

Shorter signed circuit covers of graphs

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Abstract

A signed circuit is a minimal signed graph (with respect to inclusion) that admits a nowhere-zero flow. We show that each flow-admissible signed graph on m edges can be covered by signed circuits of total length at most $(3 + 2/3) \cdot m$, improving a recent result of Cheng et al. [manuscript, 2015]. To obtain this improvement we prove several results on signed circuit covers of trees of Eulerian graphs, which are connected signed graphs such that removing all bridges results in a collection of Eulerian graphs.

1 Introduction

A *circuit cover* of a bridgeless graph G is a collection of circuits such that each edge of G belongs to at least one of them. One of the most studied problems concerning circuit covers is finding a circuit cover of the graph with small total length. A conjecture by Alon and Tarsi [1] bounds the length of the shortest circuit cover of a bridgeless graph from above.

Conjecture 1.1 (Short Cycle Cover conjecture). *Every bridgeless graph G has a circuit cover of total length at most $7/5 \cdot |E(G)|$.*

The best general upper bound on the length of a shortest cycle cover is a result obtained independently by Bermond, Jackson, and Jaeger [2] and by Alon and Tarsi [1].

Theorem 1.2. [1, 2] *If G is a bridgeless graph, then it admits a circuit cover of total length at most $5/3 \cdot |E(G)|$.*

The Short Cycle Cover Conjecture has connections to many well-known conjectures: for instance, it is implied by the Petersen Flow Conjecture (alternatively known as the Petersen Colouring Conjecture) of Jaeger [7, Section 7], while it implies the Cycle Double Cover Conjecture [8].

In parallel to the classical graph theory there is a fast-growing theory of *signed graphs*, graphs where each edge has a positive or a negative sign. More formally, a signed graph (G, σ) is a graph G endowed with a function σ called *signature* which assigns values either 1 (*positive edges*) or -1 (*negative edges*) to the edges. A circuit C in G is *balanced* if the number of negative edges in C is even, and *unbalanced* otherwise. Two signed graphs (G, σ_1) and (G, σ_2) are *equivalent* if they have the same set of balanced circuits. Equivalent signed graphs are considered to be the same, and one

can freely replace the signature of a signed graph with the signature of any equivalent signed graph. We say that a signature σ is a *minimum signature* of (G, σ) if there is no signed graph equivalent to (G, σ) with fewer negative edges.

Many fundamental concepts in graphs, such as colourings, flows, homomorphisms, or surface embeddings found their analogues in signed graphs [18, 3, 14, 15]. In this paper we will study a problem of short circuit covers for signed graphs introduced by Máčajová et al. in [12]. We refer the reader to [9] for a survey on circuit covers and nowhere-zero flows in signed graphs, and to [19] for an extensive bibliography on signed graphs.

According to [12], a *signed circuit* of a signed graph is one of the following subgraphs: (1) a balanced circuit, (2) the union of two unbalanced circuits which meet at a single vertex — a *short barbell*, or (3) the union of two disjoint unbalanced circuits with a path which meets the circuits only at its ends—a *long barbell*. A *barbell* is either a short or a long barbell.

The definition of signed circuit is chosen so that several important correspondences are preserved. A signed circuit in a signed graph G forms a signed circuit in the signed graphic matroid $\mathcal{M}(G)$ and vice versa [17, Theorem 5.1]. Moreover, a signed graph is *flow-admissible*, that is it admits a signed nowhere-zero flow, if and only if every edge of the signed graph belongs to a signed circuit [3, Proposition 3.1]. Furthermore, signed circuits are the minimal graphs that are flow-admissible, a property that circuits satisfy in ordinary graphs.

Let \mathcal{C} be a collection of subgraphs of a signed graph G . We say that \mathcal{C} is a *cover* if each edge of G is *covered by* \mathcal{C} (i.e. it is contained in at least one subgraph from \mathcal{C}), and it is a *signed circuit cover* if each element of \mathcal{C} is a signed circuit. The *length* of \mathcal{C} is the sum $\sum_{F \in \mathcal{C}} |E(F)|$. The *width* $\omega_{\mathcal{C}}(e)$ of an edge e with respect to \mathcal{C} is defined as the number of subgraphs from \mathcal{C} containing e . If $\omega_{\mathcal{C}}(e) = k$, then we also say that e is *covered k times* by \mathcal{C} . The *width* of the collection \mathcal{C} , denoted by $\omega(\mathcal{C})$, is the maximum width of an edge with respect to \mathcal{C} over all edges of G .

If $\mathcal{C}_1, \dots, \mathcal{C}_\ell$ are collections of subgraphs of G , then the *total length* of $\{\mathcal{C}_1, \dots, \mathcal{C}_\ell\}$ is the sum of lengths of all the \mathcal{C}_i ($i \in \{1, \dots, \ell\}$). The *total width* of an edge e with respect to the collection $\{\mathcal{C}_1, \dots, \mathcal{C}_\ell\}$ is defined as

$$\omega_{\mathcal{C}_1, \dots, \mathcal{C}_\ell}(e) = \sum_{i=1}^{\ell} \omega_{\mathcal{C}_i}(e).$$

Again, the *total width* of the collection $\{\mathcal{C}_1, \dots, \mathcal{C}_\ell\}$, denoted by $\omega(\mathcal{C}_1, \dots, \mathcal{C}_\ell)$, is the maximum width of an edge with respect to this collection over all edges of G .

Máčajová et al. [12] proved that every flow-admissible signed graph with m edges has a signed circuit cover of length at most $11 \cdot m$. Cheng, Lu, Luo and Zhang [4] announced in 2015 an improvement of the bound to $14/3 \cdot m - 5/3 \cdot \varepsilon_N - 4$ for flow-admissible signed graphs with ε_N negative edges in a minimum signature. A refinement of their proof idea leads to the following improvement.

Theorem 1.3. *Let (G, σ) be a flow-admissible signed graph with m edges and let ε_N be the number of negative edges in a minimum signature of (G, σ) . Then there is a signed circuit cover of (G, σ) of length at most $11/3 \cdot m - 5/3 \cdot \varepsilon_N$.*

In Section 5 we provide several variants of Theorem 1.3 comparable to the results provided by Cheng et al. [4]. We note that for bridgeless cubic signed graphs, it is possible to obtain even better bounds on the length of a short signed circuit cover as suggested by a recently announced result by Wu and Ye [16].

Despite these improvements, the bounds on the length of the shortest signed circuit cover are probably far from tight. Indeed, we are not aware of any flow-admissible signed graph (G, σ) with a shortest cycle cover of length exceeding $5/3 \cdot |E(G)|$. An example that attains this value is the Petersen graph with five negative edges forming a 5-cycle [12].

2 Covering auxiliary structures

In this section we will introduce the necessary notions and prove several key lemmas. Graphs in this paper may have parallel edges and loops. For convenience, we define the *subtraction* of two subgraphs A_1 and A_2 of a graph, denoted by $A_1 - A_2$, as follows: from A_1 , we remove all edges that are contained in A_2 and delete isolated vertices. A *cut-vertex* is a vertex whose deletion increases the number of components of a graph. Note that a vertex incident with a loop is not necessarily a cut-vertex. A graph is *2-connected* if it has no cut-vertex. Note that graphs consisting either of a vertex with loops incident to it or of an edge with loops incident to its end-vertices are both 2-connected. A *cycle* is a graph with all degrees even. A connected cycle is an *Eulerian graph* (in particular, the graph with a single vertex is an Eulerian graph, and we refer to it as *trivial*). A *circuit* is a connected 2-regular graph (note that a loop is a circuit). Recall that a *signed circuit* is a signed graph that is either a balanced circuit or a barbell.

Let H be a connected signed graph. Delete all bridges of H to obtain H' . We let $\varepsilon(H)$ denote the number of negative edges in H , and let $\varepsilon'(H)$ denote the number of negative edges in H' .

If each component of H' is an Eulerian graph, then H is a *tree of Eulerian graphs*. In the further text the notion of a balloon will be crucial. A *balloon of H* (or simply a *balloon* if H is clear from the context) is either a negative loop of H' or a component of H' from which all the loops have been deleted. If each balloon of H' is either an isolated vertex or a circuit, then H is a *tree of circuits*. Note that a tree of circuits is a special case of a tree of Eulerian graphs.

Let B be a balloon of a tree of Eulerian graphs H . We say that B is *trivial* if B is a single vertex, and *non-trivial* otherwise. The *valency of B* is 1 if B is a negative loop, and otherwise it is the number of its incident bridges and loops. The balloon B is a *leaf balloon* if its valency is 1, and an *inner balloon* otherwise. In the case that B is a circuit, we use the terms *leaf circuit* or *inner circuit*. The balloon B is *even* or *odd* according to $\varepsilon(B)$ being even or odd, respectively. An *endblock* of H is an inclusion-wise maximal subgraph of H which contains at most one cut-vertex and is different from a loop.

The following definitions and two lemmas are based on the ideas of [10] and [11]. Suppose for now that H is a tree of circuits. Let $c(H)$ and $u(H)$ denote the number of circuits and the number of unbalanced circuits of H , respectively. Suppose that

H is a tree of circuits such that every leaf circuit is unbalanced. Note that if H has at least two leaf circuits, it is flow-admissible. Let \vec{H} be a digraph obtained from H by the following operations:

- orienting the edges of each circuit in H in such a way that it becomes a directed circuit,
- replacing all bridges by directed 2-circuits.

Since \vec{H} is an Eulerian digraph, it admits an Eulerian trail W . Traverse W starting from some vertex of \vec{H} and enumerate all circuits of H as

$$C_0, C_1, \dots, C_{k-1},$$

in the order in which W visits them for the first time. (Thus, each circuit of H appears exactly once in the sequence.) Let us say that two leaf circuits C_i, C_j are *consecutive leaf circuits* if none of C_{i+1}, \dots, C_{j-1} is a leaf circuit (where the counting is modulo k). *Consecutive unbalanced circuits* are defined in an analogous way. For any two indices $s, t \in \{0, \dots, k-1\}$, let W_{st} be a shortest subtrail of W starting at a vertex of C_s and ending at a vertex of C_t . Define B_{st} to be the barbell in H obtained as the union of C_s, C_t and the path corresponding to W_{st} .

Lemma 2.1. *Let H be a tree of circuits with $u(H) \neq 1$ such that every leaf circuit is unbalanced. Then H admits a signed circuit cover \mathcal{C} such that the width of the edges of leaf circuits and bridges with respect to \mathcal{C} is 2, and the width of the edges of inner circuits with respect to \mathcal{C} is 1.*

Proof. If H is balanced, then since each leaf is unbalanced, we have $c(H) = 1$. As $u(H) \neq 1$ the circuit is balanced and we set $\mathcal{C} = \{H\}$. Thus, we may assume that $u(H) \geq 2$.

Define \mathcal{C} as the collection of all barbells B_{st} such that C_s and C_t are consecutive leaf circuits. Clearly, edges of leaf circuits and bridges have width 2 with respect to \mathcal{C} while edges of an inner circuit have width 1 with respect to \mathcal{C} . \square

A collection \mathcal{C} of subgraphs of a signed graph H is a *weak signed circuit cover* if all the elements of \mathcal{C} are signed circuits and each non-bridge edge of H is covered by \mathcal{C} .

Lemma 2.2. *Let H be a tree of circuits with $u(H)$ even. Then H admits three weak signed circuit covers $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of total width at most 4 such that \mathcal{C}_1 covers all negative loops exactly twice.*

Proof. We proceed by induction on the number of edges of H . If H is balanced, then we set $\mathcal{C}_1, \mathcal{C}_2$, and \mathcal{C}_3 to be the set of circuits of H . If there is a bridge e of H such that some component of $H - e$ is balanced, then we cover each component of $H - e$ by the induction hypothesis and define each \mathcal{C}_i as the union of the two obtained respective covers. Therefore, we may assume that every leaf balloon of H is unbalanced. Note that $u(H) \geq 2$.

Let us define the signed circuit covers $\mathcal{C}_1, \mathcal{C}_2$ and \mathcal{C}_3 . Let \mathcal{C}_1 be the weak signed circuit cover defined in Lemma 2.1. The other two weak signed circuit covers \mathcal{C}_2 and \mathcal{C}_3 are defined as follows.

Let \mathcal{K} be the collection of balanced circuits of H and suppose that the unbalanced circuits of H are $C_{i_0}, \dots, C_{i_{u-1}}$, where $u = u(H)$ and $0 \leq i_0 < \dots < i_{u-1} \leq u-1$. If B_{rs}^* denotes the barbell $B_{i_r i_s}$, then

$$\begin{aligned}\mathcal{C}_2 &= \{B_{01}^*, B_{23}^*, \dots, B_{u-2, u-1}^*\} \cup \mathcal{K}, \\ \mathcal{C}_3 &= \{B_{12}^*, B_{34}^*, \dots, B_{u-1, 0}^*\} \cup \mathcal{K}.\end{aligned}$$

Let us check that the total width of \mathcal{C}_1 , \mathcal{C}_2 and \mathcal{C}_3 is at most 4. If e is an edge of a leaf circuit, it is covered twice by \mathcal{C}_1 , once by \mathcal{C}_2 and once by \mathcal{C}_3 . If e is a bridge, it is covered twice by \mathcal{C}_1 and twice by $\mathcal{C}_2 \cup \mathcal{C}_3$, because the paths $W_{i, i+1}$ ($0 \leq i \leq u-1$) are edge-disjoint (in \vec{H}). For the same reason, an edge e of an inner circuit is covered once by some $W_{i, i+1}$ and once by a circuit of each of \mathcal{C}_2 and \mathcal{C}_3 . Since by Lemma 2.1 the edges of inner circuits have width 1 in \mathcal{C}_1 , we have found the required covers. \square

We now proceed to trees of Eulerian graphs.

Lemma 2.3. *Let A be a 2-connected non-trivial signed Eulerian graph without negative loops, and let v be a vertex of A . Then either A is a circuit, or it can be edge-decomposed into two nontrivial Eulerian graphs A_1 and A_2 such that A_1 contains v and $\varepsilon(A_2)$ is even.*

Proof. Since A is Eulerian, there is a circuit C containing v . Let A' be a component of $A - C$ (recall that $A - C$ has no isolated vertices due to our definition of graph subtraction). If $\varepsilon(A')$ is even, A can be decomposed into A' and $A - A'$.

Suppose then that $\varepsilon(A')$ is odd. By the assumption of the lemma, A' is not a loop, and by the 2-connectivity of A , $|V(C) \cap V(A')| \geq 2$. Let v_1 and v_2 be vertices of $C \cap A'$. Let P be a v_1 - v_2 -path of C that does not contain v as an internal vertex. Since A is Eulerian, A' is Eulerian as well, so it admits a closed Eulerian trail W . The trail W consists of two v_1 - v_2 trails, W_1 and W_2 . Since $\varepsilon(A')$ is odd, we may assume without loss of generality that $\varepsilon(W_1)$ has the same parity as $\varepsilon(P)$. The sought decomposition is $W_1 \cup P$ and $A - (W_1 \cup P)$. \square

To deal with loops we define two operations on Eulerian graphs: compression and decompression. Let (G, σ) be a signed Eulerian graph with a non-loop edge, and let L be a set of negative loops of G . Let $f : L \rightarrow (E(G) - L)$ be a function that assigns to each loop edge an adjacent non-loop edge of G . We say that (G^*, σ^*) is a *compression* of (G, σ) with respect to f if $G^* = G - L$ and $\sigma^*(e) := \sigma(e) \cdot (-1)^{|f^{-1}(e)|}$ for every edge e of G^* . Let (G_0^*, σ^*) be a subgraph of (G^*, σ^*) . (We take the liberty of using the same symbol for the induced signature of the subgraph.) The *decompression* of (G_0^*, σ^*) with respect to f is the subgraph (G_0, σ) of (G, σ) induced by the edges $E(G_0^*) \cup f^{-1}(E(G_0^*))$. Note the following.

Observation 2.4. *Compression and decompression preserve the parity of the number of negative edges. If two subgraphs partition the compressed graph, then the decompressed subgraphs partition the original graph. If the original graph is 2-connected, then so is the compressed graph.*

The following lemma and its corollary generalise the result of [10] and [11] stating that every signed Eulerian graph G with even number of negative edges admits a signed circuit cover of total length $4/3 \cdot |E(G)|$.

Lemma 2.5. *Let H be a tree of Eulerian graphs such that $\varepsilon'(H)$ is even. Then H admits three weak signed circuit covers $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of total width at most 4 such that $\omega(\mathcal{C}_i) \leq 2$ for each $i \in \{1, 2, 3\}$ and \mathcal{C}_1 covers all negative loops exactly twice.*

Proof. Let H be a counterexample to the lemma with minimum number of edges. Clearly, H has no balanced loops.

Assume that H is not 2-connected, and consider an end-block H_1 of H incident with a cut-vertex v . Recall that by the definition of an end-block, H_1 is different from a loop. Let H_2 be obtained from H by removing $V(H_1) - v$.

Assume first that $\varepsilon'(H_1)$ is even. Then since $\varepsilon'(H)$ is even, so is $\varepsilon'(H_2)$. Since both H_1 and H_2 contain fewer edges than H , they admit the weak signed circuit covers described in the statement of the lemma. The combination of these covers gives the sought weak signed circuit covers of H , a contradiction.

Thus, both $\varepsilon'(H_1)$ and $\varepsilon'(H_2)$ are odd (hence each of H_1, H_2 has at least two edges). For $i \in \{1, 2\}$, we add to H_i a negative loop e_i at v to obtain H'_i . Both H'_1 and H'_2 contain fewer edges than H , and hence they admit the weak signed circuit covers with the requested properties. Note that e_1 and e_2 must be covered by barbells, and each of them is covered twice by the first cover and once by the two other covers. To obtain the sought weak signed circuit covers of H , we combine the found signed circuit covers of H'_1 and H'_2 by merging the barbells containing loops e_1 and e_2 . This contradicts the minimality of H .

It follows that H is 2-connected. There are several possibilities: either H is a vertex with loops incident with it, or H consists of an edge and loops incident with the end-vertices of the edge, or H is bridgeless (on at least two vertices). In the first two cases, it is easy to find the weak signed circuit covers directly. Thus, it suffices to consider the third case, where H consists of one non-loop non-trivial balloon and possibly some unbalanced loops incident with it.

Let H^* be a compression of H with respect to an arbitrary assignment f . Let v be an arbitrary vertex of H^* . By Lemma 2.3, either H^* is a circuit or it can be decomposed into two Eulerian graphs each having even number of negative edges. In the first case, Lemma 2.2 provides the desired covers. Assume thus that H^* decomposes into nontrivial Eulerian graphs H_1^* and H_2^* with $\varepsilon(H_1^*)$ and $\varepsilon(H_2^*)$ even. Let H_1 and H_2 be decompressions of H_1^* and H_2^* . By Observation 2.4, $\varepsilon(H_1)$ and $\varepsilon(H_2)$ is even and each of H_1 and H_2 has fewer edges than H . Hence they admit the sought weak signed circuit covers. Since H_1 and H_2 partition H by Observation 2.4, the covers of H_1 and H_2 combine to form the required covers of H . This contradiction concludes the proof. \square

Corollary 2.6. *Let H be a tree of Eulerian graphs with $\varepsilon'(H)$ even. Then H admits a weak signed circuit cover of length $4/3 \cdot |E(H)|$.*

Lemma 2.7. *Let H be a tree of Eulerian graphs such that it has at least two leaf balloons, and each leaf balloon is odd. Then H admits a signed circuit cover of width at most 2 that covers every loop of H exactly twice.*

Proof. Let H be a counterexample to the lemma with the minimum number of edges. Clearly, H has no balanced loops.

We prove that H is 2-connected. Suppose that this is not the case. Consider an end-block H_1 of H incident with a cut-vertex v , and let H_2 be obtained from H by removing all the vertices of H_1 except v .

Assume that H_1 contains only one balloon and that balloon is even (hence H_1 has no loops). By Lemma 2.5, H_1 admits a signed circuit cover of width at most 2. Any leaf balloon of H_2 is either a leaf balloon of H , or it is obtained from a leaf balloon of H by deleting H_1 . By the assumption that $\varepsilon(H_1)$ is even, every leaf balloon of H_2 is odd, and there are at least two of them. Since H_2 contains fewer edges than H , it admits the desired signed circuit cover and so does H , a contradiction.

Thus, either H_1 is an odd balloon, or H_1 contains at least two balloons, at least one of which is a leaf balloon of H , and hence it is odd. We conclude that H_1 contains an odd balloon. By symmetry, H_2 contains an odd balloon. Note that in each of H_1 and H_2 , every leaf balloon (except possibly for one containing v) is a leaf balloon of H . For $i \in \{1, 2\}$, we define H'_i as the graph obtained from H_i by adding a negative loop e_i at v . For $i \in \{1, 2\}$, each leaf balloon of H'_i is odd and there are at least two of them. Since H_1 and H_2 contain fewer edges than H , they admit signed circuit covers described in the statement. Note that e_1 and e_2 must be covered by barbells, and each of them is covered exactly twice. To obtain the sought signed circuit cover of H , we combine the found signed circuit covers of H_1 and H_2 by merging barbells containing loops e_1 and e_2 . This contradicts the choice of H .

Thus, H is 2-connected. That is, either H is a vertex with loops incident with it, or H consists of an edge and loops incident with end-vertices of the edge, or H is bridgeless (on at least two vertices). In the first two cases, it is easy to find the desired signed circuit cover directly. In the third case, H consists of one non-loop balloon and possibly unbalanced loops incident with vertices of the balloon. As H has at least two leaf balloons, it contains at least one loop. If H has exactly one negative loop, then the non-loop balloon is odd and the cover \mathcal{C}_1 from Lemma 2.5 provides a contradiction. Thus, H has at least two loops. Denote by e_1 and e_2 two arbitrary distinct loops of H , and by v_1 and v_2 the end-vertices of e_1 and e_2 , respectively; it may happen that $v_1 = v_2$.

Let H^* be a compression of $H - \{e_1, e_2\}$ with respect to an arbitrary assignment f . If we partition H^* into Eulerian subgraphs in such a way that one of them, denoted by H_S^* , contains v_1 and v_2 and every other either contains even number of negative edges or its decomposition contains at least two loops, then we can produce a cover of H as follows. We take the decomposition of all subgraphs. The union of the decomposition of H_S^* with e_1 and e_2 , denoted by H_S , has fewer edges than H and as H is a smallest counterexample to the lemma, H_S can be covered so that the conditions of the lemma are satisfied. The decompositions of other subgraphs either satisfy assumptions of Lemma 2.5 (those subgraphs that have even number of negative edges), or they have at least two loops and satisfy assumptions of this lemma while having fewer edges than H . Thus we can cover them as required. A cover of H is then obtained as the union of the covers of the subgraphs under consideration. Therefore, finding such a partition of H^* provides a contradiction with the choice of

H .

If H^* is a circuit, then Lemma 2.1 provides the desired cover. If $v_1 = v_2$, then by Lemma 2.3, H^* can be partitioned into Eulerian subgraphs H_S^* and H_2^* , such that H_S^* contains v_1 and $\varepsilon(H_2^*)$ is even, a contradiction. Thus, $v_1 \neq v_2$.

Since H^* is 2-connected, there exist two internally vertex-disjoint v_1 - v_2 -paths, P_1 and P_2 . As H^* is not a circuit, $H^* - P_1 - P_2$ is non-empty. Let A be a component of $H^* - P_1 - P_2$. Assume that A contains a cut-vertex r and let A_1 and A_2 be subgraphs of A such that $A_1 \cup A_2 = A$ and $A_1 \cap A_2 = \{r\}$. Since H^* is 2-connected, both A_1 and A_2 intersect $H^* - A$. If one of A_1 and A_2 , say A_1 , is even, then H^* can be decomposed into A_1 and $H^* - A_1$ which leads to a contradiction. Thus both A_1 and A_2 are odd which implies that A is even and so H^* can be decomposed into A and $H^* - A$ which are smaller graphs that fulfill the conditions of the lemma, a contradiction. Therefore we can assume that A is 2-connected.

Let $w \in V(A) \cap V(P_1 \cup P_2)$. By Lemma 2.3, either A is a circuit, or it can be decomposed into Eulerian subgraphs H_1^* and H_2^* such that H_1^* contains w and $\varepsilon(H_2^*)$ is even. In the latter case, the decomposition of H^* into $H_S^* = H^* - H_2^*$ and H_2^* provides a contradiction. Therefore A is a circuit. If A is balanced, then we get a contradiction by decomposing H^* into $H_S^* = H^* - A$ and A . Hence, A is an unbalanced circuit.

If A intersects P_1 in more than one vertex, say A intersects P_1 in distinct vertices u_1, u_2 , then we can decompose H^* as follows. One subgraph, denoted by B , is a union of a subpath P_1' of P_1 between u_1 and u_2 , and of every component of $H^* - P_1 - P_2$ that intersects P_1' and does not intersect $(P_1 \cup P_2) - P_1'$, and of a u_1 - u_2 path of A chosen in such a way that B has even number of negative edges (since A is unbalanced, this is possible). The second subgraph is defined as $H_S^* = H^* - E(B)$. Again, we find a contradictory decomposition of H^* . Thus A intersects P_1 in at most one vertex. Similarly, A intersects P_2 in at most one vertex. As H^* is 2-connected, A intersects both P_1 and P_2 in exactly one vertex (and these vertices are distinct).

Now we are ready to define the cover of H . First, we define a set of unbalanced circuits S and a closed trail T such that $S \cup T = H$. Let L be a set of loops whose intersection with $V(P_1 \cup P_2)$ is nonempty. The set S contains all loops of H and all components of $H - P_1 - P_2 - L$ that are unbalanced circuits. Note that any two unbalanced circuits of S that are not loops are vertex-disjoint, since they arise from two different components of $H^* - P_1 - P_2$. The closed trail T starts at v_1 and traverses every edge of H that is not contained in any circuit from S in such a way that edges of P_2 are the last edges of the trail. We label the elements of S according to the order in which T visits them for the first time, starting with the unbalanced loop e_1 . Thus we obtain an ordered n -tuple of unbalanced circuits $(C_1 = e_1, C_2, \dots, C_n)$. For C_i , define u_i to be the first vertex of C_i visited by T . We define barbells covering H as follows. Take C_i , C_{i+1} and the segment of T between u_i and u_{i+1} , for $i \in \{1, 2, \dots, n-1\}$. Moreover, take the barbell containing C_n , C_1 and the segment of T after u_n . Next, we will prove that the constructed elements are, indeed, barbells.

Note that between any two occurrences of the same vertex on the trail T there is a vertex that is incident to a loop. Thus u_i - u_{i+1} subtrails of T contain no circuits, for $i \in \{1, 2, \dots, n\}$. If two unbalanced circuits, say C_i and C_j , intersect, then due to

the definition of S at least one of them is a loop, for $i, j \in \{1, 2, \dots, n\}$. Moreover, the trail segments do not internally intersect the unbalanced circuits, because the components of $H - P_1 - P_2$ intersect P_1 only in one vertex and the segments of T containing edges of P_1 and P_2 are separated by the loop e_2 . Therefore the constructed cover consists of barbells. It covers the edges of circuits in S , which contains all loops, exactly twice, while the edges in T are covered once. Thus the cover satisfies the statement of the lemma which contradicts the fact that H is a counterexample. \square

3 Proof of Theorem 1.3

In this section, we follow ideas of [4] to provide the framework for the proof of Theorem 1.3. We will show a straightforward proof of the bound $4 \cdot |E(G)| - 5/3 \cdot \varepsilon_N$ instead of $11/3 \cdot |E(G)| - 5/3 \cdot \varepsilon_N$. Further, we will analyse the straightforward proof. We identify the places that require an improvement in order to prove Theorem 1.3, and postpone the technical details to Section 4. The following is an easy observation.

Lemma 3.1. *Let σ be a minimum signature of (G, σ) . Then for every edge-cut of (G, σ) the number of negative edges does not exceed the number of positive ones.*

Given a set \mathcal{E} of vertex-disjoint Eulerian subgraphs of a signed graph (G, σ) , *connecting \mathcal{E} into a tree of Eulerian graphs* means taking the disjoint union of the graphs in \mathcal{E} and adding the minimum number of edges of G needed to get a tree of Eulerian graphs.

We are going to prove Theorem 1.3. Thus, let (G, σ) be a flow-admissible signed graph; without loss of generality, we assume that it is connected and that σ is a minimum signature. Let X be the set of negative edges of (G, σ) . If $X = \emptyset$, the result follows by Theorem 1.2. Furthermore, we may assume that $|X| \geq 2$ for if $|X| = 1$, then (G, σ) is not flow-admissible.

Let B be the set of such edges b of (G, σ) that $G - b$ has two components, each of them being unbalanced. Note that since (G, σ) is flow-admissible, B is the set of all bridges of G . Moreover, as σ is minimum, by Lemma 3.1, we have $B \cap X = \emptyset$. Let S be the set of such edges s of (G, σ) that there exists a 2-edge-cut $\{s, t\}$, where $t \in X$. Note that by Lemma 3.1 and the minimality of σ , $S \cap X = \emptyset$. Furthermore, S is exactly the set of bridges in $G - X - B$. This implies the following claim.

Claim 1. *$G - X - B - S$ is a bridgeless balanced graph.* \square

By Claim 1 and Theorem 1.2, $G - X - B - S$ has a signed circuit cover of length at most $5/3 \cdot |E(G - X - B - S)|$. We will construct a collection of signed circuits \mathcal{C} that covers the edges of $X \cup B \cup S$ with total length at most $2 \cdot |E(G)|$. This will prove Theorem 1.3.

Let T be a spanning tree of the connected graph $G - X$, and we denote $T \cup X$ by (G', σ') , where σ' is the restriction of σ to G' . Note that σ' may not be a minimum signature of (G', σ') . Let X', B', S' be defined on (G', σ') in the same way as X, B, S on (G, σ) , respectively.

Claim 2. *$X' = X$, $B' \supseteq B$ and $S' \supseteq S$.* \square

This straightforward claim guarantees that it is enough to find a collection of signed circuits of (G', σ') that covers $X' \cup B' \cup S'$. To finish the proof of Theorem 1.3, we will prove the following lemma.

Lemma 3.2. *Let (G', σ') be a signed graph such that $G' - X'$ is a spanning tree of G' , where X' denotes the set of negative edges of G' . Let B' be the set of bridges of G' which separate two unbalanced subgraphs, and let S' be the set of such edges of (G', σ') that there exists a 2-edge-cut $\{s', t'\}$, where $t' \in X'$. If $|X'| \geq 2$, then there exists a collection of signed circuits \mathcal{C}' of (G', σ') that covers the edges of $X' \cup B' \cup S'$ with total length at most $2 \cdot |E(G')|$. Moreover, \mathcal{C}' covers every negative loop of G' exactly twice.*

Proof. For the sake of a contradiction, suppose that (G', σ') is a counterexample with the minimum number of edges. Clearly, (G', σ') has no balanced loops. Also (G', σ') has no vertex of degree 1 as the incident bridge does not belong to $X' \cup B' \cup S'$.

We prove that G' is 2-connected. Suppose it is not. Let G'_1 be an end-block of G' incident with a cut-vertex v , and let G'_2 be obtained by removing all vertices of G'_1 except v . Each of these graphs inherits a signature from G' that will be omitted from the notation. Since G' contains no vertex of degree 1, both $G'_1 - T$ and $G'_2 - T$ are non-empty and thus G'_1 and G'_2 are unbalanced. For $i \in \{1, 2\}$, we add a negative loop e_i to G'_i at v and denote the resulting graph by G''_i . The signed graphs G''_1 and G''_2 satisfy the conditions of the lemma, because they have fewer edges than G' and each of them has at least two negative edges. Note that e_1 and e_2 must be covered by barbells, and that each of them is covered exactly twice. To obtain the sought collection of signed circuits for (G', σ') , we combine the found signed circuit covers of G''_1 and G''_2 by merging the barbells containing loops e_1 and e_2 . Taking into account that the e_1 and e_2 are covered exactly twice, we can bound the length of the cover of (G', σ') by

$$2 \cdot |E(G''_1)| - 2 + 2 \cdot |E(G''_2)| - 2 = 2 \cdot |E(G'_1)| + 2 \cdot |E(G'_2)| = 2 \cdot |E(G')|,$$

a contradiction.

Thus, G' is 2-connected as claimed. This means that G' is either a vertex with loops incident with it, or an edge with loops incident to its end-vertices, or a bridgeless graph with at least two vertices. In the first two cases, the required collections are easy to find directly. It follows that G' is bridgeless (with $|V(G')| \geq 2$) and hence $B' = \emptyset$, so it suffices to cover the edges of $X' \cup S'$.

For $x \in X'$, let C_x be the unique circuit of $T \cup x$. For $A \subseteq X'$, let C_A be the symmetric difference of all circuits C_a for $a \in A$. Given $x \in X'$, let S'_x be the set of such edges s of S' that belong to a 2-edge-cut $\{s, x\}$ of (G', σ') . For $A \subseteq X'$, let S'_A be the union of S'_a over all $a \in A$. Since X' contains no 2-edge-cut, $S'_x \cap S'_y = \emptyset$ for $x \neq y$.

Claim 3. *For $A \subseteq X'$, C_A contains every edge of $A \cup S'_A$.* □

Since the intersection of two paths in a tree is either a path (possibly trivial) or the empty graph, the following holds.

Claim 4. *If $|A| = 2$, then C_A is either a balanced circuit, or a short barbell, or a union of two vertex-disjoint unbalanced circuits.* \square

We will distinguish three cases based on the number of loops of G' .

Case A: *The graph G' contains at least two loops.*

Consider $C_{X'}$ and note that it contains all loops of G' as well as $X' \cup S'$ according to Claim 3. We connect $C_{X'}$ into a tree of Eulerian graphs H . If H contains an even leaf balloon F , then by Corollary 2.6, F can be covered by a signed circuit cover of total length at most $4/3 \cdot |E(F)|$. Thus we may consider the signed graph $H - E(F)$, which has fewer edges than H (and at least two negative loops). We repeat this process until we obtain a signed graph H^* that does not contain any even leaf balloon. By Lemma 2.7, H^* admits a signed circuit cover \mathcal{C} of length at most $2 \cdot |E(H^*)|$ such that every loop of H^* is covered exactly twice. The combination of this cover with the ones found for even leaf balloons provides a signed circuit cover of G' that satisfies the lemma, which is a contradiction with the choice of G' .

Case B: *The graph G' contains exactly one loop.*

Let $\ell \in X'$ be the unique loop of G' . Suppose first that $|X'|$ is even. Connecting $C_{X'}$ into a tree of Eulerian graphs H and applying Lemma 2.5, we find a collection of signed circuits of total length at most $2 \cdot |E(H)|$ covering $C_{X'}$ such that ℓ is covered twice. By Claim 3, this cover of $X' \cup S'$ provides a counterexample.

Thus, $|X'|$ is odd. Let a and b be distinct negative edges different from ℓ . Consider $C_{X' - \{a\}}$, which has an even number of negative edges, and connect it into a tree of Eulerian graphs H . By Corollary 2.6, H admits a collection of signed circuits \mathcal{C} of total length at most $4/3 \cdot |E(H)|$ that covers the edges of $C_{X' - \{a\}}$ and has width at most 2. We now augment \mathcal{C} by another signed circuit C_2 to cover a and cover ℓ for a second time if necessary.

If \mathcal{C} covers ℓ once, then we find C_2 by connecting $C_{\{\ell, a\}}$ into a barbell. If \mathcal{C} covers ℓ twice, then we apply Claim 4 and define C_2 either as $C_{\{a, b\}}$ (if it is a balanced circuit or a short barbell), or as a barbell obtained by joining the components of $C_{\{a, b\}}$ (if it is a disjoint union of unbalanced circuits). In either case, the length of C_2 is at most $|E(G')|$, and adding C_2 to the collection \mathcal{C} , we obtain a signed circuit cover of $X' \cup S'$ of total length at most $(2 + 1/3) \cdot |E(G')|$ covering ℓ twice, a contradiction (if we consider the statement of Lemma 3.2 where $2 \cdot |E(G')|$ is replaced by $(2 + 1/3) \cdot |E(G')|$).

In Section 4.2, we refine the analysis as follows. We show how to choose the edges a and b , and one of the three signed circuit covers of $C_{X' - \{a\}}$ given by Lemma 2.5, in such a way that total length of the cover is at most $2 \cdot |E(G')|$. The only case when this is not possible is when G' is homeomorphic to a certain graph. In the latter case, it is easy to find the necessary cover directly.

Case C: *The graph G' contains no loops.*

Suppose first that $|X'|$ is even. Connecting $C_{X'}$ into a tree of Eulerian graphs H and using Corollary 2.6, we find a collection of signed circuits of total length at most $4/3 \cdot |E(H)|$ covering $C_{X'}$, which covers $X' \cup S'$ by Claim 3 and leads to a contradiction.

Thus, $|X'|$ is odd. We apply an argument similar to the one used in Case B. Let a and b be distinct negative edges. Connect $C_{X' - \{a\}}$ into a tree of Eulerian graphs H and note that $\varepsilon'(H)$ is even. By Corollary 2.6, H admits a collection of signed circuits \mathcal{C} of total length at most $4/3 \cdot |E(H)|$ covering the edges of $C_{X' - \{a\}}$. Using Claim 4, we find a signed circuit C_2 containing $C_{\{a,b\}}$ (of length at most $|E(G')|$), and we add it to \mathcal{C} . The resulting collection $\mathcal{C} \cup \{C_2\}$ covers $X' \cup S'$ and its length is at most $(2 + 1/3) \cdot |E(G')|$, which is a contradiction (if we consider the statement of Lemma 3.2 where $2 \cdot |E(G')|$ is replaced by $(2 + 1/3) \cdot |E(G')|$).

In Section 4.1, we refine the analysis as follows. We show how to choose the edges a and b in such a way that the length of $C_{\{a,b\}}$ is at most $2/3 \cdot |E(G')|$. We characterise the case when this is not possible, and show how to find the cover explicitly in this case. \square

4 Finishing Cases B and C

In this section, we finish the proof of Lemma 3.2, retaining the notation introduced in Section 3. We will first introduce some further notation and a lemma that will be used to finish the arguments of Case B and Case C. Recall that (G', σ') is a minimal counterexample to Lemma 3.2 and has been found to be 2-connected and bridgeless with at least two vertices, and to contain at most one loop. Furthermore, X' is the set of negative edges of G' , and $G' - X'$ is a spanning tree T of G' .

For $A \subseteq X'$, let D_A be the graph obtained as a union of C_a over all $a \in A$. Note that $C_x = D_{\{x\}}$ for any $x \in X'$. Due to the construction of G' , $D_A - A$ is a forest and has the same number of components as D_A .

In the following lemma we will show how to extend a 2-connected graph D_A by adding a suitable edge x to A .

Lemma 4.1. *Let $A \subseteq X'$ be such that D_A is 2-connected. If $X' - A$ contains no loop, then either*

- (i) $A = X'$ and $D_A = G'$, or
- (ii) *there exists $x \in X' - A$ such that $D_A \cap C_x$ is a non-trivial path and hence $D_{A \cup \{x\}}$ is 2-connected, or*
- (iii) A contains only loops.

Proof. Since D_A is connected, $D_A - A$ is a subtree of T . If D_A contains a single vertex, then A contains only loops and (iii) holds. Therefore, D_A contains at least two vertices. If $D_A = G'$, then (i) holds. Assume thus that there is an edge e in $E(G') - E(D_A)$. If every edge of $E(G') - E(D_A)$ belongs to X' , then $T \subseteq D_A$, and thus (ii) holds. Hence we may assume that $e \in T$. Since G' is connected, we may further choose e whose end-vertex belongs to $V(D_A)$. We may assume that exactly one end-vertex of e (say v) belongs to D_A , since otherwise $e \in X'$. Consider a component K of $T - D_A$ that contains e . Since G' is 2-connected, there is an edge x connecting a vertex from $K - \{v\}$ to a vertex from $V(G') - V(K)$. Observe that $x \in X'$, because T is a tree. It is easy to see that $D_A \cap C_x$ is a non-trivial path. \square

In proofs of both Claim B and Claim C, we will need to define certain barbells. The unbalanced circuits of the barbells can be defined easily using elements of X' , since C_x is a uniquely defined elementary circuit of G' for any $x \in X'$. If C_x and C_y are disjoint, then we define the barbell $B_{x,y}^Y$, where $Y \subseteq X'$, as follows. We contract C_x and C_y into two new vertices v_x and v_y . Positive edges of the graph still form a tree. Consider the path $P_{x,y}$ consisting of edges of T that connects v_x and v_y . If the symmetric difference of $P_{x,y}$ and all elementary circuits containing edges from Y is a path, denoted by $P_{x,y}^Y$, then we set $B_{x,y}^Y = P_{x,y}^Y \cup C_x \cup C_y$.

4.1 Case C

Recall that the graph G' has odd number of negative edges (and more than one negative edge) and no loops. We want to find edges $a, b \in X$ such that $C_{\{a,b\}}$ can be covered by a signed circuit of length at most $2/3 \cdot |E(G')|$. If this is not possible, we want to find a cover of G' that satisfies Lemma 3.2.

Lemma 4.2. *Suppose that there are $x, y, z \in X'$ such that $C_x \cap C_y \neq \emptyset$, $C_x \cap C_z \neq \emptyset$, and $C_y \cap C_z \neq \emptyset$. Then there are two edges $a, b \in X'$ such that $C_{\{a,b\}}$ is either a balanced circuit or a short barbell of length at most $2/3 \cdot |E(G')|$.*

Proof. By Claim 4, the subgraphs $C_{\{x,y\}}$, $C_{\{x,z\}}$, and $C_{\{y,z\}}$ are balanced circuits or short barbells. Each edge e of $D_{\{x,y,z\}}$ belongs to one, two or three circuits among C_x , C_y , and C_z . If e belongs to exactly one of the circuits, say C_x , then $e \notin C_{\{y,z\}}$. If e belongs to two of the circuits, say C_x and C_y (and possibly C_z), then $e \notin C_{\{x,y\}}$. Thus $C_{\{x,y\}} \cap C_{\{x,z\}} \cap C_{\{y,z\}}$ has no edge, and $|E(C_{\{x,y\}})| + |E(C_{\{x,z\}})| + |E(C_{\{y,z\}})| \leq 2 \cdot |E(G')|$. The lemma follows. \square

Let $x \in X'$. Since $|X'| \geq 3$, by Lemma 4.1 there is $y \in X$ such that $C_x \cap C_y$ is a non-trivial path. Again by Lemma 4.1, there exists z such that $C_z \cap D_{\{x,y\}}$ is a non-trivial path. Due to Lemma 4.2 we may assume, without loss of generality, that $C_x \cap C_z = \emptyset$. An example of the graph $D_{\{x,y,z\}}$ is depicted in Fig. 1.

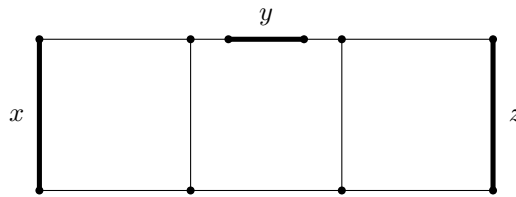


Figure 1: Possible configuration of graph $D_{\{x,y,z\}}$. As in the subsequent figures, the edges of T are represented by thin lines and the edges of X' are shown by thick lines.

If $G' = D_{\{x,y,z\}}$, then we can cover the edges of G' by two barbells $B_{x,z}^\emptyset$ and $B_{x,z}^{\{y\}}$. This cover has the property stated in Lemma 3.2. Therefore, we may assume that $G' \neq D_{\{x,y,z\}}$. By Lemma 4.1 there is an edge $w \in X' - \{x, y, z\}$ such that $C_w \cap D_{\{x,y,z\}}$ is a non-trivial path. If some three of the circuits C_x , C_y , C_z , C_w have pairwise non-empty intersection, then we can apply Lemma 4.2.

First, let us assume that one of the circuits C_x, C_y, C_z, C_w , say C_y , intersects all the other circuits. Due to Lemma 4.2, $C_w \cap C_x = \emptyset$ and $C_w \cap C_z = \emptyset$. Therefore, either $C_w \cap C_y \subseteq B_{xz}^\emptyset$ or $C_w \cap C_y \subseteq B_{xz}^{\{y\}}$. We deal with the first case, which is depicted in Fig. 2. The latter case is similar.

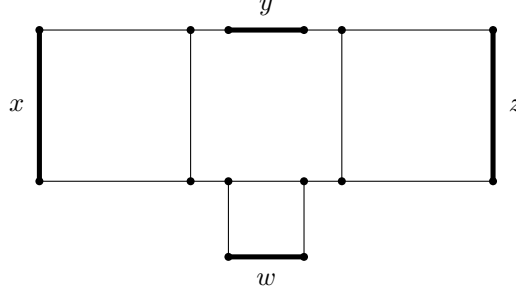


Figure 2: Possible configuration of $D_{\{x,y,z,w\}}$ in case C_y intersects all other circuits.

The barbells $B_{x,z}^{\{y\}}, B_{x,w}^\emptyset, B_{z,w}^\emptyset$ do not cover any edge of G' more than twice in total, therefore one of them has length at most $2/3 \cdot |E(G')|$.

It remains to consider that none of the circuits C_x, C_y, C_z, C_w intersects all the other circuits. Without loss of generality suppose that $C_x \cap C_y \neq \emptyset, C_y \cap C_z \neq \emptyset$, and $C_z \cap C_w \neq \emptyset$. An example of the graph $D_{\{x,y,z,w\}}$ is depicted in Fig. 4.

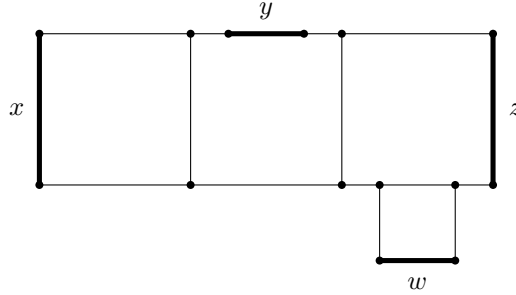


Figure 3: Possible configuration of $D_{\{x,y,z,w\}}$ in case that none of C_x, C_y, C_z and C_w intersects all other circuits.

Note that $C_{\{x,y\}}, C_{\{y,z\}},$ and $C_{\{z,w\}}$ are balanced circuits and no edge is contained in all three of them. Therefore, one of them has length at most $2/3 \cdot |E(G')|$, which gives the desired choice of a and b .

4.2 Case B

Recall that in Case B, the graph G' is 2-connected, $\varepsilon(G') \geq 3$ is odd, and G' contains one (unbalanced) loop ℓ . A *solution* \mathcal{S} in G' is a quadruple (a, b, C_1, C_2) such that $a, b \in X', C_1$ is a signed circuit that contains $C_{\{\ell,a\}}$, and C_2 is a signed circuit that contains $C_{\{a,b\}}$. The *cover corresponding to the solution* (a, b, C_1, C_2) is the collection $\{C_1, C_1, C_2\}$.

Lemma 4.3. *There are no k solutions in G' such that the union of covers corresponding to these k solutions has width at most $2k$.*

Proof. For the sake of a contradiction, assume that the k solutions exist. Let $\mathcal{S} = (a, b, C_1, C_2)$ be one of them. Consider $C_{X' - \{a\}}$, which has even number of negative edges, and connect it into a tree of Eulerian graphs H . By Lemma 2.5, there are three weak signed circuit covers $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of H that cover edges of $C_{X' - \{a\}}$ with total width at most 4. Moreover, \mathcal{C}_1 covers the loop ℓ exactly twice, and each of \mathcal{C}_2 and \mathcal{C}_3 covers ℓ exactly once. We define three collections of signed circuits as follows: $\mathcal{C}_1 \cup \{C_2\}$, $\mathcal{C}_2 \cup \{C_1\}$ and $\mathcal{C}_3 \cup \{C_1\}$.

We follow this procedure for all k solutions to obtain $3k$ collections that cover edges of $X' \cup B' \cup S'$. The total width of these $3k$ collections is obtained as the sum of total width of covers corresponding to the k solutions, which is at most $2k$ by the assumption, and the total width of $3k$ covers obtained from k uses of Lemma 2.5, which is at most $4k$; hence the total width of the $3k$ collections thus defined is at most $6k$. We conclude that one of the collections must have length at most $2 \cdot |E(G')|$. Since each of the collections covers the loop ℓ exactly twice, we obtained a contradiction with the assumption that G' is a counterexample to Lemma 3.2. \square

Lemma 4.4. *There are no two edges $x, y \in X'$ such that $C_x \cap C_y \cap C_\ell \neq \emptyset$.*

Proof. Suppose that edges $x, y \in X'$ have the stated property. We construct two solutions of width 4:

$$\begin{aligned} \mathcal{S}_1 &= (x, y, C_{\{\ell, x\}}, C_{\{x, y\}}), \\ \mathcal{S}_2 &= (y, x, C_{\{\ell, y\}}, C_{\{y, x\}}). \end{aligned}$$

These solutions provide a contradiction with Lemma 4.3. \square

Choose x such that the graph $D_{\{\ell, x\}}$ is connected and note that $D_{\{\ell, x\}}$ is 2-connected. Since $|X'| \geq 3$, by Lemma 4.1, there exists an edge $y \in X'$ such that $D_{\{\ell, x, y\}}$ is 2-connected. By Lemma 4.4, $C_\ell \cap C_x \cap C_y = \emptyset$. An example showing $D_{\{\ell, x, y\}}$ is given in Fig. 4.

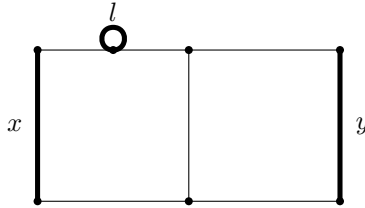


Figure 4: Possible configuration of $D_{\{\ell, x, y\}}$.

If $G' = D_{\{\ell, x, y\}}$, then a collection of signed circuits required by Lemma 3.2 that provides a contradiction is $\{C_{\{\ell, x\}}, B_{\ell, y}^\emptyset\}$. Otherwise, by Lemma 4.1, there exists an edge $z \in X'$ such that $C_z \cap D_{\{\ell, x, y\}}$ is a non-trivial path P . We will discuss the possible positions of P with respect to the segments of $D_{\{\ell, x, y\}}$.

By Lemma 4.4, $C_z \cap C_x \cap C_\ell = \emptyset$. Suppose first that P intersects $C_x - C_y$. Then there exist two edge-disjoint paths P_1 and P_2 in $D_{\{\ell, x, y, z\}} - C_y - C_z$, P_1 connecting ℓ to C_y and P_2 connecting ℓ to C_z . Then the following collections of solutions satisfy Lemma 4.3, which is a contradiction. Define C^* to be either a barbell containing $C_{\{y, z\}}$ and not containing $V(\ell)$ (for $E(C_y) \cap E(C_z) = \emptyset$) or a balanced circuit $C_{\{y, z\}}$ (otherwise). We set $S_1 = (y, z, C_{\{\ell, y\}} \cup P_1, C^*)$ and $S_2 = (z, y, C_{\{\ell, z\}} \cup P_2, C^*)$. Note that when $C_{\{y, z\}}$ is a circuit, the edges of $C_y \cap C_z$ are covered by both barbells but not by C^* .

Suppose finally that P does not intersect $C_x - C_y$. Then $P \subseteq C_y$ and there are three possible subcases:

- (i) $P \subseteq C_y - C_x$ and $B_{l, z}^\emptyset$ does not contain any edge of $C_x \cap C_y$,
- (ii) $P \subseteq C_y - C_x$ and $B_{l, z}^\emptyset$ does contain an edge of $C_x \cap C_y$,
- (iii) $P \subseteq C_x \cap C_y$.

Examples for these subcases are depicted in Fig. 5.

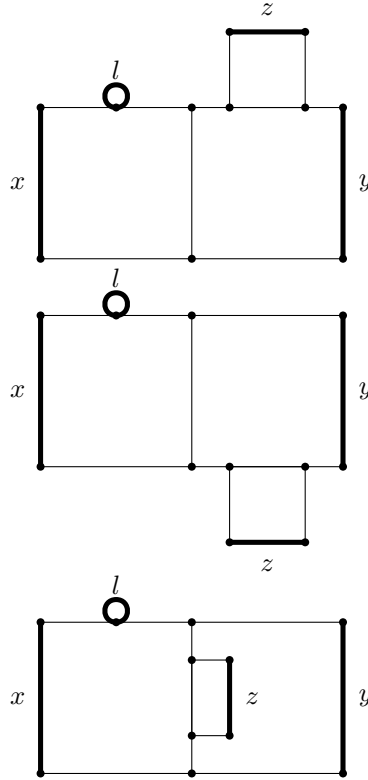


Figure 5: Possible configurations of $D_{\{\ell, x, y, z\}}$.

For each of the subcases, there are three solutions such that the corresponding covers have total width at most 6. These solutions are given in Table 1. This contradiction to Lemma 4.3 finishes the discussion of Case B and hence also the proof of Lemma 3.2. \square

case	solutions
(i)	$(x, y, C_{\{\ell,x\}}, C_{\{x,y\}}), (y, z, B_{\ell,y}^\emptyset, C_{\{y,z\}}), (z, x, B_{\ell,z}^{\{x,y\}}, B_{x,z}^\emptyset)$
(ii)	$(x, y, C_{\{\ell,x\}}, C_{\{x,y\}}), (y, z, B_{\ell,y}^\emptyset, C_{\{y,z\}}), (z, x, B_{\ell,z}^{\{x\}}, B_{x,z}^{\{y\}})$
(iii)	$(x, y, C_{\{\ell,x\}}, C_{\{x,y\}}), (z, x, B_{\ell,z}^\emptyset, C_{\{z,x\}}), (z, y, B_{\ell,z}^{\{x\}}, C_{\{z,y\}})$

Table 1: Solutions for subcases (i)–(iii) when P does not intersect $C_x - C_y$.

5 Alternative bounds

In this section we provide several alternative bounds on the length of a shortest signed circuit cover of a flow-admissible signed graph. We compare these bounds to results of Cheng et al. [4]. We will follow notation from Section 3.

The graph $G - X - B - S$ has at most $|E(G)| - |X|$ edges. The graph G' has at most $|V(G)| - 1 + |X|$ edges. By Theorem 1.2, we can cover $G - X - B - S$ with a circuit cover of length at most $5/3 \cdot (|E(G)| - |X|)$ and by Lemma 3.2 we can cover the edges $X \cup B \cup S$ in G' with signed circuits of total length at most $2 \cdot |V(G)| + 2|X| - 2$. Altogether, the constructed signed circuit cover has length at most $5/3 \cdot |E(G)| + 2|V(G)| + 1/3 \cdot |X| - 2$, while the bound obtained by Cheng et al. [4] is $5/3 \cdot |E(G)| + 3|V(G)| + 4/3 \cdot |X| - 7$. Due to minimality of X and connectivity of G we have $|E| \geq |X| + |V| - 1$. Using this inequality we obtain the bound $11/3 \cdot |E(G)| - 5/3 \cdot |X|$ from Theorem 1.3.

For dense bridgeless graphs a result of Fan [6] provides a better bound on shortest circuit covers of bridgeless graphs than Theorem 1.2. Fan showed that a bridgeless graph on n vertices and m edges can be covered with circuits of total length at most $n + m - 1$. This implies that $G - X - B - S$ can be covered by circuits of total length at most $|E(G)| - |X| + |V(G)| - 1$. Together with the cover of $X \cup B \cup S$ in G' we obtain a signed circuit cover of G of length at most $|E(G)| + 3 \cdot |V(G)| + |X| - 3$, while the bound obtained by Cheng et al. [4] is $|E(G)| + 4 \cdot |V(G)| + 2 \cdot |X| - 8$.

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