# The minimum number of Hamilton cycles in a hamiltonian threshold graph of a prescribed order* 

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#### Abstract

We prove that the minimum number of Hamilton cycles in a hamiltonian threshold graph of order $n$ is $2^{\lfloor(n-3) / 2\rfloor}$ and this minimum number is attained uniquely by the graph with degree sequence $n-1, n-1, n-2, \ldots,\lceil n / 2\rceil,\lceil n / 2\rceil, \ldots, 3,2$ of $n-2$ distinct degrees. This graph is also the unique graph of minimum size among all hamiltonian threshold graphs of order $n$.


Key words. Threshold graph; Hamilton cycle; minimum size

## 1 Introduction

There are few results concerning the precise value of the minimum or maximum number of Hamilton cycles of graphs in a special class with a prescribed order. For example, it is known that the minimum number of Hamilton cycles in a simple hamiltonian cubic graph of order $n$ is 3 , which follows from Smith's theorem [1, p.493] and an easy construction [10, p.479], but the maximum number of Hamilton cycles is not known; even the conjectured upper bound $2^{n / 3}$ [2, p.312] has not been proved. Another example is Sheehan's conjecture that every simple hamiltonian 4-regular graph has at least two Hamilton cycles [10] (see also [1, p. 494 and p.590]), which is still unsolved.

[^0]In this paper we will determine the minimum number of Hamilton cycles in a hamiltonian threshold graph of order $n$ and the unique minimizing graph. Threshold graphs were introduced by Chvátal and Hammer [3] in 1973. Besides the original definition, seven equivalent characterizations are given in the book [7].

Definition 1. A finite simple graph $G$ is called a threshold graph if there exists a nonnegative real-valued function $f$ defined on the vertex set of $G, f: V(G) \rightarrow \mathbb{R}$ and a nonnegative real number $t$ such that for any two distinct vertices $u$ and $v, u$ and $v$ are adjacent if and only if $f(u)+f(v)>t$.

The class of threshold graphs play a special role for many reasons, some of which are the following: 1) They have geometrical significance. Let $\Omega_{n}$ be the convex hull of all degree sequences of the simple graphs of order $n$. Then the extreme points of the polytope $\Omega_{n}$ are exactly the degree sequences of threshold graphs of order $n$ [6] (for another proof see [9]). 2) A nonnegative integer sequence is graphical if and only if it is majorized by the degree sequence of some threshold graph [9]. 3) A graphical sequence has a unique labeled realization if and only if it is the degree sequence of a threshold graph [7, p.72].

For terminology and notations we follow the textbooks $[1,11]$. The order of a graph is its number of vertices, and the size its number of edges. We regard isomorphic graphs as the same graph. Thus for two graphs $G$ and $H, G=H$ means that $G$ and $H$ are isomorphic. $N(v)$ and $N[v]$ denote the neighborhood and closed neighborhood of a vertex $v$ respectively. For a real number $r,\lfloor r\rfloor$ denotes the largest integer less than or equal to $r$, and $\lceil r\rceil$ denotes the least integer larger than or equal to $r$. The notation $|S|$ denotes the cardinality of a set $S$.

## 2 Main Results

Let $G=(V, E)$ be a graph whose distinct positive vertex-degrees are $\delta_{1}<\cdots<\delta_{m}$ and let $\delta_{0}=0$. Denote $D_{i}=\left\{v \in V \mid \operatorname{deg}(v)=\delta_{i}\right\}$ for $i=0,1, \ldots, m$. The sequence $D_{0}, D_{1}, \ldots, D_{m}$ is called the degree partition of $G$. Each $D_{i}$ is called a degree set. Sometimes when $D_{0}$ is empty it may be omitted. These notations will be used throughout. We will need the following characterization [7, p.11] which describes the basic structure of a threshold graph.

Lemma 1. $G$ is a threshold graph if and only if for each $v \in D_{k}$,

$$
\begin{gathered}
N(v)=\bigcup_{j=1}^{k} D_{m+1-j} \quad \text { if } k=1, \ldots,\lfloor m / 2\rfloor \\
N[v]=\bigcup_{j=1}^{k} D_{m+1-j} \quad \text { if } k=\lfloor m / 2\rfloor+1, \ldots, m .
\end{gathered}
$$

In other words, for $x \in D_{i}$ and $y \in D_{j}, x$ is adjacent to $y$ if and only if $i+j>m$.
Clearly, Lemma 1 not only implies another characterization that the vicinal preorder of a threshold graph is a total preorder, but also indicates that every threshold graph is determined uniquely by its degree sequence [7, p.72].

The following lemma can be found in [7, pp.11-13].
Lemma 2. For any threshold graph,

$$
\begin{gathered}
\delta_{k+1}=\delta_{k}+\left|D_{m-k}\right| \text { for } k=0,1, \ldots, m, k \neq\lfloor m / 2\rfloor \\
\delta_{k+1}=\delta_{k}+\left|D_{m-k}\right|-1 \quad \text { for } k=\lfloor m / 2\rfloor
\end{gathered}
$$

For two subsets $S$ and $T$ of the vertex set of a graph $G$, the notation $[S, T]$ denotes the set of edges of $G$ with one end-vertex in $S$ and the other end-vertex in $T$. Here $S$ and $T$ need not be disjoint. In the case $T=S,[S, S]$ is just the edge set of the subgraph $G[S]$ of $G$ induced by $S$. Next we define a new concept which will be used in the proofs.

Definition 2. An edge of a threshold graph $G$ with degree partition $D_{0}, D_{1}, \ldots, D_{m}$ is called a key edge of $G$ if it lies in $\left[D_{k}, D_{m+1-k}\right]$ for some $k$ with $1 \leq k \leq\lceil m / 2\rceil$.

Thus when $m$ is even we have only one type of key edges, and when $m$ is odd ( $m \geq 3$ ) we have two types of key edges. For example, if $m=4$ then the set of key edges is $\left[D_{1}, D_{4}\right] \cup\left[D_{2}, D_{3}\right]$ while if $m=5$ then the set of key edges is $\left[D_{1}, D_{5}\right] \cup\left[D_{2}, D_{4}\right] \cup\left[D_{3}, D_{3}\right]$. We will need the following two lemmas concerning properties of key edges.

Lemma 3. If e is a key edge of a threshold graph $G$, then $G-e$ is a threshold graph.
Proof. Denote $G^{\prime}=G-e$ and let $m^{\prime}$ be the number of distinct positive vertex-degrees of $G^{\prime}$. Let $e=x y$. First suppose that $x \in D_{j}$ and $y \in D_{m+1-j}$ for some $1 \leq j \leq\lfloor m / 2\rfloor$. We write TPO for the conditions in Lemma 1 (suggesting total preorder). To prove that $G^{\prime}$ is a threshold graph, by Lemma 1 it suffices to show that the degree sets of $G^{\prime}$ satisfy

TPO. The structural change of the degree partitions depends on the sizes of the two sets $D_{j}$ and $D_{m+1-j}$. We distinguish four cases.

Case 1. $\left|D_{j}\right|=1$ and $\left|D_{m+1-j}\right|=1$. The condition $\left|D_{j}\right|=1$ implies that $j=\lfloor m / 2\rfloor$ is possible only if $m$ is odd, since if $m$ is even then $\left|D_{m / 2}\right| \geq 2$. Hence $m-j>j$, implying that $D_{m-j}$ and $D_{j}$ are two distinct sets. By Lemma 2,

$$
\delta_{j}=\delta_{j-1}+\left|D_{m+1-j}\right|=\delta_{j-1}+1 \quad \text { and } \quad \delta_{m+1-j}=\delta_{m-j}+\left|D_{j}\right|=\delta_{m-j}+1
$$

After deleting $e$, the two sets $D_{j}$ and $D_{m+1-j}$ become empty, and they disappear in $G^{\prime}$. $x$ goes to $D_{j-1}$ and $y$ goes to $D_{m-j}$. Now $m^{\prime}=m-2$ and the adjacency relations among the vertices of $G^{\prime}$ still satisfy TPO.

Case 2. $\left|D_{j}\right|=1$ and $\left|D_{m+1-j}\right| \geq 2$. As in case $1, D_{m-j}$ and $D_{j}$ are two distinct sets. By Lemma 2, we have

$$
\delta_{j}=\delta_{j-1}+\left|D_{m+1-j}\right| \geq \delta_{j-1}+2 \quad \text { and } \quad \delta_{m+1-j}=\delta_{m-j}+\left|D_{j}\right|=\delta_{m-j}+1
$$

When deleting $e, x$ stays in $D_{j}$ and $y$ goes to $D_{m-j}$. Thus $m^{\prime}=m$ and $G^{\prime}$ satisfies TPO.
Case 3. $\left|D_{j}\right| \geq 2$ and $\left|D_{m+1-j}\right|=1$. We have $\delta_{j}=\delta_{j-1}+\left|D_{m+1-j}\right|=\delta_{j-1}+1$. When deleting $e, x$ goes to $D_{j-1}$. If $m$ is even, $j=m / 2$ and $\left|D_{j}\right|=2$, then $\delta_{m+1-j}=$ $\delta_{m / 2}+\left|D_{m / 2}\right|-1=\delta_{j}+1$. When deleting $e, y$ goes to $D_{j}$ and the set $D_{m+1-j}$ disappears. Thus $m^{\prime}=m-1$. In all other cases, we have $\delta_{m+1-j} \geq \delta_{m-j}+2$. In fact, if $m$ is odd or $m$ is even and $j<m / 2$, we have $\delta_{m+1-j}=\delta_{m-j}+\left|D_{j}\right| \geq \delta_{m-j}+2$, while if $m$ is even, $j=m / 2$ and $\left|D_{j}\right| \geq 3$, we have $\delta_{m+1-j}=\delta_{m-j}+\left|D_{j}\right|-1 \geq \delta_{m-j}+2$. When deleting $e, y$ remains in $D_{m+1-j}$. Thus $m^{\prime}=m$. In each case, $G^{\prime}$ satisfies TPO.

Case 4. $\left|D_{j}\right| \geq 2$ and $\left|D_{m+1-j}\right| \geq 2$. We have $\delta_{j}=\delta_{j-1}+\left|D_{m+1-j}\right| \geq \delta_{j-1}+2$. If $m$ is even, $j=m / 2$ and $\left|D_{j}\right|=2$, then $\delta_{m+1-j}=\delta_{j+1}=\delta_{j}+\left|D_{j}\right|-1=\delta_{j}+1$. When deleting $e, x$ remains in $D_{j}$ (but with degree $\delta_{j}-1$ ) and a new degree set $\{y\} \cup\left(D_{j} \backslash\{x\}\right)$ appears. Now $m^{\prime}=m+1$. In all other cases, two new degree sets appear, one containing only $x$ and the other containing only $y$, so that $m^{\prime}=m+2$. In either case, $G^{\prime}$ satisfies TPO and hence it is a threshold graph.

Now suppose that $m$ is odd and $x, y \in D_{t}$ where $t=\lfloor m / 2\rfloor+1=\lceil m / 2\rceil$. Apply Lemma 2. If $\left|D_{t}\right|=2$, when deleting $e$, both $x$ and $y$ go to $D_{\lfloor m / 2\rfloor}$ and the degree set $D_{t}$ disappears. Then $m^{\prime}=m-1$ and $G^{\prime}$ satisfies TPO. Otherwise $\left|D_{t}\right| \geq 3$. When deleting $e$, a new degree set $\{x, y\}$ appears, where $x$ and $y$ are nonadjacent. In this case $m^{\prime}=m+1$ and $G^{\prime}$ again satisfies TPO.

Lemma 4. Every key edge of a hamiltonian threshold graph lies in at least one Hamilton cycle.

Proof. Let $G$ be a hamiltonian threshold graph with degree partition $D_{1}, \ldots, D_{m}$. Let $e=x y$ be a key edge of $G$ with $x \in D_{j}$ and $y \in D_{m+1-j}$ for some $1 \leq j \leq\lceil m / 2\rceil$. Choose any Hamilton cycle $C$ of $G$. If $e$ lies in C, we are done. Otherwise let $C=(x, s, \ldots, y, t, \ldots)$. Then $s$ and $x$ are adjacent, and $t$ and $y$ are adjacent. Applying Lemma 1 we deduce that $s$ and $t$ are adjacent. Now the classical cycle exchange [1, p.485] with $x^{+}=s$ and $y^{+}=t$ yields a new Hamilton cycle containing the edge $e$.

Different necessary and sufficient conditions for a threshold graph to be hamiltonian are given by Golumbic [4], Harary and Peled [5], and Mahadev and Peled [8]. What we need is the following one by Golumbic [4, p.231] whose proof can be found in [7, p.25].

Lemma 5. Let $G$ be a threshold graph of order at least 3 with the degree partition $D_{0}, D_{1}, \ldots, D_{m}$. Then $G$ is hamiltonian if and only if $D_{0}=\phi$,

$$
\sum_{j=1}^{k}\left|D_{j}\right|<\sum_{j=1}^{k}\left|D_{m+1-j}\right|, \quad k=1, \ldots,\lfloor(m-1) / 2\rfloor
$$

and if $m$ is even, then $\sum_{j=1}^{m / 2}\left|D_{j}\right| \leq \sum_{j=1}^{m / 2}\left|D_{m+1-j}\right|$.
Definition 3. For every integer $n \geq 3$, we denote by $G_{n}$ the graph with degree sequence $n-1, n-1, n-2, \ldots,\lceil n / 2\rceil,\lceil n / 2\rceil, \ldots, 3,2$ of $n-2$ distinct degrees.
$G_{n}$ is a hamiltonian threshold graph. $G_{8}$ is depicted in Figure 1.


Fig. 1 The minimizer $G_{8}$

Now we are ready to prove the main results.

Theorem 6. The minimum number of Hamilton cycles in a hamiltonian threshold graph of order $n$ is $2^{\lfloor(n-3) / 2\rfloor}$ and this minimum number is attained uniquely by the graph $G_{n}$.

Proof. We first determine the minimizing graph and then count its number of Hamilton cycles. Let $G$ be a hamiltonian threshold graph of order $n$ having the minimum number of Hamilton cycles. Let $D_{1}, \ldots, D_{m}$ be the degree partition of $G$. Note that for any threshold graph with $m \geq 1$, we have $\left|D_{\lceil m / 2\rceil}\right| \geq 2$. This follows from

$$
1 \leq \delta_{\lfloor m / 2\rfloor+1}-\delta_{\lfloor m / 2\rfloor}=\left|D_{\lceil m / 2\rceil}\right|-1
$$

by Lemma 2.
The theorem holds trivially for the case $n=3$. Next suppose $n \geq 4$. $m=1$ means that $G$ is a complete graph, which is impossible. Thus $m \geq 2$. We claim that $\left|D_{m}\right|=2$. Lemma 5 with $k=1$ implies $\left|D_{m}\right| \geq 2$. Hence it suffices to prove $\left|D_{m}\right| \leq 2$. To the contrary suppose $\left|D_{m}\right| \geq 3$. Let $e$ be any edge in [ $D_{1}, D_{m}$ ]. Then $e$ is a key edge by definition. By Lemma 3, $G-e$ is a threshold graph. Since $G$ is a hamiltonian threshold graph, its degree sets $D_{1}, \ldots, D_{m}$ satisfy the inequalities in Lemma 5. Analyzing the change of degree partitions from $G$ to $G-e$ as in the proof of Lemma 3, we see that the degree sets of $G-e$ also satisfy the inequalities in Lemma 5. Hence by Lemma $5, G-e$ is hamiltonian. Deleting any edge cannot increase the number of Hamilton cycles. By Lemma 4, the key edge $e$ lies in at least one Hamilton cycle of $G$. It follows that $G-e$ has fewer Hamilton cycles than $G$, contradicting the minimum property of $G$. This proves $\left|D_{m}\right|=2$.

If $m=2$, then by Lemma 5 we have $\left|D_{1}\right| \leq 2$. Since $n \geq 4$, we must have $n=4$ and $\left|D_{1}\right|=2$. Then applying Lemma 1 we deduce that $G$ has the degree sequence $3,3,2,2$, so that $G=G_{4}$. Next suppose $m \geq 3$. By Lemma $5,\left|D_{1}\right|<\left|D_{m}\right|=2$. Hence $\left|D_{1}\right|=1$. We first consider the case $m \geq 4$ (The case $m=3$ will be treated later). We claim that $\left|D_{m-1}\right|=1$. Otherwise, as argued above, deleting any key edge $f$ in $\left[D_{2}, D_{m-1}\right]$ would reduce the number of Hamilton cycles such that $G-f$ is still a hamiltonian threshold graph, a contradiction. Then using the fact that $\left|D_{1}\right|=1$ and $\left|D_{m}\right|=2$ and applying Lemma 5 we deduce that if $m$ is odd or if $m$ is even and $m \geq 6$ then $\left|D_{2}\right|=1$, and if $m=4$ then $\left|D_{2}\right|=2$. Continuing in this way, by successively deleting a key edge in $\left[D_{j}, D_{m+1-j}\right]$ for $j=2, \ldots,\lfloor m / 2\rfloor$ if $\left|D_{m+1-j}\right| \geq 2$ we conclude that $\left|D_{m+1-j}\right|=1$ for each $j=2, \ldots,\lfloor m / 2\rfloor$. Then using the fact that $\left|D_{1}\right|=1$ and $\left|D_{m}\right|=2$ and applying Lemma 5 , we conclude that $\left|D_{i}\right|=1$ for each $i=2, \ldots,\lfloor m / 2\rfloor-1$ and that if $m$ is odd
then $\left|D_{\lfloor m / 2\rfloor}\right|=1$ and if $m$ is even then $\left|D_{m / 2}\right|=2$. Thus, if $m$ is even then $n=m+2$ is even, $G$ has the degree sequence $n-1, n-1, \ldots, n / 2, n / 2, \ldots, 3,2$ and hence $G=G_{n}$.

If $m \geq 3$ and $m$ is odd, we assert that $\left|D_{\lceil m / 2\rceil}\right|=2$. As remarked at the beginning, we always have $\left|D_{\lceil m / 2\rceil}\right| \geq 2$. Thus it suffices to show $\left|D_{\lceil m / 2\rceil}\right| \leq 2$. To the contrary suppose $\left|D_{[m / 2\rceil}\right| \geq 3$. By Lemma 4, any key edge $h$ in $G\left[D_{[m / 2\rceil}\right]$ lies in at least one Hamilton cycle. With the assumption that $\left|D_{\lceil m / 2\rceil}\right| \geq 3$, applying Lemma 3 and Lemma 5 we see that $G-h$ is also a hamiltonian threshold graph with fewer Hamilton cycles than $G$, a contradiction. This shows $\left|D_{\lceil m / 2\rceil}\right|=2$. Now $n=m+2$ is odd. Combining all the above information about $G$ we deduce that $G$ has the degree sequence $n-1, n-1, \ldots,(n+$ 1) $/ 2,(n+1) / 2, \ldots, 3,2$ and hence $G=G_{n}$.

Denote the number of Hamilton cycles of $G_{n}$ by $f(n)$. Since $f(3)=f(4)=1$, to prove $f(n)=2^{\lfloor(n-3) / 2\rfloor}$ it suffices to show the following

Claim. For every integer $k \geq 2$,

$$
f(2 k-1)=f(2 k) \quad \text { and } \quad f(2 k+1)=2 f(2 k)
$$

In $G_{2 k}$, let $D_{k}=\{x, y\}$ and $D_{k+1}=\{z\}$. By Lemma 5, neither $G_{2 k}-x z$ nor $G_{2 k}-y z$ is hamiltonian. Thus the path $x z y$ must lie in every Hamilton cycle of $G_{2 k}$. Deleting the vertex $z$ and adding the edge $x y$ we obtain a graph which is isomorphic to $G_{2 k-1}$ and has the same number of Hamilton cycles as $G_{2 k}$. Hence $f(2 k-1)=f(2 k)$.

In $G_{2 k+1}$, let $D_{k+1}=\{u, v\}$. Then the edge $u v$ lies in every Hamilton cycle of $G_{2 k+1}$. Denote $G^{\prime}=G-v$. Clearly $G^{\prime}$ is isomorphic to $G_{2 k}$ and hence $G^{\prime}$ has $f(2 k)$ Hamilton cycles. Since $N(u) \backslash\{v\}=N(v) \backslash\{u\}$ in $G$, from each Hamilton cycle of $G^{\prime}$ we can obtain two distinct Hamilton cycles of $G_{2 k+1}$ by replacing the vertex $u$ by the edge $u v$ in two ways. More precisely, a Hamilton cycle ( $\ldots, s, u, t, \ldots$ ) of $G^{\prime}$ yields two Hamilton cycles $(\ldots, s, u, v, t, \ldots)$ and $(\ldots, s, v, u, t, \ldots)$ of $G$. Conversely every Hamilton cycle of $G_{2 k+1}$ can be obtained in such a vertex-to-edge expansion from a Hamilton cycle of $G^{\prime}$. Hence $f(2 k+1)=2 f(2 k)$. This shows the claim and completes the proof.

The above proof of Theorem 6 also proves that $G_{n}$ is the unique graph that has the minimum size among all hamiltonian threshold graphs of order $n$. To see this, just replace the assumption that $G$ has the minimum number of Hamilton cycles by the one that $G$ has the minimum size. Also note that the size of a threshold graph is easy to count, since it is a split graph with the clique $\bigcup_{j=\lfloor m / 2\rfloor+1}^{m} D_{j}$ and the independent set $\bigcup_{j=1}^{\lfloor m / 2\rfloor} D_{j}$. Thus we have the following result.

Theorem 7. The minimum size of a hamiltonian threshold graph of order $n$ is

$$
\begin{cases}\left(n^{2}+2 n-3\right) / 4 & \text { if } n \text { is odd } \\ \left(n^{2}+2 n-4\right) / 4 & \text { if } n \text { is even }\end{cases}
$$

and this minimum size is attained uniquely by the graph $G_{n}$.

## References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory, GTM 244, Springer, 2008.
[2] G.L. Chia and C. Thomassen, On the number of longest and almost longest cycles in cubic graphs, Ars Combinatoria, 104(2012), 307-320.
[3] V. Chvátal and P.L. Hammer, Set-packing problems and threshold graphs, CORR 73-21, University of Waterloo, Canada, August 1973.
[4] M.C. Golumbic, Algorithmic Graph Theory and Perfect Graphs, Second Edition, Elsevier, 2004.
[5] F. Harary and U.N. Peled, Hamiltonian threshold graphs, Discrete Appl. Math., 16(1987), 11-15.
[6] M. Koren, Extreme degree sequences of simple graphs, J. Combin. Theory Ser. B, 15(1973), 213-224.
[7] N.V.R. Mahadev and U.N. Peled, Threshold Graphs and Related Topics, Elsevier Science B.V., 1995.
[8] N.V.R. Mahadev and U.N. Peled, Longest cycles in threshold graphs, Discrete Math., 135(1994), 169-176.
[9] U.N. Peled and M.K. Srinivasan, The polytope of degree sequences, Linear Algebra Appl., 114/115(1989), 349-377.
[10] J. Sheehan, The multiplicity of hamiltonian circuits in a graph, in Recent Advances in Graph Theory, 477-480, Academia, Prague, 1975.
[11] D.B. West, Introduction to Graph Theory, Prentice Hall, Inc., 1996.


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