Generating Near-Bipartite Bricks *

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20 November, 2016

Abstract

A 3-connected graph G is a *brick* if, for any two vertices u and v, the graph $G - \{u, v\}$ has a perfect matching. Deleting an edge e from a brick G results in a graph with zero, one or two vertices of degree two. The *bicontraction* of a vertex of degree two consists of contracting the two edges incident with it; and the *retract* of $G - e$ is the graph J obtained from it by bicontracting all its vertices of degree two. An edge e is *thin* if J is also a brick. Carvalho, Lucchesi and Murty [How to build a brick, Discrete Mathematics 306 (2006), 2383-2410] showed that every brick, distinct from K_4, the triangular prism $\overline{C_6}$ and the Petersen graph, has a thin edge. Their theorem yields a generation procedure for bricks, using which they showed that every simple planar solid brick is an odd wheel.

A brick G is *near-bipartite* if it has a pair of edges α and β such that $G - \{\alpha, \beta\}$ is bipartite and matching covered; examples are K_4 and $\overline{C_6}$. The significance of near-bipartite graphs arises from the theory of ear decompositions of matching covered graphs.

The object of this paper is to establish a generation procedure which is specific to the class of near-bipartite bricks. In particular, we prove that if G is any near-bipartite brick, distinct from K_4 and $\overline{C_6}$, then G has a thin edge e such that the retract J of $G - e$ is also near-bipartite.

*A shorter version has been accepted for publication in the Journal of Graph Theory.
†Partially supported by NSERC grant (RGPIN-2014-04351, J. Cheriyan).

Contents

1 Matching Covered Graphs

For general graph theoretic notation and terminology, we refer the reader to Bondy and Murty [1]. All graphs considered here are loopless; however, we allow multiple edges. An edge of a graph is *admissible* if there is a perfect matching of the graph that contains it. A connected graph with two or more vertices is *matching covered* if each of its edges is admissible. For a comprehensive treatment of matching theory and its origins, we refer the reader to Lovász and Plummer [12], wherein matching covered graphs are referred to as '1-extendable' graphs.

In this section, we briefly review the relevant terminology, definitions and results from the theory of matching covered graphs.

1.1 Canonical Partition

Tutte's Theorem states that a graph G has a perfect matching if and only if $\mathsf{odd}(G - S) \leq |S|$ for each subset S of G, where $\mathsf{odd}(G - S)$ denotes the number of odd components of $G - S$. For a graph G that has a perfect matching, a nonempty subset S of its vertices is a *barrier* if it satisfies the equality $\mathsf{odd}(G - S) = |S|$. The following proposition is easily deduced from Tutte's Theorem, and yields a characterization of matching covered graphs.

Proposition 1.1 *Let G be a graph that has a perfect matching. Let u and v be distinct vertices of G. Then the graph $G - \{u, v\}$ has a perfect matching if and only if there is no barrier of G which contains both u and v.*

Corollary 1.2 *Let G be a connected graph with a perfect matching. Then G is matching covered if and only if every barrier of G is stable (that is, an independent set).*

The following fundamental theorem is due to Kotzig (see [12, page 150]).

Theorem 1.3 [The Canonical Partition Theorem] *The maximal barriers of a matching covered graph G partition its vertex set.*

For a matching covered graph G, the partition of its vertex set defined by its maximal barriers is called the *canonical partition* of $V(G)$. For instance, for a bipartite matching covered graph $H[A, B]$, the canonical partition of $V(H)$ consists of precisely two parts, namely, its color classes A and B; this is implied by the following proposition which may be derived from the well-known Hall's Theorem. (The neighbourhood of a set of vertices S is denoted by $N(S)$.)

Proposition 1.4 *Let $H[A, B]$ denote a bipartite graph with four or more vertices, where $|A| = |B|$. Then the following statements are equivalent:*

(i) H is matching covered,

(ii) $|N(S)| \geq |S| + 1$ for every nonempty proper subset S of A, and

(iii) $H - \{a, b\}$ has a perfect matching for each pair of vertices $a \in A$ and $b \in B$.

A graph G, with four or more vertices, is *bicritical* if $G - \{u, v\}$ has a perfect matching for every pair of distinct vertices u and v. A barrier is *trivial* if it has a single vertex. Proposition 1.1 implies the following characterization of bicritical graphs.

Proposition 1.5 *Let G be a connected graph with a perfect matching. Then G is bicritical if and only if every barrier of G is trivial.*

Thus, for a bicritical graph G, the canonical partition of $V(G)$ consists of $|V(G)|$ parts, each of which contains a single vertex.

1.2 Bricks and Braces

For a nonempty proper subset X of the vertices of a graph G, we denote by $\partial(X)$ the cut associated with X, that is, the set of all edges of G that have one end in X and the other end in $\overline{X} := V(G) - X$. We refer to X and $\overline{X}$ as the *shores* of $\partial(X)$. A cut is *trivial* if any of its shores is a singleton. For a cut $\partial(X)$, we denote the graph obtained by contracting the shore $\overline{X}$ to a single vertex $\overline{x}$ by $G/(\overline{X} \to \overline{x})$. In case the label of the contraction vertex $\overline{x}$ is irrelevant, we simply write $G/\overline{X}$. The two graphs G/X and $G/\overline{X}$ are called the $\partial(X)$*-contractions* of G.

Let G be a matching covered graph. A cut $\partial(X)$ is a *tight cut* if $|M \cap \partial(X)| = 1$ for every perfect matching M of G. It is easily verified that if $\partial(X)$ is a nontrivial tight cut of G, then each $\partial(X)$-contraction is a matching covered graph that has strictly fewer vertices than G. If either of the $\partial(X)$-contractions has a nontrivial tight cut, then that graph can be further decomposed into even smaller matching covered graphs. We can repeat this procedure until we obtain a list of matching covered graphs, each of which is free of nontrivial tight cuts. This procedure is known as a *tight cut decomposition* of G.

Let G be a matching covered graph free of nontrivial tight cuts. If G is bipartite then it is a *brace*; otherwise it is a *brick*. Thus, a tight cut decomposition of G results in a list of bricks and braces. In general, a matching covered graph may admit several tight cut decompositions. However, Lovász [11] proved the following remarkable result, and demonstrated its significance by using it to compute the dimension of the matching lattice.

Theorem 1.6 [The Unique Decomposition Theorem] *Any two tight cut decompositions of a matching covered graph yield the same list of bricks and braces (except possibly for multiplicities of edges).*

In particular, any two tight cut decompositions of a matching covered graph G yield the same number of bricks; this number is denoted by $b(G)$. We remark that G is bipartite if and only if $b(G) = 0$.

Let G be a matching covered graph. Observe that, if S is a barrier of G, and K is an odd component of $G - S$, then $\partial(V(K))$ is a tight cut of G. Such a tight cut is called a *barrier cut*. (For instance, if v is a vertex of degree two then $\{v\} \cup N(v)$ is the shore of a barrier cut.) In particular, if G is nonbipartite then each nontrivial barrier gives rise to a nontrivial tight cut.

Now suppose that $\{u, v\}$ is a 2-vertex-cut of G such that $G - \{u, v\}$ has an even component, say K. Then each of the sets $V(K) \cup \{u\}$ and $V(K) \cup \{v\}$ is a shore of a nontrivial tight cut of G. Such a tight cut is called a 2-separation cut. (We remark that a graph may have a tight cut which is neither a barrier cut nor a 2-separation cut.)

Since a brick is a nonbipartite matching covered graph which is free of nontrivial tight cuts, it follows from the above observations that every brick is 3-connected and bicritical. Edmonds, Lovász and Pulleyblank [8] established the converse.

Theorem 1.7 *A graph G is a brick if and only if it is* 3*-connected and bicritical.*

In particular, a brick is free of nontrivial barriers and of 2-vertex-cuts. Three cubic bricks, namely K_4, $\overline{C_6}$ and the Petersen graph, play a special role in the theory of matching covered graphs.

1.3 Removable edges

An edge e of a matching covered graph G is *removable* if $G - e$ is also matching covered; otherwise it is *non-removable*. For example, each edge of the Petersen graph is removable. The following was established by Lovász [11].

Theorem 1.8 [Removable Edge Theorem] *Every brick distinct from K_4 and $\overline{C_6}$ has a removable edge.*

We point out that, if e is a removable edge of a brick G, then $G - e$ may not be a brick. For instance, $G - e$ may have vertices of degree two.

1.3.1 Near-bricks and b-invariant edges

Recall that $b(G)$ denotes the number of bricks of a matching covered graph G (in any tight cut decomposition), and it is well-defined due to the Unique Decomposition Theorem (1.6). A *near-brick* is a matching covered graph with $b(G) = 1$. Clearly, every brick is a near-brick. However, the converse is not true. When proving theorems concerning bricks, one often needs the flexibility of dealing with the wider class of near-bricks, whose properties are akin to those of bricks.

A removable edge e of a matching covered graph G is *b-invariant* if $b(G - e) = b(G)$. In particular, if G is a brick then e is b-invariant if and only if $G - e$ is a near-brick. For instance, the graph St_8 shown in Figure 1 has a unique b-invariant edge e.

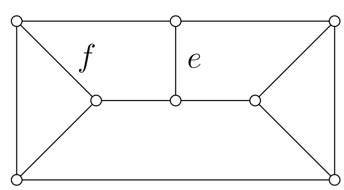


Figure 1: St_8 has a unique b-invariant edge e

It is easily verified that if G is the Petersen graph and e is any edge, then $b(G-e) = 2$. Thus each edge of the Petersen graph is removable, but none of them is b-invariant. Confirming a conjecture of Lovász, the following result was proved by Carvalho, Lucchesi and Murty [3].

Theorem 1.9 [b-invariant Edge Theorem] *Every brick distinct from K_4 and $\overline{C_6}$ and the Petersen graph has a b-invariant edge.*

1.3.2 Bicontractions, retracts and bi-splittings

Let G be a matching covered graph and let v be a vertex of degree two, with two distinct neighbours u and w. The *bicontraction* of v is the operation of contracting the two edges vu and vw incident with v. Note that $X := \{u, v, w\}$ is the shore of a tight cut of G, and that the graph resulting from the bicontraction of v is the same as the $\partial(X)$-contraction G/X, whereas the other $\partial(X)$-contraction $G/\overline{X}$ is isomorphic to C_4 (possibly with multiple edges).

The *retract* of G is the graph obtained from G by bicontracting all its degree two vertices. The above observation implies that the retract of a matching covered graph is also matching covered. Carvalho et al. [5] showed that the retract of a matching covered graph is unique up to isomorphism. It is important to note that even if G is simple, the retract of G may have multiple edges.

The operation of bi-splitting is the converse of the operation of bicontraction. Let H be a graph and let v be a vertex of H of degree at least two. Let G be a graph obtained from H by replacing the vertex v by two new vertices v_1 and v_2, distributing the edges in H incident with v between v_1 and v_2 such that each gets at least one, and then adding a new vertex v_0 and joining it to both v_1 and v_2. Then we say that G is obtained from H by *bi-splitting* v into v_1 and v_2. It is easily seen that if H is matching covered, then G is also matching covered, and that H can be recovered from G by bicontracting the vertex v_0 and denoting the contraction vertex by v.

1.3.3 Thin edges

A b-invariant edge e of a brick G is *thin* if the retract of $G - e$ is a brick. As the graph $G - e$ can have zero, one or two vertices of degree two, the retract of $G - e$ is obtained by performing at most two bicontractions, and it has at least $|V(G)| - 4$ vertices. For example, the retract of $St_8 - e$ (see Figure 1) is isomorphic to K_4 with multiple edges; thus, e is a thin edge. It should be noted that, in general, a b-invariant edge may not be thin.

The original definition of a thin edge, due to Carvalho et al. [6], was in terms of barriers; 'thin' being a reference to the fact that the barriers of $G - e$ are sparse. This viewpoint will also be useful to us in latter sections (where further explanation is provided). Carvalho, Lucchesi and Murty [6] used their b-invariant Edge Theorem (1.9) to derive the following stronger result.

Theorem 1.10 [THIN EDGE THEOREM] *Every brick distinct from K_4 and $\overline{C_6}$ and the Petersen graph has a thin edge.*

The following is an immediate consequence of the above theorem.

Theorem 1.11 [6] *Given any brick G, there exists a sequence $G_1, G_2, \ldots, G_k$ of bricks such that:*

(i) *G_1 is either K_4 or $\overline{C_6}$ or the Petersen graph,*

(ii) *$G_k := G$, and*

(iii) *for $2 \leq i \leq k$, there exists a thin edge e_i of G_i such that G_{i-1} is the retract of $G_i - e_i$.*

Carvalho et al. [6] also described four elementary 'expansion operations' which may be applied to any brick to obtain a larger brick with at most four more vertices. Each of these operations consists of bi-splitting at most two vertices and then adding a suitable edge. Given a brick J, the application of any of these four operations to J results in a brick G such that G has a thin edge e with the property that J is the retract of $G - e$. Thus, any brick may be generated from one of the three basic bricks (K_4 and $\overline{C_6}$ and the Petersen graph) by means of these four expansion operations.

1.4 Near-Bipartite Bricks

A nonbipartite matching covered graph G is *near-bipartite* if it has a pair $R := \{\alpha, \beta\}$ of edges such that the graph $H := G - R$ is bipartite and matching covered. Such a pair R is called a *removable doubleton*.

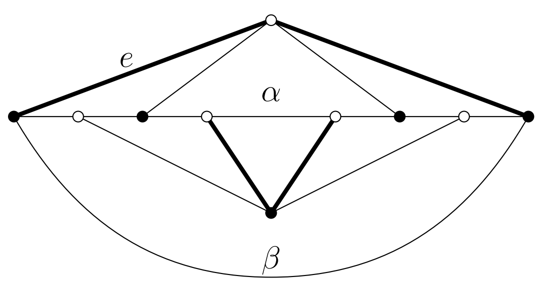
Furthermore, if G happens to be a brick, we say that G is a *near-bipartite brick*. For instance, K_4 and $\overline{C_6}$ are the smallest simple near-bipartite bricks, and each of them has three distinct removable doubletons.

Observe that the edge α joins two vertices in one color class of H, and that β joins two vertices in the other color class. Consequently, if M is any perfect matching of G then $\alpha \in M$ if and only if $\beta \in M$. (In particular, neither α nor β is a removable edge of G.) The following is an immediate consequence of [4, Theorem 5.1].

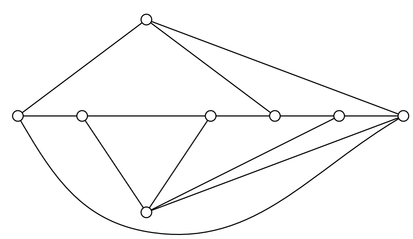
Theorem 1.12 *Every near-bipartite graph is a near-brick.*

The significance of near-bipartite graphs arises from the theory of ear decompositions of matching covered graphs; see [2] and [10]; in this context, near-bipartite graphs constitute the class of nonbipartite matching covered graphs which are 'closest' to being bipartite. Thus, certain problems which are rather difficult to solve for general nonbipartite graphs are easier to solve for the special case of near-bipartite graphs; for instance, although there has been no significant progress in characterizing Pfaffian nonbipartite graphs, Fischer and Little [9] were able to characterize Pfaffian near-bipartite graphs.

The difficulty in using Theorem 1.11 as an induction tool for studying near-bipartite bricks, is that even if $G_k := G$ is a near-bipartite brick, there is no guarantee that all of the intermediate bricks $G_1, G_2, \ldots G_{k-1}$ are also near-bipartite. For instance, the brick shown in Figure 2a is near-bipartite with a (unique) removable doubleton $R := \{\alpha, \beta\}$. Although the edge e is thin; the retract of $G - e$, as shown in Figure 2b, is not near-bipartite since it has three edge-disjoint triangles.



(a)



(b)

Figure 2: (a) A near-bipartite brick G with a thin edge e ; (b) The retract of $G - e$ is not near-bipartite

In other words, deleting an arbitrary thin edge may not preserve the property of being near-bipartite. In this sense, the Thin Edge Theorem (1.10) is inadequate for obtaining inductive proofs of results that pertain only to the class of near-bipartite bricks.

To fix this problem, we decided to look for a thin edge whose deletion preserves the property of being near-bipartite. Our main result is as follows.

Theorem 1.13 *Every near-bipartite brick G distinct from K_4 and $\overline{C_6}$ has a thin edge e such that the retract of $G - e$ is also near-bipartite.*

In fact, we prove a stronger theorem. In particular, we find it convenient to fix a removable doubleton R (of the brick under consideration), and then look for a thin edge whose deletion preserves this removable doubleton. To make this precise, we will first define a special type of removable edge which we call 'R-compatible'.

1.4.1 R-compatible edges

We use the abbreviation *R-graph* for a near-bipartite graph G with (fixed) removable doubleton R, and we shall refer to $H := G - R$ as its *underlying bipartite graph*. In the same spirit, an *R-brick* is a brick with a removable doubleton R.

A removable edge e of an R-graph G is *R-compatible* if it is removable in H as well. Equivalently, an edge e is R-compatible if $G - e$ and $H - e$ are both matching covered. For instance, the graph St_8 (see Figure 3) has two removable doubletons $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$, and its unique removable edge e is R-compatible as well as R'-compatible.

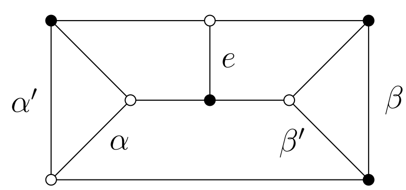


Figure 3: e is R-compatible as well as R'-compatible

Now, let G denote the R-brick shown in Figure 2a, where $R := \{\alpha, \beta\}$. The thin edge e is incident with an edge of R at a cubic vertex; consequently, $H - e$ has a vertex whose degree is only one, and so it is not matching covered. In particular, e is not R-compatible.

The brick shown in Figure 4 has two distinct removable doubletons $R := \{\alpha, \beta\}$ and $R' := \{\alpha', \beta'\}$. Its edges e and f are both R'-compatible, but neither of them is R-compatible.

Observe that, if e is an R-compatible edge of an R-graph G, then R is a removable doubleton of $G - e$, whence $G - e$ is also an R-graph; in particular, $G - e$ is near-bipartite. By Theorem 1.12, $G - e$ is a near-brick; and this proves the following.

Proposition 1.14 *Every R-compatible edge is b-invariant.* □

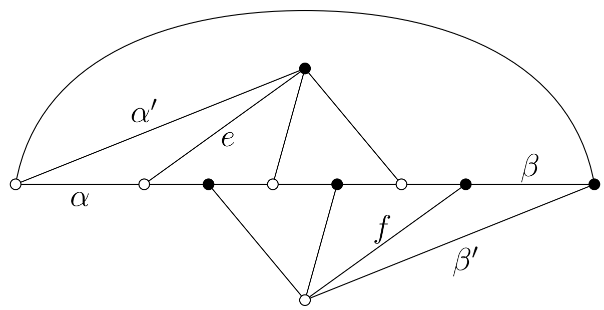


Figure 4: e and f are R'-compatible, but they are not R-compatible

Furthermore, as we will see later, if e is an R-compatible edge of an R-brick G then the unique brick J of $G-e$ is also an R-brick; in particular, J is near-bipartite. The following is a special case of a theorem of Carvalho, Lucchesi and Murty [2].

Theorem 1.15 [R-COMPATIBLE EDGE THEOREM] *Every R-brick distinct from K_4 and $\overline{C_6}$ has an R-compatible edge.*

In [2], they proved a stronger result. In particular, they showed the existence of an R-compatible edge in R-graphs with minimum degree at least three. (They did not use the term 'R-compatible'.) Using the notion of R-compatibility, we now define a thin edge whose deletion preserves the property of being near-bipartite.

1.4.2 R-thin edges

A thin edge e of an R-brick G is *R-thin* if it is R-compatible. Equivalently, an edge e is R-thin if it is R-compatible as well as thin, and in this case, the retract of $G-e$ is also an R-brick.

As noted earlier, the graph St_8, shown in Figure 3, has two removable doubletons R and R'. Its unique removable edge e is R-thin as well as R'-thin; to see this, note that the retract J of $St_8 - e$ is isomorphic to K_4 with multiple edges, and each of R and R' is a removable doubleton of J.

Using the R-compatible Edge Theorem (1.15) of Carvalho et al., we prove the following stronger result (which immediately implies Theorem 1.13).

Theorem 1.16 [R-THIN EDGE THEOREM] *Every R-brick distinct from K_4 and $\overline{C_6}$ has an R-thin edge.*

Our proof of the above theorem uses tools from the work of Carvalho et al. [6], and the overall approach is inspired by their proof of the Thin Edge Theorem (1.10). The following is an immediate consequence of Theorem 1.16.

Theorem 1.17 *Given any R-brick G, there exists a sequence $G_1, G_2, \ldots, G_k$ of R-bricks such that:*

(i) G_1 *is either* K_4 *or* $\overline{C_6}$*,*

(ii) $G_k := G$*, and*

(iii) for $2 \leq i \leq k$*, there exists an R-thin edge e_i of G_i such that G_{i-1} is the retract of $G_i - e_i$.*

It follows from the above theorem that every near-bipartite brick can be generated from one of K_4 and $\overline{C_6}$ by means of the expansion operations. Theorem 1.16 and its proof also appear in the Ph.D. thesis of Kothari [10].

2 Near-Bipartite Graphs

In this section, we examine properties of near-bipartite graphs that are relevant to our proof of Theorem 1.16. Recall that an R-graph G is a near-bipartite graph with a fixed removable doubleton R. We adopt the following notational conventions.

Notation 2.1 *For an R-graph G, we shall denote by $H[A, B]$ the underlying bipartite graph $G - R$. We let α and β denote the constituent edges of R, and we adopt the convention that $\alpha := a_1a_2$ has both ends in A, whereas $\beta := b_1b_2$ has both ends in B.*

As we will see, certain pertinent properties of G are closely related to those of H. For this reason, we also review well-known facts concerning bipartite matching covered graphs.

2.1 The exchange property

Recall that an edge of a matching covered graph is removable if its deletion results in another matching covered graph. The removable edges of a bipartite graph satisfy an 'exchange property' and its proof immediately follows from Proposition 1.4.

Proposition 2.2 [Exchange Property of Removable Edges] *Let H denote a bipartite matching covered graph, and let e denote a removable edge of H. If f is a removable edge of $H - e$, then:*

(i) f is removable in H, and

(ii) e is removable in $H - f$. □

We point out that the conclusion of Proposition 2.2 does not hold, in general, for arbitrary removable edges of nonbipartite graphs. For instance, as shown in Figure 1, the edge f is removable in the matching covered graph $St_8 - e$, but it is not removable in St_8. However, as we prove next, the exchange property does hold for R-compatible edges. Recall that an R-compatible edge of an R-graph G is one which is removable in G as well as in the underlying bipartite graph $H := G - R$; see Section 1.4.1.

Proposition 2.3 [Exchange Property of R-compatible Edges] *Let G be an R-graph, and let e denote an R-compatible edge of G. If f is an R-compatible edge of $G - e$, then:*

(i) f is R-compatible in G, and

(ii) e is R-compatible in $G - f$.

Proof: Let $H := G - R$. Since f is R-compatible in $G - e$, each of the graphs $G - e - f$ and $H - e - f$ is matching covered. To deduce *(i)*, we need to show that each of $G - f$ and $H - f$ is matching covered. Since f is removable in $H - e$, it follows from Proposition 2.2 that f is removable in H as well. That is, $H - f$ is matching covered.

Next, we note that the edge e is admissible in $H - f$. Thus e is admissible in $G - f$. As $G - e - f$ is matching covered, we conclude that $G - f$ is also matching covered. This proves *(i)*. Statement *(ii)* follows immediately, since each of $G - f - e$ and $H - f - e$ is matching covered. □

2.2 Non-removable edges of bipartite graphs

Let $H[A, B]$ denote a bipartite graph, on four or more vertices, that has a perfect matching. Using the well-known Hall's Theorem, it can be shown that an edge f of H is inadmissible (that is, f is not in any perfect matching of H) if and only if there exists a nonempty proper subset S of A such that $|N(S)| = |S|$ and f has one end in $N(S)$ and its other end is not in S.

Now suppose that H is matching covered, and let e denote a non-removable edge of H. Then some edge f of $H - e$ is inadmissible. This fact, coupled with the above observation, may be used to arrive at the following characterization of non-removable edges in bipartite matching covered graphs; see Figure 5.

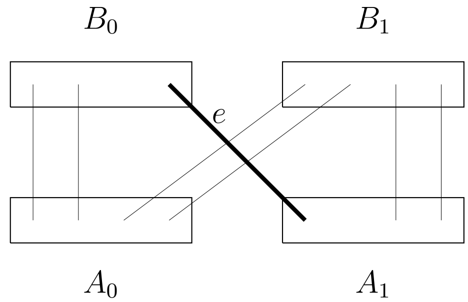


Figure 5: Non-removable edge of a bipartite graph

Proposition 2.4 [Characterization of Non-removable Edges] *Let $H[A, B]$ denote a bipartite matching covered graph on four or more vertices. An edge e of H is non-removable if and only if there exist partitions (A_0, A_1) of A and (B_0, B_1) of B such that $|A_0| = |B_0|$ and e is the only edge joining a vertex in B_0 to a vertex in A_1.* $\square$

In our work, we will often be interested in finding an R-compatible edge incident at a specified vertex v of an R-brick G. As a first step, we will upper bound the number of edges of $\partial(v)$, which are non-removable in the underlying bipartite graph $H := G - R$. For this purpose, the next lemma of Lovász and Vempala [13] is especially useful. It is an extension of Proposition 2.4. See Figure 6.

Lemma 2.5 [The Lovász-Vempala Lemma] *Let $H[A, B]$ denote a bipartite matching covered graph, and $b \in B$ denote a vertex of degree $d \geq 3$. Let $ba_1, ba_2, \ldots, ba_d$ be the edges of H incident with b. Assume that the edges $ba_1, ba_2, \ldots, ba_r$ where $0 < r \leq d$ are non-removable. Then there exist partitions $(A_0, A_1, \ldots, A_r)$ of A and $(B_0, B_1, \ldots, B_r)$ of B, such that $b \in B_0$, and for $i \in \{1, 2, \ldots, r\}$: (i) $|A_i| = |B_i|$, (ii) $a_i \in A_i$, and (iii) $N(A_i) = B_i \cup \{b\}$; in particular, ba_i is the only edge between B_0 and A_i.* $\square$

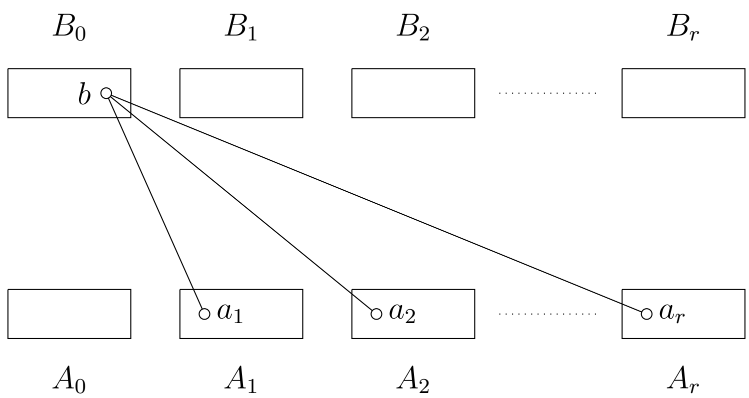


Figure 6: Non-removable edges incident at a vertex

Observe that, as per the notation in the above lemma, if ba_1 and ba_2 are non-removable edges, then the vertices a_1 and a_2 have no common neighbour distinct from b. That is, there is no 4-cycle containing edges ba_1 and ba_2 both. This proves the following corollary of Lovász and Vempala [13].

Corollary 2.6 *Let H denote a bipartite matching covered graph, and b denote a vertex of degree three or more. If e and f are two edges incident at b which lie in a 4-cycle Q then at least one of e and f is removable.* □

We conclude with an easy application of the Lovász-Vempala Lemma in the context of near-bipartite bricks.

Corollary 2.7 *Let G be an R-brick, and let $H := G - R$. Then for any vertex b, at most two edges of $\partial_H(b)$ are non-removable in H.*

Proof: We adopt Notation 2.1; assume without loss of generality that $b \in B$. If b has only two distinct neighbours in H then the assertion is easily verified. Now suppose that b has at least three distinct neighbours in H, and let d denote the degree of b in H.

Suppose instead that there are $r \geq 3$ non-removable edges incident with b; we denote these as $ba_1, ba_2, \ldots, ba_r$. Then, by the Lovász-Vempala Lemma (2.5), there exist partitions $(A_0, A_1, \ldots, A_r)$ of A and $(B_0, B_1, \ldots, B_r)$ of B, such that $b \in B_0$, and for $i \in \{1, 2, \ldots, r\}$: (i) $|A_i| = |B_i|$, (ii) $a_i \in A_i$, and (iii) $N_H(A_i) = B_i \cup \{b\}$. See Figure 6.

Observe that, for $i \in \{1, 2, \ldots, r\}$, every vertex of A_i is isolated in $H - (B_i \cup \{b\})$; consequently, $B_i \cup \{b\}$ is a nontrivial barrier of H. Since G is free of nontrivial barriers (by Theorem 1.7), adding the edges of R must kill each of these barriers. In particular, α must have an end in each A_i for $i \in \{1, 2, \ldots, r\}$. This is not possible, as $r \geq 3$; thus we have a contradiction. This completes the proof of Corollary 2.7. □

2.3 Barriers and tight cuts

We begin with a property of removable edges related to tight cuts which is easily verified; it holds for all matching covered graphs.

Proposition 2.8 *Let G be a matching covered graph, and $\partial(X)$ a tight cut of G, and e an edge of $G[X]$. Then e is removable in $G/\overline{X}$ if and only if e is removable in G.* □

Let us revisit the notion of a barrier cut. If S is a barrier of a matching covered graph G and K is an odd component of $G - S$ then $\partial(V(K))$ is a tight cut of G, and is referred to as a barrier cut. In Sections 2.3.1 and 2.3.2, among other things, we will see that every nontrivial tight cut of a bipartite or of a near-bipartite graph is a barrier cut.

2.3.1 Bipartite graphs

Suppose that X is an odd subset of the vertex set of a bipartite graph $H[A, B]$. Then, clearly one of the two sets $A \cap X$ and $B \cap X$ is larger than the other; the larger of the two sets, denoted X_+, is called the *majority part of* X; and the smaller set, denoted X_-, is called the *minority part of* X.

The following proposition is easily derived, and it provides a convenient way of visualizing tight cuts in bipartite matching covered graphs. See Figure 7.

Proposition 2.9 [Tight Cuts in Bipartite Graphs] *A cut $\partial(X)$ of a bipartite matching covered graph H is tight if and only if the following hold:*

(i) $|X|$ is odd and $|X_+| = |X_-| + 1$, consequently $|\overline{X}_+| = |\overline{X}_-| + 1$, and

(ii) there are no edges between X_- and $\overline{X}_-$. □

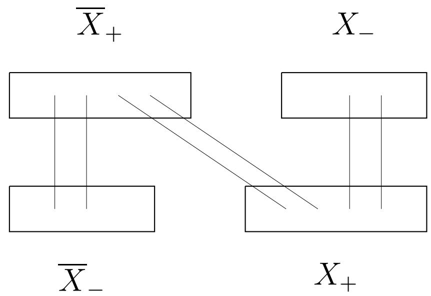


Figure 7: Tight cuts in bipartite matching covered graphs

Observe that, in the above proposition, X_+ and $\overline{X}_+$ are both barriers of H. It follows that every tight cut of a bipartite matching covered graph is a barrier cut.

Recall that, for a bipartite matching covered graph $H[A, B]$, its maximal barriers are precisely its color classes A and B. Now let S denote a nontrivial barrier of H which is not maximal, and adjust notation so that $S \subset B$. It may be inferred from Proposition 2.9 that $H - S$ has precisely $|S| - 1$ isolated vertices each of which is a member of A, and it has precisely one nontrivial odd component K which gives rise to a nontrivial barrier cut of H, namely $\partial(V(K))$.

Since braces are bipartite matching covered graphs which are free of nontrivial tight cuts, Proposition 2.9 may be used to obtain the following characterizations of braces.

Proposition 2.10 [Characterizations of Braces] *Let $H[A, B]$ denote a bipartite graph of order six or more, where $|A| = |B|$. Then the following statements are equivalent:*

(i) H is a brace,

(ii) $|N(S)| \geq |S| + 2$ for every nonempty subset S of A such that $|S| < |A| - 1$, and

(iii) $H - \{a_1, a_2, b_1, b_2\}$ has a perfect matching for any four distinct vertices $a_1, a_2 \in A$ and $b_1, b_2 \in B$. □

2.3.2 Near-Bipartite graphs

Let G denote an R-graph. We adopt Notation 2.1. For an odd subset X of $V(G)$, we define its *majority part* X_+ and its *minority part* X_- by regarding it as a subset of $V(H)$.

Observe that, if X is the shore of a tight cut in G then it is the shore of a tight cut in H as well. This observation, coupled with Proposition 2.9, may be used to derive the following characterization of tight cuts in near-bipartite graphs.

Proposition 2.11 [TIGHT CUTS IN NEAR-BIPARTITE GRAPHS] *A cut $\partial(X)$ of an R-graph G is tight if and only if the following hold:*

(i) *X is odd and $|X_+| = |X_-| + 1$, and consequently, $|\overline{X}_+| = |\overline{X}_-| + 1$,*

(ii) *there are no edges between X_- and $\overline{X}_-$; adjust notation so that $X_- \subset A$,*

(iii) *one of α and β has both ends in a majority part; adjust notation so that α has both ends in $\overline{X}_+$, and*

(iv) *β has at least one end in $\overline{X}_-$.*

Consequently, X_+ is a nontrivial barrier of G. Moreover, the $\partial(X)$-contraction G/X is near-bipartite with removable doubleton R, whereas the $\partial(X)$-contraction $G/\overline{X}$ is bipartite.

Proof: A simple counting argument shows that if all of the statements *(i)* to *(iv)* hold then $\partial(X)$ is indeed a tight cut of G. See Figure 8. Now suppose that $\partial(X)$ is a tight cut; as noted earlier, $\partial(X) - R$ is a tight cut of H. Thus *(i)* and *(ii)* follow immediately from Proposition 2.9. Adjust notation so that $X_- \subset A$.

As each perfect matching of G which contains α must also contain β, we infer that at most one of α and β lies in $\partial(X)$. Furthermore, if α has both ends in X_-, and likewise, if β has both ends in $\overline{X}_-$, then a simple counting argument shows that any perfect matching M of G containing α and β meets $\partial(X)$ in at least three edges; this is a contradiction.

The above observations imply that at least one of α and β has both ends in a majority part; this proves *(iii)*. As in the statement, adjust notation so that α has both ends in $\overline{X}_+$. Now, if β has both ends in X_+ then it is easily seen that α and β are both inadmissible. This proves *(iv)*. Note that, either β has both ends in $\overline{X}_-$ as shown in Figure 8a, or it has one end in $\overline{X}_-$ and the other end in X_+ as shown in Figure 8b.

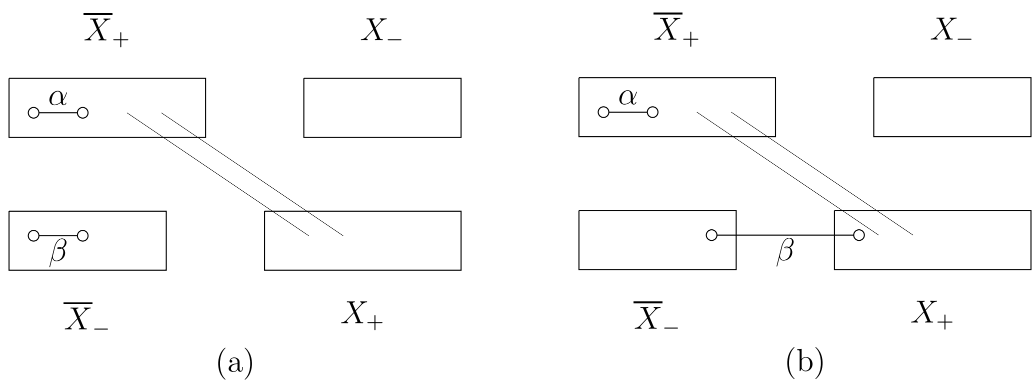


Figure 8: Tight cuts in near-bipartite graphs

Note that X_+ is a nontrivial barrier of G, and that $G/\overline{X}$ is bipartite. We let $G_1 := G/X$ denote the other $\partial(X)$-contraction. Observe that $H_1 := H/X$ is bipartite and matching covered. Furthermore, in G_1, α has both ends in one color class of H_1, and likewise, β has both ends in the other color class of H_1; this is true for each of the two cases shown in Figure 8. Since $H_1 = G_1 - R$, we infer that G_1 is near-bipartite with removable doubleton R. This completes the proof of Proposition 2.11. $\square$

Recall that a near-brick is a matching covered graph whose tight cut decomposition yields exactly one brick. The following is an immediate consequence of Proposition 2.11.

Corollary 2.12 *An R-graph G is a near-brick, and its unique brick is also near-bipartite with removable doubleton R.* $\square$

In other words, a near-bipartite graph G is a near-brick, and its unique brick, say J, inherits its removable doubletons. The *rank* of G, denoted $\mathsf{rank}(G)$, is the order of the unique brick of G. That is, $\mathsf{rank}(G) := |V(J)|$.

Proposition 2.11 shows that every tight cut of a near-bipartite graph is a barrier cut. Now, let S denote a nontrivial barrier of an R-graph G, and adjust notation so that $S \subset B$. It may be inferred from Proposition 2.11 that $G - S$ has precisely $|S| - 1$ isolated vertices each of which is a member of A, and it has precisely one nontrivial odd component K which yields a nontrivial tight cut of G, namely $\partial(V(K))$. Thus there is a bijective correspondence between the nontrivial barriers of G and its nontrivial tight cuts.

2.4 The Three Case Lemma

Recall that a removable edge e of a brick G is b-invariant if $G - e$ is a near-brick. In this section, we will discuss a lemma of Carvalho, Lucchesi and Murty [4] that pertains to the structure of such near-bricks, that is, those which are obtained from a brick by deleting a single edge. This lemma is used extensively in their works [3, 6, 7], and it will play a vital role in the proof of Theorem 1.16.

We will restrict ourselves to the case in which G is an R-brick and e is R-compatible. (By Proposition 1.14, e is b-invariant.) We adopt Notation 2.1. As the name of the lemma suggests, there will be three cases, depending on which we say that the 'index' of e is zero, one or two. In particular, the index of e (defined later) will be zero if $G - e$ is a brick.

Now consider the situation in which $G-e$ is not a brick; that is, $G-e$ has a nontrivial tight cut. By Proposition 2.11, $G - e$ has a nontrivial barrier; let S be such a barrier which is also maximal, and adjust notation so that $S \subset B$. We let I denote the set of isolated vertices of $(G - e) - S$; note that $I \subset A$. Since G itself is free of nontrivial barriers, we infer that one end of e lies in I and its other end lies in $B - S$. This observation, coupled with the Canonical Partition Theorem (1.3) and the fact that e has only two ends, implies that $G-e$ has at most two maximal nontrivial barriers; furthermore, if it is has two such barriers then one is a subset of A and the other is a subset of B.

The *index* of e, denoted $\mathsf{index}(e)$, is the number of maximal nontrivial barriers in $G - e$. It follows from the preceding paragraph that the index of e is either zero, one or two; and these form the three cases. This is the gist of the lemma; apart from this, it provides further information in the index two case which is especially useful to us. We now state the Three Case Lemma [6], as it is applicable to an R-compatible edge of an R-brick; see Figures 9 and 10. (The reason for the asymmetry in our notation in Case (2) is discussed in Section 2.4.2.)

Lemma 2.13 [The Three Case Lemma] *Let G be an R-brick, and e an R-compatible edge. Let $H[A, B] := G - R$. Then one of the following three alternatives holds:*

(0) *$G - e$ is a brick.*

(1) *$G-e$ has only one maximal nontrivial barrier, say S. Adjust notation so that $S \subset B$. Let I denote the set of isolated vertices of $(G-e)-S$. Then $I \subset A$, and e has one end in I and other end in $B-S$.*

(2) *$G-e$ has two maximal nontrivial barriers, say S_1 and S_2^*. Adjust notation so that $S_1 \subset B$ and $S_2^* \subset A$. Let I_1 denote the set of isolated vertices of $(G-e)-S_1$, and I_2^* the set of isolated vertices of $(G-e)-S_2^*$. Then the following hold:*

   (i) *$I_1 \subset A$ and $I_2^* \subset B$;*

   (ii) *e has one end in $I_1 - S_2^*$ and other end in $I_2^* - S_1$;*

   (iii) *$S_2 := S_2^* - I_1$ is the unique maximal nontrivial barrier of $(G-e)/X_1$, where $X_1 := S_1 \cup I_1$; furthermore, S_2 is a barrier of $G-e$ as well, and $I_2 := I_2^* - S_1$ is the set of isolated vertices of $(G-e)-S_2$.* □

Now, let e denote an R-compatible edge of an R-brick G. By the *rank* of e, denoted $\mathsf{rank}(e)$, we mean the rank of the R-graph $G-e$. That is, $\mathsf{rank}(e) := \mathsf{rank}(G-e)$. Recall that e is R-thin if the retract of $G-e$ is a brick. In particular, every R-compatible edge of index zero is R-thin, and these are the only edges whose rank equals $n := |V(G)|$.

In what follows, we will further discuss the cases in which the index of e is either one or two; in each case, we shall relate the rank of e with the information provided by the Three Case Lemma, and we examine the conditions under which e is R-thin. These discussions are especially relevant to Section 3.2.

We adopt Notation 2.1. Let y and z denote the ends of e such that $y \in A$ and $z \in B$. Note that, if y is cubic, then the two neighbours of y in $G-e$ constitute a barrier of $G-e$; a similar statement holds for z. It follows that if both ends of e are cubic then the index of e is two.

2.4.1 Index one

Suppose that the index of e is one. As in case (1) of the Three Case Lemma, we let S denote the unique maximal nontrivial barrier of $G-e$, and I the set of isolated vertices of $(G-e)-S$. Note that $|I| = |S| - 1$. We adjust notation so that $S \subset B$ and $I \subset A$; see Figure 9. Observe that $y \in I$ and $z \in B - S$.

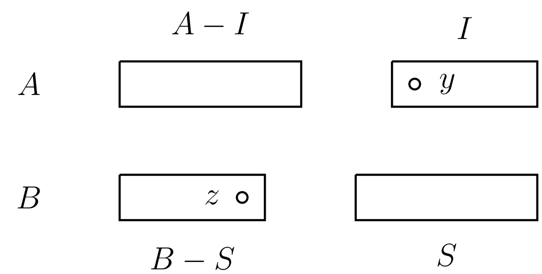


Figure 9: An R-compatible edge of index one

In this case, $G - e$ has a unique nontrivial tight cut $\partial(X)$, where $X := S \cup I$. Consequently, $(G - e)/X$ is the brick of $G - e$, and the rank of e is $|V(G) - X| + 1$. Furthermore, e is R-thin if and only if $|S| = 2$; and in this case, y is cubic, $N(y) = S \cup \{z\}$, and $\mathsf{rank}(e) = n - 2$.

2.4.2 Index two

Suppose that the index of e is two. As in case (2) of the Three Case Lemma, we let S_1 denote one of the two maximal nontrivial barriers of $G - e$, and I_1 the set of isolated vertices of $(G - e) - S_1$, adjusting notation so that $S_1 \subset B$ and $I_1 \subset A$. Note that $|I_1| = |S_1| - 1$ and that $y \in I_1$; see Figure 10.

Now, let S_2^* denote the unique maximal nontrivial barrier of $G - e$ which is a subset of A, and I_2^* the set of isolated vertices of $(G - e) - S_2^*$. As in the index one case (see Figure 9), we would like to break $V(G)$ into disjoint subsets in order to be able to compute the rank of e. However, this is complicated by the possibility that $S_2^* \cap I_1$ may be nonempty. This explains the asymmetry in our notation in case (2). Fortunately, it turns out that $S_2 := S_2^* - I_1$ is the only maximal nontrivial barrier of $(G - e)/X_1$, where $X_1 := S_1 \cup I_1$. Furthermore, S_2 is a barrier of $G - e$ as well, and $I_2 := I_2^* - S_1$ is the set of isolated vertices of $(G - e) - S_2$. Note that $|I_2| = |S_2| - 1$ and that $z \in I_2$; see Figure 10. We let $X_2 := S_2 \cup I_2$.

In this case, $\partial(X_1)$ and $\partial(X_2)$ are both tight cuts of $G - e$; more importantly, $\partial(X_2)$ is the unique tight cut of $(G - e)/X_1$. Consequently, $((G - e)/X_1)/X_2$ is the brick of $G - e$, and the rank of e is $|V(G) - X_1 - X_2| + 2$.

Furthermore, e is R-thin if and only if $|S_1| = 2 = |S_2|$; and in this case, y and z are both cubic, $N(y) = S_1 \cup \{z\}$ and $N(z) = S_2 \cup \{y\}$, and

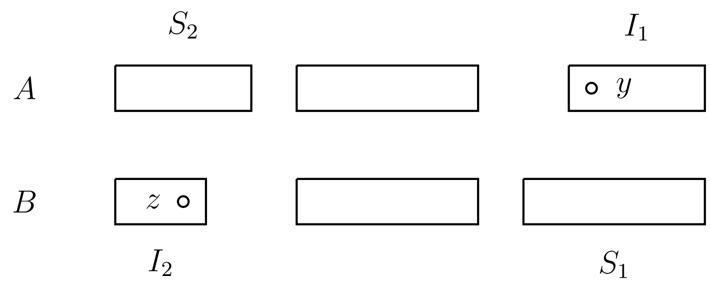


Figure 10: An R-compatible edge of index two

$\mathsf{rank}(e) = n - 4$; also, by switching the roles of S_1 and S_2^*, we infer that $|S_2^*| = 2$.

2.4.3 Index and Rank of an R-thin Edge

The following characterization of R-thin edges is immediate from our discussion in the previous two sections.

Proposition 2.14 [Characterization of R-thin Edges in terms of Barriers] *An R-compatible edge e of an R-brick G is R-thin if and only if every barrier of $G - e$ has at most two vertices.* □

In summary, if the index of e is zero then e is thin and its rank is $n := |V(G)|$. If the index of e is one then $\mathsf{rank}(e) \leq n - 2$, and equality holds if and only if e is thin. Likewise, if the index of e is two then $\mathsf{rank}(e) \leq n - 4$, and equality holds if and only if e is thin.

The following proposition gives an equivalent definition of index of an R-thin edge.

Proposition 2.15 *Let G be an R-brick, and e an R-thin edge. Then the following statements hold:*

(i) $\mathsf{index}(e) = 0$ *if and only if both ends of e have degree four or more in G;*

(ii) $\mathsf{index}(e) = 1$ *if and only if exactly one end of e has degree three in G; and*

(iii) $\mathsf{index}(e) = 2$ *if and only if both ends of e have degree three in G and e does not lie in a triangle.*

Proof: We note that $\mathsf{index}(e) = 0$ if and only if $G - e$ is free of nontrivial barriers, that is, $G - e$ is a brick; and since e is a thin edge, the latter holds if and only if both ends of e have degree four or more in G. This proves *(i)*.

Let $n := |V(G)|$. We note that $\mathsf{index}(e) = 1$ if and only if $\mathsf{rank}(e) = n-2$; and since e is a thin edge, the latter holds if and only if exactly one end of e has degree three in G.

Now suppose that $\mathsf{index}(e) = 2$, whence $\mathsf{rank}(e) = n-4$, and consequently, both ends of e have degree three in G. Conversely, if both ends of e have degree three in G then $G-e$ has two nontrivial barriers which lie in different color classes of $(G-e) - R$, and thus $\mathsf{index}(e) = 2$; furthermore, since e is R-compatible, neither end of e is incident with an edge of R and thus e does not lie in a triangle. $\square$

3 Generating Near-Bipartite Bricks

In this section, our goal is to prove the R-thin Edge Theorem (1.16). In fact, we will prove a stronger result, as described below.

Let G be an R-brick distinct from K_4 and $\overline{C_6}$. Then, by Theorem 1.15 of Carvalho et al., G has an R-compatible edge; let e be any such edge. Recall from Section 2.4 that there are two parameters associated with e: the rank of e is the order of the unique brick of $G-e$; and, the index of e is the number of maximal nontrivial barriers of $G-e$, which by the Three Case Lemma (2.13) is either zero, one or two. Using these parameters, we may state our stronger theorem as follows.

Theorem 3.1 *Let G be an R-brick which is distinct from K_4 and $\overline{C_6}$, and let e denote an R-compatible edge of G. Then one of the following alternatives hold:*

- *either e is R-thin,*
- *or there exists another R-compatible edge f such that:*
  - *(i) f has an end each of whose neighbours in $G-e$ lies in a barrier of $G-e$, and*
  - *(ii)* $\mathsf{rank}(f) + \mathsf{index}(f) > \mathsf{rank}(e) + \mathsf{index}(e)$.

Since the rank and index are bounded quantities, the above theorem immediately implies the R-thin Edge Theorem (1.16). Our proof uses tools from the work of Carvalho et al. [6], and the overall approach is inspired by their proof of the Thin Edge Theorem (1.10).

The following proposition shows that condition *(ii)* in Theorem 3.1 is implied by a weaker condition involving only the rank function.

Proposition 3.2 *Suppose that e and f denote two R-compatible edges of an R-brick G. If $\mathsf{rank}(f) > \mathsf{rank}(e)$ then $\mathsf{rank}(f) + \mathsf{index}(f) > \mathsf{rank}(e) + \mathsf{index}(e)$.*

<u>Proof</u>: Note that, since the rank of an edge is even, $\mathsf{rank}(f) > \mathsf{rank}(e) + 1$. As the index of an edge is either zero, one or two, we only need to examine the case in which $\mathsf{index}(e) = 2$ and $\mathsf{index}(f) = 0$. However, in this case, $\mathsf{rank}(f) = n$ and $\mathsf{rank}(e) \leq n - 4$ where $n := |V(G)|$, and thus the conclusion holds. $\square$

In the statement of Theorem 3.1, if the given R-compatible edge e is thin, then the assertion is vacuously true. Thus, in its proof, we may assume that e is not thin. It then follows from Proposition 2.14 that $G - e$ has a barrier with three or more vertices; let S be such a barrier. In the next section, we introduce the notion of a candidate edge (relative to e and S) which, as we will see, is an R-compatible edge that satisfies condition *(i)* in the statement of Theorem 3.1, and has rank at least that of e.

3.1 The candidate set $\mathcal{F}(e, S)$

Let G be an R-brick, and let $e := yz$ denote an R-compatible edge which is not thin. We first set up some notation and conventions which are used in the rest of this paper.

Notation 3.3 *We shall denote by $H[A, B]$ the underlying bipartite graph $G - R$. We let $R := \{\alpha, \beta\}$; and we adopt the convention that $\alpha := a_1a_2$ has both ends in A, whereas $\beta := b_1b_2$ has both ends in B. Adjust notation so that $y \in A$ and $z \in B$.*

The reader is advised to review Section 2.3.2 before proceeding further. Let S be a barrier of $G - e$ such that $|S| \geq 3$, and I the set of isolated vertices of $(G - e) - S$. Adjust notation so that $S \subset B$ and $I \subset A$, as shown in Figure 11a. Observe that $X := S \cup I$ is the shore of a tight cut in $G - e$, as well as in $H - e$. By Proposition 2.11, α has both ends in $A - I$; whereas β either has both ends in $B - S$, or it has one end in $B - S$ and another in S. We denote the bipartite matching covered graph

$$(H - e)/\overline{X} \to \overline{x}$$

by $H(e, S)$. Note that its color classes are the sets $I \cup \{\overline{x}\}$ and S; see Figure 11b.

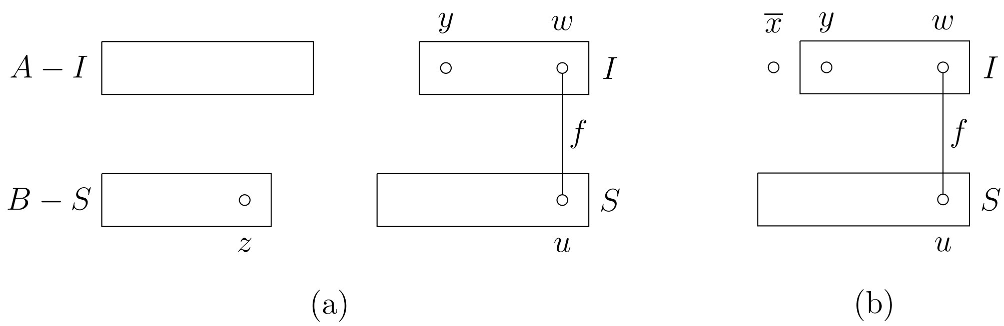


Figure 11: (a) S is a barrier of $G - e$ such that $|S| \geq 3$; (b) the bipartite graph $H(e, S)$

Definition 3.4 [The Candidate Set $\mathcal{F}(e, S)$] *We denote by $\mathcal{F}(e, S)$ the set of those removable edges of $H(e, S)$ which are not incident with the contraction vertex $\overline{x}$, and we refer to it as the candidate set (relative to e and the barrier S of $G - e$), and each member of $\mathcal{F}(e, S)$ is called a candidate edge.*

We remark that Carvalho et al. [6] used a similar notion. Since their work concerns general bricks (that is, not just near-bipartite ones), they consider the graph $(G - e)/\overline{X} \to \overline{x}$ and its removable edges which are not incident with the contraction vertex. See Lemma 23 and Theorem 24 in [6].

Now, let $f := uw$ denote a member of the candidate set $\mathcal{F}(e, S)$, as shown in Figure 11b. The end w of f lies in I, and all of the neighbours of w, in $G - e$, lie in the barrier S; consequently, f satisfies condition *(i)*, Theorem 3.1. It should be noted that e and f are adjacent if and only if w is the same as y. We now show that f is an R-compatible edge and it has rank at least that of e. The argument pertaining to ranks is the same as that in [6, Lemma 26].

Proposition 3.5 [Properties of Candidate Edges] *Every member of $\mathcal{F}(e, S)$ is an R-compatible edge of $G - e$, and of G, and has rank at least that of e. Conversely, each R-compatible edge of $G - e$, which is incident with a vertex of I, is a member of $\mathcal{F}(e, S)$.*

Proof: Let f be any member of $\mathcal{F}(e, S)$, as shown in Figure 11b. We will use Proposition 2.8 to show that f is R-compatible in $G - e$.

Observe that $H(e, S)$ is one of the C-contractions of $H - e$, where $C := \partial(X) - e - R$ is a tight cut. Since f is removable in $H(e, S)$ and $f \notin C$, Proposition 2.8 implies that f is removable in $H - e$ as well. A similar argument shows that f is removable in $G - e$. Thus, f is R-compatible in $G-e$; the exchange property (Proposition 2.3) implies that f is R-compatible in G as well.

Note that since both ends of f are in the bipartite shore X, the brick of $G-e-f$ is the same as the brick of $G-e$. In particular, $\mathsf{rank}(G-e-f) = \mathsf{rank}(G-e)$. On the other hand, note that if D is any tight cut of $G-f$ then $D - e$ is a tight cut of $G - e - f$, whence $\mathsf{rank}(G - f) \geq \mathsf{rank}(G - e - f)$. Thus $\mathsf{rank}(f) \geq \mathsf{rank}(e)$. This proves the first statement.

Now suppose that f is an R-compatible edge of $G - e$ which is incident at some vertex of I. In particular, $H - e - f$ is matching covered; that is, f is removable in $H - e$. By Proposition 2.8, f is removable in $H(e, S)$. This completes the proof of Proposition 3.5. $\square$

In summary, we have shown that every candidate edge is R-compatible; furthermore, it satisfies condition *(i)*, Theorem 3.1; and it has rank at least that of e.

The following property of candidate sets will be useful in dealing with those nontrivial barriers of $G - e$ which are not maximal.

Corollary 3.6 *Let S^* be any barrier of $G - e$. If $S \subset S^*$ then $\mathcal{F}(e, S) \subset \mathcal{F}(e, S^*)$.*

Proof: Let f be a member of $\mathcal{F}(e, S)$. Then f is incident with some vertex of I, say w. Note that w also lies in I^* which denotes the set of isolated vertices of $(G - e) - S^*$.

As f is a member of $\mathcal{F}(e, S)$, Proposition 3.5 implies that f is R-compatible in $G - e$. Consequently, since f is incident at $w \in I^*$, the last assertion of

Proposition 3.5, with S^* playing the role of S, implies that f is a member of $\mathcal{F}(e, S^*)$. Thus $\mathcal{F}(e, S) \subset \mathcal{F}(e, S^*)$. □

Now, we will prove two lemmas; each of which gives an upper bound on the number of non-removable edges incident at a vertex of the bipartite graph $H(e, S)$, which is distinct from the contraction vertex $\overline{x}$. Both of them are easy applications of the Lovász-Vempala Lemma (2.5); we will use arguments similar to those in the proof of Corollary 2.7.

Lemma 3.7 *Let u denote a vertex of S which has degree three or more in $H(e, S)$. Then at most two edges of $\partial(u) - \beta$ are non-removable in $H(e, S)$. Furthermore, if precisely two edges of $\partial(u) - \beta$ are non-removable in $H(e, S)$ and if vertices u and $\overline{x}$ are adjacent then the edge $u\overline{x}$ is non-removable in $H(e, S)$.*

Proof: Assume that there are $k \geq 1$ non-removable edges incident with the vertex u, namely, $uw_1, uw_2, \ldots, uw_k$. Then, by Lemma 2.5, there exist partitions $(A_0, A_1, \ldots, A_k)$ of $I \cup \{\overline{x}\}$, and $(B_0, B_1, \ldots, B_k)$ of S, such that $u \in B_0$, and for $j \in \{1, 2, \ldots, k\}$: (i) $|A_j| = |B_j|$, (ii) $w_j \in A_j$ and (iii) $N(A_j) = B_j \cup \{u\}$. See Figure 12.

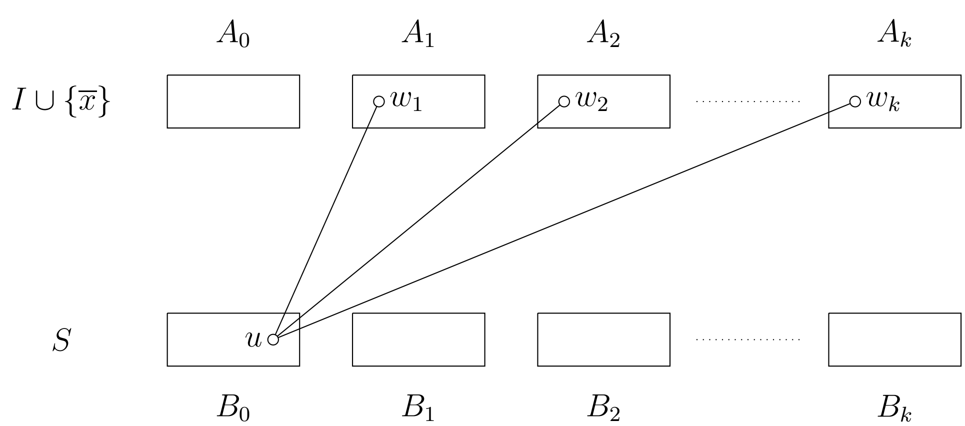


Figure 12: Illustration for Lemma 3.7

For $1 \leq j \leq k$, note that $B_j \cup \{u\}$ is a barrier of $H(e, S)$. Moreover, if the set A_j contains neither the contraction vertex $\overline{x}$ nor the end y of e, then $B_j \cup \{u\}$ is a barrier of G itself, which is not possible as G is a brick. We thus arrive at the conclusion that $k \leq 2$, which proves the first part of the assertion.

Now consider the case when $k = 2$. It follows from the above argument that one of the vertices y and $\overline{x}$ lies in the set A_1, whereas the other vertex lies in the set A_2. Adjust notation so that $y \in A_1$ and $\overline{x} \in A_2$. Observe that if u and $\overline{x}$ are adjacent, then $u\overline{x}$ is the unique edge between B_0 and A_2, and it is non-removable in $H(e, S)$ by assumption. This completes the proof of Lemma 3.7. □

Now we turn to the examination of non-removable edges of $H(e, S)$ incident with vertices in I. The proof is similar to that of Lemma 3.7, except that the roles of the color classes S and $I \cup \{\overline{x}\}$ are interchanged.

Lemma 3.8 *Let w denote a vertex of I which has degree three or more in $H(e, S)$. Then at most two edges of $\partial(w) - e$ are non-removable in $H(e, S)$. Furthermore, if precisely two edges of $\partial(w) - e$ are non-removable in $H(e, S)$ then the following hold:*

(i) an end of β lies in S; adjust notation so that $b_1 \in S$,

(ii) in $H(e, S)$, the vertices b_1 and $\overline{x}$ are nonadjacent,

(iii) if b_1 and w are adjacent then the edge b_1w is non-removable in $H(e, S)$, and

(iv) w is distinct from the end y of e.

Proof: Suppose that there exist $k \geq 1$ non-removable edges incident at the vertex w, namely, $wu_1, wu_2, \ldots, wu_k$. Then, by Lemma 2.5, there exist partitions $(A_0, A_1, \ldots, A_k)$ of the color class $I \cup \{\overline{x}\}$, and $(B_0, B_1, \ldots, B_k)$ of the color class S, such that $w \in A_0$, and for $j \in \{1, 2, \ldots, k\}$: (i) $|A_j| = |B_j|$, (ii) $u_j \in B_j$ and (iii) $N(B_j) = A_j \cup \{w\}$. See Figure 13.

For $1 \leq j \leq k$, note that $A_j \cup \{w\}$ is a barrier of $H(e, S)$. Furthermore, if the contraction vertex $\overline{x}$ is not in A_j, or if an end of the edge β is not in B_j, then $A_j \cup \{w\}$ is a barrier of G itself, which is absurd since G is a brick. Clearly, this would be the case for some $j \in \{1, 2, \ldots, k\}$ if $k \geq 3$. We conclude that $k \leq 2$, thus establishing the first part of the assertion.

Now suppose that $k = 2$. It follows from the preceding paragraph that an end of β lies in B_1 or in B_2. This proves *(i)*. Adjust notation so that

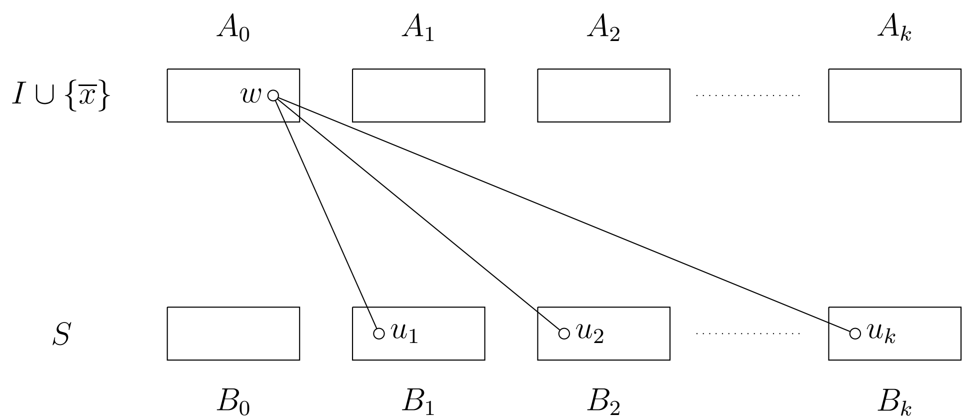


Figure 13: Illustration for Lemma 3.8

$b_1 \in B_1$. Furthermore, the contraction vertex $\overline{x}$ lies in A_2. Consequently, vertices b_1 and $\overline{x}$ are nonadjacent; this verifies *(ii)*. Note that if b_1 and w are adjacent, then the edge $b_1 w$ is the unique edge between A_0 and B_1, and it is non-removable in $H(e, S)$ by assumption. This proves *(iii)*. Finally, consider the case in which $w = y$, where y is the end of e in I. Observe that the neighbourhood of $A_0 - y$ lies in the set B_0 in the graph $H(e, S)$ as well as in G, whence B_0 is a barrier of G. We conclude that $|B_0| = 1$, and that y is the only vertex of A_0. Furthermore, the neighbourhood of A_1 lies in $B_1 \cup B_0$, and thus $B_1 \cup B_0$ is a nontrivial barrier in $H(e, S)$ as well as in G, which is absurd. We conclude that w is distinct from the end y of e; thus *(iv)* holds. This completes the proof of Lemma 3.8. □

The above lemma implies that each vertex of I, except possibly the end y of e, is incident with at least one candidate. Furthermore, if y has degree three or more in $H(e, S)$ then y is incident with at least two candidates; and likewise, if any other vertex of I, say w, has degree four or more then w is incident with at least two candidates. We thus have the following corollary which is used in the next section.

Corollary 3.9 *The candidate set $\mathcal{F}(e, S)$ has cardinality at least $|S| - 2$. (In particular, the set $\mathcal{F}(e, S)$ is nonempty.) Furthermore, if $\mathcal{F}(e, S)$ is a matching then each vertex of I is cubic in G and $|\mathcal{F}(e, S)| = |S| - 2$.* □

As we will see later, by a result of Carvalho et al. (Corollary 3.19), if the candidate set $\mathcal{F}(e, S)$ is not a matching then it has a member whose rank is strictly greater than that of e. For this reason, in the proof of Theorem 3.1, we will mainly have to deal with the case in which the candidate set is a matching.

3.1.1 When the candidate set is a matching

In this section, we suppose that the candidate set $\mathcal{F}(e, S)$ is a matching. We will make several observations, and these will be useful to us in Section 3.3 where the proof of Theorem 3.1 is presented. For all of the figures in the rest of this paper, the solid vertices are those which are known to be cubic in the brick G; the hollow vertices may or may not be cubic.

Since $\mathcal{F}(e, S)$ is a matching, Corollary 3.9 implies that every vertex of I is cubic in G, as shown in Figure 14. Furthermore, each of these vertices, except for the end y of e, is incident with exactly one candidate edge; in particular, $|\mathcal{F}(e, S)| = |I| - 1 = |S| - 2$.

Notation 3.10 *We let $w_1, w_2, \ldots, w_k$ denote the vertices of $I - y$, where $k := |S| - 2$, and for $1 \leq j \leq k$, denote the edge of $\mathcal{F}(e, S)$ incident with w_j by f_j and its end in S by u_j.*

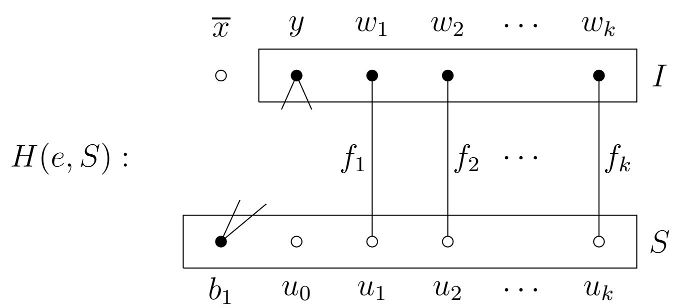


Figure 14: When $\mathcal{F}(e, S)$ is a matching

Note that, since $\mathcal{F}(e, S)$ is a matching, the vertices $u_1, u_2, \ldots, u_k$ are distinct, as shown in Figure 14. Since every vertex of I is incident with two non-removable edges of $H(e, S)$, we deduce the following by assertions *(i)*, *(ii)* and *(iii)* of Lemma 3.8, respectively:

(1) an end of β lies in S; adjust notation so that $b_1 \in S$,

(2) in $H(e, S)$, vertices b_1 and $\overline{x}$ are nonadjacent; consequently, in G, all neighbours of b_1, except b_2, lie in I, and

(3) b_1 is distinct from each of $u_1, u_2, \ldots, u_k$.

Furthermore, since b_1 is not incident with any member of $\mathcal{F}(e, S)$, Lemma 3.7 implies that it has precisely two neighbours in I; in particular, b_1 is cubic in G.

Notation 3.11 *We let u_0 denote the vertex of S which is distinct from $b_1, u_1, u_2, \ldots, u_k$. That is, $S = \{b_1, u_0, u_1, u_2, \ldots, u_k\}$. (See Figure 14.)*

As the vertex u_0 is not incident with any candidate, we conclude using Lemma 3.7 that u_0 has at most one neighbour in I. Observe that if u_0 has no neighbours in I then $(S - u_0) \cup \{z\}$ is a barrier of G (where z is the end of e which is not in I), which is absurd as G is a brick. Thus, u_0 has precisely one neighbour in I.

We note that if y is the unique neighbour of u_0 in the set I, then $S - u_0$ is a barrier of G, which leads us to the same contradiction as before. We thus conclude that u_0 has precisely one neighbour in the set $I - y$, and that its remaining neighbours lie in $\overline{X}$; see Figure 15. In particular, in $H(e, S)$, there are are least two edges between u_0 and $\overline{x}$.

$H(e, S)$:

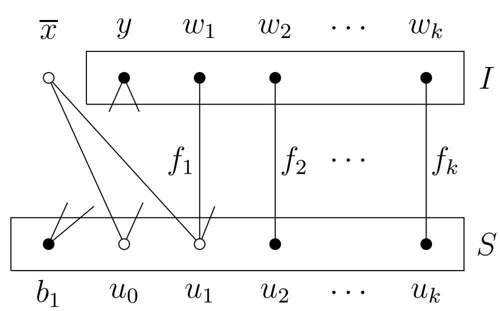


Figure 15: u_0 and u_1 are the only vertices adjacent with the contraction vertex $\overline{x}$

Finally, since each vertex u_j in the set $\{u_1, u_2, \ldots, u_k\}$ is incident with exactly one candidate, Lemma 3.7 implies that u_j must satisfy one of the following conditions:

(i) either u_j has some neighbour in the set $\overline{X}$ and it has precisely two neighbours in the set I,

(ii) or otherwise, u_j has no neighbours in the set $\overline{X}$ and it has precisely three neighbours in the set I.

Observe, by counting degrees of the vertices in I, that there are precisely $3k+2$ edges with one end in I and the other end in S. Of these $3k+2$ edges, precisely two are incident with b_1, and only one is incident with u_0. Thus there are $3k-1$ edges with one end in I and the other end in $\{u_1, u_2, \ldots, u_k\}$. It follows immediately that exactly one vertex among $u_1, u_2, \ldots, u_k$ satisfies condition (i); every other vertex satisifes condition (ii).

Notation 3.12 *We adjust notation so that u_1 is the only vertex in $\{u_1, u_2, \ldots, u_k\}$ which has neighbours in $\overline{X}$. (See Figure 15.)*

Adopting the notation introduced thus far, the next proposition summarizes our observations in terms of the brick G.

Proposition 3.13 [When the Candidate Set is a Matching] *The following hold:*

(i) each vertex of I is cubic,

(ii) b_1 is cubic and its neighbours lie in $I \cup \{b_2\}$,

(iii) u_0 has precisely one neighbour in $I - y$, and all of its remaining neighbours lie in $\overline{X}$,

(iv) u_1 has precisely two neighbours in I, and all of its remaining neighbours lie in $\overline{X}$,

(v) if $|S| \geq 4$, then each vertex of $S - \{b_1, u_0, u_1\}$ has precisely three neighbours and these neighbours lie in I. □

Observe that, if the barrier S has precisely three vertices, then the candidate set $\mathcal{F}(e, S)$ has only one edge (that is, $f_1 = u_1 w_1$); in this case, all of the edges of $G[X]$ are determined by Proposition 3.13, as listed below, and as shown in Figure 16. (Note that the underlying simple graph of $H(e, S)$ is a ladder of order six whose cubic vertices are u_1 and w_1.)

Remark 3.14 *Suppose that* $|S| = 3$. *Then the following hold:*

(i) the three neighbours of b_1 *are* y, w_1 *and* b_2,

(ii) u_0 *is adjacent with* w_1, *and all of its remaining neighbours lie in* $\overline{X}$,

(iii) u_1 *is adjacent with* y *and with* w_1, *and all of its remaining neighbours lie in* $\overline{X}$.

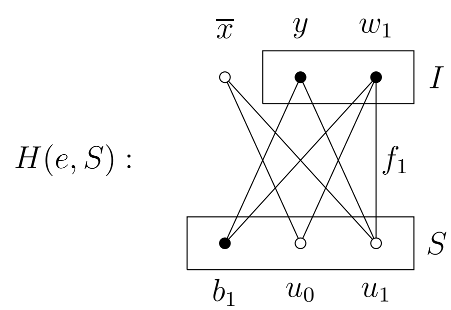


Figure 16: When $\mathcal{F}(e, S)$ is a matching, and S has only three vertices

We shall now consider the situation in which $|S| \geq 4$, that is, $k \geq 2$. Note that, as per our notation, $f_1 = u_1w_1$ is the only candidate whose end in S (that is, u_1) has a neighbour in $\overline{X}$. In this sense, f_1 is different from the remaining candidates $f_2, f_3, \ldots, f_k$. In the following proposition, we first show that b_1 is nonadjacent with the end w_1 of f_1. Consequently, b_1 is adjacent with at least one of $w_2, w_3, \ldots, w_k$; we shall assume without loss of generality that b_1 is adjacent with w_2, as shown in Figure 17. In its proof, we will apply the Lovász-Vempala Lemma (2.5) to the graph $H(e, S)$, first at w_1, and then at w_2; each of these applications is a refinement of the situation in Lemma 3.8.

Proposition 3.15 *Suppose that* $|S| \geq 4$. *Then the following hold:*

(i) b_1 *and* w_1 *are nonadjacent; adjust notation so that* b_1w_2 *is an edge of* G,

(ii) y *is adjacent with each of* b_1 *and* u_2, *and*

(iii) u_0 *and* w_2 *are nonadjacent.*

Proof: First, we apply Lemma 2.5 to the graph $H(e, S)$ at vertex w_1. Since $f_1 = u_1w_1$ is the only removable edge incident with w_1, there exist partitions (A_0, A_1, A_2) of $I \cup \{\overline{x}\}$, and (B_0, B_1, B_2) of S, such that $w_1 \in A_0$, and $|A_j| = |B_j|$ for $j \in \{0, 1, 2\}$, vertex u_1 lies in B_0, and the remaining two neighbours of w_1 lie in B_1 and in B_2, respectively. Furthermore, $N(B_1) = A_1 \cup \{w_1\}$ and $N(B_2) = A_2 \cup \{w_1\}$.

Suppose that b_1 is a neighbour of w_1, and adjust notation so that $b_1 \in B_1$. The contraction vertex $\overline{x}$ lies in A_2, since otherwise $A_2 \cup \{w_1\}$ is a nontrivial barrier in G. We will deduce that each of the sets B_0, B_1 and B_2 is a singleton, and thus the barrier S has precisely three vertices, contrary to the hypothesis.

First of all, note that the neighbourhood of $B_1 - b_1$ is contained in A_1, and thus if $|A_1| \geq 2$ then A_1 is a nontrivial barrier in G; we conclude that $|A_1| = 1$ and that $B_1 = \{b_1\}$. Observe that the contraction vertex $\overline{x}$ is only adjacent with u_1, which lies in B_0, and with u_0. Thus the neighbourhood of $B_2 - u_0$ is contained in $(A_2 - \overline{x}) \cup \{w_1\}$, whence the latter is a barrier of G; we infer that $A_2 = \{\overline{x}\}$; consequently, the unique vertex of B_2 has precisely two neighbours, namely w_1 and $\overline{x}$. It follows that $B_2 = \{u_0\}$. Since the vertex w_1 is cubic, the neighbourhood of $B_0 - u_1$ is contained in $(A_0 - w_1) \cup A_1$, whence the latter is a barrier of G; we infer that $A_0 = \{w_1\}$, thus $B_0 = \{u_1\}$. It follows that $|S| = 3$, contrary to our hypothesis. Thus b_1 and w_1 are nonadjacent; this proves *(i)*. As in the statement of the proposition, adjust notation so that b_1 and w_2 are adjacent; see Figure 17.

To deduce *(ii)* and *(iii)*, we apply Lemma 2.5 to the graph $H(e, S)$ at vertex w_2. Similar to the earlier situation, there exist partitions (A_0, A_1, A_2) of $I \cup \{\overline{x}\}$, and (B_0, B_1, B_2) of S, such that $w_2 \in A_0$, and $|A_j| = |B_j|$ for $j \in \{1, 2, 3\}$, vertex u_2 lies in B_0, and the remaining two neighbours of w_2 lie in B_1 and in B_2, respectively. Adjust notation so that b_1 lies in B_1. Also, $N(B_1) = A_1 \cup \{w_2\}$ and $N(B_2) = A_2 \cup \{w_2\}$. As before, we conclude that $\overline{x}$ lies in A_2, and that $|A_1| = |B_1| = 1$.

Observe that the unique vertex of A_1 has all of its neighbours in the set $B_0 \cup B_1$. We will show that $B_0 = \{u_2\}$; this implies that the unique vertex of A_1 has precisely two neighbours, and so it must be the end y of e; this immediately implies *(ii)*.

Note that the neighbourhood of $A_0 - w_2$ is contained in B_0. Thus, if $|A_0| \geq 2$ then y lies in A_0 (since otherwise B_0 is a barrier of G). If $|A_0| \geq 3$ then B_0 is a barrier of $G - e$ with three or more vertices. (Note that the barrier B_0 is contained in the barrier S.) Since no end of β lies in B_0, it

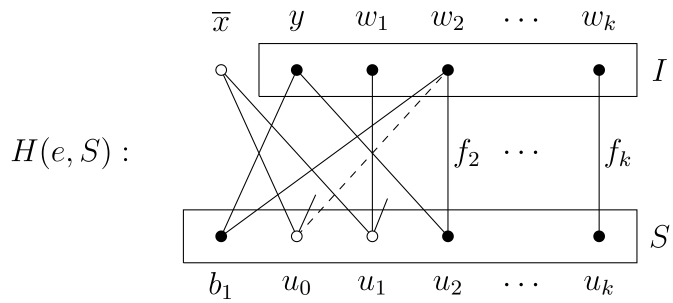


Figure 17: When $\mathcal{F}(e, S)$ is a matching, and S has four or more vertices; the vertices u_0 and w_2 are nonadjacent

follows from our earlier observations that the candidate set $\mathcal{F}(e, B_0)$ is not a matching. However, by Corollary 3.6, $\mathcal{F}(e, B_0)$ is a subset of $\mathcal{F}(e, S)$, and the latter is a matching; this is absurd. We conclude that A_0 has at most two vertices, that is, either $A_0 = \{w_2\}$ or $A_0 = \{y, w_2\}$. Now suppose that $A_0 = \{y, w_2\}$. The unique vertex of A_1 is adjacent with b_1, and thus statement *(i)* implies that $w_1 \notin A_1$. Assume without loss of generality that $A_1 = \{w_3\}$. Since w_3 is cubic, we conclude that its neighbourhood is precisely $B_0 \cup B_1$, and thus $B_0 = \{u_2, u_3\}$. Observe that $Q := w_3u_2w_2b_1w_3$ is a 4-cycle in $H(e, S)$ containing the vertex w_3, and thus by Corollary 2.6, one of the edges w_3u_2 and w_3b_1 is removable in $H(e, S)$; however, this contradicts our hypothesis since the only removable edges are the members of $\mathcal{F}(e, S)$. We thus conclude that $A_0 = \{w_2\}$. As explained earlier, $A_1 = \{y\}$, and thus y is adjacent with each of b_1 and u_2; this proves *(ii)*.

Now suppose that u_0 and w_2 are adjacent. Observe that $u_1 \in B_2$, and thus all of its neighbours lie in A_2, whence $|A_2| \geq 3$. The neighbourhood of $B_2 - \{u_0, u_1\}$ is contained in $A_2 - \overline{x}$, whence the latter is a nontrivial barrier of G, which is a contradiction. We conclude that u_0 and w_2 are nonadjacent; this proves *(iii)*, and completes the proof of Proposition 3.15. $\square$

3.2 The Equal Rank Lemma

Here, we present an important lemma which is used in the proof of Theorem 3.1. This lemma considers the situation in which G is an R-brick and $e := yz$ is an R-compatible edge of index two that is not thin, and f is a candidate relative to a barrier of $G - e$ such that f is also of index two and its rank is equal to that of e. The reader is advised to review the Three Case Lemma (2.13) and Section 2.4.2 before proceeding further.

The Equal Rank Lemma (3.17) relates the barrier structure of $G - f$ to that of $G - e$. More specifically, the lemma establishes subset/superset relationships between eight sets of vertices: the barriers S_1 and S_2 of $G-e$ (as in Case 2 of Lemma 2.13) and their corresponding sets of isolated vertices I_1 and I_2, and likewise, the barriers S_3 and S_4 of $G-f$ and their corresponding sets of isolated vertices I_3 and I_4. Among other things, the lemma shows that $S_1 \cup I_1 \cup S_2 \cup I_2 = S_3 \cup I_3 \cup S_4 \cup I_4$. We now introduce the relevant notation more precisely.

Since e is of index two, by the Three Case Lemma, $G - e$ has precisely two maximal nontrivial barriers, and since e is not thin, at least one of these barriers, say S_1, has three or more vertices (see Proposition 2.14). We adopt Notation 3.3 for the brick G and edge e. Assume without loss of generality that $S_1 \subset B$, and let I_1 denote the set of isolated vertices of $(G - e) - S_1$. We shall denote by S_2 the maximal nontrivial barrier of $(G - e)/X_1$ where $X_1 := S_1 \cup I_1$, and by I_2 the set of isolated vertices of $(G-e)-S_2$. Note that the end z of e lies in I_2 which is a subset of B, whereas the other end y of e lies in I_1 which is a subset of A. See Figure 18 (top).

By Corollary 3.9, the candidate set $\mathcal{F}(e, S_1)$ is nonempty, and by Proposition 3.5, each of its members is an R-compatible edge whose rank is at least that of e. Now, let $f := uw$ be a member of $\mathcal{F}(e, S_1)$ such that $u \in S_1$ and $w \in I_1$, and suppose that the index of f is two. The following result of Carvalho et al. [6, Lemma 32] plays a crucial role in our proof of the Equal Rank Lemma (3.17).

Lemma 3.16 *Assume that* $\mathsf{index}(e) = \mathsf{index}(f) = 2$. *If* $\mathsf{rank}(e) = \mathsf{rank}(f)$ *then* S_2 *is a subset of a barrier of* $G - f$. □

We shall let S_3 denote the maximal nontrivial barrier of $G - f$ which is contained in the color class B, and I_3 the set of isolated vertices of $(G-f)-S_3$. Furthermore, let S_4 denote the maximal nontrivial barrier of $(G-f)/(S_3 \cup I_3)$, and I_4 the set of isolated vertices of $(G - f) - S_4$. Note that the end u of f lies in I_4, and its other end w lies in I_3. See Figure 18 (bottom). We are now ready to state the Equal Rank Lemma using the notation introduced so far.

Lemma 3.17 [The Equal Rank Lemma] *Assume that* $\mathsf{index}(e) = \mathsf{index}(f) = 2$. *If* $\mathsf{rank}(e) = \mathsf{rank}(f)$ *then the following statements hold:*

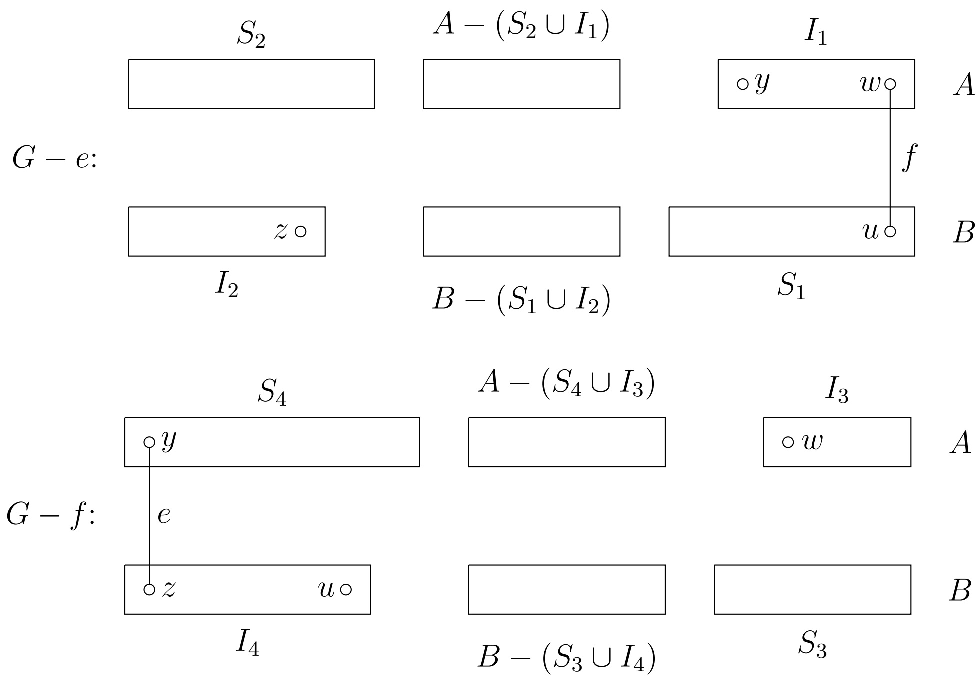


Figure 18: The Equal Rank Lemma

(i) e *and* f *are nonadjacent,*

(ii) $S_3 \subseteq S_1 - u$ *and* $I_3 \subseteq I_1 - y$*,*

(iii) $S_2 \subset S_4$ *and* $I_2 \subset I_4$*,*

(iv) $S_1 \cup I_2 = S_3 \cup I_4$ *and* $S_2 \cup I_1 = S_4 \cup I_3$*,*

(v) $N(u) \subseteq S_2 \cup I_1$*, and*

(vi) e *is a member of the candidate set* $\mathcal{F}(f, S_4)$*.*

Proof: We examine the graph $G-e-f$ in order to prove *(i)* and *(ii)*. Clearly, S_3 is a barrier of $G-e-f$. Observe that, since f has an end in S_1, every barrier of $G-e-f$ which contains S_1 is a barrier of $G-e$ as well. Since S_1 is a maximal barrier of $G-e$, we infer that S_1 is a maximal barrier of $G-e-f$

as well. By the Canonical Partition Theorem (1.3), to prove that S_3 is a subset of S_1, it suffices to show that $S_1 \cap S_3$ is nonempty. To see this, note that $w \in I_1 \cap I_3$, and thus any neighbour of w in $G - e - f$ lies in $S_1 \cap S_3$. Furthermore, since $u \notin S_3$, we conclude that $S_3 \subseteq S_1 - u$; this proves part of *(ii)*. In particular, $z \notin S_3$. Consequently, $y \notin I_3$, and thus y and w are distinct. This proves *(i)*.

Now we prove the remaining part of *(ii)*. Let $v \in I_3$, that is, v is isolated in $(G - f) - S_3$. Consequently, v is isolated in $(G - f) - S_1$. Since f has an end in S_1, we infer that v is isolated in $(G - e) - S_1$, that is, $v \in I_1$. Thus $I_3 \subseteq I_1 - y$. This proves *(ii)*.

We will now prove *(iii)* and *(iv)*. We begin by showing that S_2 is a subset of S_4. By Lemma 3.16, S_2 is a subset of the unique maximal nontrivial barrier of $G - f$ which is contained in the color class A, say S_4^*. By the Three Case Lemma (2.13), $S_4^* = S_4 \cup I'$ for some (possibly empty) subset I' of I_3. That is, S_2 is a subset of $S_4 \cup I'$. Note that S_2 and I_1 are disjoint; by *(ii)*, $S_2 \cap I' = \emptyset$. Thus, $S_2 \subseteq S_4$.

Since the ranks of e and f are equal, it follows that $|A - (S_2 \cup I_1)| = |A - (S_4 \cup I_3)|$ and likewise, $|B - (S_1 \cup I_2)| = |B - (S_3 \cup I_4)|$. In order to prove *(iv)*, it suffices to prove the following claim.

Claim 3.17.1 *$A - (S_2 \cup I_1) \subseteq A - (S_4 \cup I_3)$ and $B - (S_1 \cup I_2) \subseteq B - (S_3 \cup I_4)$.*

<u>Proof</u>: Let $v_1 \in A - (S_2 \cup I_1)$. By *(ii)*, $v_1 \notin I_3$. To prove that v_1 lies in $A - (S_4 \cup I_3)$, it suffices to show that $v_1 \notin S_4$.

Now, let v_2 be any vertex in S_2. We have already shown that $S_2 \subseteq S_4$, and thus $v_2 \in S_4$. Note that, if v_1 also belongs to the barrier S_4, then $(G - f) - \{v_1, v_2\}$ would not have a perfect matching. In the following paragraph, we will show that $(G - e - f) - \{v_1, v_2\}$ has a perfect matching, say M; consequently, $v_1 \notin S_4$.

Let H_1 be the graph $(G - e - f)/\overline{X_1} \to \overline{x_1}$, and let H_2 be the graph $(G - e - f)/\overline{X_2} \to \overline{x_2}$ where $X_2 := S_2 \cup I_2$. Note that H_1 and H_2 are bipartite matching covered graphs. Let $J := ((G - e - f)/X_1 \to x_1)/X_2 \to x_2$. Note that J is the brick of $G - e - f$. Let M_J be a perfect matching of $J - \{x_2, v_1\}$. Let g denote the edge of M_J incident with the contraction vertex x_1. Let M_1 be a perfect matching of H_1 which contains g. Let M_2 be a perfect matching of $H_2 - \{v_2, \overline{x_2}\}$. Observe that $M := M_1 + M_J + M_2$ is the desired matching.

Now, let $v \in B - (S_1 \cup I_2)$. By *(ii)*, $v \notin S_3$. To prove that v lies in $B - (S_3 \cup I_4)$, it suffices to show that $v \notin I_4$. To see this, note that since J is a brick, by Theorem 1.7, $J - \{x_1, x_2\}$ is connected; thus, v is not isolated in $(G - f) - S_4$, that is, $v \notin I_4$. $\square$

It follows from *(ii)* and *(iv)* that the end y of e lies in S_4, and thus S_2 is a proper subset of S_4. Also, we infer from *(ii)* and *(iv)* that I_2 is a subset of I_4. Furthermore, the end u of f lies in I_4, whence I_2 is a proper subset of I_4. This proves *(iii)*.

It remains to prove *(v)* and *(vi)*. As noted above, $u \in I_4$. Thus, all neighbors of u in G lie in $S_4 \cup \{w\} \subseteq S_4 \cup I_3$. It follows from *(iv)* that $N(u) \subseteq S_2 \cup I_1$. This proves *(v)*.

Finally, we prove *(vi)*. Recall that $H(f, S_4)$ denotes the bipartite matching covered graph $(H-f)/\overline{X_4} \to \overline{x_4}$ where $X_4 := S_4 \cup I_4$, and that $\mathcal{F}(f, S_4)$ is the set of those removable edges of $H(f, S_4)$ which are not incident with the contraction vertex $\overline{x_4}$. Since f is R-compatible in $G-e$ (by Proposition 3.5), the exchange property (Proposition 2.3) implies that e is R-compatible in $G - f$. Now, since the end z of e lies in I_4, the last assertion of Proposition 3.5 implies that e is a member of $\mathcal{F}(f, S_4)$. This proves *(vi)*, and finishes the proof of the Equal Rank Lemma. $\square$

3.3 Proof of Theorem 3.1

Before we proceed to prove Theorem 3.1, we state two results of Carvalho et al. [6] which are useful to us. Suppose that G is an R-brick and e is an R-compatible edge which is not thin. We let S_1 denote a maximal nontrivial barrier of $G - e$ such that $|S_1| \geq 3$, and let f denote a member of the candidate set $\mathcal{F}(e, S_1)$.

Note that, since e is not thin, its rank is at most $n-4$ where $n := |V(G)|$. If the index of f is zero then its rank is n, and in particular, it is greater than that of e. The following result of Carvalho et al. [6, Lemma 31] shows that this conclusion holds even if the index of f is one.

Lemma 3.18 *Suppose that f is a member of the candidate set $\mathcal{F}(e, S_1)$. If the index of f is one then* $\mathsf{rank}(f) > \mathsf{rank}(e)$. $\square$

The following corollary of Lemmas 3.16 and 3.18 was used implicitly by Carvalho et al. [6] in their proof of the Thin Edge Theorem (1.10). We provide its proof for the sake of completeness.

Corollary 3.19 *Assume that the index of e is two. If the candidate set $\mathcal{F}(e, S_1)$ contains two adjacent edges, say f and g, then at least one of them has rank strictly greater than* $\mathsf{rank}(e)$.

Proof: We know by Proposition 3.5 that each of f and g has rank at least $\mathsf{rank}(e)$. If either of them has rank strictly greater than that of e then there is nothing to prove. Now, suppose that $\mathsf{rank}(f) = \mathsf{rank}(g) = \mathsf{rank}(e)$. It follows from Lemma 3.18 that both f and g are of index two. We intend to arrive at a contradiction using Lemma 3.16. We let I_1 denote the set of isolated vertices of $(G-e)-S_1$, and S_2 denote the unique maximal nontrivial barrier of $(G-e)/(S_1 \cup I_1)$. By Lemma 3.16, S_2 is a subset of a barrier of $G-f$, and likewise, S_2 is a subset of a barrier of $G-g$.

Consider two distinct vertices of S_2, say v_1 and v_2. Let M be a perfect matching of the graph $G - \{v_1, v_2\}$. (Such a perfect matching exists as G is a brick.) As noted above, S_2 is a subset of a barrier of $G-f$. In particular, v_1 and v_2 lie in a barrier of $G-f$, whence $(G-f)-\{v_1, v_2\}$ has no perfect matching. Thus f lies in M. Likewise, g also lies in M. This is absurd since f and g are adjacent. We conclude that one of f and g has rank strictly greater than $\mathsf{rank}(e)$. This completes the proof of Corollary 3.19. $\square$

We now proceed to prove Theorem 3.1.

Proof of Theorem 3.1: As in the statement of the theorem, let e denote an R-compatible edge of an R-brick G. If the edge e is thin, then there is nothing to prove. Now consider the case in which e is not thin. By the Three Case Lemma (2.13), $G-e$ has either one or two maximal nontrivial barriers, and by Proposition 2.14, at least one such barrier has three or more vertices. Our goal is to establish the existence of another R-compatible edge f which satisfies conditions *(i)* and *(ii)* in the statement of Theorem 3.1.

Recall that each candidate edge (relative to e and a barrier of $G-e$ with three or more vertices) is an R-compatible edge of G which satisfies condition *(i)* of Theorem 3.1 and has rank at least $\mathsf{rank}(e)$. (See Definition 3.4 and Proposition 3.5.) Furthermore, if a candidate has rank strictly greater than $\mathsf{rank}(e)$, then by Proposition 3.2, it also satisfies condition *(ii)* of Theorem 3.1, and in this case we are done. Keeping these observations in view, we now use Lemma 3.18 to get rid of the case in which index of e is one.

Claim 3.20 *We may assume that the index of e is two.*

Proof: Suppose not. Then the index of e is one, and we let S denote the unique maximal nontrivial barrier of $G-e$. As discussed earlier, $|S| \geq 3$. Let f denote a member of the candidate set $\mathcal{F}(e,S)$, which is nonempty by Corollary 3.9. If the index of f is zero then its rank is clearly greater than $\mathsf{rank}(e)$, and by Lemma 3.18, this conclusion holds even if the index of f is one. Now consider the case in which f is of index two. Since $\mathsf{rank}(f) \geq \mathsf{rank}(e)$, we conclude that f satisfies condition *(ii)*, Theorem 3.1. Thus, irrespective of its index, the edge f satisfies both conditions *(i)* and *(ii)*, and we are done. □

We shall now invoke Corollary 3.19 to dispose of the case in which the candidate set (relative to some barrier of $G-e$) is not a matching.

Claim 3.21 *We may assume that if S is a nontrivial barrier (not necessarily maximal) of $G-e$ with three or more vertices then the corresponding candidate set $\mathcal{F}(e,S)$ is a matching.*

Proof: Suppose that the candidate set $\mathcal{F}(e,S)$ is not a matching, and thus it contains two adjacent edges, say f and g. We let S^* denote the maximal nontrivial barrier of $G-e$ such that $S \subseteq S^*$. By Corollary 3.6, edges f and g are members of $\mathcal{F}(e,S^*)$ as well. Since e is of index two (by Claim 3.20), Corollary 3.19 implies that at least one of f and g, say f, has rank strictly greater than that of e. Thus f satisfies both conditions *(i)* and *(ii)*, Theorem 3.1, and we are done. □

Now, since e is of index two (by Claim 3.20), the graph $G-e$ has precisely two maximal nontrivial barriers. Among these two, we shall denote by S_1 the barrier which is bigger (breaking ties arbitrarily if they are of equal size), and by I_1 the set of isolated vertices of $(G-e)-S_1$. Thus $|S_1| \geq 3$. Let y and z denote the ends of e. We adopt Notation 3.3. Assume without loss of generality that S_1 is a subset of B, and thus by the Three Case Lemma (2.13), the end y of e lies in I_1.

As the candidate set $\mathcal{F}(e,S_1)$ is a matching (by Claim 3.21), we invoke the observations made in Section 3.1.1, with S_1 playing the role of S, and

I_1 playing the role of I, and likewise, $X_1 := S_1 \cup I_1$ playing the role of X. In particular, we adopt Notations 3.10, 3.11 and 3.12 and we apply Proposition 3.13. See Figure 19.

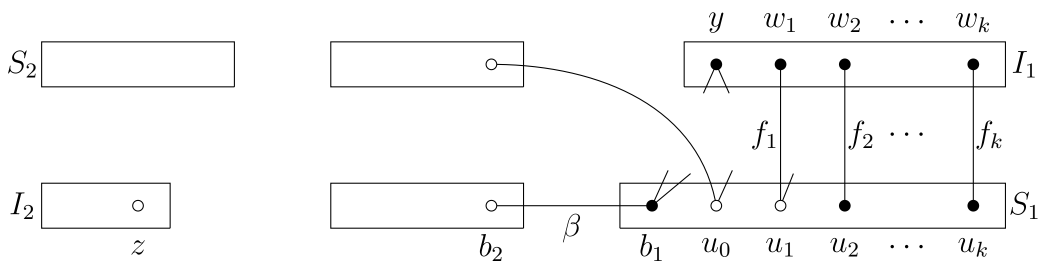


Figure 19: Index of e is two, and S_1 is the largest barrier of $G - e$

We let S_2 denote the unique maximal nontrivial barrier of $(G-e)/X_1$, and I_2 the set of isolated vertices of $(G-e)-S_2$. By the Three Case Lemma (2.13), the end z of e lies in I_2, as shown in Figure 19. Note that $|S_2| \leq |S_1|$ by the choice of S_1.

Note that, as per statements *(iv)* and *(v)* of Proposition 3.13, the edge $f_1 = u_1w_1$ is the only member of the candidate set $\mathcal{F}(e, S_1)$ whose end in the barrier S_1 (that is, vertex u_1) has some neighbour which lies in $\overline{X_1}$. Also, if $|S_1| = 3$ then f_1 is the unique member of $\mathcal{F}(e, S_1)$. For these reasons, it will play a special role.

Claim 3.22 *We may assume that* $\mathsf{rank}(f_1) = \mathsf{rank}(e)$. *Consequently, the following hold:*

(i) the index of f_1 *is two,*

(ii) all neighbours of u_1 *lie in* $S_2 \cup I_1$*, and*

(iii) the vertex u_0 *has at least one neighbour in the set* $A - (S_2 \cup I_1)$.

Proof: By Proposition 3.5, f_1 is an R-compatible edge which has rank at least that of e, and it satisfies condition *(i)*, Theorem 3.1. If $\mathsf{rank}(f_1) > \mathsf{rank}(e)$, then by Proposition 3.2, f_1 satisfies condition *(ii)* as well, and we are done. We may thus assume that $\mathsf{rank}(f_1) = \mathsf{rank}(e)$. It follows from Lemma 3.18 that the index of f_1 is two; that is, *(i)* holds. Since e and $f_1 = u_1w_1$ are of equal rank and of index two each, the Equal Rank Lemma (3.17)*(v)* implies that each neighbour of u_1 lies in the set $S_2 \cup I_1$, and this proves *(ii)*. We shall now use this fact to deduce *(iii)*.

Since H is bipartite and matching covered, Proposition 1.4*(ii)* implies that the neighbourhood of the set $A-(S_2 \cup I_1)$, in the graph H, has cardinality at least $|A-(S_2 \cup I_1)|+1$, and since $|A-(S_2 \cup I_1)| = |B-(S_1 \cup I_2)|$, we conclude that the set $A-(S_2 \cup I_1)$ has at least one neighbour which is not in $B-(S_1 \cup I_2)$; it follows from Proposition 3.13 and statement *(ii)* proved above that the only such neighbour is the vertex u_0 of barrier S_1. In other words, the vertex u_0 has at least one neighbour in the set $A-(S_2 \cup I_1)$ as shown in Figure 19; this proves *(iii)*, and completes the proof of Claim 3.22. $\square$

We shall now consider two cases depending on the cardinality of S_1.

Case 1: $|S_1| \geq 4$.

We invoke Proposition 3.15, with S_1 playing the role of S, and we adjust notation accordingly. See Figure 20. Observe that $Q := u_2w_2b_1yu_2$ is a 4-cycle of G which contains the edge $f_2 = u_2w_2$. Since f_2 is a candidate, it is an R-compatible edge whose rank is at least that of e, and it satisfies condition *(i)*, Theorem 3.1. We will use the 4-cycle Q and the Equal Rank Lemma to conclude that f_2 has rank strictly greater than that of e, and thus it satisfies condition *(ii)* as well.

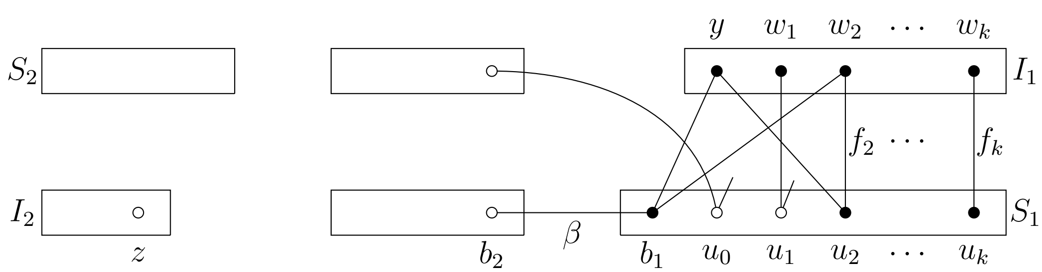


Figure 20: When $|S_1| \geq 4$

Now, let v denote the neighbour of w_2 which is distinct from u_2 and b_1. Clearly, $v \in S_1$; by Proposition 3.15*(iii)*, v is distinct from u_0.

Since each end of f_2 is cubic, it is an R-compatible edge of index two. We first set up some notation concerning the barrier structure of $G - f_2$. We denote by S_3 the maximal nontrivial barrier of $G - f_2$ which is a subset of B, and by I_3 the set of isolated vertices of $(G - f_2) - S_3$. We let S_4 denote the unique maximal nontrivial barrier of $(G - f_2)/(S_3 \cup I_3)$, and I_4 the set of isolated vertices of $(G - f_2) - S_4$. By the Three Case Lemma (2.13), the end u_2 of f_2 lies in I_4, and its end w_2 lies in I_3. Also, since $w_2 \in I_3$, $v \in S_3$.

Now, suppose for the sake of contradiction that $\mathsf{rank}(f_2) = \mathsf{rank}(e)$. Then we may apply the Equal Rank Lemma (3.17) to conclude that $S_1 \cup I_2 = S_3 \cup I_4$ and that $S_2 \cup I_1 = S_4 \cup I_3$. Furthermore, by Claim 3.22*(iii)*, the vertex u_0 has a neighbour in $A - (S_4 \cup I_3)$, and thus $u_0 \notin I_4$. We infer that $u_0 \in S_3$. We have thus shown that v and u_0 are distinct vertices of the barrier S_3 of $G - f_2$. Consequently, $(G - f_2) - \{v, u_0\}$ has no perfect matching; we will now use the 4-cycle $Q = u_2 w_2 b_1 y u_2$ to contradict this assertion.

Since G is a brick, $G - \{v, u_0\}$ has a perfect matching, say M. If f_2 is not in M then we have the desired contradiction. Now suppose that $f_2 \in M$. Since v and u_0 both lie in the color class B of H, we conclude that $\alpha \in M$ and that $\beta \notin M$. See Figure 20. Note that each of v and u_0 is distinct from b_1, and that the neighbours of b_1 are precisely b_2, w_2 and y. Since $\beta = b_1 b_2$ is not in M, and since $f_2 = u_2 w_2$ lies in M, it must be the case that $y b_1$ lies in M. Now observe that the symmetric difference of M and Q is a perfect matching of $(G - f_2) - \{v, u_0\}$, and thus we have the desired contradiction.

We conclude that $\mathsf{rank}(f_2) > \mathsf{rank}(e)$, and thus f_2 is the desired R-compatible edge which satisfies both conditions *(i)* and *(ii)*, Theorem 3.1.

<u>Case 2</u>: $|S_1| = 3$.

We note that since S_1 has precisely three vertices, by Remark 3.14, all of the edges of $G[X_1]$ are determined (where $X_1 = S_1 \cup I_1$). See Figure 21. Furthermore, f_1 is the only member of the candidate set $\mathcal{F}(e, S_1)$, and by Claim 3.22, its index is two and its rank is equal to $\mathsf{rank}(e)$. We will examine the barrier structure of $G - f_1$ using the Equal Rank Lemma (3.17), and argue that some edge adjacent with the given edge $e = yz$ (that is, either incident at y, or incident at z) is R-compatible and that its rank is strictly greater than $\mathsf{rank}(e)$. Observe that, since $\mathsf{index}(e) = 2$, each edge adjacent with e satisfies condition *(i)*, Theorem 3.1.

We let S_3 denote the unique maximal nontrivial barrier of $G - f_1$ which is a subset of B, and I_3 the set of isolated vertices of $(G - f_1) - S_3$. We denote by S_4 the unique maximal nontrivial barrier of $(G - f_1)/(S_3 \cup I_3)$, and by I_4 the set of isolated vertices of $(G - f_1) - S_4$. See Figure 21. By the Three

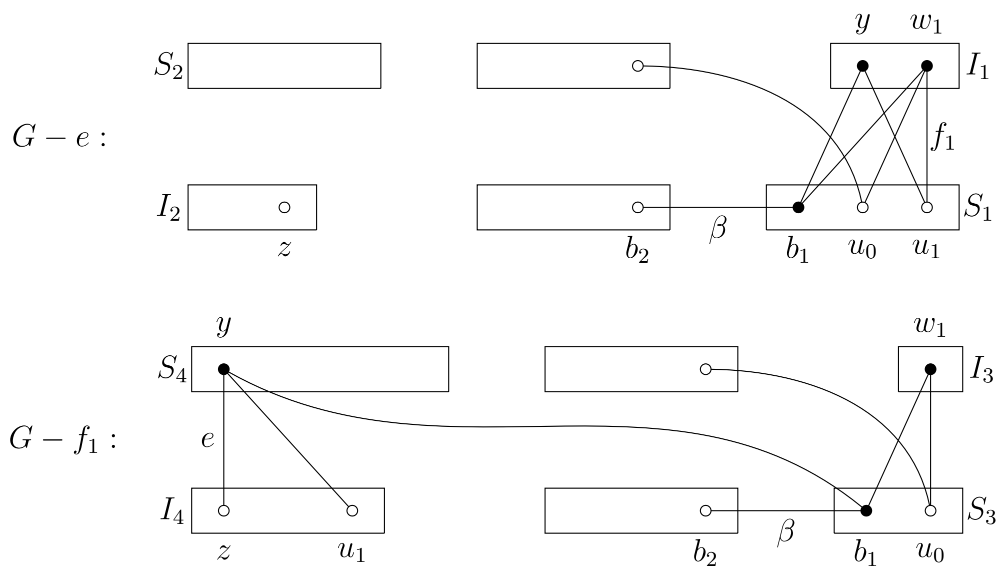


Figure 21: When $|S_1| = 3$

Case Lemma (2.13), the end u_1 of f_1 lies in I_4, and its end w_1 lies in I_3. Since each of b_1 and u_0 is a neighbour of w_1 in $G - f_1$, they both lie in the barrier S_3. By Lemma 3.17*(ii)*, with f_1 playing the role of f, we conclude that $S_3 = \{b_1, u_0\}$ and that $I_3 = \{w_1\}$, as shown in the figure.

Observe that by the choice of S_1, the barrier S_2 of $G - e$ contains either two or three vertices. However, irrespective of the cardinality of S_2, it follows from the above and from Lemma 3.17*(iv)* that $S_4 = S_2 \cup \{y\}$ and that $I_4 = I_2 \cup \{u_1\}$. In particular, the barrier S_4 of $G - f_1$ contains either three or four vertices. Note that the end z of e lies in I_2 which is a subset of I_4, and its end y lies in S_4. Furthermore, Lemma 3.17*(vi)* implies that e is a member of the candidate set $\mathcal{F}(f_1, S_4)$.

Claim 3.23 *We may assume that e is the only member of $\mathcal{F}(f_1, S_4)$ which is incident with z. Furthermore, we may assume that $|S_2| = 2$.*

Proof: Suppose there exists an edge g incident with z such that g is distinct from e and that $g \in \mathcal{F}(f_1, S_4)$. By Proposition 3.5, g is an R-compatible edge of the brick G. We now apply Corollary 3.19 (with f_1 playing the role of e, and with edges e and g playing the roles of f and g); at least one of e and g has rank strictly greater than $\mathsf{rank}(f_1)$. However, by Claim 3.22, the ranks of e and f_1 are equal; consequently, $\mathsf{rank}(g) > \mathsf{rank}(f_1) = \mathsf{rank}(e)$. By Propostion 3.2, the edge g satisifes condition *(ii)*, Theorem 3.1, and it satisfies condition *(i)* because it is adjacent with the edge e, and thus we are done. So we may assume that e is the only member of $\mathcal{F}(f_1, S_4)$ which is incident with z. Using this, we shall deduce that the barrier S_2 of $G - e$ has only two vertices.

Suppose to the contrary that $|S_2| = 3$. By Claim 3.21, the candidate set $\mathcal{F}(e, S_2)$ is a matching. Consequently, as we did in the case of S_1, we may now invoke the observations made in Section 3.1.1, with S_2 playing the role of S, and I_2 playing the role of I, and likewise, $X_2 := S_2 \cup I_2$ playing the role of X. In particular, by Remark 3.14, all of the edges of $G[X_2]$ are determined. It is worth noting that S_2 is also a maximal barrier of $G - e$ (by the choice of S_1). That is, each of S_1 and S_2 is a maximal barrier of $G - e$ with exactly three vertices. Keeping this symmetry in view, we now choose appropriate notation for those vertices of X_2 which are relevant to our argument. See Figure 22.

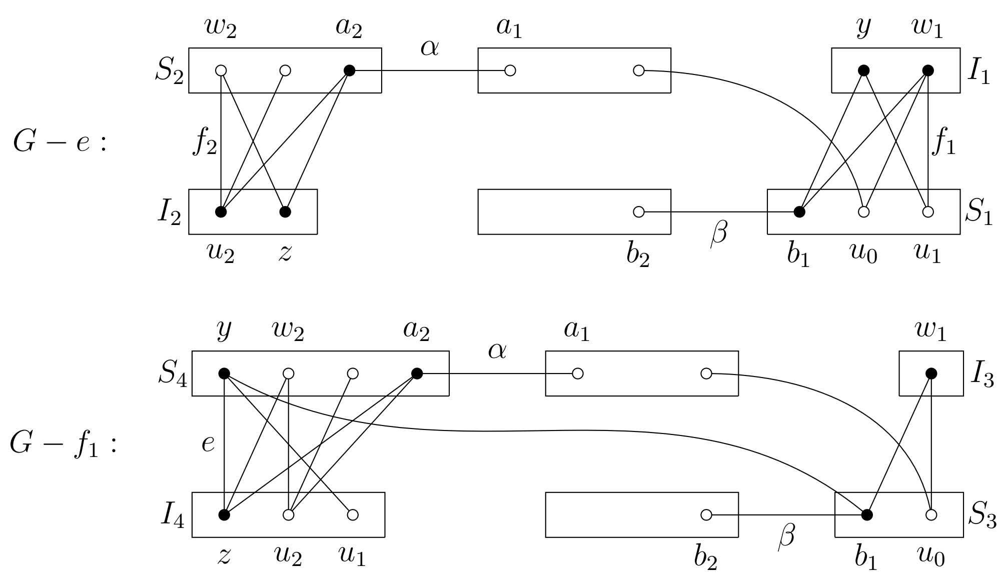


Figure 22: When $|S_1| = |S_2| = 3$

We shall let $f_2 := u_2w_2$ denote the unique member of the candidate set $\mathcal{F}(e, S_2)$, where $u_2 \in I_2$ and $w_2 \in S_2$. In particular, $I_2 = \{u_2, z\}$. One of the ends of $\alpha = a_1a_2$ lies in the barrier S_2; we adjust notation so that $a_2 \in S_2$.

Consequently, w_2 and a_2 are distinct vertices of S_2. The vertex a_2 is cubic, and its neighbours are z, u_2 and a_1. The vertex w_2 is adjacent with z and u_2, and all of its remaining neighbours lie in $\overline{X_2}$.

Observe that $Q := zw_2u_2a_2z$ is a 4-cycle of the bipartite graph $H(f_1, S_4)$ which contains the vertex z whose degree is three. Consequently, by Corollary 2.6, at least one of zw_2 and za_2 is removable in $H(f_1, S_4)$. However, since a_2 has degree two in $H(f_1, S_4)$, za_2 is non-removable; whence zw_2 is removable. It follows that zw_2 is a member of the candidate set $\mathcal{F}(f_1, S_4)$; this contradicts our first assumption. We conclude that the barrier S_2 has only two vertices, and this completes the proof of Claim 3.23. $\square$

By Proposition 2.14, an R-compatible edge of index two is thin if and only if its rank is $n-4$; where $n := |V(G)|$. Observe that, since $|S_1| = 3$ and $|S_2| = 2$, the rank of e is $n-6$, and in this sense, it is very close to being thin; the same holds for the edge f_1. We will establish a symmetry between the barrier structure of $G-e$ and that of $G-f_1$; see Figure 23. Thereafter, we will argue that the edge $g := yu_1$ is an R-thin edge of index two; in particular, it is R-compatible and its rank is $n-4$, and thus it satisfies condition *(ii)*, Theorem 3.1. Since g is adjacent with e, it satisfies condition *(i)* as well.

Since $|S_2| = 2$, the set I_2 contains only the end z of e, and the neighbourhood of z is precisely the set $S_2 \cup \{y\} = S_4$. Also, $I_4 = I_2 \cup \{u_1\} = \{z, u_1\}$, and by Claim 3.23, $e = yz$ is the only member of the candidate set $\mathcal{F}(f_1, S_4)$ which is incident with z. In other words, z is incident with only one removable edge of the bipartite graph $H(f_1, S_4)$, namely, the edge e. We now deduce some consequences of this fact using standard arguments.

First of all, by Lemma 3.8*(i)*, an end of the edge $\alpha = a_1a_2$ lies in the barrier S_4. Adjust notation so that $a_2 \in S_4$. By statement *(ii)* of the same lemma, a_2 has no neighbours in the set $\overline{X_4}$ where $X_4 := S_4 \cup I_4$. Consequently, the neighbourhood of a_2 is precisely $I_4 \cup \{a_1\} = \{z, u_1, a_1\}$. Clearly, y and a_2 are distinct vertices of S_4, and we denote by w_0 the remaining vertex of S_4. Note that $S_2 = \{w_0, a_2\}$.

Next, we observe that if the vertices u_1 and w_0 are adjacent then $Q := zw_0u_1a_2z$ is a 4-cycle of the bipartite graph $H(f_1, S_4)$ and it contains the vertex z which has degree three; by Corollary 2.6, one of the two edges zw_0 and za_2 is removable; however, this contradicts the fact that $e = yz$ is the only removable edge incident with z. Thus, the vertices u_1 and w_0 are nonadjacent. It follows that u_1 is cubic, and its neighbourhood is precisely $\{y, a_2, w_1\}$.

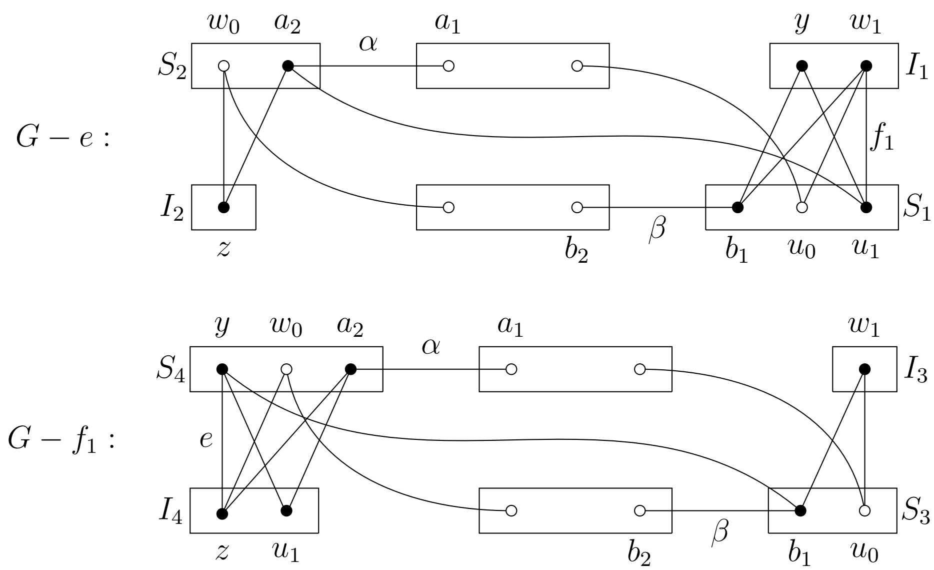


Figure 23: When $|S_1| = 3$ and $|S_2| = 2$

Observe that we have six cubic vertices whose neighbourhoods are fully determined; these are: the ends y and z of e, the ends u_1 and w_1 of f_1, the end b_1 of β, and the end a_2 of α. There is a symmetry between the barrier structure of $G - e$ and that of $G - f_1$; as is self-evident from Figure 23. We have not determined the degrees of the two vertices u_0 and w_0; observe that if these vertices are not adjacent with each other then u_0 has at least two neighbours in $A - (S_2 \cup I_1)$ and likewise, w_0 has at least two neighbours in $B-(S_1 \cup I_2)$; whereas if u_0w_0 is an edge of G then u_0 has at least one neighbour in $A - (S_2 \cup I_1)$ and likewise, w_0 has at least one neighbour in $B - (S_1 \cup I_2)$.

As mentioned earlier, we now proceed to prove that $g = yu_1$ is an R-thin edge. We let $J := ((G - e)/X_1 \to x_1)/X_2 \to x_2$ denote the unique brick of $G - e$, where $X_1 = S_1 \cup I_1$ and $X_2 := S_2 \cup I_2$. Note that J is near-bipartite with removable doubleton R.

Claim 3.24 *The edge $g = yu_1$ is R-thin. (That is, g is an R-compatible edge of index two and its rank is $n - 4$.)*

Proof: Observe that $Q := yu_1w_1b_1y$ is a 4-cycle in $H = G - R$ which contains the cubic vertex y. By Corollary 2.6, at least one of the edges $g = yu_1$ and yb_1 is removable in H. Note that yb_1 is not removable, whence g is removable in H. To conclude that g is R-compatible, it suffices to show that edges α and β are admissible in $G-g$. We shall prove something more general, which is useful in establishing the thinness of g as well.

Observe that, in $G-g$, the vertex y has neighbour set $\{z, b_1\}$, and vertex u_1 has neighbour set $\{w_1, a_2\}$. We will show that, if v_1 and v_2 are distinct vertices of the color class B such that $\{v_1, v_2\} \neq \{z, b_1\}$, then $(G-g)-\{v_1, v_2\}$ has a perfect matching, say M. This has two consequences worth noting. First of all, if $\{v_1, v_2\} = \{b_1, b_2\}$ then $M + \beta$ is a perfect matching of $G-g$ which contains α and β both, whence g is an R-compatible edge of G. Secondly, it shows that $\{z, b_1\}$ is a maximal nontrivial barrier of $G-g$. An analogous argument establishes that $\{w_1, a_2\}$ is also a maximal nontrivial barrier of $G-g$, and consequently Proposition 2.14 implies that g is indeed R-thin.

As mentioned above, suppose that v_1 and v_2 are distinct vertices of B such that $\{v_1, v_2\} \neq \{z, b_1\}$. Let N be a perfect matching of $G-\{v_1, v_2\}$. In what follows, we consider different possibilities, and in each of them, we exhibit a perfect matching M of $(G-g)-\{v_1, v_2\}$. If $g \notin N$ then clearly $M := N$. Now suppose that $g \in N$. Note that, since $v_1, v_2 \in B$, the edge α lies in N and β does not lie in N. If $b_1 \notin \{v_1, v_2\}$, then the edge b_1w_1 lies in N, and we let $M := (N - g - b_1w_1) + f_1 + yb_1$.

Now consider the case in which $b_1 \in \{v_1, v_2\}$, and adjust notation so that $b_1 = v_1$. Thus $v_2 \neq z$, whence $zw_0 \in N$. Also, w_1u_0 lies in N. Observe that v_2 lies in the set $B - (S_1 \cup I_2)$. First, we consider the case when u_0w_0 is an edge of G. Observe that the six cycle $C := u_1yzw_0u_0w_1u_1$ is N-alternating and it contains the edge g. In this case, let M denote the symmetric difference of N and C.

Finally, consider the situation in which u_0w_0 is not an edge of G. (In this case, to construct M, we will not use the matching N.) As noted earlier, since u_0 and w_0 are nonadjacent, w_0 has at least two distinct neighbours in the set $B - (S_1 \cup I_2)$. In particular, w_0 has at least one neighbour, say v', which lies in $B - (S_1 \cup I_2)$ and is distinct from v_2. Now, let M_J be a perfect matching of $J - \{v', v_2\}$. Observe that $\alpha \in M_J$ and $\beta \notin M_J$. Note that, in the matching M_J, the contraction vertex x_1 is matched with some vertex in $A - (S_2 \cup I_1)$, which is a neighbour of u_0 in the graph G. Now, we let $M := M_J + w_0v' + f_1 + e$.

In every scenario, M is a perfect matching of $(G-g)-\{v_1, v_2\}$, as desired. Thus, as discussed earlier, g is R-compatible as well as thin. This proves

Claim 3.24. □

In summary, we have shown that $g = yu_1$ is an R-compatible edge which satisfies both conditions *(i)* and *(ii)*, Theorem 3.1. This completes the proof. □

Acknowledgments: We are greatly indebted to both Joseph Cheriyan and U. S. R. Murty who have helped us throughout this work by participating in all of our discussions and making valuable suggestions.

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