# Minimum co-degree condition for perfect matchings in $k$-partite $k$-graphs 

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#### Abstract

Let $H$ be a $k$-partite $k$-graph with $n$ vertices in each partition class, and let $\delta_{k-1}(H)$ denote the minimum co-degree of $H$. We characterize those $H$ with $\delta_{k-1}(H) \geq n / 2$ and with no perfect matching. As a consequence we give an affirmative answer to the following question of Rödl and Ruciński: If $k$ is even or $n \not \equiv 2(\bmod 4)$, does $\delta_{k-1}(H) \geq$ $n / 2$ imply that $H$ has a perfect matching? We also give an example indicating that it is not sufficient to impose this degree bound on only two types of $(k-1)$-sets.


## 1 Introduction

A hypergraph $H$ consists of a vertex set $V(H)$ and an edge set $E(H)$ whose members are subsets of $V(H)$. Let $H_{1}$ and $H_{2}$ be two hypergraphs. If $V\left(H_{1}\right) \subseteq V\left(H_{2}\right)$ and $E\left(H_{1}\right) \subseteq$ $E\left(H_{2}\right)$, then $H_{1}$ is called a subgraph of $H_{2}$, denoted $H_{1} \subseteq H_{2}$. Let $k$ be a positive integer and $[k]:=\{1, \ldots, k\}$. For a set $S$, let $\binom{S}{k}:=\{T \subseteq S:|T|=k\}$. A hypergraph $H$ is $k$-uniform if $E(H) \subseteq\binom{V(H)}{k}$, and a $k$-uniform hypergraph is also called a $k$-graph. Given $T \subseteq V(H)$, let $H-T$ denote the subgraph of $H$ with vertex set $V(H)-T$ and edge set $E(H-T)=\{e \in E(H): e \subseteq V(H)-T\}$.

Let $H$ be a $k$-graph and $S \in\binom{V(H)}{l}$ with $l \in[k]$. The neighborhood of $S$ in $H$, denoted $N_{H}(S)$, is the set of all $(k-l)$-subsets $U \subseteq V(H)$ such that $S \cup U \in E(H)$. The degree of $S$ in $H$, denoted $d_{H}(S)$, is the size of $N_{H}(S)$. For $l \in[k]$, the minimum l-degree of $H$, denoted $\delta_{l}(H)$, is the minimum degree over all $l$-subsets of $V(H)$. Note that $\delta_{k-1}(H)$ is known as the minimum co-degree of $H$.

[^0]A matching in a hypergraph $H$ is a subset of $E(H)$ consisting of pairwise disjoint edges. A matching $M$ in a hypergraph $H$ is called a perfect matching if $V(M)=V(H)$. Rödl, Ruciński, and Szemerédi [9] determined the minimum co-degree threshold function that ensures a perfect matching in a $k$-graph with $n$ vertices, for $n \equiv 0(\bmod k)$ and sufficiently large. This threshold function is $\frac{n}{2}-k+C$, where $C \in\{3 / 2,2,5 / 2,3\}$, depending on the parity of $n$ and $k$. They $[9]$ also proved that, for $n \not \equiv 0(\bmod k)$, the minimum co-degree threshold that ensures a matching $M$ in a $k$-graph $H$ with $|V(M)| \geq|V(H)|-k$ is between $\lfloor n / k\rfloor$ and $n / k+O(\log n)$, and conjectured that this threshold function is $\lfloor n / k\rfloor$. This conjecture was proved recently by Han [2]. Treglown and Zhao 10, 11] determined the minimum $l$-degree threshold for perfect matchings in $k$-graphs for $k / 2 \leq l \leq k-1$.

A hypergraph $H$ is a $k$-partite $k$-graph with partition classes $V_{1}, \ldots, V_{k}$ if $V_{1}, \ldots, V_{k}$ is a partition of $V(H)$ and $\left|e \cap V_{i}\right|=1$ for all $e \in E(H)$ and $i \in[k]$. We say that a set $S \subseteq V(H)$ is legal if $\left|S \cap V_{i}\right| \leq 1$ for $i \in[k]$. For $l \in[k]$, the minimum l-degree of a $k$-partite $k$-graph $H$, also denoted $\delta_{l}(H)$, is the minimum degree over all legal $l$-subsets of $V(H)$. Again, $\delta_{k-1}(H)$ is called the minimum co-degree of $H$.

Kühn and Osthus [4] showed that the minimum co-degree threshold for the existence of a perfect matching in a $k$-partite $k$-graph with $n$ vertices in each partition class is between $n / 2$ and $n / 2+\sqrt{2 n \log n}$. Lu, Wang, and Yu [6] and, independently, Han, Zang, and Zhao [3] showed that $n / k$ is the minimum co-degree threshold for a $k$-partite $k$-graph $H$ with $n$ vertices in each partition class to admit a matching of size $|V(H)|-k$.

Aharoni, Georgakopoulos, and Sprüssel [1] obtained the following stronger result: Let $k \geq 3$ be a positive integer and $H$ be a $k$-partite $k$-graph with partition classes $V_{1}, \ldots, V_{k}$, each of size $n$. If $d_{H}(S)>n / 2$ for every legal $(k-1)$-set $S$ contained in $V-V_{1}$, and if $d_{H}(T) \geq n / 2$ for every legal $(k-1)$-set $T$ contained in $V-V_{2}$, then $H$ has a perfect matching. Example 1 in [1] (see the graph $H_{0}(k, n)$ below) shows that this bound is best possible when $k$ is odd and $n \equiv 2(\bmod 4)$. Motivated by this result, Rödl and Ruciński 8 asked the following

Question 1.1 (Rödl and Ruciński [8]) Let $k, n$ be integers with $k \geq 3$ and $n$ sufficiently large, and $H$ be a $k$-partite $k$-graph in which each partition class has size $n$. Assume that $k$ is even or $n \not \equiv 2(\bmod 4)$. Is it true that if $\delta_{k-1}(H) \geq n / 2$ then $H$ has a perfect matching? If so, is it sufficient to impose this degree bound on only two types of legal $(k-1)$-sets, similar to the above result of Aharoni, Georgakopoulos, and Sprüssel?

Note that if $n$ is odd, it follows from the above result of Aharoni, Georgakopoulos, and Sprüssel that the answer to the first part of Question 1.1 is affirmative.

We now describe an example showing the tightness of the bound in Question 1.1. Let $k, n, d_{i}, i \in[k]$, be positive integers. Let $H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$ be a $k$-partite $k$-graph with partition classes $V_{1}, \ldots, V_{k}$, and let $D_{i} \subseteq V_{i}$ for $i \in[k]$, such that $\left|V_{i}\right|=n$ and $\left|D_{i}\right|=d_{i}$ for $i \in[k]$, and $E\left(H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)\right)$ consists of those legal $k$-sets with an even number of vertices (including zero) in $\bigcup_{i \in[k]} D_{i}$. In particular, we define $H_{0}(k, n):=$ $H_{0}(\lfloor n / 2\rfloor, \ldots,\lfloor n / 2\rfloor ; k, n)$. When $k$ is odd and $n \equiv 2(\bmod 4), H_{0}(k, n)$ is Example 1 in [1]; in which case, $\delta_{k-1}\left(H_{0}(k, n)\right)=n / 2$ and $H_{0}(k, n)$ admits no perfect matching (as $\sum_{i \in[k]}\left|D_{i}\right|=k n / 2$ is odd and every edge of $H_{0}(k, n)$ has an even number of vertices in $\left.\bigcup_{i \in[k]} D_{i}\right)$.

Remark. We point out that the answer to the second part of Question 1.1 is negative. Let $k, n$ be positive integers such that $k$ is even or $n \equiv 0(\bmod 4)$. Let $J:=$ $H_{0}(n / 2, n / 2, \ldots, n / 2, n / 2+1 ; k, n)$ with partition classes $V_{1}, \ldots, V_{k}$ and let $D_{i} \subseteq V_{i}$ for $i \in[k]$ such that $\left|D_{i}\right|=n / 2$ for $i \in[k-1],\left|D_{k}\right|=n / 2+1$, and each edge of $J$ has an even number of vertices in $\bigcup_{i \in[k]} D_{i}$. Observe that all legal $(k-1)$-subsets of $V(J)$ intersecting $V_{k}$ have degree at least $n / 2$, and those legal $(k-1)$ sets contained in $V(J)-V_{k}$ and intersecting $\cup_{i \in[k]} D_{i}$ an even number of times have degree $n / 2-1$. Moreover, $J$ has no perfect matching since $\sum_{i \in[k]}\left|D_{i}\right|=k n / 2-1 \equiv 1(\bmod 2)($ as $k$ is even or $n \equiv 0(\bmod 4))$.

Our main result is the following, which implies an affirmative answer to the first part of Question 1.1 .

Theorem 1.2 Let $k, n$ be integers with $k \geq 3$ and $n$ sufficiently large, and let $H$ be a $k$ partite $k$-graph with $n$ vertices in each partition class. Suppose $\delta_{k-1}(H) \geq\lfloor n / 2\rfloor$. Then $H$ has no perfect matching if, and only if,
(i) $k$ is odd, $n \equiv 2(\bmod 4)$, and $H \cong H_{0}(k, n)$, or
(ii) $n$ is odd and there exist $d_{i} \in\{(n+1) / 2,(n-1) / 2\}$ for $i \in[k]$ such that $\sum_{i=1}^{k} d_{i}$ is odd and $H \subseteq H_{0}\left(d_{1}, d_{2}, \ldots, d_{k} ; k, n\right)$.

Our proof of Theorem 1.2 consists of two parts by considering whether or not $H$ is "close" to $H_{0}(k, n)$, which is similar to arguments in [5, 9]. Given two hypergraphs $H_{1}, H_{2}$ with $V\left(H_{1}\right)=V\left(H_{2}\right)$, let $c\left(H_{1}, H_{2}\right)$ be the minimum of $\left|E\left(H_{1}\right) \backslash E\left(H^{\prime}\right)\right|$ taken over all isomorphic copies $H^{\prime}$ of $H_{2}$ with $V\left(H^{\prime}\right)=V\left(H_{2}\right)$. For a real number $\varepsilon>0$, we say that $H_{2}$ is $\varepsilon$-close to $H_{1}$ if $V\left(H_{1}\right)=V\left(H_{2}\right)$ and $c\left(H_{1}, H_{2}\right)$ is less than $\varepsilon$ times the maximum possible number of edges on $V\left(H_{2}\right)$ (which is, for example, $\varepsilon n^{k}$ if $H_{2}$ is a $k$-partite $k$-graph with $n$ vertices in each partition class).

In Section 2, we deal with the case when $H$ is $\varepsilon$-close to $H_{0}(k, n)$ for some sufficiently small $\varepsilon$. In Section 3, we deal with the case when $H$ is not $\varepsilon$-close to $H_{0}(k, n)$, using the absorbing method from [9] and a recent result of the authors [6] (see Lemma 3.1).

## 2 Hypergraphs close to $H_{0}(k, n)$

In this section, we prove Theorem 1.2 for the case when $H$ is $\varepsilon$-close to $H_{0}(k, n)$ for some sufficiently small $\varepsilon$. Since we will be dealing with $H_{0}(k, n)$, the following notation for "even" and "odd" degrees (with respect to a given set $S$ ) will be convenient. Let $H$ be a hypergraph. For $j \in\{0,1\}, v \in V(H)$, and $S \subseteq V(H)$, we define

$$
d_{H, S}^{j}(v):=\mid\{e \in E(H): v \in e \text { and }|e \cap S| \equiv j \quad(\bmod 2)\} \mid
$$

Lemma 2.1 Let $k \geq 3$ be a positive integer, and let $\alpha, \varepsilon>0$ be small such that $\alpha<1 / 4$ and $\sqrt{\varepsilon}<\min \left\{1 /\left(100 k^{2}\right), 1 /\left(k\left(10 k^{2}\right)^{k-1}\right)\right\}$. Then for any $k$-partite $k$-graph $H$ with $n>100 k^{2}$ vertices in each partition class, the following holds: If $\delta_{k-1}(H) \geq(1 / 2-\alpha) n$, $H$ is $\varepsilon$-close to $H_{0}(k, n)$, and $H \nsubseteq H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$ for any $d_{1}, \ldots, d_{k} \in[\lceil(1 / 2-\alpha) n\rceil,\lfloor(1 / 2+\alpha) n\rfloor]$ with $\sum_{i=1}^{k} d_{i}$ odd, then $H$ has a perfect matching.

Proof. Let $H$ be a $k$-partite $k$-graph with $n$ vertices in each partition class such that $\delta_{k-1}(H) \geq(1 / 2-\alpha) n, H$ is $\varepsilon$-close to $H_{0}(k, n)$, and $H \nsubseteq H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$ for any $d_{1}, \ldots, d_{k} \in[\lceil(1 / 2-\alpha) n\rceil,\lfloor(1 / 2+\alpha) n\rfloor]$ with $\sum_{i=1}^{k} d_{i}$ odd. Let

$$
N:=\left\{v \in V(H):\left|N_{H_{0}(k, n)}(v)-N_{H}(v)\right| \geq \sqrt{\varepsilon} n^{k-1}\right\}
$$

So each vertex in $N$ is contained in at least $\sqrt{\varepsilon} n^{k-1}$ edges from $E\left(H_{0}(k, n)\right)-E(H)$. Note that

$$
|N| \leq \sqrt{\varepsilon} k n
$$

for, otherwise,

$$
\begin{aligned}
\left|E\left(H_{0}(k, n)\right)-E(H)\right| & \geq \frac{1}{k} \sum_{v \in N}\left|N_{H_{0}(k, n)}(v)-N_{H}(v)\right| \\
& >\frac{1}{k}|N| \sqrt{\varepsilon} n^{k-1} \\
& >\frac{1}{k} \sqrt{\varepsilon} k n \sqrt{\varepsilon} n^{k-1} \\
& =\varepsilon n^{k}
\end{aligned}
$$

contradicting the fact that $H$ is $\varepsilon$-close to $H_{0}(k, n)$.
The rest of our proof is organized as follows. We first find a matching $M_{1}$ in $H$ that covers all vertices in $N$ (see Claim 2). We then find a matching $M_{2}$ in $H-V\left(M_{1}\right)$ satisfying certain conditions (see Claim 3). Finally, we will show that there exists a perfect matching in $H-V\left(M_{1}\right)-V\left(M_{2}\right)$. The last part is easy when $k$ is even (see Claim 4), but needs more work when $k$ is odd (see Claims 5-8).

To find a matching in $H$ that covers all vertices in $N$, we need to fix some notation first. For $i \in[k]$, let $B_{i} \subseteq V_{i}$ such that $\left|B_{i}\right|=\lfloor n / 2\rfloor$ and each edge in $H_{0}(k, n)$ has an even number of vertices in $B:=\cup_{j \in[k]} B_{j}$. For $i \in[k]$, let $A_{i}:=V_{i}-B_{i}$. The intuition for the notation below is that the vertices $v$ in $A_{i} \cap N$ (respectively, $B_{i} \cap N$ ) with $d_{H, B}^{0}(v)<n^{k-1} / 8$ will be switched to $B_{i}^{\prime}$ (respectively, $A_{i}^{\prime}$ ). For $i \in[k]$, let

$$
A_{i}^{\prime}:=\left(A_{i}-\left\{v \in A_{i} \cap N: d_{H, B}^{0}(v)<n^{k-1} / 8\right\}\right) \cup\left\{v \in B_{i} \cap N: d_{H, B}^{0}(v)<n^{k-1} / 8\right\}
$$

and

$$
B_{i}^{\prime}:=\left(B_{i}-\left\{v \in B_{i} \cap N: d_{H, B}^{0}(v)<n^{k-1} / 8\right\}\right) \cup\left\{v \in A_{i} \cap N: d_{H, B}^{0}(v)<n^{k-1} / 8\right\}
$$

Let $A^{\prime}:=\cup_{j \in[k]} A_{j}^{\prime}$ and $B^{\prime}:=\cup_{j \in[k]} B_{j}^{\prime}$.
Since $|N| \leq \sqrt{\varepsilon} k n$ and $\left|B_{i}\right|=\lfloor n / 2\rfloor$, we have $A_{i}^{\prime} \neq \emptyset$ and $B_{i}^{\prime} \neq \emptyset$ for $i \in[k]$ (as $\left.n \geq 100 k^{2}\right)$. In fact, for $i \in[k]$,

$$
\begin{equation*}
\left|A_{i}^{\prime}\right| \geq\left|A_{i}\right|-|N| \geq(1 / 2-\sqrt{\varepsilon} k) n \tag{1}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|B_{i}^{\prime}\right| \geq\left|B_{i}\right|-|N| \geq(1 / 2-\sqrt{\varepsilon} k) n-1 \tag{2}
\end{equation*}
$$

Moreover, for each $v \in V(H)$, the number of edges in $H$ containing $v$ and intersecting $N-\{v\}$ is at most $|N| n^{k-2}$.

We now show that, for $v \in V(H)$,

$$
\begin{equation*}
d_{H-(N-\{v\}), B^{\prime}}^{0}(v) \geq(1 / 8-\sqrt{\varepsilon} k) n^{k-1} . \tag{3}
\end{equation*}
$$

First, suppose $v \in\left(A \cap A^{\prime}\right) \cup\left(B \cap B^{\prime}\right)$. Then $B^{\prime}-(N-\{v\})=B-(N-\{v\})$, and $d_{H, B}^{0}(v) \geq n^{k-1} / 8$ by definition of $A^{\prime}, B^{\prime}$. So

$$
d_{H-(N-\{v\}), B-(N-\{v\})}^{0}(v) \geq d_{H, B}^{0}(v)-|N| n^{k-2} \geq n^{k-1} / 8-\sqrt{\varepsilon} k n^{k-1} .
$$

Hence,

$$
\begin{aligned}
d_{H-(N-\{v\}), B^{\prime}}^{0}(v) & =d_{H-(N-\{v\}), B^{\prime}-(N-\{v\})}^{0}(v) \\
& =d_{H-(N-\{v\}), B-(N-\{v\})}^{0}(v) \\
& \geq(1 / 8-\sqrt{\varepsilon} k) n^{k-1} .
\end{aligned}
$$

Now assume $v \in\left(A \cap B^{\prime}\right) \cup\left(A^{\prime} \cap B\right)$. Then $v \in N$ and $d_{H, B}^{0}(v)<n^{k-1} / 8$. So $B^{\prime}-(N-\{v\})=$ $B-(N-\{v\})-\{v\}$, Since $\delta_{k-1}(H) \geq(1 / 2-\alpha) n$, it follows that $d_{H}(v) \geq(1 / 2-\alpha) n^{k-1}$. Thus, since $n \geq 100 k^{2}$ and $\alpha<1 / 4$,

$$
d_{H, B}^{1}(v) \geq(1 / 2-\alpha) n^{k-1}-d_{H, B}^{0}(v)>(1 / 2-\alpha) n^{k-1}-n^{k-1} / 8>n^{k-1} / 8
$$

Therefore, $d_{H-(N-\{v\}), B-(N-\{v\})}^{1}(v) \geq d_{H, B}^{1}(v)-|N| n^{k-2} \geq(1 / 8-\sqrt{\varepsilon} k) n^{k-1}$. Hence,

$$
\begin{aligned}
d_{H-(N-\{v\}), B^{\prime}}^{0}(v) & =d_{H-(N-\{v\}), B^{\prime}-(N-\{v\})}^{0}(v) \\
& =d_{H-(N-\{v\}), B-(N-\{v\})}^{1}(v) \\
& \geq(1 / 8-\sqrt{\varepsilon} k) n^{k-1} .
\end{aligned}
$$

We now begin our process of finding matchings $M_{1}$ and $M_{2}$. First, we need to make $\left|B^{\prime}\right|$ even.

Claim 1. Either $\left|B^{\prime}\right|$ is even (in which case let $e_{0}=\emptyset$; so $\left|B^{\prime}-e_{0}\right|$ is even), or there exists an edge $e_{0} \in E(H)$ such that $\left|B^{\prime}-e_{0}\right|$ is even.

We may assume that $\left|B^{\prime}\right|$ is odd and $\left|B^{\prime}-e\right|$ is odd for every $e \in E(H)$; as, otherwise, Claim 1 holds. Then $\left|B^{\prime} \cap e\right|$ is even for all $e \in E(H)$. Hence $H \subseteq H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$, where $d_{i}=\left|B_{i}^{\prime}\right|$ for $i \in[k], \sum_{i \in[k]} d_{i}=\left|B^{\prime}\right|$ is odd, and $B_{1}^{\prime}, \ldots, B_{k}^{\prime}$ play the roles of $D_{1}, \ldots, D_{k}$, respectively, in the definition of $H_{0}(k, n)$.

Let $v_{i} \in A_{i}^{\prime}$ and $u_{i} \in B_{i}^{\prime}$ for $i \in[k]$, and let $S:=\left\{v_{1}, \ldots, v_{k}\right\}$. Then for $i \in[k]$, since $\left|B^{\prime} \cap e\right|$ is even for all $e \in E(H)$, we have

$$
n-d_{i}=\left|A_{i}^{\prime}\right| \geq d_{H}\left(S-\left\{v_{i}\right\}\right) \geq \delta_{k-1}(H) \geq(1 / 2-\alpha) n ;
$$

so $d_{i} \leq\lfloor(1 / 2+\alpha) n\rfloor$. Moreover, for $i \in[k]$, let $j \in[k]-\{i\}$. Again, since $\left|B^{\prime} \cap e\right|$ is even for all $e \in E(H)$, we have

$$
d_{i}=\left|B_{i}^{\prime}\right| \geq d_{H}\left(\left(S \cup\left\{u_{j}\right\}\right)-\left\{v_{i}, v_{j}\right\}\right) \geq \delta_{k-1}(H) \geq(1 / 2-\alpha) n ;
$$

so $d_{i} \geq\lceil(1 / 2-\alpha) n\rceil$. This contradicts the assumption that $H \nsubseteq H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$ for any $d_{1}, \ldots, d_{k} \in[\lceil(1 / 2-\alpha) n\rceil,\lfloor(1 / 2+\alpha) n\rfloor]$ with $\sum_{i=1}^{k} d_{i}$ odd.

Note that for each $v \in N-e_{0}$, the number of edges in $H$ containing $v$ and a vertex of $e_{0}$ is at most $k n^{k-2}$. Thus by (3), we have

$$
\begin{equation*}
d_{\left(H-e_{0}\right)-(N-\{v\}), B^{\prime}-e_{0}}^{0}(v) \geq(1 / 8-2 \sqrt{\varepsilon} k) n^{k-1}-k n^{k-2}>n^{k-1} / 10, \tag{4}
\end{equation*}
$$

where the last inequality holds since $\sqrt{\varepsilon}<1 /\left(100 k^{2}\right)$ and $n \geq 100 k^{2}$.
Claim 2. There exists a matching $M_{1}$ in $H-e_{0}$ such that
(i) $\left|M_{1}\right| \leq \sqrt{\varepsilon} k n$,
(ii) $N-e_{0} \subseteq V\left(M_{1}\right)$, and
(iii) $\left|e \cap\left(B^{\prime}-e_{0}\right)\right| \equiv 0(\bmod 2)$ for all $e \in M_{1}$.

Let $M_{1}:=\emptyset$ if $N-e_{0}=\emptyset$. Now assume $N-e_{0} \neq \emptyset$, and we construct $M_{1}$ by matching vertices in $N$ greedily. Let $v_{1} \in N-\left\{e_{0}\right\}$. Since $d_{\left(H-e_{0}\right)-\left(N-\left\{v_{1}\right\}\right), B^{\prime}-e_{0}}^{0}\left(v_{1}\right)>n^{k-1} / 10$ (by (4)) and $n \geq 100 k^{2}$, there exists an edge $e_{1}$ in $H-e_{0}$, such that $v_{1} \in e_{1}$ and $\left|e_{1} \cap\left(B^{\prime}-e_{0}\right)\right| \equiv 0$ $(\bmod 2)$.

Now suppose we have found a matching $\left\{e_{1}, \ldots, e_{t}\right\}$ in $H-e_{0}$ for some $t \geq 1$, such that, for $i \in[t], e_{i} \cap\left(N-e_{0}\right) \neq \emptyset$ and $\left|e_{i} \cap\left(B^{\prime}-e_{0}\right)\right| \equiv 0(\bmod 2)$. If $N-e_{0} \subseteq \cup_{i \in[t]} e_{i}$, then $M_{1}:=\left\{e_{1}, \ldots, e_{t}\right\}$ is the desired matching (as $t<|N| \leq \sqrt{\varepsilon} k n$ ). So let $v_{t+1} \in N-e_{0}$ and $v_{t+1} \notin \cup_{i \in[t]} e_{i}$. Note that $t<|N| \leq \sqrt{\varepsilon} k n$ and that the number of edges in $H-e_{0}$ containing $v_{t+1}$ and a vertex from $\cup_{i \in[t]} e_{i}$ is at most

$$
t k n^{k-2} \leq \sqrt{\varepsilon} k^{2} n^{k-1} \leq n^{k-1} / 100
$$

as $\sqrt{\varepsilon}<1 /\left(100 k^{2}\right.$. Since $d_{\left(H-e_{0}\right)-\left(N-\left\{v_{t+1}\right\}\right), B^{\prime}-e_{0}}^{0}\left(v_{t+1}\right)>n^{k-1} / 10$ (by (4)), there exists $e_{t+1}$ in $\left(H-e_{0}\right)-\cup_{i \in[t]} e_{i}$ such that $v_{t+1} \in e_{t+1}$ and $\left|e_{t+1} \cap\left(B^{\prime}-e_{0}\right)\right| \equiv 0(\bmod 2)$.

Therefore, continuing this process (at most $\left|N-e_{0}\right|$ steps), we obtain the desired matching for Claim 2.

Let $H^{\prime}:=\left(H-e_{0}\right)-V\left(M_{1}\right)$. For $i \in[k]$, let $C_{i}:=A_{i}^{\prime}-\left(V\left(M_{1}\right) \cup e_{0}\right), D_{i}:=B_{i}^{\prime}-$ $\left(V\left(M_{1}\right) \cup e_{0}\right)$ and $D:=\cup_{i \in[k]} D_{i}$. By Claim 2, $N \cap V\left(H^{\prime}\right)=\emptyset$; so for $i \in[k]$,

$$
\begin{equation*}
D_{i} \subseteq B_{i} \tag{5}
\end{equation*}
$$

Note that $|D|$ is even (by Claims 1 and 2). Since $\left|M_{1}\right| \leq \sqrt{\varepsilon} k n$, it follows from (1) and (2) that for $i \in[k]$,

$$
\begin{equation*}
\left|C_{i}\right| \geq\left|A_{i}^{\prime}\right|-\left(\left|M_{1}\right|+1\right) \geq(1 / 2-\sqrt{\varepsilon} k) n-(\sqrt{\varepsilon} k n+1) \geq(1 / 2-2 \sqrt{\varepsilon} k) n-1 \tag{6}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|D_{i}\right| \geq\left|B_{i}^{\prime}\right|-\left(\left|M_{1}\right|+1\right) \geq((1 / 2-\sqrt{\varepsilon} k) n-1)-(\sqrt{\varepsilon} k n+1) \geq(1 / 2-2 \sqrt{\varepsilon} k) n-2 . \tag{7}
\end{equation*}
$$

Claim 3. There exists a matching $M_{2}$ in $H^{\prime}$ such that
(i) $\left|M_{2}\right| \leq 8 \sqrt{\varepsilon} k^{2} n$,
(ii) $\left|D_{i}-V\left(M_{2}\right)\right|=\left|D_{1}-V\left(M_{2}\right)\right|$ for $i \in[k]$, and
(iii) $\left|D-V\left(M_{2}\right)\right|$ is even.

Without loss of generality, we may assume that $\left|D_{1}\right| \geq\left|D_{2}\right| \geq \cdots \geq\left|D_{k}\right|$. If $\left|D_{1}\right|=\left|D_{k}\right|$ then $M_{2}=\emptyset$ gives the desired matching for Claim 3. So assume $\left|D_{1}\right|-\left|D_{k}\right|>0$. We construct an auxiliary graph and use a perfect matching in this graph to find $M_{2}$.

Let $r \in\{0,1\}$ such that $\left|D_{1}\right|+r$ is even. Let $G$ be the complete $k$-partite 2 -graph and let $W_{1}, \ldots, W_{k}$ be the partition classes of $G$, such that $\left|W_{i}\right|=\left(\left|D_{i}\right|-\left|D_{k}\right|\right)+\left(\left|D_{1}\right|+r\right)-\left|D_{k}\right|$ for $i \in[k]$. Then $\left|W_{1}\right| \geq\left|W_{2}\right| \geq \ldots \geq\left|W_{k}\right|$ and

$$
|V(G)|=\sum_{i \in[k]}\left|W_{i}\right|=\left(\sum_{i \in[k]}\left|D_{i}\right|\right)+k\left(\left|D_{1}\right|+r\right)-2 k\left|D_{k}\right| .
$$

Since $\sum_{i \in[k]}\left|D_{i}\right|$ and $\left|D_{1}\right|+r$ are even, $|V(G)|$ is also even.
We now use Tutte's 1 -factor theorem to show that $G$ has a perfect matching. For $S \subseteq V(G)$, let $o(G-S)$ denote the number of connected components of $G-S$ of odd order. If $S=\emptyset$, then $o(G-S)=0 \leq|S|$. Now assume $S \neq \emptyset$. Since $G$ is a complete $k$-partite 2-graph and $\left|W_{1}\right| \geq\left|W_{2}\right| \geq \ldots \geq\left|W_{k}\right|$, if $1 \leq|S|<\sum_{i \in[k]-\{1\}}\left|W_{i}\right|$ then $o(G-S) \leq 1 \leq|S|$, and if $|S| \geq \sum_{i \in[k]-\{1\}}\left|W_{i}\right| \geq(k-1)\left(\left|D_{1}\right|-\left|D_{k}\right|+r\right)$ then $o(G-S) \leq\left|W_{1}\right| \leq 2\left|D_{1}\right|-2\left|D_{k}\right|+r \leq|S|$ (as $k \geq 3$ ). Thus, by Tutte's 1-factor theorem, $G$ has a perfect matching, say $T$.

Since $\left|C_{i}\right| \geq(1 / 2-2 \sqrt{\varepsilon} k) n-1$ (by (6) $),\left|D_{i}\right| \leq n-\left|C_{i}\right| \leq(1 / 2+2 \sqrt{\varepsilon} k) n+1$. So by (7), $\left|D_{1}\right|-\left|D_{k}\right| \leq 4 \sqrt{\varepsilon} k n+3$. Hence,

$$
|T|=|V(G)| / 2=\left(\sum_{i \in[k]}\left|W_{i}\right|\right) / 2 \leq(k / 2)\left(2\left|D_{1}\right|-2\left|D_{k}\right|+1\right) \leq 8 \sqrt{\varepsilon} k^{2} n .
$$

Let $T=\left\{f_{1}, f_{2}, \ldots, f_{|T|}\right\}$. Corresponding to each $f_{i}$ we find an edge $g_{i}$ of $H^{\prime}$ such that $\left\{g_{1}, \ldots, g_{|T|}\right\}$ gives the desired matching $M_{2}$ for Claim 3 .

Let $g_{0}=\emptyset$ and we find $g_{1}, \ldots, g_{|T|}$ in order. Suppose we have found $g_{t}$ for some $t$, with $0 \leq t \leq|T|-1$. We describe how to find $g_{t+1}$ using $f_{t+1}$. Let $f_{t+1} \subseteq W_{p} \cup W_{q}$, where $p, q \in[k]$. By (6) and (7), $\min \left\{\left|C_{j}\right|,\left|D_{j}\right|\right\} \geq(1 / 2-2 \sqrt{\varepsilon} k) n-2$ for $j \in[k]$. Then, since $|T| \leq 8 \sqrt{\varepsilon} k^{2} n, n \geq 100 k^{2}$, and $\sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right)$, we have, for $j \in[k]$,

$$
\left|C_{j}-\cup_{i \in[t]} g_{i}\right|>n / 10 \text { and }\left|D_{j}-\cup_{i \in[t]} g_{i}\right|>n / 10
$$

So let $v_{p} \in D_{p}-\cup_{i \in[t]} g_{i}$. There exist $v_{q} \in D_{q}-\cup_{i \in[t]} g_{i}$ and $v_{j} \in C_{j}-\cup_{i \in[t]} g_{i}$ for $j \in[k]-\{p, q\}$ such that $g_{t+1}:=\left\{v_{1}, \ldots, v_{k}\right\} \in E\left(H^{\prime}\right)$; for, otherwise,

$$
\left|N_{H_{0}(k, n)}\left(v_{p}\right)-N_{H}\left(v_{p}\right)\right|>(n / 10)^{k-1}>\sqrt{\varepsilon} n^{k-1}
$$

as $\sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right.$, contradicting the fact that $v_{p} \notin N$. Clearly, $\left|g_{t+1} \cap D\right|=2$.
Therefore, $M_{2}:=\left\{g_{1}, \ldots, g_{|T|}\right\}$ is a matching in $H^{\prime}$ such that, for $i \in[k]$,

$$
\left|D_{i}-V\left(M_{2}\right)\right|=\left|D_{i}\right|-\left|W_{i}\right|=2\left|D_{k}\right|-\left|D_{1}\right|-r>0
$$

where the inequality holds because of $77,\left|D_{1}\right|-\left|D_{k}\right| \leq 4 \sqrt{\varepsilon} k n+3, \sqrt{\varepsilon}<1 /\left(100 k^{2}\right)$, and $n \geq 100 k^{2}$. Moreover, $\left|g_{j} \cap D\right|=2$ for $j \in[|T|]$. Hence, since $|D|$ is even (by Claims 1 and 2), $\left|D-V\left(M_{2}\right)\right|$ is even.

Let $H^{\prime \prime}:=H^{\prime}-V\left(M_{2}\right)$ and, for $i \in[k]$, let $D_{i}^{\prime}:=D_{i}-V\left(M_{2}\right)$ and $C_{i}^{\prime}:=C_{i}-V\left(M_{2}\right)$. Let $D^{\prime}:=\cup_{i \in[k]} D_{i}^{\prime}$ and $C^{\prime}:=\cup_{i \in[k]} C_{i}^{\prime}$. Note that $\left|D^{\prime}\right|$ is even, as $\left|D-V\left(M_{2}\right)\right|$ is even (by Claim 3). Since $\left|M_{2}\right| \leq 8 \sqrt{\varepsilon} k^{2} n$ (by Claim 3), it follows from (6) and (7) that, for $i \in[k]$,

$$
\begin{equation*}
\min \left\{\left|C_{i}^{\prime}\right|,\left|D_{i}^{\prime}\right|\right\}=\min \left\{\left|C_{i}\right|,\left|D_{i}\right|\right\}-\left|M_{2}\right| \geq(1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n . \tag{8}
\end{equation*}
$$

Claim 4. We may assume that $k$ is odd.
For, suppose $k$ is even. We show that both $H^{\prime \prime}-C^{\prime}$ and $H^{\prime \prime}-D^{\prime}$ have perfect matchings; hence the assertion of the lemma holds. Below, we only show that $H^{\prime \prime}-C^{\prime}$ has a perfect matching, since the argument for $H^{\prime \prime}-D^{\prime}$ is the same (by substituting (6) for (7) and by exchanging the roles of $C_{i}^{\prime}$ and $\left.D_{i}^{\prime}\right)$.

Let $M$ be a maximum matching in $H^{\prime \prime}-C^{\prime}$. Then $\left(H^{\prime \prime}-C^{\prime}\right)-V(M)=H\left[D^{\prime}-V(M)\right]$ has no edge. We claim that $|M| \geq n / 4$. For, otherwise, $D_{1}^{\prime}-V(M) \neq \emptyset$ by (7) (as $\sqrt{\varepsilon}<1 /\left(100 k^{2}\right)$ and $\left.n \geq 100 k^{2}\right)$. Let $v \in D_{1}^{\prime}-V(M)$. Since $k$ is even and $D_{i}^{\prime} \subseteq B_{i}$ for $i \in[k]$ (by (5)), and because $H\left[D^{\prime}-V(M)\right]$ has no edge, we have

$$
\begin{aligned}
\left|N_{H_{0}(k, n)}(v)-N_{H}(v)\right| & \geq\left|D_{2}^{\prime}-V(M)\right|\left|D_{3}^{\prime}-V(M)\right| \cdots\left|D_{k}^{\prime}-V(M)\right| \\
& \geq\left((1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-n / 4\right)^{k-1} \quad(\text { by }(8)) \\
& >(n / 10)^{k-1} \quad\left(\text { since } \sqrt{\varepsilon}<1 /\left(100 k^{2}\right) \text { and } n \geq 100 k^{2}\right) \\
& >\sqrt{\varepsilon} n^{k-1} \quad\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right),\right.
\end{aligned}
$$

contradicting the fact that $v \notin N$.
Now, suppose for a contradiction, that $M$ is not a perfect matching in $H^{\prime \prime}-C^{\prime}$. Then there exists $u_{i} \in D_{i}^{\prime}-V(M)$ for $i \in[k]$. Note that $|M| \geq n / 4>k-1$ (as $n \geq 100 k^{2}$ ).

Let $\left\{e_{1}, \ldots, e_{k-1}\right\}$ be an arbitrary $(k-1)$-subset of $M$, and write $e_{i}:=\left\{v_{i, 1}, \ldots, v_{i, k}\right\}$ with $v_{i, j} \in D_{j}^{\prime}$ for $i \in[k-1]$ and $j \in[k]$. For $j \in[k]$, let $f_{j}:=\left\{u_{j}, v_{1, j+1}, v_{2, j+2}, \ldots\right.$, $\left.v_{k-1, j+k-1}\right\}$, with the addition in the subscripts modulo $k$ (except we write $k$ for 0 ). Note that $f_{1}, \ldots, f_{k}$ are pairwise disjoint. Since $D_{i}^{\prime} \subseteq B_{i}$ for $i \in[k]$ (by (5)), and $k$ is assumed to be even, it follows that $f_{j} \in E\left(H_{0}(k, n)\right)$ for $j \in[k]$.

If $f_{i} \in E\left(H^{\prime \prime}\right)$ for all $i \in[k]$ then $M^{\prime}:=\left(M \cup\left\{f_{1}, \ldots, f_{k}\right\}\right)-\left\{e_{1}, \ldots, e_{k-1}\right\}$ is a matching in $H$ and $\left|M^{\prime}\right|=|M|+1>|M|$, contradicting the maximality of $|M|$.

Hence, $f_{j} \notin E(H)$ for some $j \in[k]$. Note that there are $\binom{|M|}{k-1}$ choices of $\left\{e_{1}, \ldots, e_{k-1}\right\} \subseteq$ $M$. Hence,

$$
\begin{aligned}
& \left|\left\{e \in E\left(H_{0}(k, n)\right)-E(H):\left|e \cap\left\{u_{i}: i \in[k]\right\}\right|=1\right\}\right| \\
\geq & |M|(|M|-1) \cdots(|M|-(k-1)+1) /(k-1)! \\
> & (n / 4-k+2)^{k-1} /(k-1)! \\
> & (n / 10 k)^{k-1} \quad\left(\text { since } n \geq 100 k^{2}\right) \\
> & k \sqrt{\varepsilon} n^{k-1} \quad\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right) .\right.
\end{aligned}
$$

This implies that there exists $i \in[k]$ such that $\left|N_{H_{0}(k, n)}\left(u_{i}\right)-N_{H}\left(u_{i}\right)\right|>\sqrt{\varepsilon} n^{k-1}$, contradicting the fact that $u_{i} \notin N$.

Next claim guarantees a divisibility condition for $\left|D^{\prime}\right|$, which will be used in the proof of Claim 7.

Claim 5. There exists a matching $M_{3}$ in $H^{\prime \prime}$ such that
(i) $\left|M_{3}\right| \leq k^{2} / 2$, and
(ii) $\left|D_{i}^{\prime}-V\left(M_{3}\right)\right|=\left|D_{1}^{\prime}-V\left(M_{3}\right)\right| \equiv 0(\bmod k-1)$ for $i \in[k]$.

Let $0 \leq s \leq k-2$ be such that $\left|D_{1}^{\prime}\right| \equiv s(\bmod k-1)$. We may assume that $s \neq 0$; for, otherwise, $M_{3}=\emptyset$ gives the desired matching for Claim 5. Moreover, since $k$ is odd (by Claim 4) and $\left|D^{\prime}\right|=k\left|D_{1}^{\prime}\right|$ is even, it follows that $s$ is even.

We now construct $M_{3}$, starting with the empty matching $T_{0}=\emptyset$. Suppose for some $j \in[s / 2]$, we have constructed a matching $T_{j-1}$ in $H^{\prime \prime}$ with $\left|T_{j-1}\right|=k(j-1)$. Since $\sqrt{\varepsilon}<1 /\left(100 k^{2}\right)$ and $n \geq 100 k^{2}$, it follows from (8) that, for $i \in[k]$,

$$
\min \left\{\left|C_{i}^{\prime}-V\left(T_{j-1}\right)\right|,\left|D_{i}^{\prime}-V\left(T_{j-1}\right)\right|\right\} \geq(1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-k(j-1)>0 .
$$

For $i \in[k]$, let $v_{j, i} \in D_{i}^{\prime}-V\left(T_{j-1}\right)$. We claim that there exist $v_{j, i+1} \in D_{i+1}^{\prime}-V\left(T_{j-1}\right)$ and $u_{j, l} \in C_{l}^{\prime}-V\left(T_{j-1}\right)$ for $l \in[k]-\{i, i+1\}$, such that $e_{j, i}:=\left\{v_{j, i}, v_{j, i+1}, u_{j, l}: l \in\right.$ $[k]-\{i, i+1\}\} \in E\left(H^{\prime \prime}\right)$ (with addition in the subscripts modulo $k$ except we use $k$ for 0 ) and $\left\{e_{j, i}: i \in[k]\right\}$ is a matching in $H^{\prime \prime}$. For, otherwise, since $D_{i}^{\prime} \subseteq B_{i}$ by ((5)), we have

$$
\begin{aligned}
\left|N_{H_{0}(k, n)}\left(v_{j, i}\right)-N_{H}\left(v_{j, i}\right)\right| & \geq\left|D_{i+1}^{\prime}-V\left(T_{j-1}\right)-\cup_{i \in[k]} e_{j, i} \prod_{l \in[k]-\{i, i+1\}}\right| C_{l}^{\prime}-V\left(T_{j-1}\right) \mid \\
& \geq\left((1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-k^{2} / 2\right)^{k-1} \\
& >(n / 10)^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(100 k^{2}\right) \text { and } n \geq 100 k^{2}\right) \\
& >\sqrt{\varepsilon} n^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right),\right.
\end{aligned}
$$

contradicting the fact that $v_{j, i} \notin N$.
Let $T_{j}=T_{j-1} \cup\left\{e_{j, i}: i \in[k]\right\}$. Then $T_{j}$ is a matching in $H^{\prime \prime}$ for $j \in[s / 2]$. Let $M_{3}:=T_{s / 2}=\left\{e_{j, i}: j \in[s / 2]\right.$ and xsi $\left.\in[k]\right\}$. Then $\left|M_{3}\right| \leq k^{2} / 2$. Note that, for $i \in[k]$, the edges in $T_{j}-T_{j-1}$ uses exactly two vertices of $D_{i}^{\prime}$. Thus, for $i \in[k],\left|D_{i}^{\prime}-V\left(M_{3}\right)\right|=$ $\left|D_{1}^{\prime}-V\left(M_{3}\right)\right|=\left|D_{1}^{\prime}\right|-s \equiv 0(\bmod k-1)$.

Let $H^{*}:=H^{\prime \prime}-V\left(M_{3}\right)$ and, for $i \in[k]$, let $D_{i}^{*}:=D_{i}^{\prime}-V\left(M_{3}\right)$ and $C_{i}^{*}:=C_{i}^{\prime}-V\left(M_{3}\right)$. Let $D^{*}:=\cup_{i \in[k]} D_{i}^{*}$ and $C^{*}:=\cup_{i \in[k]} C_{i}^{*}$. Since $\left|M_{3}\right| \leq k^{2} / 2$ (by Claim 5), it follows from (8) that

$$
\begin{equation*}
\min \left\{\left|C_{i}^{*}\right|,\left|D_{i}^{*}\right|\right\} \geq \min \left\{\left|C_{i}^{\prime}\right|,\left|D_{i}^{\prime}\right|\right\}-\left|M_{3}\right| \geq(1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-k^{2} / 2 \tag{9}
\end{equation*}
$$

By Claim 5, $\left|D_{i}^{*}\right|=\left|D_{1}^{*}\right| \equiv 0(\bmod k-1)$ for $i \in[k]$.
We will show that $H^{*}$ has a perfect matching using edges of special types. For any $e \in E\left(H^{*}\right)$, if $e \subseteq C^{*}$ then we say that $e$ is of 0 -type, and if $\left|e \cap C^{*}\right|=\left|e \cap C_{j}^{*}\right|=1$ for some $j \in[k]$ then we say that $e$ is of $j$-type. For convenience, let

$$
\tau:=1 /(9 k)
$$

Claim 6. $H^{*}$ has pairwise disjoint matchings $M^{0}, M^{1}, \ldots, M^{k}$, such that for $i \in[k] \cup\{0\}$,
(i) $\left|M^{i}\right|=\lfloor\tau n\rfloor$, and
(ii) each edge in $M^{i}$ is of $i$-type.

We construct $M^{0}, M^{1}, \ldots, M^{k}$ in the order listed. Let $T^{0}$ be a matching in $H^{*}$ such that $V\left(T^{0}\right) \subseteq C^{*}$ and, subject to this, $\left|T^{0}\right|$ is maximum. Then $C^{*}-V\left(T^{0}\right)$ has no edge. We claim that $\left|T^{0}\right| \geq\lfloor\tau n\rfloor$; for, otherwise, $\left|C_{i}^{*}-V\left(T^{0}\right)\right| \geq\left|C_{i}^{*}\right|-\tau n$ for $i \in[k]$ and, hence, for any $v \in C_{1}^{*}-V\left(T^{0}\right)$,

$$
\begin{aligned}
\left|N_{H_{0}(k, n)}(v)-N_{H}(v)\right| & \geq\left|C_{2}^{*}-V\left(T^{0}\right)\right|\left|C_{3}^{*}-V\left(T^{0}\right)\right| \cdots\left|C_{k}^{*}-V\left(T^{0}\right)\right| \\
& \geq\left((1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-k^{2} / 2-\tau n\right)^{k-1}(\text { by }(9)) \\
& >(n / 10)^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(100 k^{2}\right) \text { and } n \geq 100 k^{2}\right) \\
& >\sqrt{\varepsilon} n^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right),\right.
\end{aligned}
$$

contradicting the fact that $v \notin N$. Let $M^{0}$ be a set of any $\lfloor\tau n\rfloor$ edges in $T^{0}$.
Now suppose for some $j \in[k]$, we have found matchings $M^{0}, M^{1}, \ldots, M^{j-1}$ in $H^{*}$ such that $M^{i}$ (for $i=0, \ldots, j-1$ ) consists of $\lfloor\tau n\rfloor$ edges of $i$-type. Let $T^{j}$ be a matching in $H^{*}-\cup_{i=0}^{j-1} V\left(M_{i}\right)$ such that each edge in $T_{j}$ is of $j$-type and, subject to this, $\left|T^{j}\right|$ is maximum.

We claim that $\left|T^{j}\right| \geq\lfloor\tau n\rfloor$. For, suppose $\left|T^{j}\right|<\lfloor\tau n\rfloor$. Then, since $C_{j}^{*} \cap V\left(M^{i}\right)=\emptyset$ for $i \in[j-1]$ and $\left|V\left(M^{0}\right)\right|=\lfloor\tau n\rfloor$, it follows from (9) that
$\left|C_{j}^{*}-V\left(M^{0} \cup M^{1} \cup \ldots \cup M^{j-1}\right)-V\left(T^{j}\right)\right|>(1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-k^{2} / 2-k \tau n>0$,
where the second inequality holds because $\tau=1 /(9 k), \sqrt{\varepsilon}<1 /\left(100 k^{2}\right)$, and $n \geq 100 k^{2}$. . So let $v$ be a vertex in $C_{j}^{*}-V\left(M^{0} \cup M^{1} \cup \ldots \cup M^{j-1}\right)-V\left(T^{j}\right)$. We claim that there exists an edge $f$ of $j$-type in $H^{*}-V\left(M^{0} \cup M^{1} \cup \ldots \cup M^{j-1}\right)-V\left(T^{j}\right)$ with $v \in f$; as, otherwise, since $D_{i}^{*} \subseteq B_{i}$ for $i \in[k]$ (by (5)) and $k$ is odd,

$$
\begin{aligned}
\left|N_{H_{0}(k, n)}(v)-N_{H}(v)\right| & \geq \prod_{l \in[k]-\{j\}}\left|D_{l}^{*}-V\left(M^{0} \cup M^{1} \cup \ldots \cup M^{j-1}\right)-V\left(T^{j}\right)\right| \\
& \geq\left((1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-k^{2} / 2-k \tau n\right)^{k-1}(\text { by (9) }) \\
& >(n / 10)^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(100 k^{2}\right), \tau=1 /(9 k) \text { and } n \geq 100 k^{2}\right) \\
& >\sqrt{\varepsilon} n^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right),\right.
\end{aligned}
$$

contradicting the fact that $v \notin N$.
Let $M^{j} \subseteq T^{j}$ with $\left|M^{j}\right|=\lfloor\tau n\rfloor$. Thus, this process works for all $j \in[k]$, and we see that $M^{0}, M^{1}, \ldots, M^{k}$ give the desired matchings for Claim 6.

By Claim 6, there exist pairwise disjoint matchings $M^{0}, M^{1}, \ldots, M^{k}$ in $H^{*}$ such that

- $M^{i}$ is of $i$-type for $i \in[k] \cup\{0\}$, and
- $\left|M^{i}\right|=\left|M^{1}\right| \geq\lfloor\tau n\rfloor$ for $i \in[k]$.

We choose such $M^{0}, M^{1}, \ldots, M^{k}$ that

- $\left|M^{1}\right|=\left|M^{2}\right|=\ldots=\left|M^{k}\right|$ is maximum and, subject to this,
- $\left|M^{0}\right|$ is maximum.

Let $M=\bigcup_{i \in[k] \cup\{0\}} M^{i}$. By Claim 6, we have, for $i \in[k]$,

$$
\left|D_{i}^{*} \cap V(M)\right| \equiv 0 \quad(\bmod k-1) \text { and }\left|M^{i}\right| \leq\left|D_{i}^{*}\right| /(k-1) .
$$

Claim 7. $\left|M^{0}\right| \geq \tau n$.
For, otherwise, suppose $\left|M^{0}\right|<\tau n$. Note that for $i \in[k]$,

$$
\begin{aligned}
& \left|C_{i}^{*}-V\left(M^{0} \cup M^{1} \cup \cdots \cup M^{k}\right)\right| \\
= & \left|C_{i}^{*}\right|-\left|M^{0}\right|-\left|M^{i}\right| \\
> & \left.\left|C_{i}^{*}\right|-\tau n-\left|D_{i}^{*}\right| /(k-1) \quad \text { (since }\left|M^{0}\right|<\tau n \text { and }\left|M^{i}\right| \leq\left|D_{i}^{*}\right| /(k-1)\right) \\
\geq & \left|C_{i}^{*}\right|-\tau n-\left(n-\left|C_{i}^{*}\right|\right) /(k-1) \\
= & k\left|C_{i}^{*}\right| /(k-1)-\tau n-n /(k-1) \\
\geq & \left.\left((1 / 2-2 \sqrt{\varepsilon} k) n-2-8 \sqrt{\varepsilon} k^{2} n-k^{2} / 2\right) k /(k-1)-\tau n-n /(k-1) \quad \text { (by (9) }\right) \\
> & n / 10\left(\text { since } \sqrt{\varepsilon}<1 /\left(100 k^{2}\right), \tau=1 /(9 k) \text { and } n \geq 100 k^{2}\right) .
\end{aligned}
$$

Thus there exists $v \in C_{1}^{*}-V\left(M^{0} \cup M^{1} \cup \cdots \cup M^{k}\right)$. Since $\left|M^{0}\right|$ is maximized, $C^{*}-$ $V\left(M^{0} \cup M^{1} \cup \cdots \cup M^{k}\right)$ has no edge. Therefore,

$$
\begin{aligned}
\left|N_{H_{0}(k, n)}(v)-N_{H}(v)\right| & \geq \prod_{i \in[k]-\{1\}}\left|C_{i}^{*}-V\left(M^{0} \cup M^{1} \cup \cdots \cup M^{k}\right)\right| \\
& >(n / 10)^{k-1} \\
& >\sqrt{\varepsilon} n^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right)\right.
\end{aligned}
$$

contradicting the fact that $v \notin N$.
Claim 8. $D^{*} \subseteq V(M)$.
For, otherwise, suppose that $D^{*}-V(M) \neq \emptyset$. Recall that for each $j$-type edge $f$, $\left|f \cap C^{*}\right|=\left|f \cap C_{j}\right|=1$. Since $\left|D_{i}^{*}\right| \equiv 0(\bmod k-1)\left(\right.$ by Claim 5) and $\left|D_{i}^{*} \cap V(M)\right| \equiv$ $0(\bmod k-1)$, it follows that $\left|D_{i}^{*}-V(M)\right| \geq k-1$ for $i \in[k]$. So, for $i \in[k]$, let $s_{i, 1}, s_{i, 2}, \ldots, s_{i, k-1} \in D_{i}^{*}-V(M)$ be distinct.

When $C_{i}^{*}-V(M) \neq \emptyset$ for $i \in[k]$, let $w_{i} \in C_{i}^{*}-V(M)$ for $i \in[k]$; otherwise let $\left\{w_{1}, \ldots, w_{k}\right\} \in M^{0}$ with $w_{i} \in C_{i}^{*}$ for $i \in[k]$ (by Claim 7). Let $S_{j}:=\left\{w_{j}, s_{i, j}: i \in[k]-\{j\}\right\}$ for $j \in[k-1]$, and let $S_{k}:=\left\{w_{k}, s_{i, i}: i \in[k-1]\right\}$.

Suppose for each $j \in[k]$ there exist distinct $e_{1}^{j}, \ldots, e_{k-1}^{j} \in M^{j}$ such that $H^{*}\left[e_{1}^{j} \cup \cdots \cup\right.$ $\left.e_{k-1}^{j} \cup S_{j}\right]$ contains a perfect matching $\left\{f_{1}^{j}, \ldots, f_{k}^{j}\right\}$. Then, $N^{j}:=\left(M^{j}-\left\{e_{1}^{j}, \ldots, e_{k-1}^{j}\right\}\right) \cup$ $\left\{f_{1}^{j}, \ldots, f_{k}^{j}\right\}$ is a matching in $H^{*}$ for each $j \in[k]$, and $\left|N^{j}\right|=\left|N^{1}\right|>\lfloor\tau n\rfloor$ for $j \in[k]$. Let $N^{0}=M^{0}-\left\{\left\{w_{1}, \ldots, w_{k}\right\}\right\}$. Then $N^{0}, N^{1}, \ldots, N^{k}$ are pairwise disjoint. However, $\left|N^{j}\right|=\left|M^{j}\right|+1$ for $j \in[k]$, contradicting the choice of $M^{0}, M^{1}, \ldots, M^{k}$.

Thus we may assume without loss of generality that for any $k-1$ distinct edges $e_{1}^{k}, \ldots, e_{k-1}^{k} \in M^{k}, H^{*}\left[e_{1}^{k} \cup \cdots \cup e_{k-1}^{k} \cup S_{k}\right]$ has no perfect matching. For $i \in[k-1]$, let $e_{i}^{k}:=\left\{v_{i, 1}, v_{i, 2}, \ldots, v_{i, k}\right\}$ with $v_{i, k} \in C_{k}^{*}$ and $v_{i, j} \in D_{j}^{*}$ for $j \in[k-1]$. For convenience, let $v_{k, k}:=w_{k}$ and $v_{k, j}:=s_{j, j}$ for $j \in[k-1]$. For $i \in[k]$, define $f_{i}^{k}:=$
$\left\{v_{1, i+1}, v_{2, i+2}, \ldots, v_{k-1, i+k-1}, v_{k, k+i}\right\}$, where the addition in the subscripts is modulo $k$ (except that we write $k$ for 0$)$. Then $f_{i}^{k} \notin E\left(H^{*}\right)$ for some $i \in[k]$, as otherwise, $\left\{f_{1}^{k}, \ldots, f_{k}^{k}\right\}$ would be a perfect matching in $H^{*}\left[e_{1}^{k} \cup \cdots \cup e_{k-1}^{k} \cup S_{k}\right]$. Since $e_{1}^{k}, \ldots, e_{k-1}^{k} \in M^{k}$ are chosen arbitrarily and $k$ is odd (by Claim 5), we have

$$
\begin{aligned}
& \left|\left\{e \in E\left(H_{0}(k, n)\right)-E(H):\left|e \cap\left\{v_{k, i}: i \in[k]\right\}\right|=1\right\}\right| \\
\geq & \binom{\left|M^{k}\right|}{k-1} \\
= & \left|M^{k}\right|\left|M^{k}-1\right| \cdots\left|M^{k}-(k-1)+1\right| /(k-1)! \\
> & ((\lfloor\tau n\rfloor-k) /(k-1))^{k-1} \\
> & \left(n /\left(10 k^{2}\right)\right)^{k-1}\left(\text { since } \tau=1 /(9 k) \text { and } n \geq 100 k^{2}\right) \\
> & k \sqrt{\varepsilon} n^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right) .\right.
\end{aligned}
$$

So there exists $i \in[k]$ such that $\left|N_{H_{0}(k, n)}\left(v_{k, i}\right)-N_{H}\left(v_{k, i}\right)\right|>\sqrt{\varepsilon} n^{k-1}$, contradicting the fact that $v_{k, i} \notin N$.

If $C^{*} \subseteq V(M)$ then, by Claim $8, M$ is a perfect matching in $H^{*}$; so $\left\{e_{0}\right\} \cup M_{1} \cup M_{2} \cup$ $M_{3} \cup M$ is a perfect matching in $H$.

Therefore, we may assume that $C^{*} \nsubseteq V(M)$, and let $w_{i} \in C_{i}^{*}-V(M)$ for $i \in[k]$. Note that $\left|M^{0}\right| \geq \tau n>k-1$ (by Claim 7). Let $e_{1}, \ldots, e_{k-1} \in M^{0}$ be distinct and chosen arbitrarily. Let $e_{i}:=\left\{v_{i, 1}, v_{i, 2}, \ldots, v_{i, k}\right\}$ for $i \in[k-1]$, where $v_{i, j} \in C_{j}^{*}$ for $j \in[k]$. For $i \in[k]$, define $f_{i}:=\left\{w_{i}, v_{1, i+1}, v_{2, i+2}, \ldots, v_{k-1, i+k-1}\right\}$, with the addition in the subscripts taken modulo $k$ (except we use $k$ for 0 ).

If $f_{i} \in E\left(H^{*}\right)$ for all $i \in[k]$, then $N^{0}:=\left(M^{0} \cup\left\{f_{1}, \ldots, f_{k}\right\}\right)-\left\{e_{1}, \ldots, e_{k-1}\right\}$ is a matching in $H^{*}$ with $\left|N^{0}\right|=\left|M^{0}\right|+1$; so $N^{0}, M^{1}, \ldots, M^{k}$ contradict the choice of $M^{0}, M^{1}, \ldots, M^{k}$.

Hence, $f_{i} \notin E\left(H^{*}\right)$ for some $i \in[k]$. Since $e_{1}, \ldots, e_{k-1} \in M^{0}$ are chosen arbitrarily and $k$ is odd, we have

$$
\begin{aligned}
& \left|\left\{e \in E\left(H_{0}(k, n)\right)-E(H):\left|e \cap\left\{w_{i}: i \in[k]\right\}\right|=1\right\}\right| \\
\geq & \binom{\left|M^{0}\right|}{k-1} \\
= & \left|M^{0}\right|\left|M^{0}-1\right| \cdots\left|M^{0}-(k-1)+1\right| /(k-1)! \\
> & ((\lfloor\tau n\rfloor-k) / k)^{k-1} \\
> & \left(n /\left(10 k^{2}\right)\right)^{k-1}\left(\text { since } \tau=1 /(9 k) \text { and } n \geq 100 k^{2}\right) \\
> & k \sqrt{\varepsilon} n^{k-1}\left(\text { since } \sqrt{\varepsilon}<1 /\left(k\left(10 k^{2}\right)^{k-1}\right) .\right.
\end{aligned}
$$

So there exists $i \in[k]$ such that $\left|N_{H_{0}(k, n)}\left(w_{i}\right)-N_{H}\left(w_{i}\right)\right|>\sqrt{\varepsilon} n^{k-1}$, contradicting the fact that $w_{i} \notin N$.

Corollary 2.2 Let $k \geq 3$ be a positive integer, and let $\varepsilon>0$ be such that $\sqrt{\varepsilon}<\min \left\{1 /\left(100 k^{2}\right)\right.$, $\left.1 /\left(k\left(10 k^{2}\right)^{k-1}\right)\right\}$. Let $H$ be a $k$-partite $k$-graph with $n>100 k^{2}$ vertices in each partition class, such that $\delta_{k-1}(H) \geq\lfloor n / 2\rfloor$ and $H$ is $\varepsilon$-close to $H_{0}(k, n)$. Then $H$ has no perfect matching if, and only if,
(i) $k$ is odd, $n \equiv 2(\bmod 4)$, and $H \cong H_{0}(k, n)$, or
(ii) $n$ is odd and there exist $d_{i} \in\{(n+1) / 2,(n-1) / 2\}$ for $i \in[k]$ such that $\sum_{i=1}^{k} d_{i}$ is odd and $H \subseteq H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$.

Proof. Let $H$ be a $k$-partite $k$-graph with $n$ vertices in each partition class, such that $\delta_{k-1}(H) \geq\lfloor n / 2\rfloor$ and $H$ is $\varepsilon$-close to $H_{0}(k, n)$.

Suppose ( $i$ ) or (ii) holds. Then there exist integers $d_{1}, \ldots, d_{k}$ such that $\sum_{i=1}^{k} d_{i}$ is odd, $H \subseteq H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right), d_{1}=\ldots=d_{k}=n / 2$ when $(i)$ holds, and $d_{i} \in\{(n+1) / 2,(n-1) / 2\}$ when (ii) holds. By the definition of $H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$, there exists $D \subseteq V(H)$ such that $|D|=\sum_{i=1}^{k} d_{i}$ is odd and $|e \cap D|$ is even for all $e \in E(H)$. Hence, $H$ contains no perfect matching.

Next, suppose $H$ has no perfect matching. Applying Lemma 2.1 with $\alpha=1 / 8$, we may assume that there exist $d_{i} \in[\lceil 3 n / 8\rceil,\lfloor 5 n / 8\rfloor]$ for $i \in[k]$ such that $\sum_{i=1}^{k} d_{i}$ is odd and $H \subseteq$ $H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$. Let $V_{1}, \ldots, V_{k}$ be the partition classes of $H$ and $H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)$. For $i \in[k]$, let $D_{i} \subseteq V_{i}$ be such that $\left|D_{i}\right|=d_{i}$ and $\left|e \cap\left(\cup_{j \in[k]} D_{j}\right)\right|$ is even for all $e \in E(H)$.

We claim that $\delta_{k-1}(H) \leq \min \left\{d_{i}, n-d_{i}\right\}$ for all $i \in[k]$. By symmetry, we only show $\delta_{k-1}(H) \leq \min \left\{d_{1}, n-d_{1}\right\}$. Let $S:=\left\{v_{2}, \ldots, v_{k}\right\}$ be a legal set such that $v_{2} \in D_{2}$ and $v_{i} \in V_{i}-D_{i}$ for $i \in[k]-\{1,2\}$; then, since $e \cap D_{1} \neq \emptyset$ for all $e \in E(H)$ with $S \subseteq e, \delta_{k-1}(H) \leq d_{H}(S) \leq\left|D_{1}\right|=d_{1}$. Let $T:=\left\{u_{2}, \ldots, u_{k}\right\}$ be a legal set such that $u_{i} \in V_{i}-D_{i}$ for $i \in[k]-\{1\}$; then, since $e \cap D_{1}=\emptyset$ for any $e \in E(H)$ with $T \subseteq e$, $\delta_{k-1}(H) \leq d_{H}(T) \leq\left|V_{1}-D_{1}\right|=n-d_{1}$. Hence, $\delta_{k-1}(H) \leq \min \left\{d_{1}, n-d_{1}\right\}$.

If $n$ is odd then $\delta_{k-1}(H) \geq\lfloor n / 2\rfloor=(n-1) / 2$; so by the above claim, $d_{i} \in\{(n-$ $1) / 2,(n+1) / 2\}$ for all $i \in[k]$, and (ii) holds. Thus, we may assume that $n$ is even. Then by the above claim, $d_{i}=n / 2$ for all $i \in[k]$. Recall that $\sum_{i=1}^{k} d_{i}$ is odd. Thus both $n / 2$ and $k$ are odd, and hence $n \equiv 2(\bmod 4)$. Since $H \subseteq H_{0}\left(d_{1}, \ldots, d_{k} ; k, n\right)=H_{0}(k, n)$, we have $H=H_{0}(k, n)$ and (i) holds.

## 3 Hypergraphs not close to $H_{0}(k, n)$

In this section, we prove Theorem 1.2 for hypergraphs that are not close to $H_{0}(k, n)$, see Lemma 3.6. For this, we need a result on almost perfect matchings in $k$-partite $k$-graphs.

Kühn and Osthus [4] showed that if $H$ is a $k$-partite $k$-graph with each partition classes of size $n$ and $\delta_{k-1}(H) \geq n / k$, then $H$ has a matching of size at least $n-(k-2)$. Rödl and Ruciński [8] asked the following question: Is it true that $\delta_{k-1}(H) \geq n / k$ implies that $H$ has a matching of size at least $n-1$ ? The present authors [6] and, independently, Han, Zang, and Zhao [3] answered this question affirmatively for large $n$.

Lemma 3.1 Let $k$, $n$ be positive integers with $k \geq 3$ and $n$ sufficiently large, and let $H$ be $a k$-partite $k$-graph with $n$ vertices in each partition class. If $\delta_{k-1}(H) \geq n / k$, then $H$ has a matching of size at least $n-1$.

Let $k \geq 2$ be a positive integer and $H$ be a $k$-partite $k$-graph with partition classes $V_{1}, \ldots, V_{k}$. Given $N_{i} \subseteq V_{i}$ for $i \in[k]$, let

$$
E_{H}\left(N_{1}, \ldots, N_{k}\right):=\left\{e \in E(H): e \subseteq \cup_{i \in[k]} N_{i}\right\},
$$

and

$$
e_{H}\left(N_{1}, \ldots, N_{k}\right):=\left|E_{H}\left(N_{1}, \ldots, N_{k}\right)\right| .
$$

For $j \in[k]$, let

$$
\Lambda_{j}:=\left\{\left\{v_{i} \in V_{i}: i \in[k]-\{j\}\right\}: d_{H}\left(\left\{v_{i}: i \in[k]-\{j\}\right\}\right) \geq(1 / 2+2 / \log n) n\right\}
$$

Lemma 3.2 Let $k \geq 2$ be a positive integer. For any $\varepsilon>0$, there exists $n_{0}>0$ such that the following holds. Let $H$ be a $k$-partite $k$-graph with partition classes $V_{1}, \ldots, V_{k}$ such that $\left|V_{i}\right|=n \geq n_{0}$ for $i \in[k]$ and $\delta_{k-1}(H) \geq(1 / 2-1 / \log n) n$. Suppose $H$ is not $\varepsilon$-close to $H_{0}(k, n)$. Then one of the following conclusions holds:
(i) For all $i \in[k]$ and $N_{i} \subseteq V_{i}$ with $\left|N_{i}\right| \geq(1 / 2-1 / \log n) n, e_{H}\left(N_{1}, \ldots, N_{k}\right) \geq n^{k} / \log ^{3} n$.
(ii) There exists $j \in[k]$ such that $\left|\Lambda_{j}\right| \geq n^{k-1} / \log n$.

Proof. Let $H$ be a $k$-partite $k$-graph with partition classes $V_{1}, \ldots, V_{k}$ such that $\left|V_{i}\right|=n$ for $i \in[k]$. For convenience, let $\gamma:=1 / \log n$. Then $\delta_{k-1}(H) \geq(1 / 2-\gamma) n$.

Suppose $H$ is not $\varepsilon$-close to $H_{0}(k, n)$, and assume that neither ( $i$ ) nor (ii) holds. Then there exist $N_{1}, \ldots, N_{k}$ with $N_{i} \subseteq V_{i}$ and $\left|N_{i}\right| \geq(1 / 2-\gamma) n$ for $i \in[k]$ such that

$$
\begin{equation*}
e_{H}\left(N_{1}, \ldots, N_{k}\right)<\frac{n^{k}}{\log ^{3} n}=o\left(n^{k}\right), \tag{10}
\end{equation*}
$$

and, for all $j \in[k]$,

$$
\begin{equation*}
\left|\Lambda_{j}\right|<\frac{n^{k-1}}{\log n}=\gamma n^{k-1} \tag{11}
\end{equation*}
$$

Claim 1. $\left|N_{i}\right|<(1 / 2+2 \gamma) n$ for $i \in[k]$.
For, otherwise, we may assume without loss of generality that $\left|N_{k}\right| \geq(1 / 2+2 \gamma) n$. Then $\left|V_{k}-N_{k}\right| \leq(1 / 2-2 \gamma) n$. For any legal $(k-1)$-set $\left\{v_{1}, \ldots, v_{k-1}\right\}$ with $v_{i} \in N_{i}$ for $i \in[k-1]$, we have

$$
\left|N_{H}\left(v_{1}, \ldots, v_{k-1}\right) \cap N_{k}\right| \geq \delta_{k-1}(H)-\left|V_{k}-N_{k}\right| \geq(1 / 2-\gamma) n-(1 / 2-2 \gamma) n=\gamma n .
$$

Hence, by choosing $n_{0}$ large enough, we have for $n \geq n_{0}$,

$$
e_{H}\left(N_{1}, \ldots, N_{k}\right) \geq\left|N_{1}\right| \cdots\left|N_{k-1}\right|\left|N_{H}\left(\left\{v_{1}, \ldots, v_{k-1}\right\}\right) \cap N_{k}\right| \geq((1 / 2-\gamma) n)^{k-1} \gamma n>\frac{n^{k}}{\log ^{3} n},
$$

contradicting (10).
For $i \in[k]$, let $N_{i}^{\prime}:=V_{i}-N_{i}$ and $A_{i} \in\left\{N_{i}, N_{i}^{\prime}\right\}$. Since $\left|N_{i}\right| \geq(1 / 2-\gamma) n,\left|N_{i}^{\prime}\right| \leq$ $(1 / 2+\gamma) n$. By Claim 1, $\left|N_{i}^{\prime}\right|>(1 / 2-2 \gamma) n$. Therefore, for $i \in[k]$,

$$
\begin{equation*}
(1 / 2-2 \gamma) n<\left|A_{i}\right| \leq(1 / 2+2 \gamma) n . \tag{12}
\end{equation*}
$$

Claim 2. For $i \in[k], e_{H}\left(A_{1}, \ldots, A_{i-1}, V_{i}, A_{i+1}, \ldots, A_{k}\right)=(n / 2)^{k}+o\left(n^{k}\right)$.

By symmetry, we only prove Claim 2 for the case when $i=k$. Note that

$$
\begin{aligned}
e_{H}\left(A_{1}, \ldots, A_{k-1}, V_{k}\right) & \geq\left(\prod_{i=1}^{k-1}\left|A_{i}\right|\right)(1 / 2-\gamma) n \quad\left(\text { since } \delta_{k-1}(H) \geq(1 / 2-\gamma) n\right) \\
& \left.\geq((1 / 2-2 \gamma) n)^{k-1}(1 / 2-\gamma) n \quad(\text { by } 12)\right) \\
& =(n / 2)^{k}+o\left(n^{k}\right)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
e_{H}\left(A_{1}, \ldots, A_{k-1}, V_{k}\right) & \leq\left|\Lambda_{k}\right| n+\left(\prod_{i=1}^{k-1}\left|A_{j}\right|\right)(1 / 2+2 \gamma) n \\
& \left.<\gamma n^{k}+((1 / 2+2 \gamma) n)^{k-1}(1 / 2+2 \gamma) n \quad \text { (by (11) and (12) }\right) \\
& =(n / 2)^{k}+o\left(n^{k}\right) .
\end{aligned}
$$

Claim 3. Let $I\left(A_{1}, \ldots, A_{k}\right):=\left\{i \in[k]: A_{i}=N_{i}^{\prime}\right\}$.
(i) If $\left|I\left(A_{1}, \ldots, A_{k}\right)\right|$ is odd then $e_{H}\left(A_{1}, \ldots, A_{k}\right)=(n / 2)^{k}+o\left(n^{k}\right)$, and
(ii) if $\left|I\left(A_{1}, \ldots, A_{k}\right)\right|$ is even then $e_{H}\left(A_{1}, \ldots, A_{k}\right)=o\left(n^{k}\right)$.

We apply induction on $\left|I\left(A_{1}, \ldots, A_{k}\right)\right|$. When $\left|I\left(A_{1}, \ldots, A_{k}\right)\right|=0$, we have $e_{H}\left(A_{1}, \ldots\right.$, $\left.A_{k}\right)=e_{H}\left(N_{1}, \ldots, N_{k}\right)=o\left(n^{k}\right)$ by 10$)$. When $\left|I\left(A_{1}, \ldots, A_{k}\right)\right|=1$, say $A_{i}=N_{i}^{\prime}$ and $A_{j}=N_{j}$ for $j \in[k]-\{i\}$, then we have

$$
\begin{aligned}
& e_{H}\left(A_{1}, \ldots, A_{k}\right) \\
= & e_{H}\left(A_{1}, \ldots, A_{i-1}, V_{i}, A_{i+1}, \ldots, A_{k}\right)-e_{H}\left(N_{1}, \ldots, N_{i-1}, N_{i}, N_{i+1}, \ldots, N_{k}\right) \\
= & (n / 2)^{k}+o\left(n^{k}\right)
\end{aligned}
$$

by Claim 2 and (10).
Now assume Claim 3 holds for $A_{1}, \ldots, A_{k}$ with $A_{i} \in\left\{N_{i}, N_{i}^{\prime}\right\}$ and $0 \leq\left|I\left(A_{1}, \ldots, A_{k}\right)\right|=$ $l<k$. Consider a choice of $A_{i} \in\left\{N_{i}, N_{i}^{\prime}\right\}$ for $i \in[k]$ with $\left|I\left(A_{1}, \ldots, A_{k}\right)\right|=l+1$. Let $A_{j}=N_{j}^{\prime}$ for some $j \in[k]$. Observe that
$e_{H}\left(A_{1}, \ldots, A_{k}\right)=e_{H}\left(A_{1}, \ldots, A_{j-1}, V_{j}, A_{j+1}, \ldots, A_{k}\right)-e_{H}\left(A_{1}, \ldots, A_{j-1}, N_{j}, A_{i+1}, \ldots, A_{k}\right)$.
Therefore, by (10) and Claim 2, it follows from the induction hypothesis that if $l+1$ is odd then $l$ is even and $e_{H}\left(A_{1}, \ldots, A_{k}\right)=\left((n / 2)^{k}+o\left(n^{k}\right)\right)-o\left(n^{k}\right)=(n / 2)^{k}+o\left(n^{k}\right)$, and if $l+1$ is even then $l$ is odd and $e_{H}\left(A_{1}, \ldots, A_{k}\right)=\left((n / 2)^{k}+o\left(n^{k}\right)\right)-\left((n / 2)^{k}+o\left(n^{k}\right)\right)=o\left(n^{k}\right)$.

For $i \in[k]$, let $B_{i} \subseteq V_{i}$ be such that $\left|B_{i}\right|=\lfloor n / 2\rfloor$ with $\left|B_{i} \cap N_{i}\right|$ maximal, and let $B_{i}^{\prime}:=V_{i}-B_{i}$. By (12), for $i \in[k],\left|B_{i}-N_{i}\right| \leq 2 \gamma n$ and $\left|B_{i}^{\prime}-N_{i}^{\prime}\right| \leq 2 \gamma n$. Hence, if $A_{i} \in\left\{N_{i}, N_{i}^{\prime}\right\}$ and $C_{i} \in\left\{B_{i}, B_{i}^{\prime}\right\}$ such that for $i \in[k], A_{i}=N_{i}$ iff $C_{i}=B_{i}$, then

$$
\left|E_{H}\left(A_{1}, \ldots, A_{k}\right)-E_{H}\left(C_{1}, \ldots, C_{k}\right)\right|=o\left(n^{k}\right)
$$

and

$$
\begin{aligned}
& \left|E_{H_{0}(k, n)}\left(C_{1}, \ldots, C_{k}\right)-E_{H}\left(C_{1}, \ldots, C_{k}\right)\right| \\
\leq & \left|E_{H_{0}(k, n)}\left(C_{1}, \ldots, C_{k}\right)-E_{H}\left(A_{1}, \ldots, A_{k}\right)\right|+\left|E_{H}\left(A_{1}, \ldots, A_{k}\right)-E_{H}\left(C_{1}, \ldots, C_{k}\right)\right| \\
= & \left|E_{H_{0}(k, n)}\left(C_{1}, \ldots, C_{k}\right)-E_{H}\left(A_{1}, \ldots, A_{k}\right)\right|+o\left(n^{k}\right) .
\end{aligned}
$$

For $i \in[k]$, let $B_{i}$ play the role of $D_{i}$ in the definition of $H_{0}(k, n)$. Then, for any $\varepsilon>0$,

$$
\begin{aligned}
& \leq \sum_{\substack{C_{i} \in\left\{B_{i}, B_{i}^{\prime}\right\}, i \in[k]}}\left|E\left(E_{H_{0}(k, n)}(k, n)\right)-E(H)\right| \\
& \leq \sum_{\substack{C_{i} \in\left\{B_{i}, B_{i}^{\prime},, A_{i} \in\left\{N_{i}, N_{i}^{\prime}\right\} \\
A_{i}=N_{i} \text { if } C_{i}=C_{i} \text { for } i \in[k]\right.}}\left(\left|E_{H_{0}(k, n)}\left(C_{1}, \ldots, C_{k}\right)-E_{H}\left(A_{1}, \ldots, A_{k}\right)\right|+o\left(n^{k}\right)\right) \\
& \leq \sum_{C_{i} \in\left\{B_{i}, B_{i}^{\prime}\right\} \text { for }}\left(\left(\sum_{i \in[k]} \sum_{i \in[k]} \sum_{v \in\left(B_{i}-N_{i}\right) \cup\left(B_{i}^{\prime}-N_{i}^{\prime}\right)}\left|N_{H_{0}(k, n)}(v)\right|\right)+o\left(n^{k}\right)\right) \\
& \left.\leq 2^{k}\left(k(4 \gamma n) n^{k-1}+o\left(n^{k}\right)\right) \quad \text { (since }\left|B_{i}-N_{i}\right| \leq 2 \gamma n \text { and }\left|B_{i}^{\prime}-N_{i}^{\prime}\right| \leq 2 \gamma n\right) \\
& \leq \varepsilon n^{k} \quad\left(\text { since } \gamma=1 / \log n \text { and we may choose } n_{0}\right. \text { large enough). }
\end{aligned}
$$

However, this contradicts the assumption that $H$ is not $\varepsilon$-close to $H_{0}(k, n)$.
Next, we define two "absorbing" matchings for a legal $k$-set $S$ in a $k$-partite $k$-graph. This concept was first considered by Rödl, Ruciński, and Szemerédi 9. Let $k \geq 3$ be a positive integer and $H$ be a $k$-partite $k$-graph.

Given a legal $k$-set $S=\left\{x_{1}, \ldots, x_{k}\right\}$ in a $k$-partite $k$-graph $H$, a $k$-matching $\left\{e_{1}, \ldots, e_{k}\right\}$ in $H$ is said to be $S$-absorbing if there is a $(k+1)$-matching $\left\{e_{1}^{\prime}, \ldots, e_{k}^{\prime}, f\right\}$ in $H$ with $f=\left\{y_{1}, \ldots, y_{k}\right\}$ such that

- $e_{i}^{\prime} \cap e_{j}=\emptyset$ for all $i \neq j$,
- $e_{i}^{\prime}-e_{i}=\left\{x_{i}\right\}$ and $e_{i}-e_{i}^{\prime}=\left\{y_{i}\right\}$ for $i \in[k]$.

Figure 1 illustrates an $\left\{x_{1}, x_{2}, x_{3}\right\}$-absorbing 3 -matching $\left\{e_{1}, e_{2}, e_{3}\right\}$.
Given a legal $k$-set $S=\left\{x_{1}, \ldots, x_{k}\right\}$ in a $k$-partite $k$-graph $H$, a $(k+1)$-matching $\left\{e_{0}, e_{1}, \ldots, e_{k}\right\}$ in $H$ is said to be $S$-absorbing if there is a ( $k+2$ )-matching $\left\{e_{1}^{\prime}, \ldots, e_{k}^{\prime}, f^{\prime}, f^{\prime \prime}\right\}$ in $H$, with $e_{1} \cap f^{\prime}=f^{\prime}-e_{0}=\left\{y_{1}\right\}, e_{0}-f^{\prime}=\left\{y_{0}\right\}$, and $f^{\prime \prime}:=\left\{y_{0}, y_{2}, \ldots, y_{k}\right\}$, such that

- $e_{i}^{\prime} \cap e_{j}=\emptyset$ for all $i \neq j$, and
- $e_{i}^{\prime}-e_{i}=\left\{x_{i}\right\}$ and $e_{i}-e_{i}^{\prime}=\left\{y_{i}\right\}$ for all $i \in[k]$.

Figure 2 illustrates an $\left\{x_{1}, x_{2}, x_{3}\right\}$-absorbing 4-matching $\left\{e_{0}, e_{1}, e_{2}, e_{3}\right\}$.
The next result says that no matter which conclusion of Lemma 3.2 holds, there are always many $S$-absorbing matchings in $H$ for any given legal $k$-set $S$.


Figure 1: $\left\{x_{1}, x_{2}, x_{3}\right\}$-absorbing matchings

Lemma 3.3 Let $k \geq 3$ be a positive integer. There exists $n_{1}>0$ such that the following holds. Let $H$ be a $k$-partite $k$-graph with $n \geq n_{1}$ vertices in each partition class and with $\delta_{k-1}(H) \geq(1 / 2-1 / \log n) n$. Let $S:=\left\{x_{1}, \ldots, x_{k}\right\} \subseteq V(H)$ be legal.
(i) If for all $i \in[k]$ and $N_{i} \subseteq V_{i}$ with $\left|N_{i}\right| \geq(1 / 2-1 / \log n) n$, we have $e_{H}\left(N_{1}, \ldots, N_{k}\right) \geq$ $n^{k} / \log ^{3} n$, then the number of $S$-absorbing $k$-matchings in $H$ is $\Omega\left(n^{k^{2}} / \log ^{3} n\right)$.
(ii) If there exists $j \in[k]$ such that $\left|\Lambda_{j}\right| \geq n^{k-1} / \log n$, then the number of $S$-absorbing $(k+1)$-matchings in $H$ is $\Omega\left(n^{k^{2}+k} / \log ^{3} n\right)$.

Proof. To prove ( $i$ ), we assume that, for all $i \in[k]$ and $N_{i} \subseteq V_{i}$ with $\left|N_{i}\right| \geq(1 / 2-$ $1 / \log n) n$, we have $e_{H}\left(N_{1}, \ldots, N_{k}\right) \geq n^{k} / \log ^{3} n$.

Note that, for each $i \in[k], x_{i}$ is contained in $(n-1)^{k-2}$ legal $(k-1)$-sets in $H$ that are disjoint from $S$ and one given partition class of $H$, and each such legal $(k-1)$-set is contained in at least $(1 / 2-1 / \log n) n-1$ edges in $H-S$ (since $\left.\delta_{k-1}(H) \geq(1 / 2-1 / \log n) n\right)$. Thus, there exists $n_{1}$ such that if $n \geq n_{1}$, there are at least $n^{k-1} / 3$ legal ( $k-1$ )-sets $B_{i}$ disjoint from $S$ such that $e_{i}^{\prime}:=\left\{x_{i}\right\} \cup B_{i} \in E(H)$.

By a similar argument (and choosing $n_{1}$ large enough), there are at least $\left((n-k)^{(k-1)} / 3\right)^{k} \geq$ $(1 / 3-o(1))^{k} n^{(k-1) k}$ choices of pairwise disjoint such legal $(k-1)$-sets $B_{1}, \ldots, B_{k}$.


Figure 2: $\left\{x_{1}, x_{2}, x_{3}\right\}$-absorbing $(k+1)$-matching for $k=3$

For each such choice of $B_{1}, \ldots, B_{k}$, let $N_{i}:=N_{H}\left(B_{i}\right)$ for $i \in[k]$. Then $\left|N_{i}\right| \geq \delta_{k-1}(H) \geq$ $(1 / 2-1 / \log n) n$. By assumption, $e_{H}\left(N_{1}, \ldots, N_{k}\right) \geq n^{k} / \log ^{3} n$; so there are at least $n^{k} / \log ^{3} n-k^{2} n^{k-1}$ choices of an edge $f:=\left\{y_{1}, \ldots, y_{k}\right\}$ from $H\left[\cup_{i \in[k]} N_{i}\right]-\cup_{i \in[k]} e_{i}^{\prime}$ such that $e_{i}:=B_{i} \cup\left\{y_{i}\right\} \in E(H)$ for $i \in[k]$.

Hence, the number of $S$-absorbing $k$-matchings $\left\{e_{1}, \ldots, e_{k}\right\}$ is at least

$$
(1 / 3-o(1))^{k} n^{(k-1) k}\left(n^{k} / \log ^{3} n-k^{2} n^{k-1}\right)=\Omega\left(n^{k^{2}} / \log ^{3} n\right),
$$

as claimed in $(i)$.
We now prove (ii). So assume without loss of generality that $\left|\Lambda_{1}\right| \geq n^{k-1} / \log n$. As in the previous case, since $\delta_{k-1}(H) \geq(1 / 2-1 / \log n) n$, there are at least $(1 / 3-o(1))^{k} n^{(k-1) k}=$ $\Omega\left(n^{(k-1) k}\right)$ choices of disjoint legal $(k-1)$-sets $B_{1}, \ldots, B_{k}$ such that $\left\{x_{i}\right\} \cup B_{i} \in E(H)$ for $i \in[k]$.

For $i=2, \ldots, k$, we choose $y_{i} \in N_{H}\left(B_{i}\right)-\left\{x_{i}\right\}$ and let $e_{i}:=B_{i} \cup\left\{y_{i}\right\}$. Note that we have $(1 / 2-2 / \log n) n-1=\Omega(n)$ choices for each $y_{i}$.

By assumption, there are at least $n^{k-1} / \log n-k(k+2) n^{k-2}=\Omega\left(n^{k-1} / \log n\right)$ choices for a $(k-1)$-set $T \in \Lambda_{1}$ that is disjoint from $S \cup B_{1} \cup \cdots \cup B_{k} \cup\left\{y_{2}, \ldots, y_{k}\right\}$. Since $N_{H}(T)>(1 / 2+2 / \log n) n\left(\right.$ as $\left.T \in \Lambda_{1}\right)$ and $\delta_{k-1}(H) \geq(1 / 2-1 / \log n) n$, we have $\mid N_{H}(T) \cap$
$N_{H}\left(B_{1}\right) \mid \geq n / \log n$ and $\left|N_{H}(T) \cap N_{H}\left(\left\{y_{2}, \ldots, y_{k}\right\}\right)\right| \geq n / \log n$. Consequently, there exist distinct $y_{0}$ and $y_{1}$ with $y_{1} \in\left(N_{H}(T) \cap N_{H}\left(B_{1}\right)\right)-\left(S \cup B_{1} \cup \cdots \cup B_{k}\right)$ and $y_{0} \in\left(N_{H}(T) \cap\right.$ $\left.N_{H}\left(\left\{y_{2}, \ldots, y_{k}\right\}\right)\right)-\left(S \cup B_{1} \cup \ldots \cup B_{k}\right)$, and there are at least $n / \log n-k(k+1)-1$ choices for each of $y_{0}$ and $y_{1}$.

Let $e_{0}:=\left\{y_{0}\right\} \cup T, e_{1}:=\left\{y_{1}\right\} \cup B_{1}, f^{\prime}:=\left\{y_{1}\right\} \cup T$ and $f^{\prime \prime}:=\left\{y_{0}, y_{2}, y_{3}, \ldots, y_{k}\right\}$. Then $\left\{e_{0}, \ldots, e_{k}\right\}$ is an $S$-absorbing $(k+1)$-matching (using $e_{i}^{\prime}=B_{i} \cup\left\{x_{i}\right\}$ for $i \in[k]$ ). Moreover, the number of choice for $\left\{e_{0}, \ldots, e_{k}\right\}$ is the product of the numbers of choices for $B_{1}, \ldots, B_{k}, y_{2}, \ldots, y_{k}, T, y_{0}, y_{1}$, which is at least

$$
\Omega\left(n^{k(k-1)}\right) \Omega\left(n^{k-1}\right) \Omega\left(\frac{n^{k-1}}{\log n}\right)\left(\frac{n}{\log n}-k(k+1)-1\right)^{2}=\Omega\left(\frac{n^{k^{2}+k}}{\log ^{3} n}\right)
$$

So we have (ii).
We will need to use Chernoff bounds, which can be found in (7].
Lemma 3.4 Suppose $X_{1}, \ldots, X_{n}$ are independent random variables taking values in $\{0,1\}$. Let $X$ denote their sum and $\mu=\mathbb{E}[X]$ denote the expected value of $X$. Then for any $0<\delta \leq 1$,

$$
\mathbb{P}[X \geq(1+\delta) \mu]<e^{-\frac{\delta^{2} \mu}{3}} \text { and } \mathbb{P}[X \leq(1-\delta) \mu]<e^{-\frac{\delta^{2} \mu}{2}}
$$

and for any $\delta \geq 1$,

$$
\mathbb{P}[X \geq(1+\delta) \mu]<e^{-\frac{\delta \mu}{3}}
$$

We now show that for each conclusion of Lemma 3.2, there exists a small matching $M^{\prime}$ in $H$ such that for each legal $k$-set $S$, there are at least $k$-pairwise disjoint $S$-absorbing matchings in $H$.

Lemma 3.5 Let $k \geq 3$ be a positive integer. There exists $n_{2}>0$ such that the following holds. Let $H$ be a $k$-partite $k$-graph with partition classes $V_{1}, \ldots, V_{k}$ such that $\left|V_{i}\right|=n>n_{2}$ for $i \in[k]$ and $\delta_{k-1}(H) \geq(1 / 2-1 / \log n) n$.
(i) If for all $i \in[k]$ and $N_{i} \subseteq V_{i}$ with $\left|N_{i}\right| \geq(1 / 2-1 / \log n) n$, we have $e_{H}\left(N_{1}, \ldots, N_{k}\right) \geq$ $n^{k} / \log ^{3} n$, then there exists a matching $M^{\prime}$ in $H$ such that $\left|M^{\prime}\right|=O\left(\log ^{5} n\right)$ and for every legal $k$-set $S \subseteq V(H)$, there are at least $k$ pairwise disjoint $S$-absorbing $k$-matchings in $M^{\prime}$.
(ii) If there exists $j \in[k]$ such that $\left|\Lambda_{j}\right| \geq n^{k-1} / \log n$, then there exists a matching $M^{\prime}$ in $H$ such that $\left|M^{\prime}\right|=O\left(\log ^{5} n\right)$ and for every legal $k$-set $S \subseteq V(H)$, there are at least $k$ pairwise disjoint $S$-absorbing $(k+1)$-matchings in $M^{\prime}$.

Proof. First, we prove ( $i$ ). Suppose for all $i \in[k]$ and $N_{i} \subseteq V_{i}$ with $\left|N_{i}\right| \geq(1 / 2-1 / \log n) n$, we have $e_{H}\left(N_{1}, \ldots, N_{k}\right) \geq n^{k} / \log ^{3} n$. So we can apply $(i)$ of Lemma 3.3.

For each legal $k$-set $S \subseteq V(H)$, let $\Gamma(S)$ be the set of $\left(S_{1}, \ldots, S_{k}\right)$ with $S_{i} \subseteq V_{i}$ and $\left|S_{i}\right|=k$ for $i \in[k]$ such that $H\left[\cup_{i \in[k]} S_{i}\right]$ has a perfect matching, say $M_{\left(S_{1}, \ldots, S_{k}\right)}$. Then by (i) of Lemma 3.3, $|\Gamma(S)|=\Omega\left(n^{k^{2}} / \log ^{3} n\right) / k^{k}$. So there exists $\alpha:=\alpha(k)>0$ such that $|\Gamma(S)| \geq \alpha\binom{n}{k}^{k} / \log ^{3} n$.

Let $\mathcal{F}$ be the (random) family whose members are $\left(S_{1}, \ldots, S_{k}\right)$ with $S_{i} \subseteq V_{i}$ and $\left|S_{i}\right|=k$ for $i \in[k]$, obtained by choosing each of the $\binom{n}{k}^{k}$ such $\left(S_{1}, \ldots, S_{k}\right)$ independently with probability

$$
p=\frac{\log ^{5} n}{\binom{n}{k}^{k}}
$$

Note that $p<1$ as we can choose $n_{2}$ large enough. Then

$$
\mathbb{E}(|\mathcal{F}|)=p\binom{n}{k}^{k}=\log ^{5} n
$$

and for each legal $k$-set $S \subseteq V(H)$,

$$
\mathbb{E}(|\mathcal{F} \cap \Gamma(S)|) \geq p \alpha\binom{n}{k}^{k} / \log ^{3} n=\alpha \log ^{2} n
$$

By Lemma 3.4 and by choosing $n_{2}$ large enough, we have, for $n>n_{2}$,

$$
\mathbb{P}\left[|\mathcal{F}|>2 \log ^{5} n\right]=\mathbb{P}[|\mathcal{F}|>2 \mathbb{E}(|\mathcal{F}|)] \leq e^{-\mathbb{E}(|\mathcal{F}|) / 3}=e^{-\left(\log ^{5} n\right) / 3}<1 / 10
$$

So with probability at least $9 / 10$

$$
\begin{equation*}
|\mathcal{F}| \leq 2 \log ^{5} n \tag{13}
\end{equation*}
$$

Again by Lemma 3.4 and by choosing $n_{2}$ large enough, we have, for $n>n_{2}$,

$$
\begin{aligned}
\mathbb{P}\left[|\mathcal{F} \cap \Gamma(S)| \leq\left(\alpha \log ^{2} n\right) / 2\right] & \leq \mathbb{P}[|\mathcal{F} \cap \Gamma(S)| \leq \mathbb{E}(|\mathcal{F} \cap \Gamma(S)|) / 2] \\
& \leq e^{-\mathbb{E}(|\mathcal{F} \cap \Gamma(S)|) / 8} \\
& \leq e^{-\left(\alpha \log ^{2} n\right) / 8} .
\end{aligned}
$$

So by union bound and choosing $n_{2}$ large, we have for $n>n_{2}$,

$$
\mathbb{P}\left[\exists \text { legal } S \subseteq V(H):|\mathcal{F} \cap \Gamma(S)| \leq\left(\alpha \log ^{2} n\right) / 2\right] \leq n^{k} e^{-\left(\alpha \log ^{2} n\right) / 8}=2 n^{k-(\alpha \log n) / 8}<1 / 10
$$

Thus, with probability at least $9 / 10$, for each legal $k$-set $S \subseteq V(H)$, we have

$$
\begin{equation*}
|\mathcal{F} \cap \Gamma(S)| \geq\left(\alpha \log ^{2} n\right) / 2>k \tag{14}
\end{equation*}
$$

Furthermore, the expected number of pairs of elements $\left(S_{1}, \ldots, S_{k}\right),\left(T_{1}, \ldots, T_{k}\right) \in \mathcal{F}$ satisfying $\left(\cup_{i \in[k]} S_{i}\right) \cap\left(\cup_{i \in[k]} T_{i}\right) \neq \emptyset$ is at most

$$
\binom{n}{k}^{k} k^{2}\binom{n-1}{k-1}\binom{n}{k}^{k-1} p^{2} \leq \frac{k^{3} \log ^{10} n}{n}<1 / 2
$$

Thus, with probability at least $1 / 2$ (by Markov's inequality), for all distinct $\left(S_{1}, \ldots, S_{k}\right) \in \mathcal{F}$ and $\left(T_{1}, \ldots, T_{k}\right) \in \mathcal{F}$,

$$
\begin{equation*}
\cup_{i \in[k]} S_{i} \text { and } \cup_{i \in[k]} T_{i} \text { are disjoint. } \tag{15}
\end{equation*}
$$

Hence, with positive probability, $\mathcal{F}$ satisfies (13), (14), and (15). So we may assume that $\mathcal{F}$ satisfies (13), (14), and (15). Let $M^{\prime}$ be the union of $M_{\left(S_{1}, \ldots, S_{k}\right)}$ for $\left(S_{1}, \ldots, S_{k}\right) \in \mathcal{F}$. Then $M^{\prime}$ is the desired matching for ( $i$ ).

Next we prove (ii). Suppose there exists $j \in[k]$ such that $\left|\Lambda_{j}\right| \geq n^{k-1} / \log n$; so that we can apply (ii) of Lemma 3.3.

For each legal $k$-set $S \subseteq V(H)$, let $\Gamma^{\prime}(S)$ be the set of sequences $\left(S_{1}, \ldots, S_{k}\right)$, with $S_{i} \subseteq V_{i}$ and $\left|S_{i}\right|=k+1$ for $i \in[k]$, such that $H\left[\cup_{i \in[k]} S_{i}\right]$ has a perfect matching, say $M_{\left(S_{1}, \ldots, S_{k}\right)}$. Then by (ii) of Lemma 3.3. $\left|\Gamma^{\prime}(S)\right|=\Omega\left(n^{k^{2}+k} / \log ^{3} n\right) /(k+1)^{k}$. So there exists $\alpha^{\prime}>0$ such that $\left|\Gamma^{\prime}(S)\right| \geq \alpha^{\prime}\binom{n}{k+1}^{k} / \log ^{3} n$.

We form a random family $\mathcal{G}$ consisting of sequences $\left(S_{1}, \ldots, S_{k}\right)$, with $S_{i} \subseteq V_{i}$ and $\left|S_{i}\right|=k+1$ for $i \in[k]$, by selecting each of the $\binom{n}{k+1}^{k}$ such $\left(S_{1}, \ldots, S_{k}\right)$ independently with probability

$$
p=\frac{\log ^{5} n}{\binom{n}{k+1}^{k}} .
$$

Note that $p<1$ by choosing $n_{2}$ large enough. Then

$$
\mathbb{E}(|\mathcal{G}|)=p\binom{n}{k+1}^{k}=\log ^{5} n,
$$

and for each legal $k$-set $S \subseteq V(H)$,

$$
\mathbb{E}\left(\left|\mathcal{G} \cap \Gamma^{\prime}(S)\right|\right) \geq p \alpha^{\prime}\binom{n}{k+1}^{k} / \log ^{3} n=\alpha^{\prime} \log ^{2} n
$$

By Lemma 3.4 and by choosing $n_{2}$ large enough, we have for $n>n_{2}$,

$$
\mathbb{P}\left[|\mathcal{G}|>2 \log ^{5} n\right]=\mathbb{P}[|\mathcal{G}|>2 \mathbb{E}(|\mathcal{G}|)] \leq 2 e^{-\mathbb{E}(|\mathcal{G}|) / 3}=2 e^{-\left(\log ^{5} n\right) / 3}<1 / 10
$$

So with probability at least $9 / 10$,

$$
\begin{equation*}
|\mathcal{G}| \leq 2 \log ^{5} n \tag{16}
\end{equation*}
$$

Again by Lemma 3.4 and by choosing $n_{2}$ large enough, we have for $n>n_{2}$,

$$
\begin{aligned}
\mathbb{P}\left[\left|\mathcal{G} \cap \Gamma^{\prime}(S)\right| \leq\left(\alpha^{\prime} \log ^{2} n\right) / 2\right] & \leq \mathbb{P}\left[\left|\mathcal{G} \cap \Gamma^{\prime}(S)\right| \leq \mathbb{E}\left(\left|\mathcal{G} \cap \Gamma^{\prime}(S)\right| / 2\right)\right] \\
& \leq e^{-\mathbb{E}\left(\left|\mathcal{G} \cap \Gamma^{\prime}(S)\right|\right) / 8} \\
& \leq e^{-\alpha^{\prime} \log ^{2} n / 8} .
\end{aligned}
$$

So by union bound and choosing $n_{2}$ large,

$$
\mathbb{P}\left[\exists \operatorname{legal} S \subseteq V(H):\left|\mathcal{G} \cap \Gamma^{\prime}(S)\right| \leq \alpha^{\prime} \log ^{2} n / 2\right] \leq n^{k} e^{-\left(\alpha^{\prime} \log ^{2} n\right) / 8}=n^{k-\left(\alpha^{\prime} \log n\right) / 8}<1 / 10
$$

Hence, with probability at least $9 / 10$, for each legal $k$-set $S \subseteq V(H)$,

$$
\begin{equation*}
\left|\mathcal{G} \cap \Gamma^{\prime}(S)\right| \geq\left(\alpha^{\prime} \log ^{2} n\right) / 2>k \tag{17}
\end{equation*}
$$

Furthermore, the expected number of pairs $\left(S_{1}, \ldots, S_{k}\right),\left(T_{1}, \ldots, T_{k}\right) \in \mathcal{G}$ with $\left(\cup_{i \in[k]} S_{i}\right) \cap$ $\left(\cup_{i \in[k]} T_{i}\right) \neq \emptyset$ is

$$
\binom{n}{k+1}^{k} k(k+1)\binom{n-1}{k}\binom{n}{k+1}^{k-1} p^{2} \leq \frac{(k+1)^{3} \log ^{10} n}{n}<1 / 2 .
$$

Thus, by Markov's inequality, with probability at least $1 / 2$, for all distinct $\left(S_{1}, \ldots, S_{k}\right) \in \mathcal{G}$ and $\left(T_{1}, \ldots, T_{k}\right) \in \mathcal{G}$,

$$
\begin{equation*}
\left(\cup_{i \in[k]} S_{i}\right) \cap\left(\cup_{i \in[k]} T_{i}\right)=\emptyset . \tag{18}
\end{equation*}
$$

Hence, with positive probability, $\mathcal{G}$ satisfies (16), (17), and (18). So we may assume that $\mathcal{G}$ satisfies $\sqrt{16}, 417$, and $\sqrt[18)]{ }$ Let $M^{\prime}$ be the union of $M_{\left(S_{1}, \ldots, S_{k}\right)}^{\prime}$ for all $\left(S_{1}, \ldots, S_{k}\right) \in \mathcal{G}$. Now $M^{\prime}$ gives the desired matching for (ii).

Corollary 3.6 Let $k \geq 3$ be a positive integer. For any $\varepsilon>0$, there exists $n_{3}>0$ such that the following holds. Let $H$ be a $k$-partite $k$-graph with $n>n_{3}$ vertices in each partition class. Suppose $\delta_{k-1}(H) \geq(1 / 2-1 / \log n) n$ and $H$ is not $\varepsilon$-close to $H_{0}(k, n)$. Then $H$ has a perfect matching.

Proof. Choose $n_{3}$ large enough so that we can apply Lemmas 3.1, 3.2, and 3.5.
By Lemmas 3.2 and 3.5. $H$ contains a matching $M$ such that $|M| \leq \beta \log ^{5} n$ for some constant $\beta>0$ (dependent on $k$ only) and, for every legal $k$-set $S \subseteq V(H)$, there are at least $k$ disjoint $S$-absorbing $k$-matchings in $M$, or for every legal $k$-set $S \subseteq V(H)$, there are at least $k$ disjoint $S$-absorbing $(k+1)$-matchings in $M$.

For $k \geq 3$,

$$
\delta_{k-1}(H-V(M)) \geq(1 / 2-1 / \log n) n-\beta \log ^{5} n>n / k,
$$

where the last inequality holds for $n>n_{3}$ by choosing $n_{3}$ large enough. Thus by Lemma 3.1, $H-V(M)$ contains a matching $M^{\prime}$ of size at least $n-|M|-1$. Let $S:=H-V\left(M \cup M^{\prime}\right)$. If $S=\emptyset$, then $M \cup M^{\prime}$ is a perfect matching in $H$. So assume that $S \neq \emptyset$; then $S$ is a legal $k$-set. Hence $H[S \cup V(M)]$ has a perfect matching $M^{\prime \prime}$. Now $M^{\prime} \cup M^{\prime \prime}$ is a perfect matching in $H$.

## 4 Conclusion

Proof of Theorem 1.2. First, suppose (i) or (ii) holds. Then there exist integers $d_{1}, \ldots, d_{k}$ such that $\sum_{i=1}^{k} d_{i}$ is odd and $H \subseteq H_{0}\left(d_{1}, d_{2}, \ldots, d_{k} ; k, n\right)$. By definition of $H_{0}\left(d_{1}, d_{2}, \ldots, d_{k} ; k, n\right)$, there exists $D \subseteq V(H)$ such that $|D|=\sum_{i=1}^{k} d_{i}$ is odd and $|e \cap D|$ is even for all $e \in E(H)$. Hence, $H$ contains no perfect matchings.

Now assume that $H$ has no perfect matching. Fix $\varepsilon>0$ so that

$$
\sqrt{\varepsilon}<\min \left\{1 /\left(100 k^{2}\right), 1 /\left(k\left(10 k^{2}\right)^{k-1}\right)\right\} .
$$

Then by Corollary 3.6, $H$ must be $\varepsilon$-close to $H_{0}(k, n)$. Hence by Corollary 2.2, (i) or (ii) holds.

## References

[1] R. Aharoni, A. Georgakopoulos, and P. Sprüssel, Perfect matchings in $r$-partite $r$ graphs, European J. Combin., 30 (2009), 39-42.
[2] J. Han, Near perfect matching in $k$-uniform hypergraph, Combin., Probab. and Comput., 24 (2015), 723-732.
[3] J. Han, C. Zang, and Y. Zhao, Matchings in $k$-partite $k$-uniform hypergraphs, Submitted.
[4] D. Kühn and D. Osthus, Matchings in hypergraphs of large minimum degree, J. Graph Theory, 51 (2006), 269-280.
[5] A. Lo and K. Markström, Perfect matchings in 3-partite 3-uniform hypergraphs, J. Combin. Theory Ser. A, 127 (2014), 22-57.
[6] H. Lu, Y. Wang, and X. Yu, Almost perfect matchings in $k$-partite $k$-graphs, Submitted.
[7] M. Mitzenmacher and E. Upfal, Probability and Computing: Randomized Algorithms and Probabilistic Analysis, Cambridge University Press (2005) ISBN 0-521-83540-2.
[8] V. Rödl and A. Ruciński. Dirac-type questions for hypergraphs-a survey (or more problems for Endre to solve), An Irregular Mind, Springer Berlin Heidelberg (2010), 561-590.
[9] V. Rödl, A. Ruciński, and E. Szemerédi, Perfect matchings in large uniform hypergraphs with large minimum collective degree, J. Combin. Theory Ser. A, 116 (2009), 616-636.
[10] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs, J. Combin. Theory Ser. A, 119 (2012), 1500-1522.
[11] A. Treglown and Y. Zhao, Exact minimum degree thresholds for perfect matchings in uniform hypergraphs II, J. Combin. Theory Ser. A, 120 (2013), 1463-1482.


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