# EXTENDING EDGE-COLORINGS OF COMPLETE HYPERGRAPHS INTO REGULAR COLORINGS 

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#### Abstract

Let $\binom{X}{h}$ be the collection of all $h$-subsets of an $n$-set $X \supseteq Y$. Given a coloring (partition) of a set $S \subseteq\binom{X}{h}$, we are interested in finding conditions under which this coloring is extendible to a coloring of $\binom{X}{h}$ so that the number of times each element of $X$ appears in each color class (all sets of the same color) is the same number $r$. The case $S=\varnothing, r=1$ was studied by Sylvester in the 18 th century, and remained open until the 1970 s. The case $h=2, r=1$ is extensively studied in the literature and is closely related to completing partial symmetric Latin squares.

For $S=\binom{Y}{h}$, we settle the cases $h=4,|X| \geqslant 4.847323|Y|$, and $h=5,|X| \geqslant$ $6.285214|Y|$ completely. Moreover, we make partial progress toward solving the case where $S=\binom{X}{h} \backslash\binom{Y}{h}$. These results can be seen as extensions of the famous Baranyai's theorem, and make progress toward settling a 40-year-old problem posed by Cameron.


## 1. Introduction

Suppose that we have been entrusted to color (or partition) the collection $\binom{[n]}{h}$ of all $h$-subsets of the $n$-set $[n]:=\{1, \ldots, n\}$ so that the number of times each element of [ $n$ ] appears in each color class (all sets of the same color) is exactly $r$. Such a coloring is called an $r$-factorization of $\binom{[n]}{h}$. A solution for the case $n=6, h=3, r=1$ with 10 colors is given below.

$$
\begin{array}{rllll}
\{1,4,5\},\{2,3,6\} & \{1,2,4\},\{3,5,6\} & \{1,3,6\},\{2,4,5\} & \{1,2,3\},\{4,5,6\} & \{1,2,5\},\{3,4,6\} \\
\{1,5,6\},\{2,3,4\} & \{1,3,5\},\{2,4,6\} & \{1,4,6\},\{2,3,5\} & \{1,3,4\},\{2,5,6\} & \{1,2,6\},\{3,4,5\}
\end{array}
$$

Note that the number of times each element of [n] appears in $\binom{[n]}{h}$ is $\binom{n-1}{h-1}$. Thus, for $\binom{[n]}{h}$ to be $r$-factorable, it is clear that (i) $r$ must divide $\binom{n-1}{h-1}$. In addition, a simple double counting argument shows that (ii) $h$ must divide $r n$. One may wonder if conditions (i) and (ii) are also sufficient for $\binom{[n]}{h}$ to be $r$-factorable. In the 18 th century, Sylvester considered the case $r=1$ of this problem which remained open until the 1970s when Baranyai solved this 120-year-old problem completely [5]. In fact, Baranyai proved a far more general result which, in particular, implies that $\binom{[n]}{h}$ is $r$-factorable if and only if $h \mid r n$ and $r \left\lvert\,\binom{ n-1}{h-1}\right.$.

We are interested in a Sudoku-type version of Baranyai's theorem. A partial rfactorization of a set $S \subseteq\binom{[n]}{h}$ is a coloring of $S$ with at most $\binom{n-1}{h-1} / r$ colors so that

[^0]the number of times each element of $[n]$ appears in each color class is at most $r$. Note that a color class may be empty.
Problem 1. Under what conditions can a partial r-factorization of $S \subseteq\binom{[n]}{h}$ be extended to an r-factorization of $\binom{[n]}{h}$ ?

We are given a coloring of a subset $S \subseteq\binom{[n]}{h}$, and our task is to complete the coloring. In other words, we need to color $T:=\binom{[n]}{h} \backslash S$ so that the coloring of $S \cup T$ provides an $r$-factorization of $\binom{[n]}{h}$. Baranyai's theorem settles the case when $S=\varnothing$. A partial 4-factorization of $\binom{[9]}{3}$ is given below (Here we abbreviate a set $\{a, b, c\}$ to $a b c$ ).

$$
\begin{gathered}
156,248,379,126,348,579,127,349,568,124,389,567 \\
148,267,359,168,279,345,159,278,346,134,259 \\
128,347,569,178,249,356,169,247,358,123 \\
146,239,578,137,289,456,136,257 \\
129,367,458,125,368,479,147,258,369,157 \\
189,246,357,158,237,469,138,245,679,139,268 \\
145,236,789,167,238,459,149,256,378,135,269,478
\end{gathered}
$$

It is not too difficult to extend this to the following 4 -factorization.

$$
\begin{aligned}
& 156,248,379,126,348,579,127,349,568,124,389,567 \\
& 148,267,359,168,279,345,159,278,346,134,259,678 \\
& 128,347,569,178,249,356,169,247,358,123,467,589 \\
& 146,239,578,137,289,456,136,257,489,179,235,468 \\
& 129,367,458,125,368,479,147,258,369,157,234,689 \\
& 189,246,357,158,237,469,138,245,679,139,268,457 \\
& 145,236,789,167,238,459,149,256,378,135,269,478
\end{aligned}
$$

The case $h=2, r=1$ of Problem 1 is closely related to completing partial Latin squares, (see Lindner's excellent survey [16]). A special case of Problem 1 when $r=1$, and the partial factorization is a 1-factorization of $\binom{[m]}{h}$ for some $m<n$, was studied by Cruse (for $h=2$ ) [8], Cameron [7], and Baranyai and Brouwer [6]. Baranyai and Brouwer conjectured that a 1-factorization of $\binom{[m]}{h}$ can be extended to a 1-factorization of $\binom{[n]}{h}$ if and only if $n \geqslant 2 m$ and $h$ divides $m, n$. Häggkvist and Hellgren [10] gave a beautiful proof of this conjecture. For further generalizations of Häggkvist-Hellgren's result, we refer the reader to two recent papers by the author and Newman [2, 3] in which extending $r$-factorizations of $\binom{[m]}{h}$ to $s$-factorizations of $\binom{[n]}{h}$ is studied (for $s \geqslant r$ ).

At this point, it should be clear to the reader that the 1-factorization of $\binom{[6]}{3}$ in the first example, can not be extended to a 1-factorization of $\binom{[9]}{3}$, but it can be extended to a 1-factorization of $\binom{[12]}{3}$.

Like most results in the literature, our primary focus is the case where $S=\binom{[m]}{h}$ (for some $m<n$ ). However, unlike those, here we do not require the given partial
factorization to be a factorization itself. In this case, Problem 1 was settled by Rodger and Wantland over 20 years ago for $h=2$ [18], and recently by the author and Rodger for $h=3, n \geqslant 3.414214 m$ [4]. In this paper, we settle the cases $h=4, n \geqslant 4.847323 m$ and $h=5, n \geqslant 6.285214 m$. The major obstacle from $h=2$ to $h \geqslant 3$ stems from the natural difficulty of generalizing a graph theoretic result to hypergraphs.

Note that, in order to extend a partial $r$-factorization of $\binom{[m]}{h}$ to an $r$-factorization of $\binom{[n]}{h}$ (for $n \geqslant m$ ), it is clearly necessary that $r\left|\binom{n-1}{h-1}, h\right| r n$. Let $\chi(m, h, r)$ be the smallest $n$ such that any partial $r$-factorization of $\binom{[m]}{h}$ satisfying $r \left\lvert\,\binom{ n-1}{h-1}\right.$, h|rn can be extended to an $r$-factorization of $\binom{[n]}{h}$. Combining the results of this paper with those of $[2,3,4]$, it can be easily shown that $2 m \leqslant \chi(m, 3, r) \leqslant 3.414214 m, 2 m \leqslant$ $\chi(m, 4, r) \leqslant 4.847323 m$, and $2 m \leqslant \chi(m, 5, r) \leqslant 6.285214 m$.

Last but not least, we shall consider Problem 1 in the case when $S=\binom{[n]}{h} \backslash\binom{[m]}{h}$. In this direction, we solve a variation of the problem when we allow sets of size less than $h$, and in our extension of the coloring we also extend the sets of size less than $h$ to sets of size $h$.

The paper is self-contained and all the preliminaries are given in Section 2. In section 3, we shall consider Problem 1 in the case when $S=\binom{[n]}{h} \backslash\binom{[m]}{h}$. The cases $h=4,5$ are discussed in detail in Sections 4, 5, respectively. We conclude the paper with some open problems.

## 2. Notation and Tools

A hypergraph $\mathcal{G}$ is a pair $(V(\mathcal{G}), E(\mathcal{G}))$ where $V(\mathcal{G})$ is a finite set called the vertex set, $E(\mathcal{G})$ is the edge multiset, where every edge is itself a multi-subset of $V(\mathcal{G})$. This means that not only can an edge occur multiple times in $E(\mathcal{G})$, but also each vertex can have multiple occurrences within an edge. By an edge of the form $\left\{u_{1}^{m_{1}}, u_{2}^{m_{2}}, \ldots, u_{s}^{m_{s}}\right\}$, we mean an edge in which vertex $u_{i}$ occurs $m_{i}$ times for $1 \leqslant i \leqslant r$. The total number of occurrences of a vertex $v$ among all edges of $E(\mathcal{G})$ is called the degree, $\operatorname{deg}_{\mathcal{G}}(v)$ of $v$ in $\mathcal{G}$. The multiplicity of an edge $e$ in $\mathcal{G}$, written mult $\mathcal{G}_{\mathcal{G}}(e)$, is the number of repetitions of $e$ in $E(\mathcal{G})$ (note that $E(\mathcal{G})$ is a multiset, so an edge may appear multiple times). If $\left\{u_{1}^{m_{1}}, u_{2}^{m_{2}}, \ldots, u_{s}^{m_{s}}\right\}$ is an edge in $\mathcal{G}$, then we abbreviate $\operatorname{mult}_{\mathcal{G}}\left(\left\{u_{1}^{m_{1}}, u_{2}^{m_{2}}, \ldots, u_{s}^{m_{s}}\right\}\right)$ to mult${ }_{\mathcal{G}}\left(u_{1}^{m_{1}}, u_{2}^{m_{2}}, \ldots, u_{s}^{m_{s}}\right)$. If $U_{1}, \ldots, U_{s}$ are multi-subsets of $V(\mathcal{G})$, then $\operatorname{mult}_{\mathcal{G}}\left(U_{1}, \ldots, U_{s}\right)$ means mult $\mathcal{G}\left(\bigcup_{i=1}^{s} U_{i}\right)$, where the union of $U_{i}$ s is the usual union of multisets. Whenever it is not ambiguous, we drop the subscripts; for example we write $\operatorname{deg}(v)$ and mult $(e)$ instead of $\operatorname{deg}_{\mathcal{G}}(v)$ and mult $\mathcal{G}_{\mathcal{G}}(e)$, respectively.

For $h \in \mathbb{N}, \mathcal{G}$ is said to be $h$-uniform if $|e|=h$ for each $e \in E$, and an $h$-factor in a hypergraph $\mathcal{G}$ is a spanning $h$-regular sub-hypergraph. An $h$-factorization is a partition of the edge set of $\mathcal{G}$ into $h$-factors. The hypergraph $K_{n}^{h}:=\left(V,\binom{V}{h}\right)$ with $|V|=n$ is called a complete $h$-uniform hypergraph. A $k$-edge-coloring of $\mathcal{G}$ is a mapping $f: V(\mathcal{G}) \rightarrow[k]$ and color class $i$ of $\mathcal{G}$, written $\mathcal{G}(i)$, is the sub-hypergraph of $\mathcal{G}$ induced by the edges of color $i$.

Let $\mathcal{G}$ be a hypergraph, let $U$ be some finite set, and let $\Psi: V(\mathcal{G}) \rightarrow U$ be a surjective mapping. The map $\Psi$ extends naturally to $E(\mathcal{G})$. For $A \in E(\mathcal{G})$ we define $\Psi(A)=\{\Psi(x): x \in A\}$. Note that $\Psi$ need not be injective, and $A$ may be a multiset.

Then we define the hypergraph $\mathcal{F}$ by taking $V(\mathcal{F})=U$ and $E(\mathcal{F})=\{\Psi(A): A \in$ $E(\mathcal{G})\}$. We say that $\mathcal{F}$ is an amalgamation of $\mathcal{G}$, and that $\mathcal{G}$ is a detachment of $\mathcal{F}$. Associated with $\Psi$ is a (number) function $g$ defined by $g(u)=\left|\Psi^{-1}(u)\right|$; to be more specific we will say that $\mathcal{G}$ is a $g$-detachment of $\mathcal{F}$. Then $\mathcal{G}$ has $\sum_{u \in V(\mathcal{F})} g(u)$ vertices. Note that $\Psi$ induces a bijection between the edges of $\mathcal{F}$ and the edges of $\mathcal{G}$, and that this bijection preserves the size of an edge. We adopt the convention that it preserves the color also, so that if we amalgamate or detach an edge-colored hypergraph the amalgamation or detachment preserves the same coloring on the edges. We make explicit a straightforward observation: Given $\mathcal{G}, V(\mathcal{F})$ and $\Psi$ the amalgamation is uniquely determined, but given $\mathcal{F}, V(\mathcal{G})$ and $\Psi$ the detachment is in general far from uniquely determined.

There are quite a lot of other papers on amalgamations and some highlights include $[9,11,12,13,14,15,17,18]$.

Given an edge-colored hypergraph $\mathcal{F}$, we are interested in finding a detachment $\mathcal{G}$ obtained by splitting each vertex of $\mathcal{F}$ into a prescribed number of vertices in $\mathcal{G}$ so that (i) the degree of each vertex in each color class of $\mathcal{F}$ is shared evenly among the subvertices in the same color class in $\mathcal{G}$, and (ii) the multiplicity of each edge in $\mathcal{F}$ is shared evenly among the subvertices in $\mathcal{G}$. The following theorem, which is a special case of a general result in [1], guarantees the existence of such detachment (Here $x \approx y$ means $\lfloor y\rfloor \leqslant x \leqslant\lceil y\rceil$ ).

Theorem 2.1. (Bahmanian [1, Theorem 4.1]) Let $\mathcal{F}$ be a $k$-edge-colored hypergraph and let $g: V(\mathcal{F}) \rightarrow \mathbb{N}$. Then there exists a $g$-detachment $\mathcal{G}$ (possibly with multiple edges) of $\mathcal{F}$ whose edges are all sets, with amalgamation function $\Psi: V(\mathcal{G}) \rightarrow V(\mathcal{F})$, $g$ being the number function associated with $\Psi$, such that
(F1) for each $u \in V(\mathcal{F})$, each $v \in \Psi^{-1}(u)$ and $i \in[k]$,

$$
\operatorname{deg}_{\mathcal{G}(i)}(v) \approx \frac{\operatorname{deg}_{\mathcal{F}(i)}(u)}{g(u)}
$$

(F2) for distinct $u_{1}, \ldots, u_{s} \in V(\mathcal{F})$ and $U_{i} \subseteq \Psi^{-1}\left(u_{i}\right)$ with $\left|U_{i}\right|=m_{i} \leqslant g\left(u_{i}\right)$ for $i \in[s]$,

$$
\operatorname{mult}_{\mathcal{G}}\left(U_{1}, \ldots, U_{s}\right) \approx \frac{\operatorname{mult}_{\mathcal{F}}\left(u_{1}^{m_{1}}, \ldots, u_{s}^{m_{s}}\right)}{\prod_{i=1}^{s}\binom{g\left(u_{i}\right)}{m_{i}}}
$$

Let $\widetilde{K_{m}^{h}}$ be the hypergraph obtained by adding a new vertex $u$ and new edges to $K_{m}^{h}$ so that

$$
\operatorname{mult}\left(u^{i}, W\right)=\binom{n-m}{i} \text { for each } i \in[h] \text {, and } W \subseteq V\left(K_{m}^{h}\right) \text { with }|W|=h-i
$$

In other words, $\widetilde{K_{m}^{h}}$ is an amalgamation of $K_{n}^{h}$, obtained by identifying an arbitrary set of $n-m$ vertices in $K_{n}^{h}$.

An immediate consequence of Theorem 2.1 is the following.
Corollary 2.2. Let $k:=\binom{n-1}{h-1} / r \in \mathbb{N}$. A partial $r$-factorization of $K_{m}^{h}$ can be extended to an r-factorization of $K_{n}^{h}$ if and only if the new edges of $\mathcal{F}:=\widetilde{K_{m}^{h}}$ can be colored so
that

$$
\forall i \in[k] \quad \operatorname{deg}_{\mathcal{F}(i)}(v)= \begin{cases}r & \text { if } v \neq u  \tag{1}\\ r(n-m) & \text { if } v=u\end{cases}
$$

Proof. First, suppose that a partial $r$-factorization of $K_{m}^{h}$ can be extended to an $r$ factorization of $K_{n}^{h}$. By amalgamating the new $n-m$ vertices of $K_{n}^{h}$ into a single vertex $u$, we clearly obtain $\mathcal{F}$. The $k$-edge-coloring of $K_{n}^{h}$ (in which each color class is an $r$-factor) induces a $k$-edge-coloring in $\mathcal{F}$ that satisfies (1).

Conversely, suppose that the edges of $\mathcal{F}$ are colored so that (1) is satisfied. Let $g: V(\mathcal{F}) \rightarrow \mathbb{N}$ with $g(u)=n-m$, and $g(v)=1$ for $v \neq u$. By Theorem 2.1, there exists a $g$-detachment $\mathcal{G}$ of $\mathcal{F}$ such that
(a) for each $v \in \Psi^{-1}(u)$, and $i \in[k]$

$$
\operatorname{deg}_{\mathcal{G}(i)}(v) \approx \operatorname{deg}_{\mathcal{F}(i)}(u) / g(u)=r(n-m) /(n-m)=r
$$

(b) for $U \subseteq \Psi^{-1}(u), W \subseteq V\left(K_{m}^{h}\right)$ with $|U|=i,|W|=h-i$, for $i \in[h]$.

$$
\operatorname{mult}_{\mathcal{G}}(U, W) \approx \frac{\operatorname{mult}_{\mathcal{F}}\left(u^{i}, W\right)}{\binom{g(u)}{i}}=\frac{\binom{n-m}{i}}{\binom{n-m}{i}}=1
$$

By (a), each color class is an $r$-factor, and by (b), $\mathcal{G} \cong K_{n}^{h}$.
The following observation will be quite useful throughout the paper.
Proposition 2.3. For every $n, m, h \in \mathbb{N}$ with $n \geqslant m \geqslant h$,

$$
\begin{gather*}
\binom{n}{h}=\sum_{i=0}^{h}\binom{m}{i}\binom{n-m}{h-i}  \tag{2}\\
m\left[\binom{n-1}{h-1}-\binom{m-1}{h-1}\right]=\sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i} . \tag{3}
\end{gather*}
$$

Proof. The proof of (2) is straightforward. Let $\mathcal{F}$ be a hypergraph with vertex set $\{u, v\}$ such that $\operatorname{mult}\left(u^{i}, v^{h-i}\right)=\binom{m}{i}\binom{n-m}{h-i}$ for $0 \leqslant i \leqslant h-1$. Note that $\mathcal{F}$ is an amalgamation of the hypergraph $\mathcal{G}$ with edge set $\binom{X}{h} \backslash\binom{U}{h}$ where $|X|=n,|U|=m$. Double counting the degree of $u$ proves (3):

$$
\sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i}=\operatorname{deg}_{\mathcal{F}}(u)=\sum_{u \in U} d_{\mathcal{G}}(u)=m\left[\binom{n-1}{h-1}-\binom{m-1}{h-1}\right]
$$

In order to avoid trivial cases, throughout the rest of this paper we assume that $m>h$.

## 3. Arbitrary $h$

If we replace every edge $e$ of a hypergraph $\mathcal{G}$ by $\lambda$ copies of $e$, then we denote the new hypergraph by $\lambda \mathcal{G}$. For hypergraphs $\mathcal{G}_{1}, \ldots, \mathcal{G}_{t}$ with the same vertex set $V$, we define their union, written $\bigcup_{i=1}^{t} \mathcal{G}_{i}$, to be the hypergraph with vertex set $V$ and edge set $\bigcup_{i=1}^{t} E\left(\mathcal{G}_{i}\right)$. For a hypergraph $\mathcal{G}$ and $V \subseteq V(\mathcal{G})$, let $\mathcal{G}-V$ be the hypergraph whose vertex set is $V(\mathcal{G}) \backslash V$ and whose edge set is $\{e \backslash V \mid e \in E(\mathcal{G})\}$.

Let $V$ be an arbitrary subset of vertices in $K_{n}^{h}$ with $|V|=m \leqslant n$. Then $K_{n}^{h}-V \cong$ $\bigcup_{i=0}^{h-1}\binom{m}{i} K_{n-m}^{h-i}$. A partial r-factorization of $\mathcal{H}:=K_{n}^{h}-V$ is a coloring of the edges of $K_{n}^{h}-V$ with at most $\binom{n-1}{h-1} / r$ colors so that for each color $i, \operatorname{deg}_{\mathcal{H}(i)}(v) \leqslant r$ for each vertex of $\mathcal{H}$ (Note that $\mathcal{H}$ has singleton edges). In the next result, we completely settle the problem of extending a partial $r$-factorization of $K_{n}^{h}-V$ to an $r$-factorization of $K_{n}^{h}$. Note that here we are not only extending the coloring, but also the edges of size less than $h$ to edges of size $h$. The case $h=3$ was solved in [4].

Theorem 3.1. For $V \subseteq V\left(K_{n}^{h}\right)$ with $|V|=m$, any partial $r$-factorization of $\mathcal{H}:=$ $K_{n}^{h}-V$ can be extended to an $r$-factorization of $K_{n}^{h}$ if and only if $\left.h|r n, r| \begin{array}{c}n-1 \\ h-1\end{array}\right)$, and for all $i=1,2, \ldots,\binom{n-1}{h-1} / r$,

$$
\begin{gather*}
d_{\mathcal{H}(i)}(v)=r \quad \forall v \in V(\mathcal{H}),  \tag{4}\\
|E(\mathcal{H}(i))| \leqslant \frac{r n}{h} \tag{5}
\end{gather*}
$$

Proof. To prove the necessity, suppose that a given partial $r$-factorization of $\mathcal{H}$ is extended to an $r$-factorization of $K_{n}^{h}$. For $K_{n}^{h}$ to be $r$-factorable, the two divisibility conditions are clearly necessary. By extending an edge $e$ of size $i(i<h)$ in $\mathcal{H}$ to an edge of size $h$ in $K_{n}^{h}$, the color of $e$ does not change, and so (4) is necessary. Since the number of edges in each color class of $K_{n}^{h}$ is exactly $r n / h$, the necessity of (5) is implied.

To prove the sufficiency, suppose that a partial $r$-factorization of $\mathcal{H}$ is given, $h \mid r n$, $r \left\lvert\,\binom{ n-1}{h-1}\right.$, and that (4), (5) are satisfied. Let $k=\binom{n-1}{h-1}$, and let $\mathcal{F}=\widetilde{K_{n-m}^{h}}$. For $0 \leqslant i \leqslant h$, an edge of type $u^{i}$ in $\mathcal{F}$ is an edge in $\mathcal{F}$ containing $u^{i}$ but not containing $u^{i+1}$. Note that there are $\binom{m}{i}\binom{n-m}{h-i}$ edges of type $u^{i}$ in $\mathcal{F}$.

There is a clear one-to-one correspondence between the edges of size $i$ in $\mathcal{H}$ and the edges of type $u^{h-i}$ in $\mathcal{F}$ for each $i \in[h]$. We color the edges of type $u^{i}$ in $\mathcal{F}$ with the same color as the corresponding edge in $\mathcal{H}$ for $0 \leqslant i \leqslant h-1$. By Corollary 2.2, if we can color the remaining edges of $\mathcal{F}$ (edges of type $u^{h}$ ) so that the following condition is satisfied, then we are done.

$$
\forall i \in[k] \quad \operatorname{deg}_{\mathcal{F}(i)}(v)= \begin{cases}r & \text { if } v \neq u  \tag{6}\\ r m & \text { if } v=u\end{cases}
$$

Let $\operatorname{mult}_{i}\left(u^{j},.\right)$ be the number of edges of type $u^{j}$ in $\mathcal{F}(i)$, for $i \in[k], j \in[h]$. Note that $\operatorname{mult}_{i}\left(u^{h},.\right)=\operatorname{mult}_{\mathcal{F}(i)}\left(u^{h}\right)$ for $i \in[k]$. We color the edges of type $u^{h}$ so that for
$i \in[k]$,

$$
\operatorname{mult}_{i}\left(u^{h}, .\right)=\frac{r n}{h}-r(n-m)+\sum_{j=1}^{h-1} j \operatorname{mult}_{i}\left(u^{h-j-1}, .\right)
$$

Since $h \mid r n, \operatorname{mult}_{i}\left(u^{h},.\right)$ is an integer for $i \in[k]$. The following shows that mult ${ }_{i}\left(u^{h},.\right) \geqslant$ 0 for $i \in[k]$.

$$
\begin{aligned}
\frac{r n}{h} & \stackrel{(5)}{\geqslant}|E(\mathcal{H}(i))|=\sum_{j=0}^{h-1} \operatorname{mult}_{i}\left(u^{j}, .\right) \\
& =\sum_{j=1}^{h} j \operatorname{mult}_{i}\left(u^{h-j}, .\right)-\sum_{j=1}^{h-1} j \operatorname{mult}_{i}\left(u^{h-j-1}, .\right) \\
& \stackrel{(4)}{=} r(n-m)-\sum_{j=1}^{h-1} j \operatorname{mult}_{i}\left(u^{h-j-1}, .\right)
\end{aligned}
$$

Now we show that all edges of the type $u^{h}$ will be colored, or equivalently that, $\sum_{i=1}^{k} \operatorname{mult}_{i}\left(u^{h},.\right)=\binom{m}{h}$.

$$
\left.\begin{array}{rl}
\sum_{i=1}^{k} \operatorname{mult}_{i}\left(u^{h}, .\right) & =\sum_{i=1}^{k}\left(\frac{r n}{h}-r(n-m)+\sum_{j=1}^{h-1} j \operatorname{mult}_{i}\left(u^{h-j-1}, .\right)\right) \\
& =\frac{r k n}{h}-r k(n-m)+\sum_{j=1}^{h-1} j \sum_{i=1}^{k} \operatorname{mult}_{i}\left(u^{h-j-1}, .\right) \\
& =\binom{n}{h}-(n-m)\binom{n-1}{h-1}+\sum_{j=2}^{h}(j-1)\binom{m}{h-j}\binom{n-m}{j} \\
& \stackrel{(2),(3)}{=} \sum_{j=0}^{h}\binom{m}{j}\binom{n-m}{h-j}-\sum_{j=1}^{h-1} j\binom{n-m}{j}\binom{m}{h-j} \\
& -\left(\begin{array}{c}
n-m)\binom{n-m-1}{h-1}+\sum_{j=2}^{h}(j-1)\binom{m}{h-j}\binom{n-m}{j} \\
\\
\end{array}\right. \\
& =\binom{m}{h}-(n-m)\binom{n-m-1}{h-1}+h\binom{n-m}{h} \\
h
\end{array}\right) .
$$

To complete the proof, we show that $\operatorname{deg}_{\mathcal{F}(i)}(u)=r m$ for $i \in[k]$. We have

$$
\begin{aligned}
\operatorname{deg}_{\mathcal{F}(i)}(u) & =\sum_{j=1}^{h} j \operatorname{mult}_{i}\left(u^{j}, .\right)=h \operatorname{mult}_{i}\left(u^{h}, .\right)+\sum_{j=1}^{h}(h-j) \operatorname{mult}_{i}\left(u^{h-j}, .\right) \\
& =h \operatorname{mult}_{i}\left(u^{h}, .\right)+h \sum_{j=1}^{h} \operatorname{mult}_{i}\left(u^{h-j}, .\right)-\sum_{j=1}^{h} j \operatorname{mult}_{i}\left(u^{h-j}, .\right) \\
& =h \sum_{j=0}^{h} \operatorname{mult}_{i}\left(u^{h-j}, .\right)-\sum_{j=1}^{h} j \operatorname{mult}_{i}\left(u^{h-j}, .\right) \\
& =r n-r(n-m)=r m .
\end{aligned}
$$

For a hypergraph $\mathcal{G}$ and $V \subseteq V(\mathcal{G})$, let $\mathcal{G} \backslash V$ be the hypergraph whose vertex set is $V(\mathcal{G})$ and whose edge set is $\{e \in E(\mathcal{G}) \mid e \nsubseteq V\}$.

Let $V \subseteq V\left(K_{n}^{h}\right)$ with $|V|=m \leqslant n$, and let $\mathcal{H}:=K_{n}^{h} \backslash V$. An edge $e \in E(\mathcal{H})$ is of type $i$, if $|e \cap V|=i$ (for $0 \leqslant i \leqslant h-1$ ). Let $P$ be a partial $r$-factorization of $\mathcal{H}$. Then a partial $r$-factorization $Q$ of $\mathcal{H}$ is said to be $P$-friendly if
(a) the color of each edge of type 0 is the same in $P$ and $Q$, and
(b) the number of edges of type $i$ and color $j$ is the same in $P$ and $Q$ for each $i \in[h-1]$ and each color $j$.
We are interested in finding the conditions under which a partial $r$-factorization of $\mathcal{H}$ can be extended to an $r$-factorization of $K_{n}^{h}$.
Lemma 3.2. For $V \subseteq V\left(K_{n}^{h}\right)$ with $|V|=m$, if a partial $r$-factorization of $\mathcal{H}:=K_{n}^{h} \backslash V$ can be extended to an $r$-factorization of $K_{n}^{h}$, then
(N1) $h \mid r n$,
(N2) $r\binom{n-1}{h-1}$,
(N3) $d_{\mathcal{H}(i)}(v)=r$ for each $v \in V(\mathcal{H}) \backslash V$, and $i \in[k]$,
(N4) $|E(\mathcal{H}(i))| \leqslant r n / h$ for $i \in[k]$,
where $k:=\binom{n-1}{h-1} / r$.
It remains an open question whether these conditions are sufficient. Here we prove a weaker result.
Corollary 3.3. Let $V \subseteq V\left(K_{n}^{h}\right)$ with $|V|=m$, and let $P$ be a partial $r$-factorization of $\mathcal{H}:=K_{n}^{h} \backslash V$, and assume that (N1)-(N4) are satisfied. Then there exists a $P$-friendly partial $r$-factorization of $\mathcal{H}$ that can be extended to an r-factorization of $K_{n}^{h}$.
Proof. By eliminating all the vertices in $V$, and shrinking the edges containing vertices in $V$, we obtain $K_{n}^{h}-V$. The rest of the proof follows from Theorem 3.1.

$$
\text { 4. } h=4
$$

Theorem 4.1. For $n \geqslant 4.847323 m$, any partial $r$-factorization of $K_{m}^{4}$ can be extended to an $r$-factorization of $K_{n}^{4}$ if and only if $4 \mid r n$ and $r \left\lvert\,\binom{ n-1}{3}\right.$.

Proof. For the necessary conditions, see the previous section. To prove the sufficiency, we need to show that the edges of $\mathcal{F}:=\widetilde{K_{m}^{4}}$ can be colored with $k:=\binom{n-1}{3} / r$ colors so that (6) is satisfied.

First we color the edges in $\mathcal{F}$ of the form $W \cup\{u\}$ where $W \subseteq V:=V\left(K_{m}^{4}\right)$ and $|W|=3$. We color these edges greedily so that $\operatorname{deg}_{i}(x) \leqslant r$ for each $x \in V$ and $i \in[k]$. We claim that this coloring can be done in such a way that all edges of this type are colored. Suppose by contrary that there is an edge in $\mathcal{F}$ of the form $\{x, y, z, u\}$ with $x, y, z \in V$ that can not be colored. This implies that for each $i \in[k]$ either $\operatorname{deg}_{i}(x)=r$ or $\operatorname{deg}_{i}(y)=r$ or $\operatorname{deg}_{i}(z)=r$. Thus for each $i \in[k], \operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z) \geqslant r$. On the one hand, $\sum_{i=1}^{k}\left(\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z)\right) \geqslant r k=\binom{n-1}{3}$, and on the other hand, $\sum_{i=1}^{k}\left(\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z)\right) \leqslant 3\left[\binom{m-1}{3}+(n-m)\binom{m-1}{2}-1\right]$. Thus, we have

$$
3\left[\binom{m-1}{3}+(n-m)\binom{m-1}{2}-1\right] \geqslant\binom{ n-1}{3} .
$$

which is equivalent to $f(n, m):=n^{3}-6 n^{2}-9 m^{2} n+27 m n-7 n+6 m^{3}-9 m^{2}-15 m+30 \leqslant$ 0 . Now, we show that since $n>4 m$ and $m \geqslant 5$, we have $f(n, m)>0$, which is a contradiction, and therefore, all edges in $\mathcal{F}$ of the form $W \cup\{u\}$ where $W \subseteq V$ and $|W|=3$ can be colored using the greedy approach described above.

First, note that for $m \geqslant 5$, both $7 m^{2}+3 m-7$ and $2 m^{2}-3 m-5$ are positive. Therefore,

$$
\begin{aligned}
f(n, m) & =n\left(n(n-6)-9 m^{2}+27 m-7\right)+3 m\left(2 m^{2}-3 m-5\right)+30 \\
& >n\left(4 m(4 m-6)-9 m^{2}+27 m-7\right)+3 m\left(2 m^{2}-3 m-5\right)+30 \\
& =n\left(7 m^{2}+3 m-7\right)+3 m\left(2 m^{2}-3 m-5\right)+30>0
\end{aligned}
$$

Now we greedily color all the edges of the form $W \cup\left\{u^{2}\right\}$ where $W \subseteq V$ and $|W|=2$, so that $\operatorname{deg}_{i}(x) \leqslant r$ for each $x \in V$ and $i \in[k]$. We show that this is possible. If by contrary, some edge $\left\{x, y, u^{2}\right\}$ with $x, y \in V$ remains uncolored, then for each $i \in[k]$ either $\operatorname{deg}_{i}(x)=r$ or $\operatorname{deg}_{i}(y)=r$, and so $\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y) \geqslant r$. We have

$$
\begin{aligned}
\binom{n-1}{3}=r k & \leqslant \sum_{i=1}^{k}\left(\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)\right) \\
& \leqslant 2\left[\binom{m-1}{3}+(n-m)\binom{m-1}{2}+(m-1)\binom{n-m}{2}-1\right]
\end{aligned}
$$

which is equivalent to $n^{3}-6 m n^{2}+6 m^{2} n+12 m n-7 n-2 m^{3}-6 m^{2}-4 m+18 \leqslant 0$. Using Mathematica (Wolfram Alpha) it can be shown that this inequality does not have any real solution under the constraints $m \geqslant 5, n \geqslant 4.847323 m$. Therefore, all edges of the form $W \cup\left\{u^{2}\right\}$ where $W \subseteq V$ and $|W|=2$, can be colored.

Since for each $x \in V$,

$$
\begin{aligned}
\sum_{i=1}^{k}\left(r-\operatorname{deg}_{i}(x)\right) & =r k-\left[\binom{m-1}{3}+(n-m)\binom{m-1}{2}+(m-1)\binom{n-m}{2}\right] \\
& =\binom{n-1}{3}-\binom{m-1}{3}-(n-m)\binom{m-1}{2}-(m-1)\binom{n-m}{2} \\
& \stackrel{(2)}{=}\binom{n-m}{3}
\end{aligned}
$$

we can color all the edges of the form $\left\{w, u^{3}\right\}$ where $w \in V$ so that for each $x \in V$, there are $r-\operatorname{deg}_{i}(x)$ edges of this type colored $i$ incident with $x$ for each $i \in[k]$. Note that after this coloring,

$$
\begin{equation*}
\operatorname{deg}_{i}(x)=r \text { for each } x \in V \tag{7}
\end{equation*}
$$

For $i \in[k]$, let $a_{i}, b_{i}, c_{i}, d_{i}$ be the number of edges colored $i$ of the form $W, W \cup\{u\}, W \cup$ $\left\{u^{2}\right\}, W \cup\left\{u^{3}\right\}$ where $W \subseteq V$, respectively. We color the edges of the form $\left\{u^{4}\right\}$ so that there are exactly

$$
e_{i}:=r n / 4-r m+3 a_{i}+2 b_{i}+c_{i}
$$

edges of this type colored $i$ for $i \in[k]$. Since $4 \mid r n$, and $n>4 m, e_{i}$ is a positive integer for $i \in[k]$. We claim that all edges of the form $\left\{u^{4}\right\}$ will be colored, or equivalently, $\sum_{i=1}^{k} e_{i}=\binom{n-m}{4}$.

$$
\begin{aligned}
\sum_{i=1}^{k} e_{i} & =\sum_{i=1}^{k}\left(\frac{r n}{4}-r m+3 a_{i}+2 b_{i}+c_{i}\right)=\frac{r k n}{4}-r k m+3 \sum_{i=1}^{k} a_{i}+2 \sum_{i=1}^{k} b_{i}+\sum_{i=1}^{k} c_{i} \\
& =\frac{n}{4}\binom{n-1}{3}-m\binom{n-1}{3}+3\binom{m}{4}+2(n-m)\binom{m}{3}+\binom{n-m}{2}\binom{m}{2} \\
& =\binom{n}{4}-m\binom{n-1}{3}+3\binom{m}{4}+2(n-m)\binom{m}{3}+\binom{n-m}{2}\binom{m}{2} \\
& \stackrel{(2),(3)}{=}\binom{n-m}{4} .
\end{aligned}
$$

To complete the proof, we show that $\operatorname{deg}_{i}(u)=r(n-m)$ for $i \in[k]$. First note that for $i \in[k], r m=\sum_{x \in V} \operatorname{deg}_{i}(x)=4 a_{i}+3 b_{i}+2 c_{i}+d_{i}$. Therefore,

$$
\begin{aligned}
\operatorname{deg}_{i}(u)=b_{i}+2 c_{i}+3 d_{i}+4 e_{i} & =4\left(a_{i}+b_{i}+c_{i}+d_{i}+e_{i}\right)-\left(4 a_{i}+3 b_{i}+2 c_{i}+d_{i}\right) \\
& =r n-r m=r(n-m) .
\end{aligned}
$$

Combining this with (7) implies that (6) is satisfied, and the proof is complete.

$$
\text { 5. } h=5
$$

Theorem 5.1. For $n \geqslant 6.285214 m$, any partial $r$-factorization of $K_{m}^{5}$ can be extended to an r-factorization of $K_{n}^{5}$ if and only if $5 \mid r n$ and $r \left\lvert\,\binom{ n-1}{4}\right.$.

Proof. The necessity is obvious. To prove the sufficiency, we need to show that the edges of $\mathcal{F}:=\widetilde{K_{m}^{5}}$ can be colored with $k:=\binom{n-1}{4} / r$ colors so that (6) is satisfied.

First we color the edges of the form $W \cup\{u\}$ where $W \subseteq V$ and $|W|=4$. We color these edges greedily so that $\operatorname{deg}_{i}(x) \leqslant r$ for each $x \in V$ and $i \in[k]$. We claim that this coloring can be done in such a way that all edges of this type are colored. Suppose by contrary that there is an edge of the form $\{x, y, z, w, u\}$ with $x, y, z, w \in V$ that can not be colored. This implies that for each $i \in[k]$ either $\operatorname{deg}_{i}(x)=r$ or $\operatorname{deg}_{i}(y)=r$ or $\operatorname{deg}_{i}(z)=r$ or $\operatorname{deg}_{i}(w)=r$. Thus for each $i \in[k], \operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z)+\operatorname{deg}_{i}(w) \geqslant$ $r$. On the one hand, $\sum_{i=1}^{k}\left(\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z)+\operatorname{deg}_{i}(w)\right) \geqslant r k=\binom{n-1}{4}$, and on the other hand, $\sum_{i=1}^{k}\left(\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z)+\operatorname{deg}_{i}(w)\right) \leqslant 4\left[\binom{m-1}{4}+(n-\right.$ $\left.m)\binom{m-1}{3}-1\right]$. Thus, we have

$$
4\left[\binom{m-1}{4}+(n-m)\binom{m-1}{3}-1\right] \geqslant\binom{ n-1}{4}
$$

which is equivalent to $g_{1}(n, m):=n^{4}-10 n^{3}+35 n^{2}-16 m^{3} n+96 m^{2} n-176 m n+46 n+$ $12 m^{4}-56 m^{3}+36 m^{2}+104 m+24 \leqslant 0$.

Since $n>6 m$ and $m \geqslant 6$, we have

$$
\begin{aligned}
g_{1}(n, m) & :=n\left(n^{2}(n-10)-16 m^{3}+96 m^{2}+(35 n-176 m)+46\right) \\
& +4 m\left(m^{2}(3 m-14)+9 m+26\right)+24 \\
& >9 m^{2}(3 m-10)-16 m^{3}+96 m^{2} \\
& =11 m^{3}+6 m^{2}>0
\end{aligned}
$$

which is a contradiction, and therefore, all edges in $\mathcal{F}$ of the form $W \cup\{u\}$ where $W \subseteq V$ and $|W|=4$ can be colored.

Now we greedily color all the edges of the form $W \cup\left\{u^{2}\right\}$ where $W \subseteq V$ and $|W|=3$, so that $\operatorname{deg}_{i}(x) \leqslant r$ for each $x \in V$ and $i \in[k]$. We show that this is possible. If by contrary, some edge $\left\{x, y, z, u^{2}\right\}$ with $x, y, z \in V$ remains uncolored, then for each $i \in[k]$ either $\operatorname{deg}_{i}(x)=r$ or $\operatorname{deg}_{i}(y)=r$ or $\operatorname{deg}_{i}(z)=r$, and so $\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z) \geqslant r$. We have

$$
\begin{aligned}
\binom{n-1}{4}=r k & \leqslant \sum_{i=1}^{k}\left(\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)+\operatorname{deg}_{i}(z)\right) \\
& \leqslant 3\left[\binom{m-1}{4}+(n-m)\binom{m-1}{3}+\binom{m-1}{2}\binom{n-m}{2}-1\right]
\end{aligned}
$$

which is equivalent to $g_{2}(n, m):=n^{4}-10 n^{3}-18 m^{2} n^{2}+54 m n^{2}-n^{2}+24 m^{3} n-18 m^{2} n-$ $114 m n+58 n-9 m^{4}-6 m^{3}+45 m^{2}+42 m+24 \leqslant 0$. We show that since $n>6 m$ and $m \geqslant 6$, we have $g_{2}(n, m)>0$, which is a contradiction, and therefore, all edges in $\mathcal{F}$ of the form $W \cup\left\{u^{2}\right\}$ where $W \subseteq V$ and $|W|=3$ can be colored.

First, note that for $m \geqslant 6$ we have $12 m^{3}-9 m^{2}-57 m+29>0$. Therefore,

$$
\begin{aligned}
g_{2}(n, m) & =n^{2}\left(n(n-10)-18 m^{2}+54 m-1\right) \\
& +2 n\left(12 m^{3}-9 m^{2}-57 m+29\right) \\
& -\left(9 m^{4}+6 m^{3}-45 m^{2}-42 m-24\right) \\
& >36 m^{2}\left(6 m(6 m-10)-18 m^{2}+54 m-1\right) \\
& -\left(9 m^{4}+6 m^{3}-45 m^{2}-42 m-24\right) \\
& =639 m^{4}-150 m^{3}+18 m^{2}+42 m+24>0 .
\end{aligned}
$$

Now we greedily color all the edges of the form $W \cup\left\{u^{3}\right\}$ where $W \subseteq V$ and $|W|=2$, so that $\operatorname{deg}_{i}(x) \leqslant r$ for each $x \in V$ and $i \in[k]$. We show that this is possible. If by contrary, some edge $\left\{x, y, u^{2}\right\}$ with $x, y \in V$ remains uncolored, then for each $i \in[k]$ either $\operatorname{deg}_{i}(x)=r$ or $\operatorname{deg}_{i}(y)=r$, and so $\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y) \geqslant r$. We have

$$
\begin{aligned}
\binom{n-1}{4} & \leqslant \sum_{i=1}^{k}\left(\operatorname{deg}_{i}(x)+\operatorname{deg}_{i}(y)\right) \\
& \leqslant 2\left[\binom{m-1}{4}+(n-m)\binom{m-1}{3}+\binom{m-1}{2}\binom{n-m}{2}+(m-1)\binom{n-m}{3}-1\right]
\end{aligned}
$$

Using Mathematica it can be shown that this inequality does not have any real solution under the constraints $m \geqslant 6, n \geqslant 6.285214 m$. Therefore, all edges of the form $W \cup\left\{u^{3}\right\}$ where $W \subseteq V$ and $|W|=2$, can be colored.

Since for each $x \in V$,

$$
\begin{aligned}
\sum_{i=1}^{k}\left(r-\operatorname{deg}_{i}(x)\right) & =\binom{n-1}{4}-\binom{m-1}{4}-(n-m)\binom{m-1}{3} \\
& -\binom{m-1}{2}\binom{n-m}{2}-(m-1)\binom{n-m}{3} \\
& =\binom{n-m}{4}
\end{aligned}
$$

we can color all the edges of the form $\left\{w, u^{4}\right\}$ where $w \in V$ so that for each $x \in V$, there are $r-\operatorname{deg}_{i}(x)$ edges of this type colored $i$ incident with $x$ for each $i \in[k]$.

For $i \in[k]$, let $a_{i}, b_{i}, c_{i}, d_{i}, e_{i}$ be the number of edges colored $i$ of the form $W, W \cup$ $\{u\}, W \cup\left\{u^{2}\right\}, W \cup\left\{u^{3}\right\}, W \cup\left\{u^{4}\right\}$ where $W \subseteq V$, respectively. We color the edges of the form $\left\{u^{5}\right\}$ so that there exactly

$$
f_{i}:=r n / 5-r m+4 a_{i}+3 b_{i}+2 c_{i}+d_{i}
$$

edges of this type colored $i$ for $i \in[k]$. Since $5 \mid r n$, and $n \geqslant 6.4 m>5 m$, $e_{i}$ is a positive integer for $i \in[k]$. We claim that all edges of the form $\left\{u^{5}\right\}$ will be colored, or equivalently, $\sum_{i=1}^{k} f_{i}=\binom{n-m}{5}$.

$$
\begin{aligned}
\sum_{i=1}^{k} f_{i} & =\sum_{i=1}^{k}\left(\frac{r n}{5}-r m+4 a_{i}+3 b_{i}+2 c_{i}+d_{i}\right) \\
& =\frac{r k n}{5}-r k m+4 \sum_{i=1}^{k} a_{i}+3 \sum_{i=1}^{k} b_{i}+2 \sum_{i=1}^{k} c_{i}+\sum_{i=1}^{k} d_{i} \\
& =\binom{n}{5}-m\binom{n-1}{4}+4\binom{m}{5}+3(n-m)\binom{m}{4}+2\binom{n-m}{2}\binom{m}{3}+\binom{n-m}{3}\binom{m}{2} \\
& =\binom{n-m}{5}
\end{aligned}
$$

To complete the proof, we show that $\operatorname{deg}_{i}(u)=r(n-m)$ for $i \in[k]$. First note that for $i \in[k], r m=\sum_{x \in V} \operatorname{deg}_{i}(x)=5 a_{i}+4 b_{i}+3 c_{i}+2 d_{i}+e_{i}$. Therefore,

$$
\begin{aligned}
\operatorname{deg}_{i}(u) & =b_{i}+2 c_{i}+3 d_{i}+4 e_{i}+5 f_{i} \\
& =5\left(a_{i}+b_{i}+c_{i}+d_{i}+e_{i}+f_{i}\right)-\left(5 a_{i}+4 b_{i}+3 c_{i}+2 d_{i}+e_{i}\right) \\
& =r n-r m=r(n-m)
\end{aligned}
$$

## 6. Concluding Remarks and Open Problems

(1) At this point, it is not clear to use how to extend the results of Sections 4 and 5 without dealing with heavy computation. We believe that for $n \geqslant 2 h m$, any partial $r$-factorization of $K_{m}^{h}$ can be extended to an $r$-factorization of $K_{n}^{h}$ if and only if the obvious necessary divisibility conditions are satisfied.
(2) To embed a partial $r$-factorization of $K_{n} \backslash K_{m}^{h}$ into an $r$-factorization of $K_{n}^{h}$, we believe that the conditions (N1)-(N4) of Lemma 3.2 are sufficient, but we do not know how to go beyond Corollary 3.3.
(3) A partial $r$-factorization $S \subseteq K_{n}^{h}$ is critical if it can be extended to exactly one $r$-factorization of $K_{n}^{h}$, but removal of any element of $S$ destroys the uniqueness of the extension, and $|S|$ is the size of the critical partial $r$-factorization. It is desirable to find good bounds for the smallest and largest sizes of critical partial $r$-factorizations.
(4) Another interesting problem is finding conditions under which a partial $r$ factorization of $S \subseteq\binom{[n]}{h}$ can be extended to a cyclic $r$-factorization of $\binom{[n]}{h}$.

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