# Balanced supersaturation for some degenerate hypergraphs 

Jan Corsten* Tuan Tran ${ }^{\dagger}$

A classical theorem of Simonovits from the 1980s asserts that every graph $G$ satisfying $e(G) \gg v(G)^{1+1 / k}$ must contain $\gtrsim\left(\frac{e(G)}{v(G)}\right)^{2 k}$ copies of $C_{2 k}$. Recently, Morris and Saxton established a balanced version of Simonovits' theorem, showing that such $G$ has $\gtrsim\left(\frac{e(G)}{v(G)}\right)^{2 k}$ copies of $C_{2 k}$, which are 'uniformly distributed' over the edges of $G$. Moreover, they used this result to obtain a sharp bound on the number of $C_{2 k}$-free graphs via the method of hypergraph containers. In this paper, we generalise Morris-Saxton's results for even cycles to $\Theta$-graphs. We also prove analogous results for complete $r$-partite $r$-graphs.
Keywords: Erdős-Simonovits conjecture, balanced supersaturation, theta graph, complete $r$-partite $r$-graph, hypergraph containers.

## 1. Introduction

### 1.1. Supersaturation theorems

The Turán number $\operatorname{ex}_{r}(n, H)$ of an $r$-graph $H$ is the maximum number of edges in an $n$-vertex $r$-graph which does not contain a copy of $H$. The Erdős-Stone-Simonovits theorem [15, 12] asserts that

$$
\operatorname{ex}_{2}(n, H)=\left(1-\frac{1}{\chi(H)-1}\right)\binom{n}{2}+o\left(n^{2}\right)
$$

for every graph $H$ and therefore asymptotically determines the Turán number of every nonbipartite graph $H$. For bipartite graphs, finding the Turán number is usually very challenging and even their order of magnitude is unknown for most of them. Erdős [10] further proved that $\mathrm{ex}_{r}(n, H)=o\left(n^{r}\right)$ if and only if $H$ is an $r$-partite $r$-graph. Similarly as for graphs, not much is known for the Turán number of $r$-partite $r$-graphs. It is natural to ask now how many copies of $H$ a graph on $n$ vertices with more than $\operatorname{ex}_{2}(n, H)$ edges must contain. Erdős and Simonovits [13] observed that for non-r-partite $r$-graphs a simple double-counting argument shows that once we pass the extremal number, we can already find a constant fraction of all

[^0]copies of $H$ in the complete graph. This fails in general for $r$-partite $r$-graphs with $r \geq 3$ (see [27]), but Erdős and Simonovits [14] conjectured that in the graph case (i.e. $r=2$ ) one can always find a constant fraction of the expected number of copies of $H$ in the random graph with the same number of edges.

Conjecture 1.1. For every bipartite graph $H$ with $v$ vertices and e edges, there is some $C>0$ so that every graph $G$ with $n$ vertices and $m \geq C \cdot \operatorname{ex}_{2}(n, H)$ edges contains $\Omega\left(m^{e} n^{v-2 e}\right)$ copies of $H$.

So far this conjecture has only been verified for very few graphs. In an unpublished manuscript, Simonovits proved the conjecture for even cycles $C_{2 \ell}$ provided that ex ${ }_{2}\left(n, C_{2 \ell}\right)=$ $\Theta\left(n^{1+1 / \ell}\right)$, which is known to be true only for $\ell \in\{2,3,5\}$ (see [21, 7, 28, 8, 30]). Recently, two extensions of this theorem were obtained. One by Morris and Saxton [23] who proved a balanced version of Simonovits' theorem, which (roughly speaking) additionally guarantees the copies of $C_{2 \ell}$ to be uniformly distributed in the graph. Another one by Jiang and Yepremyan [18] who extended Simonovits' theorem to linear cycles in hypergraphs. Erdős and Simonovits further proved Conjecture 1.1 for all complete bipartite graphs $K_{s, t}$ with $s \leq t$ and $\operatorname{ex}_{2}\left(n, K_{s, t}\right)=\Theta\left(n^{2-1 / s}\right)$, which is known to be true if $t$ is large enough in terms of $s$ and conjectured to be true for all $t \geq s$ (see [20, 1, 21]). Morris and Saxton obtained a balanced strengthening of this result as well [23].

In this paper we shall extend the results of Morris and Saxton to theta graphs ( $\theta_{a, b}$ is the graph consisting of $a$ internally vertex-disjoint paths of length $b$, each with the same endpoints) and complete $r$-partite $r$-graphs. The following two supersaturation results are trivial consequences of our main results (see Section 1.3 below).

Theorem 1.1. For all $a, b \geq 2$, there is some $C>0$ so that every graph $G$ with $n$ vertices and $m \geq C \cdot n^{1+1 / b}$ edges contains $\Omega\left(m^{a b} n^{2-a(b+1)}\right)$ copies of $\theta_{a, b}$.

Theorem 1.2. For all $2 \leq a_{1} \leq \ldots \leq a_{r}$, there is some $C>0$ so that every r-graph $G$ with $n$ vertices and $m \geq C \cdot n^{r-1 / a_{1} \cdots a_{r-1}}$ edges contains $\Omega\left(m^{a_{1} \cdots a_{r}} n^{a_{1}+\ldots+a_{r}-r \cdot a_{1} \cdots a_{r}}\right)$ copies of $K_{a_{1}, \ldots, a_{r}}^{(r)}$.

Note that $\operatorname{ex}_{2}\left(n, \theta_{a, b}\right)=\Theta\left(n^{1+1 / b}\right)$ if $a$ is sufficiently large with respect to $b$ (see [16, 9]), and that $\operatorname{ex}_{r}\left(n, K_{a_{1}, \ldots, a_{r}}^{(r)}\right)=\Theta\left(n^{r-1 / a_{1} \cdots a_{r-1}}\right)$ if $a_{r}$ is sufficiently large with respect to $a_{1}, \ldots, a_{r-1}$ (c.f. [10, 22]). Hence we confirm Conjecture 1.1 for 'most' theta graphs and 'most' complete $r$-partite $r$-graphs.

### 1.2. Counting $H$-free subgraphs

It is a central problem in extremal graph theory to determine the number, $F_{r}(n, H)$, of $H$-free $r$-graphs on $n$ vertices for a given fixed $r$-graph $H$ and a natural number $n$. We trivially have

$$
\begin{equation*}
2^{\operatorname{ex}_{r}(n, H)} \leq F_{r}(n, H) \leq \sum_{i \leq \operatorname{ex}_{r}(n, H)}\binom{\binom{n}{r}}{i}=n^{O\left(\mathrm{ex}_{r}(n, H)\right)} . \tag{1.1}
\end{equation*}
$$

and all existing results in the area seem to indicate that the lower bound in (1.1) is closer to the truth. The problem of estimating $F_{r}(n, H)$ is essentially solved for every non-r-partite $r$-graph $H$. Indeed, in the graph case, Erdős, Frankl and Rödl [11] showed

$$
\begin{equation*}
F_{2}(n, H)=2^{(1+o(1)) \mathrm{ex}_{2}(n, H)}, \tag{1.2}
\end{equation*}
$$

using Szemerédi's regularity lemma. The corresponding result for $r$-graphs was proved by Nagle, Rödl and Schacht [25] via the hypergraph regularity lemma.

For $r$-partite $r$-graphs on the other hand, the problem seems to be more challenging and much less is known. Morris and Saxton [23] showed that (1.2) does not hold for $C_{6}$. Even the weaker bound $F_{r}(n, H)=2^{O\left(\mathrm{ex}_{r}(n, H)\right)}$ ( a conjecture usually attributed to Erdős) has been proven in only a few special cases: for most complete bipartite graphs (see [5, 6]), for cycles of length $\ell \in\{4,6,10\}$ (see $[19,23]$ ), and for $r$-uniform linear cycles (see [24, 4]). In this paper, we confirm the weaker conjecture for most theta graphs and most complete $r$-partite $r$-graphs. More precisely we prove the following results.
Theorem 1.3. For every $a, b \geq 2$, there are at most $2^{O\left(n^{1+1 / b}\right)} \theta_{a, b}$-free graphs on $n$ vertices and at most $2^{o\left(n^{1+1 / b}\right)}$ of them have $o\left(n^{1+1 / b}\right)$ edges.

Theorem 1.4. For all $2 \leq a_{1} \leq \ldots \leq a_{r}$, there are at most $2^{O\left(n^{r-1 /\left(a_{1} \cdots a_{r-1}\right)}\right)} K_{a_{1}, \ldots, a_{r}-\text { free }}^{(r)}$ $r$-graphs on $n$ vertices and at most $2^{o\left(n^{r-1 /\left(a_{1} \cdots a_{r-1}\right)}\right)}$ of them have o $\left(n^{r-1 /\left(a_{1} \cdots a_{r-1}\right)}\right)$ edges.

In particular it follows for those $r$-graphs $H$ that there is a positive constant $c=c(H)$ such that asymptotically almost every $H$-free $r$-graph has at least $c \cdot \operatorname{ex}_{r}(n, H)$ edges. This confirms a special case of a conjecture of Balogh, Bollobás and Simonovits [2] which states that this is true for all bipartite graphs $H$ containing a cycle.

### 1.3. Balanced supersaturation theorems

The hypergraph container method, developed independently by Balogh, Morris and Samotij [3], and Saxton and Thomason [26], is one of the most successful recent developments in extremal combinatorics. In order to apply the method, we have to find a family of copies of $H$ in $G$ that are 'evenly distributed' in the following sense.

Definition 1.5 ([23, Definition 5.5]). Let $\alpha>0$. An $r$-graph $H$ is called Erdös-Simonovits $\alpha$ good for a function $m=m(n)$ if there exist positive constants $C$ and $k_{0}$ such that the following holds. Let $k \geq k_{0}$, and suppose that $G$ is an $r$-graph with $n$ vertices and $k \cdot m(n)$ edges. Then there exists a non-empty collection $\mathcal{H}$ of copies of $H$ in $G$, satisfying

$$
d_{\mathcal{H}}(\sigma) \leq \frac{C \cdot|\mathcal{H}|}{k^{(1+\alpha)(|\sigma|-1)} e(G)} \quad \text { for every } \sigma \subset E(G) \text { with } 1 \leq|\sigma| \leq e(H)
$$

where $d_{\mathcal{H}}(\sigma):=\left|\left\{H^{\prime} \in \mathcal{H}: \sigma \subset H^{\prime}\right\}\right|$ denotes the degree of $\sigma$ in $\mathcal{H}$.
Morris and Saxton [23] conjectured that every bipartite graph $H$ is Erdős-Simonovits $\alpha$ good for $m(n)=\operatorname{ex}_{2}(n, H)$ and some $\alpha=\alpha(H)>0$ (the same statement is trivially true for non-bipartite graphs). Furthermore, they expect that the family $\mathcal{H}$ can be chosen so that it contains (up to a multiplicative factor) as many copies of $H$ as the random graph $G(n, m)$ with $m=k \cdot \operatorname{ex}_{2}(n, H)$, which leads to a stronger form of Conjecture 1.1. Their motivation in making Definition 1.5 is the following proposition.

Proposition 1.6 ([23, Proposition 5.6]). Let $H$ be an $r$-graph and let $\alpha>0$. If $H$ is ErdösSimonovits $\alpha$-good for $m(n)$, then the following hold.
(1) There are at most $2^{O(m(n))} H$-free $r$-graphs on $n$ vertices,
(2) The number of $H$-free graphs with $n$ vertices and $o(m(n))$ edges is $2^{o(m(n))}$.

A proof-sketch for a similar result was given in [23]. For completeness, we provide a full proof of Proposition 1.6 in Appendix A.

We will extend the ideas from [23] to prove the following balanced supersaturation theorems, which are the main results of this paper.

Theorem 1.7. For all $a, b \geq 2$, there are positive constants $C, \delta$ and $k_{0}$ such that for all $k \geq k_{0}$ and all graphs $G$ with $n$ vertices and $k n^{1+1 / b}$ edges, there exists a family $\mathcal{H}$ of copies of $\theta_{a, b}$ in $G$ so that
(i) $|\mathcal{H}| \geq \delta k^{a b} n^{2}$ and
(ii) $d_{\mathcal{H}}(\sigma) \leq \frac{C \cdot|\mathcal{H}|}{k^{(1+\alpha)(|\sigma|-1)} e(G)}$ for all $\sigma \subset E(G)$ with $1 \leq|\sigma| \leq a b$, where $\alpha=\frac{1}{a b-1}$.

Theorem 1.8. For all $2 \leq a_{1} \leq \ldots \leq a_{r}$, there are positive constants $C, \delta$ and $k_{0}$ such that for all $k \geq k_{0}$ and all r-graphs $G$ with $n$ vertices and $k n^{r-1 /\left(a_{1} \cdots a_{r-1}\right)}$ edges, there exists a family $\mathcal{H}$ of copies of $K_{a_{1}, \ldots, a_{r}}^{(r)}$ in $G$ so that
(i) $|\mathcal{H}| \geq \delta k^{a_{1} \cdots a_{r}} n^{a_{1}+\ldots+a_{r-1}}$ and
(ii) $d_{\mathcal{H}}(\sigma) \leq \frac{C \cdot|\mathcal{H}|}{k^{(1+\alpha)(|\sigma|-1)} e(G)}$ for all $\sigma \subset E(G)$ with $1 \leq|\sigma| \leq a_{1} \cdots a_{r}$, where $\alpha=\frac{1}{a_{1} \cdots a_{r}-1}$.

We thus confirm Morris' and Saxton's conjecture for most theta graphs and the corresponding statements for hypergraphs for most complete $r$-partite $r$-graphs. Theorem 1.3 and Theorem 1.4 follow immediately from Proposition 1.6 combined with Theorem 1.7 and Theorem 1.8. The subsequent work of Ferber, McKinley and Samotji [17] establishes a weaker, but significantly easier to prove, supersaturation result that is still sufficiently strong to derive $\left.F_{r}(n, H)=2^{\operatorname{ex}_{r}(n, H}\right)$ for a much larger class of $r$-graphs $H$. However, the result of [17] is not strong enough to imply anything non-trivial for the Turán problem in random hypergraphs.

We will prove Theorem 1.7 in Section 2, Theorem 1.8 in Section 3. We will use in these sections the slightly informal notation $\varepsilon \ll \tilde{\varepsilon}$ if $\varepsilon \leq c \cdot \tilde{\varepsilon}$ for a sufficiently small constant $c>0$.

## 2. Theta graphs

For $n, k, j \in \mathbb{N}$ and $\delta>0$, let

$$
\Delta^{(j)}(\delta, k, n):=\frac{k^{a b-1} \cdot n^{1-1 / b}}{\left(\delta k^{b /(b-1)}\right)^{j-1}}
$$

Definition 2.1. Let $a, b, n, k \in \mathbb{N}$ with $a, b \geq 2$, let $\delta>0$ and let $G$ be an $n$-vertex graph with $k n^{1+1 / b}$ edges. A collection $\mathcal{H}$ of copies of $\theta_{a, b}$ in $G$ is good for $(a, b, k, n, \delta)$ (or simply good if the parameters are understood) if $d_{\mathcal{H}}(\sigma) \leq \Delta^{(|\sigma|)}(\delta, k, n)$ for every non-empty forest $\sigma \subset E(G)$.

The aim of this section is to prove the following theorem.
Theorem 2.2. For all $a, b \geq 2$, there are some positive constants $k_{0}$ and $\delta$, such that for all $k \geq k_{0}$ and all graphs $G$ with $n$ vertices and $k n^{1+1 / b}$ edges, there exists a family $\mathcal{H}$ of copies of $\theta_{a, b}$ in $G$ of size $|\mathcal{H}| \geq \delta k^{a b} n^{2}$ which is good for $(a, b, k, n, \delta)$.

Theorem 2.2 easily implies Theorem 1.7. Indeed, for every $\sigma \subset E(G)$ with $1 \leq|\sigma| \leq a b$ and $d_{\mathcal{H}}(\sigma)>0$, take a forest $\sigma^{\prime} \subset \sigma$ of maximal size and note that

$$
d_{\mathcal{H}}(\sigma) \leq d_{\mathcal{H}}\left(\sigma^{\prime}\right) \leq \Delta^{\left(\left|\sigma^{\prime}\right|\right)}(\delta, k, n) \leq \frac{k^{a b-1} \cdot n^{1-1 / b}}{\left(\delta k^{b /(b-1)}\right)^{\left|\sigma^{\prime}\right|-1}} \leq \frac{k^{a b-1} \cdot n^{1-1 / b}}{\left(\delta k^{1+\alpha}\right)^{|\sigma|-1}}
$$

where $\alpha=1 /(a b-1)$. We remark that the worst case for the last inequality is when $|\sigma|=a b$ and $\left|\sigma^{\prime}\right|=a b-a+1$. Theorem 2.2 in turn is an immediate consequence of the following proposition.

Proposition 2.3. For all $a, b \geq 2$, there are some positive constants $k_{0}$ and $\delta>0$ such that for all $k \geq k_{0}$ and all graphs $G$ with $n$ vertices and $k n^{1+1 / b}$ edges, the following is true. If $\mathcal{H}$ is a collection of copies of $\theta_{a, b}$ in $G$ which is good for $(a, b, k, n, \delta)$ and $|\mathcal{H}| \leq \delta k^{a b} n^{2}$, then there exists a copy $H \notin \mathcal{H}$ of $\theta_{a, b}$ such that $\mathcal{H} \cup\{H\}$ is good for $(a, b, k, n, \delta)$.

The rest of this section is devoted to the proof of Proposition 2.3.

### 2.1. The setup

We define all constants here and fix the important parameters. Let $a, b \geq 2$ and set $K=5 a b$, $\varepsilon(b)=1 / K^{3}, \varepsilon(t-1)=\varepsilon(t)^{t}$ for each $2 \leq t \leq b, \delta=\varepsilon(1)^{2 a b+2}$ and $k_{0}=1 / \delta$. Let $n, k \in \mathbb{N}$ with $k \geq k_{0}$, and fix a graph $G$ with $n$ vertices and $k n^{1+1 / b}$ edges. Also fix a good collection $\mathcal{H}$ of copies of $\theta_{a, b}$ in $G$ with $|\mathcal{H}| \leq \delta k^{a b} n^{2}$.

We will make the following further assumptions on $G$. Since $\delta=\varepsilon(1)^{2 a b+2}$, there are at most

$$
\frac{a b \cdot|\mathcal{H}|}{\varepsilon(1)^{2 a b+1} k^{a b-1} n^{1-1 / b}} \leq a b \cdot \varepsilon(1) \cdot e(G) \ll e(G)
$$

edges $e \in G$ with $d_{\mathcal{H}}(e) \geq \varepsilon(1)^{2 a b+1} k^{a b-1} n^{1-1 / b}$. By deleting all such edges we may assume

$$
\begin{equation*}
d_{\mathcal{H}}(e)<\varepsilon(1)^{2 a b+1} k^{a b-1} n^{1-1 / b} \text { for every } e \in E(G) \tag{2.1}
\end{equation*}
$$

(at the cost of slightly weaker constants). In particular, we have

$$
\begin{equation*}
d_{\mathcal{H}}(e)<\Delta^{(1)}(\delta, k, n) \text { for every } e \in E(G) \tag{2.2}
\end{equation*}
$$

Similarly, since there are at most $K \varepsilon(b) k n^{1+1 / b} \ll e(G)$ edges incident to vertices of degree at most $K \varepsilon(b) k n^{1 / b}$, we may assume that

$$
\begin{equation*}
\delta(G) \geq K \varepsilon(b) k n^{1 / b} \tag{2.3}
\end{equation*}
$$

Finally, we define saturated sets of edges.
Definition 2.4 (Saturated sets of edges). Given a non-empty forest $\sigma \subset E(G)$, we say that $\sigma$ is saturated if $d_{\mathcal{H}}(\sigma) \geq\left\lfloor\Delta^{(|\sigma|)}(\delta, k, n)\right\rfloor$. Let

$$
\mathcal{F}=\{\sigma \subset E(G): \sigma \text { is saturated }\}
$$

denote the collection of all saturated sets of edges.
We emphasize that in all further results $G, \mathcal{H}, \mathcal{F}$ and all parameters are fixed as above.

### 2.2. Preliminaries

For $S \subset E(G)$ and $j \in \mathbb{N}$, define the $j$-link of $S$ as

$$
L_{\mathcal{F}}^{(j)}(S):=\{\sigma \subset E(G) \backslash S:|\sigma|=j \text { and } \sigma \cup \tau \in \mathcal{F} \text { for some non-empty } \tau \subset S\}
$$

and let $L_{\mathcal{F}}(S)=\bigcup_{j \geq 1} L_{\mathcal{F}}^{(j)}(S)$. We have the following important bound on its size.
Lemma 2.5. For every $j \in \mathbb{N}$ and every $S \subset E(G)$, we have

$$
\left|L_{\mathcal{F}}^{(j)}(S)\right| \leq 2^{a b+|S|+1} \cdot\left(\delta k^{b /(b-1)}\right)^{j} .
$$

Proof. For each non-empty forest $\tau \subset S$, set

$$
\mathcal{J}(\tau)=\{\sigma \subset E(G) \backslash S:|\sigma|=j \text { and } \sigma \cup \tau \in \mathcal{F}\} .
$$

By the handshaking lemma and the definition of goodness, we obtain

$$
\frac{1}{2^{a b}} \cdot \sum_{\sigma \in \mathcal{J}(\tau)} d_{\mathcal{H}}(\sigma \cup \tau) \leq d_{\mathcal{H}}(\tau) \leq \Delta^{(|\tau|)}(\delta, k, n),
$$

as each edge of $\mathcal{H}$ is counted at most $2^{a b}$ times in the sum. Moreover,

$$
\sum_{\sigma \in \mathcal{J}(\tau)} d_{\mathcal{H}}(\sigma \cup \tau) \geq|\mathcal{J}(\tau)| \cdot\left\lfloor\Delta^{(|\tau|+j)}(\delta, k, n)\right\rfloor
$$

by the definition of $\mathcal{J}(\tau)$ and $\mathcal{F}$. Hence

$$
|\mathcal{J}(\tau)| \leq 2^{a b} \cdot \frac{\Delta^{(|\tau|)}(\delta, k, n)}{\left\lfloor\Delta^{(|\tau|+j)}(\delta, k, n)\right\rfloor} \leq 2^{a b+1} \cdot\left(\delta k^{b /(b-1)}\right)^{j} .
$$

Finally, since the sets $\mathcal{J}(\tau)$ cover $L_{\mathcal{F}}^{(j)}(S)$, we find that

$$
\left|L_{\mathcal{F}}^{(j)}(S)\right| \leq \sum_{\tau}|\mathcal{J}(\tau)| \leq 2^{a b+S+1} \cdot\left(\delta k^{b /(b-1)}\right)^{j},
$$

as desired.
The following definition and theorem summarise a series of results of Morris and Saxton (see [23, Section 3]) which we will use in a similar way to build copies of $\theta_{a, b}$.
Definition 2.6. Let $x \in V(G)$ and $2 \leq t \in \mathbb{N}$. A $t$-neighbourhood of $x$ is a pair $(\mathcal{A}, \mathcal{P})$, in which

- $\mathcal{A}=\left(A_{0}, A_{1}, \ldots, A_{t}\right)$ is a collection of (not necessarily disjoint) sets of vertices of $G$ with $A_{0}=\{x\}$,
- $\mathcal{P}$ is a collection of paths in $G$ of the form $\left(x, u_{1}, \ldots, u_{t}\right)$, with $u_{i} \in A_{i}$ for each $i \in[t]$.

For any collection $\mathcal{P}$ of paths in $G$ and any two vertices $u, v \in V(G)$, let

$$
\mathcal{P}[u \rightarrow v]:=\left\{\left(x_{1}, \ldots, x_{s}\right): x_{1}=u, x_{s}=v\right\}
$$

denote the set of paths in $\mathcal{P}$ which begin at $u$ and end at $v$.

Theorem 2.7 (Morris-Saxton [23]). Given $G, \mathcal{H}, \mathcal{F}$ and all constants as in Section 2.1, there exist $t \in\{2, \ldots, b\}$ and some vertex $x \in V(G)$, for which there is a $t$-neighbourhood $(\mathcal{B}=$ $\left.\left(B_{0}, \ldots, B_{t}\right), \mathcal{Q}\right)$ of $x$ with the following seven properties:
(P1) $\left|B_{1}\right| \leq k n^{1 / b}$ and $\left|B_{t}\right| \leq k^{(b-t) /(b-1)} n^{t / b}$.
(P2) For every $i \in\{0,1, \ldots, t-1\}$ and every $u \in B_{i}$,

$$
\left|N(u) \cap B_{i+1}\right| \geq \varepsilon(t) k n^{1 / b} .
$$

(P3) For every $v \in B_{t}$,

$$
\left|N(v) \cap B_{t-1}\right| \geq \varepsilon(t)^{2} k^{b /(b-1)}
$$

(P4) For every $v \in B_{t}$,

$$
|\mathcal{Q}[x \rightarrow v]| \geq \varepsilon(t)^{t} k^{(t-1) b /(b-1)} .
$$

(P5) $\mathcal{Q}$ avoids $\mathcal{F}$, i.e. $\sigma \not \subset Q$ for every $\sigma \in \mathcal{F}$ and every $Q \in \mathcal{Q}$.
(P6) For every $w \in B_{t}$ and $v \in V(G) \backslash\{x, w\}$, there are at most $b k^{(t-2) b /(b-1)}$ paths $Q \in$ $\mathcal{Q}[x \rightarrow w]$ containing $v$.
(P7) For every $\sigma \subset E(G)$ with $|\sigma| \leq t-1$ and every $w \in B_{t}$, there are at most $t^{t} \cdot k^{(t-|\sigma|-1) b /(b-1)}$ paths $Q \in \mathcal{Q}[x \rightarrow w]$ with $\sigma \subset E(P)$.

We shall call $(\mathcal{B}, \mathcal{Q})$ a refined $t$-neighbourhood of $x$. Property Item (P5) is slightly different here but completely analogous (in the proof of Lemma 3.6 in [23], we need to use Lemma 2.5 instead of the corresponding lemma in [23]).

### 2.3. Finding $\theta_{a, b}$ in refined $t$-neighbourhoods

Let $G, \mathcal{H}, \mathcal{F}$ and all constants be as in Section 2.1 and let $(\mathcal{B}, \mathcal{Q})$ be the refined $t$-neighbourhood for some $x \in V(G)$ and $t \in\{2, \ldots, b\}$ guaranteed by Theorem 2.7.

For technical reasons fix

$$
X_{i}(u) \subset N(u) \cap B_{i+1} \text { of size }\left|X_{i}(u)\right|=\varepsilon(t) k n^{1 / b}
$$

for each $i \in[t-1]$ and $u \in B_{i}$, and

$$
X_{t}(u) \subset N(u) \cap B_{t-1} \text { of size }\left|X_{t}(u)\right|=\varepsilon(t)^{2} k^{b /(b-1)}
$$

for each $u \in B_{t}$. Furthermore, fix a subset

$$
\mathcal{Q}(z) \subset \mathcal{Q}(x \rightarrow z) \text { of size }|\mathcal{Q}(z)|=\varepsilon(t)^{t} k^{(t-1) b /(b-1)}
$$

for every $z \in B_{t}$.
Using the following algorithm, we shall create many copies $H$ of $\theta_{a, b}$ in $G$ such that $\mathcal{H} \cup\{H\}$ is good and deduce that one of them must not be contained in $\mathcal{H}$ already.
Algorithm 1. Initially, let $\Theta:=\emptyset$. As long as possible generate new copies of $\theta_{a, b}$ and add them to $\Theta$ via the following process. To create a copy of $\theta_{a, b}$ we shall add edges and denote the subgraph of $G$ induced by the currently selected edges by $H=(V, E)$. (Note that $H, V$ and $E$ are constantly changing.)

1. Generate a path $P_{1}=\left(x=p_{0}^{1}, p_{1}^{1}, \ldots, p_{t}^{1}\right)$ as follows. For $i=0,1, \ldots, t-1$, choose $p_{i+1}^{1}$ from $X_{i}\left(p_{i}^{1}\right) \subset N\left(p_{i}^{1}\right) \cap B_{i+1}$ such that

$$
p_{i+1}^{1} \notin V \quad \text { and } \quad\left\{p_{i}^{1}, p_{i+1}^{1}\right\} \notin L_{\mathcal{F}}^{(1)}(E) .
$$

2. Create a path $Z_{1}=\left(p_{t}^{1}=z_{0}^{1}, z_{1}^{1}, \ldots, z_{b-t}^{1}=: y\right)$ as follows. Define

$$
r(i)= \begin{cases}t-1 & \text { if } 0 \leq i \leq b-t \text { and } i \text { is even, } \\ t & \text { if } 0 \leq i \leq b-t \text { and } i \text { is odd } .\end{cases}
$$

For $i=0, \ldots, b-t-1$, select $z_{i+1}^{1}$ from $X_{r(i)}\left(z_{i}^{1}\right) \subset N\left(z_{i}^{1}\right) \cap B_{r(i+1)}$ such that

$$
z_{i+1}^{1} \notin V \quad \text { and } \quad\left\{z_{i}^{1}, z_{i+1}^{1}\right\} \notin L_{\mathcal{F}}^{(1)}(E) .
$$

3. For $j=2, \ldots, a$, create a path $Z_{j}=\left(y=z_{0}^{j}, z_{1}^{j}, \ldots, z_{b-t}^{j}=: z_{j}\right)$ as follows. Let

$$
s(i)= \begin{cases}t-1 & \text { if } 0 \leq i \leq b-t \text { and } i+(b-t) \text { is even, } \\ t & \text { if } 0 \leq i \leq b-t \text { and } i+(b-t) \text { is odd. }\end{cases}
$$

Now, for $i=0, \ldots, b-t-1$, choose $z_{i+1}^{j}$ from $X_{s(i)}\left(z_{i}^{j}\right) \subset N\left(z_{i}^{j}\right) \cap B_{s(i+1)}$ with

$$
z_{i+1}^{j} \notin V \quad \text { and } \quad\left\{z_{i}^{j}, z_{i+1}^{j}\right\} \notin L_{\mathcal{F}}^{(1)}(E) .
$$

4. For $j=2, \ldots, a$, pick a path $P_{j} \in \mathcal{Q}\left(z_{j}\right)$ which uses no vertex of $V \backslash\left\{z_{j}\right\}$ and avoids $L_{\mathcal{F}}(E)$.
Join the paths $P_{1}, Z_{1}, \ldots, P_{a}, Z_{a}$ to form a copy of $\theta_{a, b}$, and add this to $\Theta$.


Figure 1: A copy of $\theta_{3, b}$ produced by Algorithm 1.
See Figure 1 for an illustration of Algorithm 1. We shall show later that $|\Theta|$ is quite large.

Claim 1. $|\Theta| \geq \varepsilon(1)^{2 a b} k^{a b} n$.
Before we proceed with the proof of Claim 1, we show how it implies Proposition 2.3.
Proof of Proposition 2.3. Since $\left|B_{1}\right| \leq k n^{1 / b}$ by property (P1) of Theorem 2.7, we have

$$
|\Theta \cap \mathcal{H}| \leq k n^{1 / b} \cdot \max _{e \in E(G)} d_{\mathcal{H}}(e) \stackrel{(2.1)}{\leq} k n^{1 / b} \cdot \varepsilon(1)^{2 a b+1} k^{a b-1} n^{1-1 / b}=\varepsilon(1)^{2 a b+1} k^{a b} n
$$

Hence, by Claim 1, there exists some $H \in \Theta \backslash \mathcal{H}$. By the construction of $\Theta$, the collection $\mathcal{H} \cup\{H\}$ is good. This finishes our proof.

The rest of this section is devoted to the proof of Claim 1. To make the counting of $|\Theta|$ easier to follow, we introduce some notation here. For $i \in[a]$, let $\mathcal{R}_{i}$ be the set of all possible choices for the paths $P_{1}, Z_{1}, \ldots, Z_{a}, P_{2}, \ldots, P_{i}$ in Algorithm 1. For $\mathbf{R}_{i}=\left(P_{1}, Z_{1}, \ldots, Z_{a}, P_{2}, \ldots, P_{i}\right) \in$ $\mathcal{R}_{i}$, let $\mathcal{R}_{i+1}\left(\mathbf{R}_{i}\right):=\left\{P_{i+1} \in Q\left(z_{i+1}\right):\left(\mathbf{R}_{i}, P_{i+1}\right) \in \mathcal{R}_{i+1}\right\}$. We call a vertex $v \in V\left(\mathbf{R}_{1}\right) \backslash\{x\}$ forward if either $v \in V\left(P_{1}\right)$ or $v \in B_{t}$, backward if $v \in V\left(Z_{2} \cup \ldots \cup Z_{a}\right) \cap B_{t-1}$. Hence we have partitioned $V\left(\mathbf{R}_{1}\right) \backslash\{x\}$ into forward and backward vertices. Let $r_{f w}$ and $r_{b w}$ denote the number of forward and backward vertices respectively of some $\mathbf{R}_{1} \in \mathcal{R}_{1}$, and let $r=r_{f w}+r_{b w}$. It is not difficult to see that $r=a b-(a-1) t$, and

$$
\begin{cases}r_{f w}=t+a(b-t) / 2, r_{b w}=a(b-t) / 2 &  \tag{2.4}\\ r_{f w}=t+a(b-t+1) / 2-1, r_{b w}=a(b-t-1) / 2+1 & \\ \text { if } b-t \text { is odd }\end{cases}
$$

To prove Claim 1, we first bound the number of graphs $\left(P_{1}, Z_{1}, \ldots, Z_{a}\right)$, chosen in Steps $1-3$, in terms of $r_{f w}$ and $r_{b w}$.

Claim 2. $\left|\mathcal{R}_{1}\right| \geq \frac{1}{2}\left(\varepsilon(t) k n^{1 / b}\right)^{r_{f w}} \cdot\left(\varepsilon(t)^{2} k^{b /(b-1)}\right)^{r_{b w}}$.
Proof. We first show that at most $a b+2^{2 a b} \delta k^{b /(b-1)}$ choices are excluded for each vertex. Recall that $H=(V, E)$ is the graph induced by the currently selected edges. Note that at most $a b$ choices are excluded by the condition that the new vertex is not in $V$. Moreover, by Lemma 2.5 we have

$$
\left|L_{\mathcal{F}}^{(1)}(E)\right| \leq 2^{a b+|E|+1} \cdot \delta k^{b /(b-1)} \leq 2^{2 a b} \delta k^{b /(b-1)}
$$

as required. Therefore, there are at least

$$
\varepsilon(t) k n^{1 / b}-\left(a b+2^{2 a b} \delta k^{b /(b-1)}\right) \geq 2^{-1 / r} \varepsilon(t) k n^{1 / b}
$$

choices for each forward vertex, where the last inequality holds since $k \leq n^{(b-1) / b}$ and $\delta \ll \varepsilon(t)^{2}$. Similarly, using the fact that $\delta \ll \varepsilon(t)^{2}$, we find that there are at most

$$
\varepsilon(t)^{2} k^{b /(b-1)}-\left(a b+2^{2 a b} \delta k^{b /(b-1)}\right) \geq 2^{-1 / r} \varepsilon(t)^{2} k^{b /(b-1)}
$$

choices for each backward vertex. The claim now follows, as we choose $r_{f w}$ forward vertices and $r_{b w}$ backward vertices.

For each $i \in[a-1]$, define $\mathcal{D}_{i}$ to be the set of all $\mathbf{R}_{i} \in \mathcal{R}_{i}$ for which there are at least

$$
\frac{1}{4} \varepsilon(t)^{t} k^{(t-1) b /(b-1)}
$$

paths $P \in \mathcal{Q}\left(z_{i+1}\right)$ with $E(P) \in L_{\mathcal{F}}^{(t)}\left(E\left(\mathbf{R}_{i}\right)\right)$. (Here we view $\mathbf{R}_{i}$ as a graph.) We now deduce Claim 1 from the following two claims. The first shows that if the graph $\mathbf{R}_{i} \in \mathcal{R}_{i}$ satisfies $\mathbf{R}_{i} \notin \mathcal{D}_{i}$, then we have many choices for the path $P_{i+1}$ in Step 4.

Claim 3. If $\mathbf{R}_{i} \in \mathcal{R}_{i} \backslash \mathcal{D}_{i}$ for some $i \in[a-1]$, then

$$
\left|\mathcal{R}_{i+1}\left(\mathbf{R}_{i}\right)\right| \geq \frac{1}{2} \cdot \varepsilon(t)^{t} k^{(t-1) b /(b-1)}
$$

The second states that $\left|\mathcal{D}_{i}\right|$ is not too large.
Claim 4. $\left|\mathcal{D}_{i}\right| \leq \frac{1}{2}\left|\mathcal{R}_{i}\right|$ for every $i \in[a-1]$.
Proof of Claim 1. From Claims 3 and 4, we find

$$
\left|\mathcal{R}_{i+1}\right| \geq \frac{1}{2} \cdot \varepsilon(t)^{t} k^{(t-1) b /(b-1)} \cdot\left|\mathcal{R}_{i} \backslash \mathcal{D}_{i}\right| \geq \frac{1}{4} \cdot \varepsilon(t)^{t} k^{(t-1) b /(b-1)} \cdot\left|\mathcal{R}_{i}\right|
$$

for every $i \in[a-1]$. Combined with Claim 2, we obtain

$$
\begin{aligned}
\left|\mathcal{R}_{a}\right| & \geq\left(\frac{1}{4} \varepsilon(t)^{t} k^{(t-1) b /(b-1)}\right)^{a-1} \cdot \frac{1}{2}\left(\varepsilon(t) k n^{1 / b}\right)^{r_{f w}} \cdot\left(\varepsilon(t)^{2} k^{b /(b-1)}\right)^{r_{b w}} \\
& =2^{-2 a+1} \varepsilon(t)^{(a-1) t+r_{f w}+2 r_{b w}} \cdot k^{(a-1)(t-1) b /(b-1)+r_{f w}+b r_{b w} /(b-1)} n^{r_{f w} / b},
\end{aligned}
$$

By (2.4), one has

$$
\begin{aligned}
(a-1) t+r_{f w}+2 r_{b w} & \leq(a-1) t+\{t+a(b-t) / 2\}+2 a(b-t) / 2 \\
& =2 a b-a t-a(b-t) / 2 \leq 2 a b-a t .
\end{aligned}
$$

Again from (2.4) we find $r_{f w} \geq t+a(b-t) / 2=b+(a-2)(b-t) / 2 \geq b$. Together with the fact that $n \geq k^{b /(b-1)}$, this yields

$$
\begin{aligned}
k^{(a-1)(t-1) b /(b-1)+r_{f w}+b r_{b w} /(b-1)} n^{r_{f w} / b} & \geq k^{(a-1)(t-1) b /(b-1)+r_{f w}+b r_{b w} /(b-1)+\left(r_{f w} / b-1\right) \cdot b /(b-1)} n \\
& =k^{\left\{(a-1)(t-1)+r_{f w}+r_{b w}-1\right\} b /(b-1)} n=k^{a b} n,
\end{aligned}
$$

where the last equality follows from the formula $r_{f w}+r_{b w}=a b-(a-1) t$. Therefore, we get

$$
\left|\mathcal{R}_{a}\right| \geq 2^{-2 a+1} \varepsilon(t)^{2 a b-a t} k^{a b} n
$$

As each copy of $\theta_{a, b}$ appears at most $a!$ times in $\mathcal{R}_{a}$, we conclude

$$
|\Theta| \geq \frac{1}{a!}\left|\mathcal{R}_{a}\right| \geq \frac{1}{a!} 2^{-2 a+1} \varepsilon(t)^{2 a b-a t} k^{a b} n \geq \varepsilon(1)^{2 a b} k^{a b} n
$$

for $\varepsilon(1)<\varepsilon(t) \ll 1$, as required.
We end this section with the proofs of Claims 3 and 4 .

Proof of Claim 3. As $\left|\mathcal{Q}\left(z_{i+1}\right)\right|=\varepsilon(t)^{t} k^{(t-1) b /(b-1)}$, in order to prove the claim, it suffices to show $\left|\mathcal{Q}\left(z_{i+1}\right) \backslash \mathcal{R}_{i+1}\left(\mathbf{R}_{i}\right)\right| \leq \frac{1}{2} \varepsilon(t)^{t} k^{(t-1) b /(b-1)}$. In other words, we wish to bound the number of paths in $\mathcal{Q}\left(z_{i+1}\right)$ which either contain a vertex of $V\left(\mathbf{R}_{i}\right) \backslash\left\{x, z_{i+1}\right\}$, or fail to avoid $L_{\mathcal{F}}\left(E\left(\mathbf{R}_{i}\right)\right)$.

By property Item (P6) of Theorem 2.7 , the number of paths in $\mathcal{Q}\left(z_{i+1}\right)$ which contain a vertex of $V\left(\mathbf{R}_{i}\right) \backslash\left\{x, z_{i+1}\right\}$ is at most

$$
a b \cdot b k^{(t-2) b /(b-1)} \leq \varepsilon(t)^{t+1} k^{(t-1) b /(b-1)}
$$

as $k \geq k_{0} \gg \varepsilon(t)^{-(t+1)}$.
Now, let $\sigma \in L_{\mathcal{F}}\left(E\left(\mathbf{R}_{i}\right)\right)$, and consider the paths in $\mathcal{Q}\left(z_{i+1}\right)$ that contain $\sigma$. We first deal with the case $1 \leq|\sigma| \leq t-1$. According to Item (P7), the number of paths in $\mathcal{Q}\left(z_{i+1}\right)$ containing $\sigma$ is at most

$$
t^{t} k^{(t-|\sigma|-1) b /(b-1)}
$$

Moreover, by Lemma 2.5, we have

$$
\left|L_{\mathcal{F}}^{(|\sigma|)}\left(E\left(\mathbf{R}_{i}\right)\right)\right| \leq 2^{2 a b}\left(\delta k^{b /(b-1)}\right)^{|\sigma|}
$$

Therefore, the number of paths in $\mathcal{Q}\left(z_{i+1}\right)$ which contain some $\sigma \in L_{\mathcal{F}}\left(E\left(\mathbf{R}_{i}\right)\right)$ with $1 \leq|\sigma| \leq$ $t-1$ is at most

$$
(2 b)^{2 a b} \cdot \delta k^{(t-1) b /(b-1)} \leq \varepsilon(t)^{t+1} k^{(t-1) b /(b-1)}
$$

as $\delta \ll \varepsilon(t)^{t+1}$.
On the other hand, since $\mathbf{R}_{i} \notin \mathcal{D}_{i}$, there are at most

$$
\frac{1}{4} \varepsilon(t)^{t} k^{(t-1) b /(b-1)}
$$

paths in $\mathcal{Q}\left(z_{i+1}\right)$ that contain some $\sigma \in L_{\mathcal{F}}\left(E\left(\mathbf{R}_{i}\right)\right)$ with $|\sigma|=t$.
Summing these estimates gives the desired inequality

$$
\left|\mathcal{Q}\left(z_{i+1}\right) \backslash \mathcal{R}_{i+1}\left(\mathbf{R}_{i}\right)\right| \leq \frac{1}{2} \cdot \varepsilon(t)^{t} k^{(t-1) b /(b-1)}
$$

Proof of Claim 4. We proceed by induction on $i$. Let $i \in[a-1]$ and assume that the claim holds up to $i-1$ (no assumption is needed in case $i=1$ ). Thus, we have

$$
\left|\mathcal{R}_{j}\right| \stackrel{\text { Claim } 3}{\geq} \frac{1}{2} \cdot \varepsilon(t)^{t} k^{(t-1) b /(b-1)} \cdot\left|\mathcal{R}_{j-1} \backslash \mathcal{D}_{j-1}\right| \geq \frac{1}{4} \cdot \varepsilon(t)^{t} k^{(t-1) b /(b-1)} \cdot\left|\mathcal{R}_{j-1}\right|
$$

for every $2 \leq j \leq i$. Combined with Claim 2, this gives

$$
\begin{equation*}
\left|\mathcal{R}_{i}\right| \geq\left(\frac{1}{4} \varepsilon(t)^{t} k^{(t-1) b /(b-1)}\right)^{i-1} \cdot \frac{1}{2}\left(\varepsilon(t) k n^{1 / b}\right)^{r_{f w}} \cdot\left(\varepsilon(t)^{2} k^{b /(b-1)}\right)^{r_{b w}} \tag{2.5}
\end{equation*}
$$

We proceed in three steps. We first give an upper bound for the number of members of $\mathcal{D}_{i}$ containing a given set of edges. This will be used in conjunction with (2.5).

Step 1. Let $z \in B_{t}$, and let $J \subset E(G)$ be a forest of size $|J|=j \in[r+(i-1)(t-1)-1]$ which does not contain an $x$-z-path. Then there are at most

$$
m(j):= \begin{cases}(a b)^{2 a b}\left(k n^{1 / b}\right)^{r_{f w}-1}\left(k^{b /(b-1)}\right)^{r_{b w}+(i-1)(t-1)-j} & \text { if } 1 \leq j \leq r_{b w}+(i-1)(t-1) \\ (a b)^{2 a b}\left(k n^{1 / b}\right)^{r+(i-1)(t-1)-j-1} & \text { otherwise }\end{cases}
$$

$\mathbf{R}_{i} \in \mathcal{R}_{i}$ with $z_{i+1}=z$ and $J \subset E\left(\mathbf{R}_{i}\right)$.

Remark: The key property about $m(j)$ is that it satisfies

$$
\begin{equation*}
m(j) \cdot\left(\delta k^{b /(b-1)}\right)^{j} \leq \frac{\left|\mathcal{R}_{i}\right|}{k n^{1 / b}} \quad \text { for every } j \in[r+(i-1)(t-1)-1] \tag{2.6}
\end{equation*}
$$

due to (2.5) and $\delta \ll \varepsilon(t)^{3 a b}$.
Proof of Step 1. Note that we have at most $(a b)^{a b}$ choices for the positions of the edges of $J$ in $\mathbf{R}_{i}$. Let's fix such a choice and count the corresponding $\mathbf{R}_{i}$. More precisely, given a partition $J=J_{1} \cup \ldots \cup J_{i}$, we shall bound from above the number of $\mathbf{R}_{i}=\left(\mathbf{R}_{1}, P_{2}, \ldots, P_{i}\right) \in \mathcal{R}_{i}$ such that $J_{1} \subset \mathbf{R}_{1}, J_{\ell} \subset P_{\ell}$ for all $2 \leq \ell \leq i$, and $z_{i+1}=z$.

We may assume $\left|J_{\ell}\right| \leq t$ for every $2 \leq \ell \leq t$ (otherwise there is no such $\mathbf{R}_{i}$ ). Let $I$ denote the set of all $\ell \in\{2, \ldots, i\}$ with $\left|J_{\ell}\right|=t$. Note that $P_{\ell}=J_{\ell}$ for all $\ell \in I$, and so $\left\{z_{\ell}: \ell \in I\right\}$ is fixed. As $J$ contains neither an $x$ - $z$-path nor a cycle, we may assume further that the subgraph induced by $J_{1}$ together with the fixed vertices $\{x, z\} \cup\left\{z_{\ell}: \ell \in I\right\}$ is a forest, in which these $|I|+2$ fixed vertices are in different components. It follows that at least $\left|J_{1}\right|+|I|+2$ vertices of $\mathbf{R}_{1}$ are fixed, and hence there are at most $r-\left|J_{1}\right|-|I|-1$ not-yet-chosen vertices in $\mathbf{R}_{1}(x$ is excluded). This shows

$$
r_{1}+r_{2} \leq r-\left|J_{1}\right|-|I|-1,
$$

where $r_{1}$ and $r_{2}$ denote the number of free forward vertices and free backward vertices respectively. In addition, as $z \in B_{t}$ is fixed, we must have

$$
r_{1} \leq r_{f w}-1
$$

Note that we have at most $\varepsilon(t) k n^{1 / b} \leq k n^{1 / b}$ choices for each forward vertex and at most $\varepsilon(t)^{2} k^{b /(b-1)} \leq k^{b /(b-1)}$ choices for each backward vertex. Moreover, $P_{\ell}=J_{\ell}$ for all $\ell \in I$, and for each $\ell \in\{2, \ldots, i\} \backslash I$, there are at most $t^{t} k^{\left(t-\left|J_{\ell}\right|-1\right) b /(b-1)}$ choices for $P_{\ell}$ by Item (P7). Hence the number of $\mathbf{R}_{i}=\left(\mathbf{R}_{1}, P_{2}, \ldots, P_{i}\right)$ such that $\left(J_{1}, J_{2}, \ldots, J_{i}\right) \subset \mathbf{R}_{i}$ is at most

$$
\left(k n^{1 / b}\right)^{r_{1}} \cdot\left(k^{b /(b-1)}\right)^{r_{2}} \cdot \prod_{\ell \in\{2, \ldots, i\} \backslash I} t^{t} k^{\left(t-\left|J_{\ell}\right|-1\right) b /(b-1)} \leq b^{a b} \cdot\left(k n^{1 / b}\right)^{r_{1}} \cdot\left(k^{b /(b-1)}\right)^{r_{3}},
$$

where $r_{3}:=r_{2}+\sum_{\ell \in\{2, \ldots, i\} \backslash I}\left(t-\left|J_{\ell}\right|-1\right)$. To estimate the above expression, we note that

$$
\begin{aligned}
r_{1}+r_{3} & =\left(r_{1}+r_{2}\right)+\sum_{\ell \in\{2, \ldots, i\} \backslash I}\left(t-\left|J_{\ell}\right|-1\right) \\
& \leq\left(r-\left|J_{1}\right|-|I|-1\right)+\left(|I|+\sum_{\ell \in\{2, \ldots, i\}}\left(t-\left|J_{\ell}\right|-1\right)\right) \\
& =r+(i-1)(t-1)-j-1,
\end{aligned}
$$

where the second line follows from the estimate $r_{1}+r_{2} \leq r-\left|J_{1}\right|-|I|-1$ and the definition of $I$, and the last equality holds since $\left|J_{1}\right|+\ldots+\left|J_{\ell}\right|=|J|=j$. Together with the inequalities $r_{1} \leq r_{f w}-1$ and $k n^{1 / b} \geq k^{b /(b-1)}$, this implies that the number of $\mathbf{R}_{i}=\left(\mathbf{R}_{1}, P_{2}, \ldots, P_{i}\right)$ with $\left(J_{1}, J_{2}, \ldots, J_{i}\right) \subset \mathbf{R}_{i}$ is bounded from above by $b^{a b} \cdot\left(k n^{1 / b}\right)^{r_{f w}-1} \cdot\left(k^{b /(b-1)}\right)^{r_{b w}+(i-1)(t-1)-j}$ in case $r_{f w}-1 \leq r+(i-1)(t-1)-j-1$, and by $b^{a b} \cdot\left(k n^{1 / b}\right)^{r+(i-1)(t-1)-j-1}$ otherwise.

Putting everything together, we conclude that there are at most $m(j)$ choices for $\mathbf{R}_{i} \in \mathcal{R}_{i+1}$ with $z_{i+1}=z$ and $J \subset E\left(\mathbf{R}_{i}\right)$.

We shall use the inequality (2.6) in the proof of Step 3 below.

Step 2. There exist $j \in[r+(i-1)(t-1)-1]$ for which there are at least

$$
2^{-a b} \varepsilon(t)^{t} k^{(t-1) b /(b-1)} \cdot \frac{\left|\mathcal{D}_{i}\right|}{a b \cdot m(j)}
$$

distinct pairs $(J, P)$ with the following properties:
(a) $P \in \mathcal{Q}(z)$ for some $z \in B_{t}$,
(b) $J$ is a set of $j$ edges of $G$ disjoint from $E(P)$,
(c) $J \cup E(P) \in \mathcal{F}$.

Proof. Recall that for each $\mathbf{R}_{i} \in \mathcal{D}_{i}$, there are at least $\frac{1}{4} \varepsilon(t)^{t} k^{(t-1) b /(b-1)}$ paths $P$ in $\mathcal{Q}\left(z_{i+1}\right)$ with $E(P) \in L_{\mathcal{F}}^{(t)}\left(E\left(\mathbf{R}_{i}\right)\right)$. By the pigeonhole principle, it follows that for each $\mathbf{R}_{i} \in \mathcal{D}_{i}$, there exists a set $\emptyset \neq f\left(\mathbf{R}_{i}\right) \subset E\left(\mathbf{R}_{i}\right)$ such that there are at least

$$
\begin{equation*}
2^{-a b} \varepsilon(t)^{t} k^{(t-1) b /(b-1)} \tag{2.7}
\end{equation*}
$$

paths $P \in \mathcal{Q}\left(z_{i+1}\right)$, each of which is disjoint from $f\left(\mathbf{R}_{i}\right)$ and with $f\left(\mathbf{R}_{i}\right) \cup E(P) \in \mathcal{F}$. Note that $f\left(\mathbf{R}_{i}\right)$ is a forest and does not contain an $x-z_{i+1}$-path (otherwise for every path $P \in \mathcal{Q}\left(z_{i+1}\right)$, $f\left(\mathbf{R}_{i}\right) \cup E(P)$ contains a cycle and thus $\left.f\left(\mathbf{R}_{i}\right) \cup E(P) \notin \mathcal{F}\right)$. In particular, it follows that $\left|f\left(\mathbf{R}_{i}\right)\right| \in[r+(i-1)(t-1)-1]$. By another application of the pigeonhole principle, there exists some $j \in[r+(i-1)(t-1)-1]$ such that $\left|f\left(\mathbf{R}_{i}\right)\right|=j$ for at least $\left|\mathcal{D}_{i}\right| / a b$ choices of $\mathbf{R}_{i} \in \mathcal{D}_{i}$.

Now, define $\mathcal{J}$ to be the set of all pairs $(J, z)$ with $z \in B_{t},|J|=j$ and $J=f\left(\mathbf{R}_{i}\right)$ for some $\mathbf{R}_{i} \in \mathcal{D}_{i}$ with $z_{i+1}=z$. We claim that $|\mathcal{J}| \geq \frac{\left|\mathcal{D}_{i}\right|}{a b \cdot m(j)}$. Indeed, there is such a pair $\left(f\left(\mathbf{R}_{i}\right), z_{i+1}\right)$ for each $\mathbf{R}_{i} \in \mathcal{D}_{i}$ with $\left|f\left(\mathbf{R}_{i}\right)\right|=j$, and we may have counted each pair $m(j)$ times, by the above discussion and Step 1.

Finally, for each $(J, z) \in \mathcal{J}$ choose some $\mathbf{R}_{i} \in \mathcal{D}_{i}$ with $f\left(\mathbf{R}_{i}\right)=J$ and $z_{i+1}=z$. Recall that there are at least (2.7) paths $P \in \mathcal{Q}\left(z_{i+1}\right)$ with $J \cup E(P) \in \mathcal{F}$, each of which is disjoint from $J$. Since $P$ determines $z$, all such generated pairs $(J, P)$ are distinct, and hence the claim follows.

We are now ready to show that $\left|\mathcal{D}_{i}\right|$ is not too large.
Step 3. $\left|\mathcal{D}_{i}\right| \leq \frac{1}{2}\left|\mathcal{R}_{i}\right|$.
Proof. Let $N$ be the number of copies of $\theta_{a, b}$ in $G$ which contain an edge between $x$ and $B_{1}$. For each pair $(J, P)$ as in Step 2, we have $|J \cup E(P)|=j+t$ and $J \cup E(P) \in \mathcal{F}$, giving

$$
d_{\mathcal{H}}(J \cup E(P)) \geq\left\lfloor\Delta^{(j+t)}(\delta, k, n)\right\rfloor \geq \frac{1}{2} \cdot \frac{\Delta^{(1)}(\delta, k, n)}{\left(\delta k^{b /(b-1)}\right)^{j+t-1}} .
$$

Thus, noting that each member of $\mathcal{H}$ contains $J \cup E(P)$ for at most $2^{2 a b}$ pairs $(J, P)$, it follows from Step 2 that

$$
\begin{aligned}
N & \geq 2^{-a b} \varepsilon(t)^{t} k^{(t-1) b /(b-1)} \cdot \frac{\left|\mathcal{D}_{i}\right|}{a b \cdot m(j)} \cdot \frac{\Delta^{(1)}(\delta, k, n)}{2^{2 a b+1}\left(\delta k^{b /(b-1)}\right)^{j+t-1}} \\
& \geq \frac{\varepsilon(t)^{t}}{2^{4 a b} \delta^{t-1}} \cdot \frac{\left|\mathcal{D}_{i}\right| \cdot \Delta^{(1)}(\delta, k, n)}{m(j) \cdot\left(\delta k^{b /(b-1)}\right)^{j}} \\
& \geq 2 k n^{1 / b} \cdot \frac{\left|\mathcal{D}_{i}\right| \cdot \Delta^{(1)}(\delta, k, n)}{\left|\mathcal{R}_{i}\right|},
\end{aligned}
$$

where the last inequality follows from (2.6), and since $t \geq 2$ and $\delta \ll \varepsilon(t)^{t}$. Now, as $\left|B_{1}\right| \leq$ $k n^{1 / b}$, there exists an edge $e \in E(G)$ with

$$
d_{\mathcal{H}}(e) \geq 2 \cdot \frac{\left|\mathcal{D}_{i}\right| \cdot \Delta^{(1)}(\delta, k, n)}{\left|\mathcal{R}_{i}\right|}
$$

Combined with (2.2), we get the desired inequality.
This finishes the proof of Claim 4.

## 3. Complete degenerate hypergraphs

In this section we will prove Theorem 1.8. We shall record the vertex partition of each copy of $K_{a_{1}, \ldots, a_{r}}^{(r)}$. Therefore, a collection $\mathcal{H}$ of copies of $K_{a_{1}, \ldots, a_{r}}^{(r)}$ in an $r$-graph $G$ will be understood as a collection of ordered $r$-tuples $\left(A_{1}, \ldots, A_{r}\right)$ with $A_{i} \in\binom{V(G)}{a_{i}}$ for every $i \in[r]$, and with $G\left[A_{1}, \ldots, A_{r}\right]=K_{a_{1}, \ldots, a_{r}}^{(r)}$.

Given a tuple $\left(S_{1}, \ldots, S_{r}\right)$ of vertex sets such that $1 \leq\left|S_{i}\right| \leq a_{i}$ for every $i \in[r]$ and $G\left[S_{1}, \ldots, S_{r}\right]$ is a complete $r$-partite $r$-graph, we define $d_{\mathcal{H}}\left(S_{1}, \ldots, S_{r}\right)$ to be the number of members of $\mathcal{H}$ containing $\left(S_{1}, \ldots, S_{r}\right)$, that is,

$$
d_{\mathcal{H}}\left(S_{1}, \ldots, S_{r}\right)=\left|\left\{\left(A_{1}, \ldots, A_{r}\right) \in \mathcal{H}:\left(S_{1}, \ldots, S_{r}\right) \subset\left(A_{1}, \ldots, A_{r}\right)\right\}\right| .
$$

Let $\delta>0$. For each $\left(b_{1}, \ldots, b_{r}\right) \in \mathbb{N}^{r}$ with $1 \leq b_{i} \leq a_{i}$ for all $i \in[r]$, define

$$
\begin{equation*}
D^{\left(b_{1}, \ldots, b_{r}\right)}(\delta, k, n)=\prod_{i=1}^{r}\left(\delta k^{a_{1} \cdots a_{i-1}} n^{1-1 / a_{i} \cdots a_{r-1}}\right)^{a_{i}-b_{i}} \tag{3.1}
\end{equation*}
$$

where $a_{1} \cdots a_{i-1}:=1$ if $i=1$ and $a_{i} \cdots a_{r-1}:=1$ if $i=r$. In particular, we have

$$
\begin{equation*}
D^{(1, \ldots, 1)}(\delta, k, n)=\delta^{a_{1}+\ldots+a_{r}-r} k^{a_{1} \cdots a_{r}-1} n^{a_{1}+\ldots+a_{r-1}-r+1 / a_{1} \cdots a_{r-1}} \tag{3.2}
\end{equation*}
$$

One can show that (see Appendix B) Theorem 1.8 is a consequence of the following result.
Theorem 3.1. For every $2 \leq a_{1} \leq \ldots \leq a_{r}$, there exist constants $\delta>0$ and $k_{0} \in \mathbb{N}$ such that the following holds for every $k \geq k_{0}$ and every $n \in \mathbb{N}$. Given an $r$-graph $G$ with $n$ vertices and $k n^{r-1 / a_{1} \cdots a_{r-1}}$ edges, there exists a collection $\mathcal{H}$ of copies of $K_{a_{1}, \ldots, a_{r}}^{(r)}$ in $G$, satisfying:
(a) $|\mathcal{H}| \geq \delta^{a_{1}+\ldots+a_{r}} k^{a_{1} \cdots a_{r}} n^{a_{1}+\ldots+a_{r-1}}$, and
(b) $d_{\mathcal{H}}\left(S_{1}, \ldots, S_{r}\right) \leq D^{\left(\left|S_{1}\right|, \ldots,\left|S_{r}\right|\right)}(\delta, k, n)$ for every $S_{1}, \ldots, S_{r} \subset V(G)$.

Fix now $2 \leq a_{1} \leq \ldots \leq a_{r}$. We shall need various constants in the proof of Proposition 3.2 below, which we will define here for convenience. Informally, they will satisfy

$$
k_{0} \gg K \gg 1 \gg \varepsilon(1) \gg \varepsilon(2) \gg \ldots \gg \varepsilon(r) \gg \varepsilon(r+1)=\delta>0
$$

More precisely, we can set $\varepsilon(1)=1 / 2, \varepsilon(i+1)=\varepsilon(i)^{a_{i}} /\left(2^{2 a_{i}+a_{1} \cdots a_{i}} a_{i}!\right)$ for each $1 \leq i \leq r$, $\delta=\varepsilon(r+1), K=a_{1} \cdots a_{r} 2^{a_{1}+\ldots+a_{r}+1}$ and $k_{0}=1 / \delta$.

Let $G$ be an $n$-vertex $r$-graph with $k n^{r-1 / a_{1} \cdots a_{r-1}}$ edges, where $k \geq k_{0}$. Let ( $S_{1}, \ldots, S_{r}$ ) be an ordered $r$-tuple of vertex sets that satisfies $1 \leq\left|S_{i}\right| \leq a_{i}$ for every $i \in[r]$, and with $G\left[S_{1}, \ldots, S_{r}\right]=K_{\left|S_{1}\right|, \ldots,\left|S_{r}\right|}^{(r)}$. We say that $\left(S_{1}, \ldots, S_{r}\right)$ is saturated if

$$
d_{\mathcal{H}}\left(S_{1}, \ldots, S_{r}\right) \geq\left\lfloor D^{\left(\left|S_{1}\right|, \ldots,\left|S_{r}\right|\right)}(\delta, k, n)\right\rfloor,
$$

and that $\left(S_{1}, \ldots, S_{r}\right)$ is good if it contains no saturated $r$-tuple. We say that $\mathcal{H}$ is good if every $\left(A_{1}, \ldots, A_{r}\right) \in \mathcal{H}$ is good.

Proposition 3.2. Suppose that $\mathcal{H}$ is a good collection of copies of $K_{a_{1}, \ldots, a_{r}}^{(r)}$ in $G$ of size $|\mathcal{H}| \leq \delta^{a_{1}+\ldots+a_{r}} k^{a_{1} \cdots a_{r}} n^{a_{1}+\ldots+a_{r-1}}$. Then, there exists a copy $\left(A_{1}, \ldots, A_{r}\right) \notin \mathcal{H}$ of $K_{a_{1}, \ldots, a_{r}}^{(r)}$ in $G$ such that $\mathcal{H} \cup\left\{\left(A_{1}, \ldots, A_{r}\right)\right\}$ is good.

Proof. Let $\mathcal{F}$ denote the collection of saturated sets, i.e.

$$
\mathcal{F}=\left\{\left(S_{1}, \ldots, S_{r}\right): \emptyset \neq S_{1}, \ldots, S_{r} \subset V(G) \text { and } d_{\mathcal{H}}\left(S_{1}, \ldots, S_{r}\right)=\left\lfloor D^{\left(\left|S_{1}\right|, \ldots,\left|S_{r}\right|\right)}(\delta, k, n)\right\rfloor\right\} .
$$

A simple double-counting argument shows that there are at most

$$
\frac{a_{1} \cdots a_{r} \cdot|\mathcal{H}|}{\left\lfloor D^{(1, \ldots, 1)}(\delta, k, n)\right\rfloor} \ll k n^{r-1 / a_{1} \cdots a_{r-1}}=e(G)
$$

saturated edges of $G$. Here we use the assumption that $|\mathcal{H}| \leq \delta^{a_{1}+\ldots+a_{r}} k^{a_{1} \cdots a_{r}} n^{a_{1}+\ldots+a_{r-1}}$ and (3.2). Thus by choosing a non-empty subhypergraph of $G$ if necessary (and weakening the bound on $e(G)$ slightly), we may assume that

$$
\begin{equation*}
\left(\left\{v_{1}\right\}, \ldots,\left\{v_{r}\right\}\right) \notin \mathcal{F} \text { for every }\left\{v_{1}, \ldots, v_{r}\right\} \in E(G) . \tag{3.3}
\end{equation*}
$$

For $S_{1}, \ldots, S_{r} \subset V(G)$ and $i \in[r]$, define $X_{i}\left(S_{1}, \ldots, S_{r}\right)$ to be the set consisting of all vertices $v \in V(G) \backslash\left(S_{1} \cup \ldots \cup S_{r}\right)$ so that $\left(S_{1}^{\prime}, \ldots, S_{i-1}^{\prime}, S_{i}^{\prime} \cup\{v\}, S_{i+1}^{\prime}, \ldots, S_{r}^{\prime}\right) \in \mathcal{F}$ for some non-empty $S_{1}^{\prime} \subset S_{1}, \ldots, S_{r}^{\prime} \subset S_{r}$.
Claim 5. Provided that $\left|S_{i}\right| \leq a_{i}$ for each $i \in[r]$, we have

$$
\left|X_{i}\left(S_{1}, \ldots, S_{r}\right)\right| \leq K \delta k^{a_{1} \cdots a_{i-1}} n^{1-1 / a_{i} \cdots a_{r-1}} .
$$

Proof. For each tuple $\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)$ with $\emptyset \neq S_{i}^{\prime} \subset S_{i}$ for every $i \in[r]$, set

$$
\mathcal{J}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)=\left\{v \in V(G) \backslash\left(S_{1} \cup \ldots \cup S_{r}\right):\left(S_{1}^{\prime}, \ldots, S_{i}^{\prime} \cup\{v\}, \ldots, S_{r}^{\prime}\right) \in \mathcal{F}\right\}
$$

By the handshaking lemma and the definition of goodness, we obtain

$$
\frac{1}{a_{1} \cdots a_{r}} \cdot \sum_{v \in \mathcal{J}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)} d_{\mathcal{H}}\left(S_{1}^{\prime}, \ldots, S_{i}^{\prime} \cup\{v\}, \ldots, S_{r}^{\prime}\right) \leq d_{\mathcal{H}}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right) \leq D^{\left(\left|S_{1}^{\prime}\right|, \ldots,\left|S_{r}^{\prime}\right|\right)}(\delta, k, n),
$$

as each edge of $\mathcal{H}$ is counted at most $a_{1} \cdots a_{r}$ times in the sum. Moreover,

$$
\sum_{v \in \mathcal{J}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)} d_{\mathcal{H}}\left(S_{1}^{\prime}, \ldots, S_{i}^{\prime} \cup\{v\}, \ldots, S_{r}^{\prime}\right) \geq\left|\mathcal{J}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)\right| \cdot\left\lfloor D^{\left(\left|S_{1}^{\prime}\right|, \ldots,\left|S_{i}^{\prime}\right|+1, \ldots,\left|S_{r}^{\prime}\right|\right)}(\delta, k, n)\right\rfloor,
$$

by the definition of $\mathcal{J}$ and $\mathcal{F}$. Hence $\left|\mathcal{J}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)\right|$ is bounded from above by

$$
\left(a_{1} \cdots a_{r}\right) \cdot \frac{D^{\left(\left|S_{1}^{\prime}\right|, \ldots,\left|S_{r}^{\prime}\right|\right)}(\delta, k, n)}{\left\lfloor D^{\left(\left|S_{1}^{\prime}\right|, \ldots,\left|S_{i}^{\prime}\right|+1, \ldots,\left|S_{r}^{\prime}\right|\right\rangle}(\delta, k, n)\right\rfloor} \leq\left(2 a_{1} \cdots a_{r}\right) \cdot \delta k^{a_{1} \cdots a_{i-1}} n^{1-1 / a_{i} \cdots a_{r-1}} .
$$

Finally, since the sets $\mathcal{J}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)$ cover $X_{i}\left(S_{1}, \ldots, S_{r}\right)$, we find that

$$
\left|X_{i}\left(S_{1}, \ldots, S_{r}\right)\right| \leq \sum\left|\mathcal{J}\left(S_{1}^{\prime}, \ldots, S_{r}^{\prime}\right)\right| \leq a_{1} \cdots a_{r} 2^{\left|S_{1}\right|+\ldots+\left|S_{r}\right|+1} \cdot \delta k^{a_{1} \cdots a_{i-1}} n^{1-1 / a_{i} \cdots a_{r-1}}
$$

as desired.
We now show that there are at least $\delta k^{a_{1} \cdots a_{r}} n^{a_{1}+\cdots+a_{r-1}}$ good $r$-tuples $\left(A_{1}, \ldots, A_{r}\right)$ with $\left|A_{1}\right|=a_{1}, \ldots,\left|A_{r}\right|=a_{r}$. From this, the proposition follows immediately, since at least one of these is not in $\mathcal{H}$.

Claim 6. There are at least $\varepsilon(r+1) k^{a_{1} \cdots a_{r}} n^{a_{1}+\cdots+a_{r-1}}$ good $r$-tuples $\left(A_{1}, \ldots, A_{r}\right)$ with $\left|A_{1}\right|=$ $a_{1}, \ldots,\left|A_{r}\right|=a_{r}$.

Proof. Let $i \in[r+1]$, and let $v_{i}, v_{i+1}, \ldots, v_{r}$ be $r+1-i$ vertices of $G$ such that

$$
d_{G}\left(v_{i}, \ldots, v_{r}\right) \geq 2^{i-r-1} \cdot k n^{i-1-1 / a_{1} \cdots a_{r-1}} .
$$

We prove by induction on $i$ that there are at least

$$
\varepsilon(i) n^{a_{1}+\ldots+a_{i-1}-(i-1) a_{1} \cdots a_{i-1}} \cdot d_{G}\left(v_{i}, \ldots, v_{r}\right)^{a_{1} \cdots a_{i-1}}
$$

good $r$-tuples $\left(A_{1}, \ldots, A_{i-1},\left\{v_{i}\right\}, \ldots,\left\{v_{r}\right\}\right)$ with $\left|A_{1}\right|=a_{1}, \ldots,\left|A_{i-1}\right|=a_{i-1}$. Here we set $a_{1}+\ldots+a_{i-1}:=0$ when $i=1$, and $d_{G}\left(v_{i}, \ldots, v_{r}\right):=k n^{r-1 / a_{1} \cdots a_{r-1}}$ if $i=r+1$. It is easy to see that Claim 6 follows from the case $i=r+1$.

The base case $i=1$ is an immediate consequence of (3.3). Suppose, then, that the result holds for some $i \in[r+1]$. Fix $v_{i+1}, \ldots, v_{r} \in V(G)$ with

$$
\begin{equation*}
d_{G}\left(v_{i+1}, \ldots, v_{r}\right) \geq 2^{i-r} \cdot k n^{i-1 / a_{1} \cdots a_{r-1}} . \tag{3.4}
\end{equation*}
$$

Let $\mathcal{M}$ denote the collection consisting of all $i$-tuples $\left(A_{1}, \ldots, A_{i-1},\{v\}\right)$ with $v \in V(G)$, $\left|A_{1}\right|=a_{1}, \ldots,\left|A_{i-1}\right|=a_{i-1}$, and such that $\left(A_{1}, \ldots, A_{i-1},\{v\},\left\{v_{i+1}\right\}, \ldots,\left\{v_{r}\right\}\right)$ is good.
Subclaim 1: $|\mathcal{M}| \geq 2^{-a_{1} \cdots a_{i-1}} \varepsilon(i) \cdot n^{1+a_{1}+\ldots+a_{i-1}-i a_{1} \cdots a_{i-1}} \cdot d_{G}\left(v_{i+1}, \ldots, v_{r}\right)^{a_{1} \cdots a_{i-1}}$.
Proof of Subclaim 1. Set $X=\left\{v \in V(G): d_{G}\left(v, v_{i+1}, \ldots, v_{r}\right) \geq \frac{1}{2 n} \cdot d_{G}\left(v_{i+1}, \ldots, v_{r}\right)\right\}$. Then

$$
\sum_{v \in X} d_{G}\left(v, v_{i+1}, \ldots, v_{r}\right) \geq \frac{1}{2} d_{G}\left(v_{i+1}, \ldots, v_{r}\right) .
$$

Fix a vertex $v \in X$. It follows from the definition of $X$ and the assumption (3.4) that $d_{G}\left(v, v_{i+1}, \ldots, v_{r}\right) \geq 2^{i-r-1} \cdot k n^{i-1-1 / a_{1} \cdots a_{r-1}}$. Hence, by the induction hypothesis, $\mathcal{M}$ contains at least $\varepsilon(i) n^{a_{1}+\ldots+a_{i-1}-(i-1) a_{1} \cdots a_{i-1}} d_{G}\left(v, v_{i+1}, \ldots, v_{r}\right)^{a_{1} \cdots a_{i-1}} i$-tuples of the form $\left(A_{1}, \ldots, A_{i-1},\{v\}\right)$.
Summing over all $v \in X$, and using Jensen's inequality give

$$
\begin{aligned}
|\mathcal{M}| & \geq \varepsilon(i) n^{a_{1}+\ldots+a_{i-1}-(i-1) a_{1} \cdots a_{i-1}} \sum_{v \in X} d_{G}\left(v, v_{i+1}, \ldots, v_{r}\right)^{a_{1} \cdots a_{i-1}} \\
& \geq \varepsilon(i) n^{a_{1}+\ldots+a_{i-1}-(i-1) a_{1} \cdots a_{i-1}} \cdot|X|^{1-a_{1} \cdots a_{i-1}}\left(\sum_{v \in X} d_{G}\left(v_{i+1}, \ldots, v_{r}\right)\right)^{a_{1} \cdots a_{i-1}} \\
& \geq 2^{-a_{1} \cdots a_{i-1}} \varepsilon(i) \cdot n^{1+a_{1}+\ldots+a_{i-1}-i a_{1} \cdots a_{i-1}} \cdot d_{G}\left(v_{i+1}, \ldots, v_{r}\right)^{a_{1} \cdots a_{i-1}} .
\end{aligned}
$$

We now use Subclaim 1 to bound the number of good tuples $\left(A_{1}, \ldots, A_{i},\left\{v_{i+1}\right\}, \ldots,\left\{v_{r}\right\}\right)$ with $\left|A_{1}\right|=a_{1}, \ldots,\left|A_{i}\right|=a_{i}$. For each $(i-1)$-tuple $\mathbf{A}=\left(A_{1}, \ldots, A_{i-1}\right)$, define

$$
\mathcal{M}(\mathbf{A}):=\{v \in V(G):(\mathbf{A},\{v\}) \in \mathcal{M}\} .
$$

Consider $(i-1)$-tuples $\mathbf{A}$ for which

$$
\begin{equation*}
|\mathcal{M}(\mathbf{A})| \geq \frac{1}{2} n^{-\left(a_{1}+\ldots+a_{i-1}\right)}|\mathcal{M}| \tag{3.5}
\end{equation*}
$$

Subclaim 2: Suppose $\mathbf{A}$ satisfies (3.5). Then there are $\frac{1}{2^{a_{i} a_{i}!}}|\mathcal{M}(\mathbf{A})|^{a_{i}}$ sets $A_{i} \in\binom{\mathcal{M}(\mathbf{A})}{a_{i}}$ so that $\left(\mathbf{A}, A_{i},\left\{v_{i+1}\right\}, \ldots,\left\{v_{r}\right\}\right)$ is good.

Proof of Subclaim 2. From (3.5) and Subclaim 1, we see that

$$
\begin{equation*}
|\mathcal{M}(\mathbf{A})| \geq 2^{(i-r-1) a_{1} \cdots a_{i-1}-1} \varepsilon(i) \cdot k^{a_{1} \cdots a_{i-1}} n^{1-1 / a_{i} \cdots a_{r-1}} \tag{3.6}
\end{equation*}
$$

For $j=1, \ldots, a_{i}$, we can pick an arbitrary vertex

$$
u_{j} \in \mathcal{M}(\mathbf{A}) \backslash\left(\left\{u_{1}, \ldots, u_{i-1}\right\} \cup X_{i}\left(\mathbf{A},\left\{u_{1}, \ldots, u_{j-1}\right\},\left\{v_{i}\right\}, \ldots,\left\{v_{r}\right\}\right)\right)
$$

and let $A_{i}=\left\{u_{1}, \ldots, u_{a_{i}}\right\}$. By choice of $u_{j}$, the tuple $\left(A,\left\{u_{1}, \ldots, u_{j}\right\},\left\{v_{i+1}\right\}, \ldots,\left\{v_{r}\right\}\right)$ is good for every $j \in\left[a_{i}\right]$, and hence the $r$-tuple $\left(\mathbf{A}, A_{i},\left\{v_{i+1}\right\}, \ldots,\left\{v_{r}\right\}\right)$ is good. From Claim 5, we deduce that the number of choices for each $u_{j}$ is at least

$$
|\mathcal{M}(\mathbf{A})|-\left(a_{i}+K \delta k^{a_{1} \cdots a_{i-1}} n^{1-1 / a_{i} \cdots a_{r-1}}\right) \stackrel{(3.6)}{\geq}|\mathcal{M}(\mathbf{A})| / 2
$$

Thus the total number of choices for $A_{i}$ is at least $\frac{1}{2^{a_{i} a_{i}!}!}|\mathcal{M}(\mathbf{A})|^{a_{i}}$.
Finally, observe that

$$
\sum_{\text {A satisfies }}|\mathcal{M}(\mathbf{A})| \geq|\mathcal{M}| / 2
$$

and that there are at most $n^{a_{1}+\ldots+a_{i-1}}$ choices for $\mathbf{A}=\left(A_{1}, \ldots, A_{i-1}\right)$. Hence, by Subclaim 2 and convexity, the number of good $r$-tuples $\left(\mathbf{A}, A_{i},\left\{v_{i+1}\right\}, \ldots,\left\{v_{r}\right\}\right)$ is at least

$$
\begin{aligned}
\frac{1}{2^{a_{i}} a_{i}!} \sum_{\text {A satisfies (3.5) }}|\mathcal{M}(\mathbf{A})|^{a_{i}} & \geq \frac{1}{2^{a_{i}} a_{i}!} \cdot\left(n^{a_{1}+\ldots+a_{i-1}}\right)^{1-a_{i}}(|\mathcal{M}| / 2)^{a_{i}} \\
& \text { Subclaim } 1_{\geq} \varepsilon(i+1) n^{a_{1}+\ldots+a_{i}-i a_{1} \cdots a_{i}} \cdot d_{G}\left(v_{i+1}, \ldots, v_{r}\right)^{a_{1} \cdots a_{i}}
\end{aligned}
$$

completing the proof of Claim 6.
This finishes our proof of Proposition 3.2.

## 4. The Turán problem in random hypergraphs

Balanced supersaturation theorems can be used to obtain results for the corresponding random hypergraph Turán problem. This was done by Morris and Saxton in [23] for even cycles and complete bipartite graphs. For theta graphs and complete $r$-partite $r$-graphs, certain difficulties arise which we shall explain in this section.

Let $G^{(r)}(n, p)$ denote the random $r$-graph on $n$ vertices where each edge is present independently with probability $p$. For some fixed $r$-graph $H$, we denote by $\operatorname{ex}_{r}\left(G^{(r)}(n, p), H\right)$ the maximum number of edges of an $H$-free subgraph of $G^{(r)}(n, p)$. If $r=2$, we will drop the subscripts and superscripts. The following result provides lower bounds for all $p$.

## Proposition 4.1.

(a) Suppose that $\operatorname{ex}\left(n,\left\{C_{3}, C_{4}, \ldots, C_{2 b}\right\}\right)=\Theta\left(n^{1+1 / b}\right)$. Then there is some positive constant $c=c(b)$ such that w.h.p. $\operatorname{ex}\left(G(n, p), C_{2 b}\right) \geq c p^{1 / b} n^{1+1 / b}$. In particular, w.h.p. we have $\operatorname{ex}\left(G(n, p), \theta_{a, b}\right) \geq c p^{1 / b} n^{1+1 / b}$ for every $a \geq 2$.
(b) Let $r \geq 2$ and $2 \leq a_{1} \leq \ldots \leq a_{r}$, and suppose that $\operatorname{ex}_{r}\left(n, K_{a_{1}, \ldots, a_{r}}^{(r)}\right)=\Theta\left(n^{r-1 / a_{1} \cdots a_{r-1}}\right)$. Then, there is some positive constant $c=c\left(a_{1}, \ldots, a_{r}\right)$ such that w.h.p.

$$
\operatorname{ex}_{r}\left(G^{(r)}(n, p), K_{a_{1}, \ldots, a_{r}}^{(r)}\right) \geq c p^{1-1 /\left(a_{1} \cdots a_{r-1}\right)} n^{r-1 / a_{1} \cdots a_{r-1}}
$$

Morris and Saxton established Part (a) in [23, Section 2.3], while Part (b) can be obtained by adapting their construction. In order to get good upper bound on the Turán number $\operatorname{ex}_{r}\left(G^{(r)}(n, p), H\right)$, it is necessary to find the largest $\alpha>0$ for which $H$ is Erdős-Simonovits $\alpha$-good. To do so, one usually has to restrict the range of $k$ in Definition 1.5. Say that $H$ is Erdős-Simonovits $\alpha$-good for $m(n)$ up to $f(n)$, if the condition in Definition 1.5 holds for every $1 \ll k \leq f(n)$. Considering integers $a, b \geq 2$, one can easily deduce from Theorem 2.2 that $\theta_{a, b}$ is $\alpha$-Erdős-Simonovits good for $m(n)=n^{1+1 / b}$ and $\alpha=1 /((a-1) b-1)$ up to $n^{(b-1)((a-1) b-1) / b(a b-1)}$. A standard application of the hypergraph container method (see [23, Section 6]) then shows that, for every $p \geq n^{-((a-1)(b-1) /(a b-1))}(\log n)^{2(a-1) b}$, we have w.h.p. $\operatorname{ex}\left(G(n, p), \theta_{a, b}\right)=O\left(p^{1 /(a-1) b} n^{1+1 / b}\right)$. This matches the lower bound from Proposition 4.1 (a) only when $a=2$. Note that, in the case $a=2$, this recovers a result of Morris and Saxton [23, Theorem 1.8].

For complete $r$-partite $r$-graphs, the situation is very different. It is likely that Theorem 3.1 is best possible and provides a matching upper bound for Proposition 4.1 (b) for large enough $p$. However, for $r>2$, finding the largest $\alpha$ so that $K_{a_{1}, \ldots, a_{r}}^{(r)}$ is $\alpha$-Erdős-Simonovits good up to some $f(n)$ from Theorem 3.1 turns into a difficult optimisation problem we could not solve. Remark. Recently, we have learned that Spiro and Verstraëte [29, Theorem 1.3] used our Theorem 3.1 to essentially determine $\operatorname{ex}_{r}\left(G^{(r)}(n, p), K_{a_{1}, \ldots, a_{r}}^{(r)}\right)$ in the regime $p \geq n^{-r / 2} \log n$.

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## References

[1] N. Alon, L. Rónyai, and T. Szabó, Norm-graphs: variations and applications, J. Combin. Theory Ser. B 76 (1999), no. 2, 280-290.
[2] J. Balogh, B. Bollobás, and M. Simonovits, The fine structure of octahedron-free graphs, J. Combin. Theory Ser. B 101 (2011), no. 2, 67-84.
[3] J. Balogh, R. Morris, and W. Samotij, Independent sets in hypergraphs, J. Amer. Math. Soc. 28 (2015), no. 3, 669-709.
[4] J. Balogh, B. Narayanan, and J. Skokan, The number of hypergraphs without linear cycles, J. Combin. Theory Ser. B 134 (2019), 309-321.
[5] J. Balogh and W. Samotij, The number of $K_{m, m}$-free graphs, Combinatorica 31 (2011), no. 2, 131-150.
[6] $\qquad$ , The number of $K_{s, t}$-free graphs, J. Lond. Math. Soc. (2) 83 (2011), no. 2, 368-388.
[7] C. T. Benson, Minimal regular graphs of girths eight and twelve, Canad. J. Math. 18 (1966), 1091-1094.
[8] J. A. Bondy and M. Simonovits, Cycles of even length in graphs, J. Combin. Theory Ser. B 16 (1974), 97-105.
[9] D. Conlon, Graphs with few paths of prescribed length between any two vertices, Bull. Lond. Math. Soc. 51 (2019), 1015-1021.
[10] P. Erdős, On extremal problems of graphs and generalized graphs, Israel J. Math. 2 (1964), 183-190.
[11] P. Erdős, P. Frankl, and V. Rödl, The asymptotic number of graphs not containing a fixed subgraph and a problem for hypergraphs having no exponent, Graphs Combin. 2 (1986), no. 1, 113-121.
[12] P. Erdős and M. Simonovits, A limit theorem in graph theory, Studia Sci. Math. Hung, Citeseer. 1 (1966), 51-57.
[13] _, Supersaturated graphs and hypergraphs, Combinatorica 3 (1983), no. 2, 181-192.
[14] , Cube-supersaturated graphs and related problems, Progress in graph theory (Waterloo, Ont., 1982), Academic Press, Toronto, ON, 1984, pp. 203-218.
[15] P. Erdős and A. H. Stone, On the structure of linear graphs, Bull. Amer. Math. Soc. 52 (1946), no. 12, 1087-1092.
[16] R. J. Faudree and M. Simonovits, On a class of degenerate extremal graph problems, Combinatorica 3 (1983), no. 1, 83-93.
[17] A. Ferber, G. McKinley, and W. Samotij, Supersaturated sparse graphs and hypergraphs, Int. Math. Res. Not. 2 (2020), 378-402.
[18] T. Jiang and L. Yepremyan, Supersaturation of even linear cycles in linear hypergraphs, Comb. Probab. Comput. 29 (2020), 698-721.
[19] D. J. Kleitman and K. J. Winston, On the number of graphs without 4-cycles, Discrete Math. 41 (1982), no. 2, 167-172.
[20] J. Kollár, L. Rónyai, and T. Szabó, Norm-graphs and bipartite Turán numbers, Combinatorica 16 (1996), no. 3, 399-406.
[21] T. Kövari, V. T. Sós, and P. Turán, On a problem of K. Zarankiewicz, Colloquium Math. 3 (1954), 50-57.
[22] J. Ma, X. Yuan, and M. Zhang, Some extremal results on complete degenerate hypergraphs, J. Combin. Theory Ser. A 154 (2018), 598-609.
[23] R. Morris and D. Saxton, The number of $C_{2 l}$-free graphs, Adv. Math. 298 (2016), 534-580.
[24] D. Mubayi and L. Wang, The number of triple systems without even cycles, Combinatorica 39 (2019), 679-704.
[25] B. Nagle, V. Rödl, and M. Schacht, Extremal hypergraph problems and the regularity method, special edition for J. Nešetril, Topics in Discrete Mathematics (M. Klazar, J. Kratochvíl, M. Loebl, J. Matoušek, R. Thomas, and P. Valtr, eds.), Algorithms Combin., vol. 26, Springer, 2006, pp. 247-278.
[26] D. Saxton and A. Thomason, Hypergraph containers, Invent. Math. 201 (2015), no. 3, 925-992.
[27] A. Sidorenko, A correlation inequality for bipartite graphs, Graphs and Combinatorics 9 (1993), 201-204.
[28] R. Singleton, On minimal graphs of maximum even girth, J. Combin. Theory 1 (1966), 306-332.
[29] S. Spiro and J. Verstraëte, Relative Turán Problems for Uniform Hypergraphs, submitted. arXiv preprint arXiv: 2009.02416.
[30] R. Wenger, Extremal graphs with no $C^{4}$ 's, $C^{6}$ 's, or $C^{10}$ 's, J. Combin. Theory Ser. B 52 (1991), no. 1, 113-116.

## A. Proof of Proposition 1.6

The following proof is very similar to that of comparable statements given in [23, 4]. We will make use of the hypergraph container method, developed in [3, 26]. For an $s$-uniform hypergraph $\mathcal{H}$, we define the maximum $j$-degree $\Delta_{j}(\mathcal{H})$ of $\mathcal{H}$ by

$$
\Delta_{j}(\mathcal{H})=\max \left\{d_{\mathcal{H}}(\sigma): \sigma \subset V(\mathcal{H}) \text { and }|\sigma|=j\right\}
$$

and the average degree $d(\mathcal{H})$ of $\mathcal{H}$ by $d(\mathcal{H})=s|E(\mathcal{H})| /|V(\mathcal{H})|$. Furthermore, for $\tau \in(0,1)$, the co-degree function $\delta(\mathcal{H}, \tau)$ of $\mathcal{H}$ is given by

$$
\delta(\mathcal{H}, \tau)=\frac{1}{d(\mathcal{H})} \sum_{j=2}^{s} \frac{\Delta_{j}(\mathcal{H})}{\tau^{j-1}}
$$

Theorem A. 1 (see $[3,26])$. For each $s \in \mathbb{N}$, there exist positive constants $c_{1}=c_{1}(s)$ and $c_{2}=c_{2}(s)$ such that the following holds for all $N \in \mathbb{N}$. For each $0<\varepsilon<c_{1}$ and each $N$-vertex s-graph $\mathcal{H}$, if $\tau \in\left(0, c_{2}\right)$ is such that $\delta(\mathcal{H}, \tau) \leq \varepsilon$, then there exists a family $\mathcal{C}$ of at most

$$
\begin{equation*}
\exp \left(\frac{\tau \log (1 / \tau) N}{\varepsilon}\right) \tag{A.1}
\end{equation*}
$$

subsets of $V(\mathcal{H})$ such that:
(1) for each independent set $I \subset V(\mathcal{H})$, there exists some $U \in \mathcal{C}$ with $I \subset U$,
(2) $e(\mathcal{H}[U]) \leq \varepsilon e(\mathcal{H})$ for each container $U \in \mathcal{C}$.

We shall establish Proposition 1.6 through iterated applications of the following consequence of Theorem A.1.

Proposition A.2. Suppose that an r-graph $H$ is Erdős-Simonovits $\alpha$-good for $m=m(n)$. Then there exist positive constants $\varepsilon$ and $k_{0}$ such that the following holds for all $n, k \in \mathbb{N}$ with $k \geq k_{0}$. Given an $r$-graph $G$ on $[n]$ with $e(G)=k \cdot m(n)$, there exists a collection $\mathcal{C}(G)$ of at most

$$
\exp \left(O\left(k^{-\alpha} \log k \cdot m(n)\right)\right)
$$

subgraphs of $G$ satisfying:
(1) Every $H$-free subgraph of $G$ is a subgraph of some $U \in \mathcal{C}$,
(2) $e(U) \leq(1-\varepsilon) e(G)$ for every $U \in \mathcal{C}$.

Proof. Since $H$ is Erdős-Simonovits $\alpha$-good for $m=m(n)$, there exists a constant $C>0$ and a (non-empty) collection $\mathcal{H}$ of copies of $H$ in $G$ such that

$$
\begin{equation*}
d_{\mathcal{H}}(\sigma) \leq \frac{C \cdot|\mathcal{H}|}{k^{(1+\alpha)(|\sigma|-1)} e(G)} \quad \text { for every } \sigma \subset E(G) \text { with } 1 \leq|\sigma| \leq e(H) \tag{A.2}
\end{equation*}
$$

We will now think of $\mathcal{H}$ as a hypergraph whose vertex set is $E(G)$ and whose edges are the copies of $H$ in $\mathcal{H}$. Set $1 / \tau=\varepsilon^{2} k^{1+\alpha}$ and observe that, if $\varepsilon$ is sufficiently small,

$$
\begin{aligned}
& \delta(\mathcal{H}, \tau)=\frac{1}{d(\mathcal{H})} \sum_{j=2}^{e(H)} \frac{\Delta_{j}(\mathcal{H})}{\tau^{j-1}} \stackrel{(\mathrm{~A} .2)}{\leq} \frac{1}{d(\mathcal{H})} \sum_{j=2}^{e(H)} \varepsilon^{2(j-1)} k^{(1+\alpha)(j-1)} \cdot \frac{C \cdot|\mathcal{H}|}{k^{(1+\alpha)(j-1)} e(G)} \\
&=\frac{C}{e(H)} \sum_{j=2}^{e(H)} \varepsilon^{2(j-1)} \leq 2 C \varepsilon^{2} / e(H) \leq \varepsilon
\end{aligned}
$$

Using Theorem A.1, we thus obtain a collection $\mathcal{C}(G)$ of at most

$$
\exp \left(\frac{\tau \log (1 / \tau) e(G)}{\varepsilon}\right) \leq \exp \left(O\left(k^{-\alpha} \log k \cdot m(n)\right)\right)
$$

subsets of $V(\mathcal{H})=E(G)$ such that:
(1') Every $H$-free subgraph of $G$ is a subgraph of some $U \in \mathcal{C}(G)$, and
(2') $e(\mathcal{H}[U]) \leq \varepsilon e(\mathcal{H})$ for all $U \in \mathcal{C}(G)$.
The only thing that remains to prove is that $e(U) \leq(1-\varepsilon) e(G)$ for every $U \in \mathcal{C}$. Consider an arbitrary container $U \in \mathcal{C}$. From (A.2) we find

$$
|E(\mathcal{H}) \backslash E(\mathcal{H}[U])| \leq|V(\mathcal{H}) \backslash U| \cdot \frac{C \cdot e(\mathcal{H})}{e(G)} .
$$

On the other hand, it follows from condition (2') that $|E(\mathcal{H}) \backslash E(\mathcal{H}[U])| \geq(1-\varepsilon) e(\mathcal{H})$. Hence $|V(\mathcal{H}) \backslash U| \geq(1-\varepsilon) e(G) / C \geq \varepsilon e(G)$, as desired.

We are now ready to prove Proposition 1.6.
Proof of Proposition 1.6. We wish to estimate the number of $H$-free subgraphs of $K_{n}^{(r)}$. We define a sequence $\{k(i)\}_{i=1}^{t}$ of positive reals, and a sequence $\left\{\mathcal{C}_{i}\right\}_{i=1}^{t}$ of families of $r$-graphs as follows. Let $\varepsilon$ and $k_{0}$ be positive constants given by Proposition A.2. We set $k(1)=\binom{n}{r} / m(n)$ and define $k(i)=(1-\varepsilon) k(i-1)$, with $k(t)$ being the first term of this sequence to satisfy $k(t) \leq k_{0}$. We take $\mathcal{C}_{0}=\left\{K_{n}^{(r)}\right\}$, and for $1 \leq i \leq t$, we obtain $\mathcal{F}_{i}$ from $\mathcal{C}_{i-1}$ by replacing each $r$-graph $G \in \mathcal{C}_{i-1}$ for which $e(G) \geq k(i) \cdot m(n)$ by the collection $\mathcal{C}(G)$ of its subgraphs guaranteed by Proposition A.2.
Let $\mathcal{C}=\mathcal{C}_{t}$. Clearly, every $H$-free $r$-graph on $[n]$ is contained in some $G \in \mathcal{C}$. Moreover, $e(G) \leq k_{0} \cdot m(n)$ for every $G \in \mathcal{C}$. Finally, from (A.1) we see that

$$
|\mathcal{F}| \leq \exp \left(\sum_{i=1}^{t} O(1) \cdot k(i)^{-\alpha} \log k(i) \cdot m(n)\right) \leq \exp \left(O(1) \cdot k_{0}^{-\alpha} \log k_{0} \cdot m(n)\right) .
$$

Therefore, the number of $H$-free $r$-graphs on $[n]$ is at most

$$
\sum_{G \in \mathcal{C}} 2^{e(G)} \leq|\mathcal{C}| 2^{k \cdot \cdot m(n)} \leq \exp \left(O(1) \cdot k_{0}^{-\alpha} \log k_{0} \cdot m(n)+k_{0} \cdot m(n)\right)=2^{O(m(n))}
$$

Finally, given any $\delta>0$, the number of $H$-free $r$-graphs with $n$ vertices and less than $m(n) / k_{0}^{3}$ edges is bounded from above by

$$
\sum_{G \in \mathcal{C}} \sum_{i=1}^{m(n) / k_{0}^{3}}\binom{e(G)}{i} \leq|\mathcal{C}| 2^{m(n) / k_{0}} \leq \exp \left(k_{0}^{-\alpha / 2} m(n)+m(n) / k_{0}\right) \leq 2^{\delta \cdot m(n)}
$$

if $k_{0}$ is large enough.

## B. Deriving Theorem 1.8 from Theorem 3.1

In this section we will deduce Theorem 1.8 from Theorem 3.1. Obviously, property (i) in Theorem 1.8 follows from property (a) in Theorem 3.1. It remains to verify property (ii) in Theorem 3.1. Without loss of generality we can assume that $\sigma$ is a complete $r$-partite $r$-graph. In light of property (b), our task becomes to justify that

$$
\begin{equation*}
D^{\left(b_{1}, \ldots, b_{r}\right)}(\delta, k, n) \leq \frac{C \cdot|\mathcal{H}|}{k^{(1+\alpha)\left(b_{1} \cdots b_{r}-1\right)} e(G)} \tag{B.1}
\end{equation*}
$$

for every $\left(b_{1}, \ldots, b_{r}\right) \in \mathbb{N}^{r}$ satisfying $b_{1} \leq a_{1}, \ldots, b_{r} \leq a_{r}$. From (3.1), we see that

$$
D^{\left(b_{1}, \ldots, b_{r}\right)}(\delta, k, n)=O(1) \cdot k^{\beta} n^{\gamma}
$$

where $\beta=\sum_{i=1}^{r}\left(a_{i}-b_{i}\right) a_{1} \cdots a_{i-1}$ and $\gamma=\sum_{i=1}^{r-1}\left(a_{i}-b_{i}\right)\left(1-1 / a_{i} \cdots a_{r-1}\right)$. On the other hand, by the assumptions on $G$ and $\mathcal{H}$, we obtain

$$
\frac{|\mathcal{H}|}{k^{(1+\alpha)\left(b_{1} \cdots b_{r}-1\right)} e(G)}=\Omega(1) \cdot k^{\beta^{\prime}} n^{\gamma^{\prime}}
$$

where $\beta^{\prime}=a_{1} \cdots a_{r}-(1+\alpha)\left(b_{1} \cdots b_{r}-1\right)-1$ and $\gamma^{\prime}=a_{1}+\ldots+a_{r-1}-r+1 / a_{1} \cdots a_{r-1}$. As $\gamma^{\prime}=\sum_{i=1}^{r-1}\left(a_{i}-1\right)\left(1-1 / a_{i} \cdots a_{r-1}\right)$, we get

$$
\begin{equation*}
\gamma^{\prime}-\gamma=\sum_{i=1}^{r-1}\left(b_{i}-1\right)\left(1-1 / a_{i} \cdots a_{r-1}\right) \geq 0 \tag{B.2}
\end{equation*}
$$

We next show that

$$
\begin{equation*}
\left(\beta^{\prime}-\beta\right)+a_{1} \cdots a_{r-1} \cdot\left(\gamma^{\prime}-\gamma\right) \geq 0 \tag{B.3}
\end{equation*}
$$

From (B.2), (B.3) and the fact that $n \geq k^{a_{1} \cdots a_{r-1}}$, we find

$$
k^{\beta^{\prime}} n^{\gamma^{\prime}}=k^{\beta} n^{\gamma} \cdot\left(k^{-a_{1} \cdots a_{r-1}} n\right)^{\gamma^{\prime}-\gamma} \cdot k^{\left(\beta^{\prime}-\beta\right)+a_{1} \cdots a_{r-1} \cdot\left(\gamma^{\prime}-\gamma\right)} \geq k^{\beta} n^{\gamma},
$$

implying (B.1) for $C$ sufficiently large.
In the remainder of this section, we shall justify (B.3). Observe that

$$
\begin{aligned}
\beta^{\prime} & =\left(a_{1} \cdots a_{r}-1\right)-\left(b_{1} \cdots b_{r}-1\right)-\left(b_{1} \cdots b_{r}-1\right) \alpha \\
& =\sum_{i=1}^{r}\left(a_{i}-1\right) a_{1} \cdots a_{i-1}-\sum_{i=1}^{r}\left(b_{i}-1\right) b_{1} \cdots b_{i-1}-\left(b_{1} \cdots b_{r}-1\right) \alpha .
\end{aligned}
$$

From this it follows that

$$
\beta^{\prime}-\beta=\sum_{i=1}^{r}\left(b_{i}-1\right)\left(a_{1} \cdots a_{i-1}-b_{1} \cdots b_{i-1}\right)-\left(b_{1} \cdots b_{r}-1\right) \alpha .
$$

Combined with (B.2), this yields

$$
\begin{align*}
\left(\beta^{\prime}-\beta\right)+a_{1} \cdots a_{r-1} \cdot\left(\gamma^{\prime}-\gamma\right)= & \sum_{i=1}^{r}\left(b_{i}-1\right)\left(a_{1} \cdots a_{i-1}-b_{1} \cdots b_{i-1}\right)-\left(b_{1} \cdots b_{r}-1\right) \alpha \\
& +a_{1} \cdots a_{r-1} \cdot \sum_{i=1}^{r-1}\left(b_{i}-1\right)\left(1-1 / a_{i} \cdots a_{r-1}\right) \tag{B.4}
\end{align*}
$$

If $b_{1}=\ldots=b_{r-1}=1$, then the RHS of (B.4) is at least $\left(b_{r}-1\right)\left(a_{1} \cdots a_{r-1}-1\right)-\left(b_{r}-1\right) \alpha \geq 0$. So (B.3) is valid in this case.

Now suppose $\left(b_{1}, \ldots, b_{r-1}\right) \neq(1, \ldots, 1)$. Since $1 \leq b_{i} \leq a_{i}$ for every $i \in[r]$, we can bound the RHS of (B.4) from below by

$$
\begin{aligned}
-\left(b_{1} \cdots b_{r}-1\right) \alpha+a_{1} \cdots a_{r-1} \cdot \sum_{i=1}^{r-1}\left(b_{i}-1\right)\left(1-1 / a_{i} \cdots a_{r-1}\right) & \geq-\left(b_{1} \cdots b_{r}-1\right) \alpha+1 \\
& \geq 0,
\end{aligned}
$$

where in the second inequality we used the fact that $a_{1} \cdots a_{r-1} \cdot \sum_{i=1}^{r-1}\left(b_{i}-1\right)\left(1-1 / a_{i} \cdots a_{r-1}\right)$ is a positive integer when $\left(b_{1}, \ldots, b_{r-1}\right) \neq(1, \ldots, 1)$, and in the last inequality we estimated $\alpha=\frac{1}{a_{1} \cdots a_{r}-1} \leq \frac{1}{b_{1} \cdot b_{r}-1}$. Hence (B.3) is true in this case as well. This completes our proof.


[^0]:    *Department of Mathematics, LSE, London WC2A 2AE, Email: j.corsten@lse.ac.uk.
    ${ }^{\dagger}$ Discrete Mathematics Group, Institute for Basic Science (IBS), Daejeon, Republic of Korea. E-mail: tuantran@ibs.re.kr. Supported by the Institute for Basic Science (IBS-R029-Y1). Much of the work was done while supported by the Czech Science Foundation, grant number GJ16-07822Y, and with institutional support RVO:67985807, while affiliated with the Institute of Computer Science, Czech Academy of Sciences.

