Nonrepetitive Colourings of Graphs Excluding a Fixed Immersion or Topological Minor

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Abstract. We prove that graphs excluding a fixed immersion have bounded nonrepetitive chromatic number. More generally, we prove that if H is a fixed planar graph that has a planar embedding with all the vertices with degree at least 4 on a single face, then graphs excluding H as a topological minor have bounded nonrepetitive chromatic number. This is the largest class of graphs known to have bounded nonrepetitive chromatic number.

1 Introduction

A vertex colouring of a graph is *nonrepetitive* if there is no path for which the first half of the path is assigned the same sequence of colours as the second half. More precisely, a *k*colouring of a graph G is a function ψ that assigns one of k colours to each vertex of G. A path $(v_1, v_2, \ldots, v_{2t})$ of even order in G is *repetitively* coloured by ψ if $\psi(v_i) = \psi(v_{t+i})$ for $i \in \{1, \ldots, t\}$. A colouring ψ of G is *nonrepetitive* if no path of G of even order is repetitively coloured by ψ . Observe that a nonrepetitive colouring is *proper*, in the sense that adjacent vertices are coloured differently. The *nonrepetitive chromatic number* $\pi(G)$ is the minimum integer k such that G admits a nonrepetitive k-colouring. We only consider simple graphs with no loops or parallel edges.

The seminal result in this area is by Thue [41], who in 1906 proved that every path is nonrepetitively 3-colourable. Thue expressed his result in terms of strings over an alphabet of three characters—Alon et al. [3] introduced the generalisation to graphs in 2002. Nonrepetitive graph colourings have since been widely studied [2–12, 21, 25–33, 35, 37–39]. The principle result of Alon et al. [3] was that graphs with maximum degree Δ are nonrepetitively $\mathcal{O}(\Delta^2)$ colourable. Several subsequent papers improved the constant [16, 26, 30]. The best known bound is due to Dujmović et al. [16].

Theorem 1 ([16]). Every graph with maximum degree Δ is nonrepetitively $(1 + o(1))\Delta^2$ colourable.

A number of other graph classes are known to have bounded nonrepetitive chromatic number. In particular, trees are nonrepetitively 4-colourable [8, 33], outerplanar graphs are nonrepet-

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itively 12-colourable [5, 33], and graphs with bounded treewidth have bounded nonrepetitive chromatic number [5, 33]. (See Section 2 for the definition of treewidth.) The best known bound is due to Kündgen and Pelsmajer [33].

Theorem 2 ([33]). Every graph with treewidth k is nonrepetitively 4^k -colourable.

The primary contribution of this paper is to provide a qualitative generalisations of Theorems 1 and 2 via the notion of graph immersions and excluded topological minors.

A graph G contains a graph H as an *immersion* if the vertices of H can be mapped to distinct vertices of G, and the edges of H can be mapped to pairwise edge-disjoint paths in G, such that each edge vw of H is mapped to a path in G whose endpoints are the images of v and w. The image in G of each vertex in H is called a *branch vertex*. Structural and colouring properties of graphs excluding a fixed immersion have been widely studied [1, 13, 14, 18–20, 22–24, 34, 36, 40, 42]. We prove that graphs excluding a fixed immersion have bounded nonrepetitive chromatic number.

Theorem 3. For every graph H with t vertices, every graph that does not contain H as an immersion is nonrepetitively $4^{t^4+O(t^2)}$ -colourable.

Since a graph with maximum degree Δ contains no star with $\Delta + 1$ leaves as an immersion, Theorem 3 implies that graphs with bounded degree have bounded nonrepetitive chromatic number (as in Theorem 1).

We strengthen Theorem 3 as follows (although without explicit bounds). A graph G contains a graph H as a *strong immersion* if G contains H as an immersion, such that for each edge vw of H, no internal vertex of the path in G corresponding to vw is a branch vertex.

Theorem 4. For every fixed graph H, there exists a constant k, such that every graph G that does not contain H as a strong immersion is nonrepetitively k-colourable.

Note that planar graphs with n vertices are nonrepetitively $\mathcal{O}(\log n)$ -colourable [15], and the same is true for graphs excluding a fixed graph as a minor or topological minor [17]. It is unknown whether any of these classes have bounded nonrepetitive chromatic number. Our final result shows that excluding a special type of topological minor gives bounded nonrepetitive chromatic number.

Theorem 5. Let H be a fixed planar graph that has a planar embedding with all the vertices of H with degree at least 4 on a single face. Then there exists a constant k, such that every graph G that does not contain H as a topological minor is nonrepetitively k-colourable.

Graphs with bounded treewidth exclude fixed walls as topological minors. Since walls are planar graphs with maximum degree 3, Theorem 5 implies that graphs of bounded treewidth

have bounded nonrepetitive chromatic number (as in Theorem 2). Similarly, for every graph H with t vertices, the 'fat star' graph (which is the 1-subdivision of the t-leaf star with edge multiplicity t) contains H as a strong immersion. Since fat stars embed in the plane with all vertices of degree at least 4 on a single face, Theorem 5 implies that graphs excluding a fixed graph as a strong immersion have bounded nonrepetitive chromatic number (as in Theorem 4). In this sense, Theorem 5 generalises all of Theorems 1 to 4.

The results of this paper, in relation to the best known bounds on the nonrepetitive chromatic number, are summarised in Figure 1.

2 Tree Decompositions

For a graph G and tree T, a tree decomposition or T-decomposition of G consists of a collection $(T_x \subseteq V(G) : x \in V(T))$ of sets of vertices of G, called bags, indexed by the nodes of T, such that for each vertex $v \in V(G)$ the set $\{x \in V(T) : v \in T_x\}$ induces a connected subtree of T, and for each edge vw of G there is a node $x \in V(T)$ such that $v, w \in T_x$. The width of a T-decomposition is the maximum, taken over the nodes $x \in V(T)$, of $|T_x| - 1$. The treewidth of a graph G is the minimum width of a tree decomposition of G. The adhesion of a tree decomposition $(T_x : x \in V(T))$ is $\max\{|T_x \cap T_y| : xy \in E(T)\}$. The torso of each node $x \in V(T)$ is the graph obtained from $G[T_x]$ by adding a clique on $T_x \cap T_y$ for each edge $xy \in E(T)$ incident to x. Dujmović et al. [17] generalised Theorem 2 as follows:

Lemma 6 ([17]). If a graph G has a tree decomposition with adhesion k such that every torso is nonrepetitively c-colourable, then G is nonrepetitively $c4^k$ -colourable.

For integers $c, d \ge 0$ a graph G has (c, d)-bounded degree if G contains at most c vertices with degree greater than d.

Lemma 7. Every graph with (c, d)-bounded degree is nonrepetitively $c+(1+o(1))d^2$ -colourable.

Proof. Assign a distinct colour to each vertex of degree at least d, and colour the remaining graph by Theorem 1. For each vertex v of degree at least d, no other vertex is assigned the same colour as v. Thus v is in no repetitively coloured path. The result then follows from Theorem 1.

Dvořák [18] proved the following structure theorem for graphs excluding a strong immersion.

Theorem 8 ([18]). For every fixed graph H, there exists a constant k, such that every graph G that does not contain H as a strong immersion has a tree decomposition such that each torso is (k, k)-bounded degree.

Lemmas 6 and 7 and Theorem 8 imply Theorem 4.



Figure 1: Upper bounds on the nonrepetitive chromatic number of various graph classes. 'Special' refers to the condition in Theorem 5.

3 Weak Immersions

The proof of Theorem 4 gives no explicit bound on the constant k. In this section we prove an explicit bound on the nonrepetitive chromatic number of graphs excluding a weak immersion. Theorem 3 follows from Lemma 6 and the following structure theorem of independent interest.

Theorem 9. For every graph H with t vertices, every graph that does not contain H as a weak immersion has a tree decomposition with adhesion at most t^2 such that every torso has

 $(t, t^4 + 2t^2)$ -bounded degree.

The starting point for the proof of Theorem 9 is the following structure theorem of Wollan [42]. For a tree T and graph G, a T-partition of G is a partition $(T_x \subseteq V(G) : x \in V(T))$ of V(G) indexed by the nodes of T. Each set T_x is called a *bag*. Note that a bag may be empty. For each edge xy of a tree T, let T(xy) and T(yx) be the components of T - xy where x is in T(xy) and y is in T(yx). For each edge $xy \in E(T)$, let $G(T, xy) := \bigcup \{T_z : z \in V(T(xy))\}$ and $G(T, yx) := \bigcup \{T_z : z \in V(T(yx))\}$. Let E(T, xy) be the set of edges in G between G(T, xy) and G(T, yx). The *adhesion* of a T-partition $(T_x : x \in V(T))$ is the maximum, taken over all edges xy of T, of |E(T, xy)|. For each node x of T, the *torso* of x (with respect to a T-partition) is the graph obtained from G by identifying G(T, yx) into a single vertex for each edge xy incident to x (deleting resulting parallel edges and loops).

Theorem 10 ([42]). For every graph H with t vertices, for every graph G that does not contain H as a weak immersion, there is a T-partition of G with adhesion at most t^2 such that each torso has (t, t^2) -bounded degree.

Proof of Theorem 9. Let G be a graph that does not contain H as a weak immersion. Consider the T-partition $(T_x : x \in V(T))$ of G from Theorem 10.

Let T' be obtained from T by orienting each edge towards some root vertex. We now define a tree decomposition $(T_x^* : x \in V(T))$ of G. Initialise $T_x^* := T_x$ for each node $x \in V(T)$. For each edge vw of G, if $v \in T_x$ and $w \in T_y$ and z is the least common ancestor of x and y in T', then add v to T_α^* for each node α on the \vec{xz} path in T', and add w to T_α^* for each node α on the \vec{yz} path in T'. Thus each vertex $v \in T_x$ is in a sequence of bags that correspond to a directed path from x to some ancestor of x in T'. By construction, the endpoints of each edge are in a common bag. Thus $(T_x^* : x \in V(T))$ is a tree decomposition of G.

Consider a vertex $v \in T_x^* \cap T_y^*$ for some edge \overrightarrow{xy} of T'. Then v has a neighbour w in G(T, yx), and $vw \in E(T, xy)$. Thus $|T_x^* \cap T_y^*| \leq |E(T, xy)| \leq t^2$. That is, the tree decomposition $(T_x^* : x \in V(T))$ has adhesion at most t^2 .

Let G_x^+ be the torso of each node $x \in V(T)$ with respect to the tree decomposition $(T_x^* : x \in V(T))$. That is, G^+ is obtained from $G[T_x^*]$ by adding a clique on $T_x^* \cap T_y^*$ for each edge xy of T. Our goal is to prove that G_x^+ has $(t, t^4 + 2t^2)$ -bounded degree.

Consider a vertex v of G_x^+ . Then v is in at most one child bag y of x, as otherwise v would belong to a set of bags that do not correspond to a directed path in T'. Since $(T_x^* : x \in V(T))$ has adhesion at most t^2 , v has at most t^2 neighbours in $T_x^* \cap T_p^*$, where p is the parent of xand v has at most t^2 neighbours in $T_x^* \cap T_y^*$. Thus the degree of v in G_x^+ is at most the degree of v in $G[T_x^*]$ plus $2t^2$. Call this property (*).

First consider the case that $v \notin T_x$. Let z be the node of T for which $v \in T_z$. Since $v \in T_x^*$,

by construction, x is an ancestor of z. Let y be the node immediately before x on the \overrightarrow{zx} path in T'. We now bound the number of neighbours of v in T_x^* . Say $w \in N_G(v) \cap T_x^*$. If w is in G(T, xy) then let e_w be the edge vw. Otherwise, w is in G(T, yx) and thus w has a neighbour u in G(T, xy) since $w \in T_x^*$; let e_w be the edge wu. Observe that $\{e_w : w \in N_G(v) \cap T_x^*\} \subseteq E(T, xy)$, and thus $|\{e_w : w \in N_G(v) \cap T_x^*\}| \leq t^2$. Since $e_u \neq e_w$ for distinct $u, w \in N_G(v) \cap T_x^*$, we have $|N_G(v) \cap T_x^*| \leq t^2$. By (\star) , the degree of v in G_x^+ is at most $3t^2$.

Now consider the case that $v \in T_x$. Suppose further that v is not one of the at most t vertices of degree greater than t^2 in the torso Q of x with respect to the given T-partition. Suppose that in Q, v has d_1 neighbours in T_x and d_2 neighbours not in T_x (the identified vertices). So $d_1 + d_2 \leq t^2$. Consider a neighbour w of v in $G[T_x^*]$ with $w \notin T_x$. Then $w \in G(T, yx)$ for some child y of x. For at most d_2 children y of x, there is a neighbour of v in G(T, yx). Furthermore, for each child y of x, v has at most t^2 neighbours in G(T, yx) since the Tpartition has adhesion at most t^2 . Thus v has degree at most $d_1 + d_2t^2 \leq t^4$ in $G[T_x^*]$. By (\star) , v has degree at most $2t^2 + t^4$ in G_x^+ .

Since
$$3t^2 \leq t^4 + 2t^2$$
, the torso G_x^+ has $(t, t^4 + 2t^2)$ -bounded degree.

4 Excluding a Topological Minor

Theorem 5 is an immediate corollary of Lemma 6 and the following structure theorem of Dvořák [18] that extends Theorem 8.

Theorem 11 ([18]). Let H be a fixed planar graph that has a planar embedding with all the vertices of H with degree at least 4 on a single face. Then there exists a constant k, such that every graph G that does not contain H as a topological minor has a tree decomposition such that each torso has (k, k)-bounded degree.

While Theorem 11 is not explicitly stated in [18], we now explain that it is in fact a special case of Theorem 3 in [18]. This result provides a structural description of graphs excluding a given topological minor in terms of the following definition. For a graph H and surface Σ , let mf (H, Σ) be the minimum, over all possible embeddings of H in Σ , of the minimum number of faces such that every vertex of degree at least 4 is incident with one of these faces. By assumption, for our graph H and for every surface Σ , we have mf $(H, \Sigma) = 1$. In this case, Theorem 3 of Dvořák [18] says that for some integer k = k(H), every graph G that does not contain H as a topological minor is a clique sum of (k, k)-bounded degree graphs. It immediately follows that G has the desired tree decomposition. See Corollary 1.4 in [34] for a closely related structure theorem.

The following natural open problem arises from this research: Do graphs excluding a fixed planar graph as a topological minor have bounded nonrepetitive chromatic number? And what is the structure of such graphs?

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References

- FAISAL N. ABU-KHZAM AND MICHAEL A. LANGSTON. Graph coloring and the immersion order. In *Computing and combinatorics*, vol. 2697 of *Lecture Notes in Comput. Sci.*, pp. 394–403. Springer, 2003. doi: 10.1007/3-540-45071-8_40.
- [2] NOGA ALON AND JAROSŁAW GRYTCZUK. Breaking the rhythm on graphs. Discrete Math., 308:1375–1380, 2008. doi: 10.1016/j.disc.2007.07.063. MR: 2392054.
- [3] NOGA ALON, JAROSŁAW GRYTCZUK, MARIUSZ HAŁUSZCZAK, AND OLIVER RIORDAN. Nonrepetitive colorings of graphs. *Random Structures Algorithms*, 21(3-4):336–346, 2002. doi: 10.1002/rsa.10057. MR: 1945373.
- [4] JÁNOS BARÁT AND JÚLIUS CZAP. Facial nonrepetitive vertex coloring of plane graphs. J. Graph Theory, 2012. doi: 10.1002/jgt.21695.
- [5] JÁNOS BARÁT AND PÉTER P. VARJÚ. On square-free vertex colorings of graphs. Studia Sci. Math. Hungar., 44(3):411–422, 2007. doi: 10.1556/SScMath.2007.1029. MR: 2361685.
- [6] JÁNOS BARÁT AND PÉTER P. VARJÚ. On square-free edge colorings of graphs. Ars Combin., 87:377–383, 2008. MR: 2414029.
- JÁNOS BARÁT AND DAVID R. WOOD. Notes on nonrepetitive graph colouring. *Electron.* J. Combin., 15:R99, 2008. http://www.combinatorics.org/Volume_15/Abstracts/ v15i1r99.html. MR: 2426162.
- [8] BOŠTJAN BREŠAR, JAROSLAW GRYTCZUK, SANDI KLAVŽAR, STANISLAW NIWCZYK, AND IZTOK PETERIN. Nonrepetitive colorings of trees. *Discrete Math.*, 307(2):163–172, 2007. doi: 10.1016/j.disc.2006.06.017. MR: 2285186.
- [9] BOŠTJAN BREŠAR AND SANDI KLAVŽAR. Square-free colorings of graphs. Ars Combin., 70:3–13, 2004. MR: 2023057.
- [10] PANAGIOTIS CHEILARIS, ERNST SPECKER, AND STATHIS ZACHOS. Neochromatica. *Comment. Math. Univ. Carolin.*, 51(3):469–480, 2010. http://www.dml.cz/dmlcz/ 140723. MR: 2741880.
- [11] JAMES D. CURRIE. There are ternary circular square-free words of length n for $n \ge 18$. Electron. J. Combin., 9(1), 2002. http://www.combinatorics.org/Volume_9/ Abstracts/v9i1n10.html. MR: 1936865.

- [12] JAMES D. CURRIE. Pattern avoidance: themes and variations. *Theoret. Comput. Sci.*, 339(1):7–18, 2005. doi: 10.1016/j.tcs.2005.01.004. MR: 2142070.
- [13] MATT DEVOS, ZDENĚK DVOŘÁK, JACOB FOX, JESSICA MCDONALD, BOJAN MOHAR, AND DIEGO SCHEIDE. A minimum degree condition forcing complete graph immersion. *Combinatorica*, 34(3):279–298, 2014. doi: 10.1007/s00493-014-2806-z.
- [14] MATT DEVOS, JESSICA MCDONALD, BOJAN MOHAR, AND DIEGO SCHEIDE. A note on forbidding clique immersions. *Electron. J. Combin.*, 20(3):#P55, 2013. http://www. combinatorics.org/ojs/index.php/eljc/article/view/v30i3p55.
- [15] VIDA DUJMOVIĆ, FABRIZIO FRATI, GWENAËL JORET, AND DAVID R. WOOD. Nonrepetitive colourings of planar graphs with O(log n) colours. *Electron. J. Combin.*, 20(1):#P51, 2013. http://www.combinatorics.org/ojs/index.php/eljc/article/ view/v20i1p51.
- [16] VIDA DUJMOVIĆ, GWENAËL JORET, JAKUB KOZIK, AND DAVID R. WOOD. Nonrepetitive colouring via entropy compression. *Combinatorica*, 36(6):661–686, 2016. doi: 10.1007/s00493-015-3070-6.
- [17] VIDA DUJMOVIĆ, PAT MORIN, AND DAVID R. WOOD. Layered separators in minorclosed graph classes with applications. 2013. arXiv: 1306.1595.
- [18] ZDENĚK DVOŘÁK. A stronger structure theorem for excluded topological minors. 2012. arXiv: 1209.0129.
- [19] ZDENĚK DVOŘÁK AND TEREZA KLIMOŠOVÁ. Strong immersions and maximum degree. SIAM J. Discrete Math., 28(1):177–187, 2014. doi: 10.1137/130915467.
- [20] ZDENĚK DVOŘÁK AND PAUL WOLLAN. A structure theorem for strong immersions. J. Graph Theory, 83(2):152–163, 2016. doi: 10.1002/jgt.21990.
- [21] FRANCESCA FIORENZI, PASCAL OCHEM, PATRICE OSSONA DE MENDEZ, AND XUDING ZHU. Thue choosability of trees. *Discrete Applied Math.*, 159(17):2045–2049, 2011. doi:10.1016/j.dam.2011.07.017. MR: 2832329.
- [22] JACOB FOX AND FAN WEI. On the number of cliques in graphs with a forbidden subdivision or immersion. 2016. arXiv: 1606.06810.
- [23] ARCHONTIA C. GIANNOPOULOU, MARCIN KAMIŃSKI, AND DIMITRIOS M. THILIKOS. Excluding graphs as immersions in surface embedded graphs. In *Graph-theoretic concepts* in computer science, vol. 8165 of Lecture Notes in Comput. Sci., pp. 274–285. Springer, 2013. doi: 10.1007/978-3-642-45043-3_24.
- [24] ARCHONTIA C. GIANNOPOULOU, MARCIN KAMIŃSKI, AND DIMITRIOS M. THILIKOS. Forbidding Kuratowski graphs as immersions. J. Graph Theory, 78(1):43–60, 2015. doi: 10.1002/jgt.21790.

- [25] JAROSŁAW GRYTCZUK. Thue-like sequences and rainbow arithmetic progressions. Electron. J. Combin., 9(1):R44, 2002. http://www.combinatorics.org/Volume_9/ Abstracts/v9i1r44.html. MR: 1946146.
- [26] JAROSŁAW GRYTCZUK. Nonrepetitive colorings of graphs—a survey. Int. J. Math. Math. Sci., 74639, 2007. doi: 10.1155/2007/74639. MR: 2272338.
- [27] JAROSŁAW GRYTCZUK. Thue type problems for graphs, points, and numbers. Discrete Math., 308(19):4419–4429, 2008. doi: 10.1016/j.disc.2007.08.039. MR: 2433769.
- [28] JAROSŁAW GRYTCZUK, JAKUB KOZIK, AND PIOTR MICEK. A new approach to nonrepetitive sequences. *Random Structures Algorithms*, 42(2):214–225, 2013. doi:10.1002/rsa.20411.
- [29] JAROSŁAW GRYTCZUK, JAKUB PRZYBYŁO, AND XUDING ZHU. Nonrepetitive list colourings of paths. *Random Structures Algorithms*, 38(1-2):162–173, 2011. doi: 10.1002/rsa.20347. MR: 2768888.
- [30] JOCHEN HARANTA AND STANISLAV JENDROL. Nonrepetitive vertex colorings of graphs. Discrete Math., 312(2):374–380, 2012. doi: 10.1016/j.disc.2011.09.027. MR: 2852595.
- [31] FRÉDÉRIC HAVET, STANISLAV JENDROĽ, ROMAN SOTÁK, AND ERIKA ŠKRABUĽÁKOVA. Facial non-repetitive edge-coloring of plane graphs. J. Graph Theory, 66(1):38–48, 2011. doi: 10.1002/jgt.20488. MR: 2742187.
- [32] STANISLAV JENDROL AND ERIKA ŠKRABUL'ÁKOVÁ. Facial non-repetitive edge colouring of semiregular polyhedra. Acta Univ. M. Belii Ser. Math., 15:37–52, 2009. http:// actamath.savbb.sk/acta1503.shtml. MR: 2589669.
- [33] ANDRE KÜNDGEN AND MICHAEL J. PELSMAJER. Nonrepetitive colorings of graphs of bounded tree-width. *Discrete Math.*, 308(19):4473–4478, 2008. doi:10.1016/j.disc.2007.08.043. MR: 2433774.
- [34] CHUN-HUNG LIU AND ROBIN THOMAS. Excluding subdivisions of bounded degree graphs. 2014. arXiv: 1407.4428.
- [35] DÁNIEL MARX AND MARCUS SCHAEFER. The complexity of nonrepetitive coloring. Discrete Appl. Math., 157(1):13–18, 2009. doi: 10.1016/j.dam.2008.04.015. MR: 2479374.
- [36] DÁNIEL MARX AND PAUL WOLLAN. Immersions in highly edge connected graphs. SIAM J. Discrete Math., 28(1):503–520, 2014. doi: 10.1137/130924056.
- [37] JAROSLAV NEŠETŘIL, PATRICE OSSONA DE MENDEZ, AND DAVID R. WOOD. Characterisations and examples of graph classes with bounded expansion. *European J. Combinatorics*, 33(3):350–373, 2011. doi: 10.1016/j.ejc.2011.09.008. MR: 2864421.

- [38] WESLEY PEGDEN. Highly nonrepetitive sequences: winning strategies from the local lemma. Random Structures Algorithms, 38(1-2):140–161, 2011. doi: 10.1002/rsa.20354. MR: 2768887
- [39] ANDRZEJ PEZARSKI AND MICHAŁ ZMARZ. Non-repetitive 3-coloring of subdivided graphs. *Electron. J. Combin.*, 16(1):N15, 2009. http://www.combinatorics.org/ Volume_16/Abstracts/v16i1n15.html. MR: 2515755.
- [40] NEIL ROBERTSON AND PAUL SEYMOUR. Graph minors XXIII. Nash-Williams' immersion conjecture. J. Combin. Theory Ser. B, 100(2):181–205, 2010. doi:10.1016/j.jctb.2009.07.003.
- [41] AXEL THUE. Über unendliche Zeichenreihen. Norske Vid. Selsk. Skr. I. Mat. Nat. Kl. Christiania, 7:1–22, 1906.
- [42] PAUL WOLLAN. The structure of graphs not admitting a fixed immersion. J. Combin. Theory Ser. B., 110:47–66, 2015. doi: 10.1016/j.jctb.2014.07.003.