# Nonrepetitive Colourings of Graphs Excluding a Fixed Immersion or Topological Minor 

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Abstract. We prove that graphs excluding a fixed immersion have bounded nonrepetitive chromatic number. More generally, we prove that if $H$ is a fixed planar graph that has a planar embedding with all the vertices with degree at least 4 on a single face, then graphs excluding $H$ as a topological minor have bounded nonrepetitive chromatic number. This is the largest class of graphs known to have bounded nonrepetitive chromatic number.

## 1 Introduction

A vertex colouring of a graph is nonrepetitive if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. More precisely, a $k$ colouring of a graph $G$ is a function $\psi$ that assigns one of $k$ colours to each vertex of $G$. A path $\left(v_{1}, v_{2}, \ldots, v_{2 t}\right)$ of even order in $G$ is repetitively coloured by $\psi$ if $\psi\left(v_{i}\right)=\psi\left(v_{t+i}\right)$ for $i \in\{1, \ldots, t\}$. A colouring $\psi$ of $G$ is nonrepetitive if no path of $G$ of even order is repetitively coloured by $\psi$. Observe that a nonrepetitive colouring is proper, in the sense that adjacent vertices are coloured differently. The nonrepetitive chromatic number $\pi(G)$ is the minimum integer $k$ such that $G$ admits a nonrepetitive $k$-colouring. We only consider simple graphs with no loops or parallel edges.

The seminal result in this area is by Thue [41], who in 1906 proved that every path is nonrepetitively 3 -colourable. Thue expressed his result in terms of strings over an alphabet of three characters-Alon et al. [3] introduced the generalisation to graphs in 2002. Nonrepetitive graph colourings have since been widely studied $[2-12,21,25-33,35,37-39]$. The principle result of Alon et al. [3] was that graphs with maximum degree $\Delta$ are nonrepetitively $\mathcal{O}\left(\Delta^{2}\right)$ colourable. Several subsequent papers improved the constant [16, 26, 30]. The best known bound is due to Dujmović et al. [16].

Theorem 1 ([16]). Every graph with maximum degree $\Delta$ is nonrepetitively $(1+o(1)) \Delta^{2}$ colourable.

A number of other graph classes are known to have bounded nonrepetitive chromatic number. In particular, trees are nonrepetitively 4 -colourable [8, 33], outerplanar graphs are nonrepet-

[^0]itively 12-colourable [5, 33], and graphs with bounded treewidth have bounded nonrepetitive chromatic number [5, 33]. (See Section 2 for the definition of treewidth.) The best known bound is due to Kündgen and Pelsmajer [33].

Theorem 2 ([33]). Every graph with treewidth $k$ is nonrepetitively $4^{k}$-colourable.

The primary contribution of this paper is to provide a qualitative generalisations of Theorems 1 and 2 via the notion of graph immersions and excluded topological minors.

A graph $G$ contains a graph $H$ as an immersion if the vertices of $H$ can be mapped to distinct vertices of $G$, and the edges of $H$ can be mapped to pairwise edge-disjoint paths in $G$, such that each edge $v w$ of $H$ is mapped to a path in $G$ whose endpoints are the images of $v$ and $w$. The image in $G$ of each vertex in $H$ is called a branch vertex. Structural and colouring properties of graphs excluding a fixed immersion have been widely studied $[1,13,14,18-$ $20,22-24,34,36,40,42]$. We prove that graphs excluding a fixed immersion have bounded nonrepetitive chromatic number.

Theorem 3. For every graph $H$ with $t$ vertices, every graph that does not contain $H$ as an immersion is nonrepetitively $4^{t^{4}+O\left(t^{2}\right)}$-colourable.

Since a graph with maximum degree $\Delta$ contains no star with $\Delta+1$ leaves as an immersion, Theorem 3 implies that graphs with bounded degree have bounded nonrepetitive chromatic number (as in Theorem 1).

We strengthen Theorem 3 as follows (although without explicit bounds). A graph $G$ contains a graph $H$ as a strong immersion if $G$ contains $H$ as an immersion, such that for each edge $v w$ of $H$, no internal vertex of the path in $G$ corresponding to $v w$ is a branch vertex.

Theorem 4. For every fixed graph $H$, there exists a constant $k$, such that every graph $G$ that does not contain $H$ as a strong immersion is nonrepetitively $k$-colourable.

Note that planar graphs with $n$ vertices are nonrepetitively $\mathcal{O}(\log n)$-colourable [15], and the same is true for graphs excluding a fixed graph as a minor or topological minor [17]. It is unknown whether any of these classes have bounded nonrepetitive chromatic number. Our final result shows that excluding a special type of topological minor gives bounded nonrepetitive chromatic number.

Theorem 5. Let $H$ be a fixed planar graph that has a planar embedding with all the vertices of $H$ with degree at least 4 on a single face. Then there exists a constant $k$, such that every graph $G$ that does not contain $H$ as a topological minor is nonrepetitively $k$-colourable.

Graphs with bounded treewidth exclude fixed walls as topological minors. Since walls are planar graphs with maximum degree 3 , Theorem 5 implies that graphs of bounded treewidth
have bounded nonrepetitive chromatic number (as in Theorem 2). Similarly, for every graph $H$ with $t$ vertices, the 'fat star' graph (which is the 1-subdivision of the $t$-leaf star with edge multiplicity $t$ ) contains $H$ as a strong immersion. Since fat stars embed in the plane with all vertices of degree at least 4 on a single face, Theorem 5 implies that graphs excluding a fixed graph as a strong immersion have bounded nonrepetitive chromatic number (as in Theorem 4). In this sense, Theorem 5 generalises all of Theorems 1 to 4 .

The results of this paper, in relation to the best known bounds on the nonrepetitive chromatic number, are summarised in Figure 1.

## 2 Tree Decompositions

For a graph $G$ and tree $T$, a tree decomposition or $T$-decomposition of $G$ consists of a collection $\left(T_{x} \subseteq V(G): x \in V(T)\right)$ of sets of vertices of $G$, called bags, indexed by the nodes of $T$, such that for each vertex $v \in V(G)$ the set $\left\{x \in V(T): v \in T_{x}\right\}$ induces a connected subtree of $T$, and for each edge $v w$ of $G$ there is a node $x \in V(T)$ such that $v, w \in T_{x}$. The width of a $T$-decomposition is the maximum, taken over the nodes $x \in V(T)$, of $\left|T_{x}\right|-1$. The treewidth of a graph $G$ is the minimum width of a tree decomposition of $G$. The adhesion of a tree decomposition $\left(T_{x}: x \in V(T)\right)$ is $\max \left\{\left|T_{x} \cap T_{y}\right|: x y \in E(T)\right\}$. The torso of each node $x \in V(T)$ is the graph obtained from $G\left[T_{x}\right]$ by adding a clique on $T_{x} \cap T_{y}$ for each edge $x y \in E(T)$ incident to $x$. Dujmović et al. [17] generalised Theorem 2 as follows:

Lemma 6 ([17]). If a graph $G$ has a tree decomposition with adhesion $k$ such that every torso is nonrepetitively c-colourable, then $G$ is nonrepetitively $c 4^{k}$-colourable.

For integers $c, d \geqslant 0$ a graph $G$ has $(c, d)$-bounded degree if $G$ contains at most $c$ vertices with degree greater than $d$.

Lemma 7. Every graph with $(c, d)$-bounded degree is nonrepetitively $c+(1+o(1)) d^{2}$-colourable.

Proof. Assign a distinct colour to each vertex of degree at least $d$, and colour the remaining graph by Theorem 1. For each vertex $v$ of degree at least $d$, no other vertex is assigned the same colour as $v$. Thus $v$ is in no repetitively coloured path. The result then follows from Theorem 1.

Dvořák [18] proved the following structure theorem for graphs excluding a strong immersion.
Theorem 8 ([18]). For every fixed graph $H$, there exists a constant $k$, such that every graph $G$ that does not contain $H$ as a strong immersion has a tree decomposition such that each torso is $(k, k)$-bounded degree.

Lemmas 6 and 7 and Theorem 8 imply Theorem 4.


Figure 1: Upper bounds on the nonrepetitive chromatic number of various graph classes. 'Special' refers to the condition in Theorem 5.

## 3 Weak Immersions

The proof of Theorem 4 gives no explicit bound on the constant $k$. In this section we prove an explicit bound on the nonrepetitive chromatic number of graphs excluding a weak immersion. Theorem 3 follows from Lemma 6 and the following structure theorem of independent interest.

Theorem 9. For every graph $H$ with $t$ vertices, every graph that does not contain $H$ as a weak immersion has a tree decomposition with adhesion at most $t^{2}$ such that every torso has
$\left(t, t^{4}+2 t^{2}\right)$-bounded degree.

The starting point for the proof of Theorem 9 is the following structure theorem of Wollan [42]. For a tree $T$ and graph $G$, a $T$-partition of $G$ is a partition $\left(T_{x} \subseteq V(G): x \in V(T)\right)$ of $V(G)$ indexed by the nodes of $T$. Each set $T_{x}$ is called a bag. Note that a bag may be empty. For each edge $x y$ of a tree $T$, let $T(x y)$ and $T(y x)$ be the components of $T-x y$ where $x$ is in $T(x y)$ and $y$ is in $T(y x)$. For each edge $x y \in E(T)$, let $G(T, x y):=\bigcup\left\{T_{z}: z \in V(T(x y))\right\}$ and $G(T, y x):=\bigcup\left\{T_{z}: z \in V(T(y x))\right\}$. Let $E(T, x y)$ be the set of edges in $G$ between $G(T, x y)$ and $G(T, y x)$. The adhesion of a $T$-partition $\left(T_{x}: x \in V(T)\right)$ is the maximum, taken over all edges $x y$ of $T$, of $|E(T, x y)|$. For each node $x$ of $T$, the torso of $x$ (with respect to a $T$-partition) is the graph obtained from $G$ by identifying $G(T, y x)$ into a single vertex for each edge $x y$ incident to $x$ (deleting resulting parallel edges and loops).

Theorem 10 ([42]). For every graph $H$ with $t$ vertices, for every graph $G$ that does not contain $H$ as a weak immersion, there is a T-partition of $G$ with adhesion at most $t^{2}$ such that each torso has $\left(t, t^{2}\right)$-bounded degree.

Proof of Theorem 9. Let $G$ be a graph that does not contain $H$ as a weak immersion. Consider the $T$-partition $\left(T_{x}: x \in V(T)\right)$ of $G$ from Theorem 10.

Let $T^{\prime}$ be obtained from $T$ by orienting each edge towards some root vertex. We now define a tree decomposition $\left(T_{x}^{*}: x \in V(T)\right)$ of $G$. Initialise $T_{x}^{*}:=T_{x}$ for each node $x \in V(T)$. For each edge $v w$ of $G$, if $v \in T_{x}$ and $w \in T_{y}$ and $z$ is the least common ancestor of $x$ and $y$ in $T^{\prime}$, then add $v$ to $T_{\alpha}^{*}$ for each node $\alpha$ on the $\overrightarrow{x z}$ path in $T^{\prime}$, and add $w$ to $T_{\alpha}^{*}$ for each node $\alpha$ on the $\overrightarrow{y z}$ path in $T^{\prime}$. Thus each vertex $v \in T_{x}$ is in a sequence of bags that correspond to a directed path from $x$ to some ancestor of $x$ in $T^{\prime}$. By construction, the endpoints of each edge are in a common bag. Thus $\left(T_{x}^{*}: x \in V(T)\right)$ is a tree decomposition of $G$.

Consider a vertex $v \in T_{x}^{*} \cap T_{y}^{*}$ for some edge $\overrightarrow{x y}$ of $T^{\prime}$. Then $v$ has a neighbour $w$ in $G(T, y x)$, and $v w \in E(T, x y)$. Thus $\left|T_{x}^{*} \cap T_{y}^{*}\right| \leqslant|E(T, x y)| \leqslant t^{2}$. That is, the tree decomposition ( $\left.T_{x}^{*}: x \in V(T)\right)$ has adhesion at most $t^{2}$.

Let $G_{x}^{+}$be the torso of each node $x \in V(T)$ with respect to the tree decomposition ( $T_{x}^{*}: x \in$ $V(T))$. That is, $G^{+}$is obtained from $G\left[T_{x}^{*}\right]$ by adding a clique on $T_{x}^{*} \cap T_{y}^{*}$ for each edge $x y$ of $T$. Our goal is to prove that $G_{x}^{+}$has $\left(t, t^{4}+2 t^{2}\right)$-bounded degree.

Consider a vertex $v$ of $G_{x}^{+}$. Then $v$ is in at most one child bag $y$ of $x$, as otherwise $v$ would belong to a set of bags that do not correspond to a directed path in $T^{\prime}$. Since $\left(T_{x}^{*}: x \in V(T)\right)$ has adhesion at most $t^{2}, v$ has at most $t^{2}$ neighbours in $T_{x}^{*} \cap T_{p}^{*}$, where $p$ is the parent of $x$ and $v$ has at most $t^{2}$ neighbours in $T_{x}^{*} \cap T_{y}^{*}$. Thus the degree of $v$ in $G_{x}^{+}$is at most the degree of $v$ in $G\left[T_{x}^{*}\right]$ plus $2 t^{2}$. Call this property ( $\star$ ).

First consider the case that $v \notin T_{x}$. Let $z$ be the node of $T$ for which $v \in T_{z}$. Since $v \in T_{x}^{*}$,
by construction, $x$ is an ancestor of $z$. Let $y$ be the node immediately before $x$ on the $\overrightarrow{z x}$ path in $T^{\prime}$. We now bound the number of neighbours of $v$ in $T_{x}^{*}$. Say $w \in N_{G}(v) \cap T_{x}^{*}$. If $w$ is in $G(T, x y)$ then let $e_{w}$ be the edge $v w$. Otherwise, $w$ is in $G(T, y x)$ and thus $w$ has a neighbour $u$ in $G(T, x y)$ since $w \in T_{x}^{*}$; let $e_{w}$ be the edge $w u$. Observe that $\left\{e_{w}: w \in\right.$ $\left.N_{G}(v) \cap T_{x}^{*}\right\} \subseteq E(T, x y)$, and thus $\left|\left\{e_{w}: w \in N_{G}(v) \cap T_{x}^{*}\right\}\right| \leqslant t^{2}$. Since $e_{u} \neq e_{w}$ for distinct $u, w \in N_{G}(v) \cap T_{x}^{*}$, we have $\left|N_{G}(v) \cap T_{x}^{*}\right| \leqslant t^{2}$. By $(\star)$, the degree of $v$ in $G_{x}^{+}$is at most $3 t^{2}$.

Now consider the case that $v \in T_{x}$. Suppose further that $v$ is not one of the at most $t$ vertices of degree greater than $t^{2}$ in the torso $Q$ of $x$ with respect to the given $T$-partition. Suppose that in $Q, v$ has $d_{1}$ neighbours in $T_{x}$ and $d_{2}$ neighbours not in $T_{x}$ (the identified vertices). So $d_{1}+d_{2} \leqslant t^{2}$. Consider a neighbour $w$ of $v$ in $G\left[T_{x}^{*}\right]$ with $w \notin T_{x}$. Then $w \in G(T, y x)$ for some child $y$ of $x$. For at most $d_{2}$ children $y$ of $x$, there is a neighbour of $v$ in $G(T, y x)$. Furthermore, for each child $y$ of $x, v$ has at most $t^{2}$ neighbours in $G(T, y x)$ since the $T$ partition has adhesion at most $t^{2}$. Thus $v$ has degree at most $d_{1}+d_{2} t^{2} \leqslant t^{4}$ in $G\left[T_{x}^{*}\right]$. By ( $\star$ ), $v$ has degree at most $2 t^{2}+t^{4}$ in $G_{x}^{+}$.

Since $3 t^{2} \leqslant t^{4}+2 t^{2}$, the torso $G_{x}^{+}$has $\left(t, t^{4}+2 t^{2}\right)$-bounded degree.

## 4 Excluding a Topological Minor

Theorem 5 is an immediate corollary of Lemma 6 and the following structure theorem of Dvořák [18] that extends Theorem 8.

Theorem 11 ([18]). Let $H$ be a fixed planar graph that has a planar embedding with all the vertices of $H$ with degree at least 4 on a single face. Then there exists a constant $k$, such that every graph $G$ that does not contain $H$ as a topological minor has a tree decomposition such that each torso has $(k, k)$-bounded degree.

While Theorem 11 is not explcitly stated in [18], we now explain that it is in fact a special case of Theorem 3 in [18]. This result provides a structural description of graphs excluding a given topological minor in terms of the following definition. For a graph $H$ and surface $\Sigma$, let $\operatorname{mf}(H, \Sigma)$ be the minimum, over all possible embeddings of $H$ in $\Sigma$, of the minimum number of faces such that every vertex of degree at least 4 is incident with one of these faces. By assumption, for our graph $H$ and for every surface $\Sigma$, we have $\operatorname{mf}(H, \Sigma)=1$. In this case, Theorem 3 of Dvořák [18] says that for some integer $k=k(H)$, every graph $G$ that does not contain $H$ as a topological minor is a clique sum of $(k, k)$-bounded degree graphs. It immediately follows that $G$ has the desired tree decomposition. See Corollary 1.4 in [34] for a closely related structure theorem.

The following natural open problem arises from this research: Do graphs excluding a fixed planar graph as a topological minor have bounded nonrepetitive chromatic number? And what is the structure of such graphs?

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