# The hamiltonicity of essentially 9-connected line graphs

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#### Abstract

Yang et al. proved that every 3-connected, essentially 11-connected line graph is Hamilton-connected. This was extended by Li and Yang to 3-connected, essentially 10-connected graphs. Strengthening their result further, we prove that 3-connected, essentially 9-connected line graphs are Hamilton-connected. We use a method based on quasigraphs in combination with the discharging technique. The result extends to claw-free graphs.

#### 1 Introduction

A conjecture of Thomassen [13] states that every 4-connected line graph is hamiltonian. Motivated by this conjecture, Lai et al. [4] studied the hamiltonicity of line graphs in relation to their essential connectivity (the definition is recalled later in this section). They proved the following result:

**Theorem 1.** Every 3-connected, essentially 11-connected line graph is hamiltonian.

In fact, Yang et al. [14] prove that under the assumption of Theorem 1, the graph is *Hamilton-connected*, i.e., any pair of its vertices is joined by a Hamilton path. The result was recently improved by Li and Yang [7] who showed that 3-connected, essentially 10-connected graphs are Hamilton-connected.

This partially answered a question of Lai et al. [4] whether the constant in Theorem 1 can be replaced by a smaller one. They note that the least possible value is 4, which is improved to 5 in [15]. We add that there are 3-connected,

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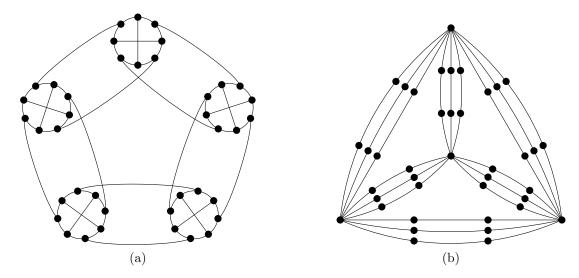


Figure 1: (a) A graph used to construct a 3-connected, essentially 4-edge-connected line graph without a Hamilton path. (b) A graph G such that L(G) is 2-connected and essentially k-connected (for k = 9) and has no Hamilton path.

essentially 4-connected line graphs that do not even have a Hamilton path, such as the line graph of a graph obtained by adding one pendant edge to each vertex of the graph in Figure 1a.

Regarding the assumption of 3-connectedness in Theorem 1, we remark that there are 2-connected line graphs of arbitrary essential connectivity that do not have any Hamilton path. A class of examples can be constructed by replacing each edge of the complete graph on 4 vertices by an odd number of internally disjoint paths of length 3, as shown in Figure 1b.

The main result of the present paper is a further strengthening of Theorem 1 as follows:

**Theorem 2.** Every 3-connected, essentially 9-connected line graph is Hamilton-connected.

Our approach uses a strengthening of the main result of [2], proved in a companion paper [3]. In particular, we use a reduction to hypergraphs, which is described in Section 2. Preliminaries on the Hamilton connectivity of line graphs (and the implications on the hypergraph side of the problem) are given in Section 3.

Section 4 describes quasigraphs, another crucial component of the approach of [2, 3], and states the main technical tool of this paper, Theorem 13 proved in [3]. A counting argument based on the outcome of Theorem 13 is given in Section 5.

Section 6 is essentially a study of small configurations in the hypergraph corresponding to a graph satisfying the hypothesis of Theorem 2. It prepares the

way for the application of a discharging argument in Section 7. The final section is devoted to an extension of the result to claw-free graphs and concluding remarks.

In the remainder of this section, we recall the necessary terminology and notation. For graph-theoretical concepts not explained here, see, for example, Diestel [1]. Unless otherwise noted, all graphs in this paper are loopless and may contain parallel edges. When speaking of a line graph of a graph G, it is understood that G may contain parallel edges (i.e., G may be a multigraph), but L(G) is by definition a simple graph.

Let G be a graph. We write V(G) and E(G) for the set of vertices and the set of edges of G, respectively. For  $i \geq 0$ ,  $V_i(G)$  is the set of vertices of degree i in G.

A vertex-cut or edge-cut X is called *essential* if G - X has at least two components which are *nontrivial* (i.e., contain more than one vertex). The graph G is *essentially k-connected* if it has more than k vertices and contains no essential vertex-cut of size less than k. Similarly, G is *essentially k-edge-connected* if it contains no essential edge-cut of size less than k. It is not hard to see that L(G) is k-connected if and only if G is essentially k-edge-connected and |E(G)| > k.

We extend the above definitions as follows. An edge-cut X in a graph G is r-essential  $(r \ge 0)$  if there at least two components of G-C, each of which contains at least r edges. The graph G is r-essentially k-edge-connected  $(k \ge 1)$  if it has no r-essential edge-cuts of size less than k. Thus, '0-essentially k-edge-connected' is the same as 'k-edge-connected', while '1-essentially k-edge-connected' is the same as 'essentially k-edge-connected'.

We note the following easy observation:

**Observation 3.** The line graph L(G) of a graph G is essentially k-connected if and only if G is 2-essentially k-edge-connected and |E(G)| > k.

The length of a path is the number of its edges. The degree of a vertex v in a graph G is denoted by  $d_G(v)$ . Given a set of vertices  $X \subseteq V(G)$ , we let  $\partial_G(X)$  denote the set of edges of G with exactly one endvertex in X. Furthermore, we extend the notation for the degree of a vertex and set  $d_G(X) = |\partial_G(X)|$ .

Besides graphs, we will also consider 3-hypergraphs, that is, hypergraphs with all hyperedges of size 2 or 3 (called 2-hyperedges or 3-hyperedges accordingly). For a hypergraph H, V(H) and E(H) denote its vertex set and its hyperedge set, respectively. The symbol  $V_i(H)$  denotes the set of vertices of degree i ( $i \geq 0$ ). In addition, for  $i \in \{2,3\}$ , we let  $E_i(H)$  denote the set of i-hyperedges of H. We define a graph G(H) whose vertex set is  $V(H) \cup E_3(H)$ , with vertices u, v joined by an edge if either u, v are neighbours in H, or  $v \in E_3(H)$  and u is a vertex contained in v.

For  $X \subseteq V(H)$ , we let  $\partial_H(X)$  be the set of hyperedges of H intersecting X but not contained in it, and define  $d_H(X) = |\partial_H(X)|$  as in the graph case.

If e is a hyperedge of H and v is a vertex contained in e, then the detachment of e from v is the operation which removes e from H and, in case |e| = 3, replaces

it with  $e - \{v\}$ .

# 2 Reduction to hypergraphs

From this point on, let G be a graph whose line graph satisfies the hypothesis of Theorem 2. By Observation 3 and the preceding discussion, G is essentially 3-edge-connected and 2-essentially 9-edge-connected.

We begin by transforming G to a graph  $G^0$  called the *core* of G. Recall that the *suppression* of a vertex v of degree 2 is the contraction of one of its incident edges (discarding any loop that may result). The graph  $G^0$  is obtained from  $G - V_1(G)$  by suppressing all the vertices of degree 2. By our assumption that G is essentially 3-edge-connected,  $G - V_1(G)$  has minimum degree at least 2 unless G is a star. (Note also that that  $V_2(G - V_1(G)) = V_2(G)$ .) Clearly, either  $G^0$  is trivial (i.e., it has only one vertex), or  $G^0$  has minimum degree at least 3. This observation is strengthened in Lemma 5 below.

Given a set  $X \subset V(G^0)$ , we define  $X^*$  as the union of X with the set of vertices  $y \in V(G) - V(G^0)$  such that  $N_G(y) \subseteq X$ .

**Observation 4.** For any  $X \subseteq V(G^0)$ , it holds that  $d_{G^0}(X) = d_G(X^*)$ . Furthermore, if  $\partial_{G^0}(X)$  is an r-essential edge-cut in  $G^0$   $(r \ge 0)$ , then  $\partial_G(X^*)$  is an r-essential edge-cut in G.

**Lemma 5.** If L(G) is 3-connected and essentially 9-connected, then  $G^0$  has the following properties:

- (i)  $G^0$  is 3-edge-connected,
- (ii)  $G^0$  is essentially 4-edge-connected,
- (iii)  $G^0$  is 2-essentially 9-edge-connected.

*Proof.* Part (i) was proved by Shao [10] (see [4, Lemma 2.2(i)]). Part (iii) of the lemma follows from Observation 4. We prove part (ii). For contradiction, let F be an essential edge-cut in  $G^0$  of size at most 3. Let K and L be two components of  $G^0 - F$  containing at least one edge each. Since F is not 2-essential, one of them (say, K) contains exactly one edge. The assumption that  $|F| \leq 3$  implies that one of the two vertices of K has degree at most 2, a contradiction with part (i).

The vertices of G which are not vertices of  $G^0$  are called *transient*. A vertex v of  $G^0$  with  $d_{G^0}(v) = 3$  is said to be *protected* if it is adjacent in G to a transient vertex.

We now turn  $G^0$  into a 3-hypergraph  $H^0$ . Let  $W_3$  be the set of vertices of degree 3 in  $G^0$ , and let  $W_3^{\times}$  be the subset of  $W_3$  consisting of the protected vertices. We choose a maximal independent subset W of  $W_3 - W_3^{\times}$ . The 3-hypergraph

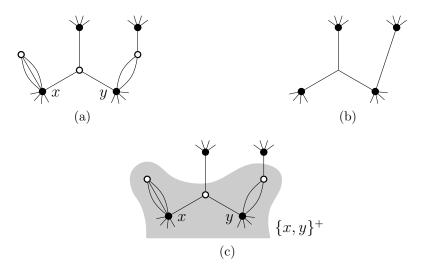


Figure 2: The transformation of  $G^0$  to  $H^0$ . (a) Part of the graph  $G^0$  with vertices in W filled white. (b) The corresponding part of  $H^0$ , where the three incident lines without a common vertex mark represent a 3-hyperedge. (c) The set  $\{x,y\}^+$  includes all of the vertices of W shown in the picture.

 $H^0$  is constructed from  $G^0-W$  by adding, for each  $w\in W$ , a hyperedge h(w) consisting of the neighbours of w in  $G^0$  provided that there are at least two such neighbours; if there are exactly two, then |h(w)|=2. The procedure is illustrated in Figure 2. In all the figures in this paper, a 3-hyperedge is represented by three lines meeting at a point which is not marked as a vertex.

The vertices in W are called *temporary*, the other vertices of  $G^0$  permanent. Thus, the set of permanent vertices is the vertex set of  $H^0$ . Note that all protected vertices are permanent.

#### **Lemma 6.** Every edge of G has a permanent endvertex.

*Proof.* Assume that an edge e of G has endvertices u and v, and that u is not permanent. If u is transient, then v is protected and hence permanent. Otherwise, u is temporary, in which case v is neither transient (as this would make u permanent) nor temporary (since temporary vertices form an independent set). Hence, v is permanent.

Similarly to the set  $X^*$  above, we introduce a set  $Y^+ \subseteq V(G^0)$ , where  $Y \subseteq V(H^0)$ . The definition is illustrated in Figure 2c. The set  $Y^+$  is defined as the union of Y with the set of all the vertices  $w \in W$  such that in  $G^0$ , w is incident with at least two edges to Y (possibly parallel). As in Observation 4, we have  $d_{H^0}(Y) = d_{G^0}(Y^+)$ . Since  $G^0$  is 3-edge-connected, we obtain the following observation:

**Observation 7.** The hypergraph  $H^0$  is 3-edge-connected.

A more detailed study of the properties of  $H^0$  is undertaken in Section 6. The results will be used in the design of a discharging procedure in Section 7.

We add one more definition. To each edge e of G, we want to assign a hyperedge k(e) of  $H^0$  which 'corresponds' to e in  $H^0$ . First, let us define a value  $k_1(e)$  which is either an edge of  $G^0$ , or the empty set:

$$k_1(e) = \begin{cases} \emptyset & \text{if } e \text{ is incident with } V_1(G), \\ e' & \text{if } e \text{ is incident with a transient vertex } z \notin V_1(G) \\ & \text{and } e' \text{ is obtained by suppressing } z, \\ e & \text{otherwise.} \end{cases}$$

Observe that  $k_1(e)$  is well-defined, because the assumption that G is essentially 3-edge-connected implies that  $V_1(G) \cup V_2(G - V_1(G))$  is an independent set.

Next, we associate a hyperedge  $k_2(f)$  of  $H^0$  with each edge f of  $G^0$ . In the definition, we allow f or  $k_2(f)$  to be the empty set.

$$k_2(f) = \begin{cases} \emptyset & \text{if } f = \emptyset, \\ h(w) & \text{if } f \text{ is incident with a temporary vertex } w, \\ f & \text{otherwise.} \end{cases}$$

Finally, for an edge e of G, we define

$$k(e) = k_2(k_1(e)).$$

## 3 Hamilton connectivity of line graphs

For the cases where the core  $G^0$  is small, we will be able to verify Theorem 2 directly using the following result [5, Lemma 3.3]:

**Lemma 8.** If L(G) is 3-connected and  $G^0$  contains two edge-disjoint spanning trees, then L(G) is Hamilton-connected.

Lemma 8 will be used in conjunction with the characterization of graphs with two disjoint spanning trees, which follows from a more general result of Tutte [11] and Nash-Williams [8]:

**Theorem 9** (Tutte and Nash-Williams). The graph  $G^0$  has two edge-disjoint spanning trees if and only if for every partition  $\mathcal{P}$  of  $V(G^0)$ , the number of edges of  $V(G^0)$  with endvertices in different classes of  $\mathcal{P}$  is at least  $2(|\mathcal{P}|-1)$ .

Using Lemma 8 and Theorem 9, we obtain the following:

**Lemma 10.** If  $G^0$  has at most 5 vertices, then L(G) is Hamilton-connected.

*Proof.* We use Theorem 9 to show that  $G^0$  admits two edge-disjoint spanning trees. Let  $\mathcal{P}$  be a partition of  $V(G^0)$  and let F be the set of edges with endvertices in different classes of  $\mathcal{P}$ . By Lemma 5(i),  $G^0$  is 3-edge-connected and thus  $|F| \geq 3 |\mathcal{P}|/2$ . Since  $|\mathcal{P}| \leq 5$ , we have  $\lceil 3 |\mathcal{P}|/2 \rceil \geq 2(|\mathcal{P}|-1)$ . The lemma follows from Theorem 9 and Lemma 8.

For the purposes of this paper, Lemma 10 enables us to restrict ourselves to the case that  $G^0$  has at least 6 vertices.

We recall a well-known necessary and sufficient condition for the line graph L(G) to be Hamilton-connected. Let  $e_1, e_2$  be edges of G. A trail T in G is an  $(e_1, e_2)$ -trail if its first edge is  $e_1$  and its last edge is  $e_2$ . The trail T is internally dominating if every edge of G is incident with an internal vertex of T. Similarly, T is internally spanning if every vertex of G appears as an internal vertex of T. The following is a folklore analogue of Harary and Nash-Williams' characterization of hamiltonian line graphs (cf. [6, Theorem 1.5]).

**Theorem 11.** Let G be a graph with at least 3 edges. Then L(G) is Hamilton-connected if and only if for every pair of edges  $e_1, e_2 \in E(G)$ , G has an internally dominating  $(e_1, e_2)$ -trail.

We will infer the existence of internally dominating trails in G using hypergraphs obtained by a small modification of  $H^0$  (the 3-hypergraph associated with G as in Section 2). First, we define a 3-hypergraph  $H^e$  and a pair of vertices  $a_1, a_2$  of  $H^e$ . The edges  $e_i$  and the vertices  $a_i$  (i = 1, 2) will be considered fixed throughout the paper.

For i = 1, 2, let  $a_i$  be a permanent vertex of  $e_i$  (which exists by Lemma 6). If possible, we choose  $a_1$  and  $a_2$  so as to be distinct. The hypergraph  $H^e$  is obtained from  $H^0$  by detaching  $k(e_1)$  from  $a_1$  and, subsequently, detaching  $k(e_2)$  from  $a_2$ . We also define  $G^e$  as the graph  $G(H^e)$  corresponding to  $H^e$ . Note that every permanent vertex of G is a vertex of  $G^e$ .

We say that a trail T in a graph G' spans a set  $X \subseteq V(G')$  if  $X \subseteq V(T)$ . If the first and last vertices of T are a and b, respectively, we say that T is an ab-trail. The case a = b is allowed in this definition.

The following lemma provides a bridge between spanning  $a_1a_2$ -trails in  $G(H^e)$  and internally dominating  $(e_1, e_2)$ -trails in G:

**Lemma 12.** If  $G^e$  admits an  $a_1a_2$ -trail spanning  $V(H^e)$ , then G contains an internally dominating  $(e_1, e_2)$ -trail.

Proof. Let  $T^e$  be an  $a_1a_2$ -trail in  $G^e$  spanning  $V(H^e)$ . Let T be the corresponding  $a_1a_2$ -trail in G (that is, whenever  $T^e$  uses an edge f obtained by suppressing a vertex w, T uses the two edges incident with w). Since  $a_i$  is an endvertex of  $e_i$  (i = 1, 2), we can construct an  $(e_1, e_2)$ -trail T' in G by prepending  $e_1$  and appending  $e_2$  to T. Since T spans all permanent vertices and every edge of G has a permanent endvertex, T' is internally dominating.

In view of Lemma 12, proving Theorem 2 reduces to finding an  $a_1a_2$ -trail spanning  $V(H^e)$  in  $G^e$  for each choice of  $e_1, e_2$ . A basic tool for this is Proposition 14 in Section 4.

# 4 Quasigraphs

Our proof relies on a strengthening of the so-called Skeletal Lemma [2, Lemma 17]. The required stronger version is proved in [3]. The formulation and proof of this result use the language and some theory of quasigraphs; in this section, we recall just the bare minimum allowing us to state Theorem 13.

Recall that a 3-hypergraph is a hypergraph whose edges have size 2 or 3. Let H be a 3-hypergraph.

A quasigraph in H is a mapping  $\pi$  that assigns to each hyperedge e of H either a subset of e of size 2, or the empty set. The hyperedges e with  $\pi(e) \neq \emptyset$  are said to be used by  $\pi$ .

Let  $\pi$  be a quasigraph in H. We let  $\pi^*$  denote the graph on V(H), obtained by considering the pairs  $\pi(e)$  ( $e \in E(H)$ ) as edges whenever  $\pi(e) \neq \emptyset$ . If  $\pi^*$  is a forest, then  $\pi$  is said to be *acyclic*. If  $\pi^*$  is the union of a circuit and a set of isolated vertices, then  $\pi$  is a *quasicycle*. The hypergraph H is *acyclic* if there exists no quasicycle in H.

Let  $X \subseteq V(H)$ . We say that the quasigraph  $\pi$  is *connected* on X if the induced subgraph of  $\pi^*$  on X is connected. A somewhat more involved notion is anticonnectedness: we say that  $\pi$  is *anticonnected* on X if for each nontrivial partition  $\mathcal{R}$  of X, there is a hyperedge f such that f intersects at least two classes of  $\mathcal{R}$  and  $\pi(f)$  is contained in one of them.

Let  $\mathcal{P}$  be a partition of V(H). If  $e \in E(H)$ , then  $e/\mathcal{P}$  is defined as the set of all classes of  $\mathcal{P}$  intersected by e. If there is more than one such class, then e is said to be  $\mathcal{P}$ -crossing. The hypergraph  $H/\mathcal{P}$  has vertex set  $\mathcal{P}$  and its hyperedges are all the sets of the form  $e/\mathcal{P}$ , where e is a  $\mathcal{P}$ -crossing hyperedge of H. A quasigraph  $\pi/\mathcal{P}$  in this hypergraph is defined by setting, for every  $\mathcal{P}$ -crossing hyperedge e of H,

$$(\pi/\mathcal{P})(e/\mathcal{P}) = \begin{cases} \pi(e)/\mathcal{P} & \text{if } \pi(e) \text{ is } \mathcal{P}\text{-crossing,} \\ \emptyset & \text{otherwise.} \end{cases}$$

The complement  $\overline{\pi}$  of  $\pi$  is the subhypergraph of H (on the same vertex set) consisting of the hyperedges not used by  $\pi$ .

A partition  $\pi$  of V(H) is  $\pi$ -skeletal if both of the following conditions hold:

- (1) for each  $X \in \mathcal{P}$ ,  $\pi$  is both connected on X and anticonnected on X,
- (2) the complement of  $\pi/\mathcal{P}$  in  $H/\mathcal{P}$  is acyclic.

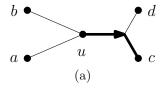


Figure 3: A bad leaf u.

For our purposes, the version of the Skeletal Lemma stated below in Theorem 13 needs to take care of a particular configuration we call 'bad leaf' since it presents a problem in our computations. Let us describe this configuration.

Recall that  $\pi$  is a quasigraph in a 3-hypergraph H. Assume now that  $\pi$  is acyclic. In each component of the graph  $\pi^*$ , we choose an arbitrary root and orient all the edges of  $\pi^*$  toward the root. A hyperedge e of H is associated with a vertex u if it is used by  $\pi$  and u is the tail of  $\pi(e)$  in the resulting oriented graph. Thus, every vertex has at most one associated hyperedge, and conversely, each hyperedge is associated with at most one vertex.

A vertex u of H is a bad leaf for  $\pi$  (and the given choice of the roots of the components of  $\pi^*$ ) if all of the following hold:

- (i) u is a leaf of  $\pi^*$ ,
- (ii) u is incident with exactly three hyperedges, exactly one of which has size 3 (say, e), and
- (iii) e is associated with u.

Bad leaves can be eliminated at the cost of performing certain local modifications in the hypergraph. More precisely, if u is a vertex of the 3-hypergraph H incident with exactly two hyperedges of size 2 and exactly one hyperedge of size 3, then a *switch* at u is the operation depicted in Figure 4. (We remark that in [3], the switch operation acts on quasigraphs in H as well, but this is not necessary for our purposes.)

We say that a 3-hypergraph  $\tilde{H}$  is related to H if it can be obtained from H by a finite sequence of switches at suitable vertices. Note that, in this case,  $G(\tilde{H})$  is isomorphic to G(H).

We can now finally state the main technical result mentioned above, a strengthening of the Skeletal Lemma proved in [3]:

**Theorem 13.** Let H be a 3-hypergraph. There exists a hypergraph H related to H and an acyclic quasigraph  $\sigma$  in  $\tilde{H}$  such that  $\sigma$  has no bad leaves (for any choice of the roots of the components of  $\sigma^*$ ) and  $V(\tilde{H})$  admits a  $\sigma$ -skeletal partition S.

Theorem 13 will be used in conjunction with the following result, implied by a special case of Lemma 28 in [2]:



Figure 4: (a) A 3-hypergraph H with a vertex u suitable for a switch. (b) The hyperedges incident with u in the 3-hypergraph resulting from the switch. The other hyperedges are not modified.

**Proposition 14.** Let H be a 3-hypergraph and  $b_1, b_2$  vertices of H. If H admits an acyclic quasigraph that is both connected and anticonnected on V(H), then the graph G(H) contains a  $b_1b_2$ -trail spanning V(H).

Proposition 14 will be useful in Section 8, where we infer that the assumption of the proposition is satisfied, which will enable us to apply Lemma 12.

# 5 Counting the hyperedges

Recall that G is a graph satisfying the assumptions of Theorem 2, and that  $H^0$  is a 3-hypergraph associated with the core  $G^0$  of G. Additionally,  $e_1, e_2$  are fixed edges of G,  $a_1, a_2$  are their permanent endvertices, and  $H^e$  is a modification of  $H^0$  defined in Section 3.

By Theorem 13, there is a 3-hypergraph  $\tilde{H}$  related to  $H^e$  and an acyclic quasigraph  $\sigma$  in  $\tilde{H}$  with no bad leaves such that  $V(\tilde{H})$  admits a  $\sigma$ -skeletal partition S. Our ultimate use of  $\sigma$  is to find a connected  $X(e_1, e_2)$ -join in the graph  $G(H^e)$  to be used in Lemma 12. Since  $G(\tilde{H})$  is isomorphic to  $G(H^e)$  if  $\tilde{H}$  and  $H^e$  are related, we may assume without loss of generality that  $\tilde{H} = H^e$ .

As we will see in Section 8, the proof of Theorem 2 is simple in the case that  $|\mathcal{S}| = 1$ . In the following calculations, we therefore assume  $\mathcal{S} \geq 2$ .

Recall that for a 3-hypergraph H, an edge e of H and a partition  $\mathcal{P}$  of V(H), the notation  $e/\mathcal{P}$  and  $H/\mathcal{P}$  has been defined in Section 4.

Let us write  $d^0(P)$  for the degree of a vertex  $P \in S$  in the hypergraph  $H^0/S$ . We proceed as in Section 8 of [2]. We set  $\tau = \sigma/S$ . Let n = |S| and let m denote the number of hyperedges of  $H^e/S$ . For  $k \in \{2,3\}$ , let  $m_k$  be the number of k-hyperedges of  $\overline{\tau}$ . Since S is  $\sigma$ -solid, the graph  $\tau^*$  is acyclic. It has n vertices and  $m_2 + m_3$  edges, and hence

$$m_2 + m_3 \le n - 1. \tag{1}$$

Similarly, the complement  $\overline{\tau}$  is an acyclic hypergraph. Consider the graph  $G(\overline{\tau})$ , defined in Section 1. Since  $\overline{\tau}$  is acyclic, so is  $G(\overline{\tau})$ . As  $G(\overline{\tau})$  has  $n + \overline{m}_3$  vertices

and  $\overline{m}_2 + 3\overline{m}_3$  edges, we get

$$\overline{m}_2 + 2\overline{m}_3 \le n - 1. \tag{2}$$

Moreover, by the assumption, either  $\tau^*$  or  $G(\overline{\tau})$  is disconnected and therefore it has at most n-2 edges. Adding (1) to (2) and using this fact, we find

$$m \le 2n - 3 - \overline{m}_3. \tag{3}$$

For any hypergraph H, let s(H) be the sum of vertex degrees in H. Let us set

$$\varepsilon = s(H^0/S) - s(H^e/S)$$

and observe that by the definition of the hypergraph  $H^e$ ,  $\varepsilon \leq 4$ . Furthermore, we have

$$s(H^e/S) = 2(m_2 + \overline{m}_2) + 3(m_3 + \overline{m}_3) = 2m + (m_3 + \overline{m}_3)$$
  
  $\leq 4n - 6 + m_3 - \overline{m}_3,$ 

where the last inequality follows by using (3) to substitute for m. Substituting  $s(H^0/\mathbb{S}) - \varepsilon$  for  $s(H^e/\mathbb{S})$ , we obtain

$$s(H^0/S) \le 4n - 6 + (m_3 - \overline{m}_3) + \varepsilon. \tag{4}$$

The quasigraph  $\tau$  in  $H^e/\mathbb{S}$  determines an (acyclic) quasigraph  $\tau^0$  in  $H^0/\mathbb{S}$  in a natural way. Let  $m_3^0$  be the number of 3-hyperedges of  $H^0/\mathbb{S}$  used by  $\tau^0$  and observe that  $m_3^0 \geq m_3$ . Furthermore, let  $\widetilde{m}_i^0$   $(i \in \{2,3\})$  be the number of *i*-hyperedges of  $H^0/\mathbb{S}$ , whether used by  $\tau^0$  or not.

Inequality (4) has two corollaries. Firstly, using the fact that  $\varepsilon \leq 4$  and  $m_3^0 \geq m_3$ , and ignoring the  $\overline{m}_3$  term, we find

$$\sum_{P \in V(H^0/\$)} (d^0(P) - 4) - m_3^0 \le -2.$$
 (5)

For the second corollary, note that since  $s(H^0/S) = 2\widetilde{m}_2^0 + 3\widetilde{m}_3^0$  and  $m_3 - \overline{m}_3 \le \widetilde{m}_3^0$ , (4) implies

$$\widetilde{m}_2^0 + \widetilde{m}_3^0 \le 2n - 3 + \frac{\varepsilon}{2}.\tag{6}$$

For later use, we record an observation concerning classes X of S such that  $|X^+| \ge 1$ ; let us call such classes nontrivial.

**Observation 15.** If X is a nontrivial class of S, then  $G^0[X^+]$  is not a matching.

*Proof.* Suppose that  $X \in S$  is nontrivial. We prove that  $G^0[X^+]$  has at least two incident edges. If |X| > 1, then this follows from the fact that  $\sigma$  is both connected and anticonnected on X by the choice of S. On the other hand, if |X| = 1, then some vertex of  $X^+ - X$  is incident with at least two edges to X and the assertion also holds.

#### 6 Structural observations

We continue to use the notation and assumptions of Section 5. The objective of this and the following section is to rule out most cases in the proof of Theorem 2 by establishing the following:

**Proposition 16.** If  $G^0$  has at least 6 vertices, then the partition S has at most 4 classes.

For the sake of a contradiction, let us assume that  $|S| \geq 5$ . We now prove several claims concerning the structure of the hypergraph  $H^0/S$  in this case.

**Lemma 17.** Suppose that  $G^0$  has at least 6 vertices. Then the following hold:

- (i)  $G^0$  contains no path of length 2 with two vertices of degree 3 and one vertex of degree at most 4,
- (ii) no permanent vertex of degree 3 in  $G^0$  is adjacent to a permanent vertex of degree at most 4.
- *Proof.* (i) We prove that if  $x_1x_2x_3$  is a path in  $G^0$ , then  $\sum_{i=1}^3 d_{G^0}(x_i) \geq 11$ . Suppose the contrary. Define  $X = \{x_1, x_2, x_3\}$ . Then  $d_{G^0}(X) \leq 6$ . Since  $G^0 X$  must be a matching on at least 3 vertices, and  $G^0$  has minimum degree at least 3, we have  $d_{G^0}(X) \geq 7$ . This is a contradiction.
- (ii) Let x, y be permanent vertices of  $G^0$  such that  $d_{G^0}(x) = 3$  and  $d_{G^0}(y) \le 4$ . By part (i), the vertex x has no temporary neighbour. Therefore, x must have a transient neighbour in G. Since G is 2-essentially 9-edge-connected,  $G^0 \{x, y\}$  must be a matching and we obtain a similar contradiction as in the proof of (i).

**Lemma 18.** Let P, Q be neighbouring vertices of the hypergraph  $H^0/S$ . If  $|S| \ge 5$  and  $G^0$  has at least 6 vertices, then the following holds:

- (i) if  $d_{H^0/S}(P) = 3$ , then  $d_{H^0/S}(Q) \ge 7$ ,
- (ii) if  $d_{H^0/\mathbb{S}}(P)=4$  and P is incident with a 3-hyperedge of  $H^0/\mathbb{S}$ , then  $d_{H^0/\mathbb{S}}(Q)\geq 6$ .

*Proof.* We prove (i). Since  $G^0$  is essentially 4-edge-connected, the class P is trivial by Observation 15; say,  $P = \{u\}$ . Being a permanent vertex of degree 3 in  $G^0$ , the vertex u is either protected, or adjacent to a temporary vertex.

We set

$$X = \begin{cases} (P \cup Q)^+ \cup \{z\} & \text{if } u \text{ is adjacent to a temporary vertex } z, \\ (P \cup Q)^+ & \text{otherwise.} \end{cases}$$

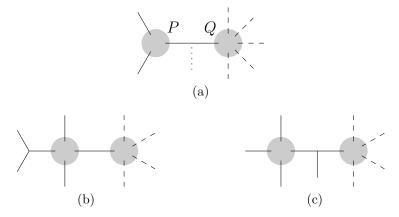


Figure 5: Forbidden configurations in  $H^0/S$ . Figure (a) illustrates part (i) of Lemma 18, (b) and (c) relate to part (ii). The gray regions represent the classes P and Q of S, dashed lines represent optional hyperedges. The dotted line in figure (a) means that the hyperedge can be of size 2 or 3.

Suppose that u is adjacent to a temporary vertex z. By Lemma 17(i), u is not adjacent to any other temporary vertex, which implies

$$d_{G^0}(X) \le 8. \tag{7}$$

Since  $G^0[X]$  is not a matching,  $G^0-X$  must be a matching as  $G^0$  is 2-essentially 9-edge-connected.

A similar argument shows that  $G^0 - X$  is a matching just as well if u is protected. In particular, in either case, no temporary vertex of  $G^0$  has two neighbours outside X.

Enumerate the classes of S other than P and Q as  $Y_1, \ldots, Y_k$ . By Observation 15, each  $Y_i$  is a trivial class, say  $Y_i = \{y_i\}$ .

Since  $k \geq 3$  and  $d_{G^0}(y_i) \geq 3$  for  $1 \leq i \leq k$ , inequality (7) implies that  $G^0$  contains an edge joining two of the vertices  $y_i$  — say,  $y_1$  and  $y_2$ . Since  $d_{G^0}(\{y_2,\ldots,y_k\}) \geq 3$ , we have  $d_{G^0}(\{y_1,y_2\}) \leq 5$ , so without loss of generality,  $d_{G^0}(y_1) = 3$  and  $d_{G^0}(y_2) \leq 4$ . This is a contradiction with Lemma 17(ii).

Part (ii) can be proved using a minor modification of the above argument, which we leave to the reader.

## 7 Discharging

We continue the discussion of Section 6 by using a discharging-type argument to prove Proposition 16. The discharging process takes place in the hypergraph  $H^0/S$ . Recall our hypothesis that the partition S has at least 5 classes.

In Section 5, we defined  $\tau^0$  as the quasigraph in  $H^0/S$  corresponding to the quasigraph  $\tau = \sigma/S$  in  $H^e/S$ . The notion of a hyperedge associated with a vertex

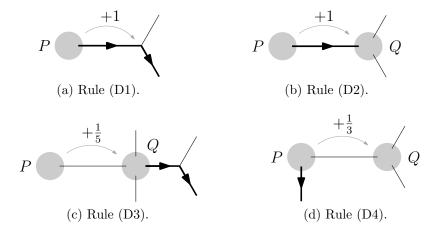


Figure 6: The rules for the redistribution of charge. Thick lines represent the quasigraph  $\tau^0$ , with the association of a hyperedge to a vertex shown by thick arrows. Gray arrows indicate the flow of the stated amount of charge. The degree of P is not represented.

will be carried over from  $H^e$  to  $H^0/S$ : by definition, a hyperedge e/S of  $H^0/S$  is associated with  $P \in S$  if the corresponding hyperedge of  $H^e$  is associated with a vertex of  $H^e$  contained in P. Note that the definition makes sense thanks to the fact that  $\sigma[P]$  is a quasitree (and hence  $\sigma[P]^*$  is connected).

fact that  $\sigma[P]$  is a quasitree (and hence  $\sigma[P]^*$  is connected). We write  $V_i = V_i(H^0/\mathbb{S})$ . Furthermore,  $V_i^{\triangle}$  denotes the subset of  $V_i$  consisting of vertices which have an associated 3-hyperedge and  $V^{\triangle}$  is the union of all  $V_i^{\triangle}$ . Given a vertex P of  $H^0/\mathbb{S}$ , the symbol  $N_*(P)$  denotes the multiset consisting of vertices Q such that P and Q are contained in a hyperedge of  $H^0/\mathbb{S}$ , with one occurrence of Q for each such hyperedge.

We begin by assigning charges to the vertices and hyperedges of  $H^0/S$ , guided by inequality (5):

- each vertex P will get a charge of  $d^0(P) 4$  units,
- a 3-hyperedge of  $H^0/\mathbb{S}$  will get a charge of -1 if it is used by  $\tau^0$  and a zero charge otherwise,
- 2-hyperedges of  $H^0/\mathbb{S}$  get zero charge.

By (5), the total charge is negative. At the same time, the only elements of  $H^0/S$  with negative charge are 3-vertices and 3-hyperedges used by  $\tau^0$ .

As usual in discharging arguments, we will describe rules for the redistribution of charge which keep the total charge unchanged and (given the assumptions about the graph G) make all the individual charges non-negative. This contradiction will show that  $\sigma$  is actually a quasitree with connected complement.

Charge will only be sent by vertices, the recipient may be either a vertex or a hyperedge. Let P be a vertex of  $H^0/S$  (that is,  $P \in S$ ). The rules (more than

one of which may apply) are in the following list. See Figure 6 for a schematic representation.

- (D1) if  $P \in V^{\triangle}$ , then P sends its associated hyperedge 1 unit of charge,
- (D2) if P has an associated 2-hyperedge whose head is a degree 3 vertex Q, then P sends 1 unit of charge to Q,
- (D3) if P has a neighbour Q in  $V_4^{\triangle}$ , then P sends Q a charge of 1/5 for each common hyperedge,
- (D4) if P has a degree 3 neighbour Q, then P sends Q a charge of 1/3 for each common hyperedge which is not associated with P.

We claim that after the redistribution of charge, all vertices and hyperedges of  $H^0/S$  will have nonnegative charge. This is clear for 2-hyperedges and for 3-hyperedges not used by  $\tau^0$ . Furthermore, each 3-hyperedge used by  $\tau^0$  will obtain 1 unit of charge by rule (D1), making the resulting charge zero.

Let us therefore investigate the ways a vertex P may be discharged. Suppose first that the degree  $d^0(P)$  of P is at least 7. Since at most one unit of charge is transferred from P based on rules (D1) and (D2), the total transfer from P is at most

$$1 + (d^{0}(P) - 1) \cdot \frac{1}{3} \le d^{0}(P) - 4,$$

where the inequality follows from the assumption that  $d^0(P) \geq 7$ . Since the right hand side is the original charge of the vertex, the resulting charge is nonnegative.

We may thus assume that  $d^0(P) \leq 6$ . By Lemma 18(i), no neighbour P' of P in  $H^0/\mathbb{S}$  has  $d^0(P') = 3$ . Suppose that  $d^0(P) = 6$ . If P is discharged according to rule (D1) or (D2), then rule (D4) does not apply to P, and the transfer from P is at most  $1 + 5 \cdot 1/5 = 2$ , the initial charge of P. On the other hand, if none of (D1) and (D2) apply, then P sends at most  $6 \cdot 1/3 = 2$  units of charge as well.

If  $d^0(v) = 5$ , then P has no neighbour in  $V_4^{\triangle}$  (Lemma 18(ii)), which rules out the use of (D3) for the discharging of P. Furthermore, the applicability of (D1), (D2) and (D4) is mutually exclusive. This means that P only sends at most a charge of 1 unit, which equals its initial charge.

We are left with the case that  $d^0(P) \leq 4$ . Suppose that  $d^0(P) = 4$  (so its initial charge is zero). Lemma 18 implies that no neighbour of P in  $H^0/\mathbb{S}$  is contained in  $V_3 \cup V_4^{\triangle}$ . Thus, if P sends any charge at all, it must be according to rule (D1). In this case,  $P \in V_4^{\triangle}$  and according to rule (D3), P receives a charge of 1/5 from each of the five vertices in  $N_*(P)$ , so its resulting charge is  $-1 + 5 \cdot 1/5 = 0$ .

It remains to consider the case that  $d^0(P) = 3$ . By Lemma 5(ii), P contains a single vertex v of  $H^0$ . By the property (Q2) of  $\sigma$ , v is not a leaf of  $\sigma^*$  in  $H^e$ , and therefore P is not a leaf of  $(\tau^0)^*$  in  $H^0/S$ . Let e be an edge of  $(\tau^0)^*$  with P as its head, and let the tail of e be denoted by t.

Similarly to the discussion in the preceding cases, P is not discharged according to any of the rules (D2), (D3) or (D4). We distinguish two cases.

If  $P \notin V_3^{\triangle}$ , then it does not send any charge to its neighbours or incident hyperedges. By Lemma 18, at most one of the four vertices in  $N_*(P)$  has an associated 3-hyperedge f such that the head of  $\tau^0(f)$  is P. Each of the remaining vertices sends either 1/3 or 1 unit of charge to P (according to rule (D4) or (D2), respectively), which accounts for a resulting charge of at least  $-1 + 3 \cdot 1/3 = 0$ .

If  $P \in V_3^{\triangle}$ , then P is discharged according to (D1), which decreases its charge from -1 to -2. On the other hand, Lemma 18 implies that e is a 2-hyperedge, so t sends one unit of charge to P according to rule (D2). Furthermore, as above, P gets at least a charge of 1/3 from each of the three remaining vertices in  $N_*(P)$ . Hence, the new charge is nonnegative again. This concludes the analysis. The contradiction establishes Proposition 16.

# 8 Completing the proof of Theorem 2

In this section, we prove Theorem 2. Before doing so, we narrow down the set of possible cases by proving that the size of the partition S is not greater than 2.

Recall from Section 5 the notation  $\widetilde{m}_i^0$  for  $|E_i(H^0/\mathbb{S})|$  (where  $i \in \{2,3\}$ ) and inequality (6):

$$\widetilde{m}_2^0 + \widetilde{m}_3^0 \le 2n - 3 + \varepsilon/2,$$

where  $n = |\mathcal{S}|$  and  $\varepsilon$  is the difference of the sum of vertex degrees in  $H^0/\mathcal{S}$  and in  $H^e/\mathcal{S}$ . Since  $\varepsilon \leq 4$  and, by Proposition 16,  $n \leq 4$ , we have

$$H^0/S$$
 has at most 7 hyperedges. (8)

Before stating the next proposition, we recall that nontrivial class of S is a class X such that  $|X^+| \ge 1$ .

**Proposition 19.** If  $G^0$  has at least 6 vertices, then  $n \leq 2$ . Moreover, if n = 2, then the following hold:

- (i) one of the classes of S is a trivial class  $\{x\}$ ,
- (ii) the degree of x in  $H^0$  is 3,
- (iii) x is incident with the hyperedges  $k(e_1)$  and  $k(e_2)$  of  $H^0$ ,
- (iv) the size of  $k(e_1)$  and  $k(e_2)$  is 2.

*Proof.* The proof consists of a series of claims.

Claim 1. The partition S has at most one nontrivial class.

By Observation 15, if S has two nontrivial classes X, Y, (8) implies that  $\partial_{G^0}(X^+)$  is a 2-essential edge-cut in  $G^0$  of size at most 7.

#### Claim 2. If S has a nontrivial class, then $H^0/S$ contains no 3-hyperedge.

Let  $X \in \mathcal{S}$  be nontrivial and let  $e/\mathcal{S}$  be a 3-hyperedge of  $H^0/\mathcal{S}$  (where e is a hyperedge of  $H^0$ ). We can choose two vertices of  $e/\mathcal{S}$  (say  $Y_1, Y_2$ ) distinct from X. For i=1,2, let  $y_i$  be the vertex of e in  $Y_i$ , and let w be the vertex of  $G^0$  such that e=h(w). Then the edge-cut  $\partial_{G_0}(X^+)$  of size at most 7 separates  $X^+$  from the path  $y_1wy_2$  in  $G^0$  and is therefore 2-essential, a contradiction.

#### Claim 3. The partition S has exactly one nontrivial class.

For contradiction, suppose that all the classes of S are trivial. Recall the parameter  $\widetilde{m}_3^0$ , introduced above equation (5). Since we assume that  $G^0$  has at least 6 vertices, we must have  $\widetilde{m}_3^0 \geq 2$ . We let the vertices of  $H^0$  be denoted by x, y, z or x, y, z, u depending on whether n equals 3 or 4. Furthermore, the 3-hyperedges of  $H^0$  are denoted by  $h(w_i)$ , where  $1 \leq i \leq \widetilde{m}_3^0$  and all the  $w_i$ 's are temporary vertices of  $G^0$ .

If  $H^0$  contains two 3-hyperedges intersecting in exactly two vertices (say, e = xyz and f = xyu), then the edge-cut  $\partial_{G^0}(\{x,z\}^+)$  is a 2-essential edge-cut of size at most 7, a contradiction. Thus, all the 3-hyperedges of  $H^0$  contain the same triple of vertices, say  $\{x,y,z\}$ .

Suppose now that  $\widetilde{m}_3^0 \geq 3$ . By (8), we may assume that x is incident with only at most three 2-hyperedges of  $H^0$ . But then the size of the 2-essential edge-cut  $\partial_{G^0}(\{x, w_1, w_2\}^+)$  in  $G^0$  is at most 8, a contradiction.

We conclude that  $\widetilde{m}_3^0 = 2$ , which implies n = 4. If uz is a 2-hyperedge in  $H^0$ , then the edge-cut  $\partial_{G^0}(\{x, w_1, y\}^+)$  in  $G^0$  separates the paths  $xw_1y$  and  $uzw_2$  and is therefore 2-essential. In addition, its size is at most 8. This contradiction concludes the proof of the claim.

#### Claim 4. The partition S has at most two classes.

Suppose that n > 2. By the above claims, S has a single nontrivial class P and all the hyperedges of  $H^0$  are of size 2. Let the vertices of  $H^0$  comprising the nontrivial parts of S be denoted by x, y or x, y, z depending on n.

Suppose first that n=4. Since the degree of each of the vertices x,y,z in  $H^0$  is at least 3, (8) implies that at least two 2-hyperedges have both endvertices in the set  $\{x,y,z\}$ . Consequently, the edge-cut  $\partial_{G^0}(P^+)$  of size at most 7 is 2-essential

We infer that n=3. By a similar argument, the vertices x and y must be adjacent vertices of degree 3 in  $G^0$ . Since they are permanent and there is no temporary vertex, they must be protected. However, if x is adjacent to a transient vertex, then it is easy to show that  $\partial_{G^0}(P^+)$  is a 2-essential 4-edge-cut in  $G^0$ .

To finish the proof of the proposition, it remains to establish properties (ii)—(iv). Let us write  $S = \{P, \{x\}\}$ , where P is the nontrivial class. By inequality (6) and the fact that  $G^0$  is 3-edge-connected, x has degree 3 in  $H^0$  and  $\varepsilon = 4$ . The latter fact means that  $k(e_1)$  and  $k(e_2)$  are 2-hyperedges of  $H^0$  incident with x. The proof is complete.

Having established Proposition 19, we can now prove the main result of this paper.

Proof of Theorem 2. If the graph  $G^0$  has at most 5 vertices, then L(G) is Hamilton-connected by Lemma 10. Assume thus that  $|V(G^0)| \ge 6$ . By Proposition 19, we have n < 2.

If n = 1, then by the choice of S,  $\sigma$  is an acyclic quasigraph in  $H^e$  that is both connected and anticonnected on  $V(H^e)$ . Proposition 14 implies that  $G(H^e)$  admits an  $a_1a_2$ -trail spanning  $V(H^e)$ . By Lemma 12, it follows that G admits an internally dominating  $(e_1, e_2)$ -trail. Since the choice of  $e_1$  and  $e_2$  is arbitrary, L(G) is Hamilton-connected by Theorem 11.

The discussion in the case that n=2 is only slightly more complicated. By Proposition 19, S contains a trivial class  $\{x\}$ , x has degree 3 in  $H^0$  and is incident in  $H^0$  with the 2-hyperedges  $k(e_1)$  and  $k(e_2)$ . Let P denote the other class of S and let  $y_1$  and  $y_2$  be the endvertex of  $k(e_1)$  and  $k(e_2)$ , respectively, in P. Furthermore, let f be the third 2-hyperedge of  $H^0$  incident with x, and let  $y_3$  be its endvertex in P.

By the definition of the temporary vertices  $a_1$  and  $a_2$ , we may assume without loss of generality that  $a_1 = x$  and  $a_2 = y_2$ . To find an  $xy_2$ -trail in  $G(H^e)$  spanning  $V(H^e)$ , we proceed as follows.

Denoting the hypergraph obtained from  $H^e$  by removing x by  $H_1$ , we observe that  $\sigma$  determines an acyclic quasigraph in  $H_1$  that is both connected and anticonnected on  $V(H_1) = P$ . By Proposition 14,  $G(H_1)$  admits an  $y_3y_2$ -trail  $T_1$  spanning P. Adding the edge f to the beginning of  $T_1$ , we obtain an  $xy_2$ -trail in  $G(H^e)$  spanning  $V(H^e)$  as desired.

## 9 Claw-free graphs

As with Theorem 1, Theorem 2 can be extended to claw-free graphs. The procedure is the same as that used in [2, Section 11]. We use the M-closure introduced in [9], namely the following result [9, Theorem 9]:

**Theorem 20.** If G is a connected claw-free graph, then there is a well-defined graph  $cl^{M}(G)$  with the following properties:

- (i) G is a spanning subgraph of  $cl^{M}(G)$ ,
- (ii)  $cl^{M}(G)$  is the line graph of a multigraph,

(iii)  $cl^{M}(G)$  is Hamilton-connected if and only if G is Hamilton-connected.

In the context of the present paper, the reference to multigraphs in Theorem 20 is not necessary, since parallel edges in graphs are allowed by default.

By condition (i) in Theorem 20, the connectivity of  $cl^M(G)$  is greater than or equal to that of G. As shown by the following observation, the closure operation does not decrease the essential connectivity either.

**Lemma 21.** If G is essentially k-connected, then  $cl^M(G)$  is also essentially k-connected.

*Proof.* Clearly,  $|V(cl^M(G))| > k$ . For contradiction, let X be a minimal essential vertex-cut in  $cl^M(G)$  of size less than k. Since X is not essential in G, there is a component K of  $cl^M(G) - X$  such that K contains some edges, but G[V(K)] is edgeless.

The operation  $cl^M$ , as defined in [9, Section 4], consists of a sequence of local completions at suitable vertices  $x_1, \ldots, x_\ell$ . Here, the local completion at  $x_i$  is the addition of all possible edges joining the neighbours of  $x_i$ . Let e be an edge of K and let  $Y = X \cap \{x_1, \ldots, x_\ell\}$ . In G, all the vertices with a neighbour in V(K) are contained in X. The fact that K contains at least one edge implies that  $Y \neq \emptyset$ . Let us say that  $x_i \in Y$ , where  $1 \leq i \leq \ell$ . By the minimality of X,  $x_i$  has a neighbour (in  $cl^M(G)$ ) in some component L of  $cl^M(G) - X$  other than K. Although the edge between them could be added by a local completion at some vertex  $x_j$  ( $1 \leq j \leq \ell$ ), this can only happen if some vertex of Y has a neighbour in V(L) prior to the local completion. We conclude that some vertex of Y, say  $x_k$ , has a neighbour in V(L) in G. But then the local completion at  $x_k$  adds an edge between a neighbour of  $x_k$  in V(K) and a neighbour of  $x_k$  in V(L), contradicting the assumption that K and L are different components of  $cl^M(G) - X$ .

Using Lemma 21, we find that  $cl^M(G)$  is a 3-connected, essentially 9-connected line graph. Thus,  $cl^M(G)$  is Hamilton-connected by Theorem 2. Condition (iii) of Theorem 20 implies that G is Hamilton-connected.

### 10 Conclusion

We have shown that every 3-connected, essentially 9-connected claw-free graph is Hamilton-connected, and that this assertion is false with 9 replaced by 4. The obvious question is left unresolved: what is the least value of k such that 3-connected, essentially k-connected claw-free graphs are Hamilton-connected (or hamiltonian)? This remains an interesting problem for further investigation.

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