# SHORT RAINBOW CYCLES IN GRAPHS AND MATROIDS 

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#### Abstract

Let $G$ be a simple $n$-vertex graph and $c$ be a colouring of $E(G)$ with $n$ colours, where each colour class has size at least 2 . We prove that ( $G, c$ ) contains a rainbow cycle of length at most $\left\lceil\frac{n}{2}\right\rceil$, which is best possible. Our result settles a special case of a strengthening of the Caccetta-Häggkvist conjecture, due to Aharoni. We also show that the matroid generalization of our main result also holds for cographic matroids, but fails for binary matroids.


## 1. Introduction

In 1978, Caccetta and Häggkvist [4] made the following conjecture.
Conjecture 1 (Caccetta-Häggkvist). For all positive integers n, r, every simple $n$-vertex digraph with minimum outdegree at least $r$ contains a directed cycle of length at most $\left\lceil\frac{n}{r}\right\rceil$.

A digraph $D$ is simple if for all $u, v \in V(D)$ there is at most one arc from $u$ to $v$. The Caccetta-Häggkvist conjecture has proven to be a notoriously difficult problem. For example, the case $r=\frac{n}{3}$ has received considerable attention [4, 5, 3, 14, 6, 12, 10, 8, but still remains open. See Sullivan [15] for a summary of partial results.

Although there has been a lot of progress on approximate versions, Conjecture 1 is known to hold exactly for only a few values of $r$. The case $r=2$ was actually proved by Caccetta and Häggkvist (4).

Theorem 2 (4). Every simple n-vertex digraph with minimum outdegree at least 2 contains a directed cycle of length at most $\left\lceil\frac{n}{2}\right\rceil$.

The case $r=3$ was settled positively by Hamidoune [7], and $r \in\{4,5\}$ by Hoàng and Reed [9].

Given a graph $G$ and a colouring $c$ of $E(G)$, we say that a subgraph $H$ of $G$ is rainbow if no two edges of $H$ are of the same colour. Aharoni (see [1) recently proposed the following strengthening of the Caccetta-Häggkvist conjecture.

Conjecture 3 (Aharoni). Let $G$ be a simple n-vertex graph and $c$ be a colouring of $E(G)$ with $n$ colours, where each colour class has size at least $r$. Then $(G, c)$ contains a rainbow cycle of length at most $\left\lceil\frac{n}{r}\right\rceil$.

In fact, we now show that the following weakening of Aharoni's conjecture implies the Caccetta-Häggkvist conjecture.

Conjecture 4. Let $G$ be a simple n-vertex graph and $c$ be a colouring of $E(G)$ with $n$ colours, where each colour class has size at least $r$. Then ( $G, c$ ) contains a cycle $C$ of length at most $\left\lceil\frac{n}{r}\right\rceil$ such that no two incident edges of $C$ are the same colour.

[^0]Proof of Conjecture 1, assuming Conjecture (4, Let $D$ be a simple digraph of order $n$ and minimum outdegree at least $r$. Let $G$ be the graph obtained from $D$ by forgetting the orientations of all arcs. Let $V(G)=[n]$ and colour $i j \in E(G)$ with colour $i$ if $(i, j) \in E(D)$. Clearly, this colouring uses $n$ colours. Moreover, since $D$ has minimum outdegree at least $r$, each colour class has size at least $r$. Therefore, by Conjecture 4, $G$ contains a properly edge-coloured cycle $C$ of length at most $\left\lceil\frac{n}{r}\right\rceil$. Let $\vec{C}$ be the subdigraph of $D$ corresponding to $C$. We claim that $\vec{C}$ is a directed cycle. If not, then there exists $i, j, k \in V(\vec{C})$ such that $(j, i) \in E(\vec{C})$ and $(j, k) \in E(\vec{C})$. Thus, the two edges of $C$ incident to vertex $j$ are the same colour. This contradicts that $C$ is properly edge-coloured.

Our main theorem is that Aharoni's conjecture holds for $r=2$.
Theorem 5. Let $G$ be a simple n-vertex graph and $c$ be a colouring of $E(G)$ with $n$ colours, where each colour class has size at least 2. Then $(G, c)$ contains a rainbow cycle of length at most $\left\lceil\frac{n}{2}\right\rceil$.
The rest of the paper is organized as follows. In Section 2, we prove our main theorem. We show that our bound is tight in Section 3, and that there is a sharp increase in the 'rainbow girth' as the number of colours decreases from $n$. In Section 4, we show that the natural matroid generalization of Theorem 5 holds for cographic matroids, but fails for binary matroids. We conclude with some open problems in Section 5 .

## 2. Proof of the Main Theorem

In this section, we prove Theorem 5. Before proceeding, we require some basic definitions. Let $G$ be a graph. A cut-vertex of $G$ is a vertex $v$ such that $G-v$ has more connected components than $G$. A block of $G$ is a maximal subgraph $H$ such that $H$ has no cut-vertices. A block is non-trivial if it has at least three vertices. An ear-decomposition of $G$ is collection of subgraphs $\left\{H_{0}, H_{1}, \ldots, H_{k}\right\}$ of $G$ satisfying $G=H_{0} \cup H_{1} \cup \cdots \cup H_{k}$, $H_{0}$ is a cycle, and $H_{i}$ is a path such that $\left|V\left(H_{i}\right)\right| \geq 2$ and $V\left(H_{i}\right) \cap \bigcup_{j=0}^{i-1} V\left(H_{j}\right)$ is the set of ends of $H_{i}$ for all $i \in[k]$. It is well-known that a graph is 2 -connected if and only if it has an ear-decomposition. The paths $H_{1}, \ldots, H_{k}$ are the ears of the ear-decomposition. A theta is a graph which has an ear-decomposition with exactly one ear.

Given an edge-coloured graph $(G, c)$, a transversal of $(G, c)$ is a subgraph $H$ of $G$ such that $V(H)=V(G)$ and $E(H)$ contains exactly one edge of each colour. In particular, a transversal is a rainbow subgraph (which may contain isolated vertices).

Proof of Theorem 55. Suppose the theorem is false and let $(G, c)$ be a counterexample with $|E(G)|$ minimum. By minimality, each colour class contains exactly two edges. We claim that $G$ contains a vertex $v$ such that all edges incident to $v$ have different colours (note that an isolated vertex satisfies this vacuously). If not, then at each vertex, there is at least one colour that appears twice. Since there are only $n$ colours, at each vertex there is exactly one colour that appears twice. For each vertex $v$, let $e_{v}$ and $f_{v}$ be the two edges incident to $v$ that have the same colour. For each $v$, we orient $e_{v}$ and $f_{v}$ away from $v$ and apply Theorem 2 to find a directed cycle of length at most $\left\lceil\frac{n}{2}\right\rceil$. This corresponds to a rainbow cycle in $G$, which contradicts that $(G, c)$ is a counterexample.

Let $H$ be an arbitrary transversal of $(G, c)$. Since $H$ has $n$ edges and $n$ vertices, it follows that $H$ contains at least one cycle and hence at least one non-trivial block. If $H$ contains two non-trivial blocks, then $H$ contains two rainbow cycles that meet in at most one vertex, and thus a cycle of length at most $\left\lceil\frac{n}{2}\right\rceil$. However, this would contradict that $(G, c)$ is a counterexample. Therefore, $H$ contains exactly one non-trivial block $B$. Suppose $B$ has an ear decomposition with at least two ears. In this case, $|V(B)| \leq n-2$, since $B$ contains at most $n$ edges. Moreover, $B$ contains a subgraph $B^{\prime}$ which is either two cycles meeting in at most two vertices, or a subdivision of $K_{4}$. If the former holds,
then $B^{\prime}$ contains a cycle of length at most $\left\lfloor\frac{\left|V\left(B^{\prime}\right)\right|+2}{2}\right\rfloor \leq\left\lceil\frac{n}{2}\right\rceil$. If the latter holds, then $B^{\prime}$ contains four cycles $C_{1}, \ldots, C_{4}$ such that $\sum_{i \in[4]}\left|V\left(C_{i}\right)\right|=2\left|V(B)^{\prime}\right|+4 \leq 2 n$. Thus, one of these four cycles has length at most $\left\lceil\frac{n}{2}\right\rceil$. Since $(G, c)$ is a counterexample, $B$ contains at most one ear. That is, $B$ is a cycle or a theta. It follows that every transversal of $(G, c)$ is either a connected graph with exactly one cycle, or the disjoint union of a tree and a graph containing a theta.

Claim 6. $(G, c)$ contains a rainbow theta.
Subproof. Since $G$ contains a vertex $v$ such that all edges incident to $v$ have different colours, there is a transversal $H$ of $(G, c)$ such that $v$ is an isolated vertex in $H$. It follows that the other component of $H$ contains a theta. In particular, $(G, c)$ contains a rainbow theta.

The rest of the proof only uses the fact that $(G, c)$ contains a rainbow theta. Let $\theta$ be a rainbow theta in $(G, c)$ with $|V(\theta)|$ minimum. Let $P_{1}, P_{2}$ and $P_{3}$ be paths in $\theta$ such that $\theta=P_{1} \cup P_{2} \cup P_{3}, V\left(P_{i}\right) \cap V\left(P_{j}\right):=\{x, y\}$ for all $i \neq j$. and $\left|V\left(P_{1}\right)\right| \leq\left|V\left(P_{2}\right)\right| \leq\left|V\left(P_{3}\right)\right|$.
Claim 7. $\theta$ contains a cycle of length at most $\left\lfloor\frac{2|V(\theta)|+2}{3}\right\rfloor$. Moreover, if $|V(\theta)|=3 k+2$, then $\theta$ contains a cycle of length at most $2 k+1$, unless $\left|V\left(P_{1}\right)\right|=\left|V\left(P_{2}\right)\right|=\left|V\left(P_{3}\right)\right|$.
Subproof. $P_{1} \cup P_{2}$ is a cycle of length at most $\left\lfloor\frac{2|V(\theta)|+2}{3}\right\rfloor$. Moreover, if $|V(\theta)|=3 k+2$, then $P_{1} \cup P_{2}$ is a cycle of length at most $2 k+1$, unless $\left|V\left(P_{1}\right)\right|=\left|V\left(P_{2}\right)\right|=\left|V\left(P_{3}\right)\right|$.

A chord of $\theta$ is an edge $e \in E(G) \backslash E(\theta)$ such that both ends of $e$ are in $V(\theta)$.
Claim 8. $\theta$ has at most two chords.
Subproof. Note that $|V(\theta)| \leq n-1$, since $\theta$ is rainbow and thus contains at most $n$ edges. Let $e$ be a chord. First suppose $e$ has both endpoints on some $P_{i}$. Let $C_{i}$ be the unique cycle in $P_{i} \cup\{e\}$. Note that $C_{i}$ is rainbow, otherwise $\left(\theta \backslash E\left(C_{i}\right)\right) \cup\{e\}$ contradicts the minimality of $|V(\theta)|$. Therefore, $\theta \cup\{e\}$ contains two rainbow cycles meeting in at most two vertices. One of these two cycles has length at most $\left\lfloor\frac{|V(\theta)|+2}{2}\right\rfloor \leq\left\lfloor\frac{n+1}{2}\right\rfloor=\left\lceil\frac{n}{2}\right\rceil$.

By symmetry, we may assume that the ends of $e$ are on $P_{1}$ and $P_{2}$. Suppose $e$ is coloured red. If $P_{1} \cup P_{2}$ does not contain a red edge, then $\theta \cup\{e\}$ contains rainbow cycles $C_{1}, \ldots, C_{4}$ such that $\sum_{i \in[4]}\left|V\left(C_{i}\right)\right|=2|V(\theta)|+4 \leq 2 n+2$. Thus, one of these cycles has length at most $\left\lceil\frac{n}{2}\right\rceil$. It follows that some edge $e^{\prime}$ of $P_{1} \cup P_{2}$ is also red. By the minimality of $|V(\theta)|$, this is only possible if $e$ and $e^{\prime}$ are incident and one end of $e^{\prime}$ is in $\{x, y\}$.

If $\theta$ has at least three chords, then by symmetry and the pigeonhole principle, we may assume there exist chords $e_{1}$ and $e_{2}$ such that $e_{1}^{\prime}$ and $e_{2}^{\prime}$ are both incident to $x$. But now $\left(\theta \cup\left\{e_{1}, e_{2}\right\}\right) \backslash\left\{e_{1}^{\prime}, e_{2}^{\prime}\right\}$ contains a rainbow theta with fewer vertices than $\theta$.

Since $G \backslash E(\theta)$ contains a transversal, there is a rainbow cycle $C$ that is edge-disjoint from $\theta$. Let $V_{1}=V(\theta) \backslash V(C), V_{2}=V(\theta) \cap V(C)$, and $V_{3}=V(C) \backslash V(\theta)$. Since $(G, c)$ is a counterexample, $|V(C)|=\left|V_{2}\right|+\left|V_{3}\right| \geq\left\lceil\frac{n}{2}\right\rceil+1$. Let $t$ be the number of chords of $\theta$. Note that $t \leq 2$ by Claim 8. Observe that if $a, b \in V_{2}$ and $a b \in E(C)$, then $a b$ is a chord of $\theta$. Therefore, $C\left[V_{2}\right]$ contains at most $t$ edges. It follows that $\left|V_{2}\right| \leq\left\lfloor\frac{|V(C)|}{2}\right\rfloor+t$, or equivalently $\left|V_{3}\right|+t \geq\left|V_{2}\right|$. Therefore, $\left|V_{3}\right| \geq \frac{\left\lceil\frac{n}{2}\right\rceil+(1-t)}{2}$, and

$$
|V(\theta)| \leq n-\left|V_{3}\right| \leq n-\frac{\left\lceil\frac{n}{2}\right\rceil+(1-t)}{2} \leq n-\frac{\left\lceil\frac{n}{2}\right\rceil-1}{2}
$$

where the last inequality follows since $t \leq 2$.
Combining the bound $|V(\theta)| \leq n-\frac{\left\lceil\frac{n}{2}\right\rceil-1}{2}$ with Claim 7. we are done unless $n \equiv 2$ $(\bmod 4)$ and all the above bounds are tight. In particular, $t=2, n=4 k+2,\left|V_{1}\right|=2 k$, $\left|V_{2}\right|=k+2,\left|V_{3}\right|=k$. Moreover, by the second part of Claim 7, each of $P_{1}, P_{2}$, and $P_{3}$
contains exactly $k+2$ vertices. Let $e$ be a chord of $\theta$ and $e^{\prime}$ be the edge of $\theta$ of the same colour as $e$. By the second part of Claim 7, $\theta^{\prime}:=(\theta \backslash\{e\}) \cup\left\{e^{\prime}\right\}$ contains a cycle of length at most $2 k+1=\frac{n}{2}$ vertices, as required.

## 3. Tightness of the Bound

We now show that our bound is tight, and that there is a dramatic change of behaviour as we decrease the number of colours from $n$. To be precise, define the rainbow girth of an edge-coloured graph $(G, c)$, denoted $\operatorname{rg}(G, c)$, to be the length of a shortest rainbow cycle in $(G, c)$. If $(G, c)$ does not contain a rainbow cycle, then $\operatorname{rg}(G, c)=\infty$. Let
$f(n, t):=\max \{\operatorname{rg}(G, c):|V(G)|=n,|E(G)|=2 t$, each colour class of $c$ has size 2$\}$.
Theorem 9. For all $n \geq 3$ and $t \leq n$,

$$
\begin{cases}f(n, t)=\infty & \\ \text { if } t \leq n-2, \\ f(n, t)=n-1 & \\ \text { if } t=n-1, \\ f(n, t)=\left\lceil\frac{n}{2}\right\rceil & \\ \text { if } t=n .\end{cases}
$$

Proof. By Theorem 5. $f(n, n) \leq\left\lceil\frac{n}{2}\right\rceil$. For the corresponding lowerbound, let $G$ be a graph with vertex set $\mathbb{Z} / n \mathbb{Z}$ and edges $i(i+1)$ and $i(i+2)$ for all $i \in V(G)$. Colour both $i(i+1)$ and $i(i+2)$ with colour $i$ for all $i \in V(G)$. See Figure 1. It is easy to check that the shortest rainbow cycle in this graph has length $\left\lceil\frac{n}{2}\right\rceil$.


Figure 1. The shortest rainbow cycle has length $\left\lceil\frac{7}{2}\right\rceil=4$.
We now show $f(n, n-1)=n-1$. For the upperbound, let $G$ be a graph with $|V(G)|=$ $n,|E(G)|=2 n-2$, and let $c$ be a colouring of $E(G)$ such that each colour class has size 2. Since there are only $n-1$ colours, there is a vertex $v$ of $G$ such that all edges incident to $v$ are coloured differently. Therefore, there is a transversal $H$ of $(G, c)$ such that $v$ is an isolated vertex in $H$. Since $H-v$ contains $n-1$ vertices and $n-1$ edges, $H-v$ contains a cycle of length at most $n-1$. For the corresponding lowerbound, let $W_{n}$ be the wheel graph on $n$ vertices. Let $c$ be a colouring of $E\left(W_{n}\right)$ such that each colour class is a path with two edges, one of which is incident to the hub vertex. See Figure 2. Observe that no rainbow cycle of ( $W_{n}, c$ ) can use the hub vertex. Therefore, the shortest rainbow cycle in ( $W_{n}, c$ ) has length $n-1$.


Figure 2. A colouring $c$ of $E\left(W_{5}\right)$, whose shortest rainbow cycle has length 4.

By deleting two edges of the same colour from $\left(W_{n}, c\right)$ we obtain a graph on $n$ vertices and $2(n-2)$ edges that does not contain a rainbow cycle. Therefore, $f(n, t)=\infty$, for all $t \leq n-2$.

We have determined $f(n, t)$ exactly for all $t \leq n$. What happens for $t>n$ ? The best general upper bound we can prove follows from a theorem of Bollobás and Szemerédi [2]. To state their result, we need some definitions. The girth of a graph $G$, denoted $g(G)$, is the length of a shortest cycle in $G$. Define

$$
g(n, k):=\max \{g(G):|V(G)|=n,|E(G)|-|V(G)|=k\}
$$

Bollobás and Szemerédi prove the following.
Theorem 10 ([2]). For all $n \geq 4$ and $k \geq 2$,

$$
g(n, k) \leq \frac{2(n+k)}{3 k}(\log k+\log \log k+4)
$$

As a corollary, we obtain the following.
Theorem 11. For all $n \geq 4$ and $k \geq 2$,

$$
f(n, n+k) \leq \frac{2(n+k)}{3 k}(\log k+\log \log k+4)
$$

Proof. Let $G$ be a simple $n$-vertex graph, with $|E(G)|=2(n+k)$ and $c$ be a colouring of $E(G)$ where each colour class has size 2 . Let $H$ be a transversal of $(G, c)$. Note that $H$ has $n$ vertices and $n+k$ edges. By Theorem 10, $H$ contains a cycle of length at most $\frac{2(n+k)}{3 k}(\log k+\log \log k+4)$. Since this cycle is necessarily rainbow, we are done.

## 4. Matroid Generalizations

In this section, we consider matroid generalizations of Aharoni's conjecture. For the reader unfamiliar with matroids, we introduce all the necessary definitions now. Note that nothing beyond basic linear algebra will be required. For a more thorough introduction to matroids, we refer the reader to Oxley [11].

A matroid is a pair $M=(E, \mathcal{C})$ where $E$ is a finite set, called the ground set of $M$, and $\mathcal{C}$ is a collection of subsets of $E$, called circuits, satisfying

- $\emptyset \notin \mathcal{C}$,
- if $C^{\prime}$ is a proper subset of $C \in \mathcal{C}$, then $C^{\prime} \notin \mathcal{C}$,
- if $C_{1}$ and $C_{2}$ are distinct members of $\mathcal{C}$ and $e \in C_{1} \cap C_{2}$, then there exists $C_{3} \subseteq$ $\left(C_{1} \cup C_{2}\right) \backslash\{e\}$.
A set $I \subseteq E$ is independent if it does not contain a circuit. The rank of $X \subseteq E$ is the size of a largest independent set contained in $X$, and is denoted $r_{M}(X)$. The rank of $M$ is
$r(M):=r_{M}(E)$. A matroid is simple it it does not contain any circuits of size 1 or 2 . We now give examples of all the matroids that appear in this paper.

Let $G$ be a graph. We will consider two different matroids with ground set $E(G)$. The circuits of the first matroid are the (edges of) cycles of $G$. This is the cycle matroid of $G$, denoted $M(G)$. A matroid is graphic if it is isomorphic to the cycle matroid of some graph. The second matroid is the dual of the cycle matroid of $G$. However, we will not define duality, opting instead to define this matroid directly. An edge-cut of $G$ is a set of edges $C^{*}$ such that $G \backslash C^{*}$ has more connected components than $G$. A cocycle is an inclusion-wise minimal edge-cut. The collection of cocycles of $G$ is also a matroid, called the cocycle matroid of $G$, and is denoted $M(G)^{*}$. A matroid is cographic if it is isomorphic to the cocycle matroid of some graph.

Let $\mathbb{F}$ be a field. An $\mathbb{F}$-matrix is a matrix with entries in $\mathbb{F}$. Let $A$ be an $\mathbb{F}$-matrix whose columns are labelled by a finite set $E$. The column matroid of $A$, denoted $M[A]$, is the matroid with ground set $E$ whose circuits correspond to the minimal (under inclusion) linearly dependent columns of $A$. A matroid is representable over $\mathbb{F}$ if it is isomorphic to $M[A]$ for some $\mathbb{F}$-matrix $A$. A matroid is binary if it representable over the two-element field, and it is regular if it is representable over every field.

Finally, for integers $0 \leq k \leq n$, the uniform matroid $U_{k, n}$ is the matroid with ground set $[n]$, whose circuits are all the subsets of $[n]$ of size $k+1$.

An attractive feature of Aharoni's conjecture as opposed to the Caccetta-Häggkvist conjecture, is that there is a natural matroid generalization. For example, the following is the matroid analogue of Theorem 5 .
Conjecture 12. Let $M$ be a simple rank- $(n-1)$ matroid and $c$ be a colouring of $E(M)$ with $n$ colours, where each colour class has size at least 2 . Then $M$ contains a rainbow circuit of size at most $\left\lceil\frac{n}{2}\right\rceil$.
Let $G$ be a simple, connected, $n$-vertex graph, and $r$ be the rank of $M(G)$. Note that $r$ is the number of edges in a spanning tree of $G$, and so $n-1=r$. Moreover, since the circuits of $M(G)$ are the cycles of $G$, Conjecture 12 holds for graphic matroids by Theorem 5 .

Unfortunately, it is easy to see that Conjecture 12 is false, since the uniform matroid $U_{n-1, m}$ does not contain any circuits of size less than $n$. On the other hand, we now prove that Conjecture 12 is true for cographic matroids.
Theorem 13. Let $N$ be a simple rank- $(n-1)$ cographic matroid and $c$ be a colouring of $E(N)$ with $n$ colours, where each colour class has size at least 2 . Then $N$ contains a rainbow circuit of size at most $\left\lceil\frac{n}{2}\right\rceil$.
Proof. Let $(N, c)$ be a counterexample with $|E(N)|$ minimum. By minimality, each colour class has size exactly 2 . Let $G$ be a graph such that $N=M(G)^{*}$. Let $G_{1}, \ldots, G_{k}$ be the connected components of $G$, and $r_{i}:=r\left(M\left(G_{i}\right)^{*}\right)$ for each $i \in[k]$. Since every cocycle of $N$ is a cocycle in some $N_{i}:=M\left(G_{i}\right)^{*}$, it follows that

$$
\begin{equation*}
\sum_{i \in[k]} r_{i}=r(N)=n-1 . \tag{1}
\end{equation*}
$$

First suppose there is some $j \in[k]$ such that $\left|E\left(G_{j}\right)\right| \geq 2\left(r_{j}+1\right)$. By merging colour classes we may assume that exactly $r_{j}+1$ colours appear in $E\left(G_{j}\right)$ and each of these colours appears at least twice in $E\left(G_{j}\right)$. By minimality, $G_{j}$ contains a rainbow cocycle $C^{*}$ of size at most $\left\lceil\frac{r_{j}+1}{2}\right\rceil \leq\left\lceil\frac{n}{2}\right\rceil$. By unmerging colours, $C^{*}$ is also a rainbow cocycle of $G$, so we are done. By (1), such an index $j$ exists unless $k=2,\left|E\left(G_{1}\right)\right|=2 r_{1}+1$, and $\left|E\left(G_{2}\right)\right|=2 r_{1}+1$. By (1) and symmetry, we may assume $r_{1} \leq\left\lfloor\frac{n-1}{2}\right\rfloor$. Since $\left|E\left(G_{1}\right)\right|=2 r_{1}+1$ and each colour appears at most twice in $E\left(G_{1}\right)$, there exists a rainbow set $A \subseteq E\left(G_{1}\right)$ such that $|A|=r_{1}+1$. Since $|A|>r_{1}, A$ contains a cocycle $C^{*}$ of $G$. Since $C^{*}$ is rainbow and $\left|C^{*}\right| \leq|A|=r_{1}+1 \leq\left\lfloor\frac{n-1}{2}\right\rfloor+1=\left\lceil\frac{n}{2}\right\rceil$, we are done.

Henceforth, we may assume that $G$ is connected. Let $N^{*}=M(G)$, the cycle matroid of $G$. We use the well-known fact that $r(N)+r\left(N^{*}\right)=|E(G)|=2 n$. Therefore, since $N$ has rank $n-1, N^{*}$ has rank $n+1$. Since $G$ is connected, $|V(G)|=n+2$.

For each vertex $v \in V(G)$, let $\delta_{G}(v)$ be the set of edges of $G$ incident to $v$. Since $\delta_{G}(v)$ is an edge-cut for each $v \in V(G)$ and $N$ is simple, $G$ has minimum degree at least 3 . Moreover, since there are exactly $n$ colours and $n+2$ vertices, there are at least two distinct vertices $x$ and $y$ of $G$ such that $\delta_{G}(x)$ and $\delta_{G}(y)$ are both rainbow. If $\operatorname{deg}_{G}(x) \leq\left\lceil\frac{n}{2}\right\rceil$ or $\operatorname{deg}_{G}(y) \leq\left\lceil\frac{n}{2}\right\rceil$, then $\delta_{G}(x)$ or $\delta_{G}(y)$ contains a rainbow cocycle of size at most $\left\lceil\frac{n}{2}\right\rceil$. Thus, $\operatorname{deg}_{G}(x), \operatorname{deg}_{G}(y) \geq\left\lceil\frac{n}{2}\right\rceil+1$. Since $4 n=2|E(G)|=\sum_{v \in V(G)} \operatorname{deg}_{G}(v)$, it follows that $\sum_{v \in V(G) \backslash\{x, y\}} \operatorname{deg}_{G}(v) \leq 3 n-2$. Therefore, some vertex $z \in V(G) \backslash\{x, y\}$ has degree at most 2, which contradicts that $G$ has minimum degree at least 3 .

We finish this section by giving an infinite family of binary matroids for which Conjecture 12 fails.

Theorem 14. For each even integer $n \geq 6$, there exists a simple rank-( $n-1$ ) binary matroid $M$ on $2 n$ elements, and a colouring of $E(M)$ where each colour class has size 2, such that all rainbow circuits of $(M, c)$ have size strictly greater than $\frac{n}{2}$.
Proof. Let $n \geq 6$ be even. For each $i \in[n-1]$, let $\mathbf{e}_{i}$ be the $i$ th standard basis vector in $\mathbb{F}_{2}^{n-1}$. Let $\mathbf{0}$ and $\mathbf{1}$ be the all-zeros and all-ones vectors in $\mathbb{F}_{2}^{n-1}$, respectively. Let $M$ be the binary matroid represented by the following $2 n$ vectors $\mathcal{V}$.

- $\mathbf{e}_{i}$, for all $i \in[n-1]$;
- $\mathbf{e}_{i}+\mathbf{e}_{i+1}$, for all $i \in[n-2]$;
- $\mathbf{1}, \mathbf{1}+\mathbf{e}_{n-2}$, and $\mathbf{e}_{1}+\mathbf{e}_{n-2}$.

Since $\left\{\mathbf{e}_{i} \mid i \in[n-1]\right\}$ are linearly independent, $M$ has rank $n-1$. Moreover, all vectors in $\mathcal{V}$ are distinct and non-zero, so $M$ is simple.

We now specify the colouring, which is just a pairing of $\mathcal{V}$. For each $i \in[n-3]$, we pair $\mathbf{e}_{i}$ with $\mathbf{e}_{i}+\mathbf{e}_{i+1}$. Finally, we pair $\mathbf{e}_{n-2}$ with $\mathbf{e}_{1}+\mathbf{e}_{n-2} ; \mathbf{e}_{n-1}$ with $\mathbf{e}_{n-2}+\mathbf{e}_{n-1} ;$ and $\mathbf{1}$ with $\mathbf{1}+\mathbf{e}_{n-2}$. To illustrate, the case $n=6$ is given by the following matrix, where column $i$ and column $6+i$ are the same colour for all $i \in[6]$.

$$
\left(\begin{array}{llllllllllll}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)
$$

Note that a subset of $\mathcal{V}$ is linearly dependent if and only if it sums to $\mathbf{0}$. Therefore, it suffices to prove that every rainbow subset of $\mathcal{V}$ summing to $\mathbf{0}$ has size more than $\frac{n}{2}$. Let $\mathcal{C} \subseteq \mathcal{V}$ be a rainbow set such that $\sum_{v \in \mathcal{C}} v=\mathbf{0}$.

We first consider the case $\mathbf{1} \in \mathcal{C}$ or $\mathbf{1}+\mathbf{e}_{n-2} \in \mathcal{C}$. Since $\mathbf{1}$ and $\mathbf{1}+\mathbf{e}_{n-2}$ are the same colour, exactly one of them, which we call $x$, is in $\mathcal{C}$. Since $\mathbf{e}_{n-1}$ and $\mathbf{e}_{n-2}+\mathbf{e}_{n-1}$ are the only other vectors in $\mathcal{V}$ that are non-zero in their $(n-1)$ th coordinate, exactly one of them, which we call $y$, is in $\mathcal{C}$. Let $\mathcal{C}^{\prime}=\mathcal{C} \backslash\{x, y\}$. In all four cases, $\sum_{v \in \mathcal{C}^{\prime}}$ is $\mathbf{1}+\mathbf{e}_{n-1}$ or $\mathbf{1}+\mathbf{e}_{n-1}+\mathbf{e}_{n-2}$. Since $n$ is even, and all vectors in $\mathcal{C}^{\prime}$ have support at most $2,\left|\mathcal{C}^{\prime}\right| \geq \frac{n}{2}-1$. Therefore, $|\mathcal{C}|>\frac{n}{2}$.

The remaining case is $\mathbf{1} \notin \mathcal{C}$ and $\mathbf{1}+\mathbf{e}_{n-2} \notin \mathcal{C}$. The only other vectors in $\mathcal{V}$ whose $(n-1)$ th coordinate is non-zero are $\mathbf{e}_{n-1}$ and $\mathbf{e}_{n-2}+\mathbf{e}_{n-1}$. Since $\mathbf{e}_{n-1}$ and $\mathbf{e}_{n-2}+\mathbf{e}_{n-1}$ are the same colour, they cannot both be in $\mathcal{C}$. Therefore, neither is in $\mathcal{C}$. Now, since $\mathbf{e}_{n-2}$ and $\mathbf{e}_{1}+\mathbf{e}_{n-2}$ are the same colour and the only other vectors in $\mathcal{V}$ whose $(n-2)$ th coordinate is non-zero, we conclude that neither $\mathbf{e}_{n-2}$ nor $\mathbf{e}_{1}+\mathbf{e}_{n-2}$ is in $\mathcal{C}$. Repeating the same argument for each of the pairs $\left\{\mathbf{e}_{i}, \mathbf{e}_{i}+\mathbf{e}_{i+1}\right\}$ for $i=1, \ldots, n-3$ (in that order), we conclude that $\mathcal{C}=\emptyset$, which is a contradiction.

A slight modification of the above construction also yields counterexamples for all odd integers $n \geq 7$. On the other hand, it is fairly easy to show that Conjecture 12 holds for binary matroids when $n \leq 5$ (it is true vacuously when $n \leq 4$ ). Thus, Conjecture 12 holds for binary matroids if and only if $n \leq 5$.

## 5. Open Problems

Note that in proving Theorem 11, we only use one fixed transversal. By considering multiple transversals, we suspect that the bound in Theorem 11 can be improved.

Problem 15. Determine $f(n, t)$ for $t>n$.
Since the Caccetta-Häggkvist conjecture is known to hold for $r \in\{3,4,5\}$, another possible direction is to prove Aharoni's conjecture for $r \in\{3,4,5\}$.

Problem 16. Prove that Conjecture 3 (or Conjecture (4) holds for $r \in\{3,4,5\}$.
Recall that our proof of Aharoni's conjecture for $r=2$ uses Theorem 2 as a blackbox. It would be interesting to find a proof of Theorem 5 that avoids using Theorem 2,

Finally, by Theorems 5 and 13, Conjecture 12 holds for both graphic and cographic matroids. Therefore, we suspect there is a proof of Conjecture 12 for regular matroids via Seymour's regular matroid decomposition theorem [13].

Conjecture 17. Let $M$ be a simple rank- $(n-1)$ regular matroid and $c$ be a colouring of $E(M)$ with $n$ colours, where each colour class has size at least 2 . Then $M$ contains a rainbow circuit of size at most $\left\lceil\frac{n}{2}\right\rceil$.

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