel. Unfortunately, despite of the other constructions like thickening and worn hex-chain, we are mostly unable to extend a clique covering of a graph to a clique covering of its extensions. On the other hand, the graphs in each of the classes of the menagerie require an exclusive method to handle their clique covering and the worse is that for most of these classes, the process is divided into several cases that should be treated separately. For these reasons, we find it best not to use the structure theorem of [4] directly and instead follow their approach in finding the structure theorem. More details will appear later on. Also, in [4], everything is stated for prismatic graphs, so we have to reformulate the definitions and statements in terms of the complements and for the sake of convenience, we maintain the titles of the classes. Furthermore, note that all three-cliqued antiprismatic graphs are orientable (see 4.2 in [3]) which have already been handled in previous section. Therefore, for simplicity, the graphs are generally assumed not to be three-cliqued.
Let us begin with recalling a couple of definitions from [4]. Consider a graph with the vertex set $\left\{v_{1}, \cdots, v_{9}\right\}$ and edges as follows; $\left\{v_{4}, v_{5}, v_{6}\right\}$ and $\left\{v_{7}, v_{8}, v_{9}\right\}$ are both triangles, and for $i=1,2,3$, $v_{i}$ is complete to $\left\{v_{4}, \ldots, v_{9}\right\} \backslash\left\{v_{i+3}, v_{i+6}\right\}$. This graph is the complement of a rotator, defined in [4] (see Figure 2). Also, a pair $(u, v)$ of vertices of a graph $G$ is said to be a square-forcer in $G$, if $u, v$ are nonadjacent and the set of vertices nonadjacent to both $u, v$ is stable. As a first step to prove Theorem 9.1, we are going to show that the complement of a counterexample to Theorem 9.1 contains an induced rotator and also excludes a square-forcer.

Theorem 9.2. Let $G$ be a counterexample to Theorem 9.1. Then $\bar{G}$ contains an induced rotator and contains no square-forcer.

The proof of Theorem 9.2, given in Subsection 9.1, uses a decomposition theorem in [4] for those non-orientable antiprismatic graphs whose complement either contain no induced rotator or contain a square-forcer. For the case of non-orientable antiprismatic graphs whose complement contain an induced rotator and no square-forcer, there is another decomposition theorem in [4]. Nevertheless, we prefer not to use this theorem, and in Subsections 9.2, 9.3, 9.4, 9.5 and 9.6 , we tackle these graphs by a structural exploration similar to the one in [4], thereby completing the proof of Theorem 9.1.

### 9.1 Proof of Theorem 9.2

In order to prove Theorem 9.2, we need a result from [4]. We begin with a definition. Given a graph $G$ and a vertex $v \in V(G)$, by replicating $v$, we mean replacing the vertex $v$ with a nonempty clique $X_{v}$ such that every vertex in $X_{v}$ is adjacent to a vertex $u \in V(G) \backslash\{v\}$ if and only if $v$ is adjacent to $u$. Also, we say that the graph $H$ is an extension of $G$, if $H$ is obtained from $G$ by replicating the vertices in $\tilde{W}(G)$ and then adding edges between some pairs of nonadjacent vertices in $\cup_{v \in \tilde{W}(G)} X_{v}$. The following theorem is a direct consequence of 2.2 and 7.2 in [4] and determines the structure of non-orientable antiprismatic graphs whose complement excludes an induced rotator (see Appendix C for the definition of the classes $\mathcal{F}_{0}, \mathcal{F}_{1}, \mathcal{F}_{2}, \mathcal{F}_{3}$ and $\mathcal{F}_{4}$ ).

Theorem 9.3. [4] Let $G$ be a non-orientable antiprismatic graph such that $\bar{G}$ contains no induced rotator. Then, $G$ is an extension of a member of $\mathcal{F}_{0} \cup \mathcal{F}_{1} \cup \mathcal{F}_{2} \cup \mathcal{F}_{3} \cup \mathcal{F}_{4}$.

Remark 9.4. In fact, Theorem 9.3 is the corrected version of 7.2 in [4], whose proof is flawed. More precisely, in Statement (6) in the proof, a number of graphs on between 12 and 18 vertices arise which are claimed to be isomorphic to a subgraph of the Schläfli graph (see Subsection 9.2 for the definition) by introducing an isomorphism map. Nevertheless, this map is actually not an isomorphism. The worse is that in some cases, these graphs are not even isomorphic to any subgraph of the Schläfli graph. To correct the proof, we introduced the additional class $\mathcal{F}_{0}$ in Appendix C , including these graphs and consequently Theorem 9.3 is valid in its present form.

Now, Theorem 9.3 together with $2.2,5.1$ and 8.2 in [4] implies the following theorem (for the definition of the graphs of parallel-square type and skew-square type, see Appendix C). Note that if $G$ is an extension of an antiprismatic graph $H$, where $\bar{G}$ contains an induced rotator and a square-forcer, then $\bar{H}$ also contains an induced rotator and a square-forcer (we leave the reader to verify it, also see the proof of 8.1 in [4]).

Theorem 9.5. [4] Let $G$ be a non-orientable antiprismatic graph such that $\bar{G}$ either contains no induced rotator or contains a square-forcer. Then $G$ is an extension of either a graph of parallelsquare type, or a graph of skew-square type, or a member of $\mathcal{F}_{0} \cup \mathcal{F}_{1} \cup \mathcal{F}_{2} \cup \mathcal{F}_{3} \cup \mathcal{F}_{4}$.

Through the following seven lemmas, we investigate the clique cover number of the extensions of the graphs in these classes, starting with the graphs of parallel-square type. Indeed, in the following lemma, we prove something more which will be also used in the proof of Lemma 9.10.

Lemma 9.6. Let $H_{0}$ be a graph of parallel-square type with $V\left(H_{0}\right)=A \cup B \cup C \cup D \cup Z \cup\{u, v, x, y\}$ as in the definition. Assume that $H$ is obtained from $H_{0}$ by possibly deleting the vertex $u$. Also, if $u \notin V(H)$, then there exist vertices $a_{i_{0}} \in A, b_{j_{0}} \in B, c_{k_{0}} \in C$ and $d_{l_{0}} \in D$ such that $a_{j_{0}}, a_{k_{0}} \notin A$, $a_{l_{0}} \notin A \backslash\left\{a_{i_{0}}\right\}, b_{i_{0}}, b_{l_{0}} \notin B, b_{k_{0}} \notin B \backslash\left\{b_{j_{0}}\right\}, c_{i_{0}}, c_{l_{0}} \notin C, c_{j_{0}} \notin C \backslash\left\{c_{k_{0}}\right\}, d_{j_{0}}, d_{k_{0}} \notin D$ and $d_{i_{0}} \notin D \backslash\left\{d_{l_{0}}\right\}$. Also, suppose that $H$ is not three-cliqued. Then, for every extension $G$ of $H$ on $n$ vertices, we have $\operatorname{cc}(G) \leq n-1$.
Proof. Define $U=V(H) \cap\{u\}$. Let $\tilde{A} \subseteq A, \tilde{B} \subseteq B, \tilde{C} \subseteq C$ and $\tilde{D} \subseteq D$ be the sets of vertices which are respectively complete to $B \cup D, A \cup \bar{C}, B$ and $A$. It is easy to check that $\tilde{W}(H) \subseteq \tilde{A} \cup \tilde{B} \cup \tilde{C} \cup \tilde{D}$ (for instance, if say $u \in \tilde{W}(H)$, then $C \cup D \cup Z \cup\{x\}$ is a clique and consequently, $\bar{V}(H)$ is the union
of three cliques $A \cup\{u, y\}, B \cup\{v\}$ and $C \cup D \cup Z \cup\{x\}$, a contradiction). For every $v \in V(H)$, let $X_{v} \subseteq V(G)$ be as in the definition of the extension, if $v \in \tilde{W}(H)$, and let $X_{v}=\{v\}$, otherwise. Also, for every $S \subseteq V(H)$, let $X_{S}=\cup_{s \in S} X_{s}$ (note that $X_{Z}=Z, X_{U}=U$ and $v, x, y \in V(G)$ ). We need the following.
(1) All the members of $\mathscr{N}_{G}\left[X_{A} ; X_{B} \cup X_{C}\right], \mathscr{N}_{G}\left[X_{B} ; X_{A} \cup X_{D}\right], \mathscr{N}_{G}\left[X_{C} ; X_{A} \cup X_{B}\right], \mathscr{N}_{G}\left[X_{C} ; X_{D} \cup X_{A}\right]$, $\mathscr{N}_{G}\left[X_{D} ; X_{A} \cup X_{B}\right]$ and $\mathscr{N}_{G}\left[X_{D} ; X_{B} \cup X_{C}\right]$ are cliques of $G$.

For if say there exists $a \in X_{A}$ adjacent to $b \in X_{B}$ and $c \in X_{C}$, where $b, c$ are nonadjacent, then $c \in W(H)$, say $c=c_{j}$, and so $a=a_{j} \in W(H)$ and $b=b_{j} \in W(H)$. This implies that $a$ is nonadjacent to $b$, a contradiction. This proves (1).

Now, we conclude
(2) If $|U|=|Z|=1$, then $\operatorname{cc}(G) \leq n-1$.

By (1), the family $\mathscr{N}_{G}\left[X_{A} ; X_{B} \cup X_{C} \cup Z\right] \cup \mathscr{N}_{G}\left[X_{B} ; X_{C} \cup Z\right] \cup \mathscr{N}_{G}\left[X_{C} ; X_{D} \cup Z\right] \cup \mathscr{N}_{G}\left[X_{D} ; X_{A} \cup X_{B} \cup\right.$ $Z] \cup\{A \cup\{u, y\}, B \cup\{u, v\}, C \cup\{v, x\}, D \cup\{x, y\}\}$ is a clique covering for $G$ of size $\left|X_{A}\right|+\left|X_{B}\right|+$ $\left|X_{C}\right|+\left|X_{D}\right|+4=n-1$. This proves (2).

Moreover, we have
(3) If either $U$ or $Z$ is empty, then $E_{H}(A, C)$ and $E_{H}(B, D)$ are both nonempty.

For if say $E_{H}(A, C)=\emptyset$, then every vertex in $B$ is complete to either $A$ or $C$, and defining $B^{\prime} \subseteq B$ as the set of vertices in $B$ that are complete to $A$, implies that $V(H)$ is the disjoint union of three cliques $A \cup B^{\prime} \cup U \cup Z,\left(B \backslash B^{\prime}\right) \cup C \cup\{v\}$ and $D \cup\{x, y\}$, a contradiction. This proves (3).

Now, by (2), we may suppose that either $U$ or $Z$ is empty. Thus, by (3), we may assume that $E_{H}(A, C)=\left\{a_{i_{1}} c_{i_{1}}, \ldots, a_{i_{\ell}} c_{i_{\ell}}\right\}$ and $E_{H}(B, D)=\left\{b_{j_{1}} d_{j_{1}}, \ldots, b_{j_{\ell^{\prime}}} d_{j_{\ell^{\prime}}}\right\}$ for some $\ell, \ell^{\prime} \geq 1$. Henceforth, w.l.o.g. assume that $\left|X_{A}\right| \leq\left|X_{B}\right|$, if $\left|X_{A}\right|=\left|X_{B}\right|$, then $\left|X_{\left\{c_{i_{1}}, \ldots, c_{i_{\ell}}\right\}}\right| \geq\left|X_{\left\{d_{j_{1}}, \ldots, d_{j_{\ell}}\right\}}\right|$, if also $\left|X_{\left\{c_{i_{1}}, \ldots, c_{i_{\ell}}\right\}}\right|=\left|X_{\left\{d_{j_{1}}, \ldots, d_{j^{\prime}}\right\}}\right|$, then $\left|X_{C} \backslash X_{\left\{c_{i_{1}}, \ldots, c_{i_{\ell}}\right\}}\right| \geq\left|X_{D} \backslash X_{\left\{d_{j_{1}}, \ldots, d_{j^{\prime}}\right\}}\right|$ and if also $\mid X_{C} \backslash$ $X_{\left\{c_{i_{1}}, \ldots, c_{i_{\ell}}\right\}}\left|=\left|X_{D} \backslash X_{\left\{d_{j_{1}}, \ldots, d_{\left.j_{\ell^{\prime}}\right\}}\right\}}\right|\right.$, then $| E\left(X_{b_{j_{0}}}, X_{C} \backslash X_{\left\{c_{i_{1}}, \ldots, c_{i_{\ell}}\right\}}\right)\left|\geq\left|E\left(X_{a_{i_{0}}}, X_{D} \backslash X_{\left\{d_{j_{1}}, \ldots, d_{j_{\ell^{\prime}}}\right\}}\right)\right|\right.$. Also, we use the following easy observation whose proof is left to the reader.
(4) For every $i \in\left\{i_{1}, \ldots, i_{\ell}\right\}$ (resp. $j \in\left\{j_{1}, \ldots, j_{\ell}\right\}$ ) and every $a \in X_{a_{i}}$ and $c \in X_{c_{i}}$ (resp. $b \in X_{b_{j}}$ and $d \in X_{d_{j}}$ ), we have $N_{G}\left[a, X_{B}\right]=N_{G}\left[c, X_{B}\right]$ and $N_{G}\left[a, X_{D}\right]=N_{G}\left[c, X_{D}\right]\left(r e s p . N_{G}\left[b, X_{A}\right]=N_{G}\left[d, X_{A}\right]\right.$ and $\left.N_{G}\left[b, X_{C}\right]=N_{G}\left[d, X_{C}\right]\right)$.
Consider the following family,

$$
\begin{aligned}
\mathscr{C}= & \mathscr{N}_{G}\left[X_{A} ; X_{B} \cup X_{C} \cup Z\right] \cup \mathscr{N}_{G}\left[X_{C} \backslash X_{\left\{c_{i_{1}}, \ldots, c_{i}\right\}} ; X_{B} \cup\{v\}\right] \cup \mathscr{N}_{G}\left[X_{D} ; X_{B} \cup X_{C} \cup Z\right] \cup \\
& \mathscr{N}_{G}\left[X_{A} ; X_{D} \cup\{y\}\right] \cup\left\{X_{A} \cup U \cup\{y\}, X_{B} \cup U \cup\{v\}, X_{C} \cup\{v, x\}, X_{D} \cup\{x, y\}\right\}
\end{aligned}
$$

whose elements by (1) are cliques of $G$. Note that by (4), the edges in $E_{G}\left(X_{\left\{c_{i_{1}}, \ldots, c_{i}\right\}}, X_{B}\right)$ are covered by the cliques in $\mathscr{N}_{G}\left[X_{A} ; X_{B} \cup X_{C} \cup Z\right]$. If $Z=\emptyset$, then $\mathscr{C}$ is a clique covering. Also, if $Z$ is nonempty, then $U=\emptyset$ and $\mathscr{C}$ is also a clique covering (because $X_{a_{i_{0}}}$ is complete to $X_{B}$ and $X_{d_{l_{0}}}$ is complete to $X_{C}$ and thus the edges in $E_{G}\left(Z, X_{B}\right)$ and $E_{G}\left(Z, X_{C}\right)$ are covered by the cliques in $\mathscr{N}_{G}\left[X_{A} ; X_{B} \cup X_{C} \cup Z\right]$ and $\left.\mathscr{N}_{G}\left[X_{D} ; X_{B} \cup X_{C} \cup Z\right]\right)$. Also, $|\mathscr{C}|=n-\left(\left|X_{B}\right|-\left|X_{A}\right|\right)-\left|X_{\left\{c_{i_{1}}, \ldots, c_{i}\right\}}\right|-|U|-|Z|+1$. Hence, if either $U \neq \emptyset$, or $Z \neq \emptyset$, or $\left|X_{B}\right|-\left|X_{A}\right| \geq 1$ or $\left|X_{\left\{c_{i_{1}}, \ldots, c_{i_{\ell}}\right\}}\right| \geq 2$, then we are done. So, we may assume that $U=Z=\emptyset,\left|X_{A}\right|=\left|X_{B}\right|$ and $\left|X_{\left\{c_{i_{1}}, \ldots, c_{i_{\ell}}\right\}}\right|=\left|X_{\left\{d_{j_{1}}, \ldots, d_{\left.j_{\ell^{\prime}}\right\}}\right\}}\right|=1$ (i.e. $\ell=\ell^{\prime}=1$ ).
In this case, let $\mathscr{C}^{\prime}$ be obtained from $\mathscr{C}$ by removing the clique $X_{B} \cup U \cup\{v\}$. Since $X_{a_{i_{0}}}$ is complete to $X_{B}$ and $X_{c_{k_{0}}}$ is complete to $X_{B} \backslash X_{b_{j_{0}}}$, the only edges of $G$ that are possibly not covered by the cliques in $\mathscr{C}^{\prime}$ are the edges in $E_{G}\left(\{v\}, X_{b_{j_{0}}}\right)$. Now, if there exists a vertex $c_{k} \in C$, where $k \notin$ $\left\{j_{0}, i_{1}\right\}$, then $X_{c_{k}}$ is complete to $X_{b_{j_{0}}}$, and thus $\mathscr{C}^{\prime}$ is a clique covering for $G$ of size $n-\left(\left|X_{B}\right|-\right.$ $\left.\left|X_{A}\right|\right)-\left|X_{c_{i_{1}}}\right| \leq n-1$. Therefore, we may assume that $j_{0}=k_{0}$ and $C=\left\{c_{i_{1}}, c_{k_{0}}\right\}$. This implies that $b_{j_{0}}, c_{k_{0}} \in W(H)$ and thus $\left|X_{C} \backslash X_{c_{i_{1}}}\right|=\left|X_{c_{k_{0}}}\right|=1$ and $E\left(X_{b_{j_{0}}}, X_{C} \backslash X_{c_{i_{1}}}\right)=\emptyset$. Therefore, $\left|X_{D} \backslash X_{d_{j_{1}}}\right|=1$ and so $E\left(X_{a_{i_{0}}}, X_{D} \backslash X_{d_{j_{1}}}\right)=\emptyset$. In particular, $X_{C}=\left\{c_{i_{1}}, c_{k_{0}}\right\}, X_{D}=\left\{d_{j_{1}}, d_{l_{0}}\right\}$,
$i_{0}=l_{0}$ and $j_{0}=k_{0}$. Now, if $\left|X_{A}\right|=\left|X_{B}\right| \geq 3$, then the family of cliques $\mathscr{N}_{G}\left[X_{C} \cup X_{D} ; X_{A} \cup X_{B}\right] \cup$ $\mathscr{N}_{G}\left[X_{C} ; X_{D}\right] \cup \mathscr{N}_{G}\left[X_{A} \backslash\left\{a_{i_{1}}\right\} ; X_{B}\right] \cup\left\{X_{A} \cup\{y\}, X_{B} \cup\{v\}, X_{C} \cup\{v, x\}, X_{D} \cup\{x, y\}\right\}$ is a clique covering for $G$ of size $9+\left|X_{A}\right|=n-\left|X_{B}\right|+2 \leq n-1$. Also, if $\left|X_{A}\right|=\left|X_{B}\right|=2$ (i.e. $X_{A}=\left\{a_{i_{0}}, a_{i_{1}}\right\}$ and $\left.X_{B}=\left\{b_{j_{0}}, b_{j_{1}}\right\}\right)$, then the family of cliques $\left\{N_{G}\left[a_{i_{1}}, X_{B} \cup X_{C}\right], N_{G}\left[a_{i_{1}}, X_{C} \cup X_{D}\right], N_{G}\left[b_{j_{1}}, X_{C} \cup\right.\right.$ $\left.\left.X_{D}\right], N_{G}\left[b_{j_{1}}, X_{D} \cup X_{A}\right],\left\{a_{i_{0}}, b_{j_{0}}\right\},\left\{c_{k_{0}}, d_{l_{0}}\right\}, X_{A} \cup\{y\}, X_{B} \cup\{v\}, X_{C} \cup\{v, x\}, X_{D} \cup\{x, y\}\right\}$ is a clique covering for $G$ of size $10=n-1$. This completes the proof of Lemma 9.6.

In the following, we examine the extensions of the graphs in the class $\mathcal{F}_{0}$.
Lemma 9.7. Let $G$ be a graph in $\mathcal{F}_{0}$ and $G^{\prime}$ be an extension of $G$ on $n$ vertices. Then $\operatorname{cc}\left(G^{\prime}\right) \leq n-1$.
Proof. Let $I_{1}, I_{2}$ and $I_{3}$ be as in the definition of the class $\mathcal{F}_{0}$. Let $R_{2}^{1}=V(G) \cap\left\{r_{2}^{1}\right\}$ and $R_{1}^{2}=$ $V(G) \cap\left\{r_{1}^{2}\right\}$. This is evident that $\tilde{W}(G)=V(G) \cap\left\{s_{1}^{3}, s_{3}^{3}, t_{3}^{2}, t_{3}^{3}\right\}$. For every $v \in \tilde{W}(G)$, let $X_{v} \subseteq V\left(G^{\prime}\right)$ be as in the definition of extension, and for every $v \in\left\{s_{1}^{3}, s_{3}^{3}, t_{3}^{2}, t_{3}^{3}\right\} \backslash V(G)$, define $X_{v}=\emptyset$. Now, let $\mathscr{C}$ be the family containing the following 12 cliques

$$
\begin{array}{lll}
\left\{r_{1}^{1}, r_{1}^{3}, s_{1}^{2}\right\} \cup R_{1}^{2}, & \left\{r_{1}^{1}, r_{1}^{3}, s_{2}^{3}\right\} \cup X_{s_{1}^{3}} \cup X_{s_{3}^{3}} \cup R_{1}^{2}, & \left\{r_{2}^{2}, r_{2}^{3}, s_{1}^{1}\right\} \cup R_{2}^{1}, \\
\left\{r_{2}^{2}, r_{2}^{3}, s_{2}^{3}\right\} \cup X_{s_{1}^{3}} \cup X_{s_{3}^{3}} \cup R_{2}^{1}, & \left\{r_{3}^{1}, r_{3}^{2}, s_{1}^{1}, s_{1}^{2}\right\} \cup X_{t_{3}^{2}} \cup X_{t_{3}^{3}}, & \left\{s_{1}^{1}, s_{1}^{2}, t_{1}^{2}, t_{2}^{2}\right\} \cup X_{s_{1}^{3}}, \\
\left\{s_{2}^{3}, t_{3}^{1}\right\} \cup X_{s_{3}^{3}}, & \left\{r_{1}^{3}, r_{2}^{3}, t_{1}^{2}, t_{2}^{2}\right\} \cup X_{s_{3}^{3}}, & \left\{r_{2}^{2}, r_{3}^{2}, t_{1}^{2}\right\} \cup R_{1}^{2}, \\
\left\{t_{2}^{2}, r_{1}^{1}, r_{3}^{1}\right\} \cup R_{2}^{1}, & \left\{t_{3}^{1}, r_{1}^{1}, r_{3}^{1}\right\} \cup X_{t_{3}^{2}} \cup X_{t_{3}^{3}} \cup R_{2}^{1}, & \left\{t_{3}^{1}, r_{2}^{2}, r_{3}^{2}\right\} \cup X_{t_{3}^{2}} \cup X_{t_{3}^{3}} \cup R_{1}^{2} .
\end{array}
$$

It can be easily seen that $\mathscr{C} \cup \mathscr{N}_{G^{\prime}}\left[X_{t_{3}^{2}} ; X_{s_{1}^{3}} \cup X_{s_{3}^{3}} \cup\left\{t_{1}^{2}, t_{2}^{2}\right\}\right] \cup \mathscr{N}_{G^{\prime}}\left[X_{t_{3}^{3}} ; X_{s_{1}^{3}} \cup X_{s_{3}^{3}} \cup\left\{s_{2}^{3}\right\}\right]$ is a clique covering for $G^{\prime}$ of size $n-\left|R_{2}^{1}\right|-\left|R_{1}^{2}\right|-\left|X_{s_{1}^{3}}\right|-\left|X_{s_{3}^{3}}\right|$. Thus, if at least one of the sets $R_{2}^{1}, R_{1}^{2}, X_{s_{1}^{3}}$ or $X_{s_{3}^{3}}$ is nonempty, then we are done. Now, assume that $R_{2}^{1}=R_{1}^{2}=X_{s_{1}^{3}}=X_{s_{3}^{3}}=\emptyset$ and in $\mathscr{C}$, replace the cliques $\left\{s_{1}^{1}, s_{1}^{2}, t_{1}^{2}, t_{2}^{2}\right\} \cup X_{s_{1}^{3}}$ and $\left\{s_{2}^{3}, t_{3}^{1}\right\} \cup X_{s_{3}^{3}}$ with the cliques $\left\{s_{1}^{1}, s_{1}^{2}, t_{1}^{2}, t_{2}^{2}\right\} \cup X_{t_{3}^{2}}$ and $\left\{s_{2}^{3}, t_{3}^{1}\right\} \cup X_{t_{3}^{3}}$ respectively, thereby providing a clique covering for $G$ of size $n-\left|X_{t_{3}^{2}}\right|-\left|X_{t_{3}^{3}}\right|$. Hence, if either $X_{t_{3}^{2}}$ or $X_{t_{3}^{3}}$ is nonempty, then we are done. Finally, assume that $X_{t_{3}^{2}}=X_{t_{3}^{3}}=\emptyset$. In this case, $\left|V\left(G^{\prime}\right)\right|=12$ and the following 11 cliques

$$
\begin{array}{llllll}
\left\{s_{1}^{2}, t_{2}^{2}, r_{1}^{1}, r_{3}^{1}\right\}, & \left\{s_{1}^{1}, t_{1}^{2}, t_{2}^{2}, r_{2}^{3}\right\}, & \left\{s_{1}^{1}, s_{1}^{2}, r_{3}^{1}, r_{3}^{2}\right\}, & \left\{s_{1}^{1}, t_{1}^{2}, r_{2}^{2}, r_{3}^{2}\right\}, & \left\{s_{1}^{2}, t_{1}^{2}, r_{1}^{3}\right\}, & \left\{t_{2}^{2}, r_{1}^{3}, r_{2}^{3}\right\}, \\
\left\{s_{2}^{3}, r_{1}^{1}, r_{1}^{3}\right\}, & \left\{t_{3}^{1}, r_{1}^{1}, r_{3}^{1}\right\}, & \left\{s_{2}^{3}, r_{2}^{2}, r_{2}^{3}\right\}, & \left\{t_{3}^{1}, r_{2}^{2}, r_{3}^{2}\right\}, & \left\{s_{2}^{3}, t_{3}^{1}\right\},
\end{array}
$$

provide a clique covering for $G^{\prime}$ of size $n-1$. This proves Lemma 9.7.
In the following, we look into the extensions of the graphs in the class $\mathcal{F}_{1}$.
Lemma 9.8. Let $G$ be a graph in $\mathcal{F}_{1}$ and $G^{\prime}$ be an extension of $G$ on $n$ vertices which is not threecliqued. Then $\operatorname{cc}\left(G^{\prime}\right) \leq n$ and equality holds if and only if $\overline{G^{\prime}}$ is isomorphic to a twister.

Proof. Let $A, B, R, s, t$ be as in the definition of the class $\mathcal{F}_{1}$. Since $G$ is antiprismatic, $R$ is complete to $A \cup B$. Let $\tilde{A} \subseteq A$ be the set of all vertices in $A$ which have no non-neighbour in $A$ and let $\tilde{B} \subseteq B$ be defined analogously. Note that from the definition, both $A$ and $B$ are the union of at most two disjoint cliques. Also, $B \cup R$ is nonempty (since otherwise $G$ and so $G^{\prime}$ is three-cliqued). Since $G$ is antiprismatic and $B \cup R$ is nonempty, $t$ is complete to $A$. Similarly, $s$ is complete to $B$. Now, if either $\tilde{A}=A$ or $\tilde{B}=B$, then $G$ and so $G^{\prime}$ is three-cliqued. This implies that $\tilde{A} \neq A$ and $\tilde{B} \neq B$. Hence, $\tilde{W}(G)=\tilde{A} \cup \tilde{B}$. Now, let $A=\tilde{A} \cup X \cup Y$, where $X=\left\{x_{1}, \ldots, x_{a}\right\}$ and $Y=\left\{y_{1}, \ldots, y_{a}\right\}$ and $x_{i} y_{i}, 1 \leq i \leq a$, are the only non-edges with both ends in $A$. Also, let $B=\tilde{B} \cup U \cup V$, where $U=\left\{u_{1}, \ldots, u_{b}\right\}$ and $V=\left\{v_{1}, \ldots, v_{b}\right\}$ and $u_{i} v_{i}, 1 \leq i \leq b$, are the only non-edges with both ends in $B$. By abuse of notation, denote the set $\cup_{v \in \tilde{A}} X_{v}$ of vertices of $G^{\prime}$ by $\tilde{A}$ and the same assumption is applied to $\tilde{B}$ (note that adjacency between $\tilde{A}$ and $\tilde{B}$ is arbitrary in $G^{\prime}$ ). If $R$ is empty, then $N_{\overline{G^{\prime}}}(s)=A \cup\{t\}$ and for every vertex $x \in A \cup\{t\}, N_{G^{\prime}}(x, B \cup\{s\})$ is a clique of $G^{\prime}$. Therefore, if $R$ is empty, then the result follows from Lemma 8.11 (by setting $u=s$ ). Hence, assume that $R$ is nonempty. Also, w.l.o.g. assume that $a \leq b$. Now, define

$$
\mathscr{C}=\mathscr{N}_{G^{\prime}}[A ; B \cup R] \cup \mathscr{N}_{G^{\prime}}[X ;(A \backslash X) \cup\{t\}] \cup \mathscr{N}_{G^{\prime}}[U ;(B \backslash U) \cup\{s\}] \cup\{X,(A \backslash X) \cup\{t\}\} .
$$

Note that since $\tilde{A} \neq A$, every vertex in $B$ has a neighbour in $A$, and so the edges in $E\left(R, V\left(G^{\prime}\right) \backslash R\right)$ are all covered by the cliques in $\mathscr{N}[A ; B \cup R]$. Also, w.l.o.g. we may assume that $U \subseteq N\left(x_{1}, B\right)$ and $V \subseteq N\left(y_{1}, B\right)$. Therefore, the cliques in $\mathscr{C}$ cover all edges of $G^{\prime}$ except possibly the edges in $E(V, \tilde{B} \cup\{s\})$. If $b \geq 2$, then these edges are covered by the cliques in $\mathscr{N}_{G^{\prime}}[U ;(B \backslash U) \cup\{s\}]$ and thus $\mathscr{C}$ is a clique covering for $G^{\prime}$ of size $|\mathscr{C}|=|A|+a+b+2 \leq|A|+|B|+2=n-1$. If $b=1$, then $a=1$ and in the collection $\mathscr{C}$, we may replace the clique $X$ with the clique $\left\{v_{1}, s\right\} \cup \tilde{B}$ to obtain a clique covering of size $|\mathscr{C}| \leq n-1$. This proves Lemma 9.8.

Prior to examination of the classes $\mathcal{F}_{2}, \mathcal{F}_{3}$ and $\mathcal{F}_{4}$, we need to recall an operation defined in [4] which is used in the definition of these classes in Appendix C.
Let $\tau=\{a, b, c\}$ be a triad of an antiprismatic graph $H$. In [4], $\tau$ is said to be a leaf triad of $H$ at $c$, if both $a$ and $b$ belong to only one triad of $H$ (namely $\tau$ ). Let $\tau=\{a, b, c\}$ be a leaf triad of $H$ at $c$. Define subsets $D_{1}, D_{2}, D_{3}$ of the set of non-neighbours of $c$ as follows. If $v$ is nonadjacent to $c$ in $H$, let

- $v \in D_{1}$ if $v$ belongs to a triad that does not contain $c$;
- $v \in D_{2}$ if $v \in W(H) \backslash \tau$, and every triad containing $v$ also contains $c$ (and hence is unique);
- $v \in D_{3}$ if $v \in \tilde{W}(H)$.

Thus, the four sets $D_{1}, D_{2}, D_{3}$ and $\{a, b\}$ are pairwise disjoint sets whose union is the set of nonneighbours of $c$ in $H$. Suppose that $D_{1}, D_{2}$ are both cliques. Let $A, B, C$ be three pairwise disjoint sets of new vertices, and let $G$ be obtained from $H$ by deleting $a, b$ and adding the new vertices in $A \cup B \cup C$, with adjacency as follows:

- $A, B$ and $C$ are cliques;
- every vertex in $A$ has at most one non-neighbour in $B$, and vice versa;
- every vertex in $V(H) \backslash\{a, b\}$ adjacent to $a$ (resp. b) in $H$ is complete to $A$ (resp. $B$ ) in $G$, and every vertex in $V(H) \backslash\{a, b\}$ nonadjacent to $a$ (resp. $b$ ) in $H$ is anticomplete to $A$ (resp. $B$ );
- every vertex in $C$ is anticomplete to $D_{1} \cup D_{3}$, and complete to $V(H) \backslash\left(D_{1} \cup D_{3} \cup\{a, b\}\right)$;
- every vertex in $C$ is nonadjacent to exactly one end of every non-edge between $A$ and $B$, and nonadjacent to every vertex in $A \cup B$ adjacent to all other vertices in $A \cup B$.

The graph $G$ is said to be obtained from $H$ by exponentiating the leaf triad $\tau=\{a, b, c\}$. We leave the reader to check that $G$ is also antiprismatic and if $H$ is three-cliqued, then so is $G$. Now, in the following, we give a method to extend a clique covering of $H$ (and its extensions) to a clique covering of $G$ (and its extensions).

Lemma 9.9. Assume that $H$ is an antiprismatic graph, $\tau=\{a, b, c\}$ is a leaf triad of $H$ at $c$ and $G$ is obtained from $H$ by exponentiating $\tau$. Also, suppose that for some fixed number $k$, every extension $H^{\prime}$ of $H$ satisfies $\operatorname{cc}\left(H^{\prime}\right) \leq\left|V\left(H^{\prime}\right)\right|-k$ and if $H+a b$ is not three-cliqued, then every extension $H^{\prime \prime}$ of $H+$ ab satisfies $\operatorname{cc}\left(H^{\prime \prime}\right) \leq\left|V\left(H^{\prime \prime}\right)\right|-k$. Let $G^{\prime}$ be an extension of $G$ which is not three-cliqued. Then, we have $\operatorname{cc}\left(G^{\prime}\right) \leq\left|V\left(G^{\prime}\right)\right|-k$.
Proof. Let $A, B, C, D_{1}, D_{2}$ and $D_{3}$ be subsets of $V(G)$ as in the definition, and also let $\tilde{A}$ (resp. $\tilde{B}$ ) be the set vertices in $A$ (resp. $B$ ) which are complete to $B$ (resp. $A$ ). Note that $\tilde{W}(G)=\tilde{W}(H) \cup \tilde{A} \cup \tilde{B} \cup C$. Now, consider an extension $G^{\prime}$ of $G$ and by abuse of notation, let $\tilde{A}=\bigcup_{v \in \tilde{A}} X_{v}, \tilde{B}=\bigcup_{v \in \tilde{B}} X_{v}$ and $C=\bigcup_{v \in C} X_{v}$ (note that adjacency between $C$ and $\tilde{A} \cup \tilde{B} \cup D_{3}$ is arbitrary in $G^{\prime}$ ). We assume that $G^{\prime}$ is not three-cliqued and we are going to provide a clique covering for $G^{\prime}$ of size at most $\left|V\left(G^{\prime}\right)\right|-k$. For every $z \in C$, define $C_{z}=D_{2} \cup N_{G^{\prime}}\left[z, A \cup B \cup D_{3}\right]$. Note that since $H$ is antiprismatic, $A \cup B$ is complete to $D_{2} \cup D_{3}$, and since $D_{3} \subseteq \tilde{W}(H) \cap N_{\bar{H}}(c)$ and $D_{2} \subseteq N_{\bar{H}}(c), D_{2} \cup D_{3}$ is a clique. Also, every vertex $z \in C$ is adjacent to exactly one end of every non-edge between $A$ and $B$. Therefore, $C_{z}$ is a clique of $G^{\prime}$.

First, assume that $\tilde{A}=A$ and thus, $\tilde{B}=B$. In this case, it is easy to see that $G^{\prime}$ is obtained from an extension $H^{\prime \prime}$ of $H+a b$ by adding the new set of vertices $C$, which is complete to $V\left(H^{\prime \prime}\right) \backslash\left(D_{1} \cup\right.$ $D_{3} \cup A \cup B$ ) and anticomplete to $D_{1}$ and adjacency between $C$ and $A \cup B \cup D_{3}$ is arbitrary. Thus, $\left|V\left(G^{\prime}\right)\right|=\left|V\left(H^{\prime \prime}\right)\right|+|C|$. Also, since $G^{\prime}$ is not three-cliqued, neither is $H+a b$ (because every vertex adjacent to $c$ in $H+a b$ is complete to $C$ in $G^{\prime}$ ). Thus, by the assumption, there exists a clique covering $\mathscr{C}$ for $H^{\prime \prime}$ of size at most $\left|V\left(H^{\prime \prime}\right)\right|-k$. For every clique in $\mathscr{C}$ containing $c$, replace $c$ with $C \cup\{c\}$, and also add the clique $C_{z}$, for all $z \in C$, thereby obtaining a clique covering for $G^{\prime}$ of size at most $\left|V\left(H^{\prime \prime}\right)\right|-k+|C|=\left|V\left(G^{\prime}\right)\right|-k$, as desired.
Next, assume that $\tilde{A} \neq A$ and then $\tilde{B} \neq B$. It is clear that $G^{\prime}$ is obtained from an extension $H^{\prime}$ of $H$ by exponentiating the leaf triad $\tau$ and then adding some edges between $C \cup \tilde{A} \cup \tilde{B}$ and $\tilde{W}\left(H^{\prime}\right)$, and between $C$ and $\tilde{A} \cup \tilde{B}$ as well. Also, $\left|V\left(G^{\prime}\right)\right|=\left|V\left(H^{\prime}\right)\right|+|A|+|B|+|C|-2$. By the assumption, there exists a clique covering $\mathscr{C}$ for $H^{\prime}$ of size at most $\left|V\left(H^{\prime}\right)\right|-k$. For every clique in $\mathscr{C}$ containing $a, b$ or $c$, replace $a, b$ and $c$ with $A, B$ and $C \cup\{c\}$, respectively to obtain a collection of cliques of $G^{\prime}$, called $\mathscr{C}^{\prime}$, covering all edges of $G^{\prime}$ except the edges between $A$ and $B$ and some edges between vertices in $\tilde{W}\left(G^{\prime}\right)$. Let $F$ be the subgraph of $G^{\prime}$ induced on the set of these uncovered edges. Since $N_{\bar{H}}(a, \tilde{W}(H))$ is a clique of $H$, for every $x \in \tilde{A}, N_{F}\left[x, \tilde{W}\left(H^{\prime}\right)\right]$ is a clique of $G^{\prime}$. Now, for every $x \in \tilde{A}$, define $C_{x}=B \cup N_{F}\left[x, \tilde{W}\left(H^{\prime}\right)\right]$ which is a clique of $G^{\prime}$ (because in $H$ every non-neighbour of $a$ in $V(H) \backslash\{c\}$ is a neighbour of $b$ ). Also, for every $x \in A \backslash \tilde{A}$, define $C_{x}=N_{G^{\prime}}[x, B \backslash \tilde{B}]$ and for every $y \in \tilde{B}$, define $C_{y}=A \cup N_{F}[y, \tilde{W}(H)]$, which are obviously cliques of $G^{\prime}$. Now, the cliques in $\left\{C_{x}: x \in A \cup \tilde{B} \cup C\right\}$ cover all the edges of $F$, and thus adding them to $\mathscr{C}^{\prime}$ yields a clique covering for $G^{\prime}$ of size at most $\left|V\left(H^{\prime}\right)\right|-k+|A|+|\tilde{B}|+|C|$. Now, if $|B \backslash \tilde{B}| \geq 2$, then we are done. Otherwise, if $|B \backslash \tilde{B}|=1$, then for $x \in A \backslash \tilde{A}$, we have $C_{x}=\{x\}$. Thus, removing $C_{x}, x \in A \backslash \tilde{A}$, yields the desired clique covering of size at most $\left|V\left(G^{\prime}\right)\right|-k$. This proves Lemma 9.9.

The following inquires into the extensions of the graphs in both classes $\mathcal{F}_{2}$ and $\mathcal{F}_{3}$.
Lemma 9.10. Let $G$ be a graph in $\mathcal{F}_{2} \cup \mathcal{F}_{3}$ and $G^{\prime}$ be an extension of $G$ on $n$ vertices which is not three-cliqued. Then $\operatorname{cc}\left(G^{\prime}\right) \leq n-1$.

Proof. First, suppose that $G \in \mathcal{F}_{2}$ and let $H$ be the graph of parallel square type with $V(H)=$ $A \cup B \cup C \cup D \cup\{u, v, x, y\}$, where $G$ is obtained from $H$ by exponentiating the leaf triad $\left\{a_{1}, b_{1}, x\right\}$, as in the definition of $\mathcal{F}_{2}$. Since $G^{\prime}$ is not three-cliqued, neither are $G$ and $H$. Thus, by Lemma 9.6, every extension $H^{\prime}$ of $H$ admits a clique covering of size at most $\left|V\left(H^{\prime}\right)\right|-1$. On the other hand, note that $H+a_{1} b_{1}$ is also a graph of parallel-square type. To see this, let $\ell=1+\max \left\{i: a_{i} \in\right.$ $A$ or $b_{i} \in B$ or $c_{i} \in C$ or $\left.d_{i} \in D\right\}$. Now, replacing the vertex $a_{1}$ with $a_{\ell}$ in the set $A$ yields a graph of parallel-square type which is isomorphic to $H+a_{1} b_{1}$. Hence, if $H+a_{1} b_{1}$ is not three-cliqued, then by Lemma 9.6, every extension $H^{\prime \prime}$ of $H+a_{1} b_{1}$ admits a clique covering of size at most $\left|V\left(H^{\prime \prime}\right)\right|-1$. Consequently, by Lemma 9.9, $\operatorname{cc}\left(G^{\prime}\right) \leq n-1$.
Moreover, for the graph $G \in \mathcal{F}_{3}$, the result follows from a similar argument with the aid of Lemmas 9.6 and 9.9 (we leave the details to the reader).

The following investigates the extensions of the graphs in $\mathcal{F}_{4}$.
Lemma 9.11. Let $G$ be a graph in $\mathcal{F}_{4}$ and $G^{\prime}$ be an extension of $G$ on $n$ vertices which is not three-cliqued. Then $\operatorname{cc}\left(G^{\prime}\right) \leq n-1$.

Proof. Let $H$ be the subgraph of Schläfli graph induced on $V(H)=Y \cup\left\{s_{j}^{i}:(i, j) \in I\right\} \cup\left\{t_{1}^{1}, t_{2}^{2}, t_{3}^{3}\right\}$, where $Y$ and $I$ are as in the definition of the class $\mathcal{F}_{4}$ and $G$ be obtained from $H$ by exponentiating the leaf triad $\left\{t_{1}^{1}, t_{2}^{2}, t_{3}^{3}\right\}$. Also, for every $j \in\{1,2,3\}$, let $R_{j}^{3}=V(H) \cap\left\{r_{j}^{3}\right\}, S_{j}=V(H) \cap\left\{s_{j}^{1}, s_{j}^{2}, s_{j}^{3}\right\}$ and $S^{j}=V(H) \cap\left\{s_{1}^{j}, s_{2}^{j}, s_{3}^{j}\right\}$. Now the following cliques

$$
\begin{array}{llllll}
S_{1} \cup\left\{t_{2}^{2}\right\}, & S_{1} \cup\left\{t_{3}^{3}\right\}, & S_{2} \cup\left\{t_{1}^{1}\right\}, & S_{2} \cup\left\{t_{3}^{3}\right\}, & S_{3} \cup\left\{t_{1}^{1}\right\}, & S_{3} \cup\left\{t_{2}^{2}\right\}, \\
R_{1}^{3} \cup R_{2}^{3} \cup S^{3}, & R_{2}^{3} \cup R_{3}^{3} \cup S^{1}, & R_{1}^{3} \cup R_{3}^{3} \cup S^{2}, & Y \cup\left\{t_{1}^{1}\right\}, & Y \cup\left\{t_{2}^{2}\right\}, &
\end{array}
$$

form a clique covering $\mathscr{C}$ for $H$ of size $11 \leq|V(H)|-1$. Also, since $|I| \geq 8$, we have $\tilde{W}(H) \subseteq Y$, which is a clique of $H$. Thus, every extension $H^{\prime}$ of $H$ is in fact a thickening of $(H, \emptyset)$, and thus by Lemma 2.2, $H^{\prime}$ admits a clique covering of size at most $\left|V\left(H^{\prime}\right)\right|-1$.
Moreover, note that $\tilde{W}\left(H+t_{1}^{1} t_{2}^{2}\right)$ is contained in $\left\{t_{1}^{1}, t_{2}^{2}, t_{3}^{3}\right\} \cup Y$. Now, let $H^{\prime \prime}$ be an extension of $H+t_{1}^{1} t_{2}^{2}$, where $X_{v}, v \in \tilde{W}\left(H+t_{1}^{1} t_{2}^{2}\right)$, are as in the definition and for every $v \in W\left(H+t_{1}^{1} t_{2}^{2}\right)$, set $X_{v}=\{v\}$. For every clique $C$ of $H+t_{1}^{1} t_{2}^{2}$, let $X_{C}=\cup_{v \in C} X_{v}$ which is a clique of $H^{\prime \prime}$ and define $\mathscr{C}^{\prime}=\left\{X_{C}: C \in \mathscr{C}\right\}$. Now, in $\mathscr{C}^{\prime}$, merge the pairs $\left(X_{S_{3} \cup\left\{t_{1}^{1}\right\}}, X_{S_{3} \cup\left\{t_{2}^{2}\right\}}\right)$ and $\left(X_{Y \cup\left\{t_{1}^{1}\right\}}, X_{Y \cup\left\{t_{2}^{2}\right\}}\right)$ and add the cliques in $\mathscr{N}\left[X_{t_{3}^{3}} ; X_{Y \cup\left\{t_{1}, t_{2}^{2}\right\}}\right]$, thereby obtaining a clique covering for $H^{\prime \prime}$ of size $9+\left|X_{t_{3}^{3}}\right| \leq$ $\left|V\left(H^{\prime \prime}\right)\right|-2$. Now, Lemma 9.11 follows immediately from Lemma 9.9.

Eventually, we consider the extensions of the graphs of skew-square type.
Lemma 9.12. Let $G$ be a non-orientable antiprismatic graph of skew-square type and $G^{\prime}$ be an extension of $G$ on $n$ vertices. Then $\operatorname{cc}\left(G^{\prime}\right) \leq n$ and equality holds if and only if $\overline{G^{\prime}}$ is isomorphic to $a$ twister.

Proof. Let $A, B, C, s, t, d_{1}, d_{2}$ and $d_{3}$ be as in the definition of the graphs of skew-square type. If $\bar{G}$ contains no induced rotator, then by Theorem 9.3, $G$ and so $G^{\prime}$ is an extension of a non-orientable antiprismatic graph in $\mathcal{F}_{0} \cup \mathcal{F}_{1} \cup \mathcal{F}_{2} \cup \mathcal{F}_{3} \cup \mathcal{F}_{4}$ and the assertion follows from Lemmas 9.7, 9.8, 9.10 and 9.11. Thus, assume that $G$ contains the complement of a rotator as an induced subgraph. Note that the only triads of $G$ are of the form $\left\{s, a_{j}, c_{j}\right\}$ or $\left\{t, b_{j}, c_{j}\right\}$, for some $j$, or $\left\{a_{i}, b_{i^{\prime}}, d_{i^{\prime \prime}}\right\}$, where $\left\{i, i^{\prime}, i^{\prime \prime}\right\}=\{1,2,3\}$. Therefore, since the complement of a rotator contains exactly three disjoint triads, (leaving the reader to check the details) one can see that the only rotators of $\bar{G}$ are induced on the subsets of $V(G)$ of the following four forms,

$$
\begin{array}{ll}
R_{1}=\left\{a_{i}, b_{i^{\prime}}, d_{i^{\prime \prime}}\right\} \cup\left\{a_{i^{\prime}}, b_{i^{\prime \prime}}, d_{i}\right\} \cup\left\{b_{i}, c_{i}, t\right\}, & R_{2}=\left\{a_{i}, b_{i^{\prime}}, d_{i^{\prime \prime}}\right\} \cup\left\{a_{i^{\prime}}, b_{i^{\prime \prime}}, d_{i}\right\} \cup\left\{a_{i^{\prime \prime}}, c_{i^{\prime \prime}}, s\right\}, \\
R_{3}=\left\{a_{i}, b_{i^{\prime}}, d_{i^{\prime \prime}}\right\} \cup\left\{a_{i^{\prime}}, c_{i^{\prime}}, s\right\} \cup\left\{b_{j}, c_{j}, t\right\}, & R_{4}=\left\{a_{i}, b_{i^{\prime}}, d_{i^{\prime \prime}}\right\} \cup\left\{a_{j}, c_{j}, s\right\} \cup\left\{b_{i}, c_{i}, t\right\},
\end{array},
$$

where $\left\{i, i^{\prime}, i^{\prime \prime}\right\}=\{1,2,3\}$ and $j \in \mathbb{N} \backslash\left\{i, i^{\prime}\right\}$. Let $\tilde{A} \subseteq A$ and $\tilde{B} \subseteq B$ be the set of vertices in $A$ and $B$, respectively, which are complete to $C$. Also, let $\tilde{C} \subseteq C$ be the set of vertices in $C$ which are complete to $A \cup B$. Note that $s, t \in W(G)$ (otherwise if say $s \in \tilde{W}(G)$, then $A$ is complete to $C$ and $V(G)$ would be the union of three cliques $A \cup C, B$ and $\left\{s, t, d_{1}, d_{2}, d_{3}\right\}$, so $G$ is orientable, a contradiction). Thus, $\tilde{W}(G) \subseteq \tilde{A} \cup \tilde{B} \cup \tilde{C} \cup\left\{d_{1}, d_{2}, d_{3}\right\}$. For $\left\{i, i^{\prime}, i^{\prime \prime}\right\}=\{1,2,3\}$, note that $a_{i} \in \tilde{W}(G)$ if and only if $a_{i} \in \tilde{A}$ and $b_{i^{\prime}}, b_{i^{\prime \prime}} \notin B$ and the similar assertion holds for $b_{i} \in \tilde{W}(G)$. Let $I=\left\{i: a_{i} \in A\right.$ and $\left.b_{i} \in B\right\}$. If $I=\emptyset$, then every vertex in $C$ is complete to either $A$ or $B$, and if $C^{\prime}$ is the set of vertices in $C$ which are complete to $A$ in $G$, then $V(G)$ is the union of three cliques $A \cup C^{\prime},\left(C \backslash C^{\prime}\right) \cup B$ and $\left\{s, t, d_{1}, d_{2}, d_{3}\right\}$, a contradiction. Thus, $I \neq \emptyset$.
Let $G^{\prime}$ be an extension of $G$ on $n$ vertices and $X_{v}, v \in \tilde{W}(G)$, be as in the definition. Also, for every $v \in W(G)$, define $X_{v}=\{v\}$ and for every $v \notin V(G)$, define $X_{v}=\emptyset$. We are going to prove that $\operatorname{cc}\left(G^{\prime}\right) \leq n-1$. Let $A^{\prime}=\cup_{k \geq 1} X_{a_{k}}, B^{\prime}=\cup_{k \geq 1} X_{b_{k}}, C^{\prime}=\cup_{k \geq 1} X_{c_{k}}$ and $D^{\prime}=X_{d_{1}} \cup X_{d_{2}} \cup X_{d_{3}}$. Also, define $I_{1}=\left\{k: 1 \leq k \leq 3, a_{k} \in \tilde{W}(G)\right\}, I_{2}=\left\{k: 1 \leq k \leq 3, b_{k} \in \tilde{W}(G)\right\}$ and $I_{3}=\left\{k: c_{k} \in \tilde{C}\right\}$ and for each $l \in\{1,2,3\}$, let $K_{l}=X_{a_{l}} \cup X_{b_{l}} \cup\left(\cup_{k \in I_{1}} X_{a_{k}}\right) \cup\left(\cup_{k \in I_{2}} X_{b_{k}}\right) \cup\left(\cup_{k \in I_{3}} X_{c_{k}}\right) \cup\left(\cup_{k \in\{1,2,3\} \backslash\{l\}} X_{c_{k}}\right)$. It is easy to verify that $K_{l}$ is a clique of $G^{\prime}$ and also for every $a \in A^{\prime}, N_{G^{\prime}}\left[a, B^{\prime} \cup C^{\prime}\right]$ is a clique of $G^{\prime}$. Therefore,

$$
\begin{aligned}
\mathscr{C}= & \left(\cup_{l=1}^{3} \mathscr{N}_{G^{\prime}}\left[X_{d_{l}} ; K_{l}\right]\right) \cup \mathscr{N}_{G^{\prime}}\left[A^{\prime} ; B^{\prime} \cup C^{\prime}\right] \cup \mathscr{N}_{G^{\prime}}\left[\cup_{l \in \mathbb{N} \backslash I} X_{b_{l}} ; C^{\prime}\right] \\
& \cup\left\{D^{\prime} \cup\{s, t\}, A^{\prime} \cup\{t\}, B^{\prime} \cup\{s\},\left(\cup_{l \geq 4} X_{a_{l}}\right) \cup D^{\prime},\left(\cup_{l \geq 4} X_{b_{l}}\right) \cup D^{\prime}\right\}
\end{aligned}
$$

is a family of cliques of $G^{\prime}$ and $|\mathscr{C}|=n-\left|C^{\prime}\right|-\left|\cup_{l \in I} X_{b_{l}}\right|+3 \leq n-|C|-|I|+3$. Note that for every $l \in I$, all the vertices in $X_{a_{l}} \cup X_{b_{l}}$ have the same set of neighbours in $C^{\prime}$. Therefore, the cliques in $\mathscr{C}$ cover all edges of $G^{\prime}$ except possibly the edges in $E\left(C^{\prime}\right)$. Now, we observe that,
(1) If $|C|+|I| \geq 4$, then $\operatorname{cc}\left(G^{\prime}\right) \leq n-1$.

Since $\bar{G}$ contains a rotator induced on a set of the form $R_{k}$ for some $k \in\{1, \ldots, 4\}$, either $\{k$ : $\left.1 \leq k \leq 3, a_{k} \in A\right\}$ or $\left\{k: 1 \leq k \leq 3, b_{k} \in B\right\}$ has cardinality at least two. Now, if both of
the sets $\left\{k: k \geq 4, a_{k} \in A\right\}$ and $\left\{k: k \geq 4, b_{k} \in B\right\}$ are nonempty, then either $A$ or $B$, say $A$, has cardinality at least three, and so the edges of $G^{\prime}$ in $E\left(C^{\prime}\right)$ are covered by the cliques of $\mathscr{C}$ in $\mathscr{N}_{G^{\prime}}\left[A^{\prime} ; B^{\prime} \cup C^{\prime}\right]$. Consequently, $\mathscr{C}$ is a clique covering for $G$ of size $|\mathscr{C}| \leq n-|C|-|I|+3 \leq n-1$. If either $\left\{k: k \geq 4, a_{k} \in A\right\}$ or $\left\{k: k \geq 4, b_{k} \in B\right\}$ is empty, then in $\mathscr{C}$, replace the clique $\left(\cup_{k \geq 4} X_{a_{k}}\right) \cup D^{\prime}$ or $\left(\cup_{k \geq 4} X_{b_{k}}\right) \cup D^{\prime}$, respectively, with the clique $C^{\prime}$ to obtain a clique covering for $G^{\prime}$ of size at most $n-1$. This proves (1).

Note that $C$ is nonempty (otherwise, $G$ would be three-cliqued). If either $|C| \geq 3$ or $|I| \geq 3$, then since $I, C \neq \emptyset$, by (1), we are done. Hence, either $|C|=1$ and $1 \leq|I| \leq 2$, or $|C|=2$ and $|I|=1$. First, suppose that $|C|=1$ and $1 \leq|I| \leq 2$. Now, $\bar{G}$ should contain a rotator induced on a subset of $V(G)$ of the form $R_{1}$ or $R_{2}$. Consequently, $C^{\prime}=C=\left\{c_{i_{0}}\right\}$, for some $i_{0} \in\{1,2,3\},|I|=2$ and at least one of the sets $\left\{k: 1 \leq k \leq 3, a_{k} \in A\right\}$ or $\left\{k: 1 \leq k \leq 3, b_{k} \in B\right\}$, say the second one, is equal to $\{1,2,3\}$. Now, if both of the sets $\left\{k: k \geq 4, a_{k} \in A\right\}$ and $\left\{k: k \geq 4, b_{k} \in B\right\}$ are nonempty, then $|B| \geq 4$, and it is enough to modify $\mathscr{C}$ by replacing the cliques in $\mathscr{N}_{G^{\prime}}\left[\cup_{k \in \mathbb{N} \backslash I} X_{b_{k}} ; C^{\prime}\right]$ with the cliques in $\mathscr{N}_{G^{\prime}}\left[C^{\prime} ; B^{\prime}\right]$, thereby obtaining a clique covering for $G^{\prime}$ of size $n-\left|B^{\prime}\right|+3 \leq n-|B|+3 \leq n-1$. If either $\left\{k: k \geq 4, a_{k} \in A\right\}$ or $\left\{k: k \geq 4, b_{k} \in B\right\}$ is empty, then removing the clique $\left(\cup_{k \geq 4} X_{a_{k}}\right) \cup D^{\prime}$ or $\left(\cup_{k \geq 4} X_{b_{k}}\right) \cup D^{\prime}$ from $\mathscr{C}$ yields a clique covering for $G^{\prime}$ of size $|\mathscr{C}|-1 \leq n-|C|-|I|+2=n-1$. Finally, suppose that $|C|=2$ and $|I|=1$. Thus, $\bar{G}$ should induce a rotator on a subset of $V(G)$ of the form $R_{3}$ or $R_{4}$. By symmetry, we may assume that $\bar{G}$ contains a rotator induced on a subset of $V(G)$ of the form $R_{3}$. Thus, $\left\{a_{i}, b_{i^{\prime}}, d_{i^{\prime \prime}}\right\} \cup\left\{a_{i^{\prime}}, c_{i^{\prime}}, t\right\} \cup\left\{b_{j}, c_{j}, s\right\} \subseteq V(G)$ and $C^{\prime}=C=\left\{c_{i^{\prime}}, c_{j}\right\}$, where $\left\{i, i^{\prime}, i^{\prime \prime}\right\}=\{1,2,3\}$ and $j \in \mathbb{N} \backslash\left\{i, i^{\prime}\right\}$. Also, since $|I|=1, b_{i} \notin B$ and $a_{j} \notin A$. Now, if either $\left\{k: k \geq 4, a_{k} \in A\right\}$ or $\left\{k: k \geq 4, b_{k} \in B\right\}$ is empty, then again removing the clique $\left(\cup_{k \geq 4} X_{a_{k}}\right) \cup D^{\prime}$ or $\left(\cup_{k \geq 4} X_{b_{k}}\right) \cup D^{\prime}$ from $\mathscr{C}$ yields a clique covering for $G^{\prime}$ of size $|\mathscr{C}|-1 \leq n-|C|-|I|+2=n-1$ (note that $E\left(C^{\prime}\right)=\left\{c_{i^{\prime}} c_{j}\right\}$ and the edge $c_{i^{\prime}} c_{j}$ is covered by the clique $\left.N_{G^{\prime}}\left[a_{i}, B^{\prime} \cup C^{\prime}\right] \in \mathscr{C}\right)$. Thus, we may assume that both $\left\{k: k \geq 4, a_{k} \in A\right\}$ and $\left\{k: k \geq 4, b_{k} \in B\right\}$ are nonempty and so $|A| \geq 3$.
For every $k \in\{1,2,3\}$, define $L_{k}=X_{d_{k}} \cup X_{c_{k+1}} \cup X_{c_{k+2}} \cup\left(\cup_{l \in \mathbb{N} \backslash\{k+1, k+2\}} X_{a_{l}}\right)$ and $M_{k}=X_{d_{k}} \cup X_{c_{k+1}} \cup$ $X_{c_{k+2}} \cup\left(\cup_{l \in \mathbb{N} \backslash\{k+1, k+2\}} X_{b_{l}}\right)$ (reading $k+1$ and $k+2$ modulo 3). Consider the following family of cliques of $G^{\prime}$

$$
\begin{aligned}
\mathscr{C}^{\prime}= & \mathscr{N}_{G^{\prime}}\left[B^{\prime} \backslash\left(X_{b_{i^{\prime}}} \cup X_{b_{i^{\prime \prime}}} \cup X_{b_{j}}\right) ; A^{\prime}\right] \cup\left\{L_{k}, M_{k}: 1 \leq k \leq 3\right\} \\
& \cup\left\{D^{\prime} \cup\{s, t\}, A^{\prime} \cup\{t\}, B^{\prime} \cup\{s\}, X_{a_{i^{\prime}}} \cup X_{b_{i^{\prime}}} \cup X_{c_{j}}\right\} .
\end{aligned}
$$

If $j=i^{\prime \prime}$, then the $\mathscr{C}^{\prime}$ is a clique covering for $G^{\prime}$ of size $10+\left|\cup_{k \in \mathbb{N} \backslash\left\{i^{\prime}, i^{\prime \prime}\right\}} X_{b_{k}}\right| \leq n-|A|+1 \leq n-2$. Also, if $j \neq i^{\prime \prime}$ and $b_{i^{\prime \prime}} \in B$, then adding the cliques $\left(A^{\prime} \cup X_{c_{i^{\prime}}} \cup X_{c_{j}}\right) \backslash X_{a_{i^{\prime}}}$ and $\left(B^{\prime} \cup X_{c_{j}}\right) \backslash X_{b_{j}}$ to $\mathscr{C}^{\prime}$ yields a clique covering of size $12+\left|\cup_{k \in \mathbb{N} \backslash\left\{i^{\prime}, i^{\prime \prime}, j\right\}} X_{b_{k}}\right| \leq n-|A|+2 \leq n-1$. Finally, if $j \neq i^{\prime \prime}$ and $b_{i^{\prime \prime}} \notin B$, then in $\mathscr{C}^{\prime}$, add the cliques $\left(A^{\prime} \cup X_{c_{i^{\prime}}} \cup X_{c_{j}}\right) \backslash X_{a_{i^{\prime}}}$ and $\left(B^{\prime} \cup X_{c_{j}}\right) \backslash X_{b_{j}}$ and merge the pair $\left(M_{i}, M_{i^{\prime \prime}}\right)$, thereby obtaining a clique covering for $G^{\prime}$ of size $11+\left|\cup_{k \in \mathbb{N} \backslash\left\{i^{\prime}, j\right\}} X_{b_{k}}\right| \leq n-|A|+2 \leq n-1$. This completes the proof of Lemma 9.12.

Now, we are in the position to prove Theorem 9.2.
Proof of Theorem 9.2. Let $G$ be a counterexample to Theorem 9.1. On the contrary, assume that $\bar{G}$ either contains no induced rotator or contains a square-forcer. By Theorem $9.5, G$ is an extension of a graph $H$, where $H$ is either a graph of parallel-square type, or a graph of skew-square type, or a member of $\mathcal{F}_{0} \cup \mathcal{F}_{1} \cup \mathcal{F}_{2} \cup \mathcal{F}_{3} \cup \mathcal{F}_{4}$. Since $G$ is non-orientable, so is $H$ (and both are not three-cliqued). Hence, by Lemmas $9.6,9.7,9.8,9.10,9.11$ and $9.12, \operatorname{cc}(G) \leq n$ and equality holds if and only if $\bar{G}$ is isomorphic to a twister, a contradiction. This proves Theorem 9.2.

### 9.2 Including a rotator

Let $G$ be a counterexample to Theorem 9.1. In view of Theorem 9.2 , henceforth we may assume that $\bar{G}$ contains an induced rotator and no square-forcer. For some reasons, in contrast to the most arguments in this paper, we do not get involved with the detailed structure of such antiprismatic graphs. Instead, with arguments similar to [4], we derive some structural properties of these graphs
required for the construction of our clique coverings. One of these graphs which play an important role in the characterization of non-orientable antiprismatic graphs is the Schläfli graph. Let us recall its definition from [4]. The Schläfli graph is a graph $\Gamma$ on the set $\left\{r_{j}^{i}, s_{j}^{i}, t_{j}^{i}: 1 \leq i, j \leq 3\right\}$ of 27 vertices, with the following adjacency. Assume that $1 \leq i, i^{\prime}, j, j^{\prime} \leq 3$.

- $s_{j}^{i}$ is adjacent to $s_{j^{\prime}}^{i^{\prime}}, t_{j}^{i}$ is adjacent to $t_{j^{\prime}}^{i^{\prime}}$ and $r_{j}^{i}$ is adjacent to $r_{j^{\prime}}^{i^{\prime}}$, if and only if either $i=i^{\prime}$ or $j=j^{\prime}$ (and not both).
- $s_{j}^{i}$ is adjacent to $t_{j^{\prime}}^{i^{\prime}}, t_{j}^{i}$ is adjacent to $r_{j^{\prime}}^{i^{\prime}}$ and $r_{j}^{i}$ is adjacent to $s_{j^{\prime}}^{i^{\prime}}$, if and only if $j \neq i^{\prime}$.

Note that, $\bar{\Gamma}$ induces a rotator on $\left\{s_{1}^{1}, s_{2}^{2}, s_{3}^{3} ; t_{3}^{1}, t_{3}^{2}, t_{3}^{3} ; r_{1}^{3}, r_{2}^{3}, r_{3}^{3}\right\}$. In the sequel, we will see that the structure of $G$, a counterexample to Theorem 9.1, is very similar to $\Gamma$. First of all, we set a couple of notations and definitions. Through the rest of the paper, read all subscripts and the superscripts modulo 3. Suppose that $G$ contains the complement of a rotator as an induced subgraph called $\rho=\left(s_{1}^{1}, s_{2}^{2}, s_{3}^{3} ; t_{3}^{1}, t_{3}^{2}, t_{3}^{3} ; r_{1}^{3}, r_{2}^{3}, r_{3}^{3}\right)$, where in $G,\left\{s_{1}^{1}, s_{2}^{2}, s_{3}^{3}\right\}$ is a triad, $T_{3}(\rho)=\left\{t_{3}^{1}, t_{3}^{2}, t_{3}^{3}\right\}$ and $R^{3}(\rho)=$ $\left\{r_{1}^{3}, r_{2}^{3}, r_{3}^{3}\right\}$ are two triangles, $T_{3}(\rho)$ is anticomplete to $R^{3}(\rho)$ and for every $i, j \in\{1,2,3\}, s_{i}^{i}$ is adjacent to $t_{3}^{j}$ and $r_{j}^{3}$, if and only if $i \neq j$. The reason for this puzzling notation is that the subgraph of $G$ induced on the vertices in $\rho$ is isomorphic to the subgraph of the Schläfli graph induced on the same vertices. Let $V_{0}(\rho)$ be the set of vertices of $G$ which are not in $\rho$. For every $i \in\{1,2,3\}$, define

- $T^{i}(\rho)$ is the set of vertices in $V_{0}(\rho)$ which are nonadjacent to $s_{i}^{i}$, complete to $\left\{r_{i+1}^{3}, r_{i+2}^{3}\right\}$ and anticomplete to $\left\{t_{3}^{i+1}, t_{3}^{i+2}\right\}$.
- $R_{i}(\rho)$ is the set of vertices in $V_{0}(\rho)$ which are nonadjacent to $s_{i}^{i}$, complete to $\left\{t_{3}^{i+1}, t_{3}^{i+2}\right\}$ and anticomplete to $\left\{r_{i+1}^{3}, r_{i+2}^{3}\right\}$.
- $S_{i+2}^{i+1}(\rho)$ is the set of vertices in $V_{0}(\rho)$ which are nonadjacent to $s_{i}^{i}$, complete to $\left\{t_{3}^{i+1}, r_{i+2}^{3}\right\}$ and anticomplete to $\left\{t_{3}^{i+2}, r_{i+1}^{3}\right\}$.
- $S_{i+1}^{i+2}(\rho)$ is the set of vertices in $V_{0}(\rho)$ which are nonadjacent to $s_{i}^{i}$, complete to $\left\{t_{3}^{i+2}, r_{i+1}^{3}\right\}$ and anticomplete to $\left\{t_{3}^{i+1}, r_{i+2}^{3}\right\}$.

Also, let $J=\{(i, j): 1 \leq i, j \leq 3, i \neq j\}$. Since $G$ is antiprismatic, it is easy to verify that $\mathcal{D}(\rho)=\left(S_{j}^{i}(\rho),(i, j) \in J ; T^{i}(\rho), R_{i}(\rho), i \in\{1,2,3\}\right)$ is a partition of $V_{0}(\rho)$ into 12 cliques (to see a proof for this fact, see 10.2 in [4]), which we call the decomposition corresponding to $\rho$ (see Figure 3). Note that the decomposition depends not only on the rotator $\rho$ but also on the ordering of vertices in $\rho$. Henceforth, in all the definitions, we omit the term $\rho$ when there is no ambiguity. For more convenience, define

$$
S=\bigcup_{(i, j) \in J} S_{j}^{i}, T=\bigcup_{i=1}^{3} T^{i}, R=\bigcup_{i=1}^{3} R_{i}, S^{\prime}=S \cup\left\{s_{1}^{1}, s_{2}^{2}, s_{3}^{3}\right\}, T^{\prime}=T \cup T_{3}, R^{\prime}=R \cup R^{3}
$$

Also for every $i \in\{1,2,3\}$, define $S^{i}=S_{i+1}^{i} \cup S_{i+2}^{i}$ and $S_{i}=S_{i}^{i+1} \cup S_{i}^{i+2}$. Subsequently, let

$$
I_{T}=\left\{i: 1 \leq i \leq 3 \text { and } T^{i} \neq \emptyset\right\} \text { and } I_{R}=\left\{j: 1 \leq j \leq 3 \text { and } R_{j} \neq \emptyset\right\}
$$

In order to provide appropriate clique covering for $G$, we need to know more about adjacency between the above sets of vertices. Firstly, we need a definition. For every graph $H$, we define a triple $(x, y, U)$ to be a quasi-triad of $H$, if $U$ is a subset of $V(H)$ (possibly empty) and $x, y$ are two distinct nonadjacent vertices in $V(H) \backslash U$ anticomplete to $U$. The following is a number of easy and useful observations about the structure of $G$ which are proved (directly or indirectly) in [4] and help us to give desired clique coverings. Here, we omit the proof.

Lemma 9.13. Let $G$ be a counterexample to Theorem 9.1. Then the following hold (Statements (iiivii) also hold with $T, R$ exchanged).


Figure 3: The decomposition corresponding to a rotator.
(i) Let $(x, y, U)$ be a quasi-triad of $G$. Then $|U| \leq 1$, and every vertex in $V(G) \backslash(U \cup\{x, y\})$ which is complete to $\{x, y\}$, is anticomplete to $U$.
(ii) For all $(i, j) \in J,\left|S_{j}^{i}\right| \leq 1$. Also, for every $k \in\{1,2,3\}, T^{k}$ and $R_{k}$ are both cliques of $G$.
(iii) For every $(i, j) \in J$, every vertex in $T^{i}$ has at most one non-neighbour in $T^{j}$.
(iv) For every $i \in\{1,2,3\}$ and every vertex $t \in T^{i}$, the sets $N_{G}\left(t, T \backslash T^{i}\right)$ and $N_{\bar{G}}\left(t, T \backslash T^{i}\right)$ are both cliques of $G$. Consequently, $G[T]$ contains no triad.
(v) For every vertex $t \in T$, the sets $N_{G}(t, R)$ and $N_{\bar{G}}(t, R)$ are both cliques of $G$.
(vi) For every pair of nonadjacent vertices $t^{1}, t^{2} \in T,\left(R^{1}, R^{2}\right)$ is a partition of $R$ into two cliques, where $R^{1}=N\left(t^{1}, R\right)$ and $R^{2}=N\left(t^{2}, R\right)$. Consequently, if $t^{1}$ and $t^{2}$ are two vertices in $T$ with a common non-neighbour $t^{3} \in T$, then $N\left(t^{1}, R\right)=N\left(t^{2}, R\right)=R \backslash N\left(t^{3}, R\right)$.
(vii) If $\left|I_{T}\right|=3$, then there exists $(i, j) \in J$ such that every vertex in $T^{i}$ has a neighbour in $T^{j}$.
(viii) For every $i \in\{1,2,3\}$, both $S^{i} \cup\left\{s_{i}^{i}\right\}$ and $S_{i} \cup\left\{s_{i}^{i}\right\}$ are cliques of $G$, and there are no more edges in $G\left[S^{\prime}\right]$.
(ix) For every $(i, j) \in J$, the sets $S_{j} \cup T^{i}$ and $S^{i} \cup R_{j}$ are both cliques of $G$.

According to Lemma 9.13 (ii), for every $(i, j) \in J,\left|S_{j}^{i}\right| \leq 1$. Let us denote the unique possible element of $S_{j}^{i}$ by $s_{j}^{i}$ (we often use $s_{j}^{i} \in S$ and $s_{j}^{i} \notin S$ instead of $S_{j}^{i} \neq \emptyset$ and $S_{j}^{i}=\emptyset$ ).
Lemma 9.13 discloses the fact that the adjacency in $G$ is very similar to the adjacency in the Schläfli graph, except the adjacency between vertices in $T \cup R$. For instance, in Figure 3, each row and column in $S^{\prime}, T^{\prime}$ and $R^{\prime}$ is a clique and for $1 \leq i, j \leq 3$, column $j$ in $S^{\prime}$ (resp. $R^{\prime}$ ) is complete to row $i$ in $T^{\prime}$ (resp. $S^{\prime}$ ) if and only if $i \neq j$ (see Lemma 9.13 (viii,ix)). In fact, as it will be seen later, the major obstacle to proving Theorem 9.1 is covering the edges of $G$ in $\mathcal{E}(G)=\bigcup_{(i, j) \in J}\left(E\left(T^{i}, T^{j}\right) \cup E\left(R_{i}, R_{j}\right)\right) \cup$ $E\left(T^{\prime}, R^{\prime}\right) \cup E\left(T_{3}\right) \cup E\left(R^{3}\right)$.
Here, we introduce two collections of cliques of $G$ covering the edges in $E(G) \backslash \mathcal{E}(G)$, and their various modified versions will be used to provide suitable clique coverings for $G$. The first one comes from a natural column-to-row clique covering of the Schläfli graph and is defined as follows. For every $(i, j) \in J$, let $X_{j}^{i}=S_{j} \cup T^{i} \cup\left\{s_{j}^{j}, t_{3}^{i}\right\}$ and $Z_{j}^{i}=R_{j} \cup S^{i} \cup\left\{s_{i}^{i}, r_{j}^{3}\right\}$ which are cliques of $G$. Define $\mathscr{O}(G)=\left\{X_{j}^{i}, Z_{j}^{i}:(i, j) \in J\right\}$ and note that $|\mathscr{O}(G)|=12$. The second collection is a bit more complicated, however enables us to conquer several sporadic cases. For every $(i, j) \in J$, define

$$
\Delta_{j}^{i}=\left(S^{i} \backslash S_{j}^{i}\right) \cup\left\{s_{i}^{i}\right\} \cup T^{j} \cup\left(R^{3} \backslash\left\{r_{i}^{3}\right\}\right), \quad \nabla_{j}^{i}=\left(S_{i} \backslash S_{i}^{j}\right) \cup\left\{s_{i}^{i}\right\} \cup\left(T_{3} \backslash\left\{t_{3}^{i}\right\}\right) \cup R_{j} .
$$

Also, for every $i \in\{1,2,3\}$, let $\Omega^{i}=S^{i} \cup T^{i} \cup\left\{t_{3}^{i}\right\}$ and $\Omega_{i}=S_{i} \cup R_{i} \cup\left\{r_{i}^{3}\right\}$. Note that by Lemma 9.13 (viii,ix) and the definitions, all the sets $\Delta_{j}^{i}, \nabla_{j}^{i}, \Omega^{i}$ and $\Omega_{i}$ are cliques of $G$. Now, let $\mathscr{P}(G)$ be the family of cliques of $G$ constructed according to the following algorithm.

P1. Let $\mathscr{P}(G)=\emptyset$.
P2. For every $i \in\{1,2,3\}$, if $T^{i} \neq \emptyset$, then add the cliques $\Delta_{i}^{i+1}$ and $\Delta_{i}^{i+2}$ to $\mathscr{P}(G)$.
P3. For every $i \in\{1,2,3\}$, if either $T^{i} \neq \emptyset$ or $\left|S^{i}\right|=2$, then add the clique $\Omega^{i}$ to $\mathscr{P}(G)$.
P4. For every $i \in\{1,2,3\}$, if $R_{i} \neq \emptyset$, then add the cliques $\nabla_{i}^{i+1}$ and $\nabla_{i}^{i+2}$ to $\mathscr{P}(G)$.

P5. For every $i \in\{1,2,3\}$, if either $R_{i} \neq \emptyset$ or $\left|S_{i}\right|=2$, then add the clique $\Omega_{i}$ to $\mathscr{P}(G)$.
P6. For every $(i, j) \in J$ with $s_{j}^{i} \in S$, add both cliques $\Delta_{6-i-j}^{i}$ and $\nabla_{6-i-j}^{j}$ to $\mathscr{P}(G)$ (if they are not added before).

P7. For every $i \in\{1,2,3\}$, add one of the cliques $\Delta_{j}^{i}, j \in\{1,2,3\} \backslash\{i\}$ and also one of the cliques $\nabla_{j}^{i}, j \in\{1,2,3\} \backslash\{i\}$ to $\mathscr{P}(G)$ (if no such cliques are added before).

Then, we have $|\mathscr{P}(G)| \leq 18$. Now, we state a simple lemma about the edges covered by the cliques in $\mathscr{O}(G)$ and $\mathscr{P}(G)$, which follows from previous arguments and we leave the proof to the reader.

Lemma 9.14. The following hold.
(i) The cliques in $\mathscr{O}(G)$ cover all the edges in $E(G) \backslash \mathcal{E}(G)$.
(ii) The cliques in $\mathscr{P}(G)$ cover all the edges in $E(G) \backslash \mathcal{E}(G)$, as well as the edges in $E\left(T, R^{3}\right) \cup$ $E\left(T_{3}, R\right) \cup E\left(T_{3}\right) \cup E\left(R^{3}\right)$.

Note that, to avoid repetition, we refuse to refer to Lemma 9.14 and it is assumed that the reader will be considering its truth throughout the rest of the paper. Furthermore, we derive more information about $G$ based on the fact that $\bar{G}$ excludes square-forcer.

Lemma 9.15. Let $G$ be a counterexample to Theorem 9.1 and $\{i, j, k\}=\{1,2,3\}$. The following hold.
(i) If $G\left[T \backslash T^{i}\right]$ is a clique and $k \notin I_{R}$, then either $s_{i}^{j} \in S$, or $s_{l}^{i} \in S$, for some $l \in I_{T} \backslash\{i\}$.
(ii) If $G\left[R \backslash R_{i}\right]$ is a clique and $k \notin I_{T}$, then either $s_{j}^{i} \in S$, or $s_{i}^{l} \in S$, for some $l \in I_{R} \backslash\{i\}$.

Proof. If (i) does not hold, then $\left(s_{i}^{i}, r_{k}^{3}\right)$ is a square-forcer in $\bar{G}$, a contradiction to Theorem 9.2. The truth of (ii) follows similarly.

We conclude this subsection with three useful lemmas which will be used in the forthcoming subsections. For simplicity, we omit the term $\rho$ within the proofs.

Lemma 9.16. Let $G$ be a counterexample to Theorem 9.1. Then for every rotator $\rho$ of $\bar{G}$, we have $\left|I_{T}(\rho)\right|+\left|I_{R}(\rho)\right| \geq 4$.

Proof. Suppse not, let $\left|I_{T}\right|+\left|I_{R}\right| \leq 3$. Then, we have
(1) Both $T$ and $R$ are nonempty.

Suppose not, by symmetry, let $R=\emptyset$. Merge the pairs $\left(Z_{2}^{1}, Z_{3}^{1}\right),\left(Z_{1}^{2}, Z_{3}^{2}\right)$ and $\left(Z_{1}^{3}, Z_{2}^{3}\right)$ in $\mathscr{O}(G)$, and call the resulting collection $\mathscr{O}_{1}(G)$. First, assume that $\left|I_{T}\right| \leq 2$, say $T^{3}=\emptyset$. By Lemma 9.15 (ii) (applying to $(i, j, k)=(1,2,3)$ and $(i, j, k)=(2,1,3))$, both $s_{2}^{1}, s_{1}^{2} \in S$. Thus, $|S| \geq 2$, and adding the cliques in $\mathscr{N}\left[T^{1} ; T^{2} \cup R^{3}\right] \cup\left\{T^{2} \cup R^{3}, T_{3}\right\}$ to $\mathscr{O}_{1}(G)$, yields a clique covering $\mathscr{C}_{1}$ for $G$ of size $11+\left|T^{1}\right|=$ $n-|S|-\left|T^{2}\right|+2 \leq n-\left|T^{2}\right|$. Thus, $T^{2}=\emptyset$ and by Lemma 9.15 (i) (applying to $(i, j, k)=(3,1,2)$ ), at least one of $s_{3}^{1}, s_{1}^{3} \in S$, and so $|S| \geq 3$, which implies that $\left|\mathscr{C}_{1}\right| \leq n-1$, a contradiction. Therefore, $\left|I_{T}\right|=3$. By Lemma 9.13 (vii) and w.l.o.g. we may assume that for every $t \in T^{3}, N\left(t, T^{2}\right) \neq \emptyset$. Thus, the cliques in $\mathscr{N}\left[T^{2} ; T^{3} \cup R^{3}\right]$ cover all the edges in $E\left(T^{3}, R^{3}\right) \cup E\left(R^{3}\right)$. Also, by Lemma 9.13 (iv), $\mathscr{N}\left[T^{1} ;\left(T \backslash T^{1}\right) \cup R^{3}\right]$ is a set of cliques of $G$. Thus, $\mathscr{O}_{1}(G) \cup \mathscr{N}\left[T^{1} ;\left(T \backslash T^{1}\right) \cup R^{3}\right] \cup \mathscr{N}\left[T^{2} ; T^{3} \cup R^{3}\right] \cup\left\{T_{3}\right\}$ is a clique covering for $G$ of size $n-|S|-\left|T^{3}\right|+1 \leq n-|S|$. Hence, $S=\emptyset$. Now, the cliques in $\left(\mathscr{P}(G) \backslash\left\{\Delta_{2}^{1}, \Delta_{3}^{1}, \Delta_{3}^{2}, \Delta_{1}^{3}\right\}\right) \cup\left(\bigcup_{i=1}^{3} \mathscr{N}\left[T^{i} ; \Delta_{i+1}^{i+2}\right]\right)$ cover all edges covered by $\mathscr{P}(G)$ as well as the edges in $E(T)$ and so this family is a clique covering for $G$ of size $8+\sum_{i=1}^{3}\left|T_{i}\right|=n-1$, a contradiction. This proves (1).
(2) We have $\left|I_{T}\right|+\left|I_{R}\right|=3$.

Suppose not, let $\left|I_{T}\right|+\left|I_{R}\right| \leq 2$. By (1), $\left|I_{T}\right|=\left|I_{R}\right|=1$, say $I_{T}=\{p\}$ and $I_{R}=\{q\}$, for some $p, q \in\{1,2,3\}$. In $\mathscr{O}(G)$, merge the pairs $\left(X_{p}^{p+1}, X_{p}^{p+2}\right)$ and $\left(Z_{q+1}^{q}, Z_{q+2}^{q}\right)$ and add the cliques in $\mathscr{N}\left[T^{p} ; R_{q}\right] \cup\left\{T^{p} \cup R^{3}, T_{3} \cup R_{q}\right\}$ to the resulting family, thereby obtaining a clique covering $\mathscr{C}$ for $G$ of size $12+\left|T^{p}\right|=n-|S|-\left|R_{q}\right|+3$. Thus $|S| \leq 2$. Now, if $p=q$, say $p=q=1$, then by Lemma 9.15 (i) and (ii) (both applying to $(i, j, k)=(1,2,3)$ and $(i, j, k)=(1,3,2)$ ), all $s_{2}^{1}, s_{3}^{1}, s_{1}^{2}, s_{1}^{3} \in S$, a contradiction.

Thus, $p \neq q$, say $p=1$ and $q=2$. Again by Lemma 9.15 (i) (applying to $(i, j, k)=(1,2,3)$ and $(i, j, k)=(3,1,2)), s_{1}^{2} \in S$ and at least one of $s_{1}^{3}, s_{3}^{2} \in S$. Thus, $|S|=2$ (i.e. either $S=\left\{s_{1}^{2}, s_{1}^{3}\right\}$ or $S=\left\{s_{1}^{2}, s_{3}^{2}\right\}$ ). Now, in $\mathscr{C}$, remove the clique $X_{3}^{1}$, replace the cliques $X_{2}^{1}$ with the clique $X_{2}^{1} \cup S_{3}^{2}$ and replace the cliques in $\mathscr{N}\left[T^{p} ; R_{q}\right]$ with the cliques in $\mathscr{N}\left[T^{1} ; R_{2} \cup\left\{s_{3}^{3}, t_{3}^{1}\right\}\right]$. This yields a clique covering for $G$ of size $11+\left|T^{1}\right|=n-\left|R_{2}\right| \leq n-1$, a contradiction. This proves (2).

By (1) and (2) and due to symmetry, we may assume that $I_{T}=\{1,2\}$ and $I_{R}=\{p\}$, for some $p \in\{1,2,3\}$. First, note that if $p \in\{1,2\}$, then by Lemma 9.15 (ii) (applying to $(i, j, k)=(p, 3-p, 3)$ ), $s_{3-p}^{p} \in S$ (and so $|S| \geq 1$ ). Also, if $p=3$, then by Lemma 9.15 (ii) (applying to $(i, j, k)=(l, 3-l, 3)$ ), for every $l \in\{1,2\},\left\{s_{3-l}^{l}, s_{l}^{3}\right\} \cap S \neq \emptyset$ (and so $|S| \geq 2$ ). Now, in $\mathscr{O}(G)$, merge the pair $\left(Z_{p+1}^{p}, Z_{p+2}^{p}\right)$ and call the resulting collection $\mathscr{O}^{\prime}(G)$. We prove the following claim.
(3) There exists $i \in I_{T}$ such that every vertex in $T^{3-i}$ has a neighbour in $T^{i}$.

Suppose not, pick $t^{1} \in T^{1}$ with no neighbour in $T^{2}$. Thus $\left(t^{1}, t_{3}^{3}, T^{2}\right)$ is a quasi-triad of $G$ and by Lemma 9.13 (i), $\left|T^{2}\right|=1$, say $T^{2}=\left\{t^{2}\right\}$. Similarly, since $t^{2}$ has no neighbour in $T^{1}$, we have $T^{1}=\left\{t^{1}\right\}$. Now, $\mathscr{C}=\mathscr{O}^{\prime}(G) \cup \mathscr{N}\left[R_{p} ; T\right] \cup\left\{T^{1} \cup R^{3}, T^{2} \cup R^{3}, T_{3} \cup R_{p}\right\}$ is a clique covering for $G$ of size $14+\left|R_{p}\right| \leq n-|S|+3$. Thus, $|S| \leq 3$.
Firstly, assume that $p=3$, and so $2 \leq|S| \leq 3$. Now, if either $|S|=2$ or $S_{3} \neq \emptyset$, then $|\mathscr{P}(G)| \leq 11+|S|$. Thus, in $\mathscr{P}(G)$, replacing the cliques $\Omega^{i}$ with the clique $\Omega^{i} \cup N\left(t^{i}, R\right)$ yields a clique covering for $G$ of size at most $11+|S|=n-\left|R_{3}\right| \leq n-1$, a contradiction. Also, if $|S|=3$ and $S_{3}=\emptyset$, then $|\mathscr{P}(G)| \leq 12+|S|$. Thus, in $\mathscr{P}(G)$, replacing the cliques $\Omega^{i}$ with the clique $\Omega^{i} \cup N\left(t^{i}, R\right)$ yields a clique covering for $G$ of size at most $12+|S|=n-\left|R_{3}\right|+1$. Hence, $\left|R_{3}\right|=1$, and by Lemma 9.13 (vi), say $R_{3}$ is complete to $t^{1}$ and anticomplete to $t^{2}$. Now, remove the cliques $Z_{3}^{1}, Z_{3}^{2}$ as well as the single clique in $\mathscr{N}\left[R_{3} ; T\right]$ from $\mathscr{C}$ and add the cliques $R_{3} \cup S_{1}^{2} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}$ and $R_{3} \cup S_{2}^{1} \cup\left\{s_{2}^{2}, t^{1}, r_{3}^{3}\right\}$, thereby obtaining a clique covering for $G$ of size $14=n-1$.
Secondly, suppose that $p \in\{1,2\}$, say $p=1$ and so $s_{2}^{1} \in S$. Now, if $|S|=\left|S_{2}^{1}\right|=1$, then in $\mathscr{P}(G)$, replace the clique $\Omega_{1}$ with the cliques in $\mathscr{N}\left[R_{1} ; T \cup\left\{r_{1}^{3}\right\}\right]$, thereby obtaining a clique covering of size $11+\left|R_{1}\right|=n-1$. If $|S|=2$, then in the case that $s_{1}^{2} \in S$, in the collection $\mathscr{O}^{\prime}(G) \cup\left\{T^{1} \cup R^{3}, T^{2} \cup\right.$ $\left.R^{3}, T_{3} \cup R_{1}\right\}$, remove the clique $Z_{1}^{3}$, replace the cliques $Z_{2}^{3}, X_{3}^{1}$ and $X_{3}^{2}$ with the cliques $Z_{2}^{3} \cup\left\{r_{1}^{3}\right\}$, $X_{3}^{1} \cup N\left(t^{1}, R_{1}\right)$ and $X_{3}^{2} \cup N\left(t^{2}, R_{1}\right)$. Since by Lemma 9.13 (vi), every vertex in $R_{1}$ is either adjacent to $t^{1}$ or $t^{2}$, this yields to a clique covering for $G$ of size $13=n-\left|R_{1}\right| \leq n-1$. Also, in the case that $s_{1}^{2} \notin S$, we have $|\mathscr{P}(G)|=12+\left|S_{3}^{1}\right|$. Now, in $\mathscr{P}(G)$, and if $S_{1}=\emptyset$, then replace the clique $\Omega_{1}$ with the cliques in $\mathscr{N}\left[R_{1} ;\left(\Omega_{1} \cup T\right) \backslash R_{1}\right]$, and if $S_{1} \neq \emptyset$ (i.e. $S_{3}^{1}=\emptyset$ ), then add the cliques in $\mathscr{N}\left[R_{1} ; T\right]$, thereby obtaining a clique covering for $G$ of size at most $12+\left|R_{1}\right|=n-1$.
Finally, in the case that $|S|=3$, if $s_{3}^{1} \notin S$, then removing the cliques in $\mathscr{N}\left[R_{1} ; T\right]$ from $\mathscr{C}$ and replacing the clique $X_{3}^{i}$ with the cliques $X_{3}^{i} \cup N\left(t^{i}, R_{1}\right)$ for every $i \in\{1,2\}$, yields a clique covering for $G$ of size $14=n-\left|R_{1}\right| \leq n-1$. Thus, $s_{3}^{1} \in S$. Also, if $S^{3}=\emptyset$, then removing the cliques in $\mathscr{N}\left[R_{1} ; T\right]$ from $\mathscr{C}$ and replacing the clique $Z_{1}^{3}$ with the cliques in $\mathscr{N}\left[R_{1} ; T \cup\left\{s_{3}^{3}, r_{1}^{3}\right\}\right]$ yields a clique covering for $G$ of size $13+\left|R_{1}\right|=n-1$. Thus, $S=\left\{s_{2}^{1}, s_{3}^{1}, s_{j}^{3}\right\}$, for some $j \in\{1,2\}$. Now, if $j=1$, then remove the clique $T^{2} \cup R^{3}$ from $\mathscr{C}$ and replace the cliques $Z_{3}^{1}, Z_{3}^{2}$ and $Z_{2}^{3}$ with the cliques $\left(Z_{3}^{1} \cup\left\{t^{2}\right\}\right) \backslash S_{2}^{1}, Z_{3}^{2} \cup S_{2}^{1}$ and $Z_{2}^{3} \cup\left\{t^{2}, r_{1}^{3}\right\}$, and if $j=2$, then remove the clique $T^{1} \cup R^{3}$ from $\mathscr{C}$ and replace the cliques $Z_{3}^{2}$ and $Z_{2}^{3}$ with the cliques $Z_{3}^{2} \cup\left\{t^{1}, r_{1}^{3}\right\}$ and $Z_{2}^{3} \cup\left\{t^{1}\right\}$, thereby obtaining a clique covering for $G$ of size $13+\left|R_{1}\right|=n-1$. This proves (3).

Now, (3) ensures that say every vertex in $T^{2}$ has a neighbour in $T^{1}$. Consequently, the collection $\mathscr{O}^{\prime}(G) \cup \mathscr{N}\left[R_{p} ; T\right] \cup \mathscr{N}\left[T^{1} ; T^{2} \cup R^{3}\right] \cup\left\{T_{3} \cup R_{p}\right\}$ is a clique covering for $G$ of size $n-|S|-\left|T^{2}\right|+3$. Thus, we have $|S|+\left|T^{2}\right| \leq 3$.
First, suppose that $p=3$. Then $|S|=2$ and $\left|T^{2}\right|=1$, say $T^{2}=\left\{t^{2}\right\}$. Let $X=N\left(t^{2}, T^{1}\right)$. Note that by the assumption, $|X| \geq 1$ and by Lemma 9.13 (iii), $\left|T^{1} \backslash X\right| \leq 1$. If $\left|T^{1} \backslash X\right|=1$, then in $\mathscr{P}(G) \cup \mathscr{N}\left[T^{2} ; T^{1}\right]$, replace the clique $\Omega_{3}$ with the cliques in $\mathscr{N}\left[R_{3} ;\left(\Omega_{3} \cup T\right) \backslash R_{3}\right]$, to obtain a clique covering for $G$ of size at most $13+\left|R_{3}\right|=n-|X| \leq n-1$. If $T^{1} \backslash X=\emptyset$ and $\left|S^{3}\right| \leq 1$, then in $\mathscr{P}(G)$, replace the clique $\Omega_{3}$ with the cliques in $\mathscr{N}\left[R_{3} ;\left(\Omega_{3} \cup T\right) \backslash R_{3}\right]$ and also choose $i \in\{1,2\}$ such that $s_{i}^{3} \notin S$ and replace the clique $\Delta_{3-i}^{3}$ with the clique $\Delta_{3-i}^{3} \cup T^{i}$, to obtain a clique covering for $G$ of size at most $12+\left|R_{3}\right|=n-|X| \leq n-1$. Also, if $T^{1} \backslash X=\emptyset$ and $\left|S^{3}\right|=2$, then in the family
$\mathscr{O}^{\prime}(G) \cup \mathscr{N}\left[R_{3} ; T\right] \cup\left\{T_{3} \cup R_{3}\right\}$, in order to cover the edges in $E\left(T \cup R^{3}\right)$, replace the cliques $X_{3}^{1}, X_{3}^{2}, Z_{2}^{1}$ and $Z_{1}^{2}$ with the cliques $T \cup\left\{s_{3}^{3}, r_{1}^{3}, r_{2}^{3}\right\},\left\{s_{3}^{3}, t_{3}^{1}, t_{3}^{2}\right\}, Z_{2}^{1} \cup T^{2} \cup\left\{r_{3}^{3}\right\}$ and $Z_{1}^{2} \cup T^{1} \cup\left\{r_{3}^{3}\right\}$, thereby having a clique covering for $G$ of size $12+\left|R_{3}\right|=n-|X| \leq n-1$.
Finally, suppose that $p \in\{1,2\}$, say $p=1$. Now, if $|S|=\left|S_{2}^{1}\right|=1$, then in $\mathscr{P}(G) \cup \mathscr{N}\left[R_{1} ; T\right]$, replace the cliques $\Delta_{1}^{3}$ and $\Delta_{2}^{3}$ with the cliques in $\mathscr{N}\left[T^{1} ; \Delta_{2}^{3}\right]$ to obtain a clique covering for $G$ of size at most $10+\left|T^{1}\right|+\left|R_{1}\right| \leq n-1$. Also, if $|S|=2$, then $\left|T^{2}\right|=1$, say $T^{2}=\left\{t^{2}\right\}$. Again, let $X=N\left(t^{2}, T^{1}\right)$, where $|X| \geq 1$ and $\left|T^{1} \backslash X\right| \leq 1$. Now, consider the collection $\mathscr{P}(G)$, in order to cover the edges in $E(T, R)$, if $S_{1}=\emptyset$, then replace the clique $\Omega_{1}$ with the cliques in $\mathscr{N}\left[R_{1} ;\left(\Omega_{1} \cup T\right) \backslash R_{1}\right]$ and if $S_{1} \neq \emptyset$, then add the cliques in $\mathscr{N}\left[R_{1} ; T\right]$. Also, in order to cover the edges in $E\left(T^{1}, T^{2}\right)$, if $S^{3}=T \backslash X=\emptyset$, then merge the pair $\left(\Delta_{1}^{3}, \Delta_{2}^{3}\right)$, if $S^{3}=\emptyset$ and $T \backslash X \neq \emptyset$, then replace the clique $\Delta_{2}^{3}$ with the clique $\Delta_{2}^{3} \cup X$, if $S^{3} \neq \emptyset$ and $T \backslash X=\emptyset$, then choose $i \in\{1,2\}$ such that $s_{i}^{3} \notin S$, and replace the clique $\Delta_{3-i}^{3}$ with the clique $\Delta_{3-i}^{3} \cup T^{i}$, and if $S^{3} \neq \emptyset$ and $T \backslash X \neq \emptyset$, then add the clique $N\left[t^{2} ; T^{1}\right]$. We leave the reader to check that this procedure yields a clique covering for $G$ of size at most $n-1$, a contradiction. This proves Lemma 9.16.

Lemma 9.17. Let $G$ be a counterexample to Theorem 9.1. Then for every rotator $\rho$ of $\bar{G}$, either $T(\rho)$ or $R(\rho)$ is not a clique.

Proof. On the contrary, assume that both $T$ and $R$ are cliques of $G$. Note that the cliques in $\mathscr{C}=\mathscr{O}(G) \cup\left\{T \cup R^{3}, T_{3} \cup R\right\}$ cover all the edges in $E(G) \backslash E(T, R)$.
(1) We have $|S| \geq 2$.

Suppose not, let $|S| \leq 1$. Let $\left\{k, k^{\prime}, k^{\prime \prime}\right\}=\{1,2,3\}$ and $S=S_{k^{\prime}}^{k}$. Considering $\mathscr{P}(G)$, for every $i \in\{1,2,3\} \backslash\{k\}$ (resp. $\left.i \in\{1,2,3\} \backslash\left\{k^{\prime}\right\}\right)$, if both $\Delta_{i+1}^{i}, \Delta_{i+2}^{i} \in \mathscr{P}(G)\left(\right.$ resp. $\left.\nabla_{i+1}^{i}, \nabla_{i+2}^{i} \in \mathscr{P}(G)\right)$, then merge the pair $\left(\Delta_{i+1}^{i}, \Delta_{i+2}^{i}\right)$ (resp. $\left(\nabla_{i+1}^{i}, \nabla_{i+2}^{i}\right)$ ) and if $T^{k^{\prime}}$ (resp. $R_{k}$ ) is nonempty, then replace the clique $\Delta_{k^{\prime}}^{k}\left(\right.$ resp. $\left.\nabla_{k}^{k^{\prime}}\right)$ with the clique $\Delta_{k^{\prime}}^{k} \cup T^{k^{\prime \prime}}$ (resp. $\nabla_{k}^{k^{\prime}} \cup R_{k^{\prime \prime}}$ ), thereby covering all the edge in $E(T) \cup E(R)$. Also, in order to cover the edges in $E(T, R)$, for every $j \in I_{T} \backslash\{k\}$, replace the clique $\Omega^{j}$ with the cliques in $\mathscr{N}\left[T^{j} ;\left(\Omega^{j} \cup R\right) \backslash T^{j}\right]$ and for every $j \in I_{R}$, replace the clique $\Omega_{j}$ with the cliques in $\mathscr{N}\left[R_{j} ;\left(\Omega_{j} \cup T^{k}\right) \backslash R_{j}\right]$. This provides a clique covering for $G$ of size at most $8+|T|+|R| \leq n-1$. This proves (1).
(2) We have $\left|I_{T}\right|,\left|I_{R}\right| \geq 2$, and thus $|T|,|R| \geq 2$.

Suppose not, w.l.o.g. let $\left|I_{R}\right| \leq 1$. By Lemma 9.16, $\left|I_{T}\right|=3$ and $I_{R}=\{p\}$, for some $p \in\{1,2,3\}$. Now, in $\mathscr{C}$, merge the pair $\left(Z_{p+1}^{p}, Z_{p+2}^{p}\right)$ and add the cliques in $\mathscr{N}\left[R_{p} ; T\right]$ to obtain a clique covering for $G$ of size $13+\left|R_{p}\right|=n-|S|-|T|+4$, which by (1) is at most $n-1$, a contradiction. This proves (2).
(3) Let $\left\{k, k^{\prime}, k^{\prime \prime}\right\}=\{1,2,3\}$. If $k, k^{\prime} \in I_{T}$, then $S^{k^{\prime \prime}} \neq \emptyset$. Similarly, if $k, k^{\prime} \in I_{R}$, then $S_{k^{\prime \prime}} \neq \emptyset$.

Suppose not, by symmetry, assume that $1,2 \in I_{T}$ and $S^{3}$ is empty. Considering $\mathscr{C}$, for every $j \in$ $I_{R} \cap\{1,2\}$, replace the clique $Z_{j}^{3}$ with the cliques in $\mathscr{N}\left[R_{j} ; T^{1} \cup T^{2} \cup\left\{s_{3}^{3}, r_{j}^{3}\right\}\right]$ and for every $j \in\{1,2\}$, replace the clique $X_{3-j}^{j}$ with the cliques in $\mathscr{N}\left[T^{j} ; R_{3} \cup S_{3-j} \cup\left\{s_{3-j}^{3-j}, t_{3}^{j}\right\}\right]$, thereby covering all the edges in $E\left(T^{1} \cup T^{2}, R\right)$. Now, add the cliques in $\mathscr{N}\left[T^{3}, R\right]$ to the resulting family to obtain a clique covering $\mathscr{C}^{\prime}$ for $G$. If $\{1,2\} \subseteq I_{R}$, then by (1), $\left|\mathscr{C}^{\prime}\right|=n-|S|-\left|R_{3}\right|+1 \leq n-1$. Otherwise, by (2), $I_{R}=\{j, 3\}$, for some $j \in\{1,2\}$ and by (1), $\left|\mathscr{C}^{\prime}\right|=n-|S|-\left|R_{3}\right|+2 \leq n-1$, a contradiction. This proves (3).
(4) We have $|S|=3,\left|I_{T}\right|=|T|=\left|I_{R}\right|=|R|=2$. Also, $T$ is neither complete nor anticomplete to $R$.

By symmetry, assume that $|T| \leq|R|$. Note that the collection $\mathscr{C} \cup \mathscr{N}[T ; R]$ is a clique covering for $G$ of size $n-|S|-|R|+5$. Thus, $|S|+|R| \leq 5$. Consequently, by (1) and (2), we have $2 \leq|S| \leq 3$. Now, if either $\left|I_{T}\right|=3$ or $\left|I_{R}\right|=3$, then by (3), $|S| \geq 3$, and so $|S|+|R| \geq 6$, which is impossible. Hence, by (2), $\left|I_{T}\right|=\left|I_{R}\right|=2$, and w.l.o.g. let $I_{T}=\{1,2\}$ and $I_{R}=\{1, p\}$, for some $p \in\{2,3\}$. Note that by Lemma 9.15 (i) (applying to $(i, j, k)=(1, p, 5-p)), S_{2}^{1} \cup S_{1}^{p} \neq \emptyset$. Now, in order to prove that $|S|=3$, note that if $p=2$, then by (3), both $S^{3}, S_{3} \neq \emptyset$ and so $|S|=3$. Also, if $p=3$ and $|S|=2$, then since
by (3) both $S^{3}, S_{2} \neq \emptyset$, we have $S \subseteq S_{2}^{1} \cup S_{1}^{3} \cup S_{2}^{3}$. Now, considering $\mathscr{C}$, replace the cliques $X_{3}^{1}$ and $X_{2}^{1}$ with the clique $X_{3}^{1} \cup S_{2}^{3}$ and the cliques in $\mathscr{N}\left[T^{1} ; R_{3} \cup S_{2}^{1} \cup\left\{s_{2}^{2}, t_{3}^{1}\right\}\right]$ to cover the edges in $E\left(T^{1}, R_{3}\right)$. Also, replace the cliques $Z_{3}^{2}$ and $Z_{3}^{1}$ with the clique $Z_{3}^{2} \cup S_{2}^{1}$ and the cliques in $\mathscr{N}\left[R_{3} ; T^{2} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}\right]$ to cover the edges in $E\left(T^{2}, R_{3}\right)$. Moreover, replace the clique $X_{3}^{2}$ with the cliques in $\mathscr{N}\left[T^{2} ; R_{1} \cup\left\{s_{3}^{3}, t_{3}^{2}\right\}\right]$ to cover the edges in $E\left(T^{2}, R_{1}\right)$ and replace the clique $Z_{1}^{2}$ with the cliques in $\mathscr{N}\left[R_{1} ; T^{1} \cup\left\{s_{2}^{2}, r_{1}^{3}\right\}\right]$ to cover the edges in $E\left(T^{1}, R_{1}\right)$. This yields a clique covering for $G$ of size $10+|T|+|R|=n-1$, a contradiction. Therefore, $|S|=3$. This, together with (2) and inequalities $|T| \leq|R|$ and $|S|+|R| \leq 5$, implies that $\left|I_{T}\right|=|T|=\left|I_{R}\right|=|R|=2$. Finally, note that if $T$ is either complete or anticomplete to $R$, then either $\mathscr{C} \cup\{T \cup R\}$ or $\mathscr{C}$ would be a clique covering for $G$ of size at most $15=n-1$, a contradiction. This proves (4).

Henceforth, by (4) and w.l.o.g. we may assume that $I_{T}=\{1,2\}, I_{R}=\{1, p\}$, for some $p \in\{1,2,3\}$, $T^{i}=\left\{t^{i}\right\}$ and $R_{j}=\left\{r_{j}\right\}$, for every $i \in\{1,2\}$ and $j \in\{1, p\}$. Now, by Lemma 9.15, $S_{2}^{1} \cup S_{1}^{p} \neq \emptyset$ and w.l.o.g. we may assume that $S_{2}^{1} \neq \emptyset$ (note that if $S_{2}^{1}=\emptyset$, then $S_{1}^{p} \neq \emptyset$ and we may consider the rotator $\rho^{\prime}=\left(s_{1}^{1}, s_{p}^{p}, s_{5-p}^{5-p} ; r_{1}^{3}, r_{p}^{3}, r_{5-p}^{3} ; t_{3}^{1}, t_{3}^{p}, t_{3}^{5-p}\right)$, where $\left.S_{2}^{1}\left(\rho^{\prime}\right) \neq \emptyset\right)$. We observe that $r_{1}$ is either complete or anticomplete to $T$ (otherwise, if $t^{i} r_{1}$ is an edge and $t^{3-i} r_{1}$ is a non-edge, then $\left\{s_{3-i}^{1}, t^{1}, t^{2}, r_{1}\right\}$ induces a claw). Now, if $r_{1}$ is anticomplete to $T$, then so is $r_{p}$ (otherwise, if $t^{i} r_{p}$ is an edge, for some $i \in\{1,2\}$, then $\left\{r_{p}, s_{i}^{1}, t^{i}, r_{1}\right\}$ induces a claw) and thus $T$ is anticomplete to $R$, which contradicts (4). Thus, $r_{1}$ is complete to $T$. Note that by (4), $r_{p}$ is nonadjacent to $t^{i_{0}}$, for some $i_{0} \in\{1,2\}$. Thus, $s_{i_{0}}^{p} \notin S$ (otherwise, $\left\{r_{1}, s_{i_{0}}^{p}, t^{i_{0}}, r_{p}\right\}$ would be a claw). Now, add the clique $T \cup R_{1}$ to $\mathscr{C}$, and if $r_{p}$ is adjacent to $t^{3-i_{0}}$, then replace the clique $X_{i_{0}}^{3-i_{0}}$ with the clique $X_{i_{0}}^{3-i_{0}} \cup\left\{r_{p}\right\}$ to cover the edge $t^{3-i_{0}} r_{p}$, thereby obtaining a clique covering for $G$ of size $15=n-|S|+2$, which by (4) is equal to $n-1$, a contradiction. This proves Lemma 9.17.

Lemma 9.18. Let $G$ be a counterexample to Theorem 9.1 with $\left|I_{T}(\rho)\right|,\left|I_{R}(\rho)\right| \geq 2$, for some rotator $\rho$ of $\bar{G}$, say $\{i, j\} \subseteq I_{T}(\rho)$ for some $i \neq j$. Also, let $t^{i} \in T^{i}(\rho)$ is nonadjacent to $t^{j} \in T^{j}(\rho)$ and $R(\rho)$ is a clique. If one of the following holds, then exactly one of $t^{i}$ and $t^{j}$ is complete to $R(\rho)$ and the other one is anticomplete to $R(\rho)$.
(i) $i \in I_{R}(\rho)$ and $S_{j}^{i}(\rho) \neq \emptyset$.
(ii) There exists $l \in I_{R}(\rho) \backslash\{i, j\}$ such that for every $l^{\prime} \in I_{R}(\rho) \cap\{i, j\}, S_{l^{\prime}}^{l}(\rho) \neq \emptyset$.

The same statement holds when $T$ is replaced with $R$ and the subscripts and the superscripts are interchanged.

Proof. First, assume that (i) holds. Note that if $t^{i}$ is nonadjacent to some $r_{i} \in R_{i}$, then $t^{i}$ is anticomplete to $R \backslash R_{i}$ (otherwise $\left\{r, s_{i}^{i}, t^{i}, r_{i}\right\}$ is a claw, for some $r \in R \backslash R_{i}$ adjacent to $t^{i}$ ). Consequently, $t^{i}$ is anticomplete to $R_{i}$ (otherwise by Lemma 9.13 (vi), $\left\{r, s_{j}^{i}, t^{j}, r_{i}^{\prime}\right\}$ would be a claw for every $r_{i}^{\prime} \in R_{i}$ adjacent to $t^{i}$ and every $r \in R \backslash R_{i}$ ). Therefore, $t^{i}$ is anticomplete to $R$ and by Lemma 9.13 (vi), $t^{j}$ is complete to $R$, as desired. Now, if $t^{i}$ is complete to $R_{i}$, then so is to $R \backslash R_{i}$ (otherwise $\left\{r, s_{j}^{i}, t^{j}, r_{i}\right\}$ would be a claw for some $r \in R \backslash R_{i}$ and every $r_{i} \in R_{i}$ ). Thus, $t^{i}$ is complete to $R$ and so by Lemma 9.13 (vi), $t^{j}$ is anticomplete to $R$.
Next, suppose that (ii) holds, and w.l.o.g. assume that $i \in I_{R}$. Note that if $t^{i}$ is nonadjacent to some $r_{i} \in R_{i}$, then $t^{i}$ is anticomplete to $R \backslash R_{i}$ (otherwise $\left\{r, s_{i}^{i}, t^{i}, r_{i}\right\}$ is a claw, for some $\left.r \in R \backslash R_{i}\right)$. Consequently, $t^{i}$ is anticomplete to $R_{i}$ (otherwise $\left\{r_{i}^{\prime}, s_{i}^{l}, t^{i}, r\right\}$ would be a claw for every $r_{i}^{\prime} \in R_{i}$ adjacent to $t^{i}$ and every $r \in R_{l}$ ), and thus anticomplete to $R$, as required. Now, if $t^{i}$ is complete to $R_{i}$, then $t^{i}$ is also complete to $R_{l}$ (otherwise, $\left\{r_{i}, s_{i}^{l}, t^{i}, r_{l}\right\}$ would be a claw for some $r_{l} \in R_{l}$ nonadjacent to $t^{i}$ and every $r_{i} \in R_{i}$ ) and so complete to $R_{j}$ (otherwise, $\left\{r_{j}, s_{j}^{l}, t^{j}, r_{l}\right\}$ would be a claw for some $r_{j} \in R_{j}$ nonadjacent to $t^{i}$ and every $r_{l} \in R_{l}$ ), as desired. This proves Lemma 9.18.

### 9.3 Types of $G[T]$ and $G[R]$

In order to construct appropriate clique coverings for $G$, we need to know more about the structure of $G[T]$ and $G[R]$. Let us begin with a definition (all definitions and theorems here are stated in terms of $G[T]$ and the analogous ones are also valid for $G[R]$, where $T$ is replaced with $R$ and the subscripts and superscripts are interchanged).

Note that if $\left|I_{T}(\rho)\right| \leq 1$, for some rotator $\rho$ of $\bar{G}$, then $T(\rho)$ is a clique of $G$. Now, assume that $\left|I_{T}(\rho)\right|=2$, say $I_{T}(\rho)=\left\{q, q^{\prime}\right\}$. We say that $G[T(\rho)]$ is of type 1 , if there exist nonadjacent vertices $t^{q} \in T^{q}(\rho)$ and $t^{q^{\prime}} \in T^{q^{\prime}}(\rho)$ such that $t^{q}$ is complete to $U_{q}(\rho)=T^{q^{\prime}}(\rho) \backslash\left\{t^{q^{\prime}}\right\}$ and $t^{q^{\prime}}$ is complete to $U_{q^{\prime}}(\rho)=T^{q}(\rho) \backslash\left\{t^{q}\right\}$. If $U_{q}(\rho)$ (resp. $\left.U_{q^{\prime}}(\rho)\right)$ is empty, then we say that $G[T(\rho)]$ is of type 1.1 at $q$ (resp. $q^{\prime}$ ). If both $U_{q}(\rho)$ and $U_{q^{\prime}}(\rho)$ are nonempty, then we say that $G[T(\rho)]$ is of type 1.2. The following lemma more or less reveals the structure of $G[T(\rho)]$ in case of $\left|I_{T}(\rho)\right|=2$. The proof is obvious and left to the reader.

Lemma 9.19. Let $G$ be a counterexample to Theorem 9.1 with $\left|I_{T}(\rho)\right|=2$, for some rotator $\rho$ of $\bar{G}$. Then, either $T(\rho)$ is a clique or $G[T(\rho)]$ is of type 1 .

Next, suppose that $\left|I_{T}(\rho)\right|=3$. Let $q \in\{1,2,3\}$. We say that $G[T(\rho)]$ is of type 2 at $q$, if $T^{q^{\prime}}(\rho)=\left\{t^{q^{\prime}}\right\}$ for every $q^{\prime} \in\{1,2,3\} \backslash\{q\}$ and there exists $t^{q} \in T^{q}(\rho)$ which is anticomplete to $T(\rho) \backslash T^{q}(\rho)$ and $T(\rho) \backslash\left\{t^{q}\right\}$ is a clique. Moreover, we say that $G[T(\rho)]$ is of type 3 at $q$, if $T^{q}(\rho)=\left\{t_{1}^{q}, t_{2}^{q}\right\}$ and there exists $t_{1}^{q+1} \in T^{q+1}(\rho)$ and $t_{2}^{q+2} \in T^{q+2}(\rho)$ such that both $T_{2}^{q+1}(\rho)=T^{q+1}(\rho) \backslash\left\{t_{1}^{q+1}\right\}$ and $T_{1}^{q+2}(\rho)=T^{q+2}(\rho) \backslash\left\{t_{2}^{q+2}\right\}$ have cardinality at most one (this implies that $4 \leq|T| \leq 6$ and we denote the unique possible members of $T_{2}^{q+1}(\rho)$ and $T_{1}^{q+2}(\rho)$ by $t_{2}^{q+1}$ and $t_{1}^{q+2}$, respectively), the sets $T_{1}(\rho)=\left\{t_{1}^{q}, t_{1}^{q+1}\right\} \cup T_{1}^{q+2}(\rho)$ and $T_{2}(\rho)=\left\{t_{2}^{q}, t_{2}^{q+2}\right\} \cup T_{2}^{q+1}(\rho)$ are both cliques of $G$ and there are no more edges in $E(T(\rho))$ except the edges in $E\left(T^{1}(\rho)\right) \cup E\left(T^{2}(\rho)\right) \cup E\left(T^{3}(\rho)\right)$. Now, we state the following lemma, which discloses the structure of $G[T(\rho)]$ when $\left|I_{T}(\rho)\right|=3$.

Lemma 9.20. Let $G$ be a counterexample to Theorem 9.1 with $\left|I_{T}(\rho)\right|=3$, for some rotator $\rho$ of $\bar{G}$. Then either $T(\rho)$ is a clique or $G[T(\rho)]$ is of type $\tau$ at $q$, for some $\tau \in\{2,3\}$ and some $q \in\{1,2,3\}$.

Proof. In the proof, we omit the term $\rho$. Suppose that $T$ is not a clique. First, assume that for every $q \in\{1,2,3\}$, there is $q^{\prime} \in\{1,2,3\} \backslash\{q\}$ and some vertex $t^{q^{\prime}} \in T^{q^{\prime}}$ anticomplete to $T^{q}$. Thus, $\left(t^{q^{\prime}}, t_{3}^{q^{\prime \prime}}, T^{q}\right)$ is a quasi-triad, where $\left\{q, q^{\prime}, q^{\prime \prime}\right\}=\{1,2,3\}$, and thus by Lemma 9.13 (i), for every $q \in\{1,2,3\}$, we have $\left|T^{q}\right|=1$, say $T^{q}=\left\{t^{q}\right\}$. On the other hand, by Lemma 9.13 (iv), the set of neighbours and non-neighbours of $t^{1}$ in $T \backslash T^{1}$ are both cliques, and also since $T$ is not a clique, $t^{1}$ is not complete to $T \backslash T^{1}$. Now, if $t^{1}$ is nonadjacent to both $t^{2}$ and $t^{3}$, then by Lemma 9.13 (iv), $t^{2}$ and $t^{3}$ are adjacent and $G[T]$ is of type 2 at 1 , and if $t^{1}$ is adjacent to $t^{2}$ (resp. $t^{3}$ ) and nonadjacent to $t^{3}$ (resp. $t^{2}$ ), $t^{2}, t^{3}$ are nonadjacent and so $G[T]$ is of type 2 at 3 (resp. 2).
Next, assume that there exists $q \in\{1,2,3\}$, say $q=1$, such that every vertex in $T \backslash T^{q}$ has a neighbour in $T^{q}$. If every vertex in $T \backslash T^{1}$ is complete to $T^{1}$, then by Lemma 9.13 (iv), $T$ is a clique, a contradiction. Thus, we may assume that there exists a vertex $t^{1} \in T^{1}$, nonadjacent to a vertex in $T \backslash T^{1}$, say $t^{2} \in T^{2}$. Define $A=N\left(t^{1}, T^{2}\right), B=N\left(t^{1}, T^{3}\right), C=N\left(t^{2}, T^{1}\right)$ and $D=N\left(t^{2}, T^{3}\right)$, where by the assumption, $C \neq \emptyset$. Then,
(1) The sets $A \cup B \cup\left\{t^{1}\right\}$ and $C \cup D \cup\left\{t^{2}\right\}$ are both cliques and there are no more edges in $E(T)$ except the edges in $E\left(T^{1}\right) \cup E\left(T^{2}\right) \cup E\left(T^{3}\right)$. Also, $|A|,|B|,|D| \leq 1$.
By Lemma 9.13 (iv), it is evident that $A \cup B \cup\left\{t^{1}\right\}$ and $C \cup D \cup\left\{t^{2}\right\}$ are both cliques, $t^{1}$ is anticomplete to $D, t^{2}$ is anticomplete to $B$, and $(B, D)$ is a partition of $T^{3}$. Also, since $\left(t^{1}, t_{3}^{2}, D\right)$ and $\left(t_{3}^{1}, t^{2}, B\right)$ are quasi-triads of $G$, by Lemma 9.13 (i), $|B|,|D| \leq 1$, and since $t^{1}$ and $t_{3}^{2}$ are complete to $A, A$ is anticomplete to $D$. Similarly, since $t_{3}^{1}$ and $t^{2}$ are complete to $C, C$ is anticomplete to $B$. Now, either $B$ or $D$ is nonempty. If $B$ is nonempty, say $B=\{b\}$, then $\left(t_{3}^{2}, b, C\right)$ is a quasi-triad and $t_{3}^{2}$ and $b$ are complete to $A$. Thus, Lemma 9.13 (i) implies that $A$ is anticomplete to $C$. Also, if $D$ is nonempty, say $D=\{d\}$, then $\left(t_{3}^{1}, d, A\right)$ is a quasi-triad and $t_{3}^{1}$ and $d$ are complete to $C$. Thus, again by Lemma 9.13 (i), $A$ is anticomplete to $C$. Finally, since for every $c \in C,\left(t_{3}^{3}, c, A\right)$ is a quasi-triad, by Lemma 9.13 (i), $|A| \leq 1$. This proves (1).
Now, by (1), if both $A$ and $B$ are empty, then $G[T]$ is of type 2 at 1 . If either $A$ or $B$ is nonempty, say $B=\{b\}$, then $\left(t_{3}^{2}, b, C\right)$ is a quasi-triad and by Lemma 9.13 (i), $|C|=1$. Hence, $G[T]$ is of type 3 at 1. This proves Lemma 9.20.

We close this subsection with the following lemma, which includes the first application of Lemma 9.20.

Lemma 9.21. Let $G$ be a counterexample to Theorem 9.1. Then for every rotator $\rho$ of $\bar{G}$, we have $\left|I_{T}(\rho)\right|,\left|I_{R}(\rho)\right| \geq 2$.

Proof. On the contrary and w.l.o.g. assume that $\left|I_{R}\right| \leq 1$. By Lemma 9.16, we have $\left|I_{T}\right|=3$ and $\left|I_{R}\right|=1$, say $I_{R}=\{p\}$, for some $p \in\{1,2,3\}$. Now, merge the pair $\left(Z_{p+1}^{p}, Z_{p+2}^{p}\right)$ in $\mathscr{O}(G)$ and call the resulting collection $\mathscr{O}^{\prime}(G)$.
(1) If $G[T]$ is of type 2 at $q$, for some $q \in\{1,2,3\}$, then $p \neq q$.

Suppose not, w.l.o.g. assume that $p=q=1$. Let $t^{1}, t^{2}$ and $t^{3}$ be as in the definition of type 2 at 1 , and $X=T^{1} \backslash\left\{t^{1}\right\}$. Note that by Lemma $9.13(\mathrm{vi}), R_{1}$ can be partitioned into $R^{1}=N\left(t^{1}, R_{1}\right)$ and $R^{2}=$ $N\left(t^{2}, R_{1}\right)=N\left(t^{3}, R_{1}\right)$. Also, the cliques in $\mathscr{O}_{1}(G)=\mathscr{O}^{\prime}(G) \cup\left\{R^{3} \cup\left\{t^{1}\right\}, R^{3} \cup T \backslash\left\{t^{1}\right\}, T_{3} \cup R_{1}\right\}$ cover all the edges in $E(G) \backslash E(T, R)$. We claim that $S^{1} \neq \emptyset$. For if $S^{1}=\emptyset$, then applying Lemma 9.15 (i) to $(i, j, k)=(1,2,3)$ and $(i, j, k)=(1,3,2)$ implies that $\left|S_{1}\right|=2$. Now in $\mathscr{O}_{1}(G)$, replace the cliques $X_{2}^{1}, X_{3}^{2}$ and $X_{2}^{3}$ with the cliques $\left(X_{2}^{1} \cup R^{1}\right) \backslash X, X_{3}^{2} \cup R^{2}$ and $X_{2}^{3} \cup R^{2}$, and also add the cliques in $\mathscr{N}\left[X ; R_{1} \cup S_{2} \cup\left\{s_{2}^{2}\right\}\right]$, thereby obtaining a clique covering for $G$ of size $14+|X|=n-|S|-|R|+2 \leq n-1$, a contradiction. This proves the claim. Thus, assume that $s_{k}^{1} \in S$, for some $k \in\{2,3\}$. Now, in $\mathscr{O}_{1}(G) \cup \mathscr{N}[R ; T]$, if $S_{1}=\emptyset$, then remove the clique $R^{3} \cup\left\{t^{1}\right\}$ and replace the cliques $Z_{3}^{2}$ and $Z_{2}^{3}$ with the cliques $Z_{3}^{2} \cup\left\{t^{1}\right\}$ and $Z_{2}^{3} \cup\left\{t^{1}, r_{1}^{3}\right\}$ and call the resulting family $\mathscr{C}$, which is a clique covering for $G$. By the claim, if either $X \neq \emptyset$, or $|S| \geq 3$, or $|S|=2$ and $S_{1}=\emptyset$, then $|\mathscr{C}| \leq n-1$, a contradiction. Thus, $X=\emptyset$ and either $|S|=1$ or $|S|=2$ and $S_{1} \neq \emptyset$. Therefore, by the above claim, $S=S_{k}^{1} \cup S_{1}^{k^{\prime}}$ for some $k^{\prime} \in\{2,3\}$ and $|\mathscr{C}|=n$. Now, if $\left|R_{1}\right|=1$, say $R_{1}=\left\{r_{1}\right\}$, then remove the single clique in $\mathscr{N}[R ; T]$ from $\mathscr{C}$ and replace the cliques $Z_{1}^{2}$ and $Z_{1}^{3}$ with the cliques $Z_{1}^{k^{\prime}} \cup N\left(r_{1}, T^{5-k^{\prime}}\right)$ and $Z_{1}^{5-k^{\prime}} \cup N\left(r_{1}, T^{1} \cup T^{k^{\prime}}\right)$ in the resulting family, and if $\left|R_{1}\right| \geq 2$, then remove the cliques in $\mathscr{N}[R ; T]$ from $\mathscr{C}$, replace the clique $X_{5-k}^{1}$ with the clique $X_{5-k}^{1} \cup R^{1}$ and add the clique $R^{2} \cup\left\{t^{2}, t^{3}\right\}$. This procedure yields a clique covering for $G$ of size at most $n-1$, a contradiction. This proves (1).
(2) $G[T]$ is of type 3 .

Suppose not, by Lemmas 9.17 and $9.20, G[T]$ is of type 2 at $q$, for some $q \in\{1,2,3\}$. By ( 1 ), $p \neq q$, say $p=2$ and $q=1$. Let $t^{1}, t^{2}$ and $t^{3}$ be as in the definition of type 2 at 1 , and $X=T^{1} \backslash\left\{t^{1}\right\}$. Note that by Lemma 9.13 (vi), $R_{2}$ can be partitioned into $R^{1}=N\left(t^{1}, R_{2}\right)$ and $R^{2}=N\left(t^{2}, R_{2}\right)=N\left(t^{3}, R_{2}\right)$. Also, the cliques in $\mathscr{O}_{2}(G)=\mathscr{O}^{\prime}(G) \cup\left\{R^{3} \cup\left\{t^{1}\right\}, R^{3} \cup T \backslash\left\{t^{1}\right\}, T_{3} \cup R_{2}\right\}$ cover all the edges in $E(G) \backslash E(T, R)$. We claim that either $S^{1} \neq \emptyset$ or $S=\left\{s_{1}^{2}\right\}$. On the contrary, assume that $S^{1}=\emptyset$ and $S \neq\left\{s_{1}^{2}\right\}$. Then, applying Lemma 9.15 (i) to $(i, j, k)=(1,2,3)$ implies that $s_{1}^{2} \in S$ and thus $|S| \geq 2$. Now, in $\mathscr{O}_{2}(G)$, replace the cliques $X_{2}^{1}$ and $X_{3}^{1}$ with the cliques $\left(X_{2}^{1} \cup S_{3}^{2}\right) \backslash S_{2}^{3},\left(X_{3}^{1} \cup S_{2}^{3} \cup R^{1}\right) \backslash\left(S_{3}^{2} \cup X\right)$, replace the clique $Z_{2}^{1}$ with the cliques in $\mathscr{N}\left[R_{2} ;\left\{r_{2}^{3}, s_{1}^{1}, t^{2}, t^{3}\right\}\right]$ and also add the cliques in $\mathscr{N}\left[X ; R_{2} \cup S_{2}^{3} \cup\left\{s_{3}^{3}\right\}\right]$, thereby obtaining a clique covering $\mathscr{C}$ for $G$ of size $13+|R|+|X|=n-|S|+1 \leq n-1$, a contradiction. This proves the claim. Now, in $\mathscr{O}_{2}(G) \cup \mathscr{N}[R ; T]$, if either $S_{1}=\emptyset$ or $S=\left\{s_{1}^{2}\right\}$, then remove the clique $R^{3} \cup\left\{t^{1}\right\}$ and replace the cliques $Z_{3}^{1}, Z_{3}^{2}$ and $Z_{1}^{3}$ with the cliques $Z_{3}^{1} \cup S_{1}^{2},\left(Z_{3}^{2} \cup\left\{t^{1}\right\}\right) \backslash S_{1}^{2}$ and $Z_{1}^{3} \cup\left\{t^{1}, r_{2}^{3}\right\}$ and call the resulting family $\mathscr{C}$, which is a clique covering for $G$. If either $X \neq \emptyset$, or $|S| \geq 3$, or $|S|=2$ and $S_{1}=\emptyset$, then $|\mathscr{C}| \leq n-1$, a contradiction. Thus, $X=\emptyset$ and either $|S|=1$ or $|S|=2$ and $S_{1} \neq \emptyset$. Therefore, by the above claim, $S=S_{1} \cup S^{1}$, and in particular, either $|S|=\left|S^{1}\right|=1$, or $|S|=\left|S_{1}^{2}\right|=1$, or $\left|S_{1}\right|=\left|S^{1}\right|=1$. Consequently, $|\mathscr{C}| \leq n$. Then, remove the cliques in $\mathscr{N}[R ; T]$ from $\mathscr{C}$ and in order to cover the edges in $E(T, R)$, replace the cliques $X_{3}^{1}$ and $X_{3}^{2}$ with the cliques $X_{3}^{1} \cup R^{1}$ and $X_{3}^{2} \cup R^{2}$. Also, if $s_{1}^{2} \notin S$, then replace the clique $X_{1}^{3}$ with the clique $X_{1}^{3} \cup R^{2}$ and if $s_{1}^{2} \in S$, then replace the cliques $X_{1}^{3}$ and $X_{2}^{3}$ with the cliques $\left(X_{1}^{3} \cup S_{2}^{1} \cup R^{2}\right) \backslash S_{1}^{2}$ and $\left(X_{2}^{3} \cup S_{1}^{2}\right) \backslash S_{2}^{1}$. This yields a clique covering of size at most $n-1$, a contradiction. This proves (2).

By (2), $G[T]$ is of type 3 at $q$, for some $q \in\{1,2,3\}$, say $q=1$. Let $T_{1}=\left\{t_{1}^{1}, t_{1}^{2}\right\} \cup T_{1}^{3}$ and $T_{2}=$ $\left\{t_{2}^{1}, t_{2}^{3}\right\} \cup T_{2}^{2}$ be as in the definition of type 3 at 1 . Note that by Lemma 9.13 (vi), $R_{p}$ can be partitioned into the sets $R^{1}$ and $R^{2}$ such that for every $i \in\{1,2\}, T_{i}$ is complete to $R^{i}$ and anticomplete to $R^{3-i}$. Now adding the cliques in $\mathscr{N}[R ; T]$ to the family of cliques $\mathscr{O}_{3}(G)=\mathscr{O}^{\prime}(G) \cup\left\{T_{1} \cup R^{3}, T_{2} \cup R^{3}, T_{3} \cup R_{p}\right\}$ provides a clique covering for $G$ of size $14+|R|=n-|S|-|T|+5$. Thus, $|S|+|T| \leq 5$. Also, if $|T|=5$, then $S=\emptyset$, and in this case, it is enough to remove the clique $\Omega_{p}$ from $\mathscr{P}(G)$ and add the cliques $T_{1} \cup R^{1} \cup\left\{r_{p}^{3}\right\}$ and $T_{2} \cup R^{2} \cup\left\{r_{p}^{3}\right\}$, thereby obtaining a clique covering for $G$ of size $14=n-|R| \leq n-1$,
which is impossible. Thus, $|T|=4$ and $|S| \leq 1$. First, assume that either $S=\emptyset$ or $|S|=1$ and both $R^{1}, R^{2} \neq \emptyset$. In this case, considering $\mathscr{P}(G)$, add the cliques in $\mathscr{N}\left[R_{p} ; T \cup\left\{r_{p}^{3}\right\}\right]$, if $S=\emptyset$, then replace the cliques $\Delta_{3}^{2}$ and $\Delta_{2}^{3}$ with the cliques $\Delta_{3}^{2} \cup\left\{t_{2}^{1}\right\}$ and $\Delta_{2}^{3} \cup\left\{t_{1}^{1}\right\}$ and if $S_{p}=\emptyset$, then remove the clique $\Omega_{p}$, to obtain a clique covering for $G$ of size at most $n-1$, a contradiction. Therefore, we have $|S|=1$ and $R_{p}=R^{i_{0}}$, for some $i_{0} \in\{1,2\}$. Now, if $S_{p+2}^{p+1} \cup S_{p+1}^{p+2}=\emptyset$, then let $\left\{p, p^{\prime}, p^{\prime \prime}\right\}=\{1,2,3\}$ such that $S_{p}^{p^{\prime}}=\emptyset$, and in $\mathscr{O}_{3}(G)$, replace the cliques $Z_{p}^{p^{\prime}}$ and $Z_{p}^{p^{\prime \prime}}$ with the cliques $Z_{p}^{p^{\prime}} \cup\left(T_{i_{0}} \cap T^{p^{\prime \prime}}\right)$ and $Z_{p}^{p^{\prime \prime}} \cup\left(T_{i_{0}} \cap\left(T^{p} \cup T^{p^{\prime}}\right)\right)$. Also, if $S=S_{p+2}^{p+1} \cup S_{p+1}^{p+2}$, then in $\mathscr{O}_{3}(G)$, replace the cliques $Z_{p}^{p+1}$ and $Z_{p}^{p+2}$ with the clique $\left(Z_{p}^{p+1} \cup S_{p+1}^{p+2} \cup\left(T_{i} \backslash T^{p+1}\right)\right) \backslash S_{p+2}^{p+1}$ and $\left(Z_{p}^{p+2} \cup S_{p+2}^{p+1} \cup\left(T_{i} \cap T^{p+1}\right)\right) \backslash S_{p+1}^{p+2}$. This provides a clique covering for $G$ of size $14=n-\left|R_{p}\right| \leq n-1$, again a contradiction. This proves Lemma 9.21.

### 9.4 Schläfli-antiprismatic graphs

Many of the classes of graphs in the "menagerie" introduced in [4] are obtained by some modifications on some induced subgraphs of the Schläfli graph $\Gamma$ (see Subsection 9.2). Our goal in this subsection is to provide appropriate clique coverings for some induced subgraphs of $\Gamma$, and obtain more properties of the graph $G$, the counterexample to Theorem 9.1. Let us begin with a definition. Let $G$ be an antiprismatic graph whose complement contains a rotator $\rho$. Suppose that for every $i \in\{1,2,3\}$, $T^{i}(\rho) \subseteq\left\{t_{1}^{i}, t_{2}^{i}\right\}$ and $R_{i}(\rho) \subseteq\left\{r_{i}^{1}, r_{i}^{2}\right\}$ and adjacency in $G[T \cup R]$ is the same as in $\Gamma[T \cup R]$. Then, $G$ is an induced subgraph of $\Gamma$ and we say that the graph $G$ is Schläfli-antiprismatic (with respect to $\rho$ ). Also, $G$ is called inflated Schläfli-antiprismatic (ISA for short) with respect to $\rho$ if $G$ is obtained from a Schläfli-antiprismatic graph by replicating vertices in the non-core.
The following is a technical lemma whose proof is lengthy and is postponed to Appendix D.
Lemma 9.22. Let $G$ be a counterexample to Theorem 9.1 such that neither $T(\rho)$ nor $R(\rho)$ is a clique of $G$, for some rotator $\rho$ of $\bar{G}$. Then $G$ is not ISA with respect to $\rho$.

Now, using the above lemma, in the following lemma we deduce more properties of the counterexample to Theorem 9.1 to be used in forthcoming subsections. We omit the term $\rho$ within the proofs.
Theorem 9.23. Let $G$ be a counterexample to Theorem 9.1 with $\left|I_{T}(\rho)\right|+\left|I_{R}(\rho)\right| \geq 5$, for some rotator $\rho$ of $\bar{G}$. Then, either $T(\rho)$ or $R(\rho)$ is a clique.

In order to prove Theorem 9.23, we need the following three lemmas.
Lemma 9.24. Let $G$ be a counterexample to Theorem 9.1 with $\left|I_{T}(\rho)\right|=3, I_{R}(\rho)=\{1,2\}$, for some rotator $\rho$ of $\bar{G}$. If $G[T(\rho)]$ is of type 2 at $q$, for some $q \in\{1,2\}$, then $R(\rho)$ is a clique of $G$.
Proof. By symmetry, assume that $G[T]$ is of type 2 at $q=1$ with $t^{1}, t^{2}$ and $t^{3}$ as in the definition. Also, let $X=T^{1} \backslash\left\{t^{1}\right\}$. By the contrary, assume that $R$ is not a clique and so by Lemma 9.19, $G[R]$ is of type 1 with $r_{1} \in R_{1}, r_{2} \in R_{2}, U^{1} \subseteq R_{2}$ and $U^{2} \subseteq R_{1}$ as in the definition. Note that by Lemma 9.15 (i)(applying to $(i, j, k)=(1,2,3)$ ), there is a vertex $s_{k^{\prime}}^{k} \in S \cap\left\{s_{1}^{2}, s_{2}^{1}, s_{3}^{1}\right\}$. First, assume that $r_{1}$ is adjacent to $t^{1}$. Thus, by Lemma 9.13 (vi), $r_{2}$ is complete to $\left\{t^{2}, t^{3}\right\}$. For every $i \in\{1,2\}$ and every $r \in U^{i}$, let $A_{i}(r)=\left\{s_{k^{\prime}}^{k}, t^{k^{\prime}}, r_{i}, r\right\}$ and $B_{i}(r)=\left\{s_{i}^{i}, t^{i}, r_{i}, r\right\}$. Now, since $A_{i}(r), i=1,2, r \in U^{i}$, is not a claw, $t^{1}$ is complete to $U^{1}$ and anticomplete to $U^{2}$ and $\left\{t^{2}, t^{3}\right\}$ is anticomplete to $U^{1}$ and complete to $U^{2}$. Also, $r_{2}$ is complete to $X$ (otherwise, for $x \in X$ nonadjacent to $r_{2}$, if $\left(k, k^{\prime}\right)=(2,1)$, then $\left\{t^{2}, s_{1}^{2}, r_{2}, x\right\}$ would be a claw and if $\left(k, k^{\prime}\right)=(1, j)$, for some $j \in\{2,3\}$, then $\left\{x, s_{j}^{1}, t^{j}, r_{1}\right\}$ would be a claw). Thus, $r_{1}$ is anticomplete to $X$. Moreover, since $\left(s_{1}^{1}, t^{1}, U^{2}\right)$ is a quasi-triad of $G$ and $U^{1}$ is complete $\left\{s_{1}^{1}, t^{1}\right\}$, by Lemma 9.13 (i), $U^{1}$ is anticomplete to $U^{2}$, and also since $\left(s_{1}^{1}, r_{1}, X\right)$ is a quasi-triad of $G$ and $U^{1}$ is complete $\left\{s_{1}^{1}, r_{1}\right\}$ (resp. $U^{2}$ is anticomplete to $s_{1}^{1}$ ), $U^{1}$ is anticomplete to $X$ (resp. $U^{2}$ is complete to $X$ ). Hence, $G$ is ISA, a contradiction with Lemma 9.22.
Now, assume that $t^{1}$ is nonadjacent to $r_{1}$. By Lemma 9.13 (vi), $r_{1}$ is complete to $\left\{t^{2}, t^{3}\right\}$ and $r_{2}$ is adjacent to $t^{1}$ and nonadjacent to $t^{2}, t^{3}$. Now, since $B_{i}(r), i=1,2, r \in U^{i}$, is not a claw, by Lemma 9.13 (vi), $t^{1}$ is anticomplete to $U^{1}$ and complete to $U^{2}$ and $\left\{t^{2}, t^{3}\right\}$ is complete to $U^{1}$ and anticomplete to $U^{2}$. Also, $r_{1}$ is complete to $X$ and thus $r_{2}$ is anticomplete to $X$ (otherwise, for every
$x \in X$ nonadjacent to $r_{1},\left\{t^{2}, s_{1}^{1}, r_{1}, x\right\}$ would be a claw). Moreover, since $\left(s_{k^{\prime}}^{k}, t^{k^{\prime}}, U^{3-k}\right)$ is a quasi-triad of $G$ and $U^{k}$ is complete to $\left\{s_{k^{\prime}}^{k}, t^{k^{\prime}}\right\}$, by Lemma 9.13 (i), $U^{1}$ is anticomplete to $U^{2}$. Furthermore, if $\left(k, k^{\prime}\right)=(1, j)$, for some $j \in\{2,3\}$, then $\left(s_{j}^{1}, t^{j}, U^{2}\right)$ is a quasi-triad and if $\left(k, k^{\prime}\right)=(2,1)$, then $\left(s_{1}^{2}, r_{2}, X\right)$ is a quasi-triad. Therefore, by Lemma 9.13 (i), $X$ is anticomplete to $U^{2}$. Now, if either $\left(k, k^{\prime}\right)=(2,1)$, or $X=\emptyset$, or $U^{1}=\emptyset$, or $U^{2} \neq \emptyset$, then $X$ is complete to $U^{1}$ and thus, $G$ is ISA, which contradicts Lemma 9.22. Hence, $S_{1}^{2}=\emptyset, S_{2}^{1} \cup S_{3}^{1} \neq \emptyset, X \neq \emptyset, U^{1} \neq \emptyset$ and $U^{2}=\emptyset$. Assume that $\left\{X, U^{1}\right\}=\{A, B\}$, where $1 \leq|A| \leq|B|$. In the sequel, we are going to give a clique covering $\mathscr{C}$ for $G$ of size at most $n-|B|+1$.
To see this, consider the family of cliques $\mathscr{O}(G) \cup\left\{T^{1} \cup R^{3},\left(T \backslash\left\{t^{1}\right\}\right) \cup R^{3}, T_{3} \cup R_{2}, T_{3} \cup\left(R \backslash\left\{r_{2}\right\}\right)\right\}$, replace the cliques $X_{1}^{2}, X_{1}^{3}$ and $Z_{1}^{2}$ with the cliques $X_{1}^{2} \cup U^{1}, X_{1}^{3} \cup U^{1}$ and $Z_{1}^{2} \cup X$ and add the clique $\left\{t^{1}, r_{2}\right\}$ as well as the cliques in $\mathscr{N}[A ; B]$, and call the resulting collection $\mathscr{O}^{\prime}(G)$. The cliques in $\mathscr{O}^{\prime}(G)$ cover all the edges in $E(G) \backslash\left\{t^{2} r_{1}, t^{3} r_{1}\right\}$. In order to cover $t^{2} r_{1}$ and $t^{3} r_{1}$, in $\mathscr{O}^{\prime}(G)$, if $S_{1}=\emptyset$, then replace the cliques $X_{1}^{2} \cup U^{1}, X_{1}^{3} \cup U^{1}, X_{3}^{2}$ and $X_{2}^{3}$ with the cliques $X_{1}^{2} \cup U^{1} \cup S_{3}^{1}, X_{1}^{3} \cup U^{1} \cup S_{2}^{1},\left(X_{3}^{2} \cup R_{1}\right) \backslash S_{3}^{1}$ and $\left(X_{2}^{3} \cup R_{1}\right) \backslash S_{2}^{1}$, if $S_{1} \neq \emptyset$ and $S_{3}^{2} \cup S_{2}^{3}=\emptyset$, then replace the cliques $Z_{1}^{2} \cup X$ and $Z_{1}^{3}$ with the cliques $Z_{1}^{2} \cup X \cup\left\{t^{3}\right\}$ and $Z_{1}^{3} \cup\left\{t^{2}\right\}$, and finally, if $S_{1} \neq \emptyset$ and $S_{3}^{2} \cup S_{2}^{3} \neq \emptyset$, then add the clique $\left\{t^{2}, t^{3}, r_{1}\right\}$ and call the resulting clique covering $\mathscr{C}$. In the latter case, $|S| \geq 3$ and $|\mathscr{C}|=18+|A| \leq n-|B|+1$, as required. Also, in the first two cases, $|\mathscr{C}|=17+|A|=n-|S|-|B|+3$, which implies the claim, when $|S| \geq 2$. Now, assume that $|S|=1$, i.e. $|S|=\left|S_{p}^{1}\right|=1$. In this case, take $\mathscr{C}$ (which is obtained by the first modification on $\left.\mathscr{O}^{\prime}(G)\right)$ and remove the clique $T_{3} \cup\left(R \backslash\left\{r_{2}\right\}\right)$, and to cover the edges in $E\left(R_{1}, T_{3} \cup U^{1}\right)$, replace the cliques $\left(X_{3}^{2} \cup R_{1}\right) \backslash S_{3}^{1}, Z_{1}^{2} \cup X$ and $Z_{3}^{2}$ with the cliques $\left(X_{3}^{2} \cup R_{1} \cup U^{1}\right) \backslash S_{3}^{1},\left\{r_{1}, s_{2}^{2}, t_{3}^{1}\right\} \cup X$ and $\left\{r_{1}^{3}, r_{3}^{3}, s_{2}^{2}\right\}$. This yields a clique covering of size $16+|A|=n-|B|+1$, as desired. Therefore, $|B|=1$ and thus $|X|=\left|U^{1}\right|=1$. Now, if $X$ is complete to $U^{1}$, then $G$ is ISA, which contradicts Lemma 9.22, and if $X$ is anticomplete to $U^{1}$, then removing the single clique in $\mathscr{N}[A ; B]$ from $\mathscr{C}$ yields a clique covering for $G$ of size $n-1$, a contradiction. This proves Lemma 9.24.

Lemma 9.25. Let $G$ be a counterexample to Theorem 9.1 with $\left|I_{T}(\rho)\right|=3, I_{R}(\rho)=\{1,2\}$, for some rotator $\rho$ of $\bar{G}$. If $G[T(\rho)]$ is of type 2 at $q=3$, then $R(\rho)$ is a clique of $G$.

Proof. Suppose that $G[T]$ is of type 2 at 3 with $t^{1}, t^{2}$ and $t^{3}$ as in the definition and let $X=T^{3} \backslash\left\{t^{3}\right\}$. Also, by the contrary, assume that $R$ is not a clique and so by Lemma 9.19, $G[R]$ is of type 1 , where $r_{1} \in R_{1}, r_{2} \in R_{2}, U^{1} \subseteq R_{2}$ and $U^{2} \subseteq R_{1}$ are as in the definition. By Lemma 9.13 (vi), either $r_{1}$ is complete to $t^{3}$ and anticomplete to $\left\{t^{1}, t^{2}\right\}$, or $r_{1}$ is anticomplete to $t^{3}$ and complete to $\left\{t^{1}, t^{2}\right\}$. By symmetry, we may assume that the former case occurs. Thus, by Lemma 9.13 (vi), $r_{2}$ is complete to $\left\{t^{1}, t^{2}\right\}$ and anticomplete to $\left\{t^{3}\right\}$. Now, since for every $i \in\{1,2\}$ and $r \in U^{i},\left\{s_{1}^{1}, t^{1}, r_{i}, r\right\}$ is not a claw, $\left\{t^{1}, t^{2}\right\}$ is anticomplete to $U^{1}$ and complete to $U^{2}$ and thus, $t^{3}$ is complete to $U^{1}$ and anticomplete to $U^{2}$. Also, since $\left\{x, s_{1}^{1}, t^{1}, r_{1}\right\}, x \in X$, is not a claw, $r_{1}$ is anticomplete to $X$ and thus $r_{2}$ is complete to $X$. Moreover, since $\left(s_{2}^{2}, t^{2}, U^{1}\right)$ is a quasi-triad of $G$ and both $X$ and $U^{2}$ are complete to $\left\{s_{2}^{2}, t^{2}\right\}$, by Lemma 9.13 (i), $X$ and $U^{2}$ are both anticomplete to $U^{1}$. Now, if $s_{3}^{1} \in S$, then $\left(s_{3}^{1}, t^{3}, U^{2}\right)$ is a quasi-triad of $G$ and $X$ is anticomplete to $\left\{s_{3}^{1}\right\}$, and thus $X$ is complete to $U^{2}$. Also, if either $X=\emptyset$, or $U^{2}=\emptyset$, or $U^{1} \neq \emptyset$, then again $X$ is complete $U^{2}$. Thus, in all these cases, $G$ is ISA, which is in contradiction with Lemma 9.22. Hence, $S_{3}^{1}=\emptyset, X \neq \emptyset, U^{2} \neq \emptyset, U^{1}=\emptyset$ and $X$ is not complete to $U^{2}$. Now, assume that $\left\{X, U^{1}\right\}=\{A, B\}$, where $1 \leq|A| \leq|B|$. In the sequel, we are going to provide a clique covering for $G$ of size at most $n-|B|+1$.
Consider the family of cliques $\mathscr{O}(G) \cup\left\{T^{3} \cup R^{3},\left(T \backslash\left\{t^{3}\right\}\right) \cup R^{3}, T_{3} \cup R_{1}, T_{3} \cup\left(R \backslash\left\{r_{1}\right\}\right)\right\}$, replace the cliques $X_{3}^{1}, X_{3}^{2}$ and $Z_{2}^{1}$ with the cliques $X_{3}^{1} \cup U^{2}, X_{3}^{2} \cup U^{2}$ and $Z_{2}^{1} \cup X$, add the clique $\left\{t^{3}, r_{1}\right\}$ and the cliques in $\mathscr{N}[A ; B]$, and call the resulting collection $\mathscr{O}^{\prime}(G)$. The cliques in $\mathscr{O}^{\prime}(G)$ cover all the edges in $E(G) \backslash\left\{t^{1} r_{2}, t^{2} r_{2}\right\}$. In order to cover $t^{1} r_{2}$ and $t^{2} r_{2}$, in $\mathscr{O}^{\prime}(G)$, if $S_{2}^{1} \cup S_{1}^{3}=\emptyset$, then replace the cliques $Z_{2}^{1} \cup X$ and $Z_{2}^{3}$ with the cliques $Z_{2}^{1} \cup X \cup\left\{t^{2}\right\}$ and $Z_{2}^{3} \cup\left\{t^{1}\right\}$, if $S_{3}^{2}=\emptyset$, then replace the cliques $X_{3}^{1} \cup U^{2}$ and $X_{3}^{2} \cup U^{2}$ with the cliques $X_{3}^{1} \cup U^{2} \cup\left\{r_{2}\right\}$ and $X_{3}^{2} \cup U^{2} \cup\left\{r_{2}\right\}$ and if $S_{2}^{1} \cup S_{1}^{3} \neq \emptyset$, $S_{3}^{2} \neq \emptyset$ and $|S| \leq 2$ (i.e. either $S=\left\{s_{2}^{1}, s_{3}^{2}\right\}$ or $S=\left\{s_{1}^{3}, s_{3}^{2}\right\}$ ), then replace the cliques $Z_{2}^{1} \cup X$ and $Z_{2}^{3}$ with the cliques $Z_{2}^{1} \cup X \cup S_{1}^{3}$ and $\left(Z_{2}^{3} \cup\left\{t^{1}, t^{2}\right\}\right) \backslash S_{1}^{3}$ and finally if $|S| \geq 3$, then add the clique $\left\{t^{1}, t^{2}, r_{2}\right\}$, thereby obtaining a clique covering $\mathscr{C}$ for $G$. If $|S| \geq 2$, then $|\mathscr{C}| \leq n-|B|+1$, as desired. Now, assume that $|S|=1$ and $S_{3}=\emptyset$. Then, in $\mathscr{C}$, remove the clique $T^{3} \cup R^{3}$ and replace the cliques
$Z_{3}^{1}$ and $Z_{3}^{2}$ with the cliques $Z_{3}^{1} \cup T^{3} \cup\left\{r_{2}^{3}\right\}$ and $Z_{3}^{2} \cup T^{3} \cup\left\{r_{1}^{3}\right\}$ to obtain a clique covering of size $n-|B|+1$. If $|B| \geq 2$, then we have a clique covering for $G$ of size at most $n-1$, a contradiction. Thus, $|B|=1$ and so $|X|=\left|U^{2}\right|=1$. Now, since $X$ is anticomplete to $U^{2}$, one may remove the single clique in $\mathscr{N}[A ; B]$ to obtain a clique covering for $G$ of size at most $n-1$. Finally, assume that $|S|=\left|S_{3}\right| \leq 1$. Then, in $\mathscr{P}(G)$, merge the pair $\left(\Delta_{1}^{3}, \Delta_{2}^{3}\right)$, replace clique $\nabla_{2}^{3}$ with the clique $\nabla_{2}^{3} \cup U^{2}$, and in order to cover the edges in $E(T, R) \cup E\left(T \backslash\left\{t^{3}\right\}\right)$, for every $i \in\{1,2\}$, replace the cliques $\Omega_{i}$ with the cliques in $\mathscr{N}\left[R_{i} ;\left(\Omega_{i} \cup T\right) \backslash R_{i}\right]$. These modifications lead to a clique covering for $G$ of size $14+\left|U^{2}\right| \leq n-|X| \leq n-1$, a contradiction. This proves Lemma 9.25.

Lemma 9.26. Let $G$ be a counterexample to Theorem 9.1 with $\left|I_{T}(\rho)\right|=3, I_{R}(\rho)=\{1,2\}$, for some rotator $\rho$ of $\bar{G}$. If $G[T(\rho)]$ is of type 3 , then $R(\rho)$ is a clique of $G$.

Proof. Suppose that $G[T]$ is of type 3 at $q$ for some $q \in\{1,2,3\}$, where $T_{1}=\left\{t_{1}^{q}, t_{1}^{q+1}\right\} \cup T_{1}^{q+2}$ and $T_{2}=\left\{t_{2}^{q}, t_{2}^{q+2}\right\} \cup T_{2}^{q+1}$ are as in the definition. Also, by the contrary, suppose that $R$ is not a clique and thus by Lemma 9.19, $G[R]$ is of type 1 with $r_{1} \in R_{1}, r_{2} \in R_{2}, U^{1} \subseteq R_{2}$ and $U^{2} \subseteq R_{1}$ as in the definition. Note that by Lemma 9.13 (vi), there is a unique $i_{0} \in\{1,2\}$ such that $r_{1}$ is complete to $T_{i_{0}}$ and anticomplete to $T_{3-i_{0}}$, and $r_{2}$ is complete to $T_{3-i_{0}}$ and anticomplete to $T_{i_{0}}$. First, assume that $q \neq 3$ and by symmetry, let $q=1$. Since for every $i \in\{1,2\}$ and $r \in U^{i},\left\{s_{1}^{1}, t_{3-i_{0}}^{1}, r_{i}, r\right\}$ is not a claw, $T_{3-i_{0}}$ is anticomplete to $U^{1}$ and complete to $U^{2}$, and thus, by Lemma 9.13 (vi), $T_{i_{0}}$ is complete to $U^{1}$ and anticomplete to $U^{2}$. Also, since $\left(s_{1}^{1}, t_{i_{0}}^{1}, U^{2}\right)$ is a quasi-triad of $G$ and $U^{1}$ is complete to $\left\{s_{1}^{1}, t_{i_{0}}^{1}\right\}$, $U^{1}$ is anticomplete to $U^{2}$. Hence, $G$ is ISA, which contradicts Lemma 9.22.
Next, suppose that $q=3$. Note that if $|R|=2$ (i.e. $U^{1}=U^{2}=\emptyset$ ), then evidently $G$ is ISA, which contradicts Lemma 9.22. Thus, $|R| \geq 3$. Now, assume that there exists $i \in\{1,2\}$ such that $S^{i} \neq \emptyset$, say $S^{1} \neq \emptyset$. If $s_{3}^{1} \in S$, then let $i^{\prime}=i^{\prime \prime}=3$, and if $s_{2}^{1} \in S$, then let $i^{\prime}=3-i_{0}$ and $i^{\prime \prime}=i_{0}$. Since for every $i \in\{1,2\}$ and every $r \in U^{i},\left\{s_{i^{\prime}}^{1}, t_{3-i_{0}}^{i^{\prime}}, r_{i}, r\right\}$ is not a claw, $T_{3-i_{0}}$ is anticomplete to $U^{1}$ and complete to $U^{2}$ and thus by Lemma 9.13 (vi), $T_{i_{0}}$ is complete to $U^{1}$ and anticomplete to $U^{2}$. Also, since $\left(s_{i^{\prime \prime}}^{1}, t_{i_{0}}^{i^{\prime \prime}}, U^{2}\right)$ is a quasi-triad of $G$ and $U^{1}$ is complete to $\left\{s_{i^{\prime \prime}}^{1}, t_{i_{0}}^{i^{\prime \prime}}\right\}, U^{1}$ is anticomplete to $U^{2}$. Hence, $G$ is ISA, again a contradiction with Lemma 9.22. Hence, $|R| \geq 3$ and $S^{1} \cup S^{2}=\emptyset$. In the sequel, we are going to provide a clique covering for $G$ of size at most $n-1$.
Note that by Lemma 9.13 (vi), $R$ can be partitioned into $R^{1}$ and $R^{2}$ such that for every $j \in\{1,2\}, R^{j}$ is complete to $T_{j}$ and anticomplete to $T_{3-j}$. Now, consider $\mathscr{P}(G)$, if $G[R]$ is of type 1.2 , then remove the cliques $\nabla_{1}^{3}$ and $\nabla_{2}^{3}$ and add the cliques in $\mathscr{N}\left[R_{1} ; \nabla_{2}^{3}\right]$, and if $G[R]$ is of type 1.1 at $i$, for some $i \in\{1,2\}$, then replace the clique $\nabla_{3-i}^{3}$ with the clique $\nabla_{3-i}^{3} \cup U^{3-i}$. Also, if $|T|=4$, then replace the cliques $\Delta_{1}^{2}$ and $\Delta_{2}^{1}$ with the cliques $\Delta_{1}^{2} \cup\left\{t_{1}^{3}\right\}$ and $\Delta_{2}^{1} \cup\left\{t_{2}^{3}\right\}$, and for every $j \in\{1,2,3\}$, replace the clique $\Omega^{j}$ with the cliques in $\mathscr{N}\left[T^{j} ;\left(\Omega^{j} \cup R\right) \backslash T^{j}\right]$, and if $|T| \geq 5$, then add the cliques $T_{1} \cup R^{1}$ and $T_{2} \cup R^{2}$, thereby covering the edges in $E(T, R) \cup E\left(T_{1}\right) \cup E\left(T_{2}\right)$. Now, if $G[R]$ is of type 1.2 , then the resulting collection is a clique covering for $G$ of size at most $10+|T|+\left|R_{1}\right|=n-|S|-\left|R_{2}\right|+1 \leq n-1$. Thus, $G[R]$ is of type 1.1 at $i$, for some $i \in\{1,2\}$, and the resulting collection is a clique covering for $G$ of size at most $12+|T|=n-|S|-|R|+3$. Hence, $|S|+|R| \leq 3$, and since $|R| \geq 3$, we have $S=\emptyset$ and $\left|U^{3-i}\right|=1$, say $U^{3-i}=\{u\}$. Now, let $r_{i} \in R^{j}, j \in\{1,2\}$. Thus, by Lemma 9.13 (vi), $r_{3-i} \in R^{3-j}$. If $U^{3-i} \subseteq R^{3-j}$, then $G$ is ISA, a contradiction with Lemma 9.22. Thus, since $\left|U^{3-i}\right|=1$, we have $R^{j}=R_{i}$ and $R^{3-j}=R_{3-i}$. Now, in $\mathscr{P}(G)$, replace the clique $\nabla_{3-i}^{3}$ with the clique $\nabla_{3-i}^{3} \cup U^{3-i}$ to cover the edges in $E\left(R_{1}, R_{2}\right)$ and also replace the cliques $\Omega_{i}$ and $\Omega_{3-i}$ with the cliques $\Omega_{i} \cup T_{j}$ and $\Omega_{3-i} \cup T_{3-j}$, thereby obtaining a clique covering for $G$ of size $15=n-|T|+3 \leq n-1$, a contradiction. This proves Lemma 9.26.

Now, we are ready to prove Theorem 9.23.
Proof of Theorem 9.23. If $\left|I_{T}\right|+\left|I_{R}\right|=5$, then the result follows from Lemmas 9.24, 9.25 and 9.26. Thus, suppose that $\left|I_{T}\right|=|R|=3$. Also, on the contrary, suppose that both $T$ and $R$ are not clique.
(1) Both $G[T]$ and $G[R]$ are of type 2.

On the contrary, by Lemma 9.20 and due to the symmetry, assume that $G[R]$ is of type 3 with $R^{1}$ and $R^{2}$ as in the definition. Note that if $G[T]$ is also of type 3 , then by Lemma 9.13 (vi), $G$
is ISA, which is impossible by Lemma 9.22. Thus, assume that $G[T]$ is of type 2 at $q$, for some $q \in\{1,2,3\}$, with $t^{q}, t^{q+1}$ and $t^{q+2}$ as in the definition. Note that since $G[R]$ is of type 3 , again by Lemma 9.13 (vi), for every $t \in T$, there exists a unique $\beta(t) \in\{1,2\}$ such that $N(t, R)=R^{\beta(t)}$. Thus, $\beta\left(t^{q+1}\right)=\beta\left(t^{q+2}\right)=3-\beta\left(t^{q}\right)$. We claim that for every $t \in T^{q} \backslash\left\{t^{q}\right\}, \beta(t)=\beta\left(t^{q+1}\right)=\beta\left(t^{q+2}\right)$. To see the claim, on the contrary assume that $\beta(t)=\beta\left(t^{q}\right)$, for some $t \in T^{q} \backslash\left\{t^{q}\right\}$. Thus, since $R^{\beta(t)} \cap\left(R \backslash R_{q}\right) \neq \emptyset$, there exists $j \in\{1,2\}$ and a vertex $r_{j} \in R_{q+j}$ adjacent to both $t$ and $t^{q}$. Consequently, $\left\{t, s_{q+j}^{q+j}, t^{q+j}, r_{j}\right\}$ induces a claw, which is impossible. This proves the claim, which immediately implies that $G$ is ISA, a contradiction with Lemma 9.22. This proves (1).
By (1), suppose that both $G[T(\rho)]$ and $G[R(\rho)]$ are of type 2 at $q$ and $q^{\prime}$ respectively, for some $q, q^{\prime} \in\{1,2,3\}$, with $t^{1}, t^{2}, t^{3}, r_{1}, r_{2}$ and $r_{3}$ as in the definition. Also, let $X=T^{q}(\rho) \backslash\left\{t^{q}\right\}$ and $X^{\prime}=R_{q^{\prime}}(\rho) \backslash\left\{r_{q^{\prime}}\right\}$. Define $\mathscr{O}^{\prime}(G)=\mathscr{O}(G) \cup\left\{T^{q} \cup R^{3},\left(T \backslash\left\{t^{q}\right\}\right) \cup R^{3}, T_{3} \cup R_{q^{\prime}}, T_{3} \cup\left(R \backslash\left\{r_{q^{\prime}}\right\}\right)\right\}$ whose cliques cover all the edges in $E(G) \backslash E(T, R)$.
(2) If $q=q^{\prime}$, then $t^{q}$ is nonadjacent to $r_{q}$.

By symmetry, let $q=q^{\prime}=1$ and on the contrary, suppose that $t^{1}$ is adjacent to $r_{1}$ and thus by Lemma 9.13 (vi), $t^{1}$ is anticomplete to $\left\{r_{2}, r_{3}\right\}$ and $\left\{t^{2}, t^{3}\right\}$ is anticomplete to $\left\{r_{1}\right\}$ and complete to $\left\{r_{2}, r_{3}\right\}$. Note that if $X=X^{\prime}=\emptyset$, then by Lemma 9.13 (vi), $G$ is ISA, which is impossible by Lemma 9.22. First, assume that $S^{1} \neq \emptyset$, say $s_{p}^{1} \in S$, for some $p \in\{2,3\}$. Thus, $r_{1}$ is anticomplete to $X$ (otherwise, for some $x \in X$ adjacent to $r_{1},\left\{x, s_{p}^{1}, t^{p}, r_{1}\right\}$ would be a claw) and thus by Lemma 9.13 (vi), $X$ is complete to $\left\{r_{2}, r_{3}\right\}$. Also, $X^{\prime}$ is complete to $\left\{t^{2}, t^{3}\right\}$ and anticomplete to $\left\{t^{1}\right\}$ (otherwise, for $x^{\prime} \in X^{\prime}$ nonadjacent to $t^{p},\left\{r_{p}, s_{p}^{1}, t^{p}, x^{\prime}\right\}$ would be a claw). Further, since $\left(s_{1}^{1}, t^{1}, X^{\prime}\right)$ is a quasi-triad of $G$ and $s_{1}^{1}$ is anticomplete to $X$, by Lemma 9.13 (i), $X$ is complete to $X^{\prime}$, and so $G$ is ISA, which is impossible by Lemma 9.22. Therefore, $S^{1}=\emptyset$ and by symmetry, $S_{1}=\emptyset$ (indeed, for the rotator $\rho^{\prime}=\left(s_{1}^{1}, s_{2}^{2}, s_{3}^{3} ; r_{1}^{3}, r_{2}^{3}, r_{3}^{3} ; t_{3}^{1}, t_{3}^{2}, t_{3}^{3}\right)$, we have $\left.S_{1}(\rho)=S^{1}\left(\rho^{\prime}\right)\right)$. Also, w.l.o.g. assume that $X \neq \emptyset$.
Now, consider $\mathscr{O}^{\prime}(G)$, replace the clique $Z_{1}^{2}$ with the cliques in $\mathscr{N}\left[R_{1} ;\left(Z_{2}^{1} \cup T^{1}\right) \backslash R_{1}\right]$ to cover the edges in $E\left(T^{1}, R_{1}\right)$, for every $i \in\{2,3\}$, replace the cliques $X_{5-i}^{i}$ and $Z_{5-i}^{i}$ with the cliques $X_{5-i}^{i} \cup N\left(t^{i}, X^{\prime}\right)$ and $Z_{5-i}^{i} \cup N\left(r_{i}, X\right)$ to cover the edges in $E\left(T^{i}, R_{1}\right)$ and $E\left(T^{1}, R_{i}\right)$, respectively, replace the cliques $X_{1}^{2}, X_{1}^{3}, Z_{2}^{1}$ and $Z_{3}^{1}$ with the cliques $\left\{s_{1}^{1}, t_{3}^{2}, t_{3}^{3}\right\},\left\{s_{1}^{1}, r_{2}^{3}, r_{3}^{3}\right\}$ and $\left\{s_{1}^{1}, t^{2}, t^{3}, r_{2}, r_{3}\right\}$, to cover the edges in $E\left(T \backslash T^{1}, R \backslash R^{1}\right)$. This yields a clique covering for $G$ of size $15+\left|X^{\prime}\right|=n-|S|-|X| \leq n-1$, a contradiction. This proves (2).
(3) We have $q \neq q^{\prime}$.

Suppose not, let $q=q^{\prime}=1$. By (2), $t^{1}$ is nonadjacent to $r_{1}$ and thus by Lemma 9.13 (vi), $t^{1}$ is complete to $\left\{r_{2}, r_{3}\right\}$ and $\left\{t^{2}, t^{3}\right\}$ is complete to $r_{1}$ and anticomplete to $\left\{r_{2}, r_{3}\right\}$. Then, $X$ is anticomplete to $\left\{r_{2}, r_{3}\right\}$ and $X^{\prime}$ is anticomplete to $\left\{t^{2}, t^{3}\right\}$ (otherwise, for $x \in X \cup X^{\prime}$ adjacent to $t^{2}, r_{2},\left\{x, s_{2}^{2}, t^{2}, r_{2}\right\}$ would be a claw) and thus by Lemma 9.13 (vi), $X$ is complete to $\left\{r_{1}\right\}$ and $X^{\prime}$ is complete to $\left\{t^{1}\right\}$. First, assume that $S^{1} \neq \emptyset$, say $s_{p}^{1} \in S$, for some $p \in\{2,3\}$. Since $\left(s_{p}^{1}, t^{p}, X^{\prime}\right)$ is a quasi-triad of $G$ and $X$ is complete to $\left\{s_{p}^{1}, t^{p}\right\}$, by Lemma 9.13 (i), $X$ is anticomplete to $X^{\prime}$, and so $G$ is ISA, which is impossible by Lemma 9.22. Therefore, $S^{1}=\emptyset$. Now, $\rho^{\prime}=\left(s_{1}^{1}, s_{2}^{2}, s_{3}^{3} ; r_{1}, t_{3}^{2}, t_{3}^{3} ; t^{1}, r_{2}^{3}, r_{3}^{3}\right)$ is a rotator of $\bar{G}$ and it is easy to see that $T^{2}\left(\rho^{\prime}\right) \cup T^{3}\left(\rho^{\prime}\right)=S^{1}=\emptyset$. Thus, $\left|I_{T}\left(\rho^{\prime}\right)\right| \leq 1$, which contradicts Lemma 9.21. This proves (3).
Now, by (3) and due to symmetry, we may assume that $q=1$ and $q^{\prime}=2$. First, suppose that $t^{1}$ is nonadjacent to $r_{2}$. Then, $X$ is anticomplete to $\left\{r_{1}, r_{3}\right\}$ and $X^{\prime}$ is anticomplete to $\left\{t^{2}, t^{3}\right\}$ (otherwise, for $x \in X \cup X^{\prime}$ adjacent to $t^{3}, r_{3},\left\{x, s_{3}^{3}, t^{3}, r_{3}\right\}$ would be a claw) and thus by Lemma 9.13 (vi), $X$ is complete to $r_{2}$ and $X^{\prime}$ is complete to $\left\{t^{1}\right\}$. Also, since $\left(s_{2}^{2}, t^{2}, X^{\prime}\right)$ is a quasi-triad of $G$ and $X$ is complete to $\left\{s_{2}^{2}, t^{2}\right\}$, by Lemma 9.13 (i), $X$ is anticomplete to $X^{\prime}$, and thus $G$ is ISA, which is impossible by Lemma 9.22 .
Next, assume that $t^{1}$ is adjacent to $r_{2}$. Then, $X$ is anticomplete to $\left\{r_{2}\right\}$ and complete to $\left\{r_{1}, r_{3}\right\}$ (otherwise, for $x \in X$ adjacent to $r_{2},\left\{x, s_{2}^{2}, t^{2}, r_{2}\right\}$ would be a claw). Also, $X^{\prime}$ is anticomplete to $\left\{t^{1}\right\}$ and complete to $\left\{t^{2}, t^{3}\right\}$ (otherwise, for $x^{\prime} \in X^{\prime}$ adjacent to $t^{1},\left\{x^{\prime}, s_{1}^{1}, t^{1}, r_{1}\right\}$ would be a claw). Now, if either $X=\emptyset$, or $X^{\prime}=\emptyset$, then $G$ is ISA, which contradicts Lemma 9.22 . Also, if $s_{1}^{2} \in S$, then since $\left(s_{1}^{2}, t^{1}, X^{\prime}\right)$ is a quasi-triad of $G$ and $X$ is anticomplete to $\left\{s_{1}^{2}\right\}, X$ is complete to $X^{\prime}$, and so $G$
is ISA, again a contradiction with Lemma 9.22. Therefore, $X \neq \emptyset, X^{\prime} \neq \emptyset$ and $S_{1}^{2}=\emptyset$. Also, w.l.o.g. assume that $|X| \geq\left|X^{\prime}\right| \geq 1$. Now, $\rho^{\prime}=\left(s_{1}^{1}, s_{2}^{2}, s_{3}^{3} ; r_{1}, t_{3}^{2}, t_{3}^{3} ; t^{1}, r_{2}^{3}, r_{3}^{3}\right)$ is a rotator of $\bar{G}$ and it is easy to see that $T^{2}\left(\rho^{\prime}\right) \cup T^{3}\left(\rho^{\prime}\right)=S^{1}$. Thus, if $S^{1}=\emptyset$, then $\left|I_{T}\left(\rho^{\prime}\right)\right| \leq 1$ which contradicts Lemma 9.21. Therefore, $|S| \geq\left|S^{1}\right| \geq 1$. First, suppose that $|S|=\left|S^{1}\right|=1$. Then, in $\mathscr{O}^{\prime}(G)$, replace the cliques $X_{1}^{2}$, $X_{1}^{3}, Z_{1}^{2}, Z_{1}^{3}, Z_{3}^{1}$ and $Z_{3}^{2}$ with the cliques $X_{1}^{2} \cup X^{\prime}, X_{1}^{3} \cup X^{\prime}, Z_{1}^{2} \cup X \cup T^{3}, Z_{1}^{3} \cup T^{2},\left(Z_{3}^{1} \cup T^{2}\right) \backslash S_{2}^{1}$ and $Z_{3}^{2} \cup X \cup T^{3} \cup S_{2}^{1}$. Also, remove the clique $Z_{2}^{3}$ and add the cliques in $\mathscr{N}\left[R_{2} ; T^{1} \cup\left\{s_{3}^{3}, r_{2}^{3}\right\}\right]$. This yields a clique covering for $G$ of size $16+\left|X^{\prime}\right|=n-|X| \leq n-1$. Finally, suppose that $|S| \geq 2$. Now, in $\mathscr{O}^{\prime}(G)$, replace the cliques $X_{1}^{2}, X_{1}^{3}, Z_{1}^{2}$ and $Z_{3}^{2}$ with the cliques $X_{1}^{2} \cup X^{\prime}, X_{1}^{3} \cup X^{\prime}, Z_{1}^{2} \cup X$ and $Z_{3}^{2} \cup X$, and add the cliques in $\mathscr{N}\left[X^{\prime} ; X\right]$ as well as the cliques $\left\{t^{1}, r_{2}\right\}$ and $\left\{t^{2}, t^{3}, r_{1}, r_{3}\right\}$, thereby obtaining a clique covering $\mathscr{C}$ for $G$ of size at most $18+\left|X^{\prime}\right|=n-|S|-|X|+3$. Thus, $|S|=2$ and $|X|=\left|X^{\prime}\right|=1$. Now, if $X$ is complete to $X^{\prime}$, then $G$ is ISA, a contradiction with Lemma 9.22 , and if $X$ is anticomplete to $X^{\prime}$, then removing the single clique in $\mathscr{N}\left[X^{\prime} ; X\right]$ from $\mathscr{C}$, yields a clique covering for $G$ of size $n-1$, a contradiction. This completes the proof of Theorem 9.23.

### 9.5 Obtaining $\left|I_{T}\right|=\left|I_{R}\right|=2$

As the penultimate step of the proof of Theorem 9.1, the aim of this subsection is the following theorem, whose proof will be given at end of this subsection.

Theorem 9.27. Let $G$ be a counterexample to Theorem 9.1. Then for every rotator $\rho$ of $\bar{G}$, we have $\left|I_{T}(\rho)\right|=\left|I_{R}(\rho)\right|=2$.

We need the following three lemmas to prove Theorem 9.27. For simplicity, throughout the proofs, we omit the term $\rho$.

Lemma 9.28. Let $G$ be a counterexample to Theorem 9.1, with $\left|I_{T}(\rho)\right|+\left|I_{R}(\rho)\right|=5$, say $\left|I_{T}(\rho)\right|=3$ and $\left|I_{R}(\rho)\right|=2$ for some rotator $\rho$ of $\bar{G}$. Also, let $R(\rho)$ be a clique of $G$. Then $G[T(\rho)]$ is of type 3 .

Proof. On the contrary, by lemmas 9.17 and $9.20, G[T]$ is of type 2 at $q$, for some $q \in\{1,2,3\}$. Also, w.l.o.g. assume that $I_{R}=\{1,2\}$. Let $t^{q} \in T^{q}, t^{q+1} \in T^{q+1}$ and $t^{q+2} \in T^{q+2}$ be as in the definition of type 2 at $q$, and let $X=T^{q} \backslash\left\{t^{q}\right\}$. Note that the cliques in $\mathscr{O}^{\prime}(G)=\mathscr{O}(G) \cup\left\{T^{q} \cup R^{3},\left(T \backslash\left\{t^{q}\right\}\right) \cup\right.$ $\left.R^{3}, T_{3} \cup R\right\}$ cover all the edges in $E(G) \backslash E(T, R)$.
(1) If $q \in\{1,2\}$, then $s_{k^{\prime}}^{k} \in S$, for some $\left(k, k^{\prime}\right) \in\{(1,2),(2,1),(q, 3)\}$. Also, exactly one of $t^{q}$ or $\left\{t^{q+1}, t^{q+2}\right\}$ is complete to $R$ (and the other one is anticomplete to $R$ ).

The first assertion follows from Lemma 9.15 (i) (applying to $(i, j, k)=(q, 3-q, 3)$ ). Consequently, the second assertion follows from Lemma 9.18 (i) (applying to $(i, j)=\left(k, k^{\prime}\right)$ ) and Lemma 9.13 (vi).

Henceforth, let $\left(k, k^{\prime}\right)$ be as in (1).
(2) If $q \in\{1,2\}$, then $\left\{t^{q+1}, t^{q+2}\right\}$ is complete to $R$ and $t^{q}$ is anticomplete to $R$.

By symmetry, let $q=1$. Suppose not, by (1), $t^{1}$ is complete to $R$ and $\left\{t^{2}, t^{3}\right\}$ is anticomplete to $R$. We observe that if $\left(k, k^{\prime}\right)=(2,1)$, then $\left(s_{1}^{2}, x, R_{2}\right), x \in X$, is a quasi-triad of $G$ and since $R_{1}$ is complete to $R_{2} \cup\left\{s_{1}^{2}\right\}$, by Lemma 9.13 (i), $X$ is anticomplete to $R_{1}$, and if $k=1$, then $\left(s_{k^{\prime}}^{1}, t^{k^{\prime}}, R_{1}\right)$ is a quasi-triad of $G$ and $X$ is complete to $\left\{s_{k^{\prime}}^{1}, t^{k^{\prime}}\right\}$, again by Lemma 9.13 (i), $X$ is anticomplete to $R_{1}$. Also, $\left(s_{2}^{2}, t^{2}, R_{2}\right)$ is a quasi-triad of $G$ and $X$ is complete to $\left\{s_{2}^{2}, t^{2}\right\}$. Thus by Lemma 9.13 (i), $X$ is anticomplete to $R_{2}$. Thus, $X$ is anticomplete to $R$. In the sequel, first we claim that $s_{1}^{3} \in S$. Suppose not, in $\mathscr{O}^{\prime}(G)$, for every $i \in\{1,2\}$, replace the clique $Z_{i}^{3}$ with the clique $Z_{i}^{3} \cup\left\{t^{1}\right\}$ to cover the edges in $E\left(\left\{t^{1}\right\}, R\right)$, thereby producing a clique covering $\mathscr{C}$ for $G$ of size $15=n-|S|-|X|-|R|+3$. Thus, $|S|+|X|+|R| \leq 3$, i.e. $|S|=1, X=\emptyset$ and $|R|=2$. Now, remove the clique $T^{1} \cup R^{3}$ from $\mathscr{C}$ and replace the cliques in $Z_{3}^{1}$ and $Z_{3}^{2}$ with the cliques $\left(Z_{3}^{1} \cup S_{1}^{2}\right) \backslash S_{2}^{1}$ and $\left(Z_{3}^{2} \cup S_{2}^{1} \cup\left\{t^{1}\right\}\right) \backslash S_{1}^{2}$ to cover the edge $t^{1} r_{3}^{3}$ (note that for $i=1,2$, the edge $t^{1} r_{i}^{3}$ has been already covered by the clique $\left.Z_{i}^{3} \cup\left\{t^{1}\right\}\right)$. This yields a clique covering for $G$ of size $14=n-1$, a contradiction. Thus, $s_{1}^{3} \in S$, and consequently by $(1),|S| \geq 2$. Now, the collection $\mathscr{O}^{\prime}(G) \cup\left\{\left\{t^{1}\right\} \cup R\right\}$ is a clique covering for $G$ of size $16=n-|S|-|R|-|\bar{X}|+4$. Therefore $|S|+|R|+|X| \leq 4$, i.e. $|S|=2,|R|=2$ and $X=\emptyset$.

Consequently, choose $l \in\{2,3\}$ for which $S_{l}^{1}=\emptyset$, and replace the cliques $X_{2}^{1}$ and $X_{3}^{1}$ with the cliques $X_{l}^{1} \cup N\left(t^{1}, R_{1} \cup R_{5-l}\right)$ and $X_{5-l}^{1} \cup R_{l}$ in the collection $\mathscr{O}^{\prime}(G)$, thereby obtaining a clique covering for $G$ of size $15=n-1$, again a contradiction. This proves (2).
(3) If $q \in\{1,2\}$, then $S_{3} \neq \emptyset$ and $|S| \geq 2$.

On the contrary, suppose that $S_{3}=\emptyset$. By symmetry, let $q=1$. Due to (2), $\left\{t^{2}, t^{3}\right\}$ is complete to $R$ and $t^{1}$ is anticomplete $R$. Now, in $\mathscr{O}^{\prime}(G)$, replace the clique $X_{3}^{2}$ with the clique $X_{3}^{2} \cup R$, for every $i \in\{1,2\}$, replace the clique $Z_{i}^{3-i}$ with the clique $Z_{i}^{3-i} \cup T^{3}$ and also add the cliques in $\mathscr{N}[X ; R]$, thereby covering the edges in $E(T, R)$ and obtaining a clique covering $\mathscr{C}$ for $G$ of size $15+|X|=n-|S|-|R|+3$. Thus, $|S|+|R| \leq 3$, i.e. $|S|=1$ and $|R|=2$ and so by (1), $|S|=\left|S_{i}^{3-i}\right|=1$ for some $i \in\{1,2\}$. Now, remove the clique $T^{1} \cup R^{3}$ from $\mathscr{C}$, replace the cliques $Z_{1}^{3}$ and $Z_{2}^{3}$ with the cliques $R \cup\left\{s_{3}^{3}\right\}$ and $\left\{s_{3}^{3}, t^{1}, r_{1}^{3}, r_{2}^{3}\right\}$ to cover the edges $t^{1} r_{1}^{3}$ and $t^{1} r_{2}^{3}$, and replace the cliques $Z_{3}^{1}$ and $Z_{3}^{2}$ with the cliques $\left(Z_{3}^{1} \cup S_{1}^{2}\right) \backslash S_{2}^{1}$ and $\left(Z_{3}^{2} \cup S_{2}^{1} \cup\left\{t^{1}\right\}\right) \backslash S_{1}^{2}$, to cover the edge $t^{1} r_{3}^{3}$. This yields a clique covering of size $14+|X|=n-1$, a contradiction. This proves that $S_{3} \neq \emptyset$. Now, assume that $|S| \leq 1$, and thus by (1), $|S|=\left|S_{3}^{1}\right|=1$. In this case, considering $\mathscr{O}^{\prime}(G) \cup \mathscr{N}[X ; R]$, remove the clique $T^{1} \cup R^{3}$, replace the cliques $X_{1}^{3}, X_{2}^{3}, Z_{1}^{3}, Z_{2}^{3}$ and $Z_{3}^{2}$ with the cliques $X_{1}^{3} \cup R_{2}, X_{2}^{3} \cup R_{1}, R \cup\left\{s_{3}^{3}, t^{2}\right\},\left\{s_{3}^{3}, t^{1}, r_{1}^{3}, r_{2}^{3}\right\}$ and $Z_{3}^{2} \cup\left\{t^{1}\right\}$ to cover the edges in $E(T, R) \cup E\left(\left\{t^{1}\right\}, R^{3}\right)$, thereby comprising a clique covering for $G$ of size $14+|X| \leq n-1$, which is impossible. This proves (3).
(4) We have $q=3$.

Suppose not, w.l.o.g. let $q=1$. By (2), $\left\{t^{2}, t^{3}\right\}$ is complete to $R$ and $t^{1}$ is anticomplete $R$ and by (3), $S_{3} \neq \emptyset$ and $|S| \geq 2$. Note that $\mathscr{O}^{\prime}(G) \cup \mathscr{N}[X ; R] \cup\left\{R \cup\left\{t^{2}, t^{3}\right\}\right\}$ is a clique covering for $G$ size $16+|X|=n-|S|-|R|+4$. Hence, $|S|=|R|=2$. Since $S_{3} \neq \emptyset$, either $S^{3}=\emptyset$ or $S=S^{3} \cup S_{3}$. Now, in $\mathscr{O}^{\prime}(G) \cup \mathscr{N}[X ; R]$, replace the cliques $X_{1}^{3}$ and $X_{2}^{3}$ with the cliques $\left(X_{1}^{3} \cup S_{2}^{1} \cup R_{2}\right) \backslash S_{1}^{2}$ and $\left(X_{2}^{3} \cup S_{1}^{2} \cup R_{1}\right) \backslash S_{2}^{1}$ to cover the edges in $E\left(T^{3}, R\right)$, if $s_{2}^{3} \in S$ (i.e. $\left.\left(k, k^{\prime}\right)=(1,3)\right)$, then replace the cliques $X_{1}^{2}$ and $X_{3}^{2}$ with the cliques $\left(X_{1}^{2} \cup R_{2} \cup S_{3}^{1}\right)$ and $\left(X_{3}^{2} \cup R_{1}\right) \backslash S_{3}^{1}$ and if $s_{2}^{3} \notin S$, then replace the cliques $Z_{1}^{3}$ and $Z_{2}^{3}$ with the cliques $Z_{1}^{3} \cup T^{2}$ and $Z_{2}^{3} \cup T^{2}$ to cover the edges in $E\left(T^{2}, R\right)$. This yields a clique covering for $G$ of size $15+|X|=n-1$, a contradiction This proves (4).
(5) If $S_{3} \neq \emptyset$, then $\left\{t^{q+1}, t^{q+2}\right\}$ is complete to $R$ and $t^{q}$ is anticomplete to $R$.

Let $s_{3}^{p} \in S$ for some $p \in\{1,2\}$. Note by (4), we have $q=3$. By Lemma 9.18 (i) (applying to $(i, j)=(p, 3)$ ), either $t^{3}$ or $\left\{t^{1}, t^{2}\right\}$ is complete to $R$ (and the other one is anticomplete to $R$ ). Thus if (5) does not hold, then $t^{3}$ is complete to $R$ and $\left\{t^{1}, t^{2}\right\}$ is anticomplete to $R$. Note that since for every $j \in\{1,2\},\left(s_{j}^{j}, t^{j}, R_{j}\right)$ is a quasi-triad of $G$ and $X$ is complete to $\left\{s_{j}^{j}, t^{j}\right\}$, by Lemma 9.13 (i), $X$ is anticomplete to $R$. Thus, adding the clique $\left\{t^{3}\right\} \cup R$ to the collection $\mathscr{O}^{\prime}(G)$ provides a clique covering for $G$ of size $16=n-|S|-|R|-|X|+4$. Thus, $|S|+|R|+|X| \leq 4$. Also, if $|S|=\left|S_{3}^{p}\right|=1$, i.e. $S=\left\{s_{3}^{p}\right\}$, then in $\mathscr{O}^{\prime}(G)$, remove the clique $T^{3} \cup R^{3}$ and replace the cliques $Z_{p}^{3-p}, Z_{3}^{3-p}, Z_{3-p}^{p}$ and $Z_{3-p}^{3}$ with the cliques $Z_{p}^{3-p} \cup\left\{t^{3}\right\}, Z_{3}^{3-p} \cup\left\{t^{3}\right\},\left(Z_{3-p}^{p} \cup\left\{t^{3}\right\}\right) \backslash S_{3}^{p}$ and $Z_{3-p}^{3} \cup S_{3}^{p}$ to cover the edges in $E\left(\left\{t^{3}\right\} ; R^{\prime}\right)$, thereby comprising a clique covering for $G$ of size $14 \leq n-1$, a contradiction. Thus, $|S|=|R|=2$ and $X=\emptyset$. Now, in $\mathscr{O}^{\prime}(G)$, replace the cliques $X_{1}^{3}$ and $X_{2}^{3}$ with the cliques $\left(X_{1}^{3} \cup S_{2}^{1} \cup R_{2}\right) \backslash S_{1}^{2}$ and $\left(X_{2}^{3} \cup S_{1}^{2} \cup R_{1}\right) \backslash S_{2}^{1}$ to cover the edges in $E\left(\left\{t^{3}\right\}, R\right)$, thereby obtaining a clique covering for $G$ of size $15=n-1$, a contradiction. This proves (5).
(6) We have $S_{3}=\emptyset$.

By (4), we have $q=3$. If (6) does not hold, say $s_{3}^{p} \in\{1,2\}$ then by (5), $\left\{t^{1}, t^{2}\right\}$ is complete to $R$ and thus $t^{3}$ is anticomplete to $R$. We claim that $S^{3} \neq \emptyset$. On the contrary, considering $\mathscr{O}^{\prime}(G)$, for every $i \in\{1,2\}$, replace the clique $Z_{i}^{3}$ with the clique $Z_{i}^{3} \cup\left\{t^{1}, t^{2}\right\}$ to cover the edges in $E\left(\left\{t^{1}, t^{2}\right\}, R\right)$ and add the cliques in $\mathscr{N}\left[X ; R \cup\left\{t^{1}, t^{2}\right\}\right]$, thereby obtaining a clique covering $\mathscr{C}$ for $G$ of size $15+|X|=n-|S|-|R|+3$. Thus, $|S|=1$, say $|S|=\left|S_{3}^{p}\right|=1$, for some $p \in\{1,2\}$, and $|R|=2$. Now, remove the clique $\left(T \backslash\left\{t^{3}\right\} \cup R^{3}\right)$ from $\mathscr{C}$ and note that the only uncovered edges are $t^{1} r_{3}^{3}$ and $t^{2} r_{3}^{3}$. In order to cover them, for every $i \in\{1,2\}$, replace the clique $Z_{3}^{3-i}$ with the clique $Z_{3}^{3-i} \cup\left\{t^{i}\right\}$. This yields a clique covering for $G$ of size $14+|X|=n-1$. Therefore, $S^{3} \neq \emptyset$, and consequently, $|S| \geq 2$. Now, the family $\mathscr{O}^{\prime}(G) \cup \mathscr{N}[X ; R] \cup\left\{\left\{t^{1}, t^{2}\right\} \cup R\right\}$ is a clique covering for $G$ of size a most $16+|X|=n-|S|-|R|+4$.

Hence, $|S|+|R| \leq 4$, i.e. $|S|=|R|=2$. In this case, in $\mathscr{O}^{\prime}(G)$, if $S_{3}^{1}=\emptyset$ replace the cliques $X_{2}^{1}, X_{3}^{1}, X_{1}^{2}$ and $X_{3}^{2}$ with the cliques $\left(X_{2}^{1} \cup S_{3}^{2} \cup R_{1}\right) \backslash S_{2}^{3},\left(X_{3}^{1} \cup S_{2}^{3} \cup R_{2}\right) \backslash S_{3}^{2}, X_{1}^{2} \cup R_{2}$ and $X_{3}^{2} \cup R_{1}$ and if $S_{3}^{1} \neq \emptyset$ replace the cliques $X_{2}^{1}, X_{3}^{1}, X_{1}^{2}$ and $X_{3}^{2}$ with the cliques $X_{2}^{1} \cup R_{1}, X_{3}^{1} \cup R_{2},\left(X_{1}^{2} \cup S_{3}^{1} \cup R_{2}\right) \backslash S_{1}^{3}$ and $\left(X_{3}^{2} \cup S_{1}^{3} \cup R_{1}\right) \backslash S_{3}^{1}$, thereby covering the edges in $E\left(\left\{t^{1}, t^{2}\right\}, R\right)$ and also add the cliques in $\mathscr{N}[X ; R]$, thereby producing a clique covering for $G$ of size $15+|X|=n-1$, a contradiction. This proves (6).

Note that by (4), $q=3$, and by (6), $S_{3}$ is empty. Now, in the collection $\mathscr{O}^{\prime}(G)$, for every $i \in\{1,2\}$, replace the cliques $X_{3}^{i}$ with the clique $X_{3}^{i} \cup N\left(t^{i}, R\right)$ and replace the clique $Z_{i}^{3-i}$ with the cliques in $\mathscr{N}\left[R_{i} ;\left(Z_{i}^{3-i} \cup T^{3}\right) \backslash R_{i}\right]$, thereby covering the edges in $E(T, R)$ and comprising a clique covering $\mathscr{C}$ for $G$ of size $13+|R|=n-|S|-|X|+1$. Thus, $|S|+|X| \leq 1$. If $|S|+|X|=1$, then removing the clique $T^{3} \cup R^{3}$ from $\mathscr{C}$ and replacing the cliques $Z_{3}^{1}$ and $Z_{3}^{2}$ with the cliques $Z_{3}^{1} \cup\left\{t^{3}, r_{2}^{3}\right\}$ and $Z_{3}^{2} \cup\left\{t^{3}, r_{1}^{3}\right\}$ to cover the edges in $E\left(\left\{t^{3}\right\}, R^{3}\right)$, provides a clique covering for $G$ of size $12+|R|=n-1$, a contradiction. Thus, $S=X=\emptyset$. Now, consider the family $\mathscr{P}(G)$, merge the pairs $\left(\Delta_{1}^{3}, \Delta_{2}^{3}\right)$ and $\left(\nabla_{1}^{3}, \nabla_{2}^{3}\right)$ to cover the edges in $E\left(T^{1}, T^{2}\right)$ and $E\left(R_{1}, R_{2}\right)$, respectively, and also for every $i \in\{1,2,3\}$, replace the clique $\Omega^{i}$ with the cliques $\Omega^{i} \cup N\left(t^{i}, R\right)$ to cover the edges in $E(T, R)$, thereby obtaining a clique covering for $G$ of size $13=n-|R|+1 \leq n-1$, again a contradiction. This proves Lemma 9.28.

Lemma 9.29. Let $G$ be a counterexample to Theorem 9.1, with $\left|I_{T}(\rho)\right|+\left|I_{R}(\rho)\right|=5$, say $\left|I_{T}(\rho)\right|=3$ and $\left|I_{R}(\rho)\right|=2$ for some rotator $\rho$ of $\bar{G}$. Then $G[R(\rho)]$ is of type 1 .

Proof. Suppose not, by Lemma 9.19, $R$ is a clique. By Lemma 9.28, $G[T]$ is of type 3 at $q$, for some $q \in\{1,2,3\}$, with $T_{1}=\left\{t_{1}^{q}, t_{1}^{q+1}\right\} \cup T_{1}^{q+2}$ and $T_{2}=\left\{t_{2}^{q}, t_{2}^{q+2}\right\} \cup T_{2}^{q+1}$ as in the definition. Note that the cliques in $\mathscr{O}^{\prime}(G)=\mathscr{O}(G) \cup\left\{T_{1} \cup R^{3}, T_{2} \cup R^{3}, T_{3} \cup R\right\}$ cover all the edges in $E(G) \backslash E(T, R)$.
(1) We have $S \neq \emptyset$. Also, if $S=S^{3} \neq \emptyset$, then there exists a unique $i_{0} \in\{1,2\}$ such that $T_{i_{0}}$ is complete $R$ and $T_{3-i_{0}}$ is anticomplete to $R$.
Note that by Lemma 9.13 (vi), for every vertex $r \in R$, there exists $i \in\{1,2\}$ such that $r$ is complete to $T_{i}$ and anticomplete to $T_{3-i}$. Thus, if (1) does not hold, then either $S=\emptyset$, or $S=S^{3} \neq \emptyset$ and for every $i \in\{1,2\}$, there is a vertex in $R$ complete to $T_{i}$ (and anticomplete to $T_{3-i}$ ). Consider the collection $\mathscr{P}(G)$, merge the pair $\left(\nabla_{1}^{3}, \nabla_{2}^{3}\right)$ to cover the edges in $E\left(R_{1}, R_{2}\right)$, and for every $i \in\{1,2\}$, if $S_{i}=\emptyset$, then replace the clique $\Omega_{i}$ with the cliques in $\left.\mathscr{N}\left[R_{i} ;\left(\Omega_{i} \cup T\right) \backslash R_{i}\right\}\right]$, and if $S_{i} \neq \emptyset$, then add the cliques in $\mathscr{N}\left[R_{i} ; T\right]$ to cover the edges in $E(T, R)$. This procedure yields a family $\mathscr{C}_{1}$ of cliques of $G$ covering all the edges in $E(G) \backslash\left(E\left(T_{1}\right) \cup E\left(T_{2}\right)\right)$. If $S=S^{3} \neq \emptyset$, then by the assumption, the latter modification in producing $\mathscr{C}_{1}$ also covers the edges in $E\left(T_{1}\right) \cup E\left(T_{2}\right)$ and so cc $(G) \leq 12+|S|+|R| \leq n-1$, a contradiction. Also, if $S=\emptyset$, then there exists at most one $j \in\{1,2\}$ such that the edges in $E\left(T_{j}\right)$ are not covered. Now, in $\mathscr{C}_{1}$, if $|T|=4$, then replace the clique $\Delta_{q+j}^{q-j+3}$ with the clique $\Delta_{q+j}^{q-j+3} \cup\left\{t_{j}^{q}\right\}$ and if $|T| \geq 5$, then add the clique $T_{j}$. This yields a clique covering for $G$ of size at most $n-1$. This proves (1).
(2) There exists a unique $i_{0} \in\{1,2\}$ such that $T_{i_{0}}$ is complete $R$ and $T_{3-i_{0}}$ is anticomplete to $R$.

For if $S=S^{3}$, then (2) follows from (1). Otherwise, there exists $s_{p^{\prime}}^{p} \in S$, for some $\left(p, p^{\prime}\right) \in J$ such that $p \neq 3$. Also, note that there is a vertex in $T^{p}$ nonadjacent to a vertex in $T^{p^{\prime}}$. Thus, (2) follows from Lemma 9.18 (i) (applying to $(i, j)=\left(p, p^{\prime}\right)$ ).
Finally, assuming $i_{0}$ to be as in (2), $\mathscr{C}=\mathscr{O}^{\prime}(G) \cup\left\{T_{i_{0}} \cup R\right\}$ is a clique covering for $G$ of size $16=$ $n-|S|-|T|-|R|+7$. Thus, $|S|+|T|+|R| \leq 7$, which by (1), yields $|S|=1,|T|=4$ and $|R|=2$, say $R_{1}=\left\{r_{1}\right\}$ and $R_{2}=\left\{r_{2}\right\}$. Now, apply the following modifications to $\mathscr{O}^{\prime}(G)$ to cover the edges in $E(T, R)$. If $S^{3}=S_{3}=\emptyset$, then replace the cliques $Z_{1}^{2}, Z_{1}^{3}, Z_{2}^{1}$ and $Z_{2}^{3}$ with the cliques $Z_{1}^{2} \cup N\left(r_{1}, T^{3}\right), Z_{1}^{3} \cup N\left(r_{1}, T^{1} \cup T^{2}\right), Z_{2}^{1} \cup N\left(r_{2}, T^{3}\right)$ and $Z_{2}^{3} \cup N\left(r_{2}, T^{1} \cup T^{2}\right)$, if $S_{3} \neq \emptyset$, then replace the cliques $Z_{1}^{2}, Z_{1}^{3}, Z_{2}^{1}$ and $Z_{2}^{3}$ with the cliques $\left(Z_{1}^{2} \cup N\left(r_{1}, T^{1} \cup T^{3}\right)\right) \backslash S_{3}^{2}, Z_{1}^{3} \cup S_{3}^{2} \cup N\left(r_{1}, T^{2}\right),\left(Z_{2}^{1} \cup\right.$ $\left.N\left(r_{2}, T^{2} \cup T^{3}\right)\right) \backslash S_{3}^{1}$ and $Z_{2}^{3} \cup S_{3}^{1} \cup N\left(r_{2}, T^{1}\right)$ and if $S^{3} \neq \emptyset$, then replace the cliques $Z_{1}^{2}, Z_{1}^{3}, Z_{2}^{1}$ and $Z_{2}^{3}$ with the cliques $Z_{1}^{2} \cup S_{2}^{3} \cup N\left(r_{1}, T^{1} \cup T^{3}\right),\left(Z_{1}^{3} \cup N\left(r_{1}, T^{2}\right)\right) \backslash S_{2}^{3}, Z_{2}^{1} \cup S_{1}^{3} \cup N\left(r_{2}, T^{2} \cup T^{3}\right)$ and $\left(Z_{2}^{3} \cup N\left(r_{2}, T^{1}\right)\right) \backslash S_{1}^{3}$. This provides a clique covering for $G$ of size $15=n-1$, a contradiction. This proves Lemma 9.29.

Lemma 9.30. Let $G$ be a counterexample to Theorem 9.1. Then, for every rotator $\rho$ of $\bar{G},\left|I_{T}(\rho)\right|=$ $\left|I_{R}(\rho)\right| \geq 2$.
Proof. By Lemma 9.21, $\left|I_{T}\right|,\left|I_{R}\right| \geq 2$. On the contrary and w.l.o.g. assume that $\left|I_{T}\right|=3$ and $\left|I_{R}\right|=\{1,2\}$. By Lemma 9.29, $G[R]$ is of type 1 , with $r_{1} \in R_{1}, r_{2} \in R_{2}, U^{1} \subseteq R_{2}$ and $U^{2} \subseteq R_{1}$ as in the definition. Thus, by Theorem 9.23, $T$ is a clique.
(1) Either $s_{3-p}^{p} \in S$, for some $p \in\{1,2\}$, or $\left|S_{3}\right|=2$.

This follows from Lemma 9.15 (i) (applying to $(i, j, k)=(1,2,3)$ and $(i, j, k)=(2,1,3))$.
(2) There exists a unique $i_{0} \in\{1,2\}$, for which $r_{i_{0}}$ is complete to $T$ and $r_{3-i_{0}}$ is anticomplete to $T$. Also, $U^{3-i_{0}}$ is anticomplete to $T$.
By (1), either $s_{3-p}^{p} \in S$, for some $p \in\{1,2\}$ or $\left|S_{3}\right|=2$, and the first claim follows from Lemma 9.18 (i) or (ii) (applying to $(i, j)=(3-p, p)$ or $\{i, j\}=\{1,2\}$ and $l=3$, respectively). Now, note that by (1), there exists $k \in\{1,2,3\}$ such that both $s_{k}^{1}, s_{k}^{2} \in V(G)$. Since $\left(s_{k}^{3-i_{0}}, r_{3-i_{0}}, T^{k}\right)$ is a quasi-triad of $G$ and $U^{3-i_{0}}$ is complete to $\left\{s_{k}^{3-i_{0}}, r_{3-i_{0}}\right\}$, by Lemma 9.13 (i), $U^{3-i_{0}}$ is anticomplete to $T^{k}$. Thus, for every $t \in T^{k},\left(s_{k}^{i_{0}}, t, U^{3-i_{0}}\right)$ is a quasi-triad of $G$, and since $T \backslash T^{k}$ is complete to $\left\{s_{k}^{i_{0}}, t\right\}$, by Lemma 9.13 (i), $U^{3-i_{0}}$ is anticomplete to $T \backslash T^{k}$, as desired. This proves (2).

Henceforth, let $i_{0}$ be as in (2).
(3) If $G[R]$ is of type 1.1 , then $|S|+|T| \leq 5$, i.e. $1 \leq|S| \leq 2$.

By symmetry, we may assume that $G[R]$ is of type 1.1 at 1 (i.e. $U^{1}=\emptyset$ ). Thus, $\mathscr{C}(G)=\mathscr{O}(G) \cup \mathscr{N}[R \backslash$ $\left.\left\{r_{3-i_{0}}\right\} ; T\right] \cup\left\{T \cup R^{3}, T_{3} \cup R_{1}, T_{3} \cup\left(R \backslash\left\{r_{1}\right\}\right)\right\}$ is a clique covering for $G$ of size $14+|R|=n-|S|-|T|+5$. This, with (1), implies (3).
(4) If $G[R]$ is of type 1.1 , then $|S|=1$.

Again, by symmetry, we may assume that $G[R]$ is of type 1.1 at 1 (i.e. $U^{1}=\emptyset$ ). Also, let $\mathscr{C}$ be the family of cliques defined in (3). If (4), does not hold, then by (3), we have $|S|=2$ and $|T|=3$. Thus by (1), either $S^{3}$ or $S_{3}$ is empty. We claim that $R_{i_{0}}=\left\{r_{i_{0}}\right\}$. For this is trivial if $i_{0}=2$, and if $i_{0}=1$, then by (2), $U^{2}$ is anticomplete to $T$, and removing the cliques in $\mathscr{N}\left[U^{2} ; T\right]$ from $\mathscr{C}$, yields a clique covering for $G$ of size $n-\left|U^{2}\right|$. Thus, $U^{2}=\emptyset$. This proves the claim. Now, remove the clique $N\left[r_{i_{0}} ; T\right]$ from $\mathscr{C}$, and in order to cover the edges in $E\left(\left\{r_{i_{0}}\right\}, T\right)$, in the case that $S^{3}=\emptyset$, replace the cliques $X_{1}^{3}, X_{2}^{3}$ and $Z_{i_{0}}^{3}$ with the cliques $\left(X_{i_{0}}^{3} \cup S_{3-i_{0}}^{i_{0}}\right) \backslash S_{i_{0}}^{3-i_{0}},\left(X_{3-i_{0}}^{3} \cup S_{i_{0}}^{3-i_{0}} \cup\left\{r_{i_{0}}\right\}\right) \backslash S_{3-i_{0}}^{i_{0}}$ and $Z_{i_{0}}^{3} \cup T^{1} \cup T^{2}$ and in the case that $S_{3}=\emptyset$, replace the cliques $X_{3}^{1}, X_{3}^{2}$ and $Z_{i_{0}}^{3-i_{0}}$ with the cliques $X_{3}^{1} \cup\left\{r_{i_{0}}\right\}, X_{3}^{2} \cup\left\{r_{i_{0}}\right\}$ and $Z_{i_{0}}^{3-i_{0}} \cup T^{3}$, thereby obtaining a clique covering for $G$ of size $n-1$, a contradiction. This proves (4).
(5) $G[R]$ is of type 1.2 .

For if $G[R]$ is of type 1 say at 1 , then by (4), $|S|=1$, and so by (1), $S=\left\{s_{3-p}^{p}\right\}$, for some $p \in\{1,2\}$. Consider the collection $\mathscr{P}(G)$, merge the pairs $\left\{\Delta_{p}^{3-p}, \Delta_{3}^{3-p}\right\}$ and $\left\{\Delta_{1}^{3}, \Delta_{2}^{3}\right\}$, replace the cliques $\Delta_{3-p}^{p}$ and $\nabla_{2}^{3}$ with the cliques $\Delta_{3-p}^{p} \cup T_{3}$ and $\nabla_{2}^{3} \cup U^{2}$, thereby covering the edges in $E(T) \cup E(R)$, Also, on order to cover the edges in $E(T, R)$, in the resulting family, if $i_{0}=p$, then replace the clique $\Omega_{i_{0}}$ with the cliques in $\mathscr{N}\left[R_{i_{0}} ;\left(\Omega_{i} \cup T\right) \backslash R_{i_{0}}\right]$, and if $i_{0}=3-p$, then replace the cliques $\Omega^{i_{0}}$ and $\Omega_{i_{0}}$ with the cliques in $\mathscr{N}\left[T^{i_{0}} ;\left(\Omega^{i_{0}} \cup R_{i_{0}}\right) \backslash T^{i_{0}}\right]$ and $\mathscr{N}\left[R_{i_{0}} ;\left(\Omega_{i_{0}} \cup T\right) \backslash\left(T^{i_{0}} \cup R_{i_{0}}\right)\right]$ (moreover, if $i_{0}=2$, then add the cliques in $\mathscr{N}\left[U^{2} ; T\right]$ ), thereby obtaining a clique covering for $G$ of size at most $13+\left|T^{i_{0}}\right|+\left|U^{2}\right|=n-\left|T \backslash T^{i_{0}}\right|+1 \leq n-1$, a contradiction. This proves (5).
Henceforth, by (5), we assume that $G[R]$ is of type 1.2 and w.l.o.g. $2 \leq\left|R_{1}\right| \leq\left|R_{2}\right|$. Note that the cliques in $\mathscr{O}^{\prime}(G)=\mathscr{O}(G) \cup \mathscr{N}\left[R_{1} ; T_{3} \cup R_{2}\right] \cup\left\{T \cup R^{3}\right\}$ cover all the edges in $E(G) \backslash E(T, R)$.
(6) We have $S_{3} \neq \emptyset$

Suppose not, in $\mathscr{O}^{\prime}(G)$, for every $i \in\{1,2\}$, replace the clique $X_{3}^{i}$ with the cliques in $\mathscr{N}\left[T^{i} ;\left(X_{3}^{i} \cup R\right) \backslash T^{i}\right]$ and add the cliques in $\mathscr{N}\left[T^{3} ; R\right]$, thereby obtaining a clique covering for $G$ of size $11+|T|+\left|R_{1}\right|=$ $n-|S|-\left|R_{2}\right|+2$, which by (1) is at most $n-1$. This proves (6).
(7) We have $|S|=\left|R_{1}\right|=\left|R_{2}\right|=2$. In particular, either $|S|=\left|S_{3}\right|=2$ or $S=\left\{s_{3-p}^{p}, s_{3}^{p^{\prime}}\right\}$, for some $p, p^{\prime} \in\{1,2\}$.
Note that $\mathscr{O}^{\prime}(G) \cup \mathscr{N}[T ; R]$ is a clique covering for $G$ of size $13+|T|+\left|R_{1}\right|=n-|S|-\left|R_{2}\right|+4$. Thus, $|S|+\left|R_{2}\right| \leq 4$, i.e. $|S| \leq 2$. Also, by (6), $\left|S_{3}\right| \geq 1$. This, together with (1), proves (7).
Note that by (7), we have $n=15+|T|$ and $\left|\mathscr{O}^{\prime}(G)\right|=15$.
(8) We have $|T| \geq 5$.

First, we claim that $\left|T^{3}\right| \geq 2$. Suppose not, let $T^{3}=\left\{t^{3}\right\}$. In the family $\mathscr{O}^{\prime}(G) \cup \mathscr{N}\left[T \backslash T^{3} ; R\right]$, replace the cliques $X_{1}^{3}$ and $X_{2}^{3}$ with the cliques $\left(X_{1}^{3} \cup S_{2}^{1} \cup N\left(t^{3}, R_{2}\right)\right) \backslash S_{1}^{2}$ and $\left(X_{2}^{3} \cup S_{1}^{2} \cup N\left(t^{3}, R_{1}\right)\right) \backslash S_{2}^{1}$ (in fact, the truth of (7) ensures that these two are cliques of $G$ ) to cover the edges in $E\left(T^{3}, R\right)$. This yields a clique covering for $G$ of size $14+|T|=n-1$, a contradiction. This proves the claim. Now, if $|T| \leq 4$, then for every $i \in\{1,2\},\left|T^{i}\right|=1$, say $T^{i}=\left\{t^{i}\right\}$, and also $\left|T^{3}\right|=2$. By (7), either $|S|=\left|S_{3}\right|=2$ or $S=\left\{s_{3-p}^{p}, s_{3}^{p^{\prime}}\right\}$, for some $p, p^{\prime} \in\{1,2\}$. In the former case, consider $\mathscr{O}^{\prime}(G) \cup \mathscr{N}\left[T \backslash T^{1} ; R\right]$ and replace the cliques $X_{2}^{1}$ and $X_{3}^{1}$ with the cliques $X_{2}^{1} \cup S_{3}^{2} \cup N\left(t^{1}, R_{1}\right)$ and $\left(X_{3}^{1} \cup N\left(t^{1}, R_{2}\right)\right) \backslash S_{3}^{2}$ to cover the edges in $E\left(T^{1}, R\right)$, thereby comprising a clique covering for $G$ of size $14+|T|=n-1$. In the latter case, in $\mathscr{O}^{\prime}(G) \cup \mathscr{N}\left[T \backslash T^{3-p} ; R\right]$, replace the cliques $X_{p}^{3-p}$ and $X_{3}^{3-p}$ with the cliques $X_{p}^{3-p} \cup S_{3}^{p} \cup N\left(t^{3-p}, R_{3-p}\right)$ and $\left(X_{3}^{3-p} \cup N\left(t^{3-p}, R_{p}\right)\right) \backslash S_{3}^{p}$. This provides a clique covering for $G$ of size $14+|T|=n-1$, a contradiction. This proves (8).

Finally, the collection $\mathscr{O}^{\prime}(G) \cup \mathscr{N}[R ; T]$ is a clique covering for $G$ of size $19=n-|T|+4$, which by (8), is at most $n-1$, a contradiction. This proves Lemma 9.30.

Now, we are ready to prove Theorem 9.27.
Proof of Theorem 9.27. By Lemma 9.30, $\left|I_{T}\right|=\left|I_{R}\right| \geq 2$. Now, on the contrary, assume that $\left|I_{T}\right|=\left|I_{R}\right|=3$. By Theorem 9.23 and w.l.o.g. assume that $R$ is a clique.
(1) We have $S \neq \emptyset$.

Suppose not, let $S=\emptyset$. Consider the family $\mathscr{P}(G)$, merge the pairs $\left(\nabla_{2}^{1}, \nabla_{3}^{1}\right),\left(\nabla_{1}^{2}, \nabla_{3}^{2}\right)$ and $\left(\nabla_{1}^{3}, \nabla_{2}^{3}\right)$ to cover the edges in $E(R)$ and for every $i \in\{1,2,3\}$, replace the clique $\Omega_{i}$ with the cliques in $\mathscr{N}\left[R_{i} ;\left(\Omega_{i} \cup T\right) \backslash R_{i}\right]$ to cover the edges in $E(T, R)$, and call the resulting collection $\mathscr{P}^{\prime}(G)$, the clique in which cover all the edges in $E(G) \backslash\left(\cup_{i=1}^{3} E\left(T^{i}, T^{i+1}\right)\right)$. By Lemma $9.17, T$ is not a clique. Also, if $G[T]$ is of type 2 at $q$, for some $q \in\{1,2,3\}$ with $t^{q}, t^{q+1}$ and $t^{q+2}$, as in the definition, then in $\mathscr{P}^{\prime}(G)$, merge the pair $\left(\Delta_{q+1}^{q}, \Delta_{q+2}^{q}\right)$ and replace the cliques $\Delta_{q+1}^{q+2}$ and $\Delta_{q+2}^{q+1}$ with the cliques $\Delta_{q+1}^{q+2} \cup\left(T^{q} \backslash\left\{t^{q}\right\}\right)$ and $\Delta_{q+2}^{q+1} \cup\left(T^{q} \backslash\left\{t^{q}\right\}\right)$, thereby obtaining a clique covering for $G$ of size $11+|R| \leq n-1$, a contradiction. Thus, by Lemma $9.20, G[T]$ is of type 3 at $q$, for some $q \in\{1,2,3\}$, with $T_{1}=\left\{t_{1}^{q}, t_{1}^{q+1}\right\} \cup T_{1}^{q+2}$ and $T_{2}=\left\{t_{2}^{q}, t_{2}^{q+2}\right\} \cup T_{2}^{q+1}$ as in the definition. Now, in $\mathscr{P}^{\prime}(G)$, if $|T|=4$, then replace the cliques $\Delta_{q+1}^{q+2}$ and $\Delta_{q+2}^{q+1}$ with the cliques $\Delta_{q+1}^{q+2} \cup\left\{t_{1}^{q}\right\}$ and $\Delta_{q+2}^{q+1} \cup\left\{t_{2}^{q}\right\}$, if $|T|=5$, say $\left|T_{i}^{q-i+3}\right|=1$ for some $i \in\{1,2\}$, then add the clique $T_{i}$ and replace the cliques $\Delta_{q}^{q+i}$ and $\Delta_{q-i+3}^{q+i}$ with the cliques $\left(\Delta_{q}^{q+i} \cup\left\{t_{1}^{q-i+3}\right\}\right) \backslash\left\{t_{2}^{q}\right\}$ and $\left(\Delta_{q-i+3}^{q+i} \cup\left\{t_{2}^{q}\right\}\right) \backslash\left\{t_{1}^{q-i+3}\right\}$, and if $|T|=6$, then add the cliques $T_{1}$ and $T_{2}$, thereby obtaining a clique covering of size $12+\left|T_{2}^{q+1}\right|+\left|T_{1}^{q+2}\right|+|R|=n-1$, a contradiction. This proves (1).
(2) $G[T]$ is of type 3 .

Suppose not, by Lemmas 9.17 and $9.20, G[T]$ is of type 2 at $q$, for some $q \in\{1,2,3\}$, and w.l.o.g. we may assume that $q=1$. Let $X=T^{1} \backslash\left\{t^{1}\right\}$. By (1), $|S| \geq 1$, say $s_{p^{\prime}}^{p} \in S$, for some $\left(p, p^{\prime}\right) \in J$. Note that the cliques $\mathscr{O}^{\prime}(G)=\mathscr{O}(G) \cup\left\{T^{1} \cup R^{3},\left(T \backslash\left\{t^{1}\right\}\right) \cup R^{3}, T_{3} \cup R\right\}$ cover all the edges in $E(G) \backslash E(T, R)$. If either $p=1$ or $p^{\prime}=1$, then by Lemma 9.18 (i) (applying to $(i, j)=\left(p, p^{\prime}\right)$ ), either $t^{1}$ or $\left\{t^{2}, t^{3}\right\}$ is complete to $R$ (and the other one is anticomplete to $R$ ), and so add the clique $\left\{t^{1}\right\} \cup R$ or $\left\{t^{2}, t^{3}\right\} \cup R$ as well as the cliques in $\mathscr{N}[X ; R]$ to $\mathscr{O}^{\prime}(G)$. Also, if $S_{1}=S^{1}=\emptyset$, then in $\mathscr{O}^{\prime}(G)$, for every $i \in\{2,3\}$, replace the clique $X_{1}^{i}$ and $X_{5-i}^{i}$ with the clique $X_{1}^{i} \cup N\left(t^{i}, R \backslash R_{1}\right)$ and $X_{5-i}^{i} \cup N\left(t^{i}, R_{1}\right)$ and also add the cliques in $\mathscr{N}\left[T^{1} ; R\right]$. This yields a clique covering for $G$ of size $16+|X|=n-|S|-|R|+4$. Thus, $|S|=\left|S_{p^{\prime}}^{p}\right|=1$ and $|R|=3$, say $R_{i}=\left\{r_{i}\right\}, i=1,2,3$. Let $\left\{p, p^{\prime}, p^{\prime \prime}\right\}=\{1,2,3\}$. Now, in $\mathscr{O}^{\prime}(G)$,
replace the cliques $Z_{p}^{p^{\prime}}, Z_{p}^{p^{\prime \prime}}, Z_{p^{\prime}}^{p}, Z_{p^{\prime}}^{p^{\prime \prime}}, Z_{p^{\prime \prime}}^{p}$ and $Z_{p^{\prime \prime}}^{p^{\prime}}$ with the cliques $Z_{p}^{p^{\prime}} \cup N\left(r_{p}, T^{p^{\prime \prime}}\right), Z_{p}^{p^{\prime \prime}} \cup N\left(r_{p}, T^{p} \cup\right.$ $\left.\left.T^{p^{\prime}}\right), Z_{p^{\prime}}^{p} \cup N\left(r_{p}, T^{p^{\prime \prime}}\right), Z_{p^{\prime}}^{p^{\prime \prime}} \cup N\left(r_{p}, T^{p} \cup T^{p^{\prime}}\right), Z_{p^{\prime \prime}}^{p} \cup N\left(r_{p^{\prime \prime}}, T \backslash T^{p}\right)\right) \backslash S_{p^{\prime}}^{p}$ and $Z_{p^{\prime \prime}}^{p^{\prime}} \cup S_{p^{\prime}}^{p} \cup N\left(r_{p^{\prime \prime}}, T^{p}\right)$, thereby comprising a clique covering for $G$ of size $15 \leq n-1$, a contradiction. This proves (2).
By (2), $G[T]$ is of type 3 at $q$, for some $q \in\{1,2,3\}$, with $T_{1}=\left\{t_{1}^{q}, t_{1}^{q+1}\right\} \cup T_{1}^{q+2}$ and $T_{2}=\left\{t_{2}^{q}, t_{2}^{q+2}\right\} \cup$ $T_{2}^{q+1}$ as in the definition. By $(1), s_{p^{\prime}}^{p} \in S$, for some $\left(p, p^{\prime}\right) \in J$, and since there is vertex in $T^{p}$ nonadjacent to a vertex in $T^{p^{\prime}}$, by Lemma 9.18(i) (applying to $(i, j)=\left(p, p^{\prime}\right)$ ) and also Lemma 9.13 (vi), there exists $i_{0} \in\{1,2\}$ such that $T_{i_{0}}$ is complete to $R T_{3-i_{0}}$ is anticomplete to $R$. Now, the collection $\mathscr{O}(G) \cup\left\{T_{i_{0}} \cup R, T_{1} \cup R^{3}, T_{2} \cup R^{3}, T_{3} \cup R\right\}$ is a clique covering for $G$ of size $16=n-|S|-|T|-|R|+7$, which by (1) is at most $n-1$, a contradiction. This proves Theorem 9.27.

### 9.6 Proof of Theorem 9.1

In this subsection, using Theorem 9.27, we complete the proof of Theorem 9.1. We need the following lemma. As usual, we omit the term $\rho$ within the proofs.

Lemma 9.31. Let $G$ be a counterexample to Theorem 9.1. Then for every rotator $\rho$ of $\bar{G}$, both $G[T(\rho)]$ and $G[R(\rho)]$ are of type 1 .

Proof. By Theorem 9.27, we have $\left|I_{T}\right|=\left|I_{R}\right|=2$. On the contrary, by Lemma 9.19 and w.l.o.g. let $R$ be a clique. By symmetry, suppose that $I_{T}=\{1,2\}$ and $I_{R}=\{1, p\}$, for some $p \in\{2,3\}$. By Lemma $9.17, T$ is not a clique, and thus by Lemma $9.19, G[T]$ is of type 1 , with $t^{1} \in T^{1}, t^{2} \in T^{2}, U_{1} \subseteq$ $T^{2}$ and $U_{2} \subseteq T^{1}$ as in the definition. Note that by Lemma 9.15 (i) (applying to $(i, j, k)=(1, p, 5-p)$ ), either $s_{2}^{1}$ or $s_{1}^{p}$ belongs to $S$. Also, if $s_{2}^{1} \in S$ or $p=2$, then apply Lemma 9.18 (i) for $\{i, j\}=\{1,2\}$, and if $s_{1}^{p} \in S$ and $p=3$, then apply Lemma 9.18 (ii) for $\left(l, l^{\prime}\right)=(p, 1)$ to deduce that there is a unique $i_{0} \in\{1,2\}$ such that $t^{i_{0}}$ is complete to $R$ and $t^{3-i_{0}}$ is anticomplete to $R$.
(1) If $i_{0}=2$, then $s_{2}^{1} \in S, s_{3}^{1}, s_{1}^{p} \notin S$ and $G[T]$ is of type 1.1 at 1 .

Since $i_{0}=2, t^{1}$ is anticomplete to $R_{1}$. Choose $r_{1} \in R_{1}$. It can be checked that $\bar{G}$ induces a rotator on $\rho^{\prime}=\left(s_{1}^{1}, s_{2}^{2}, s_{3}^{3} ; r_{1}, t_{3}^{2}, t_{3}^{3} ; t^{1}, r_{2}^{3}, r_{3}^{3}\right)$. Note that $r_{1}^{3} \in T^{1}\left(\rho^{\prime}\right), t_{3}^{1} \in R_{1}\left(\rho^{\prime}\right), R_{p} \subseteq R_{p}\left(\rho^{\prime}\right)$ and $S_{1}^{p} \subseteq R_{5-p}\left(\rho^{\prime}\right)$. Now, if $s_{1}^{p} \in S$, then $\left|I_{R}\left(\rho^{\prime}\right)\right|=3$ which contradicts Theorem 9.27. Therefore, $s_{1}^{p} \notin S$ and thus $s_{2}^{1} \in S$. Also, $s_{2}^{1} \in T^{3}\left(\rho^{\prime}\right)$ and $S_{3}^{1} \cup U_{1} \subseteq T^{2}\left(\rho^{\prime}\right)$. Now, if $S_{3}^{1} \cup U_{1} \neq \emptyset$, then $\left|I_{T}\left(\rho^{\prime}\right)\right|=3$ which again contradicts Theorem 9.27. Hence, $S_{3}^{1}=U^{1}=\emptyset$ and so $G[T]$ is of type 1.1 at 1 . This proves (1).
(2) Let $G[T]$ be of type 1.1 at $i$, for some $i \in\{1,2\}$. Then $|S| \geq 2$. Also, if $i_{0}=1$, then $S \neq\left\{s_{1}^{2}, s_{1}^{3}\right\}$ and if $i_{0}=2$, then $S \neq\left\{s_{2}^{1}, s_{1}^{5-p}\right\}$.
Suppose not, assume that either $|S|=\left|S_{2}^{1}\right|=1$, or $|S|=\left|S_{1}^{p}\right|=1$, or $i_{0}=1$ and $S \neq\left\{s_{1}^{2}, s_{1}^{3}\right\}$ or $i_{0}=2$ and $S \neq\left\{s_{2}^{1}, s_{1}^{5-p}\right\}$.
and if the third assertion does not hold, then by (1), $i=1$. Considering $\mathscr{P}(G)$, if $s_{i}^{3} \notin S$, then replace the clique $\Delta_{3-i}^{3}$ with the clique $\Delta_{3-i}^{3} \cup U_{3-i}$. Otherwise, add the clique $T \backslash\left\{t^{i}\right\}$, there by covering all edges in $E\left(T^{1}, T^{2}\right)$. Also, if $S_{5-p}=\emptyset$ (i.e. the case that either $p=2$ or $p=3$ and $s_{2}^{1} \notin S$ ), then merge the pair $\left(\nabla_{1}^{5-p}, \nabla_{p}^{5-p}\right)$. Otherwise (i.e. the case that $p=3$ and $s_{2}^{1} \in S$ ), then replace the clique $\nabla_{1}^{2}$ with the clique $\nabla_{1}^{2} \cup R_{p}$, there by covering all edges in $E\left(R_{1}, R_{p}\right)$. Thus, it remains to cover the edges in $E(T, R)$. For this purpose, add the cliques in $\mathscr{N}\left[U^{i_{0}} ; R\right]$, if either $S^{i_{0}}=\emptyset$ or $i_{0} \notin I_{R}$, then replace the clique $\Omega^{i_{0}}$ with the cliques in $\mathscr{N}\left[T^{i_{0}} ; R \cup\left\{t_{3}^{i_{0}}\right\}\right]$, and if either $S^{i_{0}} \neq \emptyset$ and $i_{0} \in I_{R}$, then replace the cliques $\Omega^{i_{0}}$ and $\Omega_{i_{0}}$ with the cliques in $\mathscr{N}\left[T^{i_{0}} ;\left(\Omega^{i_{0}} \cup R\right) \backslash\left(T^{i_{0}} \cup R_{i_{0}}\right)\right]$ and $\mathscr{N}\left[R_{i_{0}} ;\left(\Omega_{i_{0}} \cup T^{i_{0}}\right) \backslash R_{i_{0}}\right]$. We leave the reader to check that this procedure yields a clique covering for $G$ of size at most $n-1$, a contradiction. This proves (2).
If $G[T]$ is of type 1.1 at $i$, for some $i \in\{1,2\}$, then define $\mathscr{O}^{\prime}(G)=\mathscr{O}(G) \cup\left\{T^{i} \cup R^{3},\left(T \backslash\left\{t^{i}\right\}\right) \cup R^{3}, T_{3} \cup R\right\}$, which is a collection of cliques of $G$ covering all the edges in $E(G) \backslash E(T, R)$. Therefore, by (1), $\mathscr{O}^{\prime}(G) \cup \mathscr{N}\left[T \backslash\left\{t^{3-i_{0}}\right\} ; R\right]$ is a clique covering of size $14+|T|=n-|S|-|R|+5$ which is at most $n-1$, whenever $|S| \geq 4$. Hence, by (2), the following statement holds.
(3) If $G[T]$ is of type 1.1 at $i$, for some $i \in\{1,2\}$, then $2 \leq|S| \leq 3$.

Now, we deduce more information about $G$.
(4) We have $i_{0}=1, p=3$ and $\left|S_{2}^{1} \cup S_{1}^{2}\right| \leq 1$.

Suppose not, we may assume that $i_{0}=2$ (indeed, if $i_{0}=1$ and either $p=2$ or both $s_{1}^{2}, s_{2}^{1} \in S$, then in order to assume that $i_{0}=2$, we may consider the decompositions corresponding to the rotators $\left(s_{2}^{2}, s_{1}^{1}, s_{3}^{3} ; t_{3}^{2}, t_{3}^{1}, t_{3}^{3} ; r_{2}^{3}, r_{1}^{3}, r_{3}^{3}\right)$ or $\left(s_{2}^{1}, s_{1}^{2}, s_{3}^{3} ; t_{3}^{2}, t_{3}^{1}, t_{3}^{3} ; r_{1}^{3}, r_{2}^{3}, r_{3}^{3}\right)$, respectively). Thus by (1), $s_{2}^{1} \in$ $S, s_{3}^{1}, s_{1}^{p} \notin S$ and $G[T]$ is of type 1.1 at 1 . Now, considering $\mathscr{O}^{\prime}(G)$, replace the cliques $X_{1}^{2}$ and $X_{3}^{2}$ with the cliques $X_{1}^{2} \cup R_{p}$ and $X_{3}^{2} \cup R_{1}$ as well as adding the cliques in $\mathscr{N}\left[U_{2} ; R\right]$ to cover the edges in $E(T, R)$, thereby obtaining a clique covering $\mathscr{C}$ for $G$ of size $15+\left|U_{2}\right|=n-|S|-|R|+4$. Thus by (3), $|S|=|R|=2$ and since $i_{0}=2$ and $s_{2}^{1} \in S$, by (2), we have $s_{1}^{5-p} \notin S$ (i.e. $S_{1}=\emptyset$ ). Now, remove the clique $T^{1} \cup R^{3}$ from $\mathscr{C}$ and replace the cliques $Z_{1}^{5-p}, Z_{p}^{5-p}$ and $Z_{5-p}^{p}$ with the cliques $R \cup S^{5-p} \cup\left\{s_{5-p}^{5-p}\right\},\left\{s_{5-p}^{5-p}, r_{1}^{3}, r_{p}^{3}\right\} \cup S^{5-p} \cup T^{1}$ and $Z_{5-p}^{p} \cup T^{1}$ to cover the edges in $E\left(T^{1}, R^{3}\right)$, thereby obtaining clique covering for $G$ of size $14+\left|U_{2}\right|=n-1$, a contradiction. This proves (4).
(5) If $G[T]$ is of type 1.1 at $i$, for some $i \in\{1,2\}$, then we have $s_{1}^{3} \in S$.

By (4), $i_{0}=1$ and $p=3$. Note that if (5) does not hold, then $s_{2}^{1} \in S$ and again by (4), $s_{1}^{2} \notin S$. Now, in the collection $\mathscr{O}^{\prime}(G)$, if $i=1$ (resp. $i=2$ ), then remove the clique $T^{1} \cup R^{3}$ (resp. $\left.\left(T \backslash\left\{t^{2}\right\}\right) \cup R^{3}\right)$, replace the cliques $Z_{1}^{3}, Z_{2}^{3}$ and $Z_{3}^{2}$ with the cliques $Z_{1}^{3} \cup\left\{t^{1}\right\}, Z_{2}^{3} \cup\left\{t^{1}\right\}$ and $Z_{3}^{2} \cup\left\{t^{1}\right\}$ to cover the edges in $E\left(\left\{t^{1}\right\}, R^{\prime}\right)$ and add the cliques $\mathscr{N}\left[U_{3-i} ; R \cup\left\{t^{1}\right\}\right]$ to cover the edges in $E\left(T \backslash\left\{t^{1}\right\}, R\right) \cup E\left(T^{1}, T^{2}\right)$, thereby obtaining a clique covering for $G$ of size $14+\left|U_{3-i}\right|=n-|S|-|R|+3$, which by (3) is at most $n-1$. This proves (5).
(6) If $G[T]$ is of type 1.1 at $i$, for some $i \in\{1,2\}$, then $S_{2}^{1} \cup S_{1}^{2} \cup S_{2}^{3} \neq \emptyset$.

Suppose not, by (4), we have $i_{0}=1$ and $p=3$ and by (5), $s_{1}^{3} \in S$. Considering $\mathscr{O}^{\prime}(G)$, if $i=1$ (resp. $i=2$ ), remove the clique $T^{1} \cup R^{3}$ (resp. $\left.\left(T \backslash\left\{t^{2}\right\}\right) \cup R^{3}\right)$, replace the cliques $Z_{1}^{2}, Z_{3}^{2}, Z_{2}^{1}$ and $Z_{2}^{3}$ with the cliques $Z_{1}^{2} \cup\left\{t^{1}\right\}, Z_{3}^{2} \cup\left\{t^{1}\right\},\left(Z_{2}^{1} \cup S_{1}^{3}\right) \backslash S_{3}^{1}$ and $\left(Z_{2}^{3} \cup S_{3}^{1} \cup\left\{t^{1}\right\}\right) \backslash S_{1}^{3}$ to cover the edges in $E\left(\left\{t^{1}\right\}, R^{\prime}\right)$. Also, add the cliques in $\mathscr{N}\left[U_{3-i} ; R \cup\left\{t^{1}\right\}\right]$ to cover the edges in $E\left(T \backslash\left\{t^{1}\right\}, R\right) \cup E\left(T^{1}, T^{2}\right)$, thereby obtaining a clique covering for $G$ of size $14+\left|U_{3-i}\right|=n-|S|-|R|+3$, which by (2) is at most $n-1$. This proves (6).
(7) If $G[T]$ is of type 1.1 at $i$, for some $i \in\{1,2\}$, then $\left\{s_{2}^{j}, s_{1}^{3}\right\} \subseteq S \subseteq\left\{s_{2}^{j}, s_{1}^{2}, s_{1}^{3}\right\}$, for some $j \in\{1,3\}$.

By (4), $i_{0}=1$ and $p=3$, and by (5), $s_{1}^{3} \in S$. If $S_{2}=\emptyset$, then by (6), $s_{1}^{2} \in S$. Considering $\mathscr{O}^{\prime}(G)$, in the case that $i=1$ (resp. $i=2$ ), remove the clique $\left(T \backslash\left\{t^{1}\right\}\right) \cup R^{3}$ (resp. $T^{2} \cup R^{3}$ ) and replace the cliques $Z_{2}^{1}$ and $Z_{2}^{3}$ with the cliques $Z_{2}^{1} \cup\left\{t^{2}, r_{3}^{3}\right\}$ and $Z_{2}^{3} \cup\left\{t^{2}, r_{1}^{3}\right\}$ to cover the edges in $E\left(\left\{t^{2}\right\}, R^{3}\right)$. Also, let $U_{2}^{\prime} \subseteq U_{2}$ be the set of vertices of $U_{2}$ complete to $R$. Then $U_{2} \backslash U_{2}^{\prime}$ is anticomplete to $R$ (indeed, if $t \in U_{2}$ is nonadjacent to some $r \in R_{j}, j \in\{1,3\}$, then, $t$ is anticomplete to $R_{4-j}$, since otherwise, $\left\{r^{\prime}, s_{j}^{1}, t, r\right\}$ would be a claw, for some $r^{\prime} \in R_{4-j}$ ). Thus adding the clique $\left\{t^{1}\right\} \cup U_{2}^{\prime} \cup R$ as well as the cliques in $\mathscr{N}\left[U_{1} ; R\right] \cup \mathscr{N}\left[U_{2} ;\left\{t^{2}\right\}\right]$ to the resulting family to cover the edges in $E(T, R) \cup E\left(T^{1}, T^{2}\right)$, thereby providing a clique covering for $G$ of size $13+|T|=n-|S|-|R|+4$. Thus, $|S|=2$, i.e. $S=\left\{s_{1}^{2}, s_{1}^{3}\right\}$, and since $i_{0}=1$, this contradicts (2). Thus, $S_{2} \neq \emptyset$ and so $\left\{s_{2}^{j}, s_{1}^{3}\right\} \subseteq S$, for some $j \in\{1,3\}$.
Also, if $s_{1}^{2} \notin S$, then in $\mathscr{O}^{\prime}(G)$, replacing the cliques $Z_{1}^{2}$ and $Z_{3}^{2}$ with the cliques $Z_{1}^{2} \cup\left\{t^{1}\right\}$ and $Z_{3}^{2} \cup\left\{t^{1}\right\}$ and adding the cliques in $\mathscr{N}\left[U_{3-i} ; R\right]$ to cover the edges in $E(T, R)$, yield a clique covering for $G$ of size $13+|T|=n-|S|-|R|+4$. Hence, $|S|=2$. This, together with (3), proves (7).
(8) $G[T]$ is of type 1.2 .

Suppose not, let $G[T]$ be of type 1.1 at $i$ for some $i \in\{1,2\}$. By (4), $i_{0}=1$ and $p=3$. Also by (7), either $S=\left\{s_{2}^{j}, s_{1}^{3}\right\}$ or $S=\left\{s_{2}^{j}, s_{1}^{2}, s_{1}^{3}\right\}$, for some $j \in\{1,3\}$. If $S=\left\{s_{2}^{j}, s_{1}^{3}\right\}$, for some $j \in\{1,3\}$, then considering $\mathscr{O}^{\prime}(G)$ in the case that $i=1$ (resp. $i=2$ ), remove the cliques $T^{1} \cup R$ (resp. $\left(T \backslash\left\{t^{2}\right\}\right) \cup R$ ), replace the cliques $Z_{1}^{2}$ and $Z_{3}^{2}$ with the cliques $Z_{1}^{2} \cup\left\{t^{1}\right\}$ and $Z_{3}^{2} \cup\left\{t^{1}\right\}$ and add the cliques in $\mathscr{N}\left[U_{3-i} ; R \cup\left\{t^{1}\right\}\right]$ to cover the edges in $E(T, R) \cup E\left(\left\{t^{1}\right\},\left\{r_{1}^{3}, r_{3}^{3}\right\}\right) \cup E\left(T^{1}, T^{2}\right)$. Also, in order to cover the edge $t^{1} r_{2}^{3}$, if $j=1$, then replace the cliques $X_{3}^{1}, X_{3}^{2}$ and $Z_{2}^{3}$ with the cliques $\left(X_{3}^{1} \cup\left\{r_{2}^{3}\right\}\right) \backslash\left\{t_{3}^{1}\right\}$, $\left(X_{3}^{2} \cup\left\{t_{3}^{1}\right\}\right) \backslash T^{2}$ and $Z_{2}^{3} \cup T^{2}$ and if $j=3$, then replace the cliques $Z_{2}^{1}$ and $Z_{2}^{3}$ with the cliques $Z_{2}^{1} \cup S_{1}^{3}$ and $\left(Z_{2}^{3} \cup\left\{t^{1}\right\}\right) \backslash S_{1}^{3}$. This yields a clique covering for $G$ of size $12+|T|=n-|R|+1 \leq n-1$, a
contradiction. Therefore, assume that $S=\left\{s_{2}^{j}, s_{1}^{2}, s_{1}^{3}\right\}$, for some $j \in\{1,3\}$. In this case, note that $U_{2}$ is anticomplete to $R$ (indeed, $U_{2}$ is anticomplete to $R_{j}$, since otherwise $\left\{t, s_{2}^{j}, t^{2}, r\right\}$ would be a claw for some $t \in U_{2}, r \in R_{j}$, and consequently, $U_{2}$ is anticomplete to $R_{4-j}$, since otherwise $\left\{r^{\prime \prime}, s_{1}^{j}, t, r^{\prime}\right\}$ would be a claw for every $r^{\prime} \in R_{j}$ and some $\left.t \in U_{2}, r^{\prime \prime} \in R_{4-j}\right)$. Now, in $\mathscr{O}^{\prime}(G)$, replace the clique $Z_{3}^{1}$ and $Z_{3}^{2}$ with the cliques $\left(Z_{3}^{1} \cup S_{1}^{2}\right) \backslash S_{2}^{1}$ and $\left(Z_{3}^{2} \cup S_{2}^{1} \cup\left\{t^{1}\right\}\right) \backslash S_{1}^{2}$ to cover the edges in $E\left(\left\{t^{1}\right\}, R_{3}\right)$, replace the clique $X_{3}^{1}$ with the clique $\left(X_{3}^{1} \cup R_{1}\right) \backslash U_{2}$ to cover the edges in $E\left(\left\{t^{1}\right\}, R_{1}\right)$, and add the cliques in $\mathscr{N}\left[U_{1} ; R\right] \cup \mathscr{N}\left[U_{2} ;\left\{s_{3}^{3}\right\}\right]$, thereby obtaining a clique covering for $G$ of size $13+|T|=n-|R|+1 \leq n-1$, again a contradiction. This proves (8).
By (8), $G[T]$ is of type 1.2 , and $\mathscr{C}=\mathscr{O}(G) \cup\left\{T_{3} \cup R\right\} \cup \mathscr{N}\left[T^{1} ; T^{2} \cup R^{3}\right] \cup \mathscr{N}[R ; T]$ is a clique covering for $G$ of size $13+\left|T^{1}\right|+|R|=n-|S|-\left|T^{2}\right|+4$. Thus, $1 \leq|S| \leq 2$. Now, we complete the proof through the following three cases.
If $|S|=2$ and $S^{3}=\emptyset$, then remove the clique $Z_{1}^{3}$ from $\mathscr{C}$ and replace the cliques in $\mathscr{N}\left[R_{1} ; T\right]$ with the cliques in $\mathscr{N}\left[R_{1} ; T \cup\left\{s_{3}^{3}, r_{1}^{3}\right\}\right]$, thereby obtaining a clique covering for $G$ of size $12+\left|T^{1}\right|+|R|=$ $n-|S|-\left|T^{2}\right|+3 \leq n-1$, a contradiction.
Also, if $|S|=\left|S^{3}\right|=2$ and $p=3$, then every vertex in $T^{1}$ is either complete or anticomplete to $R$ (indeed, if a vertex $t \in T^{1}$ is nonadjacent to some vertex $r \in R_{i}$ for some $i \in I_{R}$, then $t$ is anticomplete to $R_{4-i}$, since otherwise $\left\{r^{\prime}, s_{1}^{j}, t, r\right\}$ would be a claw for every vertex $r^{\prime} \in R_{4-j}$ adjacent to $t$ ). Now, let $T^{1} \subseteq T^{1}$ be the set of vertices in $T^{1}$ which are complete to $R$. In $\mathscr{C}$, remove the cliques in $\mathscr{N}[R, T]$ and in order to cover the edges in $E(T, R)$, add the cliques $T^{1} \cup R$, replace the clique $Z_{1}^{2}$ with the clique $Z_{1}^{2} \cup S_{2}^{3}$ and also replace the cliques $Z_{3}^{1}$ and $Z_{1}^{3}$ with the cliques in $\mathscr{N}\left[R_{3} ;\left(Z_{3}^{1} \cup T^{2}\right) \backslash R_{3}\right]$ and the cliques in $\mathscr{N}\left[R_{1} ;\left(Z_{1}^{3} \cup T^{2}\right) \backslash S_{2}^{3} \cup R_{3}\right]$, thereby obtaining a clique covering for $G$ of size $12+\left|T^{1}\right|+|R|=n-|S|-\left|T^{2}\right|+3 \leq n-1$ which is impossible.
Finally assume that either $|S|=1$, or $|S|=2$ and $\left|S^{3}\right|=1$, or $|S|=\left|S^{3}\right|=2$ and $p=2$. Then, considering $\mathscr{P}(G)$, in order to cover the edges $E\left(T^{1}, T^{2}\right)$, for every $i \in\{1,2\}$ such that $s_{3-i}^{3} \notin S$, remove the clique $\Delta_{i}^{3}$, if $\left|S^{3}\right|=1$, then choose $j \in\{1,2\}$ satisfying $s_{j}^{3} \notin S$ and add the cliques in $\mathscr{N}\left[T^{j} ; \Delta_{3-j}^{3}\right]$ and if $\left|S^{3}\right|=2$, then add the cliques in $\mathscr{N}\left[T^{1} ; T^{2}\right]$. Also, in order to cover the edges in $E\left(R_{1}, R_{p}\right)$, if $S_{5-p}=\emptyset$, then merge the pair $\left(\nabla_{1}^{5-p}, \nabla_{p}^{5-p}\right)$, if $\left|S_{5-p}\right|=1$, then choose $i \in\{1, p\}$ satisfying $s_{5-p}^{p-i+1} \notin S$ and replace the clique $\nabla_{i}^{5-p}$ with the clique $\nabla_{i}^{5-p} \cup R_{p-i+1}$, and if $\left|S_{5-p}\right|=2$, then add the clique $R$. Finally, for every $i \in I_{R}$, in order to cover the edges $E\left(T, R_{i}\right)$, if $S_{i}=\emptyset$ or $i=3$, then replace the clique $\Omega_{i}$ and add clique $\mathscr{N}\left[R_{i} ;\left(T \cup \Omega_{i}\right) \backslash R_{i}\right]$ and if $S_{i} \neq \emptyset$ and $i \neq 3$, then add the cliques in $\mathscr{N}\left[R_{i} ; T\right]$. We leave it to the reader to check that this procedure produces a clique covering for $G$ of size at most $n-1$, a contradiction. This proves Lemma 9.31.

Now, we are ready to establish Theorem 9.1.
Proof of Theorem 9.1. Let $G$ be a counterexample. Due to Theorem $9.2, \bar{G}$ contains an induced rotator $\rho$ and no square-forcer. Hence, by Theorem 9.27 , we have $\left|I_{T}(\rho)\right|=\left|I_{R}(\rho)\right|=2$, say $I_{T}(\rho)=$ $\{1,2\}, I_{R}(\rho)=\{1, p\}, p \in\{2,3\}$. Also, by Lemma 9.31 , both $G[T(\rho)]$ and $G[R(\rho)]$ are of type 1 with $t^{1} \in T^{1}, U_{2} \subseteq T^{1}, t^{2} \in T^{2}, U_{1} \subseteq T^{2}, r_{1} \in R_{1}, U^{p} \subseteq R_{1}, r_{p} \in R_{p}$ and $U^{1} \subseteq R_{p}$ as in the definition. By Lemma 9.15 (i) (applying to $(i, j, k)=(1, p, 5-p)$ ), $s_{k^{\prime}}^{k} \in S$, for some $\left(k, k^{\prime}\right) \in\{(1,2),(p, 1)\}$. For every $i \in\{1,2\}$ and every $t \in U_{i}$, let $A_{i}(t)=\left\{s_{k^{\prime}}^{k}, t^{i}, r_{k}, t\right\}$ and $B_{i}(t)=\left\{s_{1}^{1}, t^{i}, r_{1}, t\right\}$. Also, for every $j \in\{1, p\}$ and every $r \in U^{j}$, let $C_{j}(r)=\left\{s_{k^{\prime}}^{k}, t^{k^{\prime}}, r_{j}, r\right\}$ and $D_{j}(r)=\left\{s_{1}^{1}, t^{1}, r_{j}, r\right\}$.
First, assume that $t^{1}$ is nonadjacent to $r_{p}$. Then, by Lemma 9.13 (vi), $t^{1}$ is adjacent to $r_{1}$ and $t^{2}$ is adjacent to $r_{p}$ and nonadjacent to $r_{1}$. Since $A_{1}(t), t \in U_{1}$, is not a claw, $r_{1}$ is complete to $U_{1}$ and $r_{p}$ is anticomplete to $U_{1}$. Also, since $A_{2}(t), t \in U_{2}$, is not a claw, $r_{1}$ is anticomplete to $U_{2}$ and $r_{p}$ is complete to $U_{2}$. Further, since $C_{1}(r), r \in U^{1}$, is not a claw, $t^{2}$ is anticomplete to $U^{1}$ and $t^{1}$ is complete to $U^{1}$ and since $C_{p}(r), r \in U^{p}$, is not a claw, $t^{1}$ is anticomplete to $U^{p}$ and $t^{2}$ is complete to $U^{p}$. Moreover, since $\left(s_{1}^{1}, r_{1}, U_{2}\right)$ is a quasi-triad of $G$ and $U_{1}$ is complete to $\left\{s_{1}^{1}, r_{1}\right\}$, by Lemma 9.13 (i), we have $\left|U_{2}\right| \leq 1$ and $U_{1}$ is anticomplete to $U_{2}$. Similarly, since $\left(s_{1}^{1}, t^{1}, U^{p}\right)$ is a quasi-triad of $G$ and $U^{1}$ is complete to $\left\{s_{1}^{1}, t^{1}\right\}$, we have $\left|U^{p}\right| \leq 1$ and $U^{1}$ is anticomplete to $U^{p}$. Also, $U_{2}$ is anticomplete to $U^{1}$ (otherwise, for $t \in U_{2}$ adjacent to $r \in U^{1},\left(B_{2}(t) \cup\{r\}\right) \backslash\left\{t^{2}\right\}$ would be a claw) and complete to $U^{p}$, (otherwise, for $t \in U_{2}$ nonadjacent to $r \in U^{p},\left(B_{2}(t) \cup\{r\}\right) \backslash\left\{r_{1}\right\}$ would be a claw). Finally,
$U_{1}$ is anticomplete to $U^{p}$ (otherwise, for $t \in U_{1}$ adjacent to $r \in U^{p},\left(D_{p}(r) \cup\{t\}\right) \backslash\left\{r_{p}\right\}$ would be a claw). Now, if either $U^{1}=\emptyset$, or $U_{1}=\emptyset$, or $U^{p} \neq \emptyset$, or $U_{2} \neq \emptyset$, then by Lemma 9.13 (vi), $U^{1}$ is complete to $U_{1}$ and so $G$ is ISA, which contradicts Lemma 9.22. Thus, both $U^{1}$ and $U_{1}$ are nonempty, say $0<\left|U_{1}\right| \leq\left|U^{1}\right|$, and $U^{p}=U_{2}=\emptyset$. Also, if either $p=2$ or $p=3$ and $s_{2}^{3} \in S$, then $U^{1}$ is complete to $U_{1}$ (otherwise, for $t \in U_{1}$ nonadjacent to $r \in U^{1},\left\{s_{2}^{p}, t^{1}, t, r\right\}$ would be a claw) and so $G$ is ISA, which contradicts Lemma 9.22 . Hence, $p=3$ and $s_{2}^{3} \notin S$. Subsequently, in the collection $\mathscr{O}^{\prime}(G)=\mathscr{O}(G) \cup\left\{T^{2} \cup R^{3},\left(T \backslash\left\{t^{2}\right\}\right) \cup R^{3}, T_{3} \cup R_{1}, T_{3} \cup\left(R \backslash\left\{r_{1}\right\}\right)\right\}$ (the cliques in which cover all the edges in $E(G) \backslash E(T, R)$ ), add the cliques in $\mathscr{N}\left[T^{2} ; R\right]$, replace the clique $X_{2}^{1}$ with the clique $X_{2}^{1} \cup U^{1}$ to cover the edges in $E\left(T^{1}, R_{3}\right)$. Also, in order to cover the edge $t^{1} r_{1}$, if both $S^{1}$ and $S_{1}$ have cardinality two, then add the clique $\left\{t^{1}, r_{1}\right\}$, if $\left|S^{1}\right| \leq 1$ (resp. $\left|S_{1}\right| \leq 1$ ), say $s_{i}^{1} \notin S$ (resp. $s_{1}^{i} \notin S$ ), for some $i \in\{2,3\}$, then replace the clique $X_{i}^{1}$ or possibly $X_{i}^{1} \cup U^{1}$ (resp. $Z_{1}^{i}$ ) with clique $X_{i}^{1} \cup\left\{r_{1}\right\}$ or possibly $X_{i}^{1} \cup U^{1} \cup\left\{r_{1}\right\}$ (resp. $Z_{1}^{i} \cup\left\{t^{1}\right\}$ ). It is easy to check that in the case that either $\left|S^{1}\right| \leq 1$ or $\left|S_{1}\right| \leq 1$, the resulting family $\mathscr{C}_{1}$ is a clique covering for $G$ of size $16+\left|T^{2}\right|=n-|S|-\left|U^{1}\right|+4$, and in the case that both $S^{1}$ and $S_{1}$ have cardinality two, it is a clique covering for $G$ of size $17+\left|T^{2}\right|=n-|S|-\left|U^{1}\right|+5$. Hence, $|S| \leq 4$ and if $|S| \geq 3$, then $\left|U_{1}\right|=\left|U^{1}\right|=1$. First, assume that $|S| \geq 3$. If $U_{1}$ is complete to $U^{1}$, then $G$ is ISA, a contradiction with Lemma 9.22, and if $U_{1}$ is anticomplete to $U^{1}$, then remove the single clique in $\mathscr{N}\left[U_{1} ; R\right]$ from the $\mathscr{C}_{1}$ and replace the clique $Z_{1}^{3}$ or possibly $Z_{1}^{3} \cup\left\{t^{1}\right\}$ with the clique $Z_{1}^{3} \cup U_{1} \cup\left\{t^{1}\right\}$ to obtain a clique covering for $G$ of size $n-1$, a contradiction. Therefore, $|S| \leq 2$. Now, considering $\mathscr{P}(G)$, replace the cliques $\Delta_{1}^{3}$ and $\nabla_{1}^{2}$ with the cliques $\Delta_{1}^{3} \cup U_{1}$ and $\nabla_{1}^{2} \cup U^{1}$ to cover the edges in $E\left(T^{1}, T^{2}\right) \cup E\left(R_{1}, R_{3}\right)$, replace the clique $\Omega^{1}$ with the clique $\Omega^{1} \cup U^{1}$ to cover the edges in $E\left(T^{1}, R_{3}\right)$, replace the clique $\Omega_{1}$ with the clique $\Omega_{1} \cup U_{1}$ to cover the edges in $E\left(T^{2}, R_{1}\right)$, replace the clique $\Omega^{2}$ with the cliques in $\mathscr{N}\left[T^{2} ;\left(R_{3} \cup \Omega^{2}\right) \backslash T^{2}\right]$ to cover the edges in $E\left(T^{2}, R_{3}\right)$. Finally, add the clique $T^{1} \cup R_{1}$ to cover the edges in $E\left(T^{1}, R_{1}\right)$, thereby obtaining a clique covering for $G$ of size at most $12+|S|+\left|T^{2}\right|=n-\left|U^{1}\right| \leq n-1$, again a contradiction.
Now, assume that $t^{1}$ is adjacent to $r_{p}$ and so by Lemma 9.13 (vi), $t^{1}$ is nonadjacent to $r_{1}$ and $t^{2}$ is adjacent to $r_{1}$ and nonadjacent to $r_{p}$. Since $B_{1}(t), t \in U_{1}$, is not a claw, $r_{1}$ is anticomplete to $U_{1}$ and thus $r_{p}$ is complete to $U_{1}$. Also, since $B_{2}(t), t \in U_{2}$, is not a claw, $r_{1}$ is complete to $U_{2}$ and thus $r_{p}$ is anticomplete to $U_{2}$. Also, since $D_{1}(r), r \in U^{1}$, is not a claw, $t^{1}$ is anticomplete to $U^{1}$ and thus $t^{2}$ is complete to $U^{1}$ and since $D_{p}(r), r \in U^{p}$, is not a claw, $t^{1}$ is complete to $U^{p}$ and so $t^{2}$ is anticomplete to $U^{p}$. Moreover, since $\left(s_{k^{\prime}}^{k}, r_{k}, U_{3-k^{\prime}}\right)$ is a quasi-triad of $G$ and $U_{k^{\prime}}$ is complete to $\left\{s_{k^{\prime}}^{k}, r_{k}\right\}$, by Lemma 9.13 (i), we have $\left|U_{3-k^{\prime}}\right| \leq 1$ and $U_{1}$ is anticomplete to $U_{2}$. Similarly, since ( $s_{k^{\prime}}^{k}, k^{k^{\prime}}, U^{p-k+1}$ ) is a quasi-triad of $G$ and $U^{k}$ is complete to $\left\{s_{k^{\prime}}^{k}, t^{k^{\prime}}\right\}$, by Lemma 9.13 (i), we have $\left|U^{p-k+1}\right| \leq 1$ and $U^{1}$ is anticomplete to $U^{p}$. Also, $U_{1}$ is anticomplete to $U^{1}$ (otherwise, for $t \in U_{1}$ adjacent to $r \in U^{1}$, if $\left(k, k^{\prime}\right)=(1,2)$, then $\left(C_{1}(r) \cup\{t\}\right) \backslash\left\{t^{k^{\prime}}\right\}$ and if $\left(k, k^{\prime}\right)=(p, 1)$, then $\left(C_{1}(r) \cup\{t\}\right) \backslash\left\{r_{1}\right\}$ would be a claw). Finally, $U_{2}$ is anticomplete to $U^{p}$ (otherwise, for $t \in U_{2}$ adjacent to $r \in U^{p}$, if $\left(k, k^{\prime}\right)=(1,2)$, then $\left(C_{p}(r) \cup\{t\}\right) \backslash\left\{r_{p}\right\}$ is a claw, and if $\left(k, k^{\prime}\right)=(p, 1)$, then $\left(C_{p}(r) \cup\{t\}\right) \backslash\left\{t^{1}\right\}$ would be a claw). Now, if either both $U^{1}$ and $U_{1}$ are nonempty, or both $U^{p}$ and $U_{2}$ are nonempty, or $U_{1}=U_{2}=\emptyset$, or $U^{1}=U^{p}=\emptyset$, then by Lemma $9.13(\mathrm{vi}), U^{1}$ is complete to $U_{1}$ and so $G$ is ISA, which contradicts Lemma 9.22. Thus, assume that either $U_{1}=U^{p}=\emptyset$ and both $U^{1}$ and $U_{2}$ are nonempty, or $U^{1}=U_{2}=\emptyset$ and both $U^{p}$ and $U_{1}$ are nonempty, say the former occurs (otherwise, we may consider the rotator $\left(s_{1}^{1}, s_{p}^{p}, s_{5-p}^{5-p} ; r_{1}^{3}, r_{p}^{3}, r_{5-p}^{3} ; t_{3}^{1}, t_{3}^{p}, t_{3}^{5-p}\right)$ of $\bar{G}$ as the initial rotator), say $0<\left|U_{2}\right| \leq\left|U^{1}\right|$. Also, note that $\rho^{\prime}=\left(s_{1}^{1}, s_{2}^{2}, s_{3}^{3} ; r_{1}, t_{3}^{2}, t_{3}^{3} ; t^{1}, r_{2}^{3}, r_{3}^{3}\right)$ is a rotator of $\bar{G}$ and we have $t_{3}^{1} \in R_{1}\left(\rho^{\prime}\right), U^{1} \subseteq R_{p}\left(\rho^{\prime}\right)$ and $S_{1}^{p} \subseteq R_{5-p}\left(\rho^{\prime}\right)$. Thus, if $s_{1}^{p} \in S$, then $\left|I_{R}\left(\rho^{\prime}\right)\right|=3$, which contradicts Theorem 9.27. Therefore, $s_{1}^{p} \notin S$ and so by Lemma 9.15 (i) (applying to $\left.(i, j, k)=(1, p, 5-p)\right), s_{2}^{1} \in S$. Also, $r_{1}^{3} \in T^{1}\left(\rho^{\prime}\right)$, $s_{2}^{1} \in T^{3}\left(\rho^{\prime}\right)$ and $S_{3}^{1} \subseteq T^{2}\left(\rho^{\prime}\right)$. Now, if $s_{3}^{1} \in S$ is not empty, then $\left|I_{T}\left(\rho^{\prime}\right)\right|=3$ which again contradicts Theorem 9.27. Therefore, $s_{3}^{1} \notin S$.
Now, in the collection $\mathscr{O}^{\prime \prime}(G)=\mathscr{O}(G) \cup\left\{T^{1} \cup R^{3},\left(T \backslash\left\{t^{1}\right\}\right) \cup R^{3}, T_{3} \cup R_{p}, T_{3} \cup\left(R \backslash\left\{r_{p}\right\}\right)\right\}$ (the cliques in which cover all the edges in $E(G) \backslash E(T, R)$ ), add the cliques in $\mathscr{N}\left[T^{1} ; R_{p}\right]$, replace the cliques $X_{1}^{2}, X_{3}^{2}$ and $Z_{1}^{p}$ with the cliques $X_{1}^{2} \cup U^{1}, X_{3}^{2} \cup\left\{r_{1}\right\}$ and $Z_{1}^{p} \cup U_{2}$, thereby covering the edges in $E(T, R)$. The resulting family $\mathscr{C}_{2}$ is a clique covering for $G$ of size $17+\left|U_{2}\right|=n-|S|-\left|U^{1}\right|+4$. Thus, $|S|+\left|U^{1}\right| \leq 4$. If $|S|=3$, then $\left|U_{2}\right|=\left|U^{1}\right|=1$, in the case that $U_{2}$ is complete to $U^{1}, G$ is ISA, again a contradiction with Lemma 9.22, and in the case that $U_{2}$ is anticomplete to $U^{1}$, removing the clique $U_{2} \in \mathscr{N}\left[t^{1}, R_{p}\right]$
from $\mathscr{C}_{2}$ implies that $\operatorname{cc}(G) \leq n-1$, a contradiction. Therefore, $|S| \leq 2$. Now, considering $\mathscr{P}(G)$, remove the clique $\Omega^{1}$, add the cliques in $\mathscr{N}\left[T^{1} ;\left(\Omega^{1} \cup R_{p}\right) \backslash T^{1}\right]$ to cover the edges in $E\left(T^{1}, R_{p}\right)$. Also, in order to cover the edges in $E\left(\left\{t^{2}\right\}, R\right) \cup E\left(R_{1}, R_{p}\right)$, if $S^{2}=\emptyset$ or $p=3$, then replace the clique $\Omega^{2}$ with the clique $\Omega^{2} \cup U^{1} \cup\left\{r_{1}\right\}$ and if $S^{2} \neq \emptyset$ and $p=2$, then add the clique $U^{1} \cup\left\{t^{2}, r_{1}\right\}$, and in order to cover the edges in $E\left(\left\{t^{2}, r_{1}\right\}, U_{2}\right)$, if $S_{1}=\emptyset$, then replace the clique $\Omega_{1}$ with the clique $\Omega_{1} \cup U_{2} \cup\left\{t^{2}\right\}$ and if $S_{1} \neq \emptyset$, then add the clique $U_{2} \cup\left\{t^{2}, r_{1}\right\}$. It is easy to see that these modifications lead to a clique covering for $G$ of size at most $12+|S|+\left|T^{1}\right|=n-\left|U^{1}\right| \leq n-1$, again a contradiction. This completes the proof of Theorem 9.1.

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## A Basic classes of three-cliqued claw-free graphs

The following five classes of three-cliqued claw-free graphs are the building blocks in the structure theorem of all three-cliqued claw-free graphs.
$\mathcal{T} \mathcal{C}_{1}$ Line graphs. Let $v_{1}, v_{2}, v_{3}$ be distinct pairwise nonadjacent vertices of a graph $H$, such that every edge of H is incident with one of $v_{1}, v_{2}, v_{3}$. Let $v_{1}, v_{2}, v_{3}$ be of degree at least three, and let all other vertices of $H$ be of degree at least one. Moreover, for all distinct $i, j \in\{1,2,3\}$, let there be at most one vertex different from $v_{1}, v_{2}, v_{3}$ that is adjacent to $v_{i}$ and not to $v_{j}$ in $H$. Let $G$ be the line graph of $H$ and let $A, B, C$ be the sets of edges of $H$ incident with $v_{1}, v_{2}, v_{3}$, respectively. Then $(G, A, B, C)$ is a three-cliqued claw-free graph. Moreover, let $F^{\prime}$ be the set of all pairs $\{e, f\}$, where $e, f \in E(H)$ have a common endpoint in $H$ with degree exactly two and let $F$ be a valid subset of $F^{\prime}$. Define $\mathcal{T \mathcal { C } _ { 1 }}$ to be the class of all such three-cliqued graphs with the additional property that every vertex of $G$ is in a triad of $G \backslash F$.
$\mathcal{T} \mathcal{C}_{2}$ Long circular interval graphs. Let $G$ be a long circular interval graph with $\Sigma$ and $\mathcal{I}=\left\{I_{1}, \ldots, I_{k}\right\}$ as in the definition of long circular interval graph. By a line we mean either a subset $X \subseteq V(G)$ with $|X| \leq 1$, or a subset of some $I_{i}$ homeomorphic to the interval $[0,1]$, with both endpoints in $V(G)$. Let $L_{1}, L_{2}, L_{3}$ be pairwise disjoint lines with $V(G) \subseteq L_{1} \cup L_{2} \cup L_{3}$ and define $A=$ $V(G) \cap L_{1}, B=V(G) \cap L_{2}$ and $C=V(G) \cap L_{3}$. Then $(G, A, B, C)$ is a three-cliqued claw-free graph. Moreover, let $F^{\prime}$ be the set of all pairs $\{u, v\}$ such that $u, v \in V(G)$ are distinct endpoints of $I_{i}$ for some $i$, there is no $j \neq i$ for which $u, v \in I_{j}$ and $u, v$ are not in the same set $A, B$ or $C$. Also, let $F$ be a subset of $F^{\prime}$. We denote by $\mathcal{T} \mathcal{C}_{2}$ the class of all such three-cliqued graphs with the additional property that every vertex of $G$ is in a triad of $G \backslash F$.
$\mathcal{T C}_{3}$ Near-antiprismatic graphs. Let $m \geq 2$ and construct a graph $H$ as follows. Its vertex set is disjoint union of three sets $\tilde{A}, \tilde{B}, \tilde{C}$, where $|\tilde{A}|=|\tilde{B}|=m+1$ and $|\tilde{C}|=m$, say $\tilde{A}=$ $\left\{a_{0}, a_{1}, \ldots, a_{m}\right\}, \tilde{B}=\left\{b_{0}, b_{1}, \ldots, b_{m}\right\}$ and $\tilde{C}=\left\{c_{1}, \ldots, c_{m}\right\}$. Adjacency is as follows. $\tilde{A}, \tilde{B}, \tilde{C}$ are three cliques. For $i, j \in\{0, \ldots, m\}$ with $(i, j) \neq(0,0)$, let $a_{i}, b_{j}$ be adjacent if and only if $i=j$, and for $i \in\{1, \ldots, m\}$ and $j \in\{0, \ldots, m\}$, let $c_{i}$ be adjacent to $a_{j}, b_{j_{\tilde{~}}}$ if and only if $i \neq j \neq 0$. All other pairs not specified so far are nonadjacent. Now let $X \subseteq \tilde{A} \cup \tilde{B} \cup \tilde{C} \backslash\left\{a_{0}, b_{0}\right\}$ with $|\tilde{C} \backslash X| \geq 2$. Also, let $G=H \backslash X, A=\tilde{A} \backslash X, B=\tilde{B} \backslash X$ and $C=\tilde{C} \backslash X$. Then $(G, A, B, C)$ is a three-cliqued claw-free graph called near-antiprismatic. Moreover, let $F^{\prime}=\left\{\left\{a_{i}, b_{i}\right\}: 0 \leq\right.$ $i \leq m\} \cup\left\{\left\{a_{i}, c_{i}\right\}: 1 \leq i \leq m\right\} \cup\left\{\left\{b_{i}, c_{i}\right\}: 1 \leq i \leq m\right\}$ and $F \subseteq F^{\prime}$ such that

- $\left\{a_{i}, c_{i}\right\} \in F$ for at most one value of $i \in\{1, \ldots, m\}$, and if so then $b_{i} \in X$,
$-\left\{b_{i}, c_{i}\right\} \in F$ for at most one value of $i \in\{1, \ldots, m\}$, and if so then $a_{i} \in X$,
$-\left\{a_{i}, b_{i}\right\} \in F$ for at most one value of $i \in\{1, \ldots, m\}$, and if so then $c_{i} \in X$. Moreover, $\left\{a_{0}, b_{0}\right\}$ may belong to $F$ with no restriction.

We denote by $\mathcal{T C}_{3}$ the class of all such three-cliqued graphs with the additional property that every vertex of $G$ is in a triad of $G \backslash F$.
$\mathcal{T} \mathcal{C}_{4}$ Antiprismatic graphs. Let $G$ be an antiprismatic graph and let $A, B, C$ be a partition of $V(G)$ into three cliques. We denote by $\mathcal{T C}_{4}$ the class of all such three-cliqued graphs. Note that in this case $G$ may have vertices that are in no triad. Also, let $F$ be a valid set of changeable pairs of $G$.

- Let $H$ be the graph with vertex set $\left\{v_{1}, \ldots, v_{8}\right\}$ and adjacency as follows: $v_{i}, v_{j}$ are adjacent for $1 \leq i<j \leq 6$ with $j-i \leq 2 ; v_{1}, v_{6}, v_{7}$ are pairwise adjacent, and $v_{7}, v_{8}$ are adjacent. All other pairs not specified so far are nonadjacent. Let $X \subseteq\left\{v_{3}, v_{4}\right\}, A=\left\{v_{1}, v_{2}, v_{3}\right\} \backslash X$, $B=\left\{v_{4}, v_{5}, v_{6}\right\} \backslash X, C=\left\{v_{7}, v_{8}\right\}$ and $G=H \backslash X$; then $(G, A, B, C)$ is a three-cliqued clawfree graph, and all its vertices are in triads. Moreover, let $F^{\prime}=\left\{\left\{v_{1}, v_{4}\right\},\left\{v_{3}, v_{6}\right\},\left\{v_{2}, v_{5}\right\}\right\}$ and $F \subseteq F^{\prime}$ such that $\left\{v_{1}, v_{4}\right\},\left\{v_{3}, v_{6}\right\} \in F$.
- Let $H$ be the graph with vertex set $\left\{v_{1}, \ldots, v_{9}\right\}$, and adjacency as follows: the sets $A=$ $\left\{v_{1}, v_{2}\right\}, \tilde{B}=\left\{v_{3}, v_{4}, v_{5}, v_{6}, v_{9}\right\}$ and $C=\left\{v_{7}, v_{8}\right\}$ are cliques; $v_{1}, v_{8}, v_{9}$ are pairwise adjacent; $v_{2}$ is adjacent to $v_{3}$ and $v_{6}$ is adjacent to $v_{7}$; the adjacency between the pairs $v_{2} v_{4}$ and $v_{5} v_{7}$ is arbitrary. All other pairs not specified so far are nonadjacent. Let $X \subseteq\left\{v_{3}, v_{4}, v_{5}, v_{6}\right\}$. Also, let $G=H \backslash X$ and $B=\tilde{B} \backslash X$. Then $(G, A, B, C)$ is a three-cliqued claw-free graph. Moreover, let $F^{\prime}=\left\{\left\{v_{2}, v_{4}\right\},\left\{v_{5}, v_{7}\right\},\left\{v_{1}, v_{3}\right\},\left\{v_{6}, v_{8}\right\}\right\}$ and $F \subseteq F^{\prime}$ such that $\left\{v_{1}, v_{3}\right\},\left\{v_{6}, v_{8}\right\} \in F$ and
* if $v_{3} \in X$, then either $v_{2}, v_{4}$ are adjacent, or $\left\{v_{2}, v_{4}\right\} \in F$,
* if $v_{6} \in X$, then either $v_{5}, v_{7}$ are adjacent, or $\left\{v_{5}, v_{7}\right\} \in F$,
* if $v_{4}, v_{5} \notin X$, then for every $\{x, y\} \in\left\{\left\{v_{2}, v_{4}\right\},\left\{v_{5}, v_{7}\right\}\right\}$, either $x$ is adjacent to $y$, or $\{x, y\} \in F$.

We denote by $\mathcal{T \mathcal { C } _ { 5 }}$ the class of such three-cliqued graphs (given by one of these two constructions) with the additional property that every vertex of $G$ is in a triad of $G \backslash F$.

## B Basic classes of orientable antiprismatic graphs

The following are the basic classes of orientable antiprismatic graphs which are appeared in the characterization of antiprismatic graphs. Note that they are recalled from [3], where the description of orientable prismatic graphs is given (the graphs whose complements are antiprismatic). Thus, we reformulate all the definitions in terms of the complements.

The complement of mantled $L\left(K_{3,3}\right)$. Let $G$ be a graph whose vertex set is the union of seven disjoint sets $W=\left\{a_{j}^{i}: 1 \leq i, j \leq 3\right\}, V^{1}, V^{2}, V^{3}, V_{1}, V_{2}, V_{3}$, with adjacency as follows. For $i, j, i^{\prime}, j^{\prime} \in\{1,2,3\}, a_{j}^{i}$ and $a_{j^{\prime}}^{i^{\prime}}$ are adjacent if and only if $i=i^{\prime}$ or $j=j^{\prime}$. For $i \in\{1,2,3\}, V^{i}$ and $V_{i}$ are cliques; $V^{i}$ is anticomplete to $\left\{a_{1}^{i}, a_{2}^{i}, a_{3}^{i}\right\}$ and complete to the remainder of $W$; and $V_{i}$ is anticomplete to $\left\{a_{i}^{1}, a_{i}^{2}, a_{i}^{3}\right\}$ and complete to the remainder of $W$. Moreover, $V^{1} \cup V^{2} \cup V^{3}$ is complete to $V_{1} \cup V_{2} \cup V_{3}$, and there is no triad included in $V^{1} \cup V^{2} \cup V^{3}$ or in $V_{1} \cup V_{2} \cup V_{3}$. We call such a graph $G$ the complement of a mantled $L\left(K_{3,3}\right)$.

The complement of a ring of five. Let $G$ be a graph with $V(G)$ the union of the disjoint sets $W=\left\{a_{1}, \ldots, a_{5}, b_{1}, \ldots, b_{5}\right\}$ and $V_{0}, V_{1}, \ldots, V_{5}$. Let adjacency be as follows (reading subscripts modulo 5). For $i \in\{1, \ldots, 5\},\left\{a_{i}, b_{i+1}, a_{i+2}\right\}$ is a triangle, and $\left\{b_{1}, \ldots, b_{5}\right\}$ is a clique; $V_{0}$ is anticomplete to $\left\{b_{1}, \ldots, b_{5}\right\}$ and complete to $\left\{a_{1}, \ldots, a_{5}\right\} ; V_{0}, V_{1}, \ldots, V_{5}$ are all cliques; for $i \in\{1, \ldots, 5\}, V_{i}$ is anticomplete to $\left\{a_{i-1}, b_{i}, a_{i+1}\right\}$ and complete to the remainder of $W ; V_{0}$ is complete to $V_{1} \cup \ldots \cup V_{5}$; for $i \in\{1, \ldots, 5\}, V_{i}$ is complete to $V_{i+2}$; and the adjacency between $V_{i}, V_{i+1}$ is arbitrary. We call such a graph $G$ the complement of a ring of five.

The complement of a path of triangles graph. Let $m \geq 1$ be some integer and $G$ be a graph where $V(G)$ is the union of pairwise disjoint cliques $X_{1}, \ldots, X_{2 m+1}$, satisfying the following conditions (P1)-(P7).
(P1) For $1 \leq i \leq m$, there is a nonempty subset $\hat{X}_{2 i} \subseteq X_{2 i} ;\left|\hat{X}_{2}\right|=\left|\hat{X}_{2 m}\right|=1$ and for $0<i<m$, at least one of $\hat{X}_{2 i}, \hat{X}_{2 i+2}$ has cardinality 1 .
(P2) For $1 \leq i<j \leq 2 m+1$
(1) if $j-i=2$ modulo 3 and there exist $u \in X_{i}$ and $v \in X_{j}$, adjacent, then either $i, j$ are odd and $j=i+2$, or $i, j$ are even and $u \notin \hat{X}_{i}$ and $v \notin \hat{X}_{j}$;
(2) if $j-i \neq 2$ modulo 3 , then either $j=i+1$ or $X_{i}$ is complete to $X_{j}$.
(P3) For $1 \leq i \leq m+1, X_{2 i-1}$ is the union of three pairwise disjoint sets $L_{2 i-1}, M_{2 i-1}, R_{2 i-1}$, where $L_{1}=M_{1}=M_{2 m+1}=R_{2 m+1}=\emptyset$.
(P4) If $R_{1}=\emptyset$ then $m \geq 2$ and $\left|\hat{X}_{4}\right|>1$, and if $L_{2 m+1}=\emptyset$, then $m \geq 2$ and $\left|\hat{X}_{2 m-2}\right|>1$.
(P5) For $1 \leq i \leq m, X_{2 i}$ is complete to $L_{2 i-1} \cup R_{2 i+1} ; X_{2 i} \backslash \hat{X}_{2 i}$ is complete to $M_{2 i-1} \cup M_{2 i+1}$; and every vertex in $X_{2 i} \backslash \hat{X}_{2 i}$ is adjacent to exactly one end of every non-edge between $R_{2 i-1}$ and $L_{2 i+1}$.
(P6) For $1 \leq i \leq m$, if $\left|\hat{X}_{2 i}\right|=1$, then
(1) $R_{2 i-1}, L_{2 i+1}$ are anti-matched, and every non-edge between $M_{2 i-1} \cup R_{2 i-1}$ and $L_{2 i+1} \cup M_{2 i+1}$ is between $R_{2 i-1}$ and $L_{2 i+1}$;
(2) the vertex in $\hat{X}_{2 i}$ is anticomplete to $R_{2 i-1} \cup M_{2 i-1} \cup L_{2 i+1} \cup M_{2 i+1}$;
(3) $L_{2 i-1}$ is anticomplete to $X_{2 i+1}$ and $X_{2 i-1}$ is anticomplete to $R_{2 i+1}$;
(4) if $i>1$, then $M_{2 i-1}, \hat{X}_{2 i-2}$ are anti-matched, and if $i<m$, then $M_{2 i+1}, \hat{X}_{2 i+2}$ are antimatched.
(P7) For $1<i<m$, if $\left|\hat{X}_{2 i}\right|>1$, then
(1) $R_{2 i-1}=L_{2 i+1}=\emptyset$;
(2) if $u \in X_{2 i-1}$ and $v \in X_{2 i+1}$, then $u, v$ are adjacent if and only if they have the same neighbour in $\hat{X}_{2 i}$.

Such a graph $G$ is called the complement of a path of triangles graph. For each $k \in\{1,2,3\}$, define

$$
A_{k}=\bigcup\left(X_{i}: 1 \leq i \leq 2 m+1 \text { and } i=k \text { modulo } 3\right)
$$

Then, $\left(G, A_{1}, A_{2}, A_{3}\right)$ is a three-cliqued graph. A permutation of $\left(G, A_{1}, A_{2}, A_{3}\right)$ is called the complement of a canonically-coloured path of triangles graph.

The complement of a cycle of triangles graph. Let $m \geq 5$ be some integer with $m=2$ modulo 3 and $G$ be a graph where $V(G)$ is the union of pairwise disjoint cliques $X_{1}, \ldots, X_{2 m}$, satisfying the following conditions (C1)-(C6) (reading subscripts modulo $2 m$ ):
(C1) For $1 \leq i \leq m$, there is a nonempty subset $\hat{X}_{2 i} \subseteq X_{2 i}$, and at least one of $\hat{X}_{2 i}, \hat{X}_{2 i+2}$ has cardinality 1 .
(C2) For $1 \leq i \leq 2 m$ and all $k$ with $2 \leq k \leq 2 m-2$, let $j \in\{1, \ldots, 2 m\}$ with $j=i+k$ modulo $2 m$ :
(1) if $k=2$ modulo 3 and there exist $u \in X_{i}$ and $v \in X_{j}$, adjacent, then either $i, j$ are odd and $k \in\{2,2 m-2\}$, or $i, j$ are even and $u \notin \hat{X}_{i}$ and $v \notin \hat{X}_{j}$;
(2) if $k \neq 2$ modulo 3 , then $X_{i}$ is complete to $X_{j}$.
(C3) For $1 \leq i \leq m+1, X_{2 i-1}$ is the union of three pairwise disjoint sets $L_{2 i-1}, M_{2 i-1}, R_{2 i-1}$.
(C4) For $1 \leq i \leq m, X_{2 i}$ is complete to $L_{2 i-1} \cup R_{2 i+1} ; X_{2 i} \backslash \hat{X}_{2 i}$ is complete to $M_{2 i-1} \cup M_{2 i+1}$; and every vertex in $X_{2 i} \backslash \hat{X}_{2 i}$ is adjacent to exactly one end of every non-edge between $R_{2 i-1}$ and $L_{2 i+1}$.
(C5) For $1 \leq i \leq m$, if $\left|\hat{X}_{2 i}\right|=1$, then
(1) $R_{2 i-1}, L_{2 i+1}$ are anti-matched, and every non-edge between $M_{2 i-1} \cup R_{2 i-1}$ and $L_{2 i+1} \cup M_{2 i+1}$ is between $R_{2 i-1}$ and $L_{2 i+1}$;
(2) the vertex in $\hat{X}_{2 i}$ is anticomplete to $R_{2 i-1} \cup M_{2 i-1} \cup L_{2 i+1} \cup M_{2 i+1}$;
(3) $L_{2 i-1}$ is anticomplete to $X_{2 i+1}$ and $X_{2 i-1}$ is anticomplete to $R_{2 i+1}$;
(4) $M_{2 i-1}, \hat{X}_{2 i-2}$ are anti-matched and $M_{2 i+1}, \hat{X}_{2 i+2}$ are anti-matched.
(C6) For $1 \leq i \leq m$, if $\left|\hat{X}_{2 i}\right|>1$, then
(1) $R_{2 i-1}=L_{2 i+1}=\emptyset$;
(2) if $u \in X_{2 i-1}$ and $v \in X_{2 i+1}$, then $u, v$ are adjacent if and only if they have the same neighbour in $\hat{X}_{2 i}$.

Such a graph $G$ is called the complement of a cycle of triangles graph.

## C Basic classes of non-orientable antiprismatic graphs

The following classes of graphs are the building blocks in the structure theorems of non-orientable antiprismatic graphs. They are recalled from [4], where the description of non-orientable prismatic graphs is given (the graphs whose complements are antiprismatic). Thus, we reformulate all the definitions in terms of the complements, maintaining the titles of the classes for simplicity.

## - Graphs of parallel-square type

Let $A \subset\left\{a_{1}, a_{2}, a_{3}, \ldots\right\}, B \subset\left\{b_{1}, b_{2}, b_{3}, \ldots\right\}, C \subset\left\{c_{1}, c_{2}, c_{3}, \ldots\right\}$ and $D \subset\left\{d_{1}, d_{2}, d_{3}, \ldots\right\}$ be four nonempty finite sets and $Z$ is a set with $|Z| \leq 1$. Define $G$ as the graph on the vertex set $V(G)=A \cup B \cup C \cup D \cup Z \cup\{u, v, x, y\}$, with adjacency as follows. The sets $A \cup\{u, y\}, B \cup\{u, v\}, C \cup\{v, x\}$ and $D \cup\{x, y\}$ are cliques, $Z$ is complete to $A \cup B \cup C \cup D$ and anticomplete to $\{u, v, x, y\}$, for every $i, j$, the vertex $a_{i}$ is adjacent to $b_{j}$ and $d_{j}$ if and only if $i \neq j$ and is adjacent to $c_{j}$ if and only if $i=j$, the vertex $b_{i}$ is adjacent to $c_{j}$ if and only if $i \neq j$ and is adjacent to $d_{j}$ if and only if $i=j$, the vertex $c_{i}$ is adjacent to $d_{j}$ if and only if $i \neq j$ and there are no more edges in $G$. Any such graph $G$ is antiprismatic and is called a graph of parallel-square type.

## - Graphs of skew-square type

Let $A \subset\left\{a_{1}, a_{2}, a_{3}, \ldots\right\}, B \subset\left\{b_{1}, b_{2}, b_{3}, \ldots\right\}$ and $C \subset\left\{c_{1}, c_{2}, c_{3}, \ldots\right\}$ be three nonempty finite sets and define $G$ as the graph on the vertex set $V(G)=A \cup B \cup C \cup\left\{s, t, d_{1}, d_{2}, d_{3}\right\}$, with adjacency as follows. The sets $A, B, C$ and $\left\{s, t, d_{1}, d_{2}, d_{3}\right\}$ are cliques, $s$ is complete to $B$ and $t$ is complete to $A$. For every $i, j, a_{i} \in A$ is adjacent to $b_{j} \in B$ if and only if $i=j$. Also, $a_{i} \in A$ and $b_{i} \in B$ are adjacent to $c_{j} \in C$ if and only if $i \neq j$. For $1 \leq i \leq 3$ and every $j, d_{i}$ is adjacent to $a_{j} \in A$ and $b_{j} \in B$, if and only if either $i=j$ or $j \geq 4$. Also, $d_{i}$ is adjacent to $c_{j}$ if and only if $i \neq j$ and $1 \leq j \leq 3$ and there are no more edges in $G$. Any such graph is antiprismatic and is called a graph of skew-square type.

## - The Class $\mathcal{F}_{0}$

Take the Schläfli graph with the vertices numbered $r_{j}^{i}, s_{j}^{i}$ and $t_{j}^{i}$ as usual (for the definition of Schläfli graph, see Subsection 9.2). Suppose that $H$ be the subgraph induced on

$$
\left\{r_{j}^{i}:(i, j) \in I_{1}\right\} \cup\left\{s_{j}^{i}:(i, j) \in I_{2}\right\} \cup\left\{t_{j}^{i}:(i, j) \in I_{3}\right\},
$$

where $I=\{(i, j): 1 \leq i, j \leq 3\}$ and

$$
\begin{aligned}
& -I \backslash\{(1,2),(2,1),(3,3)\} \subseteq I_{1} \subseteq I \backslash\{(3,3)\} \\
& -\{(1,1),(2,1),(3,2)\} \subseteq I_{2} \subseteq\{(1,1),(2,1),(3,1),(3,2),(3,3)\} \\
& -\{(2,1),(2,2),(1,3)\} \subseteq I_{3} \subseteq\{(2,1),(2,2),(1,3),(2,3),(3,3)\}
\end{aligned}
$$

Let $G$ be the graph obtained from $H$ by deleting the edges in $E(H) \cap\left\{s_{1}^{3} t_{3}^{2}, s_{1}^{3} t_{3}^{3}, s_{3}^{3} t_{3}^{2}\right\}$. We define $\mathcal{F}_{0}$ to be the class of all such graphs $G$ (they are all antiprismatic).

## - The class $\mathcal{F}_{1}$

Let $G$ be a graph with vertex set the disjoint union of sets $\{s, t\}, R, A, B$, where $|R| \leq 1$, and with edges as follows:

$$
\text { - } s, t \text { are nonadjacent, and both are anticomplete to } R \text {; }
$$

$-s$ is anticomplete to $A ; t$ is anticomplete to $B$;

- every vertex in $A$ has at most one non-neighbour in $A$, and every vertex in $B$ has at most one non-neighbour in $B$;
- if $a, a^{\prime} \in A$ are nonadjacent and $b, b^{\prime} \in B$ are nonadjacent, then the in the subgraph induced on $\left\{a, a^{\prime}, b, b^{\prime}\right\}$ is a matching;
- if $a, a^{\prime} \in A$ are nonadjacent, and $b \in B$ is adjacent to all other vertices of $B$, then $b$ is adjacent to exactly one of $a, a^{\prime}$.
- if $b, b^{\prime} \in B$ are nonadjacent, and $a \in A$ is adjacent to all other vertices of $A$, then $a$ is adjacent to exactly one of $b, b^{\prime}$.
- if $a \in A$ is adjacent to all other vertices of $A$, and $b \in B$ is adjacent to all other vertices of $B$, then $a, b$ are nonadjacent.

We define $\mathcal{F}_{1}$ to be the class of all such graphs $G$ (they are all antiprismatic).

## - The class $\mathcal{F}_{2}$

Let $H$ be a graph of parallel-square type and $A, B, C, D, Z, u, v, x, y$ be as in its definition. Assume that $Z$ is nonempty and $a_{1} \in A, b_{1} \in B, c_{1} \notin C, d_{1} \notin D$. Also, for every $c_{i} \in C$ and $d_{j} \in D$, we have $i \neq j$. Let $G$ be obtained from $H$ by exponentiating the leaf triad $\left\{a_{1}, b_{1}, x\right\}$ (see the definition of exponentiating in Subsection 9.1). We define $\mathcal{F}_{2}$ to be the class of all such graphs $G$ (they are all antiprismatic).

- The class $\mathcal{F}_{3}$

Let $H$ be obtained from a graph of parallel-square type with $A, B, C, D, Z, u, v, x, y$ as in its definition by removing the vertex $u$. Also, assume that $a_{1} \notin A, a_{2} \in A, b_{1} \in B, b_{2} \notin B, c_{1} \in$ $C, c_{2} \notin C, d_{1} \notin D, d_{2} \in D$. Let $G$ be obtained from $H$ by exponentiating the leaf triads $\left\{b_{1}, c_{1}, y\right\}$ and $\left\{a_{2}, d_{2}, v\right\}$. We define $\mathcal{F}_{3}$ to be the class of all such graphs $G$ (they are all antiprismatic).

- The class $\mathcal{F}_{4}$

Take the Schläfli graph, with vertices numbered $r_{j}^{i}, s_{j}^{i}, t_{j}^{i}$ as usual (for the definition of Schläfli graph, see Subsection 9.2). Let $H$ be the subgraph induced on

$$
Y \cup\left\{s_{j}^{i}:(i, j) \in I\right\} \cup\left\{t_{1}^{1}, t_{2}^{2}, t_{3}^{3}\right\}
$$

where $\emptyset \neq Y \subseteq\left\{r_{1}^{3}, r_{2}^{3}, r_{3}^{3}\right\}$ and $I \subseteq\{(i, j): 1 \leq i, j \leq 3\}$ with $|I| \geq 8$, where

$$
\{(i, j): 1 \leq i \leq 3 \text { and } 1 \leq j \leq 2\} \subseteq I
$$

Let $G$ be obtained from $H$ by exponentiating the leaf triad $\left\{t_{1}^{1}, t_{2}^{2}, t_{3}^{3}\right\}$. We define $\mathcal{F}_{4}$ to be the class of all such graphs $G$ (they are all antiprismatic).

## D Proof of Lemma 9.22

In this section, we give a proof for Lemma 9.22. Let $G$ be Schläfli-antiprismatic and let $\mathscr{S}(G)$ be the collection obtained from $\mathscr{O}(G)$ by adding the cliques $Y_{j}^{i}=V(G) \cap\left\{t_{j}^{1}, t_{j}^{2}, t_{j}^{3}, r_{1}^{i}, r_{2}^{i}, r_{3}^{i}\right\},(i, j) \in J$. Note that $\mathscr{S}(G)$ is a clique covering for $G$ of size 18. Also, let $\mu$ be an automorphism of $\Gamma$ defined as follows (we leave to the reader to check that $\mu$ is in fact an automorphism).

$$
\begin{array}{llll}
r_{1}^{1} \mapsto s_{1}^{1}, s_{2}^{2} \mapsto s_{2}^{1}, t_{3}^{3} \mapsto s_{3}^{1}, & r_{3}^{2} \mapsto t_{1}^{1}, s_{2}^{1} \mapsto t_{2}^{1}, t_{1}^{3} \mapsto t_{3}^{1}, & r_{2}^{3} \mapsto r_{1}^{1}, t_{2}^{3} \mapsto r_{2}^{1}, s_{2}^{3} \mapsto r_{3}^{1}, \\
s_{3}^{3} \mapsto s_{1}^{2}, t_{1}^{1} \mapsto s_{2}^{2}, r_{2}^{2} \mapsto s_{3}^{2}, & t_{3}^{2} \mapsto t_{1}^{2}, r_{2}^{1} \mapsto t_{2}^{2}, s_{1}^{3} \mapsto t_{3}^{2}, & t_{1}^{2} \mapsto r_{1}^{2}, s_{1}^{2} \mapsto r_{2}^{2}, r_{1}^{2} \mapsto r_{3}^{2}, \\
t_{2}^{2} \mapsto s_{1}^{3}, & r_{3}^{3} \mapsto s_{2}^{3}, s_{1}^{1} \mapsto s_{3}^{3}, & s_{3}^{2} \mapsto t_{1}^{3}, t_{2}^{1} \mapsto t_{2}^{3}, r_{1}^{3} \mapsto t_{3}^{3}, & s_{3}^{1} \mapsto r_{1}^{3}, r_{3}^{1} \mapsto r_{2}^{3}, t_{3}^{1} \mapsto r_{3}^{3}
\end{array}
$$

Also for every $(i, j) \in J$, define

$$
K_{j}^{i}=\mu^{-1}\left(X_{j}^{i}\right), \quad L_{j}^{i}=\mu^{-1}\left(Y_{j}^{i}\right), \quad M_{j}^{i}=\mu^{-1}\left(Z_{j}^{i}\right)
$$



Figure 4: The automorphism $\mu$.
Therefore, $\mathscr{M}(G)=\left\{K_{j}^{i}, L_{j}^{i}, M_{j}^{i}:(i, j) \in J\right\}$ is a clique covering for $G$ of size 18 (see Figure 4), and we use its modified versions to provide suitable clique coverings for some Schläfli-antiprismatic graphs. We need a lemma beforehand which enables us to handle Schläfli-antiprismatic graphs with at least 17 vertices. First, we need a definition. Let $k$ be a positive integer. A graph $H$ is said to be $k$-ultrahomogeneous, if every isomorphism between two of its induced subgraphs on at most $k$ vertices can be extended to an automorphism of the whole graph. It is proved in [1] that every 5 -ultrahomogeneous graph is $k$-ultrahomogeneous for every $k$ (nevertheless, there are only a few finite graphs that are $k$-ultrahomogeneous for every $k$ ). Also, the Schläfli graph $\Gamma$ and its complement $\bar{\Gamma}$ are the only connected finite graphs that are 4 -ultrahomogeneous, but not 5 -ultrahomogeneous. Thus, every isomorphism between two induced subgraphs of $\Gamma$ on at most 4 vertices can be extended to an automorphism of $\Gamma$, and this is what we are going to use in the proof of the following lemma. To learn more about this concept, the reader may refer to $[1,10]$. For simplicity, we omit the term $\rho$ within the proofs.

Lemma D.1. Let $G$ be a counterexample to Theorem 9.1, which is Schläfli-antiprismatic with respect to some rotator $\rho$ of $\bar{G}$. Then, $|S(\rho)|+|T(\rho)|+|R(\rho)| \leq 7$.

Proof. First we observe that,
(1) We have $n \leq 18$. Also, if there exists a triangle $\{x, y, z\}$ in $\Gamma$ disjoint from $V(G)$, then $n \leq 16$.

The first assertion follows from the fact that $\mathscr{S}(G)$ is a clique covering for $G$ of size 18. For the second, note that since $\Gamma$ is 4 -ultrahomogeneous, there exists an automorphism $\sigma$ of $\Gamma$, mapping the vertices $x, y$ and $z$ to the vertices $s_{1}^{1}, s_{1}^{2}$ and $s_{1}^{3}$, respectively. Now, for every $(i, j) \in J$, define

$$
A_{j}^{i}=\sigma^{-1}\left(X_{j}^{i}\right), \quad B_{j}^{i}=\sigma^{-1}\left(Y_{j}^{i}\right), \quad C_{j}^{i}=\sigma^{-1}\left(Z_{j}^{i}\right)
$$

It is easy to see that the family of cliques $\left\{A_{j}^{i}, B_{j}^{i}, C_{j}^{i}:(i, j) \in J\right\} \backslash\left\{A_{1}^{2}, A_{1}^{3}\right\}$ is a clique covering for $G$ of size 16 , and thus $n \leq 16$. This proves (1).

Now, on the contrary, assume that $|S|+|T|+|R| \geq 8$, and so $n \geq 17$. If $|T|+|R| \leq 7$, then there exists $i \in\{1,2\}$ such that $\left|Y_{i}^{3-i}\right| \leq 3$, and so we may choose $\{x, y, z\} \subseteq\left\{t_{i}^{1}, t_{i}^{2}, t_{i}^{3}, r_{1}^{3-i}, r_{2}^{3-i}, r_{3}^{3-i}\right\} \backslash V(G)$ as a triangle of $\Gamma$ disjoint from $V(G)$, which contradicts (1). Thus, $|T|+|R| \geq 8$ and w.l.o.g. we may assume that $|T| \leq|R|$. Also, if either $|S| \geq 2$ or $|T|+|R| \geq 10$, then $n \geq 19$, which is impossible by (1). Thus, $|S| \leq 1$ and $|T| \leq 4$. Now, choose $t_{j}^{i} \notin T$, for some $i \in\{1,2,3\}$ and $j \in\{1,2\}$ and choose $i^{\prime} \in\{1,2,3\} \backslash\{i\}$ such that $S_{i^{\prime}}=\emptyset$. Therefore, $\left\{i_{i^{\prime}}^{i^{\prime}+1}, s_{i^{\prime}}^{i^{\prime}+2}, t_{j}^{i}\right\}$ is a triangle of $\Gamma$ disjoint from $V(G)$, again a contradiction with (1). This proves Lemma D.1.

Now, we are ready to prove Lemma 9.22.
Proof of Lemma 9.22. Suppose not, assume that $G$ is ISA. By Lemma $9.21,\left|I_{T}\right|,\left|I_{R}\right| \geq 2$. We will provide a clique covering for $G$ of size at most $n-1$ through the following three cases. Since a clique covering for a graph can be trivially extended to a clique covering of the same size for its replication, we may assume that $G$ is Schläfli-antiprismatic.

Case 1. $\left|I_{T}\right|=\left|I_{R}\right|=2$.
By symmetry, we may assume that $I_{T}=\{1,2\}, I_{R}=\{1, p\}, p \in\{2,3\}$ and $|T| \geq|R|$. By Lemma 9.15 (ii) (applying to $(i, j, k)=(1,2,3)$ ), either $S_{2}^{1}$ or $S_{1}^{p}$ is nonempty.
(1.1) We have $|T| \geq 3$.

On the contrary, assume that $|T|=|R|=2$, say $T=\left\{t_{1}^{1}, t_{2}^{2}\right\}$ and $R=\left\{r_{i}^{1}, r_{p-i+1}^{2}\right\}$, $i \in\{1, p\}$. By Lemma D.1, $1 \leq|S| \leq 3$. Through the following four cases, we apply suitable modifications to $\mathscr{M}(G)$ which yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(p, i)=(2,1)$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}, K_{3}^{1}, L_{2}^{1}, L_{2}^{3}$ and $M_{2}^{1}$ and replace the cliques $L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $\left(L_{1}^{2} \cup S_{1}^{3} \cup\left\{t_{3}^{3}, r_{1}^{1}\right\}\right) \backslash S_{3}^{2}$ and $\left.L_{3}^{2} \cup S_{3}^{2} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{1}^{3}$. Also, if $S_{1}^{2}=\emptyset$, then remove the cliques $L_{1}^{2}, L_{3}^{2}, M_{2}^{1}, M_{2}^{3}$ and $K_{2}^{1}$ and replace the cliques $L_{2}^{1}$ and $L_{2}^{3}$ with the cliques $\left(L_{2}^{1} \cup S_{3}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}\right) \backslash S_{2}^{3}$ and $\left.L_{2}^{3} \cup S_{2}^{3} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{3}^{1}$. Now, assume that $\left|S_{2}^{1}\right|=\left|S_{1}^{2}\right|=1$. Since $|S| \leq 3$, either $S_{3}^{1} \cup S_{2}^{3}=\emptyset$, or $S_{3}^{2} \cup S_{1}^{3}=\emptyset$. In the former case, remove the cliques $K_{2}^{1}, K_{3}^{1}, M_{3}^{1}, M_{1}^{2}$ and $M_{2}^{3}$, add the clique $\left\{s_{1}^{1}, s_{2}^{1}, r_{3}^{3}\right\}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{t_{1}^{1}, r_{2}^{2}\right\}, L_{3}^{1} \cup\left\{s_{3}^{3}\right\},\left(L_{1}^{2} \cup S_{1}^{3} \cup\left\{s_{1}^{1}, t_{2}^{2}\right\}\right) \backslash S_{3}^{2},\left(L_{3}^{2} \cup S_{3}^{2} \cup\left\{r_{3}^{3}\right\}\right) \backslash S_{1}^{3}, L_{1}^{3} \cup\left\{r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}$. In the latter case, remove the cliques $K_{2}^{1}, K_{3}^{2}, K_{1}^{3}, M_{2}^{1}$ and $M_{2}^{3}$, add the clique $\left\{s_{1}^{2}, s_{2}^{2}, r_{3}^{3}\right\}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup\left\{s_{1}^{1}, t_{3}^{3}\right\}, L_{1}^{3} \cup\left\{r_{2}^{2}\right\},\left(L_{2}^{1} \cup S_{3}^{1} \cup\right.$ $\left.\left\{t_{1}^{1}, r_{3}^{3}\right\}\right) \backslash S_{2}^{3},\left(L_{2}^{3} \cup S_{2}^{3} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{3}^{1}, L_{3}^{1} \cup\left\{s_{3}^{3}\right\}$ and $L_{3}^{2} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\}$.
- $(p, i)=(2,2)$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and if $S_{2}^{1} \neq \emptyset$, then remove the clique $K_{3}^{1}$ and replace the cliques $L_{2}^{1}$ and $L_{2}^{3}$ with the cliques $\left(L_{2}^{1} \cup S_{3}^{1} \cup\left\{s_{1}^{1}\right\}\right) \backslash S_{2}^{3}$ and $\left(L_{2}^{3} \cup S_{2}^{3} \cup\left\{t_{3}^{3}\right\}\right) \backslash S_{3}^{1}$. If $S_{1}^{2}=\emptyset$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and if $S_{1}^{2} \neq \emptyset$, then remove the clique $M_{2}^{1}$ and replace the cliques $L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $\left(L_{1}^{2} \cup S_{1}^{3} \cup\left\{t_{3}^{3}\right\}\right) \backslash S_{3}^{2}$ and $\left(L_{3}^{2} \cup S_{3}^{2} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{1}^{3}$. Also, if $S_{3}^{1} \cup S_{1}^{3}=\emptyset$, then merge the pair $\left(L_{3}^{1}, M_{1}^{2}\right)$ and if $S_{3}^{1} \cup S_{1}^{3} \neq \emptyset$ and $S_{3}^{2} \cup S_{2}^{3}=\emptyset$, then merge the pair $\left(K_{1}^{3}, L_{3}^{1}\right)$, and call the resulting family $\mathscr{C}$. Now, if $S \neq S_{2}^{1} \cup S_{1}^{2}$, then $|\mathscr{C}| \leq n-1$, as required. Thus, we may assume that $S=S_{2}^{1} \cup S_{1}^{2}$. Now, if $|S|=\left|S_{2}^{1} \cup S_{1}^{2}\right|=2$, then $|\mathscr{C}|=n$, so remove the clique $K_{3}^{2}$ from $\mathscr{C}$ and replace the cliques $\left(L_{1}^{2} \cup S_{1}^{3} \cup\left\{t_{3}^{3}\right\}\right) \backslash S_{3}^{2}$ and $L_{1}^{3}$ with the cliques $\left(L_{1}^{2} \cup S_{1}^{3} \cup\left\{s_{1}^{1}, t_{3}^{3}\right\}\right) \backslash\left(S_{3}^{2} \cup\left\{r_{1}^{2}\right\}\right)$ and $L_{1}^{3} \cup\left\{r_{1}^{2}\right\}$, if $|S|=\left|S_{2}^{1}\right|=1$, then merge the pair $\left(L_{1}^{2}, L_{1}^{3}\right)$ and if $|S|=\left|S_{1}^{2}\right|=1$, then merge the pair $\left(L_{1}^{3}, L_{2}^{3}\right)$.
- $(p, i)=(3,1)$. If $S_{1}^{2}=\emptyset$, then remove the cliques $L_{1}^{2}, L_{3}^{2}, M_{2}^{1}$ and $M_{2}^{3}$ and in the resulting family, if $|S|=\left|S_{2}^{1}\right|=1$, then merge the pair $\left(L_{3}^{1}, M_{1}^{2}\right)$ and if $|S|=\left|S_{1}^{3}\right|=1$, then remove the cliques $L_{2}^{1}$ and $L_{2}^{3}$. Also, if $S_{1}^{2} \neq \emptyset$ (i.e. $|S| \geq 2$ ) and $S_{2}^{1}=\emptyset$, then remove the cliques $K_{1}^{2}, K_{1}^{3}, L_{2}^{1}, L_{2}^{3}$, add the clique $S_{3}^{2} \cup\left\{s_{3}^{3}, t_{2}^{2}, t_{3}^{2}, r_{1}^{1}\right\}$, replace the cliques $L_{3}^{1}, L_{3}^{2}$ with the cliques $L_{3}^{1} \cup\left\{s_{3}^{3}\right\}, L_{3}^{2} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\}$ and in the resulting family, if $|S|=\left|S_{1}\right|=2$, then merge the pair $\left(K_{3}^{1}, L_{1}^{2}\right)$. Finally, assume that both $S_{1}^{2}, S_{2}^{1} \neq \emptyset$. In this case, remove the cliques $K_{1}^{2}, K_{1}^{3}$, add the clique $S_{3}^{2} \cup\left\{s_{3}^{3}, t_{2}^{2}, t_{3}^{2}, r_{1}^{1}\right\}$ and replace the cliques $L_{3}^{1}, L_{3}^{2}$ with the cliques $L_{3}^{1} \cup\left\{s_{3}^{3}\right\}, L_{3}^{2} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\}$, and call the resulting family $\mathscr{C}$. Since $|S| \leq 3$, either $S_{3}^{1} \cup S_{2}^{3}=\emptyset$, or $S_{3}^{2} \cup S_{1}^{3}=\emptyset$. In the former case, remove the cliques $M_{1}^{3}$ and $M_{3}^{1}$ from $\mathscr{C}$, replace the cliques $L_{2}^{1}, L_{3}^{1} \cup\left\{s_{3}^{3}\right\}, L_{1}^{3}, L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}, L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\},\left(L_{1}^{3} \cup\left\{r_{1}^{1}\right\}\right) \backslash\left\{r_{3}^{2}\right\}, L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}, r_{3}^{2}\right\}$ and in the resulting family, if $|S|=2$, then remove the clique $K_{3}^{1}$ and replace the clique $K_{3}^{2}$ with the clique $K_{3}^{2} \cup\left\{r_{3}^{2}\right\}$ (the edges $s_{1}^{1} s_{2}^{1}$ and $s_{2}^{1} t_{3}^{3}$ are already covered by the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}$ and $\left.L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}, r_{3}^{2}\right\}\right)$. In the latter case, assuming $S_{3}^{1} \cup S_{2}^{3} \neq \emptyset$ (i.e.n $|S|=3$ ), remove the cliques $M_{1}^{2}, M_{3}^{2}$ and replace the cliques $L_{3}^{1}, L_{1}^{3}, L_{2}^{3}$ with the cliques $L_{3}^{1} \cup\left\{s_{3}^{3}, t_{1}^{1}\right\},\left(L_{1}^{3} \cup\left\{s_{3}^{3}\right\}\right) \backslash\left\{r_{3}^{2}\right\}, L_{2}^{3} \cup\left\{t_{1}^{1}, r_{3}^{2}\right\}$.
- $(p, i)=(3,3)$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}, K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$, and in the resulting family, if $|S|=\left|S_{1}^{3}\right|=1$, then merge the pair $\left(K_{1}^{3}, L_{3}^{1}\right)$. Now, assume that $S_{2}^{1} \neq \emptyset$. First, suppose that $S_{2}^{3}=\emptyset$. Remove the cliques $K_{2}^{1}, K_{1}^{2}, K_{1}^{3}, M_{2}^{1}$ and $M_{3}^{1}$, add the clique $S_{3}^{2} \cup\left\{s_{3}^{3}, t_{2}^{2}, t_{3}^{2}\right\}$, replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup S_{3}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}, L_{3}^{1} \cup\left\{t_{2}^{2}, s_{3}^{3}\right\},\left(L_{1}^{2} \cup\right.$ $\left.S_{1}^{3} \cup\left\{t_{3}^{3}\right\}\right) \backslash S_{3}^{2},\left(L_{3}^{2} \cup S_{3}^{2} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{1}^{3}$ and $\left(L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}\right) \backslash S_{3}^{1}$ and in the resulting family, if $|S|=\left|S_{2}^{1}\right|=1$, then remove the clique $M_{1}^{2}$ (the edges in $E\left(M_{1}^{2}\right)$ are already covered by the cliques $L_{2}^{1} \cup S_{3}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}, L_{3}^{1} \cup\left\{t_{2}^{2}, s_{3}^{3}\right\}$ and $\left.M_{3}^{2}\right)$.

Next, assume that $\left|S_{2}^{3}\right|=1$. Remove the cliques $K_{2}^{1}, K_{3}^{1}, K_{1}^{2}$ and $K_{1}^{3}$, add the clique $\left\{s_{1}^{1}, s_{2}^{1}, r_{3}^{3}\right\}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}$ and $L_{2}^{1}$ with the cliques $\left(L_{1}^{2} \cup\left\{t_{3}^{1}, s_{3}^{3}\right\}\right) \backslash S_{1}^{2},\left(L_{1}^{3} \cup S_{1}^{2} \cup\left\{t_{2}^{2}\right\}\right) \backslash\left\{t_{3}^{1}\right\}$ and $L_{2}^{1} \cup\left\{t_{1}^{1}\right\}$, if $S_{3}^{1}=\emptyset$, then replace the clique $L_{2}^{3}$ with the clique $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}$, if $S_{3}^{1} \neq \emptyset$, then add the cliques $\left\{s_{2}^{1}, s_{2}^{2}, t_{3}^{3}\right\}$, if $S_{1}^{2}=\emptyset$, then replace the cliques $L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $\left(L_{3}^{1} \cup\left\{t_{2}^{2}, s_{3}^{3}\right\}\right) \backslash S_{2}^{3}$ and $L_{3}^{2} \cup S_{2}^{3}$ and if $S_{1}^{2} \neq \emptyset$, then add the clique $S_{1}^{3} \cup\left\{s_{3}^{3}, t_{2}^{2}, r_{1}^{3}\right\}$, and finally if $|S|=\left|S_{2}\right|=2$, then remove the clique $M_{1}^{2}$ (the edges in $E\left(M_{1}^{2}\right)$ are already covered by the cliques are already covered by other cliques $L_{2}^{1}\left\{t_{1}^{1}\right\},\left(L_{3}^{1} \cup\left\{t_{2}^{2}, s_{3}^{3}\right\}\right) \backslash S_{2}^{3}$ and $\left.M_{3}^{2}\right)$.

This proves (1.1).
(1.2) If $|R|=2$, then $|T|=4$.

Suppose not, by (1.1), $|T|=3$, and w.l.o.g. we may assume that $T=\left\{t_{1}^{1}, t_{3-i}^{i}, t_{2}^{2}\right\}$ and $R=$ $\left\{r_{j}^{1}, r_{p-j+1}^{2}\right\}$, for some $i \in\{1,2\}$ and $j \in\{1, p\}$. By Lemma D. $1 \leq|S| \leq 2$. Through the following six cases, we apply suitable modifications to $\mathscr{M}(G)$ which yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(p, j)=(2,1)$. By symmetry, we may assume that $i=1$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}, K_{3}^{1}, K_{1}^{2}$ and $K_{1}^{3}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{1}$ with the cliques $L_{1}^{2} \cup$ $\left\{t_{2}^{2}, r_{1}^{1}\right\}, L_{1}^{3} \cup\left\{s_{3}^{3}\right\},\left(L_{2}^{1} \cup S_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}\right) \backslash S_{2}^{3},\left(L_{2}^{3} \cup S_{2}^{3} \cup\left\{s_{3}^{3}, r_{1}^{1}\right\}\right) \backslash S_{3}^{1}, L_{3}^{1} \cup\left\{s_{3}^{3}\right\}$ and $L_{3}^{1} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\}$. Also, if $S_{1}^{2}=\emptyset$, then remove the cliques $L_{1}^{2}, L_{3}^{2}, M_{2}^{1}$ and $L_{2}^{3}$. Finally, if $|S|=\left|S_{2}^{1} \cup S_{1}^{2}\right|=2$, then remove the cliques $K_{2}^{1}, K_{1}^{2}$ and $K_{2}^{3}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\}, L_{1}^{3} \cup\left\{s_{3}^{3}\right\}, L_{2}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}, L_{2}^{3} \cup\left\{s_{2}^{2}, t_{1}^{1}\right\}, L_{3}^{1} \cup\left\{t_{1}^{1}\right\}$ and $L_{3}^{1} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}$.
- $(p, j)=(2,2)$. By symmetry, we may assume that $i=1$. First, suppose that $S_{2}^{1}=\emptyset$, and remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$. Now, in the resulting family, if $S_{3}^{1} \cup S_{2}^{3}=\emptyset$, then remove the cliques $K_{1}^{2}$ and $K_{1}^{3}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}$ and $L_{1}^{3}$ with the cliques $L_{2}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}, L_{3}^{1} \cup$ $\left\{s_{3}^{3}, t_{2}^{2}\right\},\left(L_{1}^{2} \cup\left\{t_{2}^{2}\right\}\right) \backslash\left\{r_{1}^{2}\right\}$ and $L_{1}^{3} \cup\left\{s_{3}^{3}, r_{1}^{2}\right\}$, and if $S_{3}^{2} \cup S_{1}^{3}=\emptyset$, then remove the cliques $M_{2}^{1}$ and $M_{1}^{2}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup\left\{t_{3}^{3}\right\}, L_{1}^{3} \cup\left\{r_{2}^{1}\right\},\left(L_{2}^{3} \cup\right.$ $\left.\left\{s_{3}^{3}, t_{1}^{1}\right\}\right) \backslash\left\{r_{2}^{1}\right\}, L_{3}^{1} \cup\left\{s_{3}^{3}, t_{1}^{1}\right\}$ and $L_{3}^{2} \cup\left\{s_{2}^{2}\right\}$.
Next, assume that $S_{1}^{2}=\emptyset$, and remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$. Now, in the resulting family, if $S_{3}^{1} \cup S_{1}^{3}=\emptyset$, then merge the pairs $\left(L_{1}^{2}, L_{1}^{3}\right)$ and $\left(L_{3}^{1}, M_{1}^{2}\right)$, if $|S|=\left|S^{1}\right|=2$, then remove the clique $M_{1}^{2}$ and replace the cliques $L_{1}^{3}, L_{2}^{3}$ and $L_{3}^{1}$ with the cliques $L_{1}^{3} \cup\left\{s_{3}^{3}, r_{2}^{1}\right\},\left(L_{2}^{3} \cup\left\{t_{1}^{1}\right\}\right) \backslash\left\{r_{2}^{1}\right\}$ and $L_{3}^{1} \cup\left\{s_{3}^{3}, t_{1}^{1}\right\}$, and if $|S|=\left|S_{2}^{1} \cup S_{1}^{3}\right|=2$, then merge the pair $\left(L_{1}^{2}, L_{1}^{3}\right)$.
Finally, assume that $|S|=\left|S_{2}^{1} \cup S_{1}^{2}\right|=2$. Merge the pair $\left(K_{1}^{3}, L_{3}^{1}\right)$, remove the cliques $K_{3}^{2}$ and $M_{2}^{1}$ and replace the cliques $K_{3}^{1}, L_{1}^{2}, L_{1}^{3}$ and $L_{3}^{2}$ with the cliques $K_{3}^{1} \cup\left\{r_{2}^{1}\right\},\left(L_{1}^{2} \cup\left\{s_{1}^{1}, t_{3}^{3}\right\}\right) \backslash\left\{r_{1}^{2}\right\}, L_{1}^{3} \cup\left\{r_{1}^{2}\right\}$ and $L_{3}^{2} \cup\left\{s_{2}^{2}\right\}$.
- $(p, j)=(3,1)$ and $i=1$. If $S_{1}^{2}=\emptyset$, then remove cliques $L_{1}^{2}, L_{3}^{2}, M_{2}^{1}$ and $M_{2}^{3}$. Now, assume that $S_{1}^{2} \neq \emptyset$, i.e. either $S=\left\{s_{2}^{1}, s_{1}^{2}\right\}$ and $S=\left\{s_{1}^{2}, s_{1}^{3}\right\}$. In the former case, merge the pair $\left(L_{3}^{1}, M_{1}^{2}\right)$, remove the cliques $M_{3}^{1}, M_{3}^{2}$ and $M_{2}^{3}$, add the clique $\left\{s_{3}^{3}, t_{3}^{1}, r_{1}^{1}\right\}$ and replace the cliques $L_{1}^{2}, L_{3}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{1}^{2} \cup\left\{s_{1}^{1}\right\}, L_{3}^{2} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}, L_{1}^{3} \cup\left\{t_{3}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{1}^{1}\right\}$. In the latter case, merge the pairs $\left(L_{2}^{1}, M_{1}^{2}\right)$ and $\left(L_{2}^{3}, M_{3}^{2}\right)$, remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$, add the clique $S_{1}^{2} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}$ and replace the cliques $L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup\left\{s_{1}^{1}, t_{3}^{3}\right\}$ and $L_{3}^{2} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\}$.
- $(p, j)=(3,1)$ and $i=2$. First, suppose that either $S_{2}^{1}$ or $S_{1}^{2}$ is empty. If $S_{2}^{1}=\emptyset$, then remove the cliques $L_{2}^{1}$ and $L_{2}^{3}$, if $S_{1}^{2}=\emptyset$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$, if $S_{2}^{1} \cup S_{1}^{3}=\emptyset$, then merge the pair $\left(K_{3}^{1}, K_{3}^{2}\right)$, if $S_{1}^{2} \cup S_{2}^{3}=\emptyset$, then merge the pair $\left(L_{3}^{1}, L_{3}^{2}\right)$, if $S_{2}^{3} \neq \emptyset$ and $S_{3}^{1} \cup S_{1}^{2} \cup S_{1}^{3}=\emptyset$, then merge the pair $\left(L_{3}^{2}, M_{1}^{2}\right)$, if $S_{3}^{1} \cup S_{1}^{3}=\emptyset$, then remove the clique $K_{3}^{2}$ and replace the cliques $L_{1}^{2}$ and $L_{1}^{3}$ with the cliques $L_{1}^{2} \cup\left\{s_{1}^{1}\right\}$ and $L_{1}^{3} \cup\left\{t_{3}^{3}\right\}$ and if $S_{1}^{3} \neq \emptyset$ and $S_{2}^{1} \cup S_{3}^{1} \cup S_{3}^{2}=\emptyset$, then remove the clique $K_{3}^{1}$ and replace the cliques $L_{1}^{2}$ and $L_{1}^{3}$ with the cliques $L_{1}^{2} \cup\left\{s_{1}^{1}\right\}$ and $L_{1}^{3} \cup\left\{t_{3}^{3}\right\}$. Next, assume that $|S|=\left|S_{2}^{1} \cup S_{1}^{2}\right|=1$. Merge the pair $\left(L_{3}^{1}, M_{1}^{2}\right)$, remove the cliques $K_{3}^{1}$ and $M_{2}^{3}$ and replace the cliques $K_{3}^{2}, L_{2}^{1}, L_{2}^{3}, L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $K_{3}^{2} \cup\left\{r_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{2}^{3} \cup$ $\left\{t_{3}^{3}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}\right\}$ and $L_{3}^{2} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}$.
- $(p, j)=(3,3)$ and $i=1$. First, suppose that $S_{2}^{1}=\emptyset$. Remove the cliques $K_{2}^{1}, K_{3}^{1}, K_{1}^{2}$ and $K_{1}^{3}$, replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $\left(L_{1}^{2} \cup\left\{s_{3}^{3}, t_{3}^{1}\right\}\right) \backslash S_{1}^{2},\left(L_{1}^{3} \cup S_{1}^{2} \cup\left\{t_{2}^{2}\right\}\right) \backslash\left\{t_{3}^{1}\right\},\left(L_{3}^{1} \cup\right.$ $\left.\left\{s_{3}^{3}, t_{2}^{2}\right\}\right) \backslash S_{2}^{3}$ and $L_{3}^{2} \cup S_{2}^{3}$, and in th resulting family, if $S_{2}^{3}=\emptyset$, then replace the clique $L_{2}^{1}$ with the clique $L_{2}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}$ and if $S_{2}^{3} \neq \emptyset$, then add the clique $\left\{s_{3}^{3}, t_{2}^{1}, t_{2}^{2}\right\}$. Next, assume that $S_{2}^{1} \neq \emptyset$. In the case that $S_{2}^{3}=\emptyset$, remove the cliques $K_{2}^{1}, K_{1}^{2}, K_{1}^{3}$ and $M_{1}^{2}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{2}^{1}, L_{2}^{3}$ and $L_{3}^{1}$ with the cliques $\left(L_{1}^{2} \cup\left\{s_{3}^{3}, t_{3}^{1}\right\}\right) \backslash S_{1}^{2},\left(L_{1}^{3} \cup S_{1}^{2} \cup\left\{t_{2}^{2}\right\}\right) \backslash\left\{t_{3}^{1}\right\}, L_{2}^{1} \cup S_{3}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\},\left(L_{2}^{3} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{3}^{1}$ and $L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}$, and in the resulting family, if $S_{1}^{3}=\emptyset$, then replace the clique $L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}$ with the clique $L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{1}, t_{2}^{2}\right\}$, if $S_{1}^{3} \neq \emptyset$, then add the clique $\left\{s_{3}^{3}, t_{2}^{1}, t_{2}^{2}\right\}$ and $S_{3}^{1} \neq \emptyset$, then remove the clique $M_{1}^{2}$ (the edges in $E\left(M_{1}^{2}\right)$ are already covered by the cliques $L_{2}^{1} \cup S_{3}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}, L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}$ and $M_{3}^{2}$ ). Also, in the case that $|S|=\left|S_{2}\right|=2$, merge the pair ( $M_{1}^{2}, L_{3}^{1}$ ), remove the cliques $K_{3}^{2}$ and $M_{2}^{1}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{1}^{2} \cup\left\{t_{3}^{1}\right\},\left(L_{1}^{3} \cup\left\{s_{1}^{1}, t_{3}^{3}\right\}\right) \backslash\left\{t_{3}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$.
- $(p, j)=(3,3)$ and $i=2$. If $S_{2}^{1}=\emptyset$, remove the cliques $K_{2}^{1}, K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$. Now, assume that $S_{2}^{1} \neq \emptyset$. In the case that $S_{3}^{1} \cup S_{1}^{2} \cup S_{2}^{3}=\emptyset$, merge the pair ( $L_{2}^{3}, M_{2}^{1}$ ), remove the cliques $K_{2}^{1}, K_{1}^{2}$ and $K_{1}^{3}$, add the clique $S_{3}^{2} \cup\left\{s_{3}^{3}, t_{2}^{2}, t_{3}^{2}\right\}$, replace the cliques $L_{2}^{1}, L_{2}^{3}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}, L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$ and $L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}$ and in the resulting family, if $|S|=\left|S_{2}^{1}\right|=1$, then remove the clique $M_{1}^{2}$ and replace the clique $L_{3}^{2}$ with the clique $L_{3}^{2} \cup\left\{s_{3}^{3}, t_{1}^{1}\right\}$ (the edges $s_{3}^{3} r_{2}^{3}$ and $t_{1}^{1} r_{2}^{3}$ are already covered by the cliques $L_{2}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}$ and $\left.L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}\right)$. Also, in the case that $S_{3}^{1} \cup S_{1}^{2} \cup S_{2}^{3} \neq \emptyset$, if $S_{3}^{1} \neq \emptyset$, then remove the cliques $K_{3}^{1}, M_{1}^{2}$ and $M_{3}^{2}$, add the clique $\left\{s_{3}^{1}, s_{3}^{3}, t_{1}^{1}, t_{3}^{3}\right\}$, merge the pair ( $K_{1}^{3}, L_{3}^{1}$ ) and replace the cliques $L_{2}^{1}, L_{2}^{3}$ and $L_{3}^{2}$ with the cliques $L_{2}^{1} \cup S_{3}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\},\left(L_{2}^{3} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{3}^{1}$ and $L_{3}^{2} \cup\left\{s_{3}^{3}, t_{1}^{1}\right\}$, if $S_{1}^{2} \neq \emptyset$, then remove the cliques $K_{3}^{1}, M_{2}^{1}$ and $M_{3}^{1}$, add the clique $\left\{s_{1}^{2}, s_{2}^{2}, t_{3}^{3}, r_{1}^{2}\right\}$, merge the pair $\left(L_{3}^{1}, M_{1}^{2}\right)$ and replace the cliques $L_{2}^{1}, L_{2}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}$, and if $S_{2}^{3} \neq \emptyset$, then remove the cliques $K_{1}^{2}, K_{3}^{2}$ and $M_{1}^{2}$, add the clique $\left\{s_{1}^{1}, t_{2}^{2}, t_{3}^{2}\right\}$, merge the pair $\left(L_{2}^{3}, M_{2}^{1}\right)$ and replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup\left\{s_{3}^{3}\right\}, L_{1}^{3} \cup\left\{t_{3}^{3}\right\}, L_{3}^{1} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}$ and $L_{3}^{2} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}$.

This proves (1.2).
(1.3) We have $|R| \geq 3$.

Suppose not, assume that $|R|=2$. By (1.2), $|T|=4$ and w.l.o.g. we may assume that $T=\left\{t_{1}^{1}, t_{2}^{2}, t_{1}^{2}, t_{2}^{2}\right\}$ and $R=\left\{r_{1}^{1}, r_{p}^{2}\right\}$. By Lemma D.1, $|S|=1$. Now, in $\mathscr{M}(G)$, if either $p=2$ and $|S|=\left|S_{2}^{1}\right|=1$, or $p=3$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and merge the pair ( $L_{3}^{1}, L_{3}^{2}$ ), and if $p=2$ and $|S|=\left|S_{1}^{2}\right|=1$, then remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and merge the pair ( $L_{2}^{1}, L_{3}^{1}$ ), thereby obtaining a clique covering for $G$ of size $15=n-1$, a contradiction. This proves (1.3).
(1.4) We have $|T|=4$.

For if this is not the case, then by (1.3) and the assumption $|T| \geq|R|$, we have $|T|=|R|=3$. Due to symmetry, let $T=\left\{t_{1}^{1}, t_{3-i}^{i}, t_{2}^{2}\right\}$, for some $i \in\{1,2\}$. Also, for every $(j, k) \in\{(1,1),(2,1),(1, p),(2, p)\}$, let $R_{k}^{j}=V(G) \cap\left\{r_{k}^{j}\right\}$, and also let $\left(j_{0}, k_{0}\right)$ be the unique element of $\{(1,1),(2,1),(1, p),(2, p)\}$ such that $R_{k_{0}}^{j 0}=\emptyset$. By Lemma D.1, $|S|=1$. Through the following three cases, we apply suitable modifications to $\mathscr{M}(G)$ that yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $p=2$. By symmetry, we may assume that $i=1$. In the case that $|S|=\left|S_{2}^{1}\right|=1$, remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and merge the pair ( $L_{1}^{2}, L_{1}^{3}$ ), and in the case that $|S|=\left|S_{1}^{2}\right|=1$, remove the cliques $K_{2}^{1}, K_{3}^{1}$ and $M_{2}^{1}$ and replace the cliques $L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup R_{1}^{1} \cup\left\{t_{3}^{3}\right\}$ and $L_{3}^{2} \cup\left\{s_{2}^{2}\right\}$.
- $p=3$ and $i=1$. In the case that $\left(j_{0}, k_{0}\right)=(1,1)$, remove the cliques $M_{2}^{1}$ and $M_{1}^{2}$, replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{t_{1}^{1}\right\}, L_{3}^{1} \cup\left\{s_{3}^{3}\right\}, L_{1}^{3} \cup\left\{t_{3}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$ and in the resulting family, if $|S|=\left|S_{2}^{1}\right|=1$, then merge the pair ( $K_{1}^{3}, L_{3}^{1} \cup\left\{s_{3}^{3}\right\}$ ), and if $|S|=\left|S_{1}^{3}\right|=1$, then remove the clique $K_{3}^{1}$ and replace the cliques $L_{1}^{2}$ and $L_{1}^{3} \cup\left\{t_{3}^{3}\right\}$ with the cliques $L_{1}^{2} \cup\left\{t_{3}^{1}\right\}$ and $\left(L_{1}^{3} \cup\left\{s_{1}^{1}, t_{3}^{3}\right\}\right) \backslash\left\{t_{3}^{1}\right\}$. Also, in the case that $\left(j_{0}, k_{0}\right) \in\{(2,1),(1,3)\}$, remove the clique $M_{1}^{2}$, replace the cliques $L_{2}^{1}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{t_{1}^{1}\right\}$ and $L_{3}^{1} \cup\left\{s_{3}^{3}\right\}$ and in the resulting family,
if $\left(j_{0}, k_{0}\right)=(2,1)$, then remove the cliques $L_{1}^{2}$ and $L_{3}^{2}$ and if $\left(j_{0}, k_{0}\right)=(1,3)$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$. Moreover, in the case that $\left(j_{0}, k_{0}\right)=(2,3)$, remove the clique $M_{2}^{1}$, replace the cliques $L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{1}^{3} \cup\left\{t_{3}^{3}, r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$, and in the resulting family, if $|S|=\left|S_{2}^{1}\right|=1$, then remove the clique $K_{2}^{1}$, merge the pair $\left(L_{3}^{1}, M_{1}^{2}\right)$ and replace the clique $L_{2}^{1}$ with the clique $L_{2}^{1} \cup\left\{t_{1}^{1}, r_{3}^{3}\right\}$, and if $|S|=\left|S_{1}^{3}\right|=1$, then remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$.
- $p=3$ and $i=2$. First assume that $\left(j_{0}, k_{0}\right)=(2,3)$. If $|S|=\left|S_{2}^{1}\right|=1$, then merge the pair $\left(L_{3}^{1}, M_{1}^{2}\right)$, remove the cliques $K_{3}^{1}$ and $M_{2}^{1}$ and replace the cliques $L_{2}^{1}, L_{1}^{3}$ and $L_{2}^{3}$ and the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{1}^{3} \cup\left\{r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}$, and if $|S|=\left|S_{1}^{3}\right|=1$, then remove the cliques $K_{2}^{1}, K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$. Now, suppose $\left(j_{0}, k_{0}\right) \neq(2,3)$. In the case that $|S|=\left|S_{2}^{1}\right|=1$, remove the cliques $K_{3}^{1}$ and $K_{2}^{3}$, replace the cliques $K_{3}^{2}, L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $K_{3}^{2} \cup\left\{r_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{2}^{3} \cup$ $\left\{t_{3}^{3}\right\}, L_{3}^{1} \cup\left\{t_{1}^{1}, t_{1}^{2}, r_{3}^{3}\right\}$ and $\left(L_{3}^{2} \cup\left\{s_{2}^{2}\right\}\right) \backslash\left\{t_{1}^{2}\right\}$ and in the resulting family, if $\left(j_{0}, k_{0}\right)=(1,1)$, then remove the clique $M_{2}^{1}$ and replace the clique $L_{2}^{3} \cup\left\{t_{3}^{3}\right\}$ with $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}$, if $\left(j_{0}, k_{0}\right)=(2,1)$, then merge the pair $\left(L_{3}^{1}, L_{3}^{2}\right)$ and if $\left(j_{0}, k_{0}\right)=(1,3)$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$. Also, in the case that $|S|=\left|S_{1}^{3}\right|=1$, remove the cliques $L_{2}^{1}$ and $L_{2}^{3}$ and in the resulting family, if $\left(j_{0}, k_{0}\right)=(1,1)$, then merge the pair $\left(K_{1}^{3}, L_{3}^{1}\right)$, if $\left(j_{0}, k_{0}\right)=(2,1)$, then merge the pair $\left(L_{3}^{1}, L_{3}^{2}\right)$ and if $\left(j_{0}, k_{0}\right)=(1,3)$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$.

This proves (1.4).
Now (1.3) and (1.4) together imply that $|S|+|T|+|R| \geq 8$, which contradicts to Lemma D.1.
Case 2. $\left|I_{T}\right|+\left|I_{R}\right|=5$.
By symmetry, assume that $\left|I_{T}\right|=3$ and $I_{R}=\{1,2\}$, and also $\left\{r_{1}^{1}, r_{2}^{2}\right\} \subseteq R \subseteq\left\{r_{1}^{1}, r_{2}^{1}, r_{1}^{2}, r_{2}^{2}\right\}$. Let $R_{3-i}^{i}=V(G) \cap\left\{r_{3-i}^{i}\right\}, i \in\{1,2\}$.
(2.1) We have $|T| \geq 4$.

Suppose not, assume that $|T|=3$, say $T=\left\{t_{j}^{i}, t_{3-j}^{i+1}, t_{3-j}^{i+2}\right\}$. By symmetry, we may assume that $(i, j) \in\{(1,1),(1,2),(3,1)\}$. Through the following three cases, we apply some modifications to $\mathscr{M}(G)$ that yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(i, j)=(1,1)$. Note that by Lemma 9.15 (applying to $(i, j, k)=(1,2,3)), S_{2}^{1} \cup S_{3}^{1} \cup S_{1}^{2} \neq \emptyset$. By Lemma D.1, $|S|+|R| \leq 4$, i.e. $1 \leq|S| \leq 2$ and $2 \leq|R| \leq 3$. First, assume that $|R|=2$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}, K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$. Now, suppose that $S_{2}^{1} \neq \emptyset$ and $S_{1}^{2}=\emptyset$. Remove the cliques $L_{1}^{2}$ and $L_{2}^{3}$ and in the resulting family, if $S_{3}^{1}=\emptyset$, then merge the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$, remove the clique $M_{3}^{1}$ and replace the cliques $L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{1}^{3} \cup\left\{r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}$, and if $S_{3}^{1} \neq \emptyset$, then remove the clique $M_{2}^{3}$ and replace the cliques $L_{2}^{1}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}$ and $L_{3}^{1} \cup\left\{t_{2}^{2}\right\}$. Thus, we may assume that $|S|=\left|S_{2}^{1} \cup S_{1}^{2}\right|=2$, and in this case, remove the cliques $K_{2}^{1}, M_{3}^{2}$ and $M_{2}^{3}$ and replace the cliques $L_{2}^{1}, L_{1}^{2}, L_{3}^{2}, L_{1}^{3}, L_{2}^{3}$ and $M_{1}^{3}$ with the cliques $L_{2}^{1} \cup\left\{r_{3}^{3}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}, t_{2}^{2}\right\}, L_{3}^{2} \cup\left\{r_{3}^{3}\right\}, L_{1}^{3} \cup\left\{s_{3}^{3}, r_{2}^{2}\right\}, L_{2}^{3} \cup\left\{s_{2}^{2}, t_{1}^{1}\right\}$ and $M_{1}^{3} \cup\left\{t_{2}^{3}\right\}$. Next, assume that $|R|=3$. Thus, $|S|=1$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and merge the pair $\left(L_{1}^{3}, L_{2}^{3}\right)$. Also, if $|S|=\left|S_{2}^{1}\right|=1$, then merge the pairs $\left(L_{1}^{2}, L_{1}^{3}\right)$ and $\left(M_{1}^{3}, M_{2}^{3}\right)$, remove the clique $K_{1}^{3}$ and replace the cliques $L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{3}^{1} \cup\left\{t_{2}^{2}\right\}$ and $L_{3}^{2} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}$.
- $(i, j)=(1,2)$. Again by Lemma 9.15 (applying to $(i, j, k)=(1,2,3)), S_{2}^{1} \cup S_{3}^{1} \cup S_{1}^{2} \neq \emptyset$. By Lemma D.1, $|S|+|R| \leq 4$, i.e. $1 \leq|S| \leq 2$ and $2 \leq|R| \leq 3$. First, assume that $|R|=2$. In the case that $S_{1}^{2}=\emptyset$, remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and in the resulting family, if $S_{2}^{3}=\emptyset$, then merge the pair $\left(L_{2}^{1}, L_{3}^{1}\right)$, if $S_{3}^{2} \cup S_{2}^{3}=\emptyset$, then merge the pair $\left(L_{1}^{3}, M_{3}^{2}\right)$ and if $S_{2}^{3} \neq \emptyset$ (and so exactly one of $S_{2}^{1}, S_{3}^{1}=\emptyset$ ), then remove the clique $K_{1}^{2}$ and replace the cliques $L_{1}^{2}$ and $L_{1}^{3}$ with the cliques $L_{1}^{2} \cup S_{3}^{1} \cup\left\{s_{3}^{3}\right\}$ and $\left(L_{1}^{3} \cup\left\{r_{1}^{1}\right\}\right) \backslash S_{3}^{1}$. Now, assume that $S_{1}^{2} \neq \emptyset$. If $|S|=2$ and $S_{3}^{2} \cup S_{2}^{3}=\emptyset$, then merge the pair $\left(L_{1}^{3}, M_{3}^{2}\right)$, remove the cliques $K_{2}^{1}$ and $M_{2}^{3}$ and replace the cliques $K_{2}^{3}, L_{2}^{1}, L_{2}^{3}, L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $\left(K_{2}^{3} \cup\left\{t_{1}^{3}\right\}\right) \backslash\left\{t_{2}^{1}\right\}, L_{2}^{1} \cup S_{3}^{1} \cup\left\{r_{3}^{3}\right\},\left(L_{2}^{3} \cup\left\{s_{2}^{2}\right\}\right) \backslash S_{3}^{1},\left(L_{1}^{2} \cup S_{1}^{3} \cup\left\{s_{1}^{1}\right\}\right.$ and $\left(L_{3}^{2} \cup\left\{r_{3}^{3}\right\}\right) \backslash S_{1}^{3}$. Also, if $|S|=2$ and $S_{3}^{2} \cup S_{2}^{3} \neq \emptyset$, then merge the pair $\left(K_{1}^{2}, L_{1}^{3}\right)$ and in the resulting family, in the case that $|S|=\left|S^{2}\right|=2$, remove the cliques $K_{2}^{1}$ and $M_{1}^{3}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{3}^{2}$ and $M_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{r_{1}^{3}\right\},\left(L_{3}^{1} \cup\left\{s_{1}^{1}, t_{1}^{2}, r_{3}^{3}\right\}\right) \backslash\left\{r_{1}^{3}\right\},\left(L_{3}^{2} \cup\left\{s_{2}^{2}\right\}\right) \backslash\left\{t_{1}^{2}\right\}$
and $M_{2}^{3} \cup\left\{t_{1}^{2}\right\}$ and in the case that $|S|=\left|S_{1}^{2} \cup S_{2}^{3}\right|=1$, remove the cliques $K_{2}^{3}$ and $M_{2}^{3}$ and replace the cliques $K_{2}^{1}, L_{2}^{1}, L_{2}^{3}, L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $K_{2}^{1} \cup\left\{r_{1}^{3}\right\},\left(L_{2}^{1} \cup\left\{r_{3}^{3}\right\}\right) \backslash S_{2}^{3}, L_{2}^{3} \cup S_{2}^{3} \cup$ $\left\{s_{2}^{2}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}\right\}$ and $L_{3}^{2} \cup\left\{r_{3}^{3}\right\}$. Finally, if $|S|=\left|S_{1}^{2}\right|=1$, then merge the pair $\left(L_{1}^{3}, M_{3}^{2}\right)$, remove the cliques $K_{2}^{3}, M_{1}^{3}$ and $M_{2}^{3}$ and replace the cliques $K_{2}^{1}, L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}$ and $L_{2}^{3}$ with the cliques $K_{2}^{1} \cup\left\{r_{1}^{3}\right\}, L_{2}^{1} \cup\left\{r_{1}^{3}, r_{3}^{3}\right\},\left(L_{3}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}\right) \backslash\left\{r_{1}^{3}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}\right\}, L_{3}^{2} \cup\left\{r_{3}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$.
Next, assume that $|R|=3$. Thus, $|S|=1$. If $S_{1}^{2}=\emptyset$, then remove the cliques $M_{2}^{1}, M_{2}^{3}$ and $M_{3}^{2}$ and replace the cliques $L_{1}^{2}$ and $L_{1}^{3}$ with the cliques $L_{1}^{2} \cup\left\{s_{3}^{3}, r_{2}^{2}\right\}$ and $L_{1}^{3} \cup\left\{s_{3}^{3}, r_{2}^{2}\right\}$. Also, if $|S|=\left|S_{1}^{2}\right|=1$, then merge the pair $\left(K_{1}^{2}, L_{1}^{3}\right)$ and in the resulting family, if $R_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{3}$ and $M_{1}^{3}$ and replace the cliques $K_{2}^{1}, L_{2}^{1}, L_{2}^{3}$ and $L_{3}^{1}$ with the cliques $K_{2}^{1} \cup\left\{r_{1}^{3}\right\}, L_{2}^{1} \cup\left\{r_{1}^{3}, r_{3}^{3}\right\},\left(L_{3}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}\right) \backslash\left\{r_{1}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$, and if $R_{1}^{2}=\emptyset$, then remove the cliques $K_{2}^{1}$ and $M_{2}^{3}$ and replace the cliques $L_{1}^{2}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup\left\{s_{1}^{1}\right\}, L_{3}^{1} \cup\left\{t_{1}^{2}\right\}$ and $\left(L_{3}^{2} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}\right) \backslash\left\{t_{1}^{2}\right\}$.
- $(i, j)=(3,1)$. By Lemma D.1, $|S|+|R| \leq 4$, i.e. $|S| \leq 2$ and $2 \leq|R| \leq 4$. First, assume that $|R|=2$. If $S_{1}^{2} \neq \emptyset$, then remove the cliques $L_{1}^{2}, L_{3}^{2}, M_{2}^{1}$ and $M_{2}^{3}$ and if $S=\emptyset$, then merge the pair $\left(L_{1}^{3}, M_{3}^{2}\right)$. Now, assume that $S_{1}^{2} \neq \emptyset$. If $|S|=2$, then remove the cliques $M_{1}^{2}, M_{2}^{1}, M_{2}^{3}$ and $M_{3}^{2}$, add the clique $\left\{s_{1}^{1}, s_{1}^{2}, t_{3}^{3}\right\}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $\left(L_{2}^{1} \cup\left\{s_{3}^{3}, r_{1}^{3}\right\}\right) \backslash S_{2}^{1},\left(L_{3}^{1} \cup S_{2}^{1} \cup\left\{r_{2}^{2}\right\}\right) \backslash\left\{r_{1}^{3}\right\}, L_{1}^{2} \cup S_{1}^{3} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\},\left(L_{3}^{2} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}\right) \backslash S_{1}^{3},\left(L_{1}^{3} \cup\left\{s_{3}^{3}, r_{2}^{2}\right\}\right) \backslash S_{3}^{2}$ and $L_{2}^{3} \cup S_{3}^{2}$. Also, if $|S|=\left|S_{1}^{2}\right|=1$, then merge the pair $\left(L_{1}^{3}, M_{3}^{2}\right)$, remove the cliques $K_{2}^{3}, M_{1}^{3}, M_{2}^{3}$ and replace the cliques $K_{2}^{1}, L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}$ and $L_{2}^{3}$ with the cliques $K_{2}^{1} \cup\left\{r_{1}^{3}\right\}, L_{2}^{1} \cup\left\{t_{2}^{2}, r_{1}^{3}, r_{3}^{3}\right\},\left(L_{3}^{1} \cup\right.$ $\left.\left\{s_{1}^{1}, r_{3}^{3}\right\}\right) \backslash\left\{r_{1}^{3}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}, t_{2}^{2}\right\}, L_{3}^{2} \cup\left\{r_{3}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$. Next, assume that $3 \leq|R| \leq 4$. Thus, $|S| \leq 1$. If $|S|=1$ and $S_{1}^{2}=\emptyset$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and in the resulting family, if $S_{3}^{1} \neq \emptyset$, then merge the pair $\left(L_{1}^{2}, L_{1}^{3}\right)$ and if $S_{3}^{1} \neq \emptyset$, then remove the clique $K_{2}^{1}$ and replace the cliques $L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{3}^{1} \cup\left\{r_{3}^{3}\right\}$ and $L_{3}^{2} \cup\left\{s_{2}^{2}\right\}$. Also, if $|S|=\left|S_{1}^{2}\right|=1$, then merge the pair $\left(L_{1}^{3}, M_{3}^{2}\right)$, remove the cliques $K_{2}^{1}$ and $M_{2}^{1}$ and replace the cliques $L_{3}^{1}, L_{1}^{2}$ and $L_{3}^{2}$ with the cliques $L_{3}^{1} \cup\left\{r_{3}^{3}\right\}, L_{1}^{2} \cup\left\{t_{3}^{3}, r_{1}^{1}\right\}$ and $L_{3}^{2} \cup\left\{s_{2}^{2}\right\}$. Finally, if $S=\emptyset$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and merge the pairs $\left(L_{1}^{2}, L_{1}^{3}\right)$ and $\left(L_{1}^{3}, M_{3}^{2}\right)$.

This proves (2.1).
(2.2) We have $|T|=5$.

Note that if $|T|=6$, then since $|R| \geq 2$, we have $|S|+|T|+|R| \geq 8$, which contradicts Lemma D.1. Thus, if (2.2) does not hold, then by (1.2), $|T|=4$, and so by Lemma D.1, $|S|+|R| \leq 3$, i.e. $|S| \leq 1$ and $2 \leq|R| \leq 3$. In the sequel, first assume that $T=\left\{t_{1}^{i}, t_{2}^{i}, t_{j}^{i+1}, t_{j}^{i+2}\right\}$, for some $i \in\{1,2,3\}$ and $j \in\{1,2\}$. By symmetry, we may assume that $(i, j) \in\{(1,1),(1,2),(3,2)\}$. Through the following three cases, we apply some modifications to $\mathscr{M}(G)$ that yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(i, j)=(1,1)$. Note that by Lemma 9.15 (applying to $(i, j, k)=(1,2,3)), S_{1}^{2} \cup S_{2}^{1} \cup S_{3}^{1} \neq \emptyset$. Thus, $|S|=\left|S_{1}^{2} \cup S_{2}^{1} \cup S_{3}^{1}\right|=1$ and $|R|=2$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and merge the pair $\left(L_{2}^{1}, L_{3}^{1}\right)$, and if $S_{2}^{1} \neq \emptyset$, then remove the cliques $L_{1}^{2}$ and $L_{3}^{2}$ and merge the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$.
- $(i, j)=(1,2)$. Again by Lemma 9.15 (applying to $(i, j, k)=(1,2,3)$ ), we have $|S|=\mid S_{1}^{2} \cup S_{2}^{1} \cup$ $S_{3}^{1} \mid=1$ and $|R|=2$. If $S_{1}^{2}=\emptyset$, then remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and merge the pair $\left(L_{3}^{1}, L_{3}^{2}\right)$, and if $S_{1}^{2} \neq \emptyset$, then remove the cliques $M_{1}^{3}, M_{2}^{3}$ and $M_{3}^{1}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{r_{1}^{3}\right\},\left(L_{3}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}\right) \backslash\left\{r_{1}^{3}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}\right\}, L_{3}^{2} \cup\left\{r_{3}^{3}\right\}, L_{1}^{3} \cup\left\{t_{3}^{3}, r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$.
- $(i, j)=(3,1)$. First, assume that $|R|=2$. Thus, $|S| \leq 1$. In the case that $S_{1}^{2}=\emptyset$, remove the cliques $L_{1}^{2}$ and $L_{3}^{2}$ and in the resulting family, if $S_{3}^{1}=\emptyset$, then merge the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$, remove the clique $M_{3}^{1}$ and replace the cliques $L_{1}^{3}, L_{2}^{3}$ and $M_{2}^{1}$ with the cliques $\left(L_{1}^{3} \cup\left\{t_{3}^{3}\right\}\right) \backslash S_{3}^{2}, L_{2}^{3} \cup S_{3}^{2} \cup\left\{s_{2}^{2}, r_{1}^{1}\right\}$ and $M_{2}^{1} \cup S_{2}^{3}$, and if $S_{3}^{1} \neq \emptyset$, then remove the clique $M_{2}^{3}$ and replace the cliques $L_{2}^{1}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{t_{2}^{2}, r_{1}^{3}, r_{3}^{3}\right\}$ and $\left(L_{3}^{1} \cup\left\{s_{1}^{1}\right\}\right) \backslash\left\{r_{1}^{3}\right\}$. Also, in the case that $S_{1}^{2} \neq \emptyset$, merge the pair
$\left(K_{2}^{1}, L_{3}^{2}\right)$, remove the cliques $K_{1}^{2}$ and $K_{2}^{3}$ and replace the cliques $K_{2}^{1}, L_{1}^{2}, L_{1}^{3}, L_{2}^{1}$ and $L_{2}^{3}$ with the cliques $K_{2}^{1} \cup\left\{r_{1}^{3}\right\}, L_{1}^{2} \cup\left\{t_{2}^{2}, r_{1}^{1}\right\}, L_{1}^{3} \cup\left\{s_{3}^{3}\right\}, L_{2}^{1} \cup\left\{r_{3}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$.
Next, assume that $|R|=3$ and thus $S=\emptyset$, and in this case, merge the triple $\left(L_{1}^{2}, L_{1}^{3}, M_{3}^{2}\right)$ and the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$.

Next, suppose that $T=\left\{t_{1}^{i}, t_{2}^{i}, t_{j}^{i+1}, t_{3-j}^{i+2}\right\}$, for some $i \in\{1,2,3\}$ and $j \in\{1,2\}$. By symmetry, we may assume that $(i, j) \in\{(1,1),(1,2),(3,1),(3,2)\}$. Through the following possible cases, we apply some modifications to $\mathscr{M}(G)$ that yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(i, j)=(1,1)$. First, assume that $|R|=2$. Thus, $|S| \leq 1$. If $S_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and in the resulting family, in the case that $S_{1}^{3}=\emptyset$, merge the pair $\left(L_{2}^{1}, L_{3}^{1}\right)$ and in the case that $S \backslash S_{1}^{3}=\emptyset$, remove the cliques $M_{3}^{1}$ and replace the cliques $L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{1}^{3} \cup\left\{t_{3}^{3}, r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$. Also, if $S_{2}^{1} \neq \emptyset$, then merge the pair $\left(K_{1}^{2}, L_{1}^{3}\right)$, remove the cliques $K_{2}^{1}$ and $M_{3}^{1}$ and replace the cliques $L_{2}^{1}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{r_{3}^{3}\right\}, L_{1}^{3} \cup\left\{t_{3}^{3}, r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{1}^{1}\right\}$.
Next, assume that $|R|=3$ and thus $S=\emptyset$, and in this case, remove the cliques $K_{2}^{1}, K_{3}^{1}$ and $M_{3}^{2}$ and replace the cliques $L_{1}^{2}, L_{3}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{1}^{2} \cup\left\{s_{3}^{3}, r_{2}^{2}\right\}, L_{3}^{2} \cup\left\{t_{1}^{1}\right\}, L_{1}^{3} \cup R_{2}^{1} \cup\left\{s_{3}^{3}, r_{2}^{2}\right\}$ and $\left(L_{2}^{3} \cup\left\{t_{1}^{1}\right\}\right) \backslash R_{2}^{1}$.
- $(i, j)=(1,2)$. First, assume that $|R|=2$. Thus, $|S| \leq 1$. If $S_{1}^{2}=\emptyset$, then remove the cliques $L_{1}^{2}, L_{1}^{3}, M_{2}^{1}$ and $M_{2}^{3}$. Also, if $S_{1}^{2} \neq \emptyset$, then remove the cliques $M_{1}^{2}, M_{2}^{3}$ and $M_{3}^{1}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{s_{3}^{3}, r_{1}^{3}\right\},\left(L_{3}^{1} \cup\left\{t_{1}^{1}, r_{2}^{2}\right\}\right) \backslash\left\{r_{1}^{3}\right\}, L_{1}^{2} \cup$ $\left\{s_{1}^{1}, t_{2}^{2}\right\}, L_{3}^{2} \cup\left\{r_{3}^{3}\right\}, L_{1}^{3} \cup\left\{t_{3}^{3}, r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$.
Next, assume that $|R|=3$ and thus $S=\emptyset$, and in this case, remove the cliques $M_{2}^{1}$ and $M_{2}^{3}$ and merge the pair $\left(L_{1}^{2}, L_{1}^{3}\right)$.
- $(i, j)=(3,1)$. First, assume that $|R|=2$. Thus, $|S| \leq 1$. In the case that $S_{2}^{1} \cup S_{3}^{2}=\emptyset$, remove the cliques $L_{2}^{1}$ and $L_{2}^{3}$ and merge the pair $\left(K_{2}^{1}, K_{2}^{3}\right)$ and if $S_{3}^{1} \cup S_{1}^{2}=\emptyset$, then remove the cliques $L_{1}^{2}$ and $L_{3}^{2}$ and merge the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$.
Next, assume that $|R|=3$ and thus $S=\emptyset$, and in this case, merge the pairs $\left(K_{2}^{1}, K_{2}^{3}\right),\left(L_{1}^{3}, L_{2}^{3}\right)$ and $\left(M_{1}^{3}, M_{2}^{3}\right)$.
- $(i, j)=(3,2)$ and $|R|=2$. First, assume that $S_{1}^{2} \cup S_{2} \neq \emptyset$. Merge the pair $\left(K_{1}^{2}, L_{1}^{3}\right)$, if $S_{2}^{3}=\emptyset$, then remove the cliques $K_{2}^{1}$ and $K_{2}^{3}$ and replace the cliques $L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $\left(L_{2}^{1} \cup\left\{r_{3}^{3}\right\}\right) \backslash S_{2}^{3}, L_{2}^{3} \cup S_{2}^{3} \cup\left\{s_{2}^{2}\right\},\left(L_{3}^{1} \cup S_{1}^{2} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}\right) \backslash\left\{r_{2}^{3}\right\}$ and $\left(L_{3}^{2} \cup\left\{r_{2}^{3}\right\}\right) \backslash S_{1}^{2}$ and if $S_{2}^{3} \neq \emptyset$, then remove the cliques $K_{2}^{3}$ and $M_{1}^{3}$ and replace the cliques $K_{2}^{1}, L_{2}^{1}, L_{2}^{3}, L_{1}^{2}, L_{3}^{2}$ and $M_{2}^{3}$ with the cliques $K_{2}^{1} \cup\left\{r_{1}^{3}\right\},\left(L_{2}^{1} \cup\left\{r_{3}^{3}\right\}\right) \backslash S_{2}^{3}, L_{2}^{3} \cup S_{2}^{3} \cup\left\{s_{2}^{2}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}\right\}, L_{3}^{2} \cup\left\{r_{3}^{3}\right\}$ and $M_{2}^{3} \cup\left\{r_{2}^{3}\right\}$.
Next, assume that $S_{1}^{2} \cup S_{2} \neq \emptyset$. In this case, if $|S|=\left|S_{3}^{1}\right|=1$, then remove the cliques $K_{2}^{1}, K_{3}^{2}$ and $M_{3}^{2}$ and replace the cliques $L_{3}^{1}, L_{3}^{2}, L_{1}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $\left(L_{3}^{1} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}\right) \backslash$ $\left\{r_{2}^{3}\right\}, L_{3}^{2} \cup\left\{r_{2}^{3}\right\}, L_{1}^{2} \cup S_{3}^{1} \cup\left\{s_{1}^{1}\right\},\left(L_{1}^{3} \cup\left\{t_{3}^{3}, r_{2}^{2}\right\}\right) \backslash S_{3}^{1}$ and $L_{2}^{3} \cup\left\{s_{3}^{3}\right\}$, if $|S|=\left|S_{1}^{3}\right|=1$, then remove the cliques $K_{3}^{2}, M_{3}^{1}$ and $M_{2}^{3}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{r_{1}^{3}, r_{3}^{3}\right\},\left(L_{3}^{1} \cup\left\{s_{1}^{1}\right\}\right) \backslash\left\{r_{1}^{3}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}, r_{2}^{2}\right\}, L_{1}^{3} \cup\left\{t_{3}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, r_{1}^{1}\right\}$, and if $S=S_{3}^{2}$, then merge the pair $\left(K_{1}^{2}, L_{1}^{3}\right)$, remove the cliques $K_{2}^{1}, M_{3}^{1}$ and $M_{3}^{2}$, add the clique $S_{2}^{3}\left\{t_{3}^{1}, t_{3}^{3}, r_{2}^{2}\right\}$, replace the cliques $L_{3}^{1}, L_{3}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $\left(L_{3}^{1} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}\right) \backslash\left\{r_{2}^{3}\right\}, L_{3}^{2} \cup\left\{r_{2}^{3}\right\}, L_{1}^{3} \cup\left\{s_{3}^{3}, r_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$ and in the resulting family, if $S=\emptyset$, then remove the clique $K_{2}^{3}$ and replace the clique $L_{2}^{1}$ with the clique $L_{2}^{1} \cup\left\{r_{3}^{3}\right\}$.
- $(i, j)=(3,2)$ and $|R|=3$. Thus, $S=\emptyset$. Merge the pair $\left(L_{1}^{3}, M_{3}^{2}\right)$ and in the resulting family, if $R_{2}^{1}=\emptyset$, then remove the cliques $K_{2}^{3}$ and $M_{2}^{3}$ and replace the cliques $K_{2}^{1}, L_{2}^{1}, L_{3}^{1}$ and $L_{2}^{3}$ with the cliques $K_{2}^{1} \cup\left\{r_{1}^{3}\right\}, L_{2}^{1} \cup\left\{r_{1}^{3}, r_{3}^{3}\right\},\left(L_{3}^{1} \cup\left\{s_{1}^{1}\right\}\right) \backslash\left\{r_{1}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}\right\}$, and if $R_{1}^{2}=\emptyset$, then remove the cliques $K_{2}^{1}$ and $M_{1}^{3}$ and replace the cliques $L_{3}^{1}, L_{1}^{2}, L_{3}^{2}$ and $M_{2}^{3}$ with the cliques $\left(L_{3}^{1} \cup\left\{s_{2}^{2}\right\}\right) \backslash\left\{r_{2}^{3}\right\}, L_{1}^{2} \cup\left\{s_{1}^{1}\right\}, L_{3}^{2} \cup\left\{r_{2}^{3}, r_{3}^{3}\right\}$ and $M_{2}^{3} \cup\left\{r_{2}^{3}\right\}$.

This proves (2.2).
By (2.2), $|T|=5$, say $t_{j}^{i} \notin T$, for some $i \in\{1,2,3\}$ and $j \in\{1,2\}$, and thus by Lemma D.1, $S=\emptyset$ and $|R|=2$. By symmetry, we may assume that $(i, j) \in\{(1,1),(1,2),(3,1)\}$. Through the following three cases, we apply some modifications to $\mathscr{M}(G)$ that yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(i, j)=(1,1)$. Remove the cliques $K_{1}^{2}, K_{2}^{1}$ and $K_{2}^{3}$ and replace the cliques $L_{1}^{2}, L_{1}^{3}, L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{1}^{2} \cup\left\{s_{3}^{3}, t_{2}^{2}\right\}, L_{1}^{3} \cup\left\{r_{1}^{1}\right\}, L_{2}^{1} \cup\left\{r_{3}^{3}\right\}, L_{2}^{3} \cup\left\{s_{2}^{2}\right\},\left(L_{3}^{1} \cup\left\{s_{2}^{2}, r_{3}^{3}\right\}\right) \backslash\left\{r_{2}^{3}\right\}$ and $L_{3}^{2} \cup\left\{r_{2}^{3}\right\}$.
- $(i, j)=(1,2)$. Remove the cliques $L_{2}^{1}$ and $L_{2}^{3}$ and merge the pair $\left(K_{2}^{1}, K_{2}^{3}\right)$.
- $(i, j)=(3,1)$. Remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and merge the pair $\left(L_{2}^{1}, L_{3}^{1}\right)$.

Case 3. $\left|I_{T}\right|=\left|I_{R}\right|=3$.
Due to symmetry, assume that $|T| \leq|R|$.
(3.1) We have $|R| \geq 4$.

On the contrary, assume that $|T|=|R|=3$, and w.l.o.g, let $T=\left\{t_{1}^{1}, t_{2}^{2}, t_{2}^{3}\right\}$ and $R=\left\{r_{j}^{i}, r_{j+1}^{3-i}, r_{j+2}^{3-i}\right\}$, for some $i \in\{1,2\}$ and $j \in\{1,2,3\}$. By Lemma D.1, $|S| \leq 1$. Through the following six cases, we apply suitable modifications to $\mathscr{M}(G)$, which yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(i, j)=(1,1)$. In the case $S_{2}^{1} \cup S_{1}^{2}=\emptyset$, remove the cliques $L_{2}^{1}, L_{1}^{2}, L_{3}^{2}$ and $L_{2}^{3}$, in the case $|S|=$ $\left|S_{2}^{1}\right|=1$, remove the cliques $L_{1}^{2}, L_{3}^{2}$ and merge the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$ and in the case $|S|=\left|S_{1}^{2}\right|=1$, remove the cliques $L_{2}^{1}, L_{2}^{3}$ and merge the pair $\left(K_{3}^{1}, K_{3}^{2}\right)$.
- $(i, j)=(1,2)$. If $|S|=\left|S_{3}^{1}\right|=1$, then remove the cliques $K_{3}^{1}, K_{1}^{3}$ and $M_{2}^{3}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{3}^{2}$ and $L_{1}^{3}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{3}^{1} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}, L_{1}^{2} \cup\left\{t_{3}^{1}, t_{3}^{3}\right\}, L_{3}^{2} \cup\left\{s_{3}^{3}\right\}$ and $\left(L_{1}^{3} \cup\left\{s_{1}^{1}\right\}\right) \backslash\left\{t_{3}^{1}\right\}$. If $|S|=\left|S_{1}^{2}\right|=1$, then $K_{3}^{1}, K_{3}^{2}$ and $M_{1}^{3}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{1}^{2}, L_{1}^{3}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{3}^{1} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\},\left(L_{1}^{2} \cup\left\{s_{1}^{1}, t_{3}^{3}\right\}\right) \backslash\left\{r_{1}^{2}\right\}, L_{1}^{3} \cup\left\{r_{1}^{2}\right\}$ and $L_{2}^{3} \cup\left\{t_{3}^{3}\right\}$. If $|S|=\left|S_{2}^{3}\right|=1$, then merge the pairs $\left(L_{1}^{2}, L_{1}^{3}\right)$ and $\left(M_{1}^{3}, M_{2}^{3}\right)$, remove the clique $K_{3}^{2}$ and replace the cliques $K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$ with the cliques $K_{3}^{1} \cup\left\{t_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{t_{3}^{3}\right\}$. Finally, if $S_{3}^{1} \cup S_{1}^{2} \cup S_{2}^{3}=\emptyset$, then merge the pairs $\left(L_{1}^{2}, L_{1}^{3}\right)$ and $\left(M_{1}^{3}, M_{2}^{3}\right)$, and in the resulting family, if $S_{2}^{1} \cup S_{1}^{3}=\emptyset$, then remove the clique $K_{3}^{2}$ and replace the cliques $K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$ with the cliques $K_{3}^{1} \cup\left\{t_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{t_{3}^{3}\right\}$, and if $S_{3}^{2}=\emptyset$, then remove the clique $K_{1}^{3}$ and replace the cliques $L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{3}^{1} \cup\left\{t_{2}^{2}\right\}$ and $L_{3}^{2} \cup\left\{s_{3}^{3}\right\}$.
- $(i, j)=(1,3)$. In the case that $S_{2}^{1}=\emptyset$, remove the cliques $K_{2}^{1}, K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$ and in the case $|S|=\left|S_{2}^{1}\right|=1$, remove the cliques $K_{2}^{1}, K_{3}^{1}, K_{1}^{3}$ and $M_{1}^{3}$, add the clique $\left\{s_{2}^{1}, t_{1}^{1}, r_{2}^{2}\right\}$ and replace the cliques $L_{2}^{1}, L_{3}^{1}, L_{3}^{2}$ and $L_{2}^{3}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}, L_{3}^{1} \cup\left\{t_{2}^{2}\right\}, L_{3}^{2} \cup\left\{s_{3}^{3}\right\}$ and $L_{2}^{3} \cup\left\{s_{2}^{2}, t_{3}^{3}\right\}$.
- $(i, j)=(2,1)$. In the case that $S_{2}^{1} \cup S_{3}^{2}=\emptyset$, remove the cliques $K_{2}^{1}, K_{3}^{1}$ and merge the pair $\left(L_{1}^{3}, L_{2}^{3}\right)$, and in the resulting family, if $S=\emptyset$, then remove the clique $K_{1}^{3}$ and replace the cliques $L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{3}^{1} \cup\left\{t_{2}^{2}\right\}$ and $L_{3}^{2} \cup\left\{s_{3}^{3}\right\}$. Also, in the case $|S|=\left|S_{2}^{1}\right|=1$, remove the cliques $K_{1}^{3}, K_{3}^{1}$ and $M_{1}^{3}$ and replace the cliques $L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{2}^{3} \cup\left\{t_{3}^{3}\right\}, L_{3}^{1} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}$ and $L_{3}^{2} \cup\left\{s_{3}^{3}\right\}$ and in the case $|S|=\left|S_{3}^{2}\right|=1$, remove the cliques $K_{2}^{1}, K_{3}^{1}$ and $M_{1}^{3}$ and replace the cliques $L_{2}^{1}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{3}^{1} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}$.
- $(i, j)=(2,2)$. In the case $S_{2}^{1}=\emptyset$, remove the cliques $K_{2}^{1}, K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$ and in the case $|S|=\left|S_{2}^{1}\right|=1$, remove the cliques $L_{1}^{2}, L_{3}^{2}$ and $M_{1}^{3}$ and replace the cliques $L_{2}^{1}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}, r_{3}^{3}\right\}$ and $L_{3}^{1} \cup\left\{t_{2}^{2}\right\}$.
- $(i, j)=(2,3)$. In the case $S_{1}^{2} \cup S_{3}^{1}=\emptyset$, remove the cliques $L_{1}^{2}$ and $L_{3}^{2}$ and merge the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$, and if $S=\emptyset$, then remove the clique $K_{3}^{2}$ and replace the cliques $K_{3}^{1}, L_{2}^{1}, L_{2}^{3}$ with the cliques $K_{3}^{1} \cup\left\{t_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{2}^{3} \cup\left\{t_{3}^{3}\right\}$. Also, in the case $|S|=\left|S_{1}^{2}\right|=1$, remove the clique $K_{3}^{2}$, replace the cliques $L_{2}^{1}, L_{2}^{3}, L_{3}^{1}$ and $L_{3}^{2}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{2}^{3} \cup\left\{t_{3}^{3}\right\}, L \frac{1}{1} \backslash\left\{t_{2}^{3}\right\}$ and $L_{3}^{2} \cup\left\{t_{2}^{3}\right\}$
and merge the pairs $\left(K_{3}^{1}, L_{1}^{2}\right)$ and $\left(L_{3}^{1} \backslash\left\{t_{2}^{3}\right\}, M_{1}^{2}\right)$ in the resulting family. Finally, in the case $|S|=\left|S_{3}^{1}\right|=1$, remove the cliques $L_{1}^{2}, L_{3}^{2}$ and $M_{2}^{3}$ and replace the cliques $L_{2}^{1}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{3}^{1} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}$.

This proves (3.1).
(3.2) If $|T|=3$, then $|R| \geq 5$.

On the contrary, let $|T|=3$ and $|R| \leq 4$. By (3.1), $|R|=4$. Also, w.l.o.g. we may assume that $T=\left\{t_{1}^{1}, t_{2}^{2}, t_{2}^{3}\right\}$. By Lemma D.1, $S=\emptyset$. In the sequel, first suppose that $R=\left\{r_{j}^{1}, r_{j}^{2}, r_{j+1}^{i}, r_{j+2}^{i}\right\}$ for some $i \in\{1,2\}$ and some $j \in\{1,2,3\}$. Through the following four cases, we apply suitable modifications to $\mathscr{M}(G)$, which yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(i, j) \in\{(2,1),(2,3)\}$. Remove the cliques $L_{2}^{1}$ and $L_{2}^{3}$ and merge the pair $\left(K_{3}^{1}, K_{3}^{2}\right)$.
- $(i, j) \in\{(1,1),(1,2)\}$. Remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and merge the pair $\left(L_{1}^{3}, L_{2}^{3}\right)$.
- $(i, j)=(2,2)$. Merge the pairs $\left(L_{1}^{2}, L_{1}^{3}\right)$ and $\left(M_{1}^{3}, M_{2}^{3}\right)$, remove the clique $K_{3}^{2}$ and replace the cliques $K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$ with the cliques $K_{3}^{1} \cup\left\{t_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{t_{3}^{3}, r_{2}^{2}\right\}$.
- $(i, j)=(1,3)$. Remove the cliques $L_{1}^{2}, L_{3}^{2}$ and $M_{1}^{3}$, and replace the cliques $L_{2}^{1}$ and $L_{3}^{1}$ with the cliques $L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{3}^{1} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}$.

Next, suppose that $R=\left\{r_{j}^{1}, r_{j}^{2}, r_{j+1}^{i}, r_{j+2}^{3-i}\right\}$ for some $i \in\{1,2\}$ and some $j \in\{1,2,3\}$. Through the four cases, we apply some modifications to $\mathscr{M}(G)$, which yields a clique covering for $G$ of size at most $n-1$, a contradiction.

- $(i, j) \in\{(1,2),(2,1)\}$. Remove the cliques $K_{2}^{1}$ and $K_{3}^{1}$ and merge the pair $\left(L_{1}^{3}, L_{2}^{3}\right)$.
- $(i, j) \in\{(1,3),(2,2)\}$. Remove the cliques $L_{1}^{2}$ and $L_{3}^{2}$, if $(i, j)=(1,3)$, then merge the pair $\left(K_{3}^{1}, K_{3}^{2}\right)$ and if $(i, j)=(2,2)$, then merge the pair $\left(M_{1}^{3}, M_{2}^{3}\right)$.
- $(i, j)=(1,1)$. Merge the pairs $\left(L_{1}^{2}, L_{1}^{3}\right)$ and $\left(M_{1}^{3}, M_{2}^{3}\right)$, remove the clique $K_{3}^{2}$ and replace the cliques $K_{3}^{1}, L_{2}^{1}$ and $L_{2}^{3}$ with the cliques $K_{3}^{1} \cup\left\{t_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{t_{3}^{3}\right\}$.
- $(i, j)=(2,3)$. Remove the cliques $K_{3}^{2}, K_{2}^{3}$ and $M_{1}^{3}$ and replace the cliques $K_{3}^{1}, L_{2}^{1}, L_{3}^{1}, L_{3}^{2}$ and $L_{2}^{3}$ with the cliques $K_{3}^{1} \cup\left\{t_{3}^{2}\right\}, L_{2}^{1} \cup\left\{s_{1}^{1}\right\}, L_{3}^{1} \cup\left\{t_{2}^{2}, r_{3}^{3}\right\}, L_{3}^{2} \cup\left\{s_{2}^{2}, t_{1}^{1}\right\}$ and $L_{2}^{3} \cup\left\{t_{3}^{3}\right\}$.

This proves (3.2).
Now, (3.2) and the assumption $|T| \leq|R|$ implies that $|T|+|R| \geq 8$, which is impossible by Lemma D.1.
This completes the proof of Lemma 9.22.


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