# EXTENDING EDGE-COLORINGS OF COMPLETE HYPERGRAPHS INTO REGULAR COLORINGS

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ABSTRACT. Let  $\binom{X}{h}$  be the collection of all *h*-subsets of an *n*-set  $X \supseteq Y$ . Given a coloring (partition) of a set  $S \subseteq \binom{X}{h}$ , we are interested in finding conditions under which this coloring is extendible to a coloring of  $\binom{X}{h}$  so that the number of times each element of X appears in each color class (all sets of the same color) is the same number r. The case  $S = \emptyset, r = 1$  was studied by Sylvester in the 18th century, and remained open until the 1970s. The case h = 2, r = 1 is extensively studied in the literature and is closely related to completing partial symmetric Latin squares.

For  $S = \binom{Y}{h}$ , we settle the cases  $h = 4, |X| \ge 4.847323|Y|$ , and  $h = 5, |X| \ge 6.285214|Y|$  completely. Moreover, we make partial progress toward solving the case where  $S = \binom{X}{h} \setminus \binom{Y}{h}$ . These results can be seen as extensions of the famous Baranyai's theorem, and make progress toward settling a 40-year-old problem posed by Cameron.

### 1. INTRODUCTION

Suppose that we have been entrusted to color (or partition) the collection  $\binom{[n]}{h}$  of all *h*-subsets of the *n*-set  $[n] := \{1, \ldots, n\}$  so that the number of times each element of [n] appears in each color class (all sets of the same color) is exactly *r*. Such a coloring is called an *r*-factorization of  $\binom{[n]}{h}$ . A solution for the case n = 6, h = 3, r = 1 with 10 colors is given below.

Note that the number of times each element of [n] appears in  $\binom{[n]}{h}$  is  $\binom{n-1}{h-1}$ . Thus, for  $\binom{[n]}{h}$  to be *r*-factorable, it is clear that (i) *r* must divide  $\binom{n-1}{h-1}$ . In addition, a simple double counting argument shows that (ii) *h* must divide *rn*. One may wonder if conditions (i) and (ii) are also sufficient for  $\binom{[n]}{h}$  to be *r*-factorable. In the 18th century, Sylvester considered the case r = 1 of this problem which remained open until the 1970s when Baranyai solved this 120-year-old problem completely [5]. In fact, Baranyai proved a far more general result which, in particular, implies that  $\binom{[n]}{h}$  is *r*-factorable if and only if h|rn and  $r|\binom{n-1}{h-1}$ .

We are interested in a Sudoku-type version of Baranyai's theorem. A partial rfactorization of a set  $S \subseteq {\binom{[n]}{h}}$  is a coloring of S with at most  ${\binom{n-1}{h-1}}/r$  colors so that

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the number of times each element of [n] appears in each color class is at most r. Note that a color class may be empty.

**Problem 1.** Under what conditions can a partial r-factorization of  $S \subseteq {\binom{[n]}{h}}$  be extended to an r-factorization of  ${\binom{[n]}{h}}$ ?

We are given a coloring of a subset  $S \subseteq {\binom{[n]}{h}}$ , and our task is to complete the coloring. In other words, we need to color  $T := {\binom{[n]}{h}} S$  so that the coloring of  $S \cup T$  provides an *r*-factorization of  ${\binom{[n]}{h}}$ . Baranyai's theorem settles the case when  $S = \emptyset$ . A partial 4-factorization of  ${\binom{[9]}{h}}$  is given below (Here we abbreviate a set  $\{a, b, c\}$  to abc).

 $\begin{array}{c} 156,\, 248,\, 379,\, 126,\, 348,\, 579,\, 127,\, 349,\, 568,\, 124,\, 389,\, 567\\ 148,\, 267,\, 359,\, 168,\, 279,\, 345,\, 159,\, 278,\, 346,\, 134,\, 259\\ 128,\, 347,\, 569,\, 178,\, 249,\, 356,\, 169,\, 247,\, 358,\, 123\\ 146,\, 239,\, 578,\, 137,\, 289,\, 456,\, 136,\, 257\\ 129,\, 367,\, 458,\, 125,\, 368,\, 479,\, 147,\, 258,\, 369,\, 157\\ 189,\, 246,\, 357,\, 158,\, 237,\, 469,\, 138,\, 245,\, 679,\, 139,\, 268\\ 145,\, 236,\, 789,\, 167,\, 238,\, 459,\, 149,\, 256,\, 378,\, 135,\, 269,\, 478\\ \end{array}$ 

It is not too difficult to extend this to the following 4-factorization.

 $156, 248, 379, 126, 348, 579, 127, 349, 568, 124, 389, 567 \\148, 267, 359, 168, 279, 345, 159, 278, 346, 134, 259, 678 \\128, 347, 569, 178, 249, 356, 169, 247, 358, 123, 467, 589 \\146, 239, 578, 137, 289, 456, 136, 257, 489, 179, 235, 468 \\129, 367, 458, 125, 368, 479, 147, 258, 369, 157, 234, 689 \\189, 246, 357, 158, 237, 469, 138, 245, 679, 139, 268, 457 \\145, 236, 789, 167, 238, 459, 149, 256, 378, 135, 269, 478 \\$ 

The case h = 2, r = 1 of Problem 1 is closely related to completing partial Latin squares, (see Lindner's excellent survey [16]). A special case of Problem 1 when r = 1, and the partial factorization is a 1-factorization of  $\binom{[m]}{h}$  for some m < n, was studied by Cruse (for h = 2) [8], Cameron [7], and Baranyai and Brouwer [6]. Baranyai and Brouwer conjectured that a 1-factorization of  $\binom{[m]}{h}$  can be extended to a 1-factorization of  $\binom{[n]}{h}$  if and only if  $n \ge 2m$  and h divides m, n. Häggkvist and Hellgren [10] gave a beautiful proof of this conjecture. For further generalizations of Häggkvist-Hellgren's result, we refer the reader to two recent papers by the author and Newman [2, 3] in which extending r-factorizations of  $\binom{[m]}{h}$  to s-factorizations of  $\binom{[n]}{h}$  is studied (for  $s \ge r$ ).

At this point, it should be clear to the reader that the 1-factorization of  $\binom{[6]}{3}$  in the first example, can not be extended to a 1-factorization of  $\binom{[9]}{3}$ , but it can be extended to a 1-factorization of  $\binom{[12]}{3}$ .

Like most results in the literature, our primary focus is the case where  $S = {\binom{[m]}{h}}$  (for some m < n). However, unlike those, here we do not require the given partial

factorization to be a factorization itself. In this case, Problem 1 was settled by Rodger and Wantland over 20 years ago for h = 2 [18], and recently by the author and Rodger for  $h = 3, n \ge 3.414214m$  [4]. In this paper, we settle the cases  $h = 4, n \ge 4.847323m$ and  $h = 5, n \ge 6.285214m$ . The major obstacle from h = 2 to  $h \ge 3$  stems from the natural difficulty of generalizing a graph theoretic result to hypergraphs.

Note that, in order to extend a partial *r*-factorization of  $\binom{[m]}{h}$  to an *r*-factorization of  $\binom{[n]}{h}$  (for  $n \ge m$ ), it is clearly necessary that  $r | \binom{n-1}{h-1}$ , h | rn. Let  $\chi(m, h, r)$  be the smallest *n* such that any partial *r*-factorization of  $\binom{[m]}{h}$  satisfying  $r | \binom{n-1}{h-1}$ , h | rn can be extended to an *r*-factorization of  $\binom{[n]}{h}$ . Combining the results of this paper with those of [2, 3, 4], it can be easily shown that  $2m \le \chi(m, 3, r) \le 3.414214m, 2m \le \chi(m, 4, r) \le 4.847323m$ , and  $2m \le \chi(m, 5, r) \le 6.285214m$ .

Last but not least, we shall consider Problem 1 in the case when  $S = {\binom{[n]}{h}} \setminus {\binom{[m]}{h}}$ . In this direction, we solve a variation of the problem when we allow sets of size less than h, and in our extension of the coloring we also extend the sets of size less than h to sets of size h.

The paper is self-contained and all the preliminaries are given in Section 2. In section 3, we shall consider Problem 1 in the case when  $S = {\binom{[n]}{h}} \setminus {\binom{[m]}{h}}$ . The cases h = 4, 5 are discussed in detail in Sections 4, 5, respectively. We conclude the paper with some open problems.

### 2. NOTATION AND TOOLS

A hypergraph  $\mathcal{G}$  is a pair  $(V(\mathcal{G}), E(\mathcal{G}))$  where  $V(\mathcal{G})$  is a finite set called the *vertex* set,  $E(\mathcal{G})$  is the *edge* multiset, where every edge is itself a multi-subset of  $V(\mathcal{G})$ . This means that not only can an edge occur multiple times in  $E(\mathcal{G})$ , but also each vertex can have multiple occurrences within an edge. By an edge of the form  $\{u_1^{m_1}, u_2^{m_2}, \ldots, u_s^{m_s}\}$ , we mean an edge in which vertex  $u_i$  occurs  $m_i$  times for  $1 \leq i \leq r$ . The total number of occurrences of a vertex v among all edges of  $E(\mathcal{G})$  is called the *degree*,  $\deg_{\mathcal{G}}(v)$  of v in  $\mathcal{G}$ . The *multiplicity* of an edge e in  $\mathcal{G}$ , written  $\operatorname{mult}_{\mathcal{G}}(e)$ , is the number of repetitions of e in  $E(\mathcal{G})$  (note that  $E(\mathcal{G})$  is a multiset, so an edge may appear multiple times). If  $\{u_1^{m_1}, u_2^{m_2}, \ldots, u_s^{m_s}\}$  is an edge in  $\mathcal{G}$ , then we abbreviate  $\operatorname{mult}_{\mathcal{G}}(\{u_1^{m_1}, u_2^{m_2}, \ldots, u_s^{m_s}\})$  to  $\operatorname{mult}_{\mathcal{G}}(u_1^{m_1}, u_2^{m_2}, \ldots, u_s^{m_s})$ . If  $U_1, \ldots, U_s$  are multi-subsets of  $V(\mathcal{G})$ , then  $\operatorname{mult}_{\mathcal{G}}(U_1, \ldots, U_s)$  means  $\operatorname{mult}_{\mathcal{G}}(\bigcup_{i=1}^s U_i)$ , where the union of  $U_i$ s is the usual union of multisets. Whenever it is not ambiguous, we drop the subscripts; for example we write  $\deg(v)$  and  $\operatorname{mult}(e)$  instead of  $\deg_{\mathcal{G}}(v)$  and  $\operatorname{mult}_{\mathcal{G}}(e)$ , respectively.

For  $h \in \mathbb{N}$ ,  $\mathcal{G}$  is said to be *h*-uniform if |e| = h for each  $e \in E$ , and an *h*-factor in a hypergraph  $\mathcal{G}$  is a spanning *h*-regular sub-hypergraph. An *h*-factorization is a partition of the edge set of  $\mathcal{G}$  into *h*-factors. The hypergraph  $K_n^h := (V, \binom{V}{h})$  with |V| = n is called a *complete h*-uniform hypergraph. A *k*-edge-coloring of  $\mathcal{G}$  is a mapping  $f : V(\mathcal{G}) \to [k]$ and color class *i* of  $\mathcal{G}$ , written  $\mathcal{G}(i)$ , is the sub-hypergraph of  $\mathcal{G}$  induced by the edges of color *i*.

Let  $\mathcal{G}$  be a hypergraph, let U be some finite set, and let  $\Psi : V(\mathcal{G}) \to U$  be a surjective mapping. The map  $\Psi$  extends naturally to  $E(\mathcal{G})$ . For  $A \in E(\mathcal{G})$  we define  $\Psi(A) = \{\Psi(x) : x \in A\}$ . Note that  $\Psi$  need not be injective, and A may be a multiset.

Then we define the hypergraph  $\mathcal{F}$  by taking  $V(\mathcal{F}) = U$  and  $E(\mathcal{F}) = \{\Psi(A) : A \in E(\mathcal{G})\}$ . We say that  $\mathcal{F}$  is an *amalgamation* of  $\mathcal{G}$ , and that  $\mathcal{G}$  is a *detachment* of  $\mathcal{F}$ . Associated with  $\Psi$  is a (number) function g defined by  $g(u) = |\Psi^{-1}(u)|$ ; to be more specific we will say that  $\mathcal{G}$  is a g-detachment of  $\mathcal{F}$ . Then  $\mathcal{G}$  has  $\sum_{u \in V(\mathcal{F})} g(u)$  vertices. Note that  $\Psi$  induces a bijection between the edges of  $\mathcal{F}$  and the edges of  $\mathcal{G}$ , and that this bijection preserves the size of an edge. We adopt the convention that it preserves the color also, so that if we amalgamate or detach an edge-colored hypergraph the amalgamation or detachment preserves the same coloring on the edges. We make explicit a straightforward observation: Given  $\mathcal{G}$ ,  $V(\mathcal{F})$  and  $\Psi$  the amalgamation is uniquely determined, but given  $\mathcal{F}$ ,  $V(\mathcal{G})$  and  $\Psi$  the detachment is in general far from uniquely determined.

There are quite a lot of other papers on amalgamations and some highlights include [9, 11, 12, 13, 14, 15, 17, 18].

Given an edge-colored hypergraph  $\mathcal{F}$ , we are interested in finding a detachment  $\mathcal{G}$  obtained by splitting each vertex of  $\mathcal{F}$  into a prescribed number of vertices in  $\mathcal{G}$  so that (i) the degree of each vertex in each color class of  $\mathcal{F}$  is shared evenly among the subvertices in the same color class in  $\mathcal{G}$ , and (ii) the multiplicity of each edge in  $\mathcal{F}$  is shared evenly among the subvertices in  $\mathcal{G}$ . The following theorem, which is a special case of a general result in [1], guarantees the existence of such detachment (Here  $x \approx y$  means  $|y| \leq x \leq [y]$ ).

**Theorem 2.1.** (Bahmanian [1, Theorem 4.1]) Let  $\mathcal{F}$  be a k-edge-colored hypergraph and let  $g: V(\mathcal{F}) \to \mathbb{N}$ . Then there exists a g-detachment  $\mathcal{G}$  (possibly with multiple edges) of  $\mathcal{F}$  whose edges are all sets, with amalgamation function  $\Psi: V(\mathcal{G}) \to V(\mathcal{F})$ , g being the number function associated with  $\Psi$ , such that

(F1) for each  $u \in V(\mathcal{F})$ , each  $v \in \Psi^{-1}(u)$  and  $i \in [k]$ ,

$$\deg_{\mathcal{G}(i)}(v) \approx \frac{\deg_{\mathcal{F}(i)}(u)}{g(u)};$$

(F2) for distinct  $u_1, \ldots, u_s \in V(\mathcal{F})$  and  $U_i \subseteq \Psi^{-1}(u_i)$  with  $|U_i| = m_i \leq g(u_i)$  for  $i \in [s]$ ,

$$\operatorname{mult}_{\mathcal{G}}(U_1,\ldots,U_s) \approx \frac{\operatorname{mult}_{\mathcal{F}}(u_1^{m_1},\ldots,u_s^{m_s})}{\prod_{i=1}^s \binom{g(u_i)}{m_i}}.$$

Let  $\widetilde{K^h_m}$  be the hypergraph obtained by adding a new vertex u and new edges to  $K^h_m$  so that

$$\operatorname{mult}(u^{i}, W) = \binom{n-m}{i} \text{ for each } i \in [h], \text{ and } W \subseteq V(K_{m}^{h}) \text{ with } |W| = h - i.$$

In other words,  $\widetilde{K_m^h}$  is an amalgamation of  $K_n^h$ , obtained by identifying an arbitrary set of n-m vertices in  $K_n^h$ .

An immediate consequence of Theorem 2.1 is the following.

**Corollary 2.2.** Let  $k := \binom{n-1}{h-1}/r \in \mathbb{N}$ . A partial r-factorization of  $K_m^h$  can be extended to an r-factorization of  $K_n^h$  if and only if the new edges of  $\mathcal{F} := \widetilde{K_m^h}$  can be colored so that

(1) 
$$\forall i \in [k] \qquad \deg_{\mathcal{F}(i)}(v) = \begin{cases} r & \text{if } v \neq u, \\ r(n-m) & \text{if } v = u. \end{cases}$$

*Proof.* First, suppose that a partial *r*-factorization of  $K_m^h$  can be extended to an *r*-factorization of  $K_n^h$ . By amalgamating the new n - m vertices of  $K_n^h$  into a single vertex u, we clearly obtain  $\mathcal{F}$ . The *k*-edge-coloring of  $K_n^h$  (in which each color class is an *r*-factor) induces a *k*-edge-coloring in  $\mathcal{F}$  that satisfies (1).

Conversely, suppose that the edges of  $\mathcal{F}$  are colored so that (1) is satisfied. Let  $g: V(\mathcal{F}) \to \mathbb{N}$  with g(u) = n - m, and g(v) = 1 for  $v \neq u$ . By Theorem 2.1, there exists a g-detachment  $\mathcal{G}$  of  $\mathcal{F}$  such that

(a) for each  $v \in \Psi^{-1}(u)$ , and  $i \in [k]$ 

$$\deg_{\mathcal{G}(i)}(v) \approx \deg_{\mathcal{F}(i)}(u)/g(u) = r(n-m)/(n-m) = r.$$

(b) for  $U \subseteq \Psi^{-1}(u), W \subseteq V(K_m^h)$  with |U| = i, |W| = h - i, for  $i \in [h]$ .

$$\operatorname{mult}_{\mathcal{G}}(U,W) \approx \frac{\operatorname{mult}_{\mathcal{F}}(u^{i},W)}{\binom{g(u)}{i}} = \frac{\binom{n-m}{i}}{\binom{n-m}{i}} = 1$$

By (a), each color class is an *r*-factor, and by (b),  $\mathcal{G} \cong K_n^h$ .

The following observation will be quite useful throughout the paper.

**Proposition 2.3.** For every  $n, m, h \in \mathbb{N}$  with  $n \ge m \ge h$ ,

(2) 
$$\binom{n}{h} = \sum_{i=0}^{h} \binom{m}{i} \binom{n-m}{h-i}.$$

(3) 
$$m[\binom{n-1}{h-1} - \binom{m-1}{h-1}] = \sum_{i=1}^{h-1} i\binom{m}{i}\binom{n-m}{h-i}.$$

*Proof.* The proof of (2) is straightforward. Let  $\mathcal{F}$  be a hypergraph with vertex set  $\{u, v\}$  such that  $\operatorname{mult}(u^i, v^{h-i}) = \binom{m}{i}\binom{n-m}{h-i}$  for  $0 \leq i \leq h-1$ . Note that  $\mathcal{F}$  is an amalgamation of the hypergraph  $\mathcal{G}$  with edge set  $\binom{X}{h} \setminus \binom{U}{h}$  where |X| = n, |U| = m. Double counting the degree of u proves (3):

$$\sum_{i=1}^{h-1} i \binom{m}{i} \binom{n-m}{h-i} = \deg_{\mathcal{F}}(u) = \sum_{u \in U} d_{\mathcal{G}}(u) = m \left[\binom{n-1}{h-1} - \binom{m-1}{h-1}\right].$$

In order to avoid trivial cases, throughout the rest of this paper we assume that m > h.

## 3. Arbitrary h

If we replace every edge e of a hypergraph  $\mathcal{G}$  by  $\lambda$  copies of e, then we denote the new hypergraph by  $\lambda \mathcal{G}$ . For hypergraphs  $\mathcal{G}_1, \ldots, \mathcal{G}_t$  with the same vertex set V, we define their *union*, written  $\bigcup_{i=1}^t \mathcal{G}_i$ , to be the hypergraph with vertex set V and edge set  $\bigcup_{i=1}^t E(\mathcal{G}_i)$ . For a hypergraph  $\mathcal{G}$  and  $V \subseteq V(\mathcal{G})$ , let  $\mathcal{G} - V$  be the hypergraph whose vertex set is  $V(\mathcal{G}) \setminus V$  and whose edge set is  $\{e \setminus V | e \in E(\mathcal{G})\}$ .

Let V be an arbitrary subset of vertices in  $K_n^h$  with  $|V| = m \leq n$ . Then  $K_n^h - V \cong \bigcup_{i=0}^{h-1} {m \choose i} K_{n-m}^{h-i}$ . A partial r-factorization of  $\mathcal{H} := K_n^h - V$  is a coloring of the edges of  $K_n^h - V$  with at most  ${n-1 \choose h-1}/r$  colors so that for each color *i*,  $\deg_{\mathcal{H}(i)}(v) \leq r$  for each vertex of  $\mathcal{H}$  (Note that  $\mathcal{H}$  has singleton edges). In the next result, we completely settle the problem of extending a partial r-factorization of  $K_n^h - V$  to an r-factorization of  $K_n^h$ . Note that here we are not only extending the coloring, but also the edges of size less than h to edges of size h. The case h = 3 was solved in [4].

**Theorem 3.1.** For  $V \subseteq V(K_n^h)$  with |V| = m, any partial r-factorization of  $\mathcal{H} := K_n^h - V$  can be extended to an r-factorization of  $K_n^h$  if and only if  $h|rn, r|\binom{n-1}{h-1}$ , and for all  $i = 1, 2, \ldots, \binom{n-1}{h-1}/r$ ,

(4) 
$$d_{\mathcal{H}(i)}(v) = r \quad \forall v \in V(\mathcal{H}),$$

(5) 
$$|E(\mathcal{H}(i))| \leq \frac{rn}{h}$$

*Proof.* To prove the necessity, suppose that a given partial *r*-factorization of  $\mathcal{H}$  is extended to an *r*-factorization of  $K_n^h$ . For  $K_n^h$  to be *r*-factorable, the two divisibility conditions are clearly necessary. By extending an edge *e* of size *i* (*i* < *h*) in  $\mathcal{H}$  to an edge of size *h* in  $K_n^h$ , the color of *e* does not change, and so (4) is necessary. Since the number of edges in each color class of  $K_n^h$  is exactly rn/h, the necessity of (5) is implied.

To prove the sufficiency, suppose that a partial *r*-factorization of  $\mathcal{H}$  is given, h|rn,  $r|\binom{n-1}{h-1}$ , and that (4), (5) are satisfied. Let  $k = \binom{n-1}{h-1}$ , and let  $\mathcal{F} = \widetilde{K_{n-m}^h}$ . For  $0 \leq i \leq h$ , an edge of type  $u^i$  in  $\mathcal{F}$  is an edge in  $\mathcal{F}$  containing  $u^i$  but not containing  $u^{i+1}$ . Note that there are  $\binom{m}{i}\binom{n-m}{h-i}$  edges of type  $u^i$  in  $\mathcal{F}$ .

There is a clear one-to-one correspondence between the edges of size i in  $\mathcal{H}$  and the edges of type  $u^{h-i}$  in  $\mathcal{F}$  for each  $i \in [h]$ . We color the edges of type  $u^i$  in  $\mathcal{F}$  with the same color as the corresponding edge in  $\mathcal{H}$  for  $0 \leq i \leq h-1$ . By Corollary 2.2, if we can color the remaining edges of  $\mathcal{F}$  (edges of type  $u^h$ ) so that the following condition is satisfied, then we are done.

(6) 
$$\forall i \in [k] \qquad \deg_{\mathcal{F}(i)}(v) = \begin{cases} r & \text{if } v \neq u, \\ rm & \text{if } v = u. \end{cases}$$

Let  $\operatorname{mult}_i(u^j, .)$  be the number of edges of type  $u^j$  in  $\mathcal{F}(i)$ , for  $i \in [k], j \in [h]$ . Note that  $\operatorname{mult}_i(u^h, .) = \operatorname{mult}_{\mathcal{F}(i)}(u^h)$  for  $i \in [k]$ . We color the edges of type  $u^h$  so that for

 $i \in [k],$ 

$$\operatorname{mult}_{i}(u^{h},.) = \frac{rn}{h} - r(n-m) + \sum_{j=1}^{h-1} j \operatorname{mult}_{i}(u^{h-j-1},.).$$

Since h|rn,  $\operatorname{mult}_i(u^h, .)$  is an integer for  $i \in [k]$ . The following shows that  $\operatorname{mult}_i(u^h, .) \ge 0$  for  $i \in [k]$ .

$$\frac{rn}{h} \stackrel{(5)}{\geq} |E(\mathcal{H}(i))| = \sum_{j=0}^{h-1} \operatorname{mult}_i(u^j, .)$$
$$= \sum_{j=1}^{h} j \operatorname{mult}_i(u^{h-j}, .) - \sum_{j=1}^{h-1} j \operatorname{mult}_i(u^{h-j-1}, .)$$
$$\stackrel{(4)}{=} r(n-m) - \sum_{j=1}^{h-1} j \operatorname{mult}_i(u^{h-j-1}, .).$$

Now we show that all edges of the type  $u^h$  will be colored, or equivalently that,  $\sum_{i=1}^k \text{mult}_i(u^h, .) = \binom{m}{h}$ .

$$\begin{split} \sum_{i=1}^{k} \operatorname{mult}_{i}(u^{h}, .) &= \sum_{i=1}^{k} \left( \frac{rn}{h} - r(n-m) + \sum_{j=1}^{h-1} j \operatorname{mult}_{i}(u^{h-j-1}, .) \right) \\ &= \frac{rkn}{h} - rk(n-m) + \sum_{j=1}^{h-1} j \sum_{i=1}^{k} \operatorname{mult}_{i}(u^{h-j-1}, .) \\ &= \binom{n}{h} - (n-m)\binom{n-1}{h-1} + \sum_{j=2}^{h} (j-1)\binom{m}{h-j}\binom{n-m}{j} \\ \binom{(2),(3)}{=} \sum_{j=0}^{h} \binom{m}{j} \binom{n-m}{h-j} - \sum_{j=1}^{h-1} j\binom{n-m}{j} \binom{m}{h-j} \\ &- (n-m)\binom{n-m-1}{h-1} + \sum_{j=2}^{h} (j-1)\binom{m}{h-j}\binom{n-m}{j} \\ &= \binom{m}{h} - (n-m)\binom{n-m-1}{h-1} + h\binom{n-m}{h} \\ &= \binom{m}{h}. \end{split}$$

To complete the proof, we show that  $\deg_{\mathcal{F}(i)}(u) = rm$  for  $i \in [k]$ . We have

$$deg_{\mathcal{F}(i)}(u) = \sum_{j=1}^{h} j \operatorname{mult}_{i}(u^{j}, .) = h \operatorname{mult}_{i}(u^{h}, .) + \sum_{j=1}^{h} (h - j) \operatorname{mult}_{i}(u^{h - j}, .)$$
  
$$= h \operatorname{mult}_{i}(u^{h}, .) + h \sum_{j=1}^{h} \operatorname{mult}_{i}(u^{h - j}, .) - \sum_{j=1}^{h} j \operatorname{mult}_{i}(u^{h - j}, .)$$
  
$$= h \sum_{j=0}^{h} \operatorname{mult}_{i}(u^{h - j}, .) - \sum_{j=1}^{h} j \operatorname{mult}_{i}(u^{h - j}, .)$$
  
$$= rn - r(n - m) = rm.$$

For a hypergraph  $\mathcal{G}$  and  $V \subseteq V(\mathcal{G})$ , let  $\mathcal{G} \setminus V$  be the hypergraph whose vertex set is  $V(\mathcal{G})$  and whose edge set is  $\{e \in E(\mathcal{G}) | e \notin V\}$ .

Let  $V \subseteq V(K_n^h)$  with  $|V| = m \leq n$ , and let  $\mathcal{H} := K_n^h \setminus V$ . An edge  $e \in E(\mathcal{H})$  is of type *i*, if  $|e \cap V| = i$  (for  $0 \leq i \leq h - 1$ ). Let *P* be a partial *r*-factorization of  $\mathcal{H}$ . Then a partial *r*-factorization *Q* of  $\mathcal{H}$  is said to be *P*-friendly if

- (a) the color of each edge of type 0 is the same in P and Q, and
- (b) the number of edges of type i and color j is the same in P and Q for each  $i \in [h-1]$  and each color j.

We are interested in finding the conditions under which a partial r-factorization of  $\mathcal{H}$  can be extended to an r-factorization of  $K_n^h$ .

**Lemma 3.2.** For  $V \subseteq V(K_n^h)$  with |V| = m, if a partial r-factorization of  $\mathcal{H} := K_n^h \setminus V$  can be extended to an r-factorization of  $K_n^h$ , then

(N1) h|rn,  $(N2) r|\binom{n-1}{h-1},$   $(N3) d_{\mathcal{H}(i)}(v) = r \text{ for each } v \in V(\mathcal{H}) \setminus V, \text{ and } i \in [k],$   $(N4) |E(\mathcal{H}(i))| \leq rn/h \text{ for } i \in [k],$ where  $k := \binom{n-1}{h-1}/r.$ 

It remains an open question whether these conditions are sufficient. Here we prove a weaker result.

**Corollary 3.3.** Let  $V \subseteq V(K_n^h)$  with |V| = m, and let P be a partial r-factorization of  $\mathcal{H} := K_n^h \setminus V$ , and assume that (N1) - (N4) are satisfied. Then there exists a P-friendly partial r-factorization of  $\mathcal{H}$  that can be extended to an r-factorization of  $K_n^h$ .

*Proof.* By eliminating all the vertices in V, and shrinking the edges containing vertices in V, we obtain  $K_n^h - V$ . The rest of the proof follows from Theorem 3.1.

4. 
$$h = 4$$

**Theorem 4.1.** For  $n \ge 4.847323m$ , any partial r-factorization of  $K_m^4$  can be extended to an r-factorization of  $K_n^4$  if and only if 4|rn and  $r|\binom{n-1}{3}$ .

*Proof.* For the necessary conditions, see the previous section. To prove the sufficiency, we need to show that the edges of  $\mathcal{F} := \widetilde{K_m^4}$  can be colored with  $k := \binom{n-1}{3}/r$  colors so that (6) is satisfied.

First we color the edges in  $\mathcal{F}$  of the form  $W \cup \{u\}$  where  $W \subseteq V := V(K_m^4)$  and |W| = 3. We color these edges greedily so that  $\deg_i(x) \leq r$  for each  $x \in V$  and  $i \in [k]$ . We claim that this coloring can be done in such a way that all edges of this type are colored. Suppose by contrary that there is an edge in  $\mathcal{F}$  of the form  $\{x, y, z, u\}$  with  $x, y, z \in V$  that can not be colored. This implies that for each  $i \in [k]$  either  $\deg_i(x) = r$  or  $\deg_i(y) = r$  or  $\deg_i(z) = r$ . Thus for each  $i \in [k]$ ,  $\deg_i(x) + \deg_i(y) + \deg_i(z) \geq r$ . On the one hand,  $\sum_{i=1}^k (\deg_i(x) + \deg_i(y) + \deg_i(z)) \leq rk = \binom{n-1}{3}$ , and on the other hand,  $\sum_{i=1}^k (\deg_i(x) + \deg_i(y) + \deg_i(z)) \leq 3[\binom{m-1}{3} + (n-m)\binom{m-1}{2} - 1]$ . Thus, we have

$$3\begin{bmatrix} \binom{m-1}{3} + (n-m)\binom{m-1}{2} - 1 \end{bmatrix} \ge \binom{n-1}{3}.$$

which is equivalent to  $f(n,m) := n^3 - 6n^2 - 9m^2n + 27mn - 7n + 6m^3 - 9m^2 - 15m + 30 \le 0$ . Now, we show that since n > 4m and  $m \ge 5$ , we have f(n,m) > 0, which is a contradiction, and therefore, all edges in  $\mathcal{F}$  of the form  $W \cup \{u\}$  where  $W \subseteq V$  and |W| = 3 can be colored using the greedy approach described above.

First, note that for  $m \ge 5$ , both  $7m^2 + 3m - 7$  and  $2m^2 - 3m - 5$  are positive. Therefore,

$$f(n,m) = n\left(n(n-6) - 9m^2 + 27m - 7\right) + 3m(2m^2 - 3m - 5) + 30$$
  
>  $n\left(4m(4m - 6) - 9m^2 + 27m - 7\right) + 3m(2m^2 - 3m - 5) + 30$   
=  $n(7m^2 + 3m - 7) + 3m(2m^2 - 3m - 5) + 30 > 0.$ 

Now we greedily color all the edges of the form  $W \cup \{u^2\}$  where  $W \subseteq V$  and |W| = 2, so that  $\deg_i(x) \leq r$  for each  $x \in V$  and  $i \in [k]$ . We show that this is possible. If by contrary, some edge  $\{x, y, u^2\}$  with  $x, y \in V$  remains uncolored, then for each  $i \in [k]$ either  $\deg_i(x) = r$  or  $\deg_i(y) = r$ , and so  $\deg_i(x) + \deg_i(y) \geq r$ . We have

$$\binom{n-1}{3} = rk \leq \sum_{i=1}^{k} \left( \deg_i(x) + \deg_i(y) \right)$$
$$\leq 2\left[ \binom{m-1}{3} + (n-m)\binom{m-1}{2} + (m-1)\binom{n-m}{2} - 1 \right],$$

which is equivalent to  $n^3 - 6mn^2 + 6m^2n + 12mn - 7n - 2m^3 - 6m^2 - 4m + 18 \leq 0$ . Using Mathematica (Wolfram Alpha) it can be shown that this inequality does not have any real solution under the constraints  $m \ge 5, n \ge 4.847323m$ . Therefore, all edges of the form  $W \cup \{u^2\}$  where  $W \subseteq V$  and |W| = 2, can be colored.

Since for each  $x \in V$ ,

$$\begin{split} \sum_{i=1}^{k} \left( r - \deg_{i}(x) \right) &= rk - \left[ \binom{m-1}{3} + (n-m)\binom{m-1}{2} + (m-1)\binom{n-m}{2} \right] \\ &= \binom{n-1}{3} - \binom{m-1}{3} - (n-m)\binom{m-1}{2} - (m-1)\binom{n-m}{2} \\ &\stackrel{(2)}{=} \binom{n-m}{3}, \end{split}$$

we can color all the edges of the form  $\{w, u^3\}$  where  $w \in V$  so that for each  $x \in V$ , there are  $r - \deg_i(x)$  edges of this type colored *i* incident with *x* for each  $i \in [k]$ . Note that after this coloring,

(7) 
$$\deg_i(x) = r \text{ for each } x \in V.$$

For  $i \in [k]$ , let  $a_i, b_i, c_i, d_i$  be the number of edges colored *i* of the form  $W, W \cup \{u\}, W \cup \{u^2\}, W \cup \{u^3\}$  where  $W \subseteq V$ , respectively. We color the edges of the form  $\{u^4\}$  so that there are exactly

$$e_i := rn/4 - rm + 3a_i + 2b_i + c_i$$

edges of this type colored *i* for  $i \in [k]$ . Since 4|rn, and n > 4m,  $e_i$  is a positive integer for  $i \in [k]$ . We claim that all edges of the form  $\{u^4\}$  will be colored, or equivalently,  $\sum_{i=1}^{k} e_i = \binom{n-m}{4}$ .

$$\sum_{i=1}^{k} e_{i} = \sum_{i=1}^{k} \left(\frac{rn}{4} - rm + 3a_{i} + 2b_{i} + c_{i}\right) = \frac{rkn}{4} - rkm + 3\sum_{i=1}^{k} a_{i} + 2\sum_{i=1}^{k} b_{i} + \sum_{i=1}^{k} c_{i}$$
$$= \frac{n}{4} \binom{n-1}{3} - m\binom{n-1}{3} + 3\binom{m}{4} + 2(n-m)\binom{m}{3} + \binom{n-m}{2}\binom{m}{2}$$
$$= \binom{n}{4} - m\binom{n-1}{3} + 3\binom{m}{4} + 2(n-m)\binom{m}{3} + \binom{n-m}{2}\binom{m}{2}$$
$$\stackrel{(2),(3)}{=} \binom{n-m}{4}.$$

To complete the proof, we show that  $\deg_i(u) = r(n-m)$  for  $i \in [k]$ . First note that for  $i \in [k]$ ,  $rm = \sum_{x \in V} \deg_i(x) = 4a_i + 3b_i + 2c_i + d_i$ . Therefore,

$$\deg_i(u) = b_i + 2c_i + 3d_i + 4e_i = 4(a_i + b_i + c_i + d_i + e_i) - (4a_i + 3b_i + 2c_i + d_i)$$
  
=  $rn - rm = r(n - m).$ 

Combining this with (7) implies that (6) is satisfied, and the proof is complete.  $\Box$ 

5. 
$$h = 5$$

**Theorem 5.1.** For  $n \ge 6.285214m$ , any partial r-factorization of  $K_m^5$  can be extended to an r-factorization of  $K_n^5$  if and only if 5|rn and  $r|\binom{n-1}{4}$ .

*Proof.* The necessity is obvious. To prove the sufficiency, we need to show that the edges of  $\mathcal{F} := \widetilde{K_m^5}$  can be colored with  $k := \binom{n-1}{4}/r$  colors so that (6) is satisfied. First we color the edges of the form  $W \cup \{u\}$  where  $W \subseteq V$  and |W| = 4. We color

First we color the edges of the form  $W \cup \{u\}$  where  $W \subseteq V$  and |W| = 4. We color these edges greedily so that  $\deg_i(x) \leq r$  for each  $x \in V$  and  $i \in [k]$ . We claim that this coloring can be done in such a way that all edges of this type are colored. Suppose by contrary that there is an edge of the form  $\{x, y, z, w, u\}$  with  $x, y, z, w \in V$  that can not be colored. This implies that for each  $i \in [k]$  either  $\deg_i(x) = r$  or  $\deg_i(y) = r$  or  $\deg_i(z) = r$  or  $\deg_i(w) = r$ . Thus for each  $i \in [k]$ ,  $\deg_i(x) + \deg_i(y) + \deg_i(z) + \deg_i(w) \geq r$ . On the one hand,  $\sum_{i=1}^k (\deg_i(x) + \deg_i(y) + \deg_i(z) + \deg_i(w)) \geq rk = \binom{n-1}{4}$ , and on the other hand,  $\sum_{i=1}^k (\deg_i(x) + \deg_i(y) + \deg_i(z) + \deg_i(w)) \leq 4[\binom{m-1}{4} + (n - m)\binom{m-1}{3} - 1]$ . Thus, we have

$$4\begin{bmatrix} \binom{m-1}{4} + (n-m)\binom{m-1}{3} - 1 \end{bmatrix} \ge \binom{n-1}{4}.$$

which is equivalent to  $g_1(n,m) := n^4 - 10n^3 + 35n^2 - 16m^3n + 96m^2n - 176mn + 46n + 12m^4 - 56m^3 + 36m^2 + 104m + 24 \leq 0.$ 

Since n > 6m and  $m \ge 6$ , we have

$$g_1(n,m) := n \Big( n^2(n-10) - 16m^3 + 96m^2 + (35n-176m) + 46 \Big) + 4m \Big( m^2(3m-14) + 9m + 26 \Big) + 24 > 9m^2(3m-10) - 16m^3 + 96m^2 = 11m^3 + 6m^2 > 0,$$

which is a contradiction, and therefore, all edges in  $\mathcal{F}$  of the form  $W \cup \{u\}$  where  $W \subseteq V$  and |W| = 4 can be colored.

Now we greedily color all the edges of the form  $W \cup \{u^2\}$  where  $W \subseteq V$  and |W| = 3, so that  $\deg_i(x) \leq r$  for each  $x \in V$  and  $i \in [k]$ . We show that this is possible. If by contrary, some edge  $\{x, y, z, u^2\}$  with  $x, y, z \in V$  remains uncolored, then for each  $i \in [k]$ either  $\deg_i(x) = r$  or  $\deg_i(y) = r$  or  $\deg_i(z) = r$ , and so  $\deg_i(x) + \deg_i(y) + \deg_i(z) \geq r$ . We have

$$\binom{n-1}{4} = rk \quad \leq \quad \sum_{i=1}^{k} \left( \deg_i(x) + \deg_i(y) + \deg_i(z) \right)$$
  
$$\leq \quad 3\left[ \binom{m-1}{4} + (n-m)\binom{m-1}{3} + \binom{m-1}{2}\binom{n-m}{2} - 1 \right].$$

which is equivalent to  $g_2(n,m) := n^4 - 10n^3 - 18m^2n^2 + 54mn^2 - n^2 + 24m^3n - 18m^2n - 114mn + 58n - 9m^4 - 6m^3 + 45m^2 + 42m + 24 \leq 0$ . We show that since n > 6m and  $m \geq 6$ , we have  $g_2(n,m) > 0$ , which is a contradiction, and therefore, all edges in  $\mathcal{F}$  of the form  $W \cup \{u^2\}$  where  $W \subseteq V$  and |W| = 3 can be colored.

First, note that for  $m \ge 6$  we have  $12m^3 - 9m^2 - 57m + 29 > 0$ . Therefore,

$$g_{2}(n,m) = n^{2} \left( n(n-10) - 18m^{2} + 54m - 1 \right) + 2n \left( 12m^{3} - 9m^{2} - 57m + 29 \right) - (9m^{4} + 6m^{3} - 45m^{2} - 42m - 24) > 36m^{2} \left( 6m(6m - 10) - 18m^{2} + 54m - 1 \right) - (9m^{4} + 6m^{3} - 45m^{2} - 42m - 24) = 639m^{4} - 150m^{3} + 18m^{2} + 42m + 24 > 0.$$

Now we greedily color all the edges of the form  $W \cup \{u^3\}$  where  $W \subseteq V$  and |W| = 2, so that  $\deg_i(x) \leq r$  for each  $x \in V$  and  $i \in [k]$ . We show that this is possible. If by contrary, some edge  $\{x, y, u^2\}$  with  $x, y \in V$  remains uncolored, then for each  $i \in [k]$ either  $\deg_i(x) = r$  or  $\deg_i(y) = r$ , and so  $\deg_i(x) + \deg_i(y) \geq r$ . We have

$$\binom{n-1}{4} \leq \sum_{i=1}^{k} \left( \deg_{i}(x) + \deg_{i}(y) \right)$$
  
 
$$\leq 2\left[ \binom{m-1}{4} + (n-m)\binom{m-1}{3} + \binom{m-1}{2}\binom{n-m}{2} + (m-1)\binom{n-m}{3} - 1 \right].$$

Using Mathematica it can be shown that this inequality does not have any real solution under the constraints  $m \ge 6, n \ge 6.285214m$ . Therefore, all edges of the form  $W \cup \{u^3\}$ where  $W \subseteq V$  and |W| = 2, can be colored.

Since for each  $x \in V$ ,

$$\sum_{i=1}^{k} \left( r - \deg_i(x) \right) = \binom{n-1}{4} - \binom{m-1}{4} - (n-m)\binom{m-1}{3}$$
$$- \binom{m-1}{2}\binom{n-m}{2} - (m-1)\binom{n-m}{3}$$
$$= \binom{n-m}{4},$$

we can color all the edges of the form  $\{w, u^4\}$  where  $w \in V$  so that for each  $x \in V$ , there are  $r - \deg_i(x)$  edges of this type colored *i* incident with *x* for each  $i \in [k]$ .

For  $i \in [k]$ , let  $a_i, b_i, c_i, d_i, e_i$  be the number of edges colored i of the form  $W, W \cup \{u\}, W \cup \{u^2\}, W \cup \{u^3\}, W \cup \{u^4\}$  where  $W \subseteq V$ , respectively. We color the edges of the form  $\{u^5\}$  so that there exactly

$$f_i := rn/5 - rm + 4a_i + 3b_i + 2c_i + d_i$$

edges of this type colored *i* for  $i \in [k]$ . Since 5|rn, and  $n \ge 6.4m > 5m$ ,  $e_i$  is a positive integer for  $i \in [k]$ . We claim that all edges of the form  $\{u^5\}$  will be colored, or equivalently,  $\sum_{i=1}^k f_i = \binom{n-m}{5}$ .

$$\sum_{i=1}^{k} f_{i} = \sum_{i=1}^{k} \left(\frac{rn}{5} - rm + 4a_{i} + 3b_{i} + 2c_{i} + d_{i}\right)$$

$$= \frac{rkn}{5} - rkm + 4\sum_{i=1}^{k} a_{i} + 3\sum_{i=1}^{k} b_{i} + 2\sum_{i=1}^{k} c_{i} + \sum_{i=1}^{k} d_{i}$$

$$= \binom{n}{5} - m\binom{n-1}{4} + 4\binom{m}{5} + 3(n-m)\binom{m}{4} + 2\binom{n-m}{2}\binom{m}{3} + \binom{n-m}{3}\binom{m}{2}$$

$$= \binom{n-m}{5}.$$

To complete the proof, we show that  $\deg_i(u) = r(n-m)$  for  $i \in [k]$ . First note that for  $i \in [k]$ ,  $rm = \sum_{x \in V} \deg_i(x) = 5a_i + 4b_i + 3c_i + 2d_i + e_i$ . Therefore,

$$\deg_i(u) = b_i + 2c_i + 3d_i + 4e_i + 5f_i$$
  
=  $5(a_i + b_i + c_i + d_i + e_i + f_i) - (5a_i + 4b_i + 3c_i + 2d_i + e_i)$   
=  $rn - rm = r(n - m).$ 

## 6. CONCLUDING REMARKS AND OPEN PROBLEMS

- (1) At this point, it is not clear to use how to extend the results of Sections 4 and 5 without dealing with heavy computation. We believe that for  $n \ge 2hm$ , any partial *r*-factorization of  $K_m^h$  can be extended to an *r*-factorization of  $K_n^h$  if and only if the obvious necessary divisibility conditions are satisfied.
- (2) To embed a partial r-factorization of  $K_n \setminus K_m^h$  into an r-factorization of  $K_n^h$ , we believe that the conditions (N1)–(N4) of Lemma 3.2 are sufficient, but we do not know how to go beyond Corollary 3.3.
- (3) A partial r-factorization  $S \subseteq K_n^h$  is critical if it can be extended to exactly one r-factorization of  $K_n^h$ , but removal of any element of S destroys the uniqueness of the extension, and |S| is the size of the critical partial r-factorization. It is desirable to find good bounds for the smallest and largest sizes of critical partial r-factorizations.
- (4) Another interesting problem is finding conditions under which a partial *r*-factorization of  $S \subseteq {[n] \choose h}$  can be extended to a **cyclic** *r*-factorization of  ${[n] \choose h}$ .

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