The minimum number of Hamilton cycles in a hamiltonian threshold graph of a prescribed order^{*}

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Abstract

We prove that the minimum number of Hamilton cycles in a hamiltonian threshold graph of order n is $2^{\lfloor (n-3)/2 \rfloor}$ and this minimum number is attained uniquely by the graph with degree sequence $n - 1, n - 1, n - 2, \ldots, \lceil n/2 \rceil, \lceil n/2 \rceil, \ldots, 3, 2$ of n - 2 distinct degrees. This graph is also the unique graph of minimum size among all hamiltonian threshold graphs of order n.

Key words. Threshold graph; Hamilton cycle; minimum size

1 Introduction

There are few results concerning the precise value of the minimum or maximum number of Hamilton cycles of graphs in a special class with a prescribed order. For example, it is known that the minimum number of Hamilton cycles in a simple hamiltonian cubic graph of order n is 3, which follows from Smith's theorem [1, p.493] and an easy construction [10, p.479], but the maximum number of Hamilton cycles is not known; even the conjectured upper bound $2^{n/3}$ [2, p.312] has not been proved. Another example is Sheehan's conjecture that every simple hamiltonian 4-regular graph has at least two Hamilton cycles [10] (see also [1, p.494 and p.590]), which is still unsolved.

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In this paper we will determine the minimum number of Hamilton cycles in a hamiltonian threshold graph of order n and the unique minimizing graph. Threshold graphs were introduced by Chvátal and Hammer [3] in 1973. Besides the original definition, seven equivalent characterizations are given in the book [7].

Definition 1. A finite simple graph G is called a *threshold graph* if there exists a nonnegative real-valued function f defined on the vertex set of G, $f : V(G) \to \mathbb{R}$ and a nonnegative real number t such that for any two distinct vertices u and v, u and v are adjacent if and only if f(u) + f(v) > t.

The class of threshold graphs play a special role for many reasons, some of which are the following: 1) They have geometrical significance. Let Ω_n be the convex hull of all degree sequences of the simple graphs of order n. Then the extreme points of the polytope Ω_n are exactly the degree sequences of threshold graphs of order n [6] (for another proof see [9]). 2) A nonnegative integer sequence is graphical if and only if it is majorized by the degree sequence of some threshold graph [9]. 3) A graphical sequence has a unique labeled realization if and only if it is the degree sequence of a threshold graph [7, p.72].

For terminology and notations we follow the textbooks [1,11]. The order of a graph is its number of vertices, and the size its number of edges. We regard isomorphic graphs as the same graph. Thus for two graphs G and H, G = H means that G and H are isomorphic. N(v) and N[v] denote the neighborhood and closed neighborhood of a vertex v respectively. For a real number r, $\lfloor r \rfloor$ denotes the largest integer less than or equal to r, and $\lceil r \rceil$ denotes the least integer larger than or equal to r. The notation |S| denotes the cardinality of a set S.

2 Main Results

Let G = (V, E) be a graph whose distinct positive vertex-degrees are $\delta_1 < \cdots < \delta_m$ and let $\delta_0 = 0$. Denote $D_i = \{v \in V | \deg(v) = \delta_i\}$ for $i = 0, 1, \ldots, m$. The sequence D_0, D_1, \ldots, D_m is called the *degree partition* of G. Each D_i is called a *degree set*. Sometimes when D_0 is empty it may be omitted. These notations will be used throughout. We will need the following characterization [7, p.11] which describes the basic structure of a threshold graph. **Lemma 1.** G is a threshold graph if and only if for each $v \in D_k$,

$$N(v) = \bigcup_{j=1}^{k} D_{m+1-j} \quad \text{if } k = 1, \dots, \lfloor m/2 \rfloor$$
$$N[v] = \bigcup_{j=1}^{k} D_{m+1-j} \quad \text{if } k = \lfloor m/2 \rfloor + 1, \dots, m.$$

In other words, for $x \in D_i$ and $y \in D_j$, x is adjacent to y if and only if i + j > m.

Clearly, Lemma 1 not only implies another characterization that the vicinal preorder of a threshold graph is a total preorder, but also indicates that every threshold graph is determined uniquely by its degree sequence [7, p.72].

The following lemma can be found in [7, pp.11-13].

Lemma 2. For any threshold graph,

$$\delta_{k+1} = \delta_k + |D_{m-k}| \quad \text{for } k = 0, 1, \dots, m, \ k \neq \lfloor m/2 \rfloor$$
$$\delta_{k+1} = \delta_k + |D_{m-k}| - 1 \quad \text{for } k = \lfloor m/2 \rfloor.$$

For two subsets S and T of the vertex set of a graph G, the notation [S, T] denotes the set of edges of G with one end-vertex in S and the other end-vertex in T. Here S and T need not be disjoint. In the case T = S, [S, S] is just the edge set of the subgraph G[S]of G induced by S. Next we define a new concept which will be used in the proofs.

Definition 2. An edge of a threshold graph G with degree partition D_0, D_1, \ldots, D_m is called a *key edge* of G if it lies in $[D_k, D_{m+1-k}]$ for some k with $1 \le k \le \lceil m/2 \rceil$.

Thus when m is even we have only one type of key edges, and when m is odd $(m \ge 3)$ we have two types of key edges. For example, if m = 4 then the set of key edges is $[D_1, D_4] \cup [D_2, D_3]$ while if m = 5 then the set of key edges is $[D_1, D_5] \cup [D_2, D_4] \cup [D_3, D_3]$. We will need the following two lemmas concerning properties of key edges.

Lemma 3. If e is a key edge of a threshold graph G, then G - e is a threshold graph.

Proof. Denote G' = G - e and let m' be the number of distinct positive vertex-degrees of G'. Let e = xy. First suppose that $x \in D_j$ and $y \in D_{m+1-j}$ for some $1 \le j \le \lfloor m/2 \rfloor$. We write TPO for the conditions in Lemma 1 (suggesting total preorder). To prove that G' is a threshold graph, by Lemma 1 it suffices to show that the degree sets of G' satisfy TPO. The structural change of the degree partitions depends on the sizes of the two sets D_j and D_{m+1-j} . We distinguish four cases.

Case 1. $|D_j| = 1$ and $|D_{m+1-j}| = 1$. The condition $|D_j| = 1$ implies that $j = \lfloor m/2 \rfloor$ is possible only if m is odd, since if m is even then $|D_{m/2}| \ge 2$. Hence m - j > j, implying that D_{m-j} and D_j are two distinct sets. By Lemma 2,

$$\delta_j = \delta_{j-1} + |D_{m+1-j}| = \delta_{j-1} + 1$$
 and $\delta_{m+1-j} = \delta_{m-j} + |D_j| = \delta_{m-j} + 1.$

After deleting e, the two sets D_j and D_{m+1-j} become empty, and they disappear in G'. x goes to D_{j-1} and y goes to D_{m-j} . Now m' = m - 2 and the adjacency relations among the vertices of G' still satisfy TPO.

Case 2. $|D_j| = 1$ and $|D_{m+1-j}| \ge 2$. As in case 1, D_{m-j} and D_j are two distinct sets. By Lemma 2, we have

$$\delta_j = \delta_{j-1} + |D_{m+1-j}| \ge \delta_{j-1} + 2$$
 and $\delta_{m+1-j} = \delta_{m-j} + |D_j| = \delta_{m-j} + 1.$

When deleting e, x stays in D_j and y goes to D_{m-j} . Thus m' = m and G' satisfies TPO.

Case 3. $|D_j| \ge 2$ and $|D_{m+1-j}| = 1$. We have $\delta_j = \delta_{j-1} + |D_{m+1-j}| = \delta_{j-1} + 1$. When deleting e, x goes to D_{j-1} . If m is even, j = m/2 and $|D_j| = 2$, then $\delta_{m+1-j} = \delta_{m/2} + |D_{m/2}| - 1 = \delta_j + 1$. When deleting e, y goes to D_j and the set D_{m+1-j} disappears. Thus m' = m - 1. In all other cases, we have $\delta_{m+1-j} \ge \delta_{m-j} + 2$. In fact, if m is odd or m is even and j < m/2, we have $\delta_{m+1-j} = \delta_{m-j} + |D_j| \ge \delta_{m-j} + 2$, while if m is even, j = m/2 and $|D_j| \ge 3$, we have $\delta_{m+1-j} = \delta_{m-j} + |D_j| - 1 \ge \delta_{m-j} + 2$. When deleting e, y remains in D_{m+1-j} . Thus m' = m. In each case, G' satisfies TPO.

Case 4. $|D_j| \ge 2$ and $|D_{m+1-j}| \ge 2$. We have $\delta_j = \delta_{j-1} + |D_{m+1-j}| \ge \delta_{j-1} + 2$. If m is even, j = m/2 and $|D_j| = 2$, then $\delta_{m+1-j} = \delta_{j+1} = \delta_j + |D_j| - 1 = \delta_j + 1$. When deleting e, x remains in D_j (but with degree $\delta_j - 1$) and a new degree set $\{y\} \cup (D_j \setminus \{x\})$ appears. Now m' = m + 1. In all other cases, two new degree sets appear, one containing only xand the other containing only y, so that m' = m + 2. In either case, G' satisfies TPO and hence it is a threshold graph.

Now suppose that m is odd and $x, y \in D_t$ where $t = \lfloor m/2 \rfloor + 1 = \lceil m/2 \rceil$. Apply Lemma 2. If $|D_t| = 2$, when deleting e, both x and y go to $D_{\lfloor m/2 \rfloor}$ and the degree set D_t disappears. Then m' = m - 1 and G' satisfies TPO. Otherwise $|D_t| \ge 3$. When deleting e, a new degree set $\{x, y\}$ appears, where x and y are nonadjacent. In this case m' = m + 1and G' again satisfies TPO. \Box **Lemma 4.** Every key edge of a hamiltonian threshold graph lies in at least one Hamilton cycle.

Proof. Let G be a hamiltonian threshold graph with degree partition D_1, \ldots, D_m . Let e = xy be a key edge of G with $x \in D_j$ and $y \in D_{m+1-j}$ for some $1 \le j \le \lceil m/2 \rceil$. Choose any Hamilton cycle C of G. If e lies in C, we are done. Otherwise let $C = (x, s, \ldots, y, t, \ldots)$. Then s and x are adjacent, and t and y are adjacent. Applying Lemma 1 we deduce that s and t are adjacent. Now the classical cycle exchange [1, p.485] with $x^+ = s$ and $y^+ = t$ yields a new Hamilton cycle containing the edge $e.\Box$

Different necessary and sufficient conditions for a threshold graph to be hamiltonian are given by Golumbic [4], Harary and Peled [5], and Mahadev and Peled [8]. What we need is the following one by Golumbic [4, p.231] whose proof can be found in [7, p.25].

Lemma 5. Let G be a threshold graph of order at least 3 with the degree partition D_0, D_1, \ldots, D_m . Then G is hamiltonian if and only if $D_0 = \phi$,

$$\sum_{j=1}^{k} |D_j| < \sum_{j=1}^{k} |D_{m+1-j}|, \quad k = 1, \dots, \lfloor (m-1)/2 \rfloor$$

and if *m* is even, then $\sum_{j=1}^{m/2} |D_j| \le \sum_{j=1}^{m/2} |D_{m+1-j}|$.

Definition 3. For every integer $n \ge 3$, we denote by G_n the graph with degree sequence $n - 1, n - 1, n - 2, ..., \lceil n/2 \rceil, \lceil n/2 \rceil, ..., 3, 2$ of n - 2 distinct degrees.

 G_n is a hamiltonian threshold graph. G_8 is depicted in Figure 1.

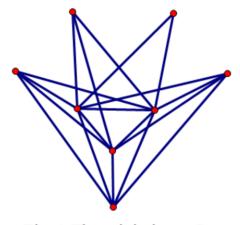


Fig. 1 The minimizer G_8

Now we are ready to prove the main results.

Theorem 6. The minimum number of Hamilton cycles in a hamiltonian threshold graph of order n is $2^{\lfloor (n-3)/2 \rfloor}$ and this minimum number is attained uniquely by the graph G_n .

Proof. We first determine the minimizing graph and then count its number of Hamilton cycles. Let G be a hamiltonian threshold graph of order n having the minimum number of Hamilton cycles. Let D_1, \ldots, D_m be the degree partition of G. Note that for any threshold graph with $m \ge 1$, we have $|D_{\lceil m/2 \rceil}| \ge 2$. This follows from

$$1 \le \delta_{\lfloor m/2 \rfloor + 1} - \delta_{\lfloor m/2 \rfloor} = |D_{\lceil m/2 \rceil}| - 1$$

by Lemma 2.

The theorem holds trivially for the case n = 3. Next suppose $n \ge 4$. m = 1 means that G is a complete graph, which is impossible. Thus $m \ge 2$. We claim that $|D_m| = 2$. Lemma 5 with k = 1 implies $|D_m| \ge 2$. Hence it suffices to prove $|D_m| \le 2$. To the contrary suppose $|D_m| \ge 3$. Let e be any edge in $[D_1, D_m]$. Then e is a key edge by definition. By Lemma 3, G - e is a threshold graph. Since G is a hamiltonian threshold graph, its degree sets D_1, \ldots, D_m satisfy the inequalities in Lemma 5. Analyzing the change of degree partitions from G to G - e as in the proof of Lemma 3, we see that the degree sets of G - e also satisfy the inequalities in Lemma 5. Hence by Lemma 5, G - eis hamiltonian. Deleting any edge cannot increase the number of Hamilton cycles. By Lemma 4, the key edge e lies in at least one Hamilton cycle of G. It follows that G - ehas fewer Hamilton cycles than G, contradicting the minimum property of G. This proves $|D_m| = 2$.

If m = 2, then by Lemma 5 we have $|D_1| \leq 2$. Since $n \geq 4$, we must have n = 4 and $|D_1| = 2$. Then applying Lemma 1 we deduce that G has the degree sequence 3, 3, 2, 2, so that $G = G_4$. Next suppose $m \geq 3$. By Lemma 5, $|D_1| < |D_m| = 2$. Hence $|D_1| = 1$. We first consider the case $m \geq 4$ (The case m = 3 will be treated later). We claim that $|D_{m-1}| = 1$. Otherwise, as argued above, deleting any key edge f in $[D_2, D_{m-1}]$ would reduce the number of Hamilton cycles such that G - f is still a hamiltonian threshold graph, a contradiction. Then using the fact that $|D_1| = 1$ and $|D_m| = 2$ and applying Lemma 5 we deduce that if m is odd or if m is even and $m \geq 6$ then $|D_2| = 1$, and if m = 4 then $|D_2| = 2$. Continuing in this way, by successively deleting a key edge in $[D_j, D_{m+1-j}]$ for $j = 2, \ldots, \lfloor m/2 \rfloor$ if $|D_{m+1-j}| \geq 2$ we conclude that $|D_{m+1-j}| = 1$ for each $j = 2, \ldots, \lfloor m/2 \rfloor$. Then using the fact that $|D_1| = 1$ and $|D_m| = 2$ and applying Lemma 5, we conclude that $|D_i| = 1$ for each $i = 2, \ldots, \lfloor m/2 \rfloor$.

then $|D_{\lfloor m/2 \rfloor}| = 1$ and if *m* is even then $|D_{m/2}| = 2$. Thus, if *m* is even then n = m + 2 is even, *G* has the degree sequence $n - 1, n - 1, \ldots, n/2, n/2, \ldots, 3, 2$ and hence $G = G_n$.

If $m \ge 3$ and m is odd, we assert that $|D_{\lceil m/2 \rceil}| = 2$. As remarked at the beginning, we always have $|D_{\lceil m/2 \rceil}| \ge 2$. Thus it suffices to show $|D_{\lceil m/2 \rceil}| \le 2$. To the contrary suppose $|D_{\lceil m/2 \rceil}| \ge 3$. By Lemma 4, any key edge h in $G[D_{\lceil m/2 \rceil}]$ lies in at least one Hamilton cycle. With the assumption that $|D_{\lceil m/2 \rceil}| \ge 3$, applying Lemma 3 and Lemma 5 we see that G - h is also a hamiltonian threshold graph with fewer Hamilton cycles than G, a contradiction. This shows $|D_{\lceil m/2 \rceil}| = 2$. Now n = m + 2 is odd. Combining all the above information about G we deduce that G has the degree sequence $n - 1, n - 1, \ldots, (n + 1)/2, (n + 1)/2, \ldots, 3, 2$ and hence $G = G_n$.

Denote the number of Hamilton cycles of G_n by f(n). Since f(3) = f(4) = 1, to prove $f(n) = 2^{\lfloor (n-3)/2 \rfloor}$ it suffices to show the following

Claim. For every integer $k \ge 2$,

$$f(2k-1) = f(2k)$$
 and $f(2k+1) = 2f(2k)$.

In G_{2k} , let $D_k = \{x, y\}$ and $D_{k+1} = \{z\}$. By Lemma 5, neither $G_{2k} - xz$ nor $G_{2k} - yz$ is hamiltonian. Thus the path xzy must lie in every Hamilton cycle of G_{2k} . Deleting the vertex z and adding the edge xy we obtain a graph which is isomorphic to G_{2k-1} and has the same number of Hamilton cycles as G_{2k} . Hence f(2k-1) = f(2k).

In G_{2k+1} , let $D_{k+1} = \{u, v\}$. Then the edge uv lies in every Hamilton cycle of G_{2k+1} . Denote G' = G - v. Clearly G' is isomorphic to G_{2k} and hence G' has f(2k) Hamilton cycles. Since $N(u) \setminus \{v\} = N(v) \setminus \{u\}$ in G, from each Hamilton cycle of G' we can obtain two distinct Hamilton cycles of G_{2k+1} by replacing the vertex u by the edge uv in two ways. More precisely, a Hamilton cycle $(\ldots, s, u, t, \ldots)$ of G' yields two Hamilton cycles $(\ldots, s, u, v, t, \ldots)$ and $(\ldots, s, v, u, t, \ldots)$ of G. Conversely every Hamilton cycle of G_{2k+1} can be obtained in such a vertex-to-edge expansion from a Hamilton cycle of G'. Hence f(2k+1) = 2f(2k). This shows the claim and completes the proof. \Box

The above proof of Theorem 6 also proves that G_n is the unique graph that has the minimum size among all hamiltonian threshold graphs of order n. To see this, just replace the assumption that G has the minimum number of Hamilton cycles by the one that G has the minimum size. Also note that the size of a threshold graph is easy to count, since it is a split graph with the clique $\bigcup_{j=\lfloor m/2 \rfloor+1}^m D_j$ and the independent set $\bigcup_{j=1}^{\lfloor m/2 \rfloor} D_j$. Thus we have the following result.

Theorem 7. The minimum size of a hamiltonian threshold graph of order n is

$$\begin{cases} (n^2 + 2n - 3)/4 & \text{if } n \text{ is odd} \\ (n^2 + 2n - 4)/4 & \text{if } n \text{ is even} \end{cases}$$

and this minimum size is attained uniquely by the graph G_n .

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