Online Ramsey theory for a triangle on F-free graphs

Hojin Choi[*¶], Ilkyoo Choi[†§], Jisu Jeong[*¶], and Sang-il Oum[*‖¶]

[§]Department of Mathematics, Hankuk University of Foreign Studies, Yongin-si, Gyeonggi-do, Republic of Korea
[‖]Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, Republic of Korea
[¶]Department of Mathematical Sciences, KAIST, Daejeon, Republic of Korea

January 11, 2019

Abstract

Given a class $\mathcal{C}$ of graphs and a fixed graph H, the *online Ramsey game for H on $\mathcal{C}$* is a game between two players Builder and Painter as follows: an unbounded set of vertices is given as an initial state, and on each turn Builder introduces a new edge with the constraint that the resulting graph must be in $\mathcal{C}$, and Painter colors the new edge either red or blue. Builder wins the game if Painter is forced to make a monochromatic copy of H at some point in the game. Otherwise, Painter can avoid creating a monochromatic copy of H forever, and we say Painter wins the game.

We initiate the study of characterizing the graphs F such that for a given graph H, Painter wins the online Ramsey game for H on F-free graphs. We characterize all graphs F such that Painter wins the online Ramsey game for C_3 on the class of F-free graphs, except when F is one particular graph. We also show that Painter wins the online Ramsey game for C_3 on the class of K_4-minor-free graphs, extending a result by Grytczuk, Hałuszczak, and Kierstead.

1 Introduction

All graphs in this paper are finite. For a host graph G and a target graph H, let $G \to H$ mean that there exists a monochromatic copy of H for every (not necessarily proper) 2-edge-coloring of G. For a graph parameter ρ, let $R_\rho(H)$ denote the minimum $\rho(G)$ where $G \to H$.

[*]Supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. NRF-2017R1A2B4005020).

[†]Corresponding author. Supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B07043049), and also by Hankuk University of Foreign Studies Research Fund.

[‡]Emails: `hojinchoi@kaist.ac.kr`, `ilkyoo@hufs.ac.kr`, `jjisu@kaist.ac.kr`, `sangil@kaist.edu`

When ρ counts the number of vertices in a graph, $R_\rho(H)$ is *the Ramsey number* of H and it is often denoted $R(H)$. The well-known Ramsey's Theorem [22] from 1930 states that $R(H)$ is finite for every graph H.

Burr, Erdős, and Lovász [3] introduced *the chromatic Ramsey number* and *the degree Ramsey number*, which arises when ρ is the chromatic number and the maximum degree, respectively. Erdős et al. [9] introduced *the size Ramsey number*, denoted $R_e(H)$, which arises when $e(G)$ is the number of edges in a graph G. We redirect the readers to a thorough survey by Conlon, Fox, and Sudakov [6] for more history regarding these parameters.

Another variant of Ramsey theory is online Ramsey theory, introduced by Beck [2] in 1993. Given a class $\mathcal{C}$ of graphs and a fixed graph H, an *online Ramsey game for H on $\mathcal{C}$* is a game between two players Builder and Painter with the following rules: an unbounded set of vertices is given as an initial state, and on each turn Builder introduces a new edge with the constraint that the resulting graph must be in $\mathcal{C}$, and Painter colors the new edge either red or blue. Builder wins if Painter is forced to make a monochromatic copy of H at some point of the game, and we say Builder wins the online Ramsey game for H on $\mathcal{C}$. Otherwise, Painter can avoid creating a monochromatic copy of H forever, and we say Painter wins the online Ramsey game for H on $\mathcal{C}$.

If no graph in $\mathcal{C}$ contains H as a subgraph, then Painter wins the online Ramsey game for H on $\mathcal{C}$ since a copy of H cannot be created, let alone a monochromatic one. Therefore it must be that H is a subgraph of at least one graph in $\mathcal{C}$ for a result to be nontrivial. If $\mathcal{C}$ is the class of graphs with bounded maximum degree, then this is the online version of the degree Ramsey number; see [4, 23, 24] for results regarding this topic.

This paper concerns the online version of the size Ramsey number. For a graph H, *the online (size) Ramsey number* of H, denoted $r(H)$, is the minimum number of rounds required for Builder to win, assuming that both Builder and Painter play optimally. When there are no restrictions on the graphs Builder can create, it is an easy consequence of Ramsey's theorem [22] that Builder always wins the online Ramsey game for every target graph H, so $r(H) \leq R_e(H)$. For a fixed graph H, studying the ratio of $r(H)$ and $R_e(H)$ was initiated in [2, 10, 14] and has drawn much attention since then [11, 12, 13, 20]. There is also a line of research trying to determine some exact online Ramsey numbers [5, 7, 8, 12, 20, 21]. Additionally, there are some results on the behavior of $r(H)$ in various random settings [1, 15, 16, 17, 19].

The investigation of online (size) Ramsey theory on specific graph classes was initiated in 2004 by Grytczuk, Hałuszczak, and Kierstead [11]. They studied online Ramsey theory on forests, k-colorable graphs, outerplanar graphs, and planar graphs. In particular, they conjectured that Builder wins the online Ramsey game for H on planar graphs if and only if H is an outerplanar graph. This conjecture was recently disproved by Petříčková [18]; she showed that one direction of the conjecture is true while the other direction is not.

Proposition 1.1 ([18])**.** *For every outerplanar graph H, Builder wins the online Ramsey game for H on planar graphs.*

Proposition 1.2 ([18])**.** *Builder wins the online Ramsey game for $K_{2,3}$ on planar graphs.*

In [11], it is shown that Painter wins the online Ramsey game for C_3 on outerplanar graphs, and the graphs containing C_3 as a subgraph are the only known graphs where Painter wins the online Ramsey game on outerplanar graphs. On the other hand, they also demonstrate that Builder wins the online Ramsey game for C_3 on 2-degenerate planar graphs.

Theorem 1.3 ([11]). *Painter wins the online Ramsey game for C_3 on outerplanar graphs.*

Proposition 1.4 ([11]). *Builder wins the online Ramsey game for C_3 on 2-degenerate planar graphs.*

We extend the class of graphs where Painter wins the online Ramsey game for C_3 from outerplanar graphs to K_4-minor-free graphs. Our proof is a generalization of the proof of Theorem 1.3 in [11].

Theorem 2.4. *Painter wins the online Ramsey game for C_3 on K_4-minor-free graphs.*

We initiate the study of characterizing the graphs F such that for a given graph H, Painter wins the online Ramsey game for H on F-free graphs. A graph class is *F-free* if every graph in the class does not contain F as a subgraph. We characterize all graphs F such that Painter wins the online Ramsey game for C_3 on F-free graphs, except when F is one special graph. We put the constraint that F has no isolated vertices because the game is defined to have infinitely many isolated vertices as the initial state. The following theorem is our main result.

Theorem 3.1. *Let $X_1, \ldots, X_5$ be the graphs in Figure 1, and let F be a graph with no isolated vertices. Given that F is not isomorphic to X_5, Painter wins the online Ramsey game for C_3 on F-free graphs if and only if F is isomorphic to a subgraph of a graph in $\{X_1, X_2, X_3, X_4\}$.*

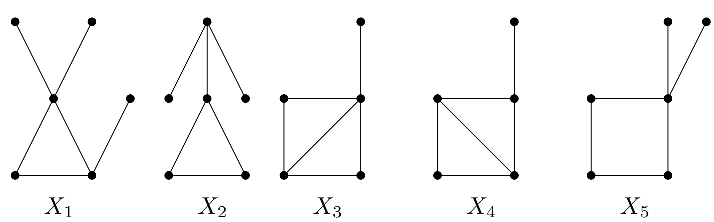

Figure 1: The graphs X_1, X_2, X_3, X_4, X_5.

This paper is organized as follows. In Section 2, we prove Theorem 2.4 and in Section 3, we prove Theorem 3.1. Section 3 is further divided into three subsections. Subsection 3.1 and Subsection 3.2 deals with the classes of graphs where Builder and Painter wins, respectively. Subsection 3.3 concludes Section 3.

For an edge e, we say that "Painter cannot color e" if there is a monochromatic copy of H whether Painter colors e red or blue; in other words, Builder wins the game no matter what color Painter uses on e. In particular, we say that "Painter cannot color e red (blue)" or "Painter must color e blue (red)", if we already observed that Painter will eventually lose (a monochromatic copy of H will appear) when Painter colors e red (blue).

2 The online Ramsey game for C_3 on K_4-minor-free graphs

Grytczuk, Hałuszczak, and Kierstead [11] proved that Builder wins the online Ramsey game for C_3 on 2-degenerate planar graphs, but Painter wins the online Ramsey game for C_3 on outerplanar graphs. We extend the class the graphs on which Painter is known to win the online Ramsey game for C_3. Since a graph is outerplanar if and only if it does not contain $K_{2,3}$ and K_4 as a minor, we focus on $K_{2,3}$-minor-free graphs and K_4-minor-free graphs. We show that Painter wins the online Ramsey game for C_3 on K_4-minor-free graphs, but Builder still wins the online Ramsey game for C_3 on $K_{2,3}$-minor-free graphs.

The following proposition shows that Builder wins the online Ramsey game for C_3 on $K_{2,3}$-minor-free graphs. Builder will use Strategy 2.1.

Strategy 2.1. *Builder draws a copy of $K_{1,5}$. Let u be the vertex of degree 5. By the pigeonhole principle, Painter will color at least three edges with the same color, say uv_1, uv_2, uv_3. Builder draws the edges v_1v_2, v_2v_3, and v_3v_1.*

Proposition 2.2. *Builder wins the online Ramsey game for C_3 on $K_{2,3}$-minor-free graphs.*

Proof. Builder uses Strategy 2.1. Assume uv_1, uv_2, uv_3 are colored red. If Painter colors one of v_1v_2, v_2v_3, v_3v_1 red, then this creates a red C_3. Therefore Painter must color all of v_1v_2, v_2v_3, v_3v_1 blue, but then this creates a blue C_3 with vertices v_1, v_2, and v_3.

The graph resulting from Strategy 2.1 has no $K_{2,3}$ as a minor. Thus Builder wins the online Ramsey game for C_3 on $K_{2,3}$-minor-free graphs. □

Now, we will prove that Painter wins the online Ramsey game for C_3 on K_4-minor-free graphs. The key idea of this proof stemmed from the proof of Theorem 1.3 in [11].

Recall that a graph G contains H as a minor if there exists a set $\mathcal{S}$ of pairwise disjoint subsets of $V(G)$ satisfying the following:

- For every vertex u of H, there is an element $S_u \in \mathcal{S}$ such that $G[S_u]$ is connected.
- For every edge uv of H, there is an edge between S_u and S_v.

We call S_u the *branch set* of u in an H-minor of G for every vertex u of H. When the branch set has one vertex, we also call it a *branch vertex*. For two vertices x, y in G, an *x, y-path* is a path in G from x to y.

Lemma 2.3. *Let xy be an edge of a K_4-minor-free graph G, and let P and Q be two x, y-paths in $G - xy$. For an integer $k \geq 3$, if $x = v_1, \dots, v_k = y$ are the common vertices of P and Q, then these vertices are in the same order on both P and Q.*

Proof. The claim is trivial when $k = 3$, so we may assume $k > 3$. By reordering the indices, let $v_1, \dots, v_k$ be the order of these vertices on P.

We claim that for $j > i + 1$, if there is a v_i, v_j-path R in G that is internally disjoint with P, then there is no path from $\{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$ to $V(P) \setminus \{v_i, v_{i+1}, \dots, v_j\}$ that is internally disjoint with P. Suppose not. Take an a, b-path P' where $a \in \{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$ and $b \in V(P) \setminus \{v_i, v_{i+1}, \dots, v_j\}$. If P' and R share a vertex z, then G has a K_4-minor where the branch vertices are z, v_i, v_j, a. If P' and R are vertex disjoint, then G has a K_4-minor where the branch vertices are a, b, v_i, v_j.

Thus, if R is a subpath of Q, then Q can never visit $v_{i+1}, v_{i+2}, \dots, v_{j-1}$ because otherwise Q will contain a subpath from $\{v_{i+1}, v_{i+2}, \dots, v_{j-1}\}$ to x or y. This is a problem since Q is an x, y-path and must go through all of $v_1, \dots, v_k$. Therefore, we conclude that $v_1, \dots, v_k$ are in the same order on both P and Q. □

Given two vertices u, v on a path P, let $P[u, v]$ denote the subpath of P from u to v. For a 2-edge-colored graph H, let $f(H)$ denote the number of red edges minus the number of blue edges in H modulo 3. A 2-edge-colored graph H is *zero*, *positive*, and *negative* if $f(H)$ is 0, 1, and 2, respectively. Given a 2-edge-colored graph G, a zero cycle C is *good* if there exist two vertices α, β on $V(C)$ such that an α, β-path on C is zero and there exists an α, β-path in G whose internal vertices are disjoint from $V(C)$.

Theorem 2.4. *Painter wins the online Ramsey game for C_3 on K_4-minor-free graphs.*

Proof. Assume Builder drew the edge $e = xy$ to the previous graph to obtain the current graph G, which is 2-edge-colored except for e. Since the initial graph has no edges, it suffices to show that if $G - e$ has a 2-edge-coloring such that every zero cycle is good, then this coloring can be extended to G so that every zero cycle is good. Note that if every zero cycle is good, then there cannot be a monochromatic C_3, since a monochromatic C_3 is a zero cycle and cannot have a zero path as a subgraph.

Suppose whenever Painter tries to color e red and blue in G, there arises a zero cycle C^r and C^b, respectively, that is not good. Let $P^r = C^r - e$ and $P^b = C^b - e$. Since C^r and C^b are zero cycles, P^r is negative and P^b is positive. Let $x = v_1, v_2, \dots, v_t = y$ be the common vertices of P^r and P^b. By Lemma 2.3, they are in the same order on P^r and P^b. Without loss of generality, let $v_1, \dots, v_t$ be the ordering of these vertices on P^r and P^b. Note that $P^r[v_j, v_{j+1}] = P^b[v_j, v_{j+1}]$ might happen for some $j \in \{1, \dots, t-1\}$, but there must exist an i where $P^r[v_i, v_{i+1}] \neq P^b[v_i, v_{i+1}]$, since P^r is negative and P^b is positive. Fix such an i, and note that $P^r[v_i, v_{i+1}]$ and $P^b[v_i, v_{i+1}]$ are internally disjoint.

We claim that both $P^r[v_i, v_{i+1}]$ and $P^b[v_i, v_{i+1}]$ are not zero. Without loss of generality, assume $P^r[v_i, v_{i+1}]$ was zero. Since $P^b[v_i, v_{i+1}]$ is a path from v_i to v_{i+1} whose internal vertices are disjoint from $V(C^r)$, this implies that C^r is a good cycle, which is a contradiction.

Now we claim that $P^r[v_i, v_{i+1}]$ and $P^b[v_i, v_{i+1}]$ are either both positive or both negative. Without loss of generality assume $P^r[v_i, v_{i+1}]$ is positive and $P^b[v_i, v_{i+1}]$ is negative. Since the cycle D formed by $P^r[v_i, v_{i+1}]$ and $P^b[v_i, v_{i+1}]$ is zero even before Builder drew e, we know that D is a good cycle by the induction hypothesis. Therefore, there are two vertices α, β on D where an α, β-path on D is zero and $G - e$ (also, G) has an α, β-path whose internal vertices are disjoint from $V(D)$. Note that this latter α, β-path cannot share its internal vertices with P^r and P^b since this would create a K_4-minor. If both α, β are on the same P^j for some $j \in \{r, b\}$, then because there are two zero α, β-paths (on C^j) and another internally disjoint α, β-path, we can conclude C^j is good, which is a contradiction. If α, β

are on different paths of P^r, P^b, then G contains K_4 as a minor where the branch vertices are $v_i, v_{i+1}, \alpha, \beta$, which is again a contradiction.

Now we know that $P^r[v_i, v_{i+1}]$ and $P^b[v_i, v_{i+1}]$ are both positive or both negative for every $i \in \{1, \dots, t-1\}$, which implies that P^r and P^b are both positive or both negative, which contradicts that P^r is negative and P^b is positive.

Thus, Painter can color e so that every zero cycle in G is good, and hence there is no monochromatic C_3 in the coloring Painter produces. $\square$

We remark that the proof of Theorem 2.4 works for not only K_4-minor-free graphs, but also K_4-topological-minor-free graphs.

3 The online Ramsey game for C_3 on F-free graphs

In this section, we attempt to characterize all graphs F such that Painter wins the online Ramsey game for C_3 on F-free graphs. We determine the winner of the game in all cases except when F is the graph X_5, which is in Figure 1. Recall that we put the constraint that F has no isolated vertices because the game is defined to have infinitely many isolated vertices as the initial state. Here is our main result.

Theorem 3.1. *Let $X_1, \dots, X_5$ be the graphs in Figure 1. Suppose that F is a graph with no isolated vertices that is not isomorphic to X_5. Painter wins the online Ramsey game for C_3 on F-free graphs if and only if F is isomorphic to a subgraph of a graph in $\{X_1, X_2, X_3, X_4\}$.*

3.1 When does Builder win the online Ramsey game for C_3 on F-free graphs?

In this subsection, we provide three different classes where Builder wins the online Ramsey game for C_3. We start by proving Lemma 3.2, which shows that we only need to consider F to be a subgraph of the graph X, which is in Figure 2. Then we investigate the classes of (1) K_4-free graphs, (2) $K_{1,5}$-free graphs, and (3) Y-free graphs where Y is the graph in Figure 5.

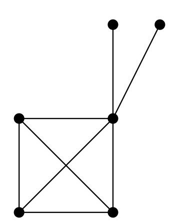

Figure 2: The graph X.

Lemma 3.2. *Let X be the graph in Figure 2. If a graph F is not isomorphic to a subgraph of X, then Builder wins the online Ramsey game for C_3 on F-free graphs.*

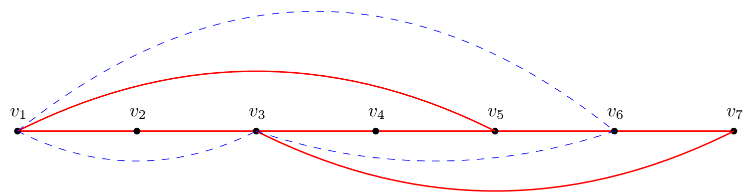

Figure 3: A strategy for Builder to win the online Ramsey game for C_3 on K_4-free graphs.

Proof. Builder uses Strategy 2.1. Assume uv_1, uv_2, uv_3 are colored red. If Painter colors one of v_1v_2, v_2v_3, v_3v_1 red, then this creates a red C_3. Therefore Painter must color all of v_1v_2, v_2v_3, v_3v_1 blue, but then this creates a blue C_3 with vertices v_1, v_2, and v_3.

There is no F as a subgraph at every step of the game since the resulting graph is X and F is not isomorphic to any of the subgraphs of X. Hence, Builder wins the online Ramsey game for C_3 on F-free graphs. $\square$

The following Proposition 3.3 is a special case of a result in [11], and a more general theorem is proved in [13]. For the sake of completeness, we include a proof of Proposition 3.3.

Proposition 3.3. *Builder wins the online Ramsey game for C_3 on K_4-free graphs.*

Proof. We will present a winning strategy for Builder.

Given a forest S, it is known that Builder wins the online Ramsey game for S on the class of all forests by [11]. Thus, we may assume that Builder has forced Painter to create a monochromatic path of length six while drawing a forest. We label the seven vertices on the path by $v_1, v_2, \ldots, v_7$ and suppose that these vertices on the path are in this order. Without loss of generality, assume the edges of the path are colored red. Note that there might be more edges incident with v_i for $i \in \{1, \ldots, 7\}$, but since the whole graph is a forest, it is K_4-free.

Next, Builder draws v_1v_5 and v_3v_7. We claim that Painter must color both v_1v_5 and v_3v_7 red. Without loss of generality assume that v_1v_5 is colored blue. Now Builder draws both v_1v_3 and v_3v_5. Painter must color v_1v_3 blue, otherwise there is a red C_3 with three vertices v_1, v_2, v_3. Now Painter cannot color v_3v_5. Therefore, both v_1v_5 and v_3v_7 must be colored red.

Finally, Builder draws three edges v_1v_3, v_3v_6, and v_6v_1. If Painter colors any of them red, then a red C_3 is created. Otherwise, Painter colors all of them blue, and this creates a blue C_3 with three vertices v_1, v_3, and v_6. See Figure 3.

Four vertices of degree at least 3 appear only in the previous paragraph. It is easy to check that K_4 does not appear as a subgraph in this case, so K_4 does not appear as a subgraph at every step of the game. Hence, Builder wins the online Ramsey game for C_3 on K_4-free graphs. $\square$

The following proposition is implied by a result in [4] (see Proposition 4.2). For completeness, we provide a proof here as well.

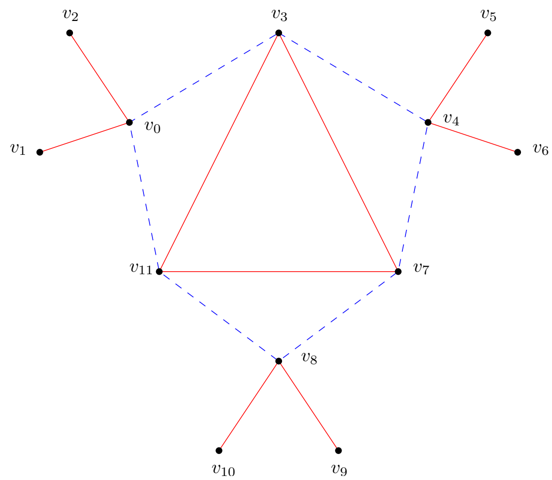

Figure 4: A strategy for Builder to win the online Ramsey game for C_3 on $K_{1,5}$-free graphs.

Proposition 3.4. *Builder wins the online Ramsey game for C_3 on $K_{1,5}$-free graphs.*

Proof. We will present a winning strategy for Builder.

Builder draws five pairwise disjoint induced copies of $K_{1,3}$. We claim that Painter must not create a monochromatic copy of $K_{1,3}$. Otherwise, without loss of generality, assume that there is a red $K_{1,3}$. Now, Builder draws K_4 containing the red $K_{1,3}$ as a subgraph. If Painter colors any of the newly drawn edges red, then a red C_3 is created. Otherwise, Painter colors all of the newly drawn edges blue, and a blue C_3 is created.

Therefore, since there is no monochromatic copy of $K_{1,3}$, we may assume that at least three of the five pairwise disjoint induced copies of $K_{1,3}$ contain two red edges and one blue edge; let these copies of $K_{1,3}$ be S_0, S_1, S_2 where $V(S_i) = \{v_{4i}, v_{4i+1}, v_{4i+2}, v_{4i+3}\}$ and $E(S_i) = \{v_{4i}v_{4i+1}, v_{4i}v_{4i+2}, v_{4i}v_{4i+3}\}$ for $i \in \{0, 1, 2\}$ while v_0v_3, v_4v_7, and v_8v_{11} are blue, and all other edges in $E(S_0) \cup E(S_1) \cup E(S_2)$ are red.

Next, Builder draws v_3v_4, v_7v_8, and $v_{11}v_0$. We claim that Painter must color all these edges blue. Suppose without loss of generality that Painter colors v_3v_4 red. Then Builder draws v_3v_5, v_5v_6, and v_6v_3. If Painter colors any of them red, then a red C_3 is created. If Painter colors all of them blue, then this creates a blue C_3 with vertices v_3, v_5, and v_6.

Therefore we may assume that Painter colors v_3v_4, v_7v_8, and $v_{11}v_0$ blue. Finally, Builder draws v_3v_7, v_7v_{11}, and $v_{11}v_3$. If Painter colors any of them blue, then a blue C_3 is created. If Painter colors all of them red, then this creates a red C_3 with vertices v_3, v_7, and v_{11}. See Figure 4.

It is easy to check that $K_{1,5}$ does not appear as a subgraph at every step of the game. Hence, Builder wins the online Ramsey game for C_3 on $K_{1,5}$-free graphs. □

Lemma 3.5. *Let Y be the graph in Figure 5. While playing the online Ramsey game for C_3 on Y-free graphs, Builder can force Painter to create either a monochromatic copy of C_3 or a blue edge xy with* $\deg(x) = 1$ *and* $\deg(y) \leq 2$.

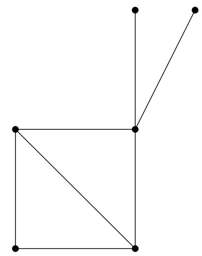

Figure 5: The graph Y.

Proof. This can be proven by letting Builder draw an edge and extend it to a path of length 4. At any moment, if Painter colors any of these edges blue, then that creates the blue edge we seek. Otherwise, we may assume Painter produced a red path of length 4. Let P be such a red path with vertices x_1, x_2, x_3, x_4, and x_5 in this order on P.

Now, Builder draws two edges x_2x_6 and x_4x_6 with a new vertex x_6. We claim that Painter must color both x_2x_6 and x_4x_6 with the color blue. Without loss of generality, suppose Painter colors x_2x_6 red. Now, Builder draws x_1x_3, x_3x_6, and x_6x_1. If Painter colors any of these edges red, then there is a red C_3. If Painter colors all of these edges blue, then this creates a blue C_3. Therefore, Painter must color x_2x_6 and x_4x_6 blue.

Finally, Builder draws x_2x_4. Whenever Painter colors x_2x_4 red or blue, this creates a monochromatic copy of C_3.

It is easy to check that Y does not appear as a subgraph at every step of the game. Hence, Builder can force Painter to create either a monochromatic copy of C_3 or a blue edge xy with $\deg(x) = 1$, $\deg(y) \leq 2$, while playing the online Ramsey game for C_3 on Y-free graphs. □

Proposition 3.6. *Let Y be the graph in Figure 5. Builder wins the online Ramsey game for C_3 on Y-free graphs.*

Proof. We will present a winning strategy for Builder.

Builder draws seven pairwise disjoint edges. By the pigeonhole principle, Painter colors at least four edges with the same color. Without loss of generality, assume v_1w_1, v_2w_2, v_3w_3, and v_4w_4 are red edges.

Next, Builder draws the four edges vv_i for $i \in \{1, 2, 3, 4\}$ with a new vertex v. We claim that Painter must color two of them red and the other two blue. Suppose Painter colors vv_1, vv_2, and vv_3 red. Now Builder draws v_1v_2, v_2v_3, and v_3v_1. If Painter colors any of them red, then a red C_3 is created. If Painter colors all of these edges blue, then this creates a blue C_3 with vertices v_1, v_2, and v_3. Therefore, we may assume that vv_1, vv_2 are red and vv_3, vv_4 are blue.

Next, Builder draws w_1w_2. Suppose Painter colors w_1w_2 blue. Now, Builder draws vw_1 and vw_2. If Painter colors any of these edges red, then a red C_3 is created. If Painter colors both vw_1 and vw_2 blue, then a blue C_3 with vertices v, w_1, and w_2 is created. Therefore we may assume that Painter colors w_1w_2 red.

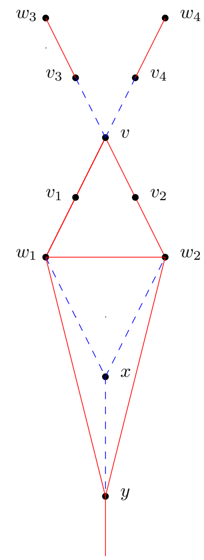

Figure 6: A strategy for Builder to win the online Ramsey game for C_3 on Y-free graphs.

Now, Builder forces Painter to create a blue edge xy with $\deg(x) = 1$ and $\deg(y) \leq 2$, which is possible by Lemma 3.5. Next, Builder draws xw_1 and xw_2. We claim that Painter must color these edges blue. Without loss of generality, suppose xw_1 is colored red. Then Builder draws two more edges xv_1 and v_1w_2. If Painter colors any of xw_2, xv_1, and v_1w_2 red, then there is a red C_3. If Painter colors all of them blue, then this creates a blue C_3 with vertices x, v_1, and w_2. Therefore, Painter must color xw_1 and xw_2 blue.

Finally, Builder draws yw_1 and yw_2. If Painter colors any of them blue, then there is a blue C_3. If Painter colors all of them red, then this creates a red C_3 with vertices y, w_1 and w_2. See Figure 6.

It is easy to check that Y never appears as a subgraph at every step of the game. Hence, Builder wins the online Ramsey game for C_3 on Y-free graphs. □

3.2 When does Painter win the online Ramsey game for C_3 on F-free graphs?

In this section, we will prove that Painter wins the online Ramsey game for C_3 on F-free graphs for various F. Recall that by Lemma 3.2, we only need to consider F to be a subgraph of the graph X, which is in Figure 2. For a fixed F, it is sufficient to provide a strategy for Painter so that a monochromatic C_3 does not appear forever on F-free graphs. We will provide three different winning strategies for Painter for three different F.

Strategy 3.7. *Painter colors each new edge red, unless doing so creates a red $K_{1,3}$, a red C_3, or a red C_4, in which case the new edge is colored blue.*

Proposition 3.8. *Let X_1 be the graph in Figure 1. Painter wins the online Ramsey game for C_3 on X_1-free graphs.*

Proof. Painter will use Strategy 3.7. We claim that Painter can always color the new edge $e = xy$ with Strategy 3.7. Let G be the new graph when Builder draws e. We will use induction on the number of edges. The base case is trivial.

By the induction hypothesis, we may assume that there is no red $K_{1,3}$, no red C_3, no red C_4, and no blue C_3 in $G-e$. The strategy fails when coloring e blue results in a blue C_3 and coloring e red results in a red $K_{1,3}$, a red C_4, or a red C_3. Let x, y, z be the vertices of the blue C_3 when e is colored blue. We will prove that if the strategy fails, then G has X_1 as a subgraph, which is a contradiction, and thus the strategy does not fail. We will divide the cases according to which red subgraph appears when Painter colors e red.

Case 1 Assume a red C_3 is created when Painter colors e red, and let w be the third vertex of this red C_3. Since Painter colored neither xz nor zy red, coloring each of xz and yz red must have created a red C_4, a red C_3, or a red $K_{1,3}$ in $G-e$. We will show that a red C_3 or a red C_4 cannot be created by coloring either xz or yz red. Without loss of generality, let us consider xz.

If coloring xz red resulted in a red C_4 with vertices x, s, t, z in cyclic order, then $t \neq y$ and $s \neq y$, since in $G-e$, the edge yz is blue and e does not exist. We also know that $t \neq w$, since otherwise $G-e$ has a red $K_{1,3}$ as a subgraph, which is a contradiction to the induction hypothesis. If $s = w$, then it must be that $t = y$ in order for $G-e$ to not have a red $K_{1,3}$, but this contradicts that $t \neq y$. This implies that $s, t \notin \{x, y, z, w\}$, which means G has X_1 as a subgraph, which is a contradiction.

If coloring xz red resulted in a red C_3 with vertices x, z, u, then $u \neq w$, since otherwise $G-e$ has a red $K_{1,3}$ as a subgraph, which contradicts the induction hypothesis. This implies that $u \notin \{x, y, z, w\}$. Now, y and z cannot have neighbors outside of $\{u, x, y, z, w\}$ since that would create a copy of X_1 in G. There is no red edge between u and w because that would create a red C_3 in $G-e$. Since either a red yu or a red zw would create a red $K_{1,3}$, neither y nor z can have more incident red edges, which means yz could have been colored red, which is a contradiction.

This boils down to the case where both xz and zy were colored blue because coloring either one red would create a red $K_{1,3}$. Since zw cannot be a red edge (creates a red $K_{1,3}$ in $G-e$) and z cannot have two neighbors outside of $\{x, y, w\}$ (creates a copy of X_1 in G), each of x and y have a neighbor x' and y', respectively, such that xx' and yy' are red. It cannot be that $x' = y'$, since this creates a red C_4 with vertices x, w, y, x' in $G-e$. If $x' \neq y'$, then this creates a copy of X_1 in G. In either case, we obtain a contradiction.

Case 2 Assume a red $K_{1,3}$ is created when Painter colors e red, and without loss of generality let x, y, u, v be the vertices of the red $K_{1,3}$ so that xy, ux, xv are red edges. Now, z and y cannot have neighbors outside of $\{x, y, z, u, v\}$ since that would create a copy of X_1. This implies that each of z and y cannot have two red edges incident to it, since that would create a red C_4, with vertices z, u, x, v. Also, uv cannot be a red edge since $G-e$ would have a red C_3, with vertices u, v, x. Since zy was not colored with red, coloring zy with red must create a red $K_{1,3}$, a red C_3, or a red C_4 in $G - e$. The only possible case is when coloring zy with red creates a red C_3, which implies that either u or v is a vertex of this red C_3, which implies the existence of a red $K_{1,3}$ in $G - e$, which is a contradiction.

Case 3 Assume a red C_4 is created when Painter colors e red, and let $xx', x'y', y'y$ be the red edges of this red C_4 other than e. Now, neither x nor y can have a neighbor outside of $\{x, y, x', y', z\}$ since this would create a copy of X_1 in G. Also, x' and y' cannot have a neighbor $v \notin \{x, x', y', y\}$ where $x'v$ and $y'v$ is red, respectively, since this would create a red copy of $K_{1,3}$ in $G - e$. Since Painter colored neither xz nor yz red, coloring each of xz and yz red must create a red $K_{1,3}$, a red C_4, or a red C_3. The only possible case is when there is a red $K_{1,3}$ centered at z when Painter colors xz or yz red. In particular, z must have two neighbors z', z'' outside of $\{x, x', y, y'\}$ where zz' and zz'' are red edges. Yet, this creates a copy of X_1, which is a contradiction.

Therefore, Strategy 3.7 works and thus Painter wins the online Ramsey game for C_3 on X_1-free graphs. □

Before starting the proof for the case of X_2-free graphs, we define some *"good"* subgraphs of a graph. We say a subgraph H of G that is isomorphic to either $K_{1,3}$ or C_4 is *good* if H is red, and there exists a subgraph I of G where H is a subgraph of I in such a way that I is isomorphic to one of the graphs in Figures 7 and 8, where the thick edges correspond to the edges of H; moreover, for $i \in \{1, \ldots, 5\}$, we say H *is good by property A_i (or B_i)* to mean that the corresponding I is isomorphic to the graph labeled A_i (or B_i) in Figures 7 and 8. We also say H is *good* if H is good because of multiple properties. For example, if H satisfies the property A_1, then H is isomorphic to $K_{1,3}$ and the vertex of degree 3 of $G[V(H)]$ has degree at least 5 in G. We say that a red subgraph H of G that is isomorphic to either $K_{1,3}$ or C_4 is *bad* if it is not good. Note that if a subgraph H is bad, then all of its edges are red.

The idea is that we want to forbid $K_{1,3}$ and C_4 in the graph as much as we can, but we allow copies of $K_{1,3}$ and C_4 if we can guarantee that there is some structure we can utilize.

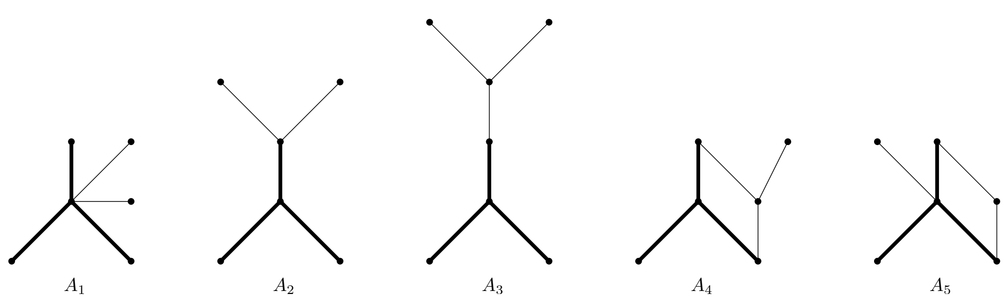

Figure 7: The five good $K_{1,3}$'s.

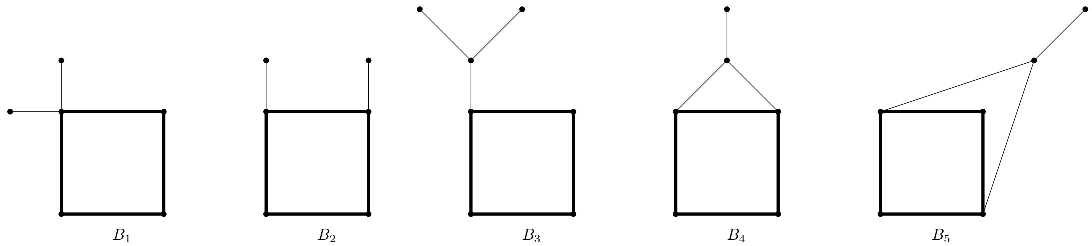

Figure 8: The five good C_4's.

Lemma 3.9. *Let X_2 be the graph in Figure 1. Let G be a graph that has a good $K_{1,3}$ with vertices v, v_1, v_2, v_3 where v is the vertex of degree 3. If v_1v_2, v_2v_3, and v_3v_1 are edges in G, then G contains X_2 as a subgraph.*

Proof. See Figure 9. It is easy to check that G has X_2 as a subgraph in each case. $\square$

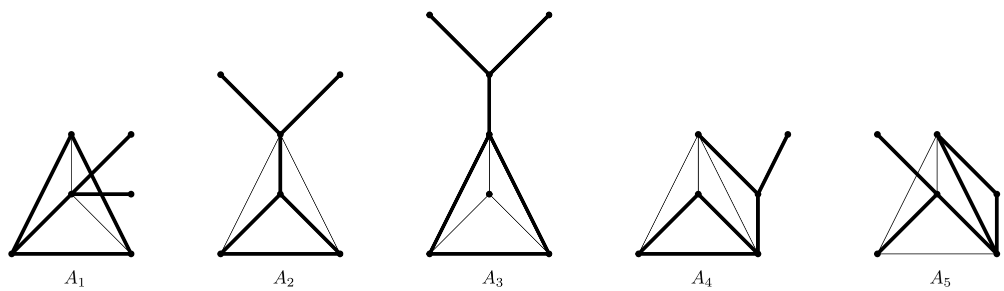

Figure 9: Observation for the proof of Lemma 3.9.

Strategy 3.10. *Painter colors each new edge red, unless doing so creates a red C_3, a bad $K_{1,3}$, or a bad C_4, in which case the new edge is colored blue.*

Proposition 3.11. *Let X_2 be the graph in Figure 1. Painter wins the online Ramsey game for C_3 on X_2-free graphs.*

Proof. Painter will use Strategy 3.10. We claim that Painter can always color the new edge $e = xy$ with Strategy 3.10. Let G be the new graph when Builder draws e. We will use induction on the number of edges. The base case is trivial.

By the induction hypothesis, we may assume that none of a red C_3, a bad $K_{1,3}$, or a bad C_4 exists in $G - e$. The strategy fails when coloring e blue results in a blue C_3 and coloring e red results in a red C_3, a bad $K_{1,3}$, or a bad C_4. Let z be the vertex of the blue C_3 so that xz and zy are blue. Note that every blue edge has at least two red edges incident with it in G while Painter uses Strategy 3.10. We will prove that if the strategy fails, then G has X_2 as a subgraph, which is a contradiction, and thus the strategy does not fail. We will divide the cases according to which red graph appears when Painter colors e red.

Case 1 Assume a red C_3 is created when Painter colors e red, and let w be the third vertex of this red C_3. Since Painter did not color xz and yz red, coloring any of xz and yz red must have created a red C_3, a bad C_4, or a bad $K_{1,3}$. By Lemma 3.9, we may assume that there is no red edge between z and w. Now, we consider three subcases where coloring xz red creates one of a red C_3, a bad $K_{1,3}$, or a bad C_4.

Subcase 1-1 Assume that coloring xz red creates a red C_3 with vertices x, z, and u. Since we assumed that there is no red edge between z and w, we know that $u \neq w$. By Lemma 3.9, we may assume that there is no red edge between y and u. Moreover, y and z cannot have neighbors outside of $\{x, y, z, u, w\}$, since G cannot have X_2 as a subgraph. However, this is a contradiction because Painter must have colored yz red (instead of blue) since this does not create any of a bad $K_{1,3}$, a bad C_4, or a red C_3. Note that although there can be an edge uw in $G - e$, Painter could not color uw red since this creates a red C_3 in $G - e$.

Subcase 1-2 Assume that coloring xz red creates a bad C_4, say R, with vertices x, u, v, and z in cyclic order. Since there is no red edge between z and w, we know that $v \neq w$.

Suppose $u \neq w$. Note that u, v, and z cannot have neighbors outside of $\{x, y, z, u, v, w\}$ and $E(G)$ has none of vy, vx, vw, and uz, otherwise G has X_2 as a subgraph. Therefore, there was no red C_3 when Painter colored yz red.

If there was a bad C_4 when Painter colored yz red, then the only possible case is when the bad C_4 consists of vertices u, v, y, and z since u, v, and z has no neighbor outside of $\{x, y, z, u, v, w\}$. Note that there is a red $K_{1,3}$ with vertices u, v, x, and y. If $\{x, y\}$ has no neighbors outside of $\{x, y, z, u, v\}$, then this red $K_{1,3}$ must be bad, which is a contradiction. Therefore, whenever xz is drawn later than yz or yz is drawn later than xz, the later one must be colored red since the corresponding red C_4 is actually good.

The only remaining reason that Painter colored yz blue is that there are two red edges incident with y so that coloring yz red creates a bad $K_{1,3}$, say S. There are two cases: when there is a red edge between y and u so that $E(S) = \{yz, yu, yw\}$ and when there is no red edge between y and u but there is a red edge ys with a new vertex s so that $E(S) = \{yz, ys, yw\}$.

- When there is a red edge between y and u so that $E(S) = \{yz, yu, yw\}$.
  - Suppose Builder drew xz later than yz. Then R is good by property B_5, which is a contradiction.
  - Suppose Builder drew yz later than xz. Then S is good by property A_2, which is a contradiction.
- When there is no red edge between y and u but there is a red edge ys with a new vertex s so that $E(S) = \{yz, ys, yw\}$.
  - Suppose Builder drew xz later than yz. Then R is good by property B_3, which is a contradiction.

- Suppose Builder drew yz later than xz. Then S is good by property A_2, which is a contradiction.

Note that for both cases, xw may not be drawn at each step of the game.

Now suppose $u = w$. It is easy to check that v, w, and z cannot have neighbors outside of $\{v, w, x, y, z\}$, since otherwise G has X_2 as a subgraph. Since a red $K_{1,3}$ with vertices x, y, w, v must be good, $\{x, y\}$ must have at least one neighbor outside of $\{v, w, x, y, z\}$. Note that this is only true for those steps of the game in which the red $K_{1,3}$ has already been drawn.

- Suppose that yz is drawn later than xz.
  - Coloring yz red cannot create a red C_3 since z cannot have neighbors outside of $\{v, w, x, y, z\}$.
  - Coloring yz red cannot create a bad C_4 since the only possible red C_4 is of vertices v, w, y, and z. Since there is a red $K_{1,3}$, $\{x, y\}$ must have at least one neighbor outside of $\{v, w, x, y, z\}$ and this implies that the red C_4 is good.
  - Coloring yz red cannot create a bad $K_{1,3}$ since z cannot have neighbors outside of $\{v, w, x, y, z\}$. Even if there are two red edges ys and yt for vertices s and t (one of s and t may be equal to w, but not to v or x), the red $K_{1,3}$ with vertices s, t, y, and z is good by property A_2.

    Note that there are no red edges between z and w, between v and y, between v and x, or between x and y.
- Suppose that xz is drawn later than yz. Now, there are two cases: when coloring yz red created a bad C_4 or when coloring yz red created a bad $K_{1,3}$. Note that coloring yz red cannot create a red C_3.
  - If coloring yz red would have created a bad C_4, then the vertices of this C_4 must be v, w, y, and z, since v, w, and z cannot have neighbors outside of $\{v, w, x, y, z\}$ and wz is not a red edge. Hence, yw must have been drawn before yz.
  - If coloring yz red would have created a bad $K_{1,3}$, then there must have been two vertices s, t such that ys and yt are red, and these are drawn earlier than yz. Whenever $w \in \{s, t\}$ or not, R is good, which is a contradiction.

    Note that there are no red edges between v and x or between v and y.

Subcase 1-3 We may assume that coloring xz red creates a bad $K_{1,3}$, say T_1. If z is the center of T_1, then since there is no red edge between z and w, G has X_2 as a subgraph, which is a contradiction. Therefore, we may assume that x is a center of T_1, and xz, xu_1, and xu_2 are the three edges of T_1 with new vertices u_1 and u_2. By symmetry, we may assume that coloring yz red creates a bad $K_{1,3}, sayT_2$, with the center y. We may also assume that yz, yv_1, and yv_2 are the three edges of T_2 with new vertices v_1 and v_2. Note that w is not necessarily distinct from u_1, u_2, v_1, v_2.

- If $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 0$, then it is easy to check that Painter can color one of xz and yz red since T_1 or T_2 must be good by property A_3, which is a contradiction.
- If $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 1$, then it is easy to check that Painter can color one of xz and yz red since T_1 or T_2 must be good by property A_4, which is a contradiction.
- If $|\{u_1, u_2\} \cap \{v_1, v_2\}| = 2$, then let $w_1 = u_1 = v_1$ and $w_2 = u_2 = v_2$. We may assume that yz is drawn later than xz, by symmetry. Then right before Builder draws yz, each of $\{x, y, z, w_1, w_2\}$ cannot have neighbors outside of $\{x, y, z, w_1, w_2\}$, since otherwise T_2 becomes good when Painter colors yz red. However, this is a contradiction since the red C_4 with vertices x, w_1, y, w_2, is bad in $G - e$. This is because a red C_4 in a component of at most five vertices is always bad.

Case 2 Assume a bad $K_{1,3}$, say S, is created when Painter colors e red, and without loss of generality let x, y, u, and v be the vertices of S so that ux, xv are red edges. Now, y cannot have neighbors outside of $\{x, z, u, v\}$ in G since otherwise S is good by property A_2 when e is colored red in G. If there is a red edge between u and y and between v and y, then this case is covered by Case 1. Therefore, we may assume that there is no red edge between u and y and between v and y in G. Since the blue edge yz must have two incident red edges in G, we may assume that the two red edges are incident with z, say zs, zt. We can check that $\{s, t\} = \{u, v\}$, since otherwise S is good by property A_4 when e is colored red in G. Since $G - e$ has a red C_4 with vertices x, u, z, v, say R, by the induction hypothesis, R must be good. This implies that the component containing R must have at least six vertices, thus, one of x, y, z, u, and v has a neighbor outside of $\{x, y, z, u, v\}$. However, this implies that S is good when e is colored red, which is a contradiction.

Case 3 Assume a bad C_4, say R, is created when Painter colors e red, and let xu, uv, vy be the red edges of R. Now each of $\{x, y, z, u, v\}$ cannot have neighbors outside of $\{x, y, z, u, v\}$, since otherwise R is good in G. This also implies that there is no red $K_{1,3}$ in this component since $K_{1,3}$ in a component of at most five vertices must be bad. Then the blue edge zx has no two red edges incident with it in G, which is a contradiction.

Therefore, Strategy 3.10 works, and thus Painter wins the online Ramsey game for C_3 on X_2-free graphs. □

We present a winning strategy for Painter that can be used for the following proposition covering two cases.

Strategy 3.12. *When Builder draws an edge e, if a blue C_3 is made when Painter colors e blue or there is no red edge incident to e, then Painter colors e red. Otherwise, Painter colors e blue.*

Proposition 3.13. *Let X_3 and X_4 be the graphs in Figure 1. Painter wins the online Ramsey game for C_3 on X_3-free graphs and Painter wins the online Ramsey game for C_3 on X_4-free graphs.*

Proof. We will prove both statements at the same time. Painter will use Strategy 3.12. We claim that Painter can always color the new edge $e = xy$ with Strategy 3.12. Let G be the new graph when Builder draws e. We will use induction on the number of edges. The base case is trivial.

By the induction hypothesis, we may assume that every blue edge is incident with at least one red edge and that there is no monochromatic C_3 in $G - e$. The strategy fails when coloring e blue and red results in a blue C_3 and a red C_3, respectively. Let z_1, z_2 be vertices such that $\{x, z_1, y\}$ and $\{x, z_2, y\}$ are vertices of the blue C_3 and the red C_3, respectively. We will prove that if the strategy fails, then G has both X_3 and X_4 as subgraphs, which is a contradiction, and thus the strategy does not fail in either game.

Without loss of generality, we may assume that Builder has drawn xz_2 later than z_2y. Consider the graph right after Builder drew xz_2. Note that xz_2 is incident to a red edge z_2y. Since Painter uses Strategy 3.12 and Painter colored xz_2 red, there must be a blue C_3 when Painter colors xz_2 blue. Let x, z_2, v be the vertices of the blue C_3. Note that xv and z_2v are drawn earlier than xz_2. If $v \neq z_1$, then G contains both X_3 and X_4 as a subgraph, and thus v must be the same as z_1.

Now consider the graph right before Builder drew xz_2. Since Builder has already drawn xz_1 and Painter colored it blue, xz_1 must have at least one incident red edge in G. This red edge is incident with either x or z_1, but in both cases G contains both X_3 and X_4 as a subgraph, which is a contradiction, and thus the strategy works.

Therefore, Strategy 3.12 works, and thus Painter wins the online Ramsey game for C_3 on both X_3-free graphs and X_4-free graphs. □

3.3 The final touch

In this subsection we prove Theorem 3.1. We need two additional lemmas to prove Theorem 3.1.

Lemma 3.14. *If Builder wins the online Ramsey game for H on I-free graphs for a graph I, then Builder wins the online Ramsey game for H on J-free graphs for every graph J that has I as a subgraph.*

Proof. Since the set of I-free graphs is a subset of the set of J-free graphs, Builder can use the same strategy used in the case of J-free graphs. □

Lemma 3.15. *If Painter wins the online Ramsey game for H on I-free graphs for a graph I, then Painter wins the online Ramsey game for H on J-free graphs for every graph J that is a subgraph of I.*

Proof. Since the set of J-free graphs is a subset of the set of I-free graphs, Painter can use the same strategy used in the case of I-free graphs. □

Finally, we prove Theorem 3.1.

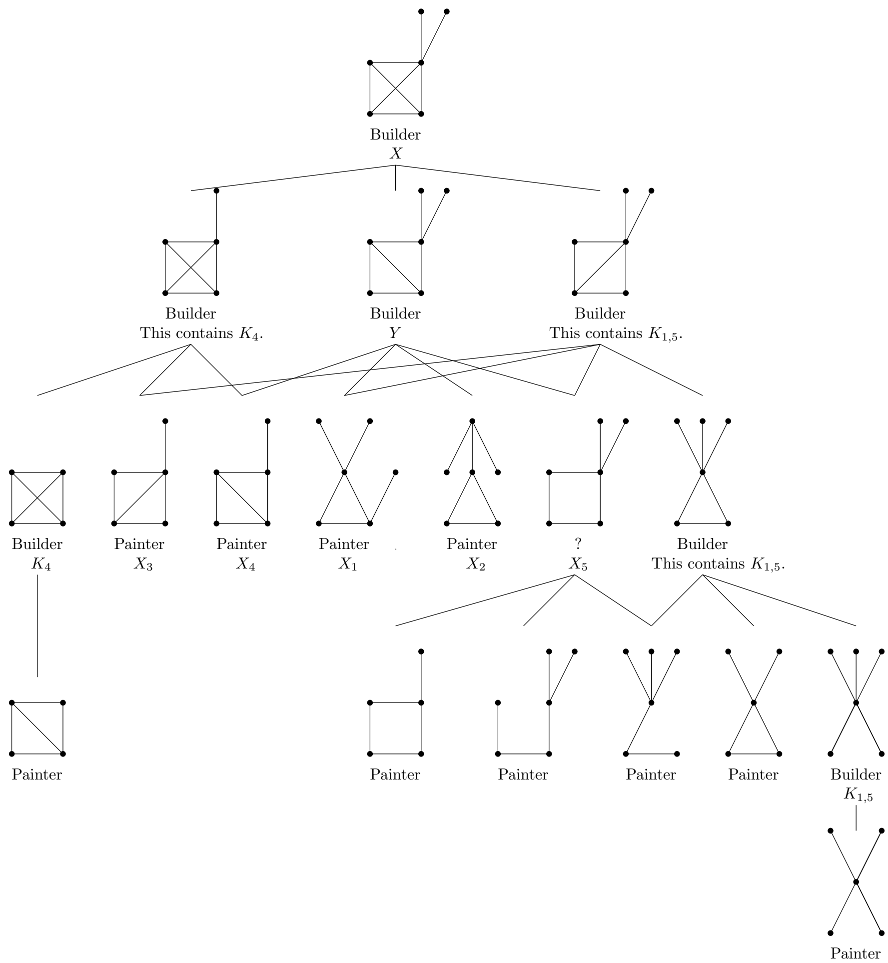

Figure 10: The lines between graphs imply that the lower graph is a subgraph of the higher graph.

Proof of Theorem 3.1. By Lemma 3.2, it is enough to consider when F is a subgraph of X.

By Propositions 3.8, 3.11, and 3.13, along with Lemma 3.15, Painter wins the online Ramsey game for C_3 on F-free graphs if F is isomorphic to a subgraph of a graph in $\{X_1, X_2, X_3, X_4\}$. By Propositions 3.3, 3.4, and 3.6, along with Lemma 3.14, Builder wins the online Ramsey game for C_3 on F-free graphs if F contains a graph in $\{X_1, X_2, X_3, X_4\}$ as a proper subgraph.

It is easy to check that all graphs without isolated vertices are covered by the above paragraph except for the graph X_5. Figure 10 shows subgraphs of X. Moreover, "Builder" and "Painter" written under some graph in Figure 10 means that Builder and Painter, respectively, wins the online Ramsey game for C_3 on F-free graphs. $\square$

We end this section with the only case that is unsolved.

Question 3.16. *Let X_5 be the graph in Figure 1. Who wins the online Ramsey game for C_3 on X_5-free graphs?*

Acknowledgments

We thank the anonymous referee for helping us improve the readability of the paper.

References

[1] J. Balogh and J. Butterfield. Online Ramsey games for triangles in random graphs. *Discrete Math.*, 310(24):3653–3657, 2010.

[2] J. Beck. Achievement games and the probabilistic method. In *Combinatorics, Paul Erdős is eighty, Vol. 1*, Bolyai Soc. Math. Stud., pages 51–78. János Bolyai Math. Soc., Budapest, 1993.

[3] S. A. Burr, P. Erdős, and L. Lovasz. On graphs of Ramsey type. *Ars Combinatoria*, 1(1):167–190, 1976.

[4] J. Butterfield, T. Grauman, W. B. Kinnersley, K. G. Milans, C. Stocker, and D. B. West. On-line Ramsey theory for bounded degree graphs. *Electron. J. Combin.*, 18(1):Paper 136, 17, 2011.

[5] D. Conlon. On-line Ramsey numbers. *SIAM J. Discrete Math.*, 23(4):1954–1963, 2009/10.

[6] D. Conlon, J. Fox, and B. Sudakov. Recent developments in graph Ramsey theory. In *Surveys in combinatorics 2015*, volume 424 of *London Math. Soc. Lecture Note Ser.*, pages 49–118. Cambridge Univ. Press, Cambridge, 2015.

[7] J. Cyman and T. Dzido. A note on on-line Ramsey numbers for quadrilaterals. *Opuscula Math.*, 34(3):463–468, 2014.

[8] J. Cyman, T. Dzido, J. Lapinskas, and A. Lo. On-line Ramsey numbers of paths and cycles. *Electron. J. Combin.*, 22(1):Paper 1.15, 32, 2015.

[9] P. Erdős, R. J. Faudree, C. C. Rousseau, and R. H. Schelp. The size Ramsey number. *Period. Math. Hungar.*, 9(1-2):145–161, 1978.

[10] E. Friedgut, Y. Kohayakawa, V. Rödl, A. Ruciński, and P. Tetali. Ramsey games against a one-armed bandit. *Combin. Probab. Comput.*, 12(5-6):515–545, 2003. Special issue on Ramsey theory.

[11] J. A. Grytczuk, M. Hałuszczak, and H. A. Kierstead. On-line Ramsey theory. *Electron. J. Combin.*, 11(1):Research Paper 60, 10, 2004. Paper number later changed by the publisher from 60 to 57.

[12] J. A. Grytczuk, H. A. Kierstead, and P. Prałat. On-line Ramsey numbers for paths and stars. *Discrete Math. Theor. Comput. Sci.*, 10(3):63–74, 2008.

[13] H. A. Kierstead and G. Konjevod. Coloring number and on-line Ramsey theory for graphs and hypergraphs. *Combinatorica*, 29(1):49–64, 2009.

[14] A. Kurek and A. Ruciński. Two variants of the size Ramsey number. *Discuss. Math. Graph Theory*, 25(1-2):141–149, 2005.

[15] M. Marciniszyn, D. Mitsche, and M. Stojaković. Online balanced graph avoidance games. *European J. Combin.*, 28(8):2248–2263, 2007.

[16] M. Marciniszyn, R. Spöhel, and A. Steger. Online Ramsey games in random graphs. *Combin. Probab. Comput.*, 18(1-2):271–300, 2009.

[17] M. Marciniszyn, R. Spöhel, and A. Steger. Upper bounds for online Ramsey games in random graphs. *Combin. Probab. Comput.*, 18(1-2):259–270, 2009.

[18] Š. Petříčková. Online Ramsey theory for planar graphs. *Electron. J. Combin.*, 21(1):Paper 1.64, 14, 2014.

[19] A. Prakash, R. Spöhel, and H. Thomas. Balanced online Ramsey games in random graphs. *Electron. J. Combin.*, 16(1):Research Paper 11, 22, 2009.

[20] P. Prałat. A note on small on-line Ramsey numbers for paths and their generalization. *Australas. J. Combin.*, 40:27–36, 2008.

[21] P. Prałat. A note on off-diagonal small on-line Ramsey numbers for paths. *Ars Combin.*, 107:295–306, 2012.

[22] F. P. Ramsey. On a Problem of Formal Logic. *Proc. London Math. Soc.*, S2-30(1):264.

[23] D. Rolnick. Trees with an on-line degree Ramsey number of four. *Electron. J. Combin.*, 18(1):Paper 173, 21, 2011.

[24] David Rolnick. The on-line degree Ramsey number of cycles. *Discrete Math.*, 313(20):2084–2093, 2013.