ISOMETRIC COPIES OF DIRECTED TREES IN ORIENTATIONS OF GRAPHS

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Abstract. The *isometric Ramsey number* $\mathsf{IR}(\vec{\mathcal{H}})$ of a family $\vec{\mathcal{H}}$ of digraphs is the smallest number of vertices in a graph G such that any orientation of the edges of G contains every member of $\vec{H}$ in the distance-preserving way. We observe that for any finite family $\vec{\mathcal{H}}$ of finite acyclic graphs the isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{H}})$ is finite, and present upper bounds for $\mathsf{IR}(\vec{\mathcal{H}})$ in some special cases. For example, we show that the isometric Ramsey number of the family of all oriented trees with n vertices is at most $n^{2n+o(n)}$.

1. Introduction

In this paper we consider the "isometric" version of the result of Cochand and Duchet [6] who proved (generalizing a result of Rödl [11]) that for every acyclic digraph $\vec{H}$ there exists a finite graph G such that every orientation of G contains an isomorphic copy of $\vec{H}$.

First we recall the necessary definitions from Graph Theory. A *graph* is a pair $G = (V_G, E_G)$ consisting of a set V_G of *vertices* and a set E_G of two-element subsets of V_G, called the *edges* of G. By a *digraph* we will mean a pair $\vec{G} = (V_{\vec{G}}, E_{\vec{G}})$ consisting of a set $V_{\vec{G}}$ of vertices and a set $E_{\vec{G}} \subset V_{\vec{G}} \times V_{\vec{G}}$ of *directed edges*, where neither loops (x, x), nor pairs of opposite arcs (x, y) and (y, x) are allowed. By an *orientation* of a graph $G = (V_G, E_G)$ we understand a function $\vec{\cdot}: E_G \to V_G^2$ assigning to each edge $e \in E_G$ an ordered pair $\vec{e} = (a, b) \in V_G^2$ such that $e = \{a, b\}$. In this case the pair $\vec{G} = (V_G, \{\vec{e}\}_{e\in E_G})$ is a digraph called an *orientation* of G.

A sequence $(v_0, \dots, v_n)$ of distinct vertices of a graph G is called a *path* in G if for every positive $i \le n$ the unordered pair $\{v_{i-1}, v_i\}$ is an edge of G. The *length* of the path $(v_0, \dots, v_n)$ is n, that is, the number of edges. The *distance* $d_G(x, y)$ between two vertices v, u of a graph G is the smallest length of a path in G connecting the vertices v and u. If u and v cannot be connected by a path, then we write $d_G(x, y) = \infty$ and assume that $\infty > n$ for all $n \in \omega$. A graph G is called *connected* if any two vertices u, v can be connected by a path in G. The distance in a digraph is taken with respect to the underlying undirected graph.

A sequence $(v_0, \dots, v_n)$ of distinct vertices of a digraph $\vec{G}$ is called a *directed path* in $\vec{G}$ if for every positive $i \le n$ the ordered pair (v_{i-1}, v_i) is an edge of G. A *directed cycle* is a sequence $(v_0, \dots, v_n)$ of distinct vertices with (x_i, x_{i+1}) being a directed edge for each residue i modulo $n+1$. A digraph $\vec{G}$ is *acyclic* if it contains no directed cycles. It is well-known that each graph G admits an acyclic orientation $\vec{G}$: take any linear order $\le$ on the set V_G of vertices and for any edge $\{u, v\} \in E_G$ put $(u, v) \in E_{\vec{G}}$ if and only if $u < v$.

Following Rado's arrow notations, for a graph G and a digraph $\vec{H}$ we write $G \to \vec{H}$ if for every orientation $\vec{G}$ of G there exists an injective function $f : V_{\vec{H}} \to V_G$ such that an ordered pair (u, v) of vertices of $\vec{H}$ is a directed edge in $\vec{H}$ if and only if $(f(u), f(v))$ is a directed edge in $\vec{G}$. (Thus we require that f induces an isomorphism of undirected graphs and preserves all edge orientations.) If, moreover, $d_{\vec{H}}(u, v) = d_G(f(u), f(v))$ for every pair of vertices $u, v \in V_{\vec{H}}$, then we write $G \Rightarrow \vec{H}$ and say that f is an *isometric embedding* of $\vec{H}$ in $\vec{G}$. Since each graph G admits an acyclic orientation, the arrow $G \to \vec{H}$ implies that the digraph $\vec{H}$ is acyclic.

Given a graph G and a class $\vec{\mathcal{H}}$ of digraphs, we write $G \to \vec{\mathcal{H}}$ (resp. $G \Rightarrow \vec{\mathcal{H}}$) if for every oriented graph $\vec{H} \in \vec{\mathcal{H}}$ we have $G \to \vec{H}$ (resp. $G \Rightarrow \vec{H}$). In this case the family $\vec{\mathcal{H}}$ necessarily consists of acyclic digraphs. For a natural number $n \in \mathbb{N}$ by $\vec{\mathcal{T}}_n$ we denote the class of oriented trees on n vertices. By a *tree* we understand a connected graph without cycles. For $n \in \mathbb{N}$, the *directed path* $\vec{I}_n$ is the digraph with $V_{\vec{I}_n} = \{0, \dots, n-1\}$ and $E_{\vec{I}_n} = \{(i-1, i) : 0 < i < n\}$.

For a class $\vec{\mathcal{H}}$ of digraphs let $\mathsf{R}(\vec{\mathcal{H}})$ (resp. $\mathsf{IR}(\vec{\mathcal{H}})$) be the smallest number of vertices of a graph G such that $G \to \vec{\mathcal{H}}$ (resp. $G \Rightarrow \vec{\mathcal{H}}$). If no graph G with $G \to \vec{\mathcal{H}}$ (resp. $G \Rightarrow \vec{\mathcal{H}}$) exists, then we put $\mathsf{R}(\vec{\mathcal{H}}) = \infty$ (resp.

1991 *Mathematics Subject Classification.* 05C20; 05C55; 05C80.
Key words and phrases. Orientation of a graph, directed tree, isometric embedding, girth, chromatic number.
O. Pikhurko was supported by ERC grant 306493 and EPSRC grant EP/K012045/1.

$\mathsf{IR}(\vec{\mathcal{H}}) = \infty$). The number $\mathsf{R}(\vec{\mathcal{H}})$ (resp. $\mathsf{IR}(\vec{\mathcal{H}})$) is called the (*isometric*) *Ramsey number* of the family $\vec{\mathcal{H}}$. If the family $\vec{\mathcal{H}}$ consists of a unique digraph $\vec{H}$, then we write $\mathsf{R}(\vec{H})$ and $\mathsf{IR}(\vec{H})$ instead of $\mathsf{R}(\{\vec{H}\})$ and $\mathsf{IR}(\{\vec{H}\})$, respectively.

By Theorem B of Cochand and Duchet [6], for every finite acyclic digraph $\vec{H}$, the Ramsey number $\mathsf{R}(\vec{H})$ is finite. This implies that for every finite family $\vec{\mathcal{H}}$ of finite acyclic digraphs the Ramsey number $\mathsf{R}(\vec{\mathcal{H}}) \le \sum_{\vec{H}\in\vec{\mathcal{H}}} \mathsf{R}(\vec{H})$ is finite, too. In Section 2 we shall apply a deep Ramsey result of Dellamonica and Rödl [7] to prove that the isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{H}})$ is finite, too.

For the family $\vec{\mathcal{T}}_n$ of oriented trees on n vertices Kohayakawa, Łuczak and Rödl [9] proved that $\mathsf{R}(\vec{\mathcal{T}}_n) = O(n^4 \log n)$. In this paper for every $n \in \mathbb{N}$ we construct a graph G_n with $< 2^{2^{n-1}}$ vertices such that $G_n \Rightarrow \vec{\mathcal{T}}_n$, showing that $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$. Using Bollobás' [3] bounds on the order of graphs of large girth and large chromatic number, we shall improve the upper bounds $\mathsf{IR}(\vec{I}_n) \le \mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$ to $\mathsf{IR}(\vec{I}_n) = o(n^{2n})$ and $\mathsf{IR}(\vec{\mathcal{T}}_n) = o(n^{4n})$. In Theorem 4.5 using random graphs we improve the latter upper bound to $\mathsf{IR}(\vec{\mathcal{T}}_n) \le (4e + o(1))^n (n^2 \ln n)^n = n^{2n+o(n)}$. The technique developed for the proof of Theorem 4.5 allows us to improve the upper bound $\mathsf{R}(\vec{\mathcal{T}}_n) \le (2500e^8 + o(1))\, n^4 \ln n$ obtained by Kohayakawa, Łuczak and Rödl [9] to the upper bound $(K + o(1))\, n^4 \ln n$, where $K = \min_{x>1} \frac{16x^2}{1-x+x\ln x} \approx 98.8249...$. In Section 5 we search for long directed paths in arbitrary orientations of graphs. In the final Section 6 we prove that every infinite graph G admits an orientation containing no directed path of infinite diameter in G. Some other results and problems related to coloring and orientations in graphs can be found in [10].

2. The isometric Ramsey number for a finite acyclic digraph

In this section we prove that each finite acyclic digraph $\vec{H}$ has finite isometric Ramsey number $\mathsf{IR}(\vec{H})$. The idea of the proof of this result was suggested to the authors by Yoshiharu Kohayakawa.

Theorem 2.1. *For any finite acyclic digraph $\vec{H} = (V, \vec{E})$, the isometric Ramsey number $\mathsf{IR}(\vec{H})$ is finite.*

Proof. Clearly, it is enough to prove the theorem when the graph $\vec{H}$ is connected. Fix any vertex h of H and consider the digraph $\vec{\Gamma}$ with

$$V_{\vec{\Gamma}} := V_{\vec{\Gamma}} \times \{0,1\} \quad\text{and}\quad \vec{E}_{\vec{\Gamma}} := \big\{\big((h,0),(h,1)\big)\big\} \cup \big\{\big((u,0),(v,0)\big),\big((v,1),(u,1)\big) : (u,v) \in E_{\vec{H}}\big\}.$$

Observe that the digraph $\vec{\Gamma}$ is acyclic, connected and contains isometric copies of $\vec{H}$ and the graph $\vec{H}$ with the opposite orientation. Being acyclic, the graph $\vec{\Gamma}$ admits a linear ordering $<$ of vertices such that $u < v$ for any directed edge $(u, v) \in \vec{E}_{\vec{\Gamma}}$.

By Theorem 1.8 of [7], there exists a finite graph G with a linear ordering of vertices such that for any 2-coloring of its edges there exists a monotone isometric embedding $f : V_{\vec{\Gamma}} \to V_G$ such that the set $\big\{\{f(u), f(v)\} : (u, v) \in E_{\vec{\Gamma}}\big\}$ is monochrome. In this case we shall say that the embedding f is *monochrome*. The monotonicity of f means that f preserves the order of vertices.

We claim that $G \Rightarrow \vec{H}$. Given any orientation $\vec{G}$ of the graph G, color an edge $\{u, v\} \in E_G$ with $u < v$ in green if $(u, v) \in E_{\vec{G}}$ and in red if $(v, u) \in E_{\vec{G}}$. By the Ramsey property of G, there exists a monochrome monotone isometric embedding $f : V_{\vec{\Gamma}} \to V_G$. If the color of the monochromatic set $C = \big\{\{f(u), f(v)\} : (u, v) \in E_{\vec{\Gamma}}\big\}$ is green, then the map $g_0 : V_{\vec{H}} \to V_G$, $g_0 : v \mapsto f(v, 0)$, is a required isometric isomorphic embedding of $\vec{H}$ into $\vec{G}$. If the color of C is red, then the map $g_1 : V_{\vec{H}} \to V_G$, $g_1 : v \mapsto f(v, 1)$, is an isometric isomorphic embedding of $\vec{H}$ into $\vec{G}$. In both cases we get $G \Rightarrow \vec{H}$. □

Corollary 2.2. *Any finite family $\vec{\mathcal{H}}$ of finite acyclic digraphs has finite isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{H}})$.* □

Corollary 2.3. *For every $n \in \mathbb{N}$ the family $\vec{\mathcal{T}}_n$ of directed trees on n vertices has finite isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{T}}_n)$.* □

Remark 2.4. The proof of [7, Theorem 1.8] proceeds by a more general induction involving amalgamation and hypergraphs, and seems to give very bad bounds on the isometric Ramsey number $\mathsf{IR}(\vec{\mathcal{A}}_n)$ for the family $\vec{\mathcal{A}}_n$ of all acyclic digraphs on n vertices. It would be interesting to get some reasonable upper bound on this function.

3. Simple bounds for the isometric Ramsey numbers $\mathsf{IR}(\vec{\mathcal{T}}_n)$

In this section we prove some simple upper bounds on the isometric Ramsey numbers $\mathsf{IR}(\vec{\mathcal{T}}_n)$ and $\mathsf{IR}(\vec{I}_n)$. First we present a simple example of a graph witnessing that $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$. The construction of this graph exploits rectangular products of graphs. By definition, the *rectangular product* $G \times H$ of two graphs G, H is a graph such that $V_{G\times H} = V_G \times V_H$ and an unordered pair $\{(g,h),(g',h')\} \subset G \times H$ is an edge of $G \times H$ if and only if either $\{g,g'\} \in E_G$ and $h = h'$ or $g = g'$ and $\{h,h'\} \in E_H$. It can be shown that for any vertices $(g,h),(g',h')$ of $G \times H$ we get

$$d_{G\times H}\big((g,h),(g',h')\big) = d_G(g,g') + d_H(h,h').$$

For an (oriented) graph G by $|G|$ we denote the cardinality of the set V_G of vertices of G. For a cardinal number m by K_m we denote the complete graph on m vertices.

Lemma 3.1. *Let $\vec{\mathcal{T}}, \vec{\mathcal{T}}'$ be two families of finite oriented trees such that for every oriented tree $\vec{T}' \in \vec{\mathcal{T}}'$ there is an oriented subtree $\vec{T} \in \vec{\mathcal{T}}$ of $\vec{T}'$ such that $|\vec{T}| = |\vec{T}'| - 1$. For any graph G with $G \Rightarrow \vec{\mathcal{T}}$ we get $G \times K_{|G|+1} \Rightarrow \vec{\mathcal{T}}'$.*

Proof. Let $G' = G \times K_{|G|+1}$. To prove that $G' \Rightarrow \vec{\mathcal{T}}'$, take any oriented tree $\vec{T}' \in \vec{\mathcal{T}}'$ and any orientation $\vec{G}'$ of the graph G'. By our assumption, for the tree $\vec{T}'$ there exists an oriented subtree $\vec{T} \in \vec{\mathcal{T}}$ of $\vec{T}'$ such that $|\vec{T}| = |\vec{T}'| - 1$. Let t' be the unique element of the set $V_{\vec{T}'} \setminus V_{\vec{T}}$ and $t \in V_{\vec{T}}$ be the unique vertex of $\vec{T}$ such that (t', t) or (t, t') is an edge of $\vec{T}'$.

For every vertex u of the complete graph $K_{|G|+1}$, consider the subgraph $G'_u = G' \times \{u\}$ of G' and its orientation $\vec{G}'_u$, inherited from the orientation $\vec{G}'$ of G'. Since $G \Rightarrow \vec{\mathcal{T}}$, there is an isometric embedding $f_u : \vec{T} \to \vec{G}'_u$. By the Pigeonhole Principle, there are two distinct vertices u, w in $K_{|G|+1}$ such that $f_u(t) = (g,u)$ and $f_w(t) = (g,w)$ for some vertex g of the graph G. Now look at the orientation of the edges $\{t,t'\}$ and $\{(g,u),(g,w)\}$ in the digraphs $\vec{T}'$ and $\vec{G}'$.

If either $(t,t') \in E_{\vec{T}'}$ and $\big((g,u),(g,w)\big) \in E_{\vec{G}'}$ or $(t',t) \in E_{\vec{T}'}$ and $\big((g,w),(g,u)\big) \in E_{\vec{G}'}$, then we define a map $f : \vec{T}' \to G'$ by $f(t') = (g,w)$ and $f|\vec{T} = f_u$ and observe that f is an isometric embedding of $\vec{T}'$ into $\vec{G}'$.

If either $(t,t') \in E_{\vec{T}'}$ and $\big((g,w),(g,u)\big) \in E_{\vec{G}'}$ or $(t',t) \in E_{\vec{T}'}$ and $\big((g,u),(g,w)\big) \in E_{\vec{G}'}$, then we define a map $f : \vec{T}' \to G'$ by $f(t') = (g,u)$ and $f|\vec{T} = f_w$ and observe that f is an isometric embedding of $\vec{T}'$ into $\vec{G}'$. □

Corollary 3.2. *If for some $n \in \mathbb{N}$ a graph G satisfies the isometric Ramsey relation $G \Rightarrow \vec{\mathcal{T}}_n$, then $G \times K_{|G|+1} \Rightarrow \vec{\mathcal{T}}_{n+1}$.* □

Theorem 3.3. *For every $n \in \mathbb{N}$ $\mathsf{IR}(\vec{\mathcal{T}}_{n+1}) \le \mathsf{IR}(\vec{\mathcal{T}}_n)(\mathsf{IR}(\vec{\mathcal{T}}_n)+1)$ and $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$.*

Proof. The inequality $\mathsf{IR}(\vec{\mathcal{T}}_{n+1}) \le \mathsf{IR}(\vec{\mathcal{T}}^n)(\mathsf{IR}(\vec{\mathcal{T}}^n)+1)$ follows from Corollary 3.2. Indeed, for every $n \in \omega$ we can choose a graph G with $|G| = \mathsf{IR}(\vec{\mathcal{T}}^n)$ vertices and $G \Rightarrow \vec{\mathcal{T}}_n$. By Corollary 3.2, the graph $G' = G \times K_{|G|+1}$ satisfies the relation $G' \Rightarrow \vec{\mathcal{T}}_{n+1}$ and hence

$$\mathsf{IR}(\vec{\mathcal{T}}_{n+1}) \le |G'| = |G|(|G|+1) = \mathsf{IR}(\vec{\mathcal{T}}_n)(\mathsf{IR}(\vec{\mathcal{T}}_n)+1).$$

It remains to prove that $\mathsf{IR}(\vec{\mathcal{T}}_n)+1 \le 2^{2^{n-1}}$ for $n \in \mathbb{N}$. For $n = 1$ we get the equality $\mathsf{IR}(\vec{\mathcal{T}}_1)+1 = 1+1 = 2^{2^0}$. Assume that for some $n \in \mathbb{N}$ we have proved that $\mathsf{IR}(\vec{\mathcal{T}}_n)+1 \le 2^{2^{n-1}}$. Then

$$\mathsf{IR}(\vec{\mathcal{T}}_{n+1})+1 \le \mathsf{IR}(\vec{\mathcal{T}}_n)(\mathsf{IR}(\vec{\mathcal{T}}_n)+1)+1 \le (2^{2^{n-1}}-1)2^{2^{n-1}}+1 = 2^{2^n} - 2^{2^{n-1}} + 1 \le 2^{2^n}.$$

□

The upper bound $\mathsf{IR}(\vec{\mathcal{T}}_n) < 2^{2^{n-1}}$ can be greatly improved using known upper bounds on the Erdős function $\mathsf{Erdős}(k,g)$, which assigns to any positive integer numbers k, g the smallest cardinality $|G|$ of a graph G with chromatic number $\chi(G) \ge k$ and girth $g(G) \ge g$. We recall that the *girth* $g(G)$ of a graph is the smallest cardinality of a cycle in G. If G contains no cycles, then we put $g(G) = \infty$. The *chromatic number* $\chi(G)$ of a graph G is the smallest number $k \in \mathbb{N}$ for which there exists a map $\chi : V_G \to \{1,\dots,k\}$ such that $\chi(x) \ne \chi(y)$ for any edge $\{x,y\} \in E_G$. The following bounds for the Erdős function $\mathsf{Erdős}(k,g)$ were proved by Erdős [8], Bollobás [3] and Spencer [12], respectively.

Proposition 3.4. (1) *For any k, g we get $\mathsf{Erdős}(k,g) \ge k^{(g-1)/2}$;*
(2) *For any $k, g \ge 4$ we get $\mathsf{Erdős}(k,g) \le \lceil h^g \rceil$ where $h = 6(k+1)\ln(k+1)$.*

(3) *There exists a constant C such that for any numbers $k, g \geq 3$ and $m = \mathsf{Erdős}(k, g)$ we have the inequality $\sqrt[g-2]{m} \cdot \ln m < Ck$, which implies that $\mathsf{Erdős}(k, g) = o(k^{g-2})$ as $\max\{k, g\} \to \infty$.* □

Write $G \rightharpoonup \vec{\mathcal{H}}$ if for every orientation $\vec{G}$ of G and every $\vec{H} \in \vec{\mathcal{H}}$ there is an injective map $f : V_{\vec{H}} \to V_G$ such that for every directed edge (x, y) of H the pair $(f(x), f(y))$ is a directed edge of $\vec{G}$. (Note that we do not require that f induces isomorphism, that is, G can have extra edges inside the set $f(V_{\vec{H}})$.) Another function related to $\mathsf{IR}(\vec{\mathcal{H}})$ is Burr's function $\mathsf{Burr}(\vec{\mathcal{H}})$ assigning to every family $\vec{\mathcal{H}}$ of oriented trees the smallest number k such that $G \rightharpoonup \vec{\mathcal{H}}$ for every graph G with chromatic number $\chi(G) \geq k$. If such number k does not exist, then we put $\mathsf{Burr}(\vec{\mathcal{H}}) = \infty$. By the Gallai-Hasse-Roy-Vitaver Theorem [13, Theorem 3.13], the chromatic number $\chi(G)$ of a finite graph G is equal to $\max\{n \in \mathbb{N} : G \rightharpoonup \vec{I}_n\}$. This equality implies that $\mathsf{Burr}(\vec{I}_n) = n$ for every $n \in \mathbb{N}$. In [5] Burr considered the numbers $\mathsf{Burr}(\vec{\mathcal{T}}_n)$ and proved that $\mathsf{Burr}(\vec{\mathcal{T}}_n) \leq (n-1)^2$. This upper bound was improved to the upper bound $\mathsf{Burr}(\vec{\mathcal{T}}_n) \leq \frac{1}{2}n^2 - \frac{1}{2}n + 1$ in [2]. According to (still unproved) Conjecture of Burr [5], the equality $\mathsf{Burr}(\vec{\mathcal{T}}_n) = 2n - 2$ holds for all $n \geq 2$.

Proposition 3.5. *For any $n \in \mathbb{N}$ and a subclass $\vec{\mathcal{H}} \subset \vec{\mathcal{T}}_n$ we get the upper bound*

$$\mathsf{IR}(\vec{\mathcal{H}}) \leq \mathsf{Erdős}(\mathsf{Burr}(\vec{\mathcal{H}}), 2n - 2).$$

Proof. Fix a graph G of cardinality $|G| = \mathsf{Erdős}(\mathsf{Burr}(\vec{\mathcal{H}}), 2n - 2)$ with chromatic number $\chi(G) \geq \mathsf{Burr}(\vec{\mathcal{H}})$ and girth $g(G) \geq 2n - 2$. Let us prove that $G \Rrightarrow \vec{\mathcal{H}}$. Take any orientation $\vec{G}$ of G and $\vec{H} \in \vec{\mathcal{H}}$. Since $G \rightharpoonup \vec{\mathcal{H}}$, there is an orientation-preserving injection $f : \vec{H} \to \vec{G}$. Since $\vec{H}$ is a connected graph with at most n vertices and $g(G) \geq 2n - 2$, the map f is an isometric embedding. So, $G \Rrightarrow \vec{\mathcal{H}}$. □

Combining Proposition 3.5 with known upper bounds $\mathsf{Burr}(\vec{I}_n) = n$ and $\mathsf{Burr}(\vec{\mathcal{T}}_n) \leq \frac{1}{2}n^2 - \frac{1}{2}n + 1$ we get the following upper bounds for the isometric Ramsey numbers $\mathsf{IR}(\vec{I}_n)$ and $\mathsf{IR}(\vec{\mathcal{T}}_n)$.

Corollary 3.6. *For every $n \in \mathbb{N}$ we get the upper bounds*

$$\mathsf{IR}(\vec{I}_n) \leq \mathsf{Erdős}(n, 2n - 2) = o(n^{2n-4}) = o(n^{2n}) \text{ and}$$
$$\mathsf{IR}(\vec{\mathcal{T}}_n) \leq \mathsf{Erdős}\big(\tfrac{1}{2}n^2 - \tfrac{1}{2}n + 1, 2n - 2\big) = o\big((\tfrac{1}{2}n^2 - \tfrac{1}{2}n + 1)^{2n-4}\big) = o(n^{4n}).$$

□

In Theorem 4.5 we shall improve the upper bound $o(n^{4n})$ for $\mathsf{IR}(\vec{\mathcal{T}}_n)$ to the upper bound $n^{2n+o(n)}$.

Remark 3.7. By Theorem 3 in [9], $\mathsf{R}(\vec{I}_n) \geq n^2/2$ for all $n \in \mathbb{N}$. This yields the lower bound

$$\tfrac{1}{2}n^2 \leq \mathsf{R}(\vec{I}_n) \leq \mathsf{IR}(\vec{I}_n) \leq \mathsf{IR}(\vec{\mathcal{T}}_n)$$

for the isometric Ramsey numbers $\mathsf{IR}(\vec{I}_n)$ and $\mathsf{IR}(\vec{\mathcal{T}}_n)$.

Remark 3.8. It can be shown that

$$\mathsf{IR}(\vec{I}_1) = \mathsf{IR}(\vec{\mathcal{T}}_1) = 1 = |K_1|,$$
$$\mathsf{IR}(\vec{I}_2) = \mathsf{IR}(\vec{\mathcal{T}}_2) = 2 = |K_2|,$$
$$\mathsf{IR}(\vec{I}_3) = 5 = |C_5|,\ \mathsf{IR}(\vec{\mathcal{T}}_3) = 6 = |K_2 \times K_3|,$$
$$\mathsf{IR}(\vec{I}_4) \leq 30 = |C_5 \times K_6|,\ \mathsf{IR}(\vec{\mathcal{T}}_4) \leq 42 = |K_2 \times K_3 \times K_7|.$$

Question 3.9. *What is the exact value of the isometric Ramsey numbers $\mathsf{IR}(\vec{I}_4)$ and $\mathsf{IR}(\vec{\mathcal{T}}_4)$? Are they distinct?*

4. Isometric copies of directed trees in orientations of random graphs

In this section we shall apply the technique of random graphs and shall improve the upper bound $\mathsf{IR}(\vec{\mathcal{T}}_n) = o(n^{4n})$ established in Corollary 3.6 to the upper bound $\mathsf{IR}(\vec{\mathcal{T}}_n) \leq (4e + o(1))^n (n^2 \ln n)^n = n^{2n+o(n)}$.

First we prove some technical lemmas. The first of them uses the idea of the proof of Theorem 1 in [9].

Lemma 4.1. *A graph $G = (V_G, E_G)$ satisfies $G \Rrightarrow \vec{\mathcal{T}}_n$ for some $n \in \mathbb{N}$ if there exist sequences $(w_k)_{k=1}^{n-1}$ and $(d_k)_{k=1}^{n-1}$ of positive real numbers such that for every $2 \leq k < n$ the following conditions hold:*

(1) *For every set $S = \{s_1, \dots, s_{k-1}\} \subset V_G$ of cardinality $k-1$ and every $v \in V_G \setminus S$, we have that $|Y| \leq d_k$, where Y consists of $y \in V_G \setminus (S \cup \{v\})$ such that $\{y, v\} \in E_G$ and $\mathrm{dist}_{G-v}(y, s_i) \leq i$ for some $1 \leq i < k$.*

(2) *Every set* $W \subset V_G$ *of cardinality* $|W| > w_k$ *spans more than* $(d_k + k - 1)w_k$ *edges in* G.
(3) $\sum_{k=1}^{n-1} w_k < |V_G|$.

Proof. For a subset $U \subset V_G$ by $G[U]$ we denote the induced subgraph $G[U] = (U, E[U])$ of G, where $E[U] = \big\{\{u,v\} \in E_G : \{u,v\} \subset U\big\}$. Also, let us write $(G,U) \Rightarrow \vec{\mathcal{T}}_k$, meaning that, for every $\vec{T} \in \vec{\mathcal{T}}_k$, every orientation $\vec{G}$ of G contains a copy of $\vec{T}$ which lies inside U and is an isometric subgraph of G.

We shall inductively prove that for every $1 \le k \le n$ and every set $U \subset V_G$ of size $|U| > \sum_{i=1}^{k-1} w_i$, we have $(G,U) \Rightarrow \vec{\mathcal{T}}_k$. The base case $k = 1$ is trivial. Suppose that this holds for some k. Take any $U \subset V_G$ with $|U| > \sum_{i=1}^{k} w_i$. Take any orientation $\vec{E}(G[U])$ of $E(G[U])$ and any directed tree $\vec{T} \in \vec{\mathcal{T}}_{k+1}$. Let u be a pendant vertex of $\vec{T}$. By symmetry, assume that (v,u) is an arc in $\vec{T}$, that is, the arc in $\vec{T}$ goes from the unique neighbor v of u to u.

Let W be the set of vertices in U whose out-degree in $G[U]$ is at most $d_k + k - 1$. We claim that $|W| \le w_k$. Suppose not. Then $|W| > w_k$ and Item 2 guarantees that W spans more than $(d_k + k - 1)w_k$ edges in G, each edge contributing to out-degree of some vertex in W. Thus $(d_k + k - 1)|W| \ge |E[W]| > (d_k + k - 1)w_k$, which is a desired contradiction showing that $|W| \le w_k$.

Thus $U' = U \setminus W$ has size $|U'| = |U| - |W| > \big(\sum_{i=1}^{k} w_i\big) - w_k = \sum_{i=1}^{k-1} w_i$. By inductive assumption, U' has a G-isometric copy $\vec{T}'$ of the oriented tree $\vec{T} - u$. Let $\{s_1, \ldots, s_{k-1}\}$ be an enumeration of the set $S := V_{\vec{T}'} \setminus \{v\} \subset U'$ such that $\mathrm{dist}(s_i, v) \le i$ for every $i < k$. Let Y be defined as in Item 1 with respect to v and $\{s_1, \ldots, s_{k-1}\}$. By Item 1, $|Y| \le d_k$. On the other hand, the neighbor $v \in V_{\vec{T}'} \subset U \setminus W$ of u must have out-degree in $U \setminus S$ greater than $d_k + k - 1 - |S| = d_k$. Thus there is an out-neighbor of v which is in $U \setminus (W \cup Y)$. Let u be mapped to this vertex. Then $(v,u) \in \vec{E}(G[U])$ is oriented from v to u, as desired. Since $d_{G-v}(u, s_i) > i$ for each $i < k$, the addition of u cannot violate the G-isometry property (since all vertices of $\vec{T} - u$ are embedded into $S \cup \{v\}$). This gives the required embedding of $\vec{T}$ and finishes the proof. □

Our next elementary lemma yields an upper bound on the sum of a geometric progression.

Lemma 4.2. *For positive real numbers* a, c *with* $a > 1 + \frac{1}{c}$ *we get* $\frac{a^n - 1}{a - 1} < (1 + c)a^{n-1}$ *for every* $n \in \mathbb{N}$.

Proof. The inequality is equivalent to $a^n - 1 < (1+c)a^{n-1}(a-1) = a^n - a^{n-1} + ca^{n-1}(a-1)$ and to $a^{n-1} - 1 < ca^{n-1}(a-1)$. The latter inequality follows from $a^{n-1} < ca^{n-1}(a-1)$, which is equivalent to $1 < c(a-1)$. □

In the proof of Lemma 4.4 we shall use the following Chernoff-type bounds; for a proof see e.g. [1, §A.1].

Lemma 4.3 (Chernoff bounds)**.** *Let* $X_1, \ldots, X_n$ *be independent random variables taking values in* $\{0,1\}$ *and let* $\mathbb{E}X$ *be the expected value of their sum* $X = \sum_{i=1}^{n} X_i$. *Then*

$$\mathbb{P}\big\{X \ge C \cdot \mathbb{E}X\big\} \le \big(\tfrac{e^{C-1}}{C^C}\big)^{\mathbb{E}X}, \ \ \mathbb{P}\big\{X \ge (1+c)\mathbb{E}X\big\} \le e^{-\frac{c^2}{3}\mathbb{E}X} \quad \text{and} \quad \mathbb{P}\big\{X \le (1-c)\mathbb{E}X\big\} \le e^{-\frac{c^2}{2}\mathbb{E}X}$$

for every $C > 1$ *and* $0 < c < 1$. □

Lemma 4.4. *For positive integers* n, N *the inequality* $\mathsf{IR}(\vec{\mathcal{T}}_n) \le N$ *holds if there exist real numbers* $c, p \in (0,1)$, $C \in (1, \infty)$ *satisfying the following inequalities:*

(1) $c^2 pN > 3\ln(3N)$;
(2) $(1 - C + C\ln C)p(1+c)^n(pN)^{n-2} > (n-1)\ln N + \ln(1+c) + \ln(3)$;
(3) $c^2C^2(1+c)^{2n}(pN)^{2n-4} > N\ln 2 + \ln(3n)$;
(4) $\frac{(n-1)(n-2)}{(1-c)p} + \frac{2C}{(1-c)}(n-1)(1+c)^n(pN)^{n-2} < N$.

Proof. Assume that the numbers n, N, p, c, C satisfy the assumptions of the lemma. Let $G = G(N,p)$ be a random graph on N vertices in which an edge $\{u,v\} \subset V_G$ appears with probability p. We shall prove that with non-zero probability the random graph G has $G \Rightarrow \vec{\mathcal{T}}_n$.

Let

$$\hbar := (1+c)^n(pN)^{n-2}.$$

For every positive integer $k < n$ let

$$d_k = Cp\hbar \ \text{ and } \ w_k = \frac{2(d_k + k - 1)}{(1-c)p}.$$

Chernoff bound implies that any fixed vertex of G has degree $\geq (1+c)p(N-1)$ with probability $< e^{-\frac{c^2}{3}p(N-1)}$. Consequently, with probability $P_1 > 1 - Ne^{-\frac{c^2}{3}p(N-1)}$ all vertices of G have degree $< (1+c)pN$. The condition (1) implies that $-\frac{c^2}{3}p(N-1) < -\ln(3N)$ and hence

$$P_1 > 1 - Ne^{-\frac{c^2}{3}p(N-1)} > 1 - Ne^{-\ln(3N)} = \tfrac{2}{3}.$$

For every $k < n$, take any pairwise distinct points $v, s_1, \dots, s_{k-1} \in V_G$. If the maximum degree of G is at most $(1+c)pN$, then for every $i < k$ the ball $B(s_i, i) = \{x \in V_G : \mathrm{dist}_G(x, s_i) \leq i\}$ has cardinality

$$|B(s_i,i)| \leq \sum_{j=0}^{i} \big((1+c)pN\big)^j = \frac{\big((1+c)pN\big)^{i+1} - 1}{(1+c)pN - 1} < (1+c)\big((1+c)pN\big)^i.$$

The latter strict inequality can be derived from Lemma 4.2 and the inequality $cpN \geq c^2pN > 3\ln(3N) \geq 3$.

By above, the set X of vertices of $G - v$ at distance at most $i < k$ in $G - v$ from some s_i has size at most $(1+c)\sum_{i=1}^{k-1}\big((1+c)pN\big)^i = (1+c)\frac{((1+c)pN)^k - 1}{(1+c)pN-1} < (1+c)^{k+1}(pN)^{k-1} \leq \hbar$.

Consider the set Y of neighbors of v that fall into the set X. The definition of X does not depend on the edges incident to v, so conditioned on X (of size at most $\hbar$) the size of Y is dominated by $Y' \sim Bin(\hbar, p)$. Chernoff bound shows that the probability that Y' is at least $Cp\hbar = C\mathbb{E}Y'$ is at most $\big(\frac{e^{C-1}}{C^C}\big)^{p\hbar}$. Since the number of possible choices of $v, s_1, \dots, s_{k-1}$ is equal to $\frac{N!}{(N-k)!} \leq N^k$, with probability

$$P_2 \geq 1 - \sum_{k=1}^{n-1} N^k \big(\tfrac{e^{C-1}}{C^C}\big)^{p\hbar} = 1 - \big(\tfrac{e^{C-1}}{C^C}\big)^{p\hbar}\frac{N^n - 1}{N-1} > 1 - (1+c)N^{n-1}\big(\tfrac{e^{C-1}}{C^C}\big)^{p\hbar}$$

the condition (1) of Lemma 4.1 is satisfied or we have a vertex of degree $\geq (1+c)pN$. We claim that $P_2 > \frac{2}{3}$. It suffices to prove that

$$\ln(1+c) + (n-1)\ln N + p\hbar(C - 1 - C\ln C) < -\ln(3).$$

But this follows from condition (2).

Next, we prove that with probability $> \frac{2}{3}$ the condition (2) of Lemma 4.1 holds. Take any positive $k < n$ and put $\bar{w}_k = \min\{m \in \mathbb{N} : w_k < m\}$. For any fixed set $W \subset V_G$ of cardinality $|W| = \bar{w}_k$, the number of edges it spans is $Bin(\binom{\bar{w}_k}{2}, p)$. By Chernoff bound, the probability that it is less than $(1-c)p\binom{\bar{w}_k}{2}$ is less that $e^{-\frac{1}{2}c^2p\binom{\bar{w}_k}{2}}$. The probability $P_{3,k}$ that some set $W \subset V_G$ of cardinality $|W| = \bar{w}_k$ spans less than $(1-c)p\binom{\bar{w}_k}{2}$ edges is $P_{3,k} < \binom{N}{\bar{w}_k}e^{-\frac{1}{2}c^2p\binom{\bar{w}_k}{2}} < 2^N e^{-\frac{1}{4}c^2p\bar{w}_k(\bar{w}_k+1)}$. We claim that $P_{3,k} < \frac{1}{3n}$ which will follow as soon as we show that $N\ln 2 - \frac{1}{4}c^2p\bar{w}_k(\bar{w}_k+1)) < -\ln(3n)$. For this it suffices to check that $\frac{1}{4}c^2p\bar{w}_k(\bar{w}_k+1) > N\ln 2 + \ln(3n)$.

This follows from the chain of the inequalities

$$\tfrac{1}{4}c^2\bar{w}_k(\bar{w}_k+1) > \tfrac{1}{4}c^2w_k^2 > c^2C^2\hbar^2 = c^2C^2(1+c)^{2n}(pN)^{2n-4} > N\ln 2 + \ln(3n),$$

the last inequality postulated in (3). Therefore, $P_{3,k} < \frac{1}{3n}$ and the probability P_3 that for every $k < n$ every set $W \subset V[G]$ of cardinality $|W| > w_k$ spans at least

$$(1-c)p\binom{\bar{w}_k}{2} > (1-c)pw_k(w_k+1)/2 = (d_k + k - 1)(w_k+1) > (d_k+k-1)w_k$$

edges is $> 1 - \sum_{k=1}^{n-1} P_{3,k} > 1 - \frac{n-1}{3n} > \frac{2}{3}$. So, with probability $> \frac{2}{3}$ the condition (2) of Lemma 4.1 holds.

Since $(1-P_1)+(1-P_2)+(1-P_3) < 1$, there is a non-zero probability that the random graph $G = G(N,p)$ satisfies the conditions (1) and (2) of Lemma 4.1.

It remains to show that the condition (3) of Lemma 4.1 holds, too. For this observe that

$$\sum_{k=1}^{n-1} w_k = \sum_{k=1}^{n-1}\frac{2(Cp\hbar + k - 1)}{(1-c)p} = \frac{2}{(1-c)p}\sum_{k=1}^{n-1}(k-1) + \frac{2C}{1-c}(n-1)\hbar =$$
$$= \frac{(n-1)(n-2)}{(1-c)p} + \frac{2C}{1-c}(n-1)(1+c)^n(pN)^{n-2} < N.$$

The last inequality follows from the condition (4) of the Lemma.

Now it is legal to apply Lemma 4.1 and conclude that $G \Rightarrow \vec{\mathcal{T}}_n$ and hence $\mathsf{IR}(\vec{\mathcal{T}}_n) \leq |G| = N$. □

Now we are able to prove the promised upper bound $\mathsf{IR}(\vec{\mathcal{T}}_n) \leq (4e + o(1))^n(n^2\ln n)^n = n^{2n+o(n)}$.

Theorem 4.5. *For every $\varepsilon \in (0,1)$ there is $n_\varepsilon \in \mathbb{N}$ such that $\mathsf{IR}(\vec{\mathcal{T}}_n) \le \big(4e(1+\varepsilon)\,n^2 \ln n\big)^n$ for all $n \ge n_\varepsilon$.*

Proof. Choose any positive $\delta, c \in (0,1)$ such that

$$(1+\delta)(1+c) < 1+\varepsilon \quad \text{and} \quad 4(1+\delta)\,\frac{1-c}{2+c} > 2+\delta.$$

For every $n \in \mathbb{N}$ let N be the smallest integer number, which is greater than

$$\tfrac{(2+c)e^n}{1-c}(n-1)(1+c)^n\big(4(1+\delta)\,n^2\ln n\big)^{n-2}$$

and let

$$p := \frac{4(1+\delta)\,n^2\ln n}{N}.$$

So, $N > \frac{(2+c)e^n}{1-c}(n-1)(1+c)^n(pN)^{n-2} \ge N-1$. It is easy to see that

$$N = o\big((4e(1+\varepsilon)\,n^2\ln n)^n\big)$$

and for $C = e^n$ the conditions (1),(3),(4) of Lemma 4.4 hold for all sufficiently large n. To verify the condition (2), observe that

$$(1-C+C\ln C)p(1+c)^n(pN)^{n-2} \ge (1-e^n+e^n\ln e^n)p\frac{(N-1)(1-c)}{(2+c)e^n(n-1)} =$$
$$\frac{1+e^n(n-1)}{e^n(n-1)}\,\frac{1-c}{2+c}\,\frac{N-1}{N}\,pN = \Big(1+\frac{1}{e^n(n-1)}\Big)\frac{N-1}{N}\,\frac{1-c}{2+c}\,4(1+\delta)n^2\ln n >$$
$$> \Big(1+\frac{1}{e^n(n-1)}\Big)\frac{N-1}{N}(2+\delta)\,n^2\ln n = (2+\delta+o(1))\,n^2\ln n.$$

On the other hand, $(n-1)\ln N + \ln(1+c) + \ln 3 = (2+o(1))\,n^2\ln n$. So, the condition (2) holds for large n. Applying Lemma 4.4, we conclude that

$$\mathsf{IR}(\vec{\mathcal{T}}_n) \le N \le (4e(1+\varepsilon)\,n^2\ln n)^n$$

for all sufficiently large n. □

By Corollary 3.6 and Theorem 4.5, $\mathsf{IR}(\vec{I}_n) = o(n^{2n})$ and $\mathsf{IR}(\vec{\mathcal{T}}_n) \le n^{2n+o(n)}$.

Question 4.6. *What is the growth rate of the sequence $\mathsf{IR}(\vec{\mathcal{T}}_n)$? Is $\mathsf{IR}(\vec{\mathcal{T}}_n) = n^{o(n)}$?*

The technique developed for the proof of Theorem 4.5 allows us to improve the upper bound

$$\mathsf{R}(\vec{\mathcal{T}}_n) \le \big(4(5e^2)^4 + o(1)\big)\,n^4\ln n,$$

obtained by Kohayakawa, Łuczak and Rödl in (the proof of) Theorem 1 of [9], and replace the constant $4(5e^2)^4 = 2500e^8 \approx 7452395.96...$ by the a much smaller constant $K \approx 98.82...$.

Theorem 4.7. *Let $K := \min_{x>1}\frac{16x^2}{1-x+x\ln x} \approx 98.8249...$ For any positive $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that $\mathsf{R}(\vec{\mathcal{T}}_n) < (K+\varepsilon)\,n^4\ln n$ for all $n \ge n_\varepsilon$. Consequently, $\mathsf{R}(\vec{\mathcal{T}}_n) < 99\,n^4\ln n$ for all sufficiently large n.*

Proof. We indicate which changes should be made in the proof of Theorem 4.5 to obtain Theorem 4.7.

In the condition (1) of Lemma 4.1 the inequality $d_{G-v}(y,s_i) \le i$ should be replaced by $d_{G-v}(y,s_i) \le 1$.

In the proof of Lemma 4.4 the constant $\hbar$ should be redefined as $\hbar := (1+c)(n-2)pN$ and the conditions (1)–(4) of Lemma 4.4 should be changed to the conditions:

(1') $c^2pN > 3\ln(3N)$;
(2') $(1-C+C\ln C)(1+c)(n-2)p^2N > (n-1)\ln N + \ln(1+c) + \ln(3)$;
(3') $(cC(1+c)(n-2)pN)^2 > N\ln 2 + \ln(3n)$;
(4') $\frac{n(n-1)}{(1-c)p} + \frac{2C(1+c)}{(1-c)}(n-1)(n-2)pN < N$.

Now we are able to prove Theorem 4.7. Let $C \approx 4.92155...$ be the unique real number in $(1,\infty)$ such that

$$\frac{16C^2}{1-C+C\ln C} = K := \min_{x>1}\frac{16x^2}{1-x+x\ln x} \approx 98.8249...$$ [1]

[1]The approximate values of C and K were found by the online WolframAlpha computational knowledge engine at www.wolframalpha.com

Given any $\varepsilon > 0$, choose real numbers $\delta, c \in (0,1)$ such that $K\delta < \varepsilon$ and

$$4(1+\delta)\frac{(1-c)^2}{(1+c)^3} > 4+\delta.$$

For every $n \in \mathbb{N}$ let $p := \frac{1-c}{2C(1+c)^2n^2}$ and let N be the smallest integer, which is greater than $K(1+\delta)n^4\ln n$. It is easy to see that $N = o\big((K+\varepsilon)\,n^4\ln n\big)$ and the conditions (1'), (3') and (4') are satisfied for all sufficiently large n. To see that (2') holds, observe that

$$(1-C+C\ln C)(1+c)(n-2)p^2N \geq \frac{(1-C+C\ln C)(1+c)(1-c)^2}{(2C(1+c)^2n^2)^2}(n-2)K(1+\delta)\,n^4\ln n =$$
$$= \frac{1-C+C\ln C}{C^2}\frac{(1-c)^2}{4(1+c)^3}(n-2)K(1+\delta)\ln n = (1+\delta)K\frac{16}{K}\frac{(1-c)^2}{4(1+c)^3}(n-2)\ln n >$$
$$> (4+\delta)(n-2)\ln n = (4+\delta+o(1))\,n\ln n.$$

On the other hand,

$$(n-1)\ln N + \ln(1+c) + \ln 3 \leq (n-1)\ln(1+K(1+\delta)\,n^4\ln n) + \ln(1+c) + \ln 3 = (4+o(1))\,n\ln n,$$

so for large n the condition $(2')$ is satisfied, too.

Applying the modified version of Lemma 4.4, we get

$$\mathsf{R}(\vec{\mathcal{T}}_n) \leq N \leq (K+\varepsilon)\,n^4\ln n$$

for all sufficiently large numbers n. □

5. Long directed paths in orientations of a graph

By the Gallai-Hasse-Roy-Vitaver Theorem [13, Theorem 3.13], each finite graph G has chromatic number

$$\chi(G) = \max\{n \in \mathbb{N} : G \rightharpoonup \vec{I}_n\},$$

where the symbol $G \rightharpoonup \vec{I}_n$ means that each orientaion of G contains a simple directed path of length n. Having in mind this characterization, for every graph G consider the numbers

$$\bar{\bar{\chi}}_I(G) = \sup\{n \in \mathbb{N} : G \Rightarrow \vec{I}_n\},\ \bar{\bar{\chi}}_T(G) = \sup\{n \in \mathbb{N} : G \Rightarrow \vec{\mathcal{T}}_n\},$$

and observe that $\bar{\bar{\chi}}_T(G) \leq \bar{\bar{\chi}}_I(G) \leq \chi(G)$ and

$$\bar{\bar{\chi}}_I(G) \leq \sup\{\mathrm{diam}(G') + 1 : G' \text{ is a connected component of } G\}.$$

Observe that $\mathsf{IR}(\vec{I}_n)$ (resp. $\mathsf{IR}(\vec{\mathcal{T}}_n)$) is equal to the smallest cardinality $|G|$ of a graph G with $\bar{\bar{\chi}}_I(G) \geq n$ (resp. $\bar{\bar{\chi}}_T(G) \geq n$). So, the characteristics $\bar{\bar{\chi}}_I$ and $\bar{\bar{\chi}}_T$ determine the isometric Ramsey numbers $\mathsf{IR}(\vec{I}_n)$ and $\mathsf{IR}(\vec{\mathcal{T}}_n)$.

We shall show that a graph G has $\bar{\bar{\chi}}_I(G) \leq 2$ if and only if G is a comparability graph. We recall that a graph G is called a *comparability graph* if G admits a *transitive* orientation $\vec{G}$ (that is, for any directed edges (x,y) and (y,z) of $\vec{G}$ the pair (x,z) is a directed edge of $\vec{G}$); equivalently, the set V_G of vertices of G admits a partial order such that a pair $\{u,v\}$ of distinct vertices of G is an edge of G if and only if u and v are comparable in the partial order. By the results of Ghouila-Houri and of Gilmore and Hoffman (see [4, Theorem 6.1.1]), comparability graphs can be characterized as graphs G whose every cycle of odd length has a triangular chord (more precisely, for every $(2n+3)$-cycle on $(v_0,\dots,v_{2n+2})$ with $n \geq 1$, there is a residue i modulo $2n+3$ such that $\{v_i, v_{i+2}\} \in E_G$). More information on comparability graphs can be found in Chapter 6 of the survey [4].

Proposition 5.1. *A graph G has $\bar{\bar{\chi}}_I(G) \leq 2$ if and only if G is a comparability graph.*

Proof. If G is comparability graph, then G has a transitive orientation $\vec{G}$. It follows that for any directed path (v_0,v_1,v_2) in $\vec{G}$ the pair (v_0,v_2) is an edge of $\vec{G}$ and hence $d_G(v_0,v_2) \leq 1$. This means that $G \not\Rightarrow \vec{I}_3$ and hence $\bar{\bar{\chi}}_I(G) \leq 2$.

If G is not a comparability graph, then G contains an odd cycle C without a triangular chord. It is easy to see that any orientation $\vec{C}$ of the cycle C contains a directed path (v_0,v_1,v_2). Since C has no triangular chords, $d_G(v_0,v_2) = 2$, which means that $\{v_0,v_1,v_2\}$ is an isometric copy of $\vec{I}_3$ in $\vec{C}$ and in G. Therefore, $\bar{\bar{\chi}}_I(G) \geq 3$. □

Problem 5.2. *Characterize graphs G with $\bar{\bar{\chi}}_I(G) \leq 3$ ($\bar{\bar{\chi}}_I(G) \leq n$ for $n \geq 4$).*

Problem 5.3. *Characterize graphs G with $\bar{\bar{\chi}}_T(G) \leq 2$ ($\bar{\bar{\chi}}_T(G) \leq n$ for $n \geq 3$).*

Remark 5.4. Any cycle C of odd length $n \ge 5$ satisfies $\bar{\bar{\chi}}_I(C) = 3$ and $\bar{\bar{\chi}}_T(C) = 2$.

Now we prove a weak 3-space property for the number $\bar{\bar{\chi}}_I(G)$. By a *weak homomorphism* $f : G \to H$ of graphs G, H we understand a function $f : V_G \to V_H$ such that for every edge $\{u, v\}$ of G we have either $f(u) = f(v)$ or $\{f(u), f(v)\}$ is an edge of H. For a weak homomorphism $f : G \to H$ and vertex y of H the preimage $f^{-1}(y)$ is a graph with the set of edges $\big\{\{u, v\} \in E_G : f(u) = y = f(v)\big\}$.

Proposition 5.5. *If $f : G \to H$ is a weak homomorphism of finite graphs, then*

$$\bar{\bar{\chi}}_I(G) \le \max\Big\{\sum_{y\in F} \bar{\bar{\chi}}_I(f^{-1}(y)) : F \subset V_H,\ |F| \le \chi(H)\Big\}.$$

Proof. By definition of the chromatic number $\chi(H)$, there exists a coloring $c : V_H \to \{1, \dots, \chi(H)\}$ of the graph H such that for every edge $\{u, v\}$ of G the colors $c(u)$ and $c(v)$ are distinct. For every $y \in H$ choose an orientation $\vec{G}_y$ of the graph $G_y = f^{-1}(y)$ such that $\vec{G}_y \not\Rightarrow \vec{I}_k$ for $k = \bar{\bar{\chi}}_I(G_y) + 1$. Let $\vec{G}$ be the orientation of the graph G such that for an edge $\{u, v\}$ of G the ordered pair (u, v) is an edge of $\vec{G}$ if and only if either $c(f(u)) < c(f(v))$ or (u, v) is an edge of $\vec{G}_y$ for some $y \in H$.

We claim that the digraph $\vec{G}$ contains no isometric copy of the graph $\vec{I}_{m+1}$, where

$$m = \max\Big\{\sum_{y\in F} \bar{\bar{\chi}}_I(G_y) : F \subset V_H,\ |F| \le \chi(H)\Big\}.$$

Suppose on the contrary that $\vec{G}$ contains a directed path $(v_0, \dots, v_m)$ such that $d_G(v_0, v_m) = m$. It follows that $\big(c(f(v_0)), \dots, c(f(v_n))\big)$ is a non-decreasing sequence of numbers in the interval $\{1, \dots, \chi(H)\}$. Consequently, for every number i in the set $C = \{c(f(v_0)), \dots, c(f(v_n))\}$ the set $J_i = \{j \in \{0, \dots, n\} : c(f(v_j)) = i\}$ coincides with some subinterval $[a_i, b_i]$ of $\{0, \dots, n\}$ and the set $\{f(v_j) : j \in [a_i, b_i]\}$ is a singleton $\{y_i\}$ for some vertex $y_i \in H$. It follows that $(v_{a_i}, \dots, v_{b_i})$ is a directed path isometric to $\vec{I}_{|[a_i,b_i]|}$ in the graph G_{y_i} and hence $|[a_i - b_i]| \le \bar{\bar{\chi}}_I(G_{y_i})$. The choice of the orientation $\vec{G}$ guarantees that the set $F = \{y_i : i \in C\}$ has cardinality $|F| = |C| \le \chi(H)$. Then

$$m + 1 = |[0, m]| = \sum_{i\in C} |[a_i, b_i]| \le \sum_{i\in C} \bar{\bar{\chi}}_I(G_{y_i}) = \sum_{y\in F} \bar{\bar{\chi}}_I(G_y) \le m,$$

which is a desired contradiction. □

6. Infinite directed paths in orientations of graphs

Now we discuss the problem of existence of infinite directed paths in orientations of graphs. Consider the infinite digraphs $\vec{I}_\omega$ and $\vec{I}_{-\omega}$ with $V_{\vec{I}_\omega} = \omega = V_{\vec{I}_{-\omega}}$, $E_{\vec{I}_\omega} = \{(i, i+1) : i \in \omega\}$, and $E_{\vec{I}_{-\omega}} = \{(i+1, i) : i \in \omega\}$.

First, observe that Theorem 3.3 implies the following:

Corollary 6.1. *There exists a countable graph G such that $G \Rightarrow \vec{I}_n$ for every $n \in \mathbb{N}$.* □

On the other hand, we shall prove that each graph G admits an orientation containing no isometric copy of the digraphs $\vec{I}_\omega$ or $\vec{I}_{-\omega}$ and, more generally, no directed paths of infinite diameter in G. (For a subset $A \subset V_G$ of a graph G its *diameter* is defined as $\mathrm{diam}(A) = \sup\{d_G(u, v) : u, v \in A\} \in \omega \cup \{\infty\}$.)

A sequence $(v_n)_{n\in\omega} \in V_G^\omega$ of distinct vertices of a graph G is called an *ω-path* in G if for every $n \in \omega$ the pair $\{v_n, v_{n+1}\}$ is an edge of G. An ω-path $(v_n)_{n\in\omega}$ in a graph G is called *$\overrightarrow{\omega}$-directed* (resp. *$\overleftarrow{\omega}$-directed*) in an orientation $\vec{G}$ of G if for every $n \in \omega$ the pair (v_n, v_{n+1}) (resp. (v_{n+1}, v_n)) is a directed edge of $\vec{G}$. An ω-path in G is called *directed* in an orientation $\vec{G}$ of G if it is either $\overrightarrow{\omega}$-directed or $\overleftarrow{\omega}$-directed.

The Ramsey Theorem implies that every orientation of the complete countable graph K_ω contains $\vec{I}_\omega$ or $\vec{I}_{-\omega}$. On the other hand, we have the following result.

Theorem 6.2. *Every graph G has an orientation $\vec{G}$ containing no directed ω-paths of infinite diameter in G. This implies that $G \not\Rightarrow \vec{I}_\omega$ and $G \not\Rightarrow \vec{I}_{-\omega}$.*

Proof. Without loss of generality, the graph G is connected. Fix any vertex o in G and for every vertex v of G let $\|v\|$ be the smallest length of a path linking the vertices v and o. Choose an orientation $\vec{G}$ of G such that for any edge $\{u, v\}$ in G with $\|v\| = \|u\| + 1$ the pair (u, v) is an edge of $\vec{G}$ if $\|u\|$ is even and (v, u) is an edge of $\vec{G}$ if $\|u\|$ is odd.

We claim that the orientation $\vec{G}$ contains no directed ω-paths of infinite diameter. To derive a contradiction, assume that $(v_n)_{n\in\omega}$ is a directed ω-path of infinite diameter. Fix any even number $n \in \omega$ such that $\|v_0\| < n$. Since the ω-path $(v_n)_{n\in\omega}$ has infinite diameter, there exists a number $k \in \omega$ such that $\|v_k\| \geq n$. We can assume that k is the smallest number with this property. Taking into account that $\big|\|v_n\| - \|v_{n+1}\|\big| \leq 1$ for all $n \in \omega$, we conclude that $\|v_k\| = n > \|v_0\|$ and $\|v_{k-1}\| = n-1$, and hence (v_{k-1}, v_k) is an edge of $\vec{G}$. Let also m be the smallest number such that $\|v_m\| \geq n+1$. For this number we get $\|v_m\| = n+1$, $\|v_{m-1}\| = n$ and hence (v_m, v_{m-1}) is a directed edge $\vec{G}$. Since both pairs (v_{k-1}, v_k) and (v_m, v_{m-1}) are directed edges of the oriented graph $\vec{G}$, the ω-path $(v_n)_{n\in\omega}$ is not directed in $\vec{G}$. Since the graphs $\vec{I}_\omega$ and $\vec{I}_{-\omega}$ have infinite diameters, the digraph $\vec{G}$ does not contain isometric copies of $\vec{I}_\omega$ or $\vec{I}_{-\omega}$. □

Remark 6.3. Theorem 6.2 implies that every locally finite graph G admits an orientation containing no directed ω-paths.

7. Acknowledgments

The authors express their sincere thanks to Yoshiharu Kohayakawa for suggesting the proof of Theorem 2.1 (which resolves a problem posed in an earlier version of this paper) and to Tomasz Łuczak for valuable discussions related to random graphs.

References

[1] N. Alon, J. Spencer, *The probabilistic method*, Fourth edition. Wiley Series in Discrete Mathematics and Optimization. John Wiley & Sons, Inc., Hoboken, NJ, 2016.
[2] L. Addario-Berry, F. Havet, C.L. Sales, B. Reed, S. Thomassé, *Oriented trees in digraphs*, Discrete Math. 313:8 (2013), 967–974.
[3] B. Bollobás, *Chromatic number, girth and maximal degree*, Discrete Math. **24**:3 (1978), 311–314.
[4] A. Brandstädt, B.V. Le, J. Spinrad, *Graph Classes: A Survey*, SIAM Monographs on Discrete Mathematics and Applications, 1999.
[5] S. Burr, *Subtrees of directed graphs and hypergraphs*, Proceedings of the Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing (Florida Atlantic Univ., Boca Raton, Fla., 1980), Vol. I. Congr. Numer. 28 (1980), 227–239.
[6] M. Cochand, P. Duchet, *A few remarks on orientation of graphs and Ramsey theory*, in: Irregularities of partitions (Fertod, 1986), 39–46, Algorithms Combin. Study Res. Texts, 8, Springer, Berlin, 1989.
[7] D.Jr. Dellamonica, V. Rödl, *Distance preserving Ramsey graphs*, Combin. Probab. Comput. **21**:4 (2012) 554–581.
[8] P. Erdős, *Graph theory and probability*, Canad. J. Math. **11** (1959) 34–38.
[9] Y. Kohayakawa, T. Łuczak, V. Rödl, *Ramsey-type results for oriented trees*, J. Graph Theory **22**:1 (1996) 1–8.
[10] I.V. Protasov, *New chromatic numbers in open problems*, Mat. Stud. **42**:2 (2016) 115–117.
[11] V. Rödl, *A generalization of Ramsey Theorem*, in: Graphs, Hypergraphs and Block Systems (Zielona Góra, 1976), 211–220.
[12] J. Spencer, *Asymptotic lower bounds for Ramsey functions*, Discrete Math. **20**:1 (1977/78), 69–76
[13] J. Nešetril, P. Ossona de Mendez, *Sparsity. Graphs, structures, and algorithms*, Algorithms and Combinatorics, 28. Springer, Heidelberg, 2012.

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