Minimum co-degree condition for perfect matchings in k-partite k-graphs

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Abstract

Let *H* be a *k*-partite *k*-graph with *n* vertices in each partition class, and let $\delta_{k-1}(H)$ denote the minimum co-degree of *H*. We characterize those *H* with $\delta_{k-1}(H) \ge n/2$ and with no perfect matching. As a consequence we give an affirmative answer to the following question of Rödl and Ruciński: If *k* is even or $n \ne 2 \pmod{4}$, does $\delta_{k-1}(H) \ge n/2$ imply that *H* has a perfect matching? We also give an example indicating that it is not sufficient to impose this degree bound on only two types of (k-1)-sets.

1 Introduction

A hypergraph H consists of a vertex set V(H) and an edge set E(H) whose members are subsets of V(H). Let H_1 and H_2 be two hypergraphs. If $V(H_1) \subseteq V(H_2)$ and $E(H_1) \subseteq$ $E(H_2)$, then H_1 is called a *subgraph* of H_2 , denoted $H_1 \subseteq H_2$. Let k be a positive integer and $[k] := \{1, \ldots, k\}$. For a set S, let $\binom{S}{k} := \{T \subseteq S : |T| = k\}$. A hypergraph H is k-uniform if $E(H) \subseteq \binom{V(H)}{k}$, and a k-uniform hypergraph is also called a k-graph. Given $T \subseteq V(H)$, let H - T denote the subgraph of H with vertex set V(H) - T and edge set $E(H - T) = \{e \in E(H) : e \subseteq V(H) - T\}$.

Let H be a k-graph and $S \in \binom{V(H)}{l}$ with $l \in [k]$. The neighborhood of S in H, denoted $N_H(S)$, is the set of all (k-l)-subsets $U \subseteq V(H)$ such that $S \cup U \in E(H)$. The degree of S in H, denoted $d_H(S)$, is the size of $N_H(S)$. For $l \in [k]$, the minimum *l*-degree of H, denoted $\delta_l(H)$, is the minimum degree over all *l*-subsets of V(H). Note that $\delta_{k-1}(H)$ is known as the minimum co-degree of H.

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A matching in a hypergraph H is a subset of E(H) consisting of pairwise disjoint edges. A matching M in a hypergraph H is called a *perfect matching* if V(M) = V(H). Rödl, Ruciński, and Szemerédi [9] determined the minimum co-degree threshold function that ensures a perfect matching in a k-graph with n vertices, for $n \equiv 0 \pmod{k}$ and sufficiently large. This threshold function is $\frac{n}{2} - k + C$, where $C \in \{3/2, 2, 5/2, 3\}$, depending on the parity of n and k. They [9] also proved that, for $n \neq 0 \pmod{k}$, the minimum co-degree threshold that ensures a matching M in a k-graph H with $|V(M)| \geq |V(H)| - k$ is between $\lfloor n/k \rfloor$ and $n/k + O(\log n)$, and conjectured that this threshold function is $\lfloor n/k \rfloor$. This conjecture was proved recently by Han [2]. Treglown and Zhao [10, 11] determined the minimum l-degree threshold for perfect matchings in k-graphs for $k/2 \leq l \leq k - 1$.

A hypergraph H is a k-partite k-graph with partition classes V_1, \ldots, V_k if V_1, \ldots, V_k is a partition of V(H) and $|e \cap V_i| = 1$ for all $e \in E(H)$ and $i \in [k]$. We say that a set $S \subseteq V(H)$ is legal if $|S \cap V_i| \leq 1$ for $i \in [k]$. For $l \in [k]$, the minimum *l*-degree of a k-partite k-graph H, also denoted $\delta_l(H)$, is the minimum degree over all legal l-subsets of V(H). Again, $\delta_{k-1}(H)$ is called the minimum co-degree of H.

Kühn and Osthus [4] showed that the minimum co-degree threshold for the existence of a perfect matching in a k-partite k-graph with n vertices in each partition class is between n/2 and $n/2 + \sqrt{2n \log n}$. Lu, Wang, and Yu [6] and, independently, Han, Zang, and Zhao [3] showed that n/k is the minimum co-degree threshold for a k-partite k-graph H with n vertices in each partition class to admit a matching of size |V(H)| - k.

Aharoni, Georgakopoulos, and Sprüssel [1] obtained the following stronger result: Let $k \geq 3$ be a positive integer and H be a k-partite k-graph with partition classes V_1, \ldots, V_k , each of size n. If $d_H(S) > n/2$ for every legal (k-1)-set S contained in $V - V_1$, and if $d_H(T) \geq n/2$ for every legal (k-1)-set T contained in $V - V_2$, then H has a perfect matching. Example 1 in [1] (see the graph $H_0(k, n)$ below) shows that this bound is best possible when k is odd and $n \equiv 2 \pmod{4}$. Motivated by this result, Rödl and Ruciński [8] asked the following

Question 1.1 (Rödl and Ruciński [8]) Let k, n be integers with $k \ge 3$ and n sufficiently large, and H be a k-partite k-graph in which each partition class has size n. Assume that kis even or $n \not\equiv 2 \pmod{4}$. Is it true that if $\delta_{k-1}(H) \ge n/2$ then H has a perfect matching? If so, is it sufficient to impose this degree bound on only two types of legal (k-1)-sets, similar to the above result of Aharoni, Georgakopoulos, and Sprüssel?

Note that if n is odd, it follows from the above result of Aharoni, Georgakopoulos, and Sprüssel that the answer to the first part of Question 1.1 is affirmative.

We now describe an example showing the tightness of the bound in Question 1.1. Let $k, n, d_i, i \in [k]$, be positive integers. Let $H_0(d_1, \ldots, d_k; k, n)$ be a k-partite k-graph with partition classes V_1, \ldots, V_k , and let $D_i \subseteq V_i$ for $i \in [k]$, such that $|V_i| = n$ and $|D_i| = d_i$ for $i \in [k]$, and $E(H_0(d_1, \ldots, d_k; k, n))$ consists of those legal k-sets with an even number of vertices (including zero) in $\bigcup_{i \in [k]} D_i$. In particular, we define $H_0(k, n) :=$ $H_0(\lfloor n/2 \rfloor, \ldots, \lfloor n/2 \rfloor; k, n)$. When k is odd and $n \equiv 2 \pmod{4}$, $H_0(k, n)$ is Example 1 in [1]; in which case, $\delta_{k-1}(H_0(k, n)) = n/2$ and $H_0(k, n)$ admits no perfect matching (as $\sum_{i \in [k]} |D_i| = kn/2$ is odd and every edge of $H_0(k, n)$ has an even number of vertices in $\bigcup_{i \in [k]} D_i$). **Remark.** We point out that the answer to the second part of Question 1.1 is negative. Let k, n be positive integers such that k is even or $n \equiv 0 \pmod{4}$. Let $J := H_0(n/2, n/2, \ldots, n/2, n/2 + 1; k, n)$ with partition classes V_1, \ldots, V_k and let $D_i \subseteq V_i$ for $i \in [k]$ such that $|D_i| = n/2$ for $i \in [k-1]$, $|D_k| = n/2 + 1$, and each edge of J has an even number of vertices in $\bigcup_{i \in [k]} D_i$. Observe that all legal (k-1)-subsets of V(J) intersecting V_k have degree at least n/2, and those legal (k-1) sets contained in $V(J) - V_k$ and intersecting $\bigcup_{i \in [k]} D_i$ an even number of times have degree n/2 - 1. Moreover, J has no perfect matching since $\sum_{i \in [k]} |D_i| = kn/2 - 1 \equiv 1 \pmod{2}$ (as k is even or $n \equiv 0 \pmod{4}$).

Our main result is the following, which implies an affirmative answer to the first part of Question 1.1.

Theorem 1.2 Let k, n be integers with $k \ge 3$ and n sufficiently large, and let H be a k-partite k-graph with n vertices in each partition class. Suppose $\delta_{k-1}(H) \ge \lfloor n/2 \rfloor$. Then H has no perfect matching if, and only if,

- (i) k is odd, $n \equiv 2 \pmod{4}$, and $H \cong H_0(k, n)$, or
- (ii) *n* is odd and there exist $d_i \in \{(n+1)/2, (n-1)/2\}$ for $i \in [k]$ such that $\sum_{i=1}^k d_i$ is odd and $H \subseteq H_0(d_1, d_2, \dots, d_k; k, n)$.

Our proof of Theorem 1.2 consists of two parts by considering whether or not H is "close" to $H_0(k, n)$, which is similar to arguments in [5,9]. Given two hypergraphs H_1, H_2 with $V(H_1) = V(H_2)$, let $c(H_1, H_2)$ be the minimum of $|E(H_1) \setminus E(H')|$ taken over all isomorphic copies H' of H_2 with $V(H') = V(H_2)$. For a real number $\varepsilon > 0$, we say that H_2 is ε -close to H_1 if $V(H_1) = V(H_2)$ and $c(H_1, H_2)$ is less than ε times the maximum possible number of edges on $V(H_2)$ (which is, for example, εn^k if H_2 is a k-partite k-graph with n vertices in each partition class).

In Section 2, we deal with the case when H is ε -close to $H_0(k, n)$ for some sufficiently small ε . In Section 3, we deal with the case when H is not ε -close to $H_0(k, n)$, using the absorbing method from [9] and a recent result of the authors [6] (see Lemma 3.1).

2 Hypergraphs close to $H_0(k, n)$

In this section, we prove Theorem 1.2 for the case when H is ε -close to $H_0(k, n)$ for some sufficiently small ε . Since we will be dealing with $H_0(k, n)$, the following notation for "even" and "odd" degrees (with respect to a given set S) will be convenient. Let H be a hypergraph. For $j \in \{0, 1\}, v \in V(H)$, and $S \subseteq V(H)$, we define

$$d_{H,S}^{j}(v) := |\{e \in E(H) : v \in e \text{ and } |e \cap S| \equiv j \pmod{2}\}|.$$

Lemma 2.1 Let $k \geq 3$ be a positive integer, and let $\alpha, \varepsilon > 0$ be small such that $\alpha < 1/4$ and $\sqrt{\varepsilon} < \min\{1/(100k^2), 1/(k(10k^2)^{k-1})\}$. Then for any k-partite k-graph H with $n > 100k^2$ vertices in each partition class, the following holds: If $\delta_{k-1}(H) \geq (1/2 - \alpha)n$, H is ε -close to $H_0(k, n)$, and $H \not\subseteq H_0(d_1, \ldots, d_k; k, n)$ for any $d_1, \ldots, d_k \in [\lceil (1/2 - \alpha)n \rceil, \lfloor (1/2 + \alpha)n \rfloor]$ with $\sum_{i=1}^k d_i$ odd, then H has a perfect matching.

Proof. Let *H* be a *k*-partite *k*-graph with *n* vertices in each partition class such that $\delta_{k-1}(H) \geq (1/2 - \alpha)n$, *H* is ε -close to $H_0(k, n)$, and $H \not\subseteq H_0(d_1, \ldots, d_k; k, n)$ for any $d_1, \ldots, d_k \in [\lceil (1/2 - \alpha)n \rceil, \lfloor (1/2 + \alpha)n \rfloor]$ with $\sum_{i=1}^k d_i$ odd. Let

$$N := \{ v \in V(H) : |N_{H_0(k,n)}(v) - N_H(v)| \ge \sqrt{\varepsilon} n^{k-1} \}.$$

So each vertex in N is contained in at least $\sqrt{\varepsilon}n^{k-1}$ edges from $E(H_0(k,n)) - E(H)$. Note that

$$|N| \le \sqrt{\varepsilon} kn;$$

for, otherwise,

$$|E(H_0(k,n)) - E(H)| \geq \frac{1}{k} \sum_{v \in N} |N_{H_0(k,n)}(v) - N_H(v)|$$

$$> \frac{1}{k} |N| \sqrt{\varepsilon} n^{k-1}$$

$$> \frac{1}{k} \sqrt{\varepsilon} kn \sqrt{\varepsilon} n^{k-1}$$

$$= \varepsilon n^k,$$

contradicting the fact that H is ε -close to $H_0(k, n)$.

The rest of our proof is organized as follows. We first find a matching M_1 in H that covers all vertices in N (see Claim 2). We then find a matching M_2 in $H - V(M_1)$ satisfying certain conditions (see Claim 3). Finally, we will show that there exists a perfect matching in $H - V(M_1) - V(M_2)$. The last part is easy when k is even (see Claim 4), but needs more work when k is odd (see Claims 5-8).

To find a matching in H that covers all vertices in N, we need to fix some notation first. For $i \in [k]$, let $B_i \subseteq V_i$ such that $|B_i| = \lfloor n/2 \rfloor$ and each edge in $H_0(k, n)$ has an even number of vertices in $B := \bigcup_{j \in [k]} B_j$. For $i \in [k]$, let $A_i := V_i - B_i$. The intuition for the notation below is that the vertices v in $A_i \cap N$ (respectively, $B_i \cap N$) with $d^0_{H,B}(v) < n^{k-1}/8$ will be switched to B'_i (respectively, A'_i). For $i \in [k]$, let

$$A'_{i} := \left(A_{i} - \{v \in A_{i} \cap N : d^{0}_{H,B}(v) < n^{k-1}/8\}\right) \cup \{v \in B_{i} \cap N : d^{0}_{H,B}(v) < n^{k-1}/8\},$$

and

$$B'_i := \left(B_i - \{ v \in B_i \cap N : d^0_{H,B}(v) < n^{k-1}/8 \} \right) \cup \{ v \in A_i \cap N : d^0_{H,B}(v) < n^{k-1}/8 \}.$$

Let $A' := \bigcup_{j \in [k]} A'_j$ and $B' := \bigcup_{j \in [k]} B'_j$.

Since $|N| \leq \sqrt{\varepsilon}kn$ and $|B_i| = \lfloor n/2 \rfloor$, we have $A'_i \neq \emptyset$ and $B'_i \neq \emptyset$ for $i \in [k]$ (as $n \geq 100k^2$). In fact, for $i \in [k]$,

$$|A'_i| \ge |A_i| - |N| \ge (1/2 - \sqrt{\varepsilon}k)n \tag{1}$$

and

$$|B'_i| \ge |B_i| - |N| \ge (1/2 - \sqrt{\varepsilon}k)n - 1.$$
 (2)

Moreover, for each $v \in V(H)$, the number of edges in H containing v and intersecting $N - \{v\}$ is at most $|N|n^{k-2}$.

We now show that, for $v \in V(H)$,

$$d^{0}_{H-(N-\{v\}),B'}(v) \ge (1/8 - \sqrt{\varepsilon}k)n^{k-1}.$$
(3)

First, suppose $v \in (A \cap A') \cup (B \cap B')$. Then $B' - (N - \{v\}) = B - (N - \{v\})$, and $d^0_{H,B}(v) \ge n^{k-1}/8$ by definition of A', B'. So

$$d^{0}_{H-(N-\{v\}),B-(N-\{v\})}(v) \ge d^{0}_{H,B}(v) - |N|n^{k-2} \ge n^{k-1}/8 - \sqrt{\varepsilon}kn^{k-1}.$$

Hence,

$$d^{0}_{H-(N-\{v\}),B'}(v) = d^{0}_{H-(N-\{v\}),B'-(N-\{v\})}(v)$$

= $d^{0}_{H-(N-\{v\}),B-(N-\{v\})}(v)$
 $\geq (1/8 - \sqrt{\varepsilon}k)n^{k-1}.$

Now assume $v \in (A \cap B') \cup (A' \cap B)$. Then $v \in N$ and $d^{0}_{H,B}(v) < n^{k-1}/8$. So $B' - (N - \{v\}) = B - (N - \{v\}) - \{v\}$, Since $\delta_{k-1}(H) \ge (1/2 - \alpha)n$, it follows that $d_H(v) \ge (1/2 - \alpha)n^{k-1}$. Thus, since $n \ge 100k^2$ and $\alpha < 1/4$,

$$d_{H,B}^{1}(v) \ge (1/2 - \alpha)n^{k-1} - d_{H,B}^{0}(v) > (1/2 - \alpha)n^{k-1} - n^{k-1}/8 > n^{k-1}/8.$$

Therefore, $d^1_{H-(N-\{v\}),B-(N-\{v\})}(v) \ge d^1_{H,B}(v) - |N|n^{k-2} \ge (1/8 - \sqrt{\varepsilon}k)n^{k-1}$. Hence,

$$d^{0}_{H-(N-\{v\}),B'}(v) = d^{0}_{H-(N-\{v\}),B'-(N-\{v\})}(v)$$

= $d^{1}_{H-(N-\{v\}),B-(N-\{v\})}(v)$
 $\geq (1/8 - \sqrt{\varepsilon}k)n^{k-1}.$

We now begin our process of finding matchings M_1 and M_2 . First, we need to make |B'| even.

Claim 1. Either |B'| is even (in which case let $e_0 = \emptyset$; so $|B' - e_0|$ is even), or there exists an edge $e_0 \in E(H)$ such that $|B' - e_0|$ is even.

We may assume that |B'| is odd and |B' - e| is odd for every $e \in E(H)$; as, otherwise, Claim 1 holds. Then $|B' \cap e|$ is even for all $e \in E(H)$. Hence $H \subseteq H_0(d_1, \ldots, d_k; k, n)$, where $d_i = |B'_i|$ for $i \in [k]$, $\sum_{i \in [k]} d_i = |B'|$ is odd, and B'_1, \ldots, B'_k play the roles of D_1, \ldots, D_k , respectively, in the definition of $H_0(k, n)$.

Let $v_i \in A'_i$ and $u_i \in B'_i$ for $i \in [k]$, and let $S := \{v_1, \ldots, v_k\}$. Then for $i \in [k]$, since $|B' \cap e|$ is even for all $e \in E(H)$, we have

$$n - d_i = |A'_i| \ge d_H(S - \{v_i\}) \ge \delta_{k-1}(H) \ge (1/2 - \alpha)n;$$

so $d_i \leq \lfloor (1/2 + \alpha)n \rfloor$. Moreover, for $i \in [k]$, let $j \in [k] - \{i\}$. Again, since $|B' \cap e|$ is even for all $e \in E(H)$, we have

$$d_i = |B'_i| \ge d_H((S \cup \{u_j\}) - \{v_i, v_j\}) \ge \delta_{k-1}(H) \ge (1/2 - \alpha)n;$$

so $d_i \ge \lceil (1/2 - \alpha)n \rceil$. This contradicts the assumption that $H \not\subseteq H_0(d_1, \ldots, d_k; k, n)$ for any $d_1, \ldots, d_k \in [\lceil (1/2 - \alpha)n \rceil, \lfloor (1/2 + \alpha)n \rfloor]$ with $\sum_{i=1}^k d_i$ odd. \Box

Note that for each $v \in N - e_0$, the number of edges in H containing v and a vertex of e_0 is at most kn^{k-2} . Thus by (3), we have

$$d^{0}_{(H-e_{0})-(N-\{v\}),B'-e_{0}}(v) \ge (1/8 - 2\sqrt{\varepsilon}k)n^{k-1} - kn^{k-2} > n^{k-1}/10, \tag{4}$$

where the last inequality holds since $\sqrt{\varepsilon} < 1/(100k^2)$ and $n \ge 100k^2$.

Claim 2. There exists a matching M_1 in $H - e_0$ such that

- (i) $|M_1| \leq \sqrt{\varepsilon} kn$,
- (*ii*) $N e_0 \subseteq V(M_1)$, and
- (*iii*) $|e \cap (B' e_0)| \equiv 0 \pmod{2}$ for all $e \in M_1$.

Let $M_1 := \emptyset$ if $N - e_0 = \emptyset$. Now assume $N - e_0 \neq \emptyset$, and we construct M_1 by matching vertices in N greedily. Let $v_1 \in N - \{e_0\}$. Since $d^0_{(H-e_0)-(N-\{v_1\}),B'-e_0}(v_1) > n^{k-1}/10$ (by (4)) and $n \ge 100k^2$, there exists an edge e_1 in $H-e_0$, such that $v_1 \in e_1$ and $|e_1 \cap (B'-e_0)| \equiv 0 \pmod{2}$.

Now suppose we have found a matching $\{e_1, \ldots, e_t\}$ in $H - e_0$ for some $t \ge 1$, such that, for $i \in [t]$, $e_i \cap (N - e_0) \neq \emptyset$ and $|e_i \cap (B' - e_0)| \equiv 0 \pmod{2}$. If $N - e_0 \subseteq \bigcup_{i \in [t]} e_i$, then $M_1 := \{e_1, \ldots, e_t\}$ is the desired matching (as $t < |N| \le \sqrt{\varepsilon}kn$). So let $v_{t+1} \in N - e_0$ and $v_{t+1} \notin \bigcup_{i \in [t]} e_i$. Note that $t < |N| \le \sqrt{\varepsilon}kn$ and that the number of edges in $H - e_0$ containing v_{t+1} and a vertex from $\bigcup_{i \in [t]} e_i$ is at most

$$tkn^{k-2} \leq \sqrt{\varepsilon}k^2n^{k-1} \leq n^{k-1}/100$$

as $\sqrt{\varepsilon} < 1/(100k^2)$. Since $d^0_{(H-e_0)-(N-\{v_{t+1}\}),B'-e_0}(v_{t+1}) > n^{k-1}/10$ (by (4)), there exists e_{t+1} in $(H-e_0) - \bigcup_{i \in [t]} e_i$ such that $v_{t+1} \in e_{t+1}$ and $|e_{t+1} \cap (B'-e_0)| \equiv 0 \pmod{2}$.

Therefore, continuing this process (at most $|N-e_0|$ steps), we obtain the desired matching for Claim 2. \Box

Let $H' := (H - e_0) - V(M_1)$. For $i \in [k]$, let $C_i := A'_i - (V(M_1) \cup e_0)$, $D_i := B'_i - (V(M_1) \cup e_0)$ and $D := \bigcup_{i \in [k]} D_i$. By Claim 2, $N \cap V(H') = \emptyset$; so for $i \in [k]$,

$$D_i \subseteq B_i. \tag{5}$$

Note that |D| is even (by Claims 1 and 2). Since $|M_1| \leq \sqrt{\varepsilon} kn$, it follows from (1) and (2) that for $i \in [k]$,

$$|C_i| \ge |A'_i| - (|M_1| + 1) \ge (1/2 - \sqrt{\varepsilon}k)n - (\sqrt{\varepsilon}kn + 1) \ge (1/2 - 2\sqrt{\varepsilon}k)n - 1$$
(6)

and

$$|D_i| \ge |B_i'| - (|M_1| + 1) \ge ((1/2 - \sqrt{\varepsilon}k)n - 1) - (\sqrt{\varepsilon}kn + 1) \ge (1/2 - 2\sqrt{\varepsilon}k)n - 2.$$
(7)

Claim 3. There exists a matching M_2 in H' such that

(i) $|M_2| \leq 8\sqrt{\varepsilon}k^2n$,

(*ii*)
$$|D_i - V(M_2)| = |D_1 - V(M_2)|$$
 for $i \in [k]$, and

(*iii*) $|D - V(M_2)|$ is even.

Without loss of generality, we may assume that $|D_1| \ge |D_2| \ge \cdots \ge |D_k|$. If $|D_1| = |D_k|$ then $M_2 = \emptyset$ gives the desired matching for Claim 3. So assume $|D_1| - |D_k| > 0$. We construct an auxiliary graph and use a perfect matching in this graph to find M_2 .

Let $r \in \{0, 1\}$ such that $|D_1| + r$ is even. Let G be the complete k-partite 2-graph and let $W_1, ..., W_k$ be the partition classes of G, such that $|W_i| = (|D_i| - |D_k|) + (|D_1| + r) - |D_k|$ for $i \in [k]$. Then $|W_1| \ge |W_2| \ge ... \ge |W_k|$ and

$$|V(G)| = \sum_{i \in [k]} |W_i| = \left(\sum_{i \in [k]} |D_i|\right) + k(|D_1| + r) - 2k|D_k|.$$

Since $\sum_{i \in [k]} |D_i|$ and $|D_1| + r$ are even, |V(G)| is also even.

We now use Tutte's 1-factor theorem to show that G has a perfect matching. For $S \subseteq V(G)$, let o(G-S) denote the number of connected components of G-S of odd order. If $S = \emptyset$, then $o(G-S) = 0 \le |S|$. Now assume $S \ne \emptyset$. Since G is a complete k-partite 2-graph and $|W_1| \ge |W_2| \ge \ldots \ge |W_k|$, if $1 \le |S| < \sum_{i \in [k] - \{1\}} |W_i|$ then $o(G-S) \le 1 \le |S|$, and if $|S| \ge \sum_{i \in [k] - \{1\}} |W_i| \ge (k-1)(|D_1| - |D_k| + r)$ then $o(G-S) \le |W_1| \le 2|D_1| - 2|D_k| + r \le |S|$ (as $k \ge 3$). Thus, by Tutte's 1-factor theorem, G has a perfect matching, say T.

Since $|C_i| \ge (1/2 - 2\sqrt{\varepsilon}k)n - 1$ (by (6)), $|D_i| \le n - |C_i| \le (1/2 + 2\sqrt{\varepsilon}k)n + 1$. So by (7), $|D_1| - |D_k| \le 4\sqrt{\varepsilon}kn + 3$. Hence,

$$|T| = |V(G)|/2 = \left(\sum_{i \in [k]} |W_i|\right)/2 \le (k/2)(2|D_1| - 2|D_k| + 1) \le 8\sqrt{\varepsilon}k^2n.$$

Let $T = \{f_1, f_2, ..., f_{|T|}\}$. Corresponding to each f_i we find an edge g_i of H' such that $\{g_1, \ldots, g_{|T|}\}$ gives the desired matching M_2 for Claim 3.

Let $g_0 = \emptyset$ and we find $g_1, \ldots, g_{|T|}$ in order. Suppose we have found g_t for some t, with $0 \le t \le |T| - 1$. We describe how to find g_{t+1} using f_{t+1} . Let $f_{t+1} \subseteq W_p \cup W_q$, where $p, q \in [k]$. By (6) and (7), $\min\{|C_j|, |D_j|\} \ge (1/2 - 2\sqrt{\varepsilon}k)n - 2$ for $j \in [k]$. Then, since $|T| \le 8\sqrt{\varepsilon}k^2n, n \ge 100k^2$, and $\sqrt{\varepsilon} < 1/(k(10k^2)^{k-1})$, we have, for $j \in [k]$,

$$|C_j - \bigcup_{i \in [t]} g_i| > n/10 \text{ and } |D_j - \bigcup_{i \in [t]} g_i| > n/10.$$

So let $v_p \in D_p - \bigcup_{i \in [t]} g_i$. There exist $v_q \in D_q - \bigcup_{i \in [t]} g_i$ and $v_j \in C_j - \bigcup_{i \in [t]} g_i$ for $j \in [k] - \{p, q\}$ such that $g_{t+1} := \{v_1, ..., v_k\} \in E(H')$; for, otherwise,

$$|N_{H_0(k,n)}(v_p) - N_H(v_p)| > (n/10)^{k-1} > \sqrt{\varepsilon} n^{k-1},$$

as $\sqrt{\varepsilon} < 1/(k(10k^2)^{k-1})$, contradicting the fact that $v_p \notin N$. Clearly, $|g_{t+1} \cap D| = 2$. Therefore, $M_2 := \{g_1, \dots, g_{|T|}\}$ is a matching in H' such that, for $i \in [k]$,

$$|D_i - V(M_2)| = |D_i| - |W_i| = 2|D_k| - |D_1| - r > 0$$

where the inequality holds because of (7), $|D_1| - |D_k| \le 4\sqrt{\varepsilon}kn + 3$, $\sqrt{\varepsilon} < 1/(100k^2)$, and $n \ge 100k^2$. Moreover, $|g_j \cap D| = 2$ for $j \in [|T|]$. Hence, since |D| is even (by Claims 1 and 2), $|D - V(M_2)|$ is even. \Box

Let $H'' := H' - V(M_2)$ and, for $i \in [k]$, let $D'_i := D_i - V(M_2)$ and $C'_i := C_i - V(M_2)$. Let $D' := \bigcup_{i \in [k]} D'_i$ and $C' := \bigcup_{i \in [k]} C'_i$. Note that |D'| is even, as $|D - V(M_2)|$ is even (by Claim 3). Since $|M_2| \leq 8\sqrt{\varepsilon}k^2n$ (by Claim 3), it follows from (6) and (7) that, for $i \in [k]$,

$$\min\{|C'_i|, |D'_i|\} = \min\{|C_i|, |D_i|\} - |M_2| \ge (1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n.$$
(8)

Claim 4. We may assume that k is odd.

For, suppose k is even. We show that both H'' - C' and H'' - D' have perfect matchings; hence the assertion of the lemma holds. Below, we only show that H'' - C' has a perfect matching, since the argument for H'' - D' is the same (by substituting (6) for (7) and by exchanging the roles of C'_i and D'_i).

Let M be a maximum matching in H'' - C'. Then (H'' - C') - V(M) = H[D' - V(M)]has no edge. We claim that $|M| \ge n/4$. For, otherwise, $D'_1 - V(M) \ne \emptyset$ by (7) (as $\sqrt{\varepsilon} < 1/(100k^2)$ and $n \ge 100k^2$). Let $v \in D'_1 - V(M)$. Since k is even and $D'_i \subseteq B_i$ for $i \in [k]$ (by (5)), and because H[D' - V(M)] has no edge, we have

$$\begin{aligned} |N_{H_0(k,n)}(v) - N_H(v)| &\geq |D'_2 - V(M)| |D'_3 - V(M)| \cdots |D'_k - V(M)| \\ &\geq ((1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n - n/4)^{k-1} \quad (by \ (8)) \\ &> (n/10)^{k-1} \quad (\text{since } \sqrt{\varepsilon} < 1/(100k^2) \text{ and } n \ge 100k^2) \\ &> \sqrt{\varepsilon}n^{k-1} \quad (\text{since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k-1}), \end{aligned}$$

contradicting the fact that $v \notin N$.

Now, suppose for a contradiction, that M is not a perfect matching in H'' - C'. Then there exists $u_i \in D'_i - V(M)$ for $i \in [k]$. Note that $|M| \ge n/4 > k - 1$ (as $n \ge 100k^2$).

Let $\{e_1, \ldots, e_{k-1}\}$ be an arbitrary (k-1)-subset of M, and write $e_i := \{v_{i,1}, \ldots, v_{i,k}\}$ with $v_{i,j} \in D'_j$ for $i \in [k-1]$ and $j \in [k]$. For $j \in [k]$, let $f_j := \{u_j, v_{1,j+1}, v_{2,j+2}, \ldots, v_{k-1,j+k-1}\}$, with the addition in the subscripts modulo k (except we write k for 0). Note that f_1, \ldots, f_k are pairwise disjoint. Since $D'_i \subseteq B_i$ for $i \in [k]$ (by (5)), and k is assumed to be even, it follows that $f_j \in E(H_0(k, n))$ for $j \in [k]$.

If $f_i \in E(H'')$ for all $i \in [k]$ then $M' := (M \cup \{f_1, \ldots, f_k\}) - \{e_1, \ldots, e_{k-1}\}$ is a matching in H and |M'| = |M| + 1 > |M|, contradicting the maximality of |M|.

Hence, $f_j \notin E(H)$ for some $j \in [k]$. Note that there are $\binom{|M|}{k-1}$ choices of $\{e_1, \ldots, e_{k-1}\} \subseteq M$. Hence,

$$\begin{split} &|\{e \in E(H_0(k,n)) - E(H) : |e \cap \{u_i : i \in [k]\}| = 1\}|\\ \geq &|M|(|M| - 1) \cdots (|M| - (k - 1) + 1)/(k - 1)!\\ > &(n/4 - k + 2)^{k - 1}/(k - 1)!\\ > &(n/10k)^{k - 1} \quad (\text{since } n \ge 100k^2)\\ > &k\sqrt{\varepsilon}n^{k - 1} \quad (\text{since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k - 1}). \end{split}$$

This implies that there exists $i \in [k]$ such that $|N_{H_0(k,n)}(u_i) - N_H(u_i)| > \sqrt{\varepsilon} n^{k-1}$, contradicting the fact that $u_i \notin N$. \Box

Next claim guarantees a divisibility condition for |D'|, which will be used in the proof of Claim 7.

Claim 5. There exists a matching M_3 in H'' such that

- (i) $|M_3| \leq k^2/2$, and
- (*ii*) $|D'_i V(M_3)| = |D'_1 V(M_3)| \equiv 0 \pmod{k-1}$ for $i \in [k]$.

Let $0 \le s \le k-2$ be such that $|D'_1| \equiv s \pmod{k-1}$. We may assume that $s \ne 0$; for, otherwise, $M_3 = \emptyset$ gives the desired matching for Claim 5. Moreover, since k is odd (by Claim 4) and $|D'| = k|D'_1|$ is even, it follows that s is even.

We now construct M_3 , starting with the empty matching $T_0 = \emptyset$. Suppose for some $j \in [s/2]$, we have constructed a matching T_{j-1} in H'' with $|T_{j-1}| = k(j-1)$. Since $\sqrt{\varepsilon} < 1/(100k^2)$ and $n \ge 100k^2$, it follows from (8) that, for $i \in [k]$,

$$\min\{|C'_i - V(T_{j-1})|, |D'_i - V(T_{j-1})|\} \ge (1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n - k(j-1) > 0.$$

For $i \in [k]$, let $v_{j,i} \in D'_i - V(T_{j-1})$. We claim that there exist $v_{j,i+1} \in D'_{i+1} - V(T_{j-1})$ and $u_{j,l} \in C'_l - V(T_{j-1})$ for $l \in [k] - \{i, i+1\}$, such that $e_{j,i} := \{v_{j,i}, v_{j,i+1}, u_{j,l} : l \in [k] - \{i, i+1\}\} \in E(H'')$ (with addition in the subscripts modulo k except we use k for 0) and $\{e_{j,i} : i \in [k]\}$ is a matching in H''. For, otherwise, since $D'_i \subseteq B_i$ by ((5)), we have

$$\begin{aligned} |N_{H_0(k,n)}(v_{j,i}) - N_H(v_{j,i})| &\geq |D'_{i+1} - V(T_{j-1}) - \bigcup_{i \in [k]} e_{j,i}| \prod_{l \in [k] - \{i,i+1\}} |C'_l - V(T_{j-1})| \\ &\geq ((1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n - k^2/2)^{k-1} \\ &> (n/10)^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(100k^2) \text{ and } n \geq 100k^2) \\ &> \sqrt{\varepsilon}n^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k-1}), \end{aligned}$$

contradicting the fact that $v_{j,i} \notin N$.

Let $T_j = T_{j-1} \cup \{e_{j,i} : i \in [k]\}$. Then T_j is a matching in H'' for $j \in [s/2]$. Let $M_3 := T_{s/2} = \{e_{j,i} : j \in [s/2] \text{ and } xsi \in [k]\}$. Then $|M_3| \leq k^2/2$. Note that, for $i \in [k]$, the edges in $T_j - T_{j-1}$ uses exactly two vertices of D'_i . Thus, for $i \in [k], |D'_i - V(M_3)| = |D'_1 - V(M_3)| = |D'_1| - s \equiv 0 \pmod{k-1}$. \Box

Let $H^* := H'' - V(M_3)$ and, for $i \in [k]$, let $D_i^* := D_i' - V(M_3)$ and $C_i^* := C_i' - V(M_3)$. Let $D^* := \bigcup_{i \in [k]} D_i^*$ and $C^* := \bigcup_{i \in [k]} C_i^*$. Since $|M_3| \le k^2/2$ (by Claim 5), it follows from (8) that

$$\min\{|C_i^*|, |D_i^*|\} \ge \min\{|C_i'|, |D_i'|\} - |M_3| \ge (1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n - k^2/2.$$
(9)

By Claim 5, $|D_i^*| = |D_1^*| \equiv 0 \pmod{k-1}$ for $i \in [k]$.

We will show that H^* has a perfect matching using edges of special types. For any $e \in E(H^*)$, if $e \subseteq C^*$ then we say that e is of 0-type, and if $|e \cap C^*| = |e \cap C_j^*| = 1$ for some $j \in [k]$ then we say that e is of j-type. For convenience, let

$$\tau := 1/(9k).$$

Claim 6. H^* has pairwise disjoint matchings M^0, M^1, \ldots, M^k , such that for $i \in [k] \cup \{0\}$,

- (i) $|M^i| = \lfloor \tau n \rfloor$, and
- (*ii*) each edge in M^i is of *i*-type.

We construct M^0, M^1, \ldots, M^k in the order listed. Let T^0 be a matching in H^* such that $V(T^0) \subseteq C^*$ and, subject to this, $|T^0|$ is maximum. Then $C^* - V(T^0)$ has no edge. We claim that $|T^0| \ge \lfloor \tau n \rfloor$; for, otherwise, $|C_i^* - V(T^0)| \ge |C_i^*| - \tau n$ for $i \in [k]$ and, hence, for any $v \in C_1^* - V(T^0)$,

$$\begin{aligned} |N_{H_0(k,n)}(v) - N_H(v)| &\geq |C_2^* - V(T^0)| |C_3^* - V(T^0)| \cdots |C_k^* - V(T^0)| \\ &\geq ((1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n - k^2/2 - \tau n)^{k-1} \text{ (by (9))} \\ &> (n/10)^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(100k^2) \text{ and } n \geq 100k^2) \\ &> \sqrt{\varepsilon}n^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k-1}), \end{aligned}$$

contradicting the fact that $v \notin N$. Let M^0 be a set of any $|\tau n|$ edges in T^0 .

Now suppose for some $j \in [k]$, we have found matchings $M^0, M^1, \ldots, M^{j-1}$ in H^* such that M^i (for $i = 0, \ldots, j - 1$) consists of $\lfloor \tau n \rfloor$ edges of *i*-type. Let T^j be a matching in $H^* - \bigcup_{i=0}^{j-1} V(M_i)$ such that each edge in T_j is of *j*-type and, subject to this, $|T^j|$ is maximum.

We claim that $|T^j| \ge \lfloor \tau n \rfloor$. For, suppose $|T^j| < \lfloor \tau n \rfloor$. Then, since $C_j^* \cap V(M^i) = \emptyset$ for $i \in [j-1]$ and $|V(M^0)| = \lfloor \tau n \rfloor$, it follows from (9) that

$$|C_{j}^{*} - V(M^{0} \cup M^{1} \cup \ldots \cup M^{j-1}) - V(T^{j})| > (1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^{2}n - k^{2}/2 - k\tau n > 0,$$

where the second inequality holds because $\tau = 1/(9k)$, $\sqrt{\varepsilon} < 1/(100k^2)$, and $n \ge 100k^2$. So let v be a vertex in $C_j^* - V(M^0 \cup M^1 \cup \ldots \cup M^{j-1}) - V(T^j)$. We claim that there exists an edge f of j-type in $H^* - V(M^0 \cup M^1 \cup \ldots \cup M^{j-1}) - V(T^j)$ with $v \in f$; as, otherwise, since $D_i^* \subseteq B_i$ for $i \in [k]$ (by (5)) and k is odd,

$$\begin{aligned} |N_{H_0(k,n)}(v) - N_H(v)| &\geq \prod_{l \in [k] - \{j\}} \left| D_l^* - V(M^0 \cup M^1 \cup \ldots \cup M^{j-1}) - V(T^j) \right| \\ &\geq ((1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n - k^2/2 - k\tau n)^{k-1} \text{ (by (9))} \\ &> (n/10)^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(100k^2), \ \tau = 1/(9k) \text{ and } n \geq 100k^2) \\ &> \sqrt{\varepsilon}n^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k-1}), \end{aligned}$$

contradicting the fact that $v \notin N$.

Let $M^j \subseteq T^j$ with $|M^j| = \lfloor \tau n \rfloor$. Thus, this process works for all $j \in [k]$, and we see that M^0, M^1, \ldots, M^k give the desired matchings for Claim 6. \Box

By Claim 6, there exist pairwise disjoint matchings M^0, M^1, \ldots, M^k in H^* such that

- M^i is of *i*-type for $i \in [k] \cup \{0\}$, and
- $|M^i| = |M^1| \ge \lfloor \tau n \rfloor$ for $i \in [k]$.

We choose such M^0, M^1, \ldots, M^k that

• $|M^1| = |M^2| = \ldots = |M^k|$ is maximum and, subject to this,

• $|M^0|$ is maximum.

Let $M = \bigcup_{i \in [k] \cup \{0\}} M^i$. By Claim 6, we have, for $i \in [k]$,

$$|D_i^* \cap V(M)| \equiv 0 \pmod{k-1}$$
 and $|M^i| \le |D_i^*|/(k-1)$.

Claim 7. $|M^0| \ge \tau n$. For otherwise suppose $|M^0| < \tau$

For, otherwise, suppose $|M^0| < \tau n$. Note that for $i \in [k]$,

$$\begin{aligned} &|C_i^* - V(M^0 \cup M^1 \cup \dots \cup M^k)| \\ &= |C_i^*| - |M^0| - |M^i| \\ &> |C_i^*| - \tau n - |D_i^*|/(k-1) \quad (\text{since } |M^0| < \tau n \text{ and } |M^i| \le |D_i^*|/(k-1))) \\ &\ge |C_i^*| - \tau n - (n - |C_i^*|)/(k-1) \\ &= k|C_i^*|/(k-1) - \tau n - n/(k-1) \\ &\ge ((1/2 - 2\sqrt{\varepsilon}k)n - 2 - 8\sqrt{\varepsilon}k^2n - k^2/2)k/(k-1) - \tau n - n/(k-1) \quad (\text{by } (9)) \\ &> n/10 \text{ (since } \sqrt{\varepsilon} < 1/(100k^2), \ \tau = 1/(9k) \text{ and } n \ge 100k^2). \end{aligned}$$

Thus there exists $v \in C_1^* - V(M^0 \cup M^1 \cup \cdots \cup M^k)$. Since $|M^0|$ is maximized, $C^* - V(M^0 \cup M^1 \cup \cdots \cup M^k)$ has no edge. Therefore,

$$\begin{aligned} |N_{H_0(k,n)}(v) - N_H(v)| &\geq \prod_{i \in [k] - \{1\}} |C_i^* - V(M^0 \cup M^1 \cup \dots \cup M^k)| \\ &> (n/10)^{k-1} \\ &> \sqrt{\varepsilon} n^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k-1}), \end{aligned}$$

contradicting the fact that $v \notin N$. \Box

Claim 8. $D^* \subseteq V(M)$.

For, otherwise, suppose that $D^* - V(M) \neq \emptyset$. Recall that for each *j*-type edge f, $|f \cap C^*| = |f \cap C_j| = 1$. Since $|D_i^*| \equiv 0 \pmod{k-1}$ (by Claim 5) and $|D_i^* \cap V(M)| \equiv 0 \pmod{k-1}$, it follows that $|D_i^* - V(M)| \geq k-1$ for $i \in [k]$. So, for $i \in [k]$, let $s_{i,1}, s_{i,2}, \ldots, s_{i,k-1} \in D_i^* - V(M)$ be distinct.

When $C_i^* - V(M) \neq \emptyset$ for $i \in [k]$, let $w_i \in C_i^* - V(M)$ for $i \in [k]$; otherwise let $\{w_1, \ldots, w_k\} \in M^0$ with $w_i \in C_i^*$ for $i \in [k]$ (by Claim 7). Let $S_j := \{w_j, s_{i,j} : i \in [k] - \{j\}\}$ for $j \in [k-1]$, and let $S_k := \{w_k, s_{i,i} : i \in [k-1]\}$.

Suppose for each $j \in [k]$ there exist distinct $e_1^j, \ldots, e_{k-1}^j \in M^j$ such that $H^*[e_1^j \cup \cdots \cup e_{k-1}^j \cup S_j]$ contains a perfect matching $\{f_1^j, \ldots, f_k^j\}$. Then, $N^j := (M^j - \{e_1^j, \ldots, e_{k-1}^j\}) \cup \{f_1^j, \ldots, f_k^j\}$ is a matching in H^* for each $j \in [k]$, and $|N^j| = |N^1| > \lfloor \tau n \rfloor$ for $j \in [k]$. Let $N^0 = M^0 - \{\{w_1, \ldots, w_k\}\}$. Then N^0, N^1, \ldots, N^k are pairwise disjoint. However, $|N^j| = |M^j| + 1$ for $j \in [k]$, contradicting the choice of M^0, M^1, \ldots, M^k .

Thus we may assume without loss of generality that for any k-1 distinct edges $e_1^k, \ldots, e_{k-1}^k \in M^k$, $H^*[e_1^k \cup \cdots \cup e_{k-1}^k \cup S_k]$ has no perfect matching. For $i \in [k-1]$, let $e_i^k := \{v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$ with $v_{i,k} \in C_k^*$ and $v_{i,j} \in D_j^*$ for $j \in [k-1]$. For convenience, let $v_{k,k} := w_k$ and $v_{k,j} := s_{j,j}$ for $j \in [k-1]$. For $i \in [k]$, define $f_i^k :=$

 $\{v_{1,i+1}, v_{2,i+2}, \ldots, v_{k-1,i+k-1}, v_{k,k+i}\}$, where the addition in the subscripts is modulo k (except that we write k for 0). Then $f_i^k \notin E(H^*)$ for some $i \in [k]$, as otherwise, $\{f_1^k, \ldots, f_k^k\}$ would be a perfect matching in $H^*[e_1^k \cup \cdots \cup e_{k-1}^k \cup S_k]$. Since $e_1^k, \ldots, e_{k-1}^k \in M^k$ are chosen arbitrarily and k is odd (by Claim 5), we have

$$\begin{split} &|\{e \in E(H_0(k,n)) - E(H) : |e \cap \{v_{k,i} : i \in [k]\}| = 1\}|\\ \geq & \binom{|M^k|}{k-1}\\ = & |M^k||M^k - 1|\cdots|M^k - (k-1) + 1|/(k-1)!\\ > & ((\lfloor \tau n \rfloor - k)/(k-1))^{k-1}\\ > & (n/(10k^2))^{k-1} \text{ (since } \tau = 1/(9k) \text{ and } n \ge 100k^2)\\ > & k\sqrt{\varepsilon}n^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k-1}). \end{split}$$

So there exists $i \in [k]$ such that $|N_{H_0(k,n)}(v_{k,i}) - N_H(v_{k,i})| > \sqrt{\varepsilon} n^{k-1}$, contradicting the fact that $v_{k,i} \notin N$. \Box

If $C^* \subseteq V(M)$ then, by Claim 8, M is a perfect matching in H^* ; so $\{e_0\} \cup M_1 \cup M_2 \cup M_3 \cup M$ is a perfect matching in H.

Therefore, we may assume that $C^* \not\subseteq V(M)$, and let $w_i \in C_i^* - V(M)$ for $i \in [k]$. Note that $|M^0| \ge \tau n > k - 1$ (by Claim 7). Let $e_1, \ldots, e_{k-1} \in M^0$ be distinct and chosen arbitrarily. Let $e_i := \{v_{i,1}, v_{i,2}, \ldots, v_{i,k}\}$ for $i \in [k-1]$, where $v_{i,j} \in C_j^*$ for $j \in [k]$. For $i \in [k]$, define $f_i := \{w_i, v_{1,i+1}, v_{2,i+2}, \ldots, v_{k-1,i+k-1}\}$, with the addition in the subscripts taken modulo k (except we use k for 0).

If $f_i \in E(H^*)$ for all $i \in [k]$, then $N^0 := (M^0 \cup \{f_1, \dots, f_k\}) - \{e_1, \dots, e_{k-1}\}$ is a matching in H^* with $|N^0| = |M^0| + 1$; so N^0, M^1, \dots, M^k contradict the choice of M^0, M^1, \dots, M^k .

Hence, $f_i \notin E(H^*)$ for some $i \in [k]$. Since $e_1, \ldots, e_{k-1} \in M^0$ are chosen arbitrarily and k is odd, we have

$$\begin{split} &|\{e \in E(H_0(k,n)) - E(H) : |e \cap \{w_i : i \in [k]\}| = 1\}|\\ \geq & \binom{|M^0|}{k-1}\\ &= & |M^0||M^0 - 1| \cdots |M^0 - (k-1) + 1|/(k-1)!\\ > & ((\lfloor \tau n \rfloor - k)/k)^{k-1}\\ > & (n/(10k^2))^{k-1} \text{ (since } \tau = 1/(9k) \text{ and } n \ge 100k^2)\\ &> & k\sqrt{\varepsilon}n^{k-1} \text{ (since } \sqrt{\varepsilon} < 1/(k(10k^2)^{k-1}). \end{split}$$

So there exists $i \in [k]$ such that $|N_{H_0(k,n)}(w_i) - N_H(w_i)| > \sqrt{\varepsilon} n^{k-1}$, contradicting the fact that $w_i \notin N$.

Corollary 2.2 Let $k \ge 3$ be a positive integer, and let $\varepsilon > 0$ be such that $\sqrt{\varepsilon} < \min\{1/(100k^2), 1/(k(10k^2)^{k-1})\}$. Let H be a k-partite k-graph with $n > 100k^2$ vertices in each partition class, such that $\delta_{k-1}(H) \ge \lfloor n/2 \rfloor$ and H is ε -close to $H_0(k, n)$. Then H has no perfect matching if, and only if,

(i) k is odd, $n \equiv 2 \pmod{4}$, and $H \cong H_0(k, n)$, or

(ii) n is odd and there exist $d_i \in \{(n+1)/2, (n-1)/2\}$ for $i \in [k]$ such that $\sum_{i=1}^k d_i$ is odd and $H \subseteq H_0(d_1, \ldots, d_k; k, n)$.

Proof. Let *H* be a *k*-partite *k*-graph with *n* vertices in each partition class, such that $\delta_{k-1}(H) \geq \lfloor n/2 \rfloor$ and *H* is ε -close to $H_0(k, n)$.

Suppose (i) or (ii) holds. Then there exist integers d_1, \ldots, d_k such that $\sum_{i=1}^k d_i$ is odd, $H \subseteq H_0(d_1, \ldots, d_k; k, n), d_1 = \ldots = d_k = n/2$ when (i) holds, and $d_i \in \{(n+1)/2, (n-1)/2\}$ when (ii) holds. By the definition of $H_0(d_1, \ldots, d_k; k, n)$, there exists $D \subseteq V(H)$ such that $|D| = \sum_{i=1}^k d_i$ is odd and $|e \cap D|$ is even for all $e \in E(H)$. Hence, H contains no perfect matching.

Next, suppose H has no perfect matching. Applying Lemma 2.1 with $\alpha = 1/8$, we may assume that there exist $d_i \in [\lceil 3n/8 \rceil, \lfloor 5n/8 \rfloor]$ for $i \in [k]$ such that $\sum_{i=1}^k d_i$ is odd and $H \subseteq$ $H_0(d_1, \ldots, d_k; k, n)$. Let V_1, \ldots, V_k be the partition classes of H and $H_0(d_1, \ldots, d_k; k, n)$. For $i \in [k]$, let $D_i \subseteq V_i$ be such that $|D_i| = d_i$ and $|e \cap (\bigcup_{j \in [k]} D_j)|$ is even for all $e \in E(H)$.

We claim that $\delta_{k-1}(H) \leq \min\{d_i, n-d_i\}$ for all $i \in [k]$. By symmetry, we only show $\delta_{k-1}(H) \leq \min\{d_1, n-d_1\}$. Let $S := \{v_2, \ldots, v_k\}$ be a legal set such that $v_2 \in D_2$ and $v_i \in V_i - D_i$ for $i \in [k] - \{1, 2\}$; then, since $e \cap D_1 \neq \emptyset$ for all $e \in E(H)$ with $S \subseteq e, \ \delta_{k-1}(H) \leq d_H(S) \leq |D_1| = d_1$. Let $T := \{u_2, \ldots, u_k\}$ be a legal set such that $u_i \in V_i - D_i$ for $i \in [k] - \{1\}$; then, since $e \cap D_1 = \emptyset$ for any $e \in E(H)$ with $T \subseteq e, \ \delta_{k-1}(H) \leq d_H(T) \leq |V_1 - D_1| = n - d_1$. Hence, $\delta_{k-1}(H) \leq \min\{d_1, n - d_1\}$.

If n is odd then $\delta_{k-1}(H) \geq \lfloor n/2 \rfloor = (n-1)/2$; so by the above claim, $d_i \in \{(n-1)/2, (n+1)/2\}$ for all $i \in [k]$, and (ii) holds. Thus, we may assume that n is even. Then by the above claim, $d_i = n/2$ for all $i \in [k]$. Recall that $\sum_{i=1}^k d_i$ is odd. Thus both n/2 and k are odd, and hence $n \equiv 2 \pmod{4}$. Since $H \subseteq H_0(d_1, \ldots, d_k; k, n) = H_0(k, n)$, we have $H = H_0(k, n)$ and (i) holds.

3 Hypergraphs not close to $H_0(k, n)$

In this section, we prove Theorem 1.2 for hypergraphs that are not close to $H_0(k, n)$, see Lemma 3.6. For this, we need a result on almost perfect matchings in k-partite k-graphs.

Kühn and Osthus [4] showed that if H is a k-partite k-graph with each partition classes of size n and $\delta_{k-1}(H) \ge n/k$, then H has a matching of size at least n - (k-2). Rödl and Ruciński [8] asked the following question: Is it true that $\delta_{k-1}(H) \ge n/k$ implies that H has a matching of size at least n - 1? The present authors [6] and, independently, Han, Zang, and Zhao [3] answered this question affirmatively for large n.

Lemma 3.1 Let k, n be positive integers with $k \ge 3$ and n sufficiently large, and let H be a k-partite k-graph with n vertices in each partition class. If $\delta_{k-1}(H) \ge n/k$, then H has a matching of size at least n-1.

Let $k \geq 2$ be a positive integer and H be a k-partite k-graph with partition classes V_1, \ldots, V_k . Given $N_i \subseteq V_i$ for $i \in [k]$, let

$$E_H(N_1,\ldots,N_k) := \{ e \in E(H) : e \subseteq \bigcup_{i \in [k]} N_i \},\$$

and

$$e_H(N_1,\ldots,N_k) := |E_H(N_1,\ldots,N_k)|.$$

For $j \in [k]$, let

$$\Lambda_j := \{\{v_i \in V_i : i \in [k] - \{j\}\} : d_H(\{v_i : i \in [k] - \{j\}\}) \ge (1/2 + 2/\log n)n\}$$

Lemma 3.2 Let $k \ge 2$ be a positive integer. For any $\varepsilon > 0$, there exists $n_0 > 0$ such that the following holds. Let H be a k-partite k-graph with partition classes V_1, \ldots, V_k such that $|V_i| = n \ge n_0$ for $i \in [k]$ and $\delta_{k-1}(H) \ge (1/2 - 1/\log n)n$. Suppose H is not ε -close to $H_0(k, n)$. Then one of the following conclusions holds:

- (i) For all $i \in [k]$ and $N_i \subseteq V_i$ with $|N_i| \ge (1/2 1/\log n)n$, $e_H(N_1, \ldots, N_k) \ge n^k/\log^3 n$.
- (ii) There exists $j \in [k]$ such that $|\Lambda_j| \ge n^{k-1}/\log n$.

Proof. Let *H* be a *k*-partite *k*-graph with partition classes V_1, \ldots, V_k such that $|V_i| = n$ for $i \in [k]$. For convenience, let $\gamma := 1/\log n$. Then $\delta_{k-1}(H) \ge (1/2 - \gamma)n$.

Suppose H is not ε -close to $H_0(k, n)$, and assume that neither (i) nor (ii) holds. Then there exist N_1, \ldots, N_k with $N_i \subseteq V_i$ and $|N_i| \ge (1/2 - \gamma)n$ for $i \in [k]$ such that

$$e_H(N_1, \dots, N_k) < \frac{n^k}{\log^3 n} = o(n^k),$$
 (10)

and, for all $j \in [k]$,

$$|\Lambda_j| < \frac{n^{k-1}}{\log n} = \gamma n^{k-1}.$$
(11)

Claim 1. $|N_i| < (1/2 + 2\gamma)n$ for $i \in [k]$.

For, otherwise, we may assume without loss of generality that $|N_k| \ge (1/2+2\gamma)n$. Then $|V_k - N_k| \le (1/2-2\gamma)n$. For any legal (k-1)-set $\{v_1, \ldots, v_{k-1}\}$ with $v_i \in N_i$ for $i \in [k-1]$, we have

$$|N_H(v_1,\ldots,v_{k-1})\cap N_k| \ge \delta_{k-1}(H) - |V_k - N_k| \ge (1/2 - \gamma)n - (1/2 - 2\gamma)n = \gamma n.$$

Hence, by choosing n_0 large enough, we have for $n \ge n_0$,

$$e_H(N_1,\ldots,N_k) \ge |N_1|\cdots|N_{k-1}||N_H(\{v_1,\ldots,v_{k-1}\})\cap N_k| \ge ((1/2-\gamma)n)^{k-1}\gamma n > \frac{n^k}{\log^3 n},$$

contradicting (10). \Box

For $i \in [k]$, let $N'_i := V_i - N_i$ and $A_i \in \{N_i, N'_i\}$. Since $|N_i| \ge (1/2 - \gamma)n$, $|N'_i| \le (1/2 + \gamma)n$. By Claim 1, $|N'_i| > (1/2 - 2\gamma)n$. Therefore, for $i \in [k]$,

$$(1/2 - 2\gamma)n < |A_i| \le (1/2 + 2\gamma)n.$$
(12)

Claim 2. For $i \in [k]$, $e_H(A_1, \ldots, A_{i-1}, V_i, A_{i+1}, \ldots, A_k) = (n/2)^k + o(n^k)$.

By symmetry, we only prove Claim 2 for the case when i = k. Note that

$$e_H(A_1, \dots, A_{k-1}, V_k) \ge \left(\prod_{i=1}^{k-1} |A_i|\right) (1/2 - \gamma)n \quad \text{(since } \delta_{k-1}(H) \ge (1/2 - \gamma)n)$$
$$\ge ((1/2 - 2\gamma)n)^{k-1} (1/2 - \gamma)n \quad \text{(by (12))}$$
$$= (n/2)^k + o(n^k).$$

On the other hand,

$$e_{H}(A_{1}, \dots, A_{k-1}, V_{k}) \leq |\Lambda_{k}| n + \left(\prod_{i=1}^{k-1} |A_{j}|\right) (1/2 + 2\gamma) n$$

$$< \gamma n^{k} + ((1/2 + 2\gamma)n)^{k-1} (1/2 + 2\gamma) n \quad (by (11) and (12))$$

$$= (n/2)^{k} + o(n^{k}). \quad \Box$$

Claim 3. Let $I(A_1, \ldots, A_k) := \{i \in [k] : A_i = N'_i\}.$

(i) If $|I(A_1, ..., A_k)|$ is odd then $e_H(A_1, ..., A_k) = (n/2)^k + o(n^k)$, and

(*ii*) if $|I(A_1, ..., A_k)|$ is even then $e_H(A_1, ..., A_k) = o(n^k)$.

We apply induction on $|I(A_1, \ldots, A_k)|$. When $|I(A_1, \ldots, A_k)| = 0$, we have $e_H(A_1, \ldots, A_k) = e_H(N_1, \ldots, N_k) = o(n^k)$ by (10). When $|I(A_1, \ldots, A_k)| = 1$, say $A_i = N'_i$ and $A_j = N_j$ for $j \in [k] - \{i\}$, then we have

$$e_H(A_1, \dots, A_k) = e_H(A_1, \dots, A_{i-1}, V_i, A_{i+1}, \dots, A_k) - e_H(N_1, \dots, N_{i-1}, N_i, N_{i+1}, \dots, N_k) = (n/2)^k + o(n^k)$$

by Claim 2 and (10).

Now assume Claim 3 holds for A_1, \ldots, A_k with $A_i \in \{N_i, N'_i\}$ and $0 \leq |I(A_1, \ldots, A_k)| = l < k$. Consider a choice of $A_i \in \{N_i, N'_i\}$ for $i \in [k]$ with $|I(A_1, \ldots, A_k)| = l + 1$. Let $A_j = N'_j$ for some $j \in [k]$. Observe that

$$e_H(A_1,\ldots,A_k) = e_H(A_1,\ldots,A_{j-1},V_j,A_{j+1},\ldots,A_k) - e_H(A_1,\ldots,A_{j-1},N_j,A_{i+1},\ldots,A_k)$$

Therefore, by (10) and Claim 2, it follows from the induction hypothesis that if l+1 is odd then l is even and $e_H(A_1, \ldots, A_k) = ((n/2)^k + o(n^k)) - o(n^k) = (n/2)^k + o(n^k)$, and if l+1is even then l is odd and $e_H(A_1, \ldots, A_k) = ((n/2)^k + o(n^k)) - ((n/2)^k + o(n^k)) = o(n^k)$. \Box

For $i \in [k]$, let $B_i \subseteq V_i$ be such that $|B_i| = \lfloor n/2 \rfloor$ with $|B_i \cap N_i|$ maximal, and let $B'_i := V_i - B_i$. By (12), for $i \in [k]$, $|B_i - N_i| \leq 2\gamma n$ and $|B'_i - N'_i| \leq 2\gamma n$. Hence, if $A_i \in \{N_i, N'_i\}$ and $C_i \in \{B_i, B'_i\}$ such that for $i \in [k]$, $A_i = N_i$ iff $C_i = B_i$, then

$$|E_H(A_1,\ldots,A_k) - E_H(C_1,\ldots,C_k)| = o(n^k)$$

and

$$\begin{aligned} & |E_{H_0(k,n)}(C_1,\ldots,C_k) - E_H(C_1,\ldots,C_k)| \\ \leq & |E_{H_0(k,n)}(C_1,\ldots,C_k) - E_H(A_1,\ldots,A_k)| + |E_H(A_1,\ldots,A_k) - E_H(C_1,\ldots,C_k)| \\ = & |E_{H_0(k,n)}(C_1,\ldots,C_k) - E_H(A_1,\ldots,A_k)| + o(n^k). \end{aligned}$$

For $i \in [k]$, let B_i play the role of D_i in the definition of $H_0(k, n)$. Then, for any $\varepsilon > 0$,

$$\begin{aligned} &|E(H_{0}(k,n)) - E(H)| \\ &\leq \sum_{\substack{C_{i} \in \{B_{i},B'_{i}\}, i \in [k] \\ A_{i} = N_{i} \text{ iff } C_{i} = B_{i} \text{ for } i \in [k]}} \left| E_{H_{0}(k,n)}(C_{1}, \dots, C_{k}) - E_{H}(C_{1}, \dots, C_{k}) \right| \\ &\leq \sum_{\substack{C_{i} \in \{B_{i},B'_{i}\}, A_{i} \in \{N_{i},N'_{i}\} \\ A_{i} = N_{i} \text{ iff } C_{i} = B_{i} \text{ for } i \in [k]}} \left(\left| E_{H_{0}(k,n)}(C_{1}, \dots, C_{k}) - E_{H}(A_{1}, \dots, A_{k}) \right| + o(n^{k}) \right) \\ &\leq \sum_{\substack{C_{i} \in \{B_{i},B'_{i}\} \text{ for } i \in [k]}} \left(\left(\sum_{i \in [k]} \sum_{v \in (B_{i} - N_{i}) \cup (B'_{i} - N'_{i})} |N_{H_{0}(k,n)}(v)| \right) + o(n^{k}) \right) \\ &\leq 2^{k} \left(k(4\gamma n)n^{k-1} + o(n^{k}) \right) \quad (\text{since } |B_{i} - N_{i}| \leq 2\gamma n \text{ and } |B'_{i} - N'_{i}| \leq 2\gamma n) \\ &\leq \varepsilon n^{k} \quad (\text{since } \gamma = 1/\log n \text{ and we may choose } n_{0} \text{ large enough}). \end{aligned}$$

However, this contradicts the assumption that H is not ε -close to $H_0(k, n)$.

Next, we define two "absorbing" matchings for a legal k-set S in a k-partite k-graph. This concept was first considered by Rödl, Ruciński, and Szemerédi [9]. Let $k \ge 3$ be a positive integer and H be a k-partite k-graph.

Given a legal k-set $S = \{x_1, \ldots, x_k\}$ in a k-partite k-graph H, a k-matching $\{e_1, \ldots, e_k\}$ in H is said to be S-absorbing if there is a (k + 1)-matching $\{e'_1, \ldots, e'_k, f\}$ in H with $f = \{y_1, \ldots, y_k\}$ such that

- $e'_i \cap e_j = \emptyset$ for all $i \neq j$,
- $e'_i e_i = \{x_i\}$ and $e_i e'_i = \{y_i\}$ for $i \in [k]$.

Figure 1 illustrates an $\{x_1, x_2, x_3\}$ -absorbing 3-matching $\{e_1, e_2, e_3\}$.

Given a legal k-set $S = \{x_1, \ldots, x_k\}$ in a k-partite k-graph H, a (k + 1)-matching $\{e_0, e_1, \ldots, e_k\}$ in H is said to be S-absorbing if there is a (k+2)-matching $\{e'_1, \ldots, e'_k, f', f''\}$ in H, with $e_1 \cap f' = f' - e_0 = \{y_1\}, e_0 - f' = \{y_0\}, \text{ and } f'' := \{y_0, y_2, \ldots, y_k\}$, such that

- $e'_i \cap e_j = \emptyset$ for all $i \neq j$, and
- $e'_i e_i = \{x_i\}$ and $e_i e'_i = \{y_i\}$ for all $i \in [k]$.

Figure 2 illustrates an $\{x_1, x_2, x_3\}$ -absorbing 4-matching $\{e_0, e_1, e_2, e_3\}$.

The next result says that no matter which conclusion of Lemma 3.2 holds, there are always many S-absorbing matchings in H for any given legal k-set S.

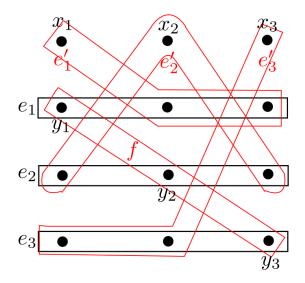


Figure 1: $\{x_1, x_2, x_3\}$ -absorbing matchings

Lemma 3.3 Let $k \ge 3$ be a positive integer. There exists $n_1 > 0$ such that the following holds. Let H be a k-partite k-graph with $n \ge n_1$ vertices in each partition class and with $\delta_{k-1}(H) \ge (1/2 - 1/\log n)n$. Let $S := \{x_1, \ldots, x_k\} \subseteq V(H)$ be legal.

- (i) If for all $i \in [k]$ and $N_i \subseteq V_i$ with $|N_i| \ge (1/2 1/\log n)n$, we have $e_H(N_1, \ldots, N_k) \ge n^k/\log^3 n$, then the number of S-absorbing k-matchings in H is $\Omega(n^{k^2}/\log^3 n)$.
- (ii) If there exists $j \in [k]$ such that $|\Lambda_j| \ge n^{k-1}/\log n$, then the number of S-absorbing (k+1)-matchings in H is $\Omega(n^{k^2+k}/\log^3 n)$.

Proof. To prove (i), we assume that, for all $i \in [k]$ and $N_i \subseteq V_i$ with $|N_i| \ge (1/2 - 1/\log n)n$, we have $e_H(N_1, \ldots, N_k) \ge n^k/\log^3 n$.

Note that, for each $i \in [k]$, x_i is contained in $(n-1)^{k-2}$ legal (k-1)-sets in H that are disjoint from S and one given partition class of H, and each such legal (k-1)-set is contained in at least $(1/2-1/\log n)n-1$ edges in H-S (since $\delta_{k-1}(H) \ge (1/2-1/\log n)n$). Thus, there exists n_1 such that if $n \ge n_1$, there are at least $n^{k-1}/3$ legal (k-1)-sets B_i disjoint from S such that $e'_i := \{x_i\} \cup B_i \in E(H)$.

By a similar argument (and choosing n_1 large enough), there are at least $((n-k)^{(k-1)}/3)^k \ge (1/3 - o(1))^k n^{(k-1)k}$ choices of pairwise disjoint such legal (k-1)-sets B_1, \ldots, B_k .

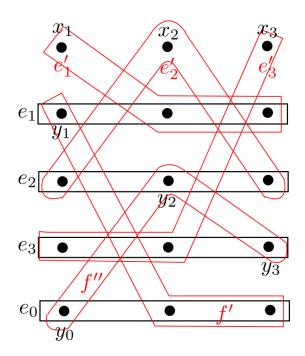


Figure 2: $\{x_1, x_2, x_3\}$ -absorbing (k + 1)-matching for k = 3

For each such choice of B_1, \ldots, B_k , let $N_i := N_H(B_i)$ for $i \in [k]$. Then $|N_i| \ge \delta_{k-1}(H) \ge (1/2 - 1/\log n)n$. By assumption, $e_H(N_1, \ldots, N_k) \ge n^k/\log^3 n$; so there are at least $n^k/\log^3 n - k^2 n^{k-1}$ choices of an edge $f := \{y_1, \ldots, y_k\}$ from $H[\bigcup_{i \in [k]} N_i] - \bigcup_{i \in [k]} e'_i$ such that $e_i := B_i \cup \{y_i\} \in E(H)$ for $i \in [k]$.

Hence, the number of S-absorbing k-matchings $\{e_1, \ldots, e_k\}$ is at least

$$(1/3 - o(1))^k n^{(k-1)k} (n^k / \log^3 n - k^2 n^{k-1}) = \Omega(n^{k^2} / \log^3 n),$$

as claimed in (i).

We now prove (*ii*). So assume without loss of generality that $|\Lambda_1| \ge n^{k-1}/\log n$. As in the previous case, since $\delta_{k-1}(H) \ge (1/2-1/\log n)n$, there are at least $(1/3-o(1))^k n^{(k-1)k} =$ $\Omega(n^{(k-1)k})$ choices of disjoint legal (k-1)-sets B_1, \ldots, B_k such that $\{x_i\} \cup B_i \in E(H)$ for $i \in [k]$.

For i = 2, ..., k, we choose $y_i \in N_H(B_i) - \{x_i\}$ and let $e_i := B_i \cup \{y_i\}$. Note that we have $(1/2 - 2/\log n)n - 1 = \Omega(n)$ choices for each y_i .

By assumption, there are at least $n^{k-1}/\log n - k(k+2)n^{k-2} = \Omega(n^{k-1}/\log n)$ choices for a (k-1)-set $T \in \Lambda_1$ that is disjoint from $S \cup B_1 \cup \cdots \cup B_k \cup \{y_2, \ldots, y_k\}$. Since $N_H(T) > (1/2+2/\log n)n$ (as $T \in \Lambda_1$) and $\delta_{k-1}(H) \ge (1/2-1/\log n)n$, we have $|N_H(T) \cap$ $N_H(B_1)| \ge n/\log n$ and $|N_H(T) \cap N_H(\{y_2, \ldots, y_k\})| \ge n/\log n$. Consequently, there exist distinct y_0 and y_1 with $y_1 \in (N_H(T) \cap N_H(B_1)) - (S \cup B_1 \cup \cdots \cup B_k)$ and $y_0 \in (N_H(T) \cap N_H(\{y_2, \ldots, y_k\})) - (S \cup B_1 \cup \ldots \cup B_k)$, and there are at least $n/\log n - k(k+1) - 1$ choices for each of y_0 and y_1 .

Let $e_0 := \{y_0\} \cup T$, $e_1 := \{y_1\} \cup B_1$, $f' := \{y_1\} \cup T$ and $f'' := \{y_0, y_2, y_3, \dots, y_k\}$. Then $\{e_0, \dots, e_k\}$ is an S-absorbing (k + 1)-matching (using $e'_i = B_i \cup \{x_i\}$ for $i \in [k]$). Moreover, the number of choice for $\{e_0, \dots, e_k\}$ is the product of the numbers of choices for $B_1, \dots, B_k, y_2, \dots, y_k, T, y_0, y_1$, which is at least

$$\Omega\left(n^{k(k-1)}\right)\Omega\left(n^{k-1}\right)\Omega\left(\frac{n^{k-1}}{\log n}\right)\left(\frac{n}{\log n}-k(k+1)-1\right)^2=\Omega\left(\frac{n^{k^2+k}}{\log^3 n}\right).$$

So we have (ii).

We will need to use Chernoff bounds, which can be found in [7].

Lemma 3.4 Suppose $X_1, ..., X_n$ are independent random variables taking values in $\{0, 1\}$. Let X denote their sum and $\mu = \mathbb{E}[X]$ denote the expected value of X. Then for any $0 < \delta \leq 1$,

$$\mathbb{P}[X \ge (1+\delta)\mu] < e^{-\frac{\delta^2\mu}{3}} \text{ and } \mathbb{P}[X \le (1-\delta)\mu] < e^{-\frac{\delta^2\mu}{2}},$$

and for any $\delta \geq 1$,

$$\mathbb{P}[X \ge (1+\delta)\mu] < e^{-\frac{\delta\mu}{3}}.$$

We now show that for each conclusion of Lemma 3.2, there exists a small matching M' in H such that for each legal k-set S, there are at least k-pairwise disjoint S-absorbing matchings in H.

Lemma 3.5 Let $k \ge 3$ be a positive integer. There exists $n_2 > 0$ such that the following holds. Let H be a k-partite k-graph with partition classes V_1, \ldots, V_k such that $|V_i| = n > n_2$ for $i \in [k]$ and $\delta_{k-1}(H) \ge (1/2 - 1/\log n)n$.

- (i) If for all $i \in [k]$ and $N_i \subseteq V_i$ with $|N_i| \ge (1/2 1/\log n)n$, we have $e_H(N_1, \ldots, N_k) \ge n^k/\log^3 n$, then there exists a matching M' in H such that $|M'| = O(\log^5 n)$ and for every legal k-set $S \subseteq V(H)$, there are at least k pairwise disjoint S-absorbing k-matchings in M'.
- (ii) If there exists $j \in [k]$ such that $|\Lambda_j| \ge n^{k-1}/\log n$, then there exists a matching M' in H such that $|M'| = O(\log^5 n)$ and for every legal k-set $S \subseteq V(H)$, there are at least k pairwise disjoint S-absorbing (k+1)-matchings in M'.

Proof. First, we prove (i). Suppose for all $i \in [k]$ and $N_i \subseteq V_i$ with $|N_i| \ge (1/2 - 1/\log n)n$, we have $e_H(N_1, \ldots, N_k) \ge n^k/\log^3 n$. So we can apply (i) of Lemma 3.3.

For each legal k-set $S \subseteq V(H)$, let $\Gamma(S)$ be the set of (S_1, \ldots, S_k) with $S_i \subseteq V_i$ and $|S_i| = k$ for $i \in [k]$ such that $H[\bigcup_{i \in [k]} S_i]$ has a perfect matching, say $M_{(S_1,\ldots,S_k)}$. Then by (i) of Lemma 3.3, $|\Gamma(S)| = \Omega(n^{k^2}/\log^3 n)/k^k$. So there exists $\alpha := \alpha(k) > 0$ such that $|\Gamma(S)| \ge \alpha {n \choose k}^k / \log^3 n$.

Let \mathcal{F} be the (random) family whose members are (S_1, \ldots, S_k) with $S_i \subseteq V_i$ and $|S_i| = k$ for $i \in [k]$, obtained by choosing each of the $\binom{n}{k}^k$ such (S_1, \ldots, S_k) independently with probability

$$p = \frac{\log^5 n}{\binom{n}{k}^k}.$$

Note that p < 1 as we can choose n_2 large enough. Then

$$\mathbb{E}(|\mathcal{F}|) = p\binom{n}{k}^{k} = \log^{5} n$$

and for each legal k-set $S \subseteq V(H)$,

$$\mathbb{E}(|\mathcal{F} \cap \Gamma(S)|) \ge p\alpha {\binom{n}{k}}^k / \log^3 n = \alpha \log^2 n.$$

By Lemma 3.4 and by choosing n_2 large enough, we have, for $n > n_2$,

$$\mathbb{P}[|\mathcal{F}| > 2\log^5 n] = \mathbb{P}[|\mathcal{F}| > 2\mathbb{E}(|\mathcal{F}|)] \le e^{-\mathbb{E}(|\mathcal{F}|)/3} = e^{-(\log^5 n)/3} < 1/10.$$

So with probability at least 9/10

$$|\mathcal{F}| \le 2\log^5 n. \tag{13}$$

Again by Lemma 3.4 and by choosing n_2 large enough, we have, for $n > n_2$,

$$\mathbb{P}[|\mathcal{F} \cap \Gamma(S)| \le (\alpha \log^2 n)/2] \le \mathbb{P}[|\mathcal{F} \cap \Gamma(S)| \le \mathbb{E}(|\mathcal{F} \cap \Gamma(S)|)/2]$$
$$\le e^{-\mathbb{E}(|\mathcal{F} \cap \Gamma(S)|)/8}$$
$$< e^{-(\alpha \log^2 n)/8}.$$

So by union bound and choosing n_2 large, we have for $n > n_2$,

$$\mathbb{P}[\exists \text{ legal } S \subseteq V(H) : |\mathcal{F} \cap \Gamma(S)| \le (\alpha \log^2 n)/2] \le n^k e^{-(\alpha \log^2 n)/8} = 2n^{k-(\alpha \log n)/8} < 1/10.$$

Thus, with probability at least 9/10, for each legal k-set $S \subseteq V(H)$, we have

$$|\mathcal{F} \cap \Gamma(S)| \ge (\alpha \log^2 n)/2 > k.$$
(14)

Furthermore, the expected number of pairs of elements $(S_1, \ldots, S_k), (T_1, \ldots, T_k) \in \mathcal{F}$ satisfying $(\bigcup_{i \in [k]} S_i) \cap (\bigcup_{i \in [k]} T_i) \neq \emptyset$ is at most

$$\binom{n}{k}^{k} k^{2} \binom{n-1}{k-1} \binom{n}{k}^{k-1} p^{2} \le \frac{k^{3} \log^{10} n}{n} < 1/2.$$

Thus, with probability at least 1/2 (by Markov's inequality), for all distinct $(S_1, \ldots, S_k) \in \mathcal{F}$ and $(T_1, \ldots, T_k) \in \mathcal{F}$,

$$\cup_{i \in [k]} S_i$$
 and $\cup_{i \in [k]} T_i$ are disjoint. (15)

Hence, with positive probability, \mathcal{F} satisfies (13), (14), and (15). So we may assume that \mathcal{F} satisfies (13), (14), and (15). Let M' be the union of $M_{(S_1,\ldots,S_k)}$ for $(S_1,\ldots,S_k) \in \mathcal{F}$. Then M' is the desired matching for (i).

Next we prove (*ii*). Suppose there exists $j \in [k]$ such that $|\Lambda_j| \ge n^{k-1}/\log n$; so that we can apply (*ii*) of Lemma 3.3.

For each legal k-set $S \subseteq V(H)$, let $\Gamma'(S)$ be the set of sequences (S_1, \ldots, S_k) , with $S_i \subseteq V_i$ and $|S_i| = k + 1$ for $i \in [k]$, such that $H[\bigcup_{i \in [k]} S_i]$ has a perfect matching, say $M_{(S_1,\ldots,S_k)}$. Then by (ii) of Lemma 3.3, $|\Gamma'(S)| = \Omega(n^{k^2+k}/\log^3 n)/(k+1)^k$. So there exists $\alpha' > 0$ such that $|\Gamma'(S)| \ge \alpha' {n \choose k+1}^k/\log^3 n$. We form a random family \mathcal{G} consisting of sequences (S_1,\ldots,S_k) , with $S_i \subseteq V_i$ and

We form a random family \mathcal{G} consisting of sequences (S_1, \ldots, S_k) , with $S_i \subseteq V_i$ and $|S_i| = k + 1$ for $i \in [k]$, by selecting each of the $\binom{n}{k+1}^k$ such (S_1, \ldots, S_k) independently with probability

$$p = \frac{\log^5 n}{\binom{n}{k+1}^k}.$$

Note that p < 1 by choosing n_2 large enough. Then

$$\mathbb{E}(|\mathcal{G}|) = p\binom{n}{k+1}^{k} = \log^{5} n,$$

and for each legal k-set $S \subseteq V(H)$,

$$\mathbb{E}(|\mathcal{G} \cap \Gamma'(S)|) \ge p\alpha' \binom{n}{k+1}^k / \log^3 n = \alpha' \log^2 n.$$

By Lemma 3.4 and by choosing n_2 large enough, we have for $n > n_2$,

$$\mathbb{P}[|\mathcal{G}| > 2\log^5 n] = \mathbb{P}[|\mathcal{G}| > 2\mathbb{E}(|\mathcal{G}|)] \le 2e^{-\mathbb{E}(|\mathcal{G}|)/3} = 2e^{-(\log^5 n)/3} < 1/10$$

So with probability at least 9/10,

$$|\mathcal{G}| \le 2\log^5 n. \tag{16}$$

Again by Lemma 3.4 and by choosing n_2 large enough, we have for $n > n_2$,

$$\mathbb{P}[|\mathcal{G} \cap \Gamma'(S)| \le (\alpha' \log^2 n)/2] \le \mathbb{P}[|\mathcal{G} \cap \Gamma'(S)| \le \mathbb{E}(|\mathcal{G} \cap \Gamma'(S)|/2)]$$
$$\le e^{-\mathbb{E}(|\mathcal{G} \cap \Gamma'(S)|)/8}$$
$$< e^{-\alpha' \log^2 n/8}.$$

So by union bound and choosing n_2 large,

 $\mathbb{P}[\exists \text{ legal } S \subseteq V(H) : |\mathcal{G} \cap \Gamma'(S)| \le \alpha' \log^2 n/2] \le n^k e^{-(\alpha' \log^2 n)/8} = n^{k - (\alpha' \log n)/8} < 1/10.$

Hence, with probability at least 9/10, for each legal k-set $S \subseteq V(H)$,

$$|\mathcal{G} \cap \Gamma'(S)| \ge (\alpha' \log^2 n)/2 > k.$$
(17)

Furthermore, the expected number of pairs $(S_1, \ldots, S_k), (T_1, \ldots, T_k) \in \mathcal{G}$ with $(\bigcup_{i \in [k]} S_i) \cap (\bigcup_{i \in [k]} T_i) \neq \emptyset$ is

$$\binom{n}{k+1}^k k(k+1)\binom{n-1}{k}\binom{n}{k+1}^{k-1} p^2 \le \frac{(k+1)^3 \log^{10} n}{n} < 1/2.$$

Thus, by Markov's inequality, with probability at least 1/2, for all distinct $(S_1, \ldots, S_k) \in \mathcal{G}$ and $(T_1, \ldots, T_k) \in \mathcal{G}$,

$$(\bigcup_{i \in [k]} S_i) \cap (\bigcup_{i \in [k]} T_i) = \emptyset.$$
(18)

Hence, with positive probability, \mathcal{G} satisfies (16), (17), and (18). So we may assume that \mathcal{G} satisfies (16), (17), and (18) Let M' be the union of $M'_{(S_1,\ldots,S_k)}$ for all $(S_1,\ldots,S_k) \in \mathcal{G}$. Now M' gives the desired matching for (*ii*).

Corollary 3.6 Let $k \ge 3$ be a positive integer. For any $\varepsilon > 0$, there exists $n_3 > 0$ such that the following holds. Let H be a k-partite k-graph with $n > n_3$ vertices in each partition class. Suppose $\delta_{k-1}(H) \ge (1/2 - 1/\log n)n$ and H is not ε -close to $H_0(k, n)$. Then H has a perfect matching.

Proof. Choose n_3 large enough so that we can apply Lemmas 3.1, 3.2, and 3.5.

By Lemmas 3.2 and 3.5, H contains a matching M such that $|M| \leq \beta \log^5 n$ for some constant $\beta > 0$ (dependent on k only) and, for every legal k-set $S \subseteq V(H)$, there are at least k disjoint S-absorbing k-matchings in M, or for every legal k-set $S \subseteq V(H)$, there are at least k disjoint S-absorbing (k + 1)-matchings in M.

For $k \geq 3$,

$$\delta_{k-1}(H - V(M)) \ge (1/2 - 1/\log n)n - \beta \log^5 n > n/k,$$

where the last inequality holds for $n > n_3$ by choosing n_3 large enough. Thus by Lemma 3.1, H-V(M) contains a matching M' of size at least n-|M|-1. Let $S := H-V(M\cup M')$. If $S = \emptyset$, then $M \cup M'$ is a perfect matching in H. So assume that $S \neq \emptyset$; then S is a legal k-set. Hence $H[S \cup V(M)]$ has a perfect matching M''. Now $M' \cup M''$ is a perfect matching in H.

4 Conclusion

Proof of Theorem 1.2. First, suppose (i) or (ii) holds. Then there exist integers d_1, \ldots, d_k such that $\sum_{i=1}^k d_i$ is odd and $H \subseteq H_0(d_1, d_2, \ldots, d_k; k, n)$. By definition of $H_0(d_1, d_2, \ldots, d_k; k, n)$, there exists $D \subseteq V(H)$ such that $|D| = \sum_{i=1}^k d_i$ is odd and $|e \cap D|$ is even for all $e \in E(H)$. Hence, H contains no perfect matchings.

Now assume that H has no perfect matching. Fix $\varepsilon > 0$ so that

$$\sqrt{\varepsilon} < \min\{1/(100k^2), 1/(k(10k^2)^{k-1})\}.$$

Then by Corollary 3.6, H must be ε -close to $H_0(k, n)$. Hence by Corollary 2.2, (i) or (ii) holds.

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