3-degenerate induced subgraph of a planar graph

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Abstract

A graph G is d-degenerate if every non-null subgraph of G has a vertex of degree at most d. We prove that every n-vertex planar graph has a 3-degenerate induced subgraph of order at least $3n/4$.

Keywords: planar graph; graph degeneracy.

1 Introduction

Graphs in this paper are simple, having no loops and no parallel edges. For a graph $G = (V, E)$, the neighbourhood of $x \in V$ is denoted by $N(x) = N_G(x)$, the degree of x is denoted by $d(x) = d_G(x)$, and the minimum degree of G is denoted by $\delta(G)$. Let $\Pi = \Pi(G)$ be the set of total orderings of V. For $L \in \Pi$, we orient each edge $vw \in E$ as (v, w) if $w <_L v$ to form a directed graph G_L. We denote the *out-neighbourhood*, also called the *back-neighbourhood*, of x by $N^L_G(x)$, the *out-degree*, or *back-degree*, of x by $d^L_G(x)$. We write $\delta^+(G_L)$ and $\Delta^+(G_L)$ to denote the minimum out-degree and the

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maximum out-degree, respectively, of G_L. We define $|G| := |V|$, called the *order* of G, and $\|G\| := |E|$.

An ordering $L \in \Pi(G)$ is d*-degenerate* if $\Delta^+(G_L) \leq d$. A graph G is d-degenerate if some $L \in \Pi(G)$ is d-degenerate. The *degeneracy* of G is $\min_{L\in\Pi(G)} \Delta^+(G_L)$. It is well known that the degeneracy of G is equal to $\max_{H\subseteq G} \delta(H)$.

Alon, Kahn, and Seymour [4] initiated the study of maximum d-degenerate induced subgraphs in a general graph and proposed the problem on planar graphs. We study maximum d-degenerate induced subgraphs of planar graphs. For a non-negative integer d and a graph G, let

$$\alpha_d(G) = \max\{|S| : S \subseteq V(G),\ G[S] \text{ is } d\text{-degenerate}\} \text{ and}$$
$$\bar{\alpha}_d = \inf\{\alpha_d(G)/|V(G)| : G \text{ is a non-null planar graph}\}.$$

Let us review known bounds for $\bar{\alpha}_d$. Suppose that $G = (V, E)$ is a planar graph. For $d \geq 5$, trivially we have $\bar{\alpha}_d = 1$ because planar graphs are 5-degenerate.

For $d = 0$, a 0-degenerate graph has no edges and therefore $\alpha_0(G)$ is the size of a maximum independent set of G. By the Four Colour Theorem, G has an independent set I with $|I| \geq |V(G)|/4$. Both K_4 and C_8^2 witness that $\bar{\alpha}_0 \leq 1/4$, so $\bar{\alpha}_0 = 1/4$. In 1968, Erdős (see [5]) asked whether this bound could be proved without the Four Colour Theorem. This question still remains open. In 1976, Albertson [2] showed that $\bar{\alpha}_0 \geq 2/9$ independently of the Four Colour Theorem. This bound was improved to $\bar{\alpha}_0 \geq 3/13$ independently of the Four Colour Theorem by Cranston and Rabern in 2016 [8].

For $d = 1$, a 1-degenerate graph is a forest. Since K_4 has no induced forest of order greater than 2, we have $\bar{\alpha}_1 \leq 1/2$. Albertson and Berman [3] and Akiyama and Watanabe [1] independently conjectured that $\bar{\alpha}_1 = 1/2$. In other words, every planar graph has an induced forest containing at least half of its vertices. This conjecture received much attention in the past 40 years; however, it remains largely open. Borodin [7] proved that the vertex set of a planar graph can be partitioned into five classes such that the subgraph induced by the union of any two classes is a forest. Taking the two largest classes yields an induced forest of order at least $2|V(G)|/5$. So $\bar{\alpha}_1 \geq 2/5$. This remains the best known lower bound on $\bar{\alpha}_1$. On the other hand, the conjecture of Albertson and Berman, Akiyama and Watanabe was verified for some subfamilies of planar graphs. For example, C_3-free, C_5-free, or C_6-free planar graphs were shown in [20, 11] to be 3-degenerate, and a greedy algorithm shows that the vertex set of a 3-degenerate graph can be partitioned into two parts, each inducing a forest. Hence C_3-free, C_5-free, or C_6-free planar graphs satisfy the conjecture. Moreover, Raspaud and Wang [16] showed that C_4-free planar graphs can be partitioned into two induced forests, thus satisfying the conjecture. In fact, many of these graphs have larger induced forests. Le [14] showed that if a planar graph G is C_3-free, then it has an induced forest with at least $5|V(G)|/9$ vertices; Kelly and Liu [12] proved that if in addition G is C_4-free, then G has an induced forest with at least $2|V(G)|/3$ vertices.

Now let us move on to the case that $d = 2$. The octahedron has 6 vertices and is 4-regular, so a 2-degenerate induced subgraph has at most 4 vertices. Thus $\bar{\alpha}_2 \leq 2/3$.

We conjecture that equality holds. Currently, we only have a more or less trivial lower bound: $\bar{\alpha}_2 \geq 1/2$, which follows from the fact that G is 5-degenerate, and hence we can greedily 2-colour G in an ordering that witnesses its degeneracy so that no vertex has three out-neighbours of the same colour, i.e., each colour class induces a 2-degenerate subgraph. Dvořák and Kelly [10] showed that if a planar graph G is C_3-free, then it has a 2-degenerate induced subgraph containing at least $4|V(G)|/5$ vertices.

For $d = 4$, the icosahedron has 12 vertices and is 5-regular, so a 4-degenerate induced subgraph has at most 11 vertices. Thus $\bar{\alpha}_4 \leq 11/12$. Again we conjecture that equality holds. The best known lower bound is $\bar{\alpha}_4 \geq 8/9$, which was obtained by Lukoťka, Mazák and Zhu [15].

In this paper, we study 3-degenerate induced subgraphs of planar graphs. Both the octahedron C_6^2 and the icosahedron witness that $\bar{\alpha}_3 \leq 5/6$. Here is our main theorem.

Theorem 1.1. *Every n-vertex planar graph has a 3-degenerate induced subgraph of order at least $3n/4$.*

We conjecture that the upper bounds for $\bar{\alpha}_d$ mentioned above are tight. We remark that it is possible to obtain infinitely many 3-connected tight examples for each d by gluing together many copies of the tight example discussed above.

Conjecture 1.1. $\bar{\alpha}_2 = 2/3, \bar{\alpha}_3 = 5/6$*, and* $\bar{\alpha}_4 = 11/12$.

The problem of colouring the vertices of a planar graph G so that colour classes induce certain degenerate subgraphs has been studied in many papers. Borodin [7] proved that every planar graph G is acyclically 5-colourable, meaning that $V(G)$ can be coloured in 5 colours so that a subgraph of G induced by each colour class is 0-degenerate and a subgraph of G induced by the union of any two colour classes is 1-degenerate. As a strengthening of this result, Borodin [6] conjectured that every planar graph has degenerate chromatic number at most 5, which means that the vertices of any planar graph G can be coloured in 5 colours so that for each $i \in \{1, 2, 3, 4\}$, a subgraph of G induced by the union of any i colour classes is $(i-1)$-degenerate. This conjecture remains open, but it was proved in [13] that the list degenerate chromatic number of a graph is bounded by its 2-colouring number, and it was proved in [9] that the 2-colouring number of every planar graph is at most 8. As consequences of the above conjecture, Borodin posed two other weaker conjectures: (1) Every planar graph has a vertex partition into two sets such that one induces a 2-degenerate graph and the other induces a forest. (2) Every planar graph has a vertex partition into an independent set and a set inducing a 3-degenerate graph. Thomassen confirmed these conjectures in [18] and [19].

This paper is organized as follows. In Section 2 we will present our notation. In Section 3 we will formulate a stronger theorem that allows us to apply induction. This will involve identifying numerous obstructions to a more direct proof. In Section 4, we will organize our proof by contradiction around the notion of an *extreme counterexample*. In Sections 5–7, we will develop properties of extreme counterexamples that eventually lead to a contradiction in Section 8.

2 Notation

For sets X and Y, define $Z = X \mathbin{\dot\cup} Y$ to mean $Z = X \cup Y$ and $X \cap Y = \emptyset$. Let $G = (V, E)$ be a graph with $v, x, y \in V$ and $X, Y \subseteq V$. Then $\|v, X\|$ is the number of edges incident with v and a vertex in X and $\|X, Y\| = \sum_{v \in X} \|v, Y\|$. When X and Y are disjoint, $\|X, Y\|$ is the number of edges xy with $x \in X$ and $y \in Y$. In general, edges in $X \cap Y$ are counted twice by $\|X, Y\|$. Let $N(X) = \bigcup_{x \in X} N(x) - X$.

We write $H \subseteq G$ to indicate that H is a subgraph of G. The subgraph of G induced by a vertex set A is denoted by $G[A]$. The path P with $V(P) = \{v_1, \dots, v_n\}$ and $E(P) = \{v_1v_2, \dots, v_{n-1}v_n\}$ is denoted by $v_1 \cdots v_n$. Similarly the cycle $C = P + v_nv_1$ is denoted by $v_1 \cdots v_nv_1$.

Now let G be a simple connected plane graph. The boundary of the infinite face is denoted by $\mathbf{B} = \mathbf{B}(G)$ and $V(\mathbf{B}(G))$ is denoted by $B = B(G)$. Then $\mathbf{B}$ is a subgraph of the outerplanar graph $G[B]$. For a cycle C in G, let $\text{int}_G[C]$ denote the subgraph of G obtained by removing all exterior vertices and edges and let $\text{ext}_G[C]$ be the subgraph of G obtained by removing all interior vertices and edges. Usually the graph G is clear from the text, and we write $\text{int}[C]$ and $\text{ext}[C]$ for $\text{int}_G[C]$ and $\text{ext}_G[C]$. Let $\text{int}(C) = \text{int}[C] - V(C)$ and $\text{ext}(C) = \text{ext}[C] - V(C)$. Let $N^\circ(x) = N(x) - B$ and $N^\circ(X) = N(X) - B$.

For $L \in \Pi$, the *up-set* of x in L is defined as $U_L(x) = \{y \in V : y >_L x\}$ and the *down-set* of x in L is defined as $D_L(x) = \{y \in V : y <_L x\}$. Note that for each $L \in \Pi$, $y <_L x$ means that $y \leq_L x$ and $y \neq x$. For two sets X and Y, we say $X \leq_L Y$ if $x \leq_L y$ for all $x \in X$, $y \in Y$.

3 Main result

In this section we phrase a stronger, more technical version of Theorem 1.1 that is more amenable to induction. This is roughly analogous to the proof of the 5-Choosability Theorem by Thomassen [17].

If $G = G_1 \cup G_2$ and $G_1 \cap G_2 = G[A]$ for a set A of vertices, then we would like to join two 3-degenerate subgraphs obtained from G_1 and G_2 by induction to form a 3-degenerate subgraph of G. The problem is that vertices from A may have neighbours in both subgraphs. Dealing with this motivates the following definitions.

Let $A \subseteq V(G)$. A subgraph H of G is (k,A)-*degenerate* if there exists an ordering $L \in \Pi(G)$ such that $A \leq_L V - A$ and $d^L_H(v) \leq k$ for every vertex $v \in V(H) - A$. Equivalently, every subgraph H' of H with $V(H') - A \neq \emptyset$ has a vertex $v \in V(H') - A$ such that $d_{H'}(v) \leq k$. A subset Y of V is A-*good* if $G[Y]$ is (3,A)-degenerate. We say a subgraph H is A-good if $V(H)$ is A-good. Thus if $A = \emptyset$ then G is A-good if and only if G is 3-degenerate. Let

$$f(G; A) = \max\{|Y| : \ Y \subseteq V(G) \text{ is } A\text{-good}\}.$$

Since $\emptyset$ is A-good, $f(G; A)$ is well defined.

For an induced subgraph H of G and a set Y of vertices of H, we say Y is *collectable* in H if the vertices of Y can be ordered as $y_1, y_2, \ldots, y_k$ such that for each $i \in \{1, 2, \ldots, k\}$, either $y_i \notin A$ and $d_{H-\{y_1,y_2,\ldots,y_{i-1}\}}(y_i) \leq 3$ or $V(H) - \{y_1, y_2, \ldots, y_{i-1}\} \subseteq A$.

In order to build an A-good subset, we typically apply a sequence of operations of deleting and collecting. *Deleting* $X \subset V$ means replacing G with $G - X$. An ordering witnessing that Y is collectable is called a *collection* order. For disjoint subsets $V_1, \ldots, V_s$ of V, if V_i is collectable in $G - \bigcup_{j=1}^{i-1} V_j$ for each $i = 1, 2, \ldots, s$, then *collecting* $V_1, \ldots, V_s$ means first putting V_1 at the end of L in a collection order for V_1, then putting V_2 at the end of $L - V_1$ in a collection order for V_2 in $G - V_1$, etc. Note that if Y is a collectable set in G and $V - Y$ is A-good, then V is A-good.

Definition 3.1. A path $v_1 v_2 \ldots v_\ell$ of a plane graph G is *admissible* if $\ell > 0$ and it is a path in $\mathbf{B}(G)$ such that for each $1 < i < \ell$, $G - v_i$ has no path from v_{i-1} to v_{i+1}.

A path of length 0 has only 1 vertex in its vertex set.

Definition 3.2. A set A of vertices of a plane graph G is *usable* in G if for each component G' of G, $A \cap V(G')$ is the empty set or the vertex set of an admissible path of G'.

Lemma 3.1. *Let G be a plane graph and let A be a usable set in G. Then for each vertex v of G, $|N_G(v) \cap A| \leq 2$.*

Proof. This is clear from the definition of an admissible path. □

Observation 3.1. *If G is outerplanar and A is a usable set in G, then G is (2,A)-degenerate.*

Observation 3.1 motivates the expectation that plane graphs with large boundaries have large 3-degenerate induced subgraphs. Roughly, we intend to prove that $f(G; A) \leq 3|V(G)|/4 + |B|/4$. This formulation provides a potential function for measuring progress as we collect and delete vertices. For example, deleting a boundary vertex with at least four interior neighbours provides a smaller graph whose potential is at least as large. Some of the bonus $|B|/4$ is needed for dealing with chords. But this does not quite work; C_6^2 is a counterexample, and there are infinitely many more. The rest of this section is devoted to formulating a more refined potential function.

A set Z of vertices is said to be *exposed* if $Z \subseteq B$. We say that a vertex z is *exposed* if $\{z\}$ is exposed. We say that deleting Y and collecting X *exposes* Z if $Z \subseteq B(G - Y - X) - B$.

Definition 3.3. Let $\mathcal{Q} = \{Q_1, Q_2, Q_2^+, Q_3, Q_4, Q_4^+, Q_4^{++}\}$ be the set of plane graphs shown in Figure 1. For a plane graph G, a cycle C of G is *special* if $G_C := \mathrm{int}_G[C]$ is isomorphic to a plane graph in $\mathcal{Q}$, where C corresponds to the boundary. In this case, G_C is also *special.*

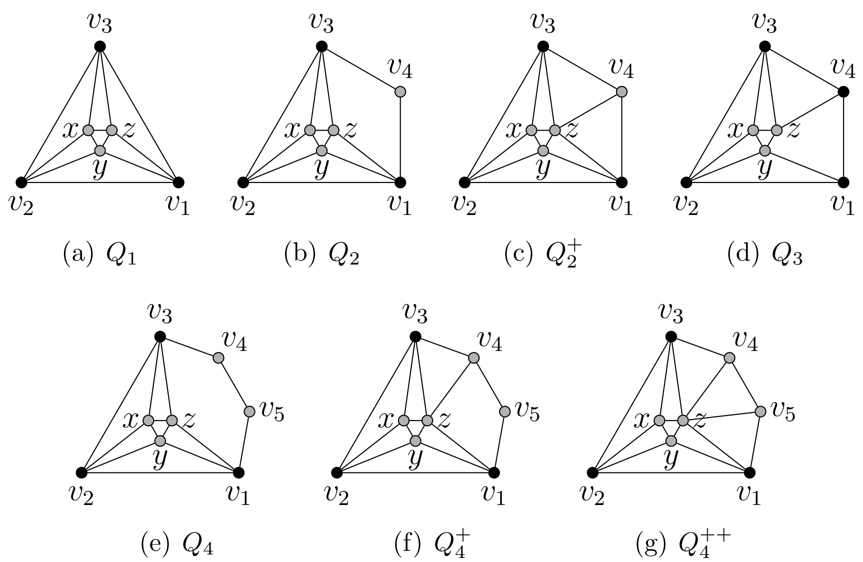


Figure 1: Plane graphs in $\mathcal{Q}$ defining special subgraphs G_C where C corresponds to the boundary cycle. Solid black vertices denote vertices in X_C.

For a special cycle C of a plane graph G, we define

$$\begin{aligned}
&T_C := \mathrm{int}_G(C), \text{ which is isomorphic to } K_3,\\
&X_C := \{v \in V(C) : \text{there is a facial cycle } D \text{ such that}\\
&\qquad\qquad v \in V(C) \cap V(D) \text{ and } |V(T_C) \cap V(D)| = 2\},\\
&V_C := V(G_C),\ Y_C := X_C \cup V(T_C), \text{ and } \overline{Y}_C := V_C - Y_C = V(C) - X_C.
\end{aligned}$$

Then $V(C) = X_C \cup \overline{Y}_C$.

Observation 3.2. *Let A be a usable set in a plane graph G. Let $C = v_1 \dots v_k v_1$ be a special cycle of G. If G_C is (not only isomorphic but also equal to a plane graph) in $\mathcal{Q}$, then the following hold.*

(a) $T_C = xyzx$ with $N_G(x) = \{y, z, v_2, v_3\}$ and $N_G(y) = \{x, z, v_1, v_2\}$.
(b) $X_C = \{v_1, v_2, v_3\}$ if $G_C \neq Q_3$ and $X_C = \{v_1, v_2, v_3, v_4\}$ if $G_C = Q_3$.
(c) Deleting any vertex in $X_C \cap B$ exposes two vertices of T_C.
(d) For each vertex $v \in X_C$, $V(T_C)$ is collectable in $G - v$, except that if $G_C = Q_4^{++}$ and $v = v_2$ then only $\{x, y\}$ is collectable in $G - v$.
(e) If $\overline{Y}_C \neq \emptyset$ then there is a facial cycle C^* containing $\overline{Y}_C \cup \{v\}$ for some $v \in V(T_C)$. Moreover, $v = z$ is unique, and if $|\overline{Y}_C| = 2$, then C^* is unique.
(f) T_C has at least two vertices v such that $d_G(v) = 4$.

Note that vertices on C may have neighbours in $\mathrm{ext}(C)$ or maybe contained in A. Thus we may not be able to collect vertices of C.

A special cycle C is called *exposed* if $X_C \subseteq B(G)$. A *special cycle packing* of G is a set of exposed special cycles $\{C_1, \dots, C_m\}$ such that $Y_{C_i} \cap Y_{C_j} = \emptyset$ for all $i \neq j$. Let

$\tau(G)$ be the maximum cardinality of a special cycle packing and

$$\partial(G) = \frac{3}{4}|V(G)| + \frac{1}{4}(|B| - \tau(G)).$$

We say that a special cycle packing of G is *optimal* if its cardinality is equal to $\tau(G)$.

Theorem 3.2. *For all plane graphs G and usable sets $A \subseteq B(G)$,*

$$f(G; A) \geq \partial(G). \tag{3.1}$$

Clearly $|B| - \tau(G) \geq 2$ for any plane graph G with at least 2 vertices. This is trivial if $\tau(G) = 0$. If $\tau(G) = k$, then each of the k exposed cycles in the maximum cardinality special cycle packing of G has at least 3 vertices in B and therefore $|B| - \tau(G) \geq 2k \geq 2$. The following consequence of Theorem 3.2 is the main result of this paper.

Corollary 3.3. *Every n-vertex planar graph G (with $n \geq 2$) has an induced 3-degenerate subgraph H with $|V(H)| \geq (3n + 2)/4$.*

4 Setup of the proof

Suppose Theorem 3.2 is not true. Among all counterexamples, choose $(G; A)$ so that

(i) $|V(G)|$ is minimum,
(ii) subject to (i), $|A|$ is maximum, and
(iii) subject to (i) and (ii), $|E(G)|$ is maximum.

We say that such a counterexample is *extreme*.

If $A' \not\subseteq V(G')$, then we may abbreviate $(G'; A' \cap V(G'))$ by $(G'; A')$, but still (ii) refers to $|A' \cap V(G')|$. We shall derive a sequence of properties of $(G; A)$ that leads to a contradiction. Trivially $|V(G)| > 2$, G is connected (if G is the disjoint union of G_1 and G_2, then $f(G; A) = f(G_1; A) + f(G_2; A)$ and $\partial(G) = \partial(G_1) + \partial(G_2)$).

Lemma 4.1. *Let G be a plane graph and X be a subset of $V(G)$. If A is usable in G, then $A - X$ is usable in $G - X$.*

Proof. We may assume that G is connected and $X = \{v\}$. If $v \notin A$, then it is trivial. Let $P = v_0 v_1 \cdots v_k$ be the admissible path in G such that $A = V(P)$. If $v = v_0$ or $v = v_k$, then again it is trivial. If $v = v_i$ for some $0 < i < k$, then by the definition of admissible paths, $G - v_i$ is disconnected, and v_{i-1} and v_{i+1} are in distinct components. Thus again $A - \{v\}$ is usable in $G - v$. □

Suppose Y is a nonempty subset of $V(G)$ and $G[Y]$ is connected. Let C be an exposed special cycle of $G' = G - Y$. Then C satisfies one of the following conditions.

(a) C is an exposed special cycle of G.
(b) C is a non-exposed special cycle of G; in this case $X_C \cap (B(G') - B) \neq \emptyset$.
(c) C is not a special cycle of G; in this case $Y \subseteq \text{int}_G(C)$, and so $Y \cap B = \emptyset$.

A cycle C is *type-a*, *-b*, *-c*, respectively, if it satisfies condition (a), (b), (c), respectively. Let

$$\delta(Y)=\begin{cases}1, & \text{if } G' \text{ has a type-c exposed special cycle,}\\ 0, & \text{otherwise.}\end{cases}$$

Lemma 4.2. *Let Y be a nonempty subset of $V(G)$ such that $G[Y]$ is connected. Let $G'=G-Y$. If C,C' are distinct exposed type-c special cycles of G', then $Y_C\cap Y_{C'}\neq\emptyset$.*

Proof. Let C, C' be distinct exposed type-c special cycles of G'. Since $B\cap Y=\emptyset$ and $G[Y]$ is connected, there exists a facial cycle D of G' such that $\mathrm{int}_G(D)=G[Y]$. Then D is a facial cycle of both G'_C and $G'_{C'}$. Arguing by contradiction, suppose $Y_C\cap Y_{C'}=\emptyset$. Since $V(D)\subseteq V(G'_C)=Y_C\cup\overline{Y}_C$ and $V(D)\subseteq V(G'_{C'})=Y_{C'}\cup\overline{Y}_{C'}$, we have

$$V(D)\subseteq V(G'_C)\cap V(G'_{C'})\subseteq\overline{Y}_C\cup\overline{Y}_{C'}.$$

By symmetry we may assume that $|\overline{Y}_C\cap V(D)|\geq|\overline{Y}_{C'}\cap V(D)|$. Using $|\overline{Y}_C|,|\overline{Y}_{C'}|\leq 2$, we deduce that

$$3\leq|V(D)|\leq|V(G'_C)\cap V(G'_{C'})|\leq 4,\ \overline{Y}_C\subseteq V(D),\ |\overline{Y}_C|=2,\ \text{and}\ \overline{Y}_{C'}\cap V(D)\neq\emptyset.$$

We will show that H is isomorphic to H_1 or H_2 in Figure 2. Since $|\overline{Y}_C|=2$, by Observation 3.2(e), D is the unique facial cycle in G'_C such that there is a vertex $\dot{z}\in V(T_C)$ with $\overline{Y}_C\cup\{\dot{z}\}\subseteq V(D)$. As $\dot{z}\in V(D)$ and $\dot{z}\in Y_C$, we have $\dot{z}\in\overline{Y}_{C'}$. Since $\overline{Y}_{C'}\neq\emptyset$, again by Observation 3.2(e), there is a unique vertex $\ddot{z}\in V(T_{C'})$ such that $\overline{Y}_{C'}\cup\{\ddot{z}\}$ is contained in a facial cycle of $G'_{C'}$. Then $\ddot{z}\in Y_{C'}\cap\overline{Y}_C\subseteq V(D)$.

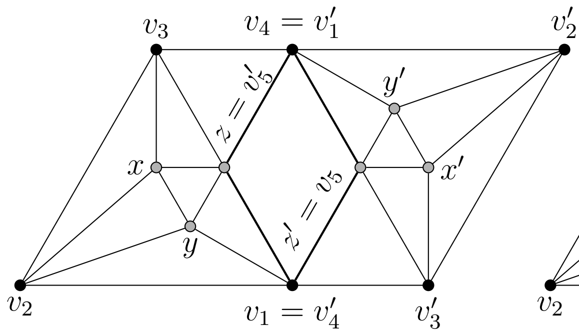
First we show that $|V(G'_C)\cap V(G'_{C'})|=4$. Assume to the contrary that $|V(G'_C)\cap V(G'_{C'})|=3$. Since $V(D)\subseteq V(G'_C)\cap V(G'_{C'})$, we conclude that $|V(D)|=3$. Then $\dot{z}\ddot{z}$ is an edge, and the two inner faces of G' incident with $\dot{z}\ddot{z}$ are contained in $V(G'_C)\cap V(G'_{C'})$. Since the intersection of any two inner faces of G'_C has at most 2 vertices, we have $|V(G'_C)\cap V(G'_{C'})|\geq 4$, a contradiction.

As $V(G'_C)\cap V(G'_{C'})=\overline{Y}_C\cup\overline{Y}_{C'}$, we conclude that $|\overline{Y}_{C'}|=2$ and $|V(C)|=5=|V(C')|$.

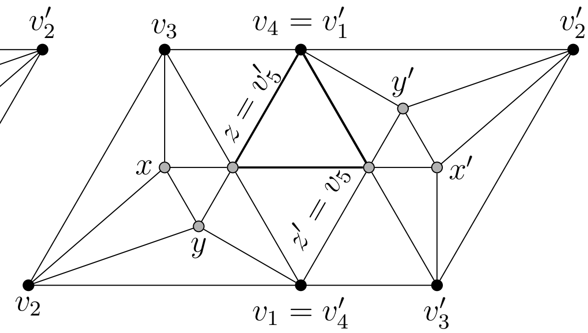
Let $Q\in\mathcal{Q}$ be the plane graph isomorphic to G'_C. By inspection of Figure 1, $G'_{C'}$ is isomorphic to Q. We may assume that $G'_C=Q$ by relabelling vertices. Let $u\mapsto u'$ be an isomorphism from G'_C to $G'_{C'}$. Using uniqueness from Observation 3.2(e), $z=\dot{z}$, $z'=\ddot{z}$, $\overline{Y}_C=\{v_4,v_5\}$ and $\overline{Y}_{C'}=\{v'_4,v'_5\}$. To prove our claim let us divide our analysis into two cases, resulting either in H_1 or H_2.

- If $|V(D)|=4$, then $Q=Q_4^+$ and $V(G'_C)\cap V(G'_C)=V(D)=\{v_4,v_5,v_1,z\}$. Since $v_4,v_5\in\overline{Y}_C$, we have $v_1,z\in\overline{Y}_{C'}$. Then $v'_4=v_1$, $v'_5=z$, and $v_5=z'$. As X_C and $X_{C'}$ are exposed in G', the cycle $v_1v_2v_3v'_1v'_2v'_3v_1$ is in $G'[B(G')]$ and so $H=H_1$ in Figure 2(a).

- If $|V(D)|=3$, then $Q=Q_4^{++}$, $V(D)=\{z,v_4,v_5\}$. By symmetry, we may assume that $z'=v_5$. Then $V(G'_C)\cap V(G'_{C'})=\{z,v_4,v_5,v_1\}$, as C' contains all common



(a) H_1



(b) H_2

Figure 2: The isomorphism types H_1 and H_2 of $H = G'[V(G'_C) \cup V(G'_{C'})]$ when $|V(D)| = 4$ or $|V(D)| = 3$ in the proof of Lemma 4.2. Solid black vertices denote boundary vertices of G and thick edges represent edges in D.

neighbors of z and z' in G', which is a property of Q_4^{++}. Since $v_4, v_5 \in \overline{Y}_C$ and $V(G'_C) \cap V(G'_{C'}) \subseteq \overline{Y}_C \cup \overline{Y}_{C'}$, we deduce that $v_1, z \in \overline{Y}_{C'}$. By symmetry in $G'_{C'}$, we may assume that $z = v'_5$ and $v_1 = v'_4$. As X_C and $X_{C'}$ are exposed in G', the cycle $v_1 v_2 v_3 v'_1 v'_2 v'_3 v_1$ is in $G'[B(G')]$. So $H = H_1 + zz' = H_2$ in Figure 2(b).

Notice that in both cases, $v_4 = v'_1 \in B(G')$ and $v_4 \in V(D)$. Set $Y' = \{v_4, x', y'\}$ and $G'' = G - Y'$. As $V(\mathrm{int}_G(D)) = Y$, in $G - v_4$, we can collect both x' and y' and at least one vertex of Y is exposed. Thus $B(G'') - B$ contains z, z' and $(B(G'') - B) \cap Y \neq \emptyset$. So $|B(G'') - B| \geq 3$.

Let $\mathcal{P}$ be an optimal special cycle packing of G'', and put

$$\mathcal{P}_0 = \{C^* \in \mathcal{P} : C^* \text{ is a non-exposed special cycle of } G\}.$$

Consider $C^* \in \mathcal{P}_0$. As $v_4 = v'_1 \in B \cap Y'$, there is no exposed type-c special cycle in G''. Thus C^* is type-b, and so $X_{C^*} \cap (B(G'') - B) \neq \emptyset$. Let $w \in X_{C^*} \cap (B(G'') - B)$. Since T_{C^*} is connected, has a neighbour of w, and has no vertex from $B(G'')$, we have $V(T_{C^*}) \subseteq Y$ and $X_{C^*} \subseteq (B(G'') - B) \cup \{v_1\}$.

As $\mathcal{P}_0$ is a packing, $3|\mathcal{P}_0| \leq |B(G'') - B| + 1$. This implies that $|\mathcal{P}_0| \leq |B(G'') - B| - 2$, because $|B(G'') - B| \geq 3$. We now deduce that

$$|\mathcal{P}_0| \leq |B(G'') - B| - 2 = |B(G'')| - (|B| - 1) - 2 = |B(G'')| - |B| - 1.$$

Therefore

$$\tau(G) \geq \tau(G'') - |\mathcal{P}_0| \geq \tau(G'') - |B(G'')| + |B| + 1.$$

Hence, using $V(G) = V(G'') \cup Y'$,

$$\partial(G) = \frac{3}{4}|V(G)| + \frac{1}{4}(|B| - \tau(G)) \leq \frac{3}{4}(|V(G'')| + 3) + \frac{1}{4}(|B(G'')| - \tau(G'') - 1) = \partial(G'') + 2.$$

Now, as we have already collected x', y', we have

$$f(G; A) \geq f(G''; A) + 2 \geq \partial(G'') + 2 \geq \partial(G).$$

This contradicts the assumption that G is a counterexample. $\square$

Lemma 4.3. *Let Y be a nonempty subset of $V(G)$ such that $G[Y]$ is connected and let $G' = G - Y$. Then*

$$\partial(G) \leq \partial(G') + \frac{3|Y|}{4} + \frac{|B - B(G')| + \delta(Y)}{4}.$$

Moreover, if G has an exposed special cycle C such that $Y_C \cap Y \neq \emptyset$ and $Y_C \cap Y_{C'} = \emptyset$ for any other exposed special cycle C' of G, then

$$\partial(G) \leq \partial(G') + \frac{3|Y|}{4} + \frac{|B - B(G')| + \delta(Y) - 1}{4}.$$

Proof. An optimal special cycle packing of G' has at most $|B(G') - B|$ type-b cycles by definition and has $\delta(Y)$ type-c cycles by Lemma 4.2. We can remove such cycles from the special cycle packing of G' to obtain a special cycle packing of G. So

$$\tau(G) \geq \tau(G') - |B(G') - B| - \delta(Y) = \tau(G') - |B(G')| + |B| - |B - B(G')| - \delta(Y).$$

Plugging this into the definition of $\partial(G)$, we obtain

$$\partial(G) \leq \partial(G') + \frac{3|Y|}{4} + \frac{|B - B(G')| + \delta(Y)}{4}.$$

If G has an exposed special cycle C such that C is not a special cycle of G' and Y_C is disjoint from $Y_{C'}$ for any other exposed special cycle C' of G, then we can add cycle C to the special cycle packing of G obtained above. So

$$\tau(G) \geq \tau(G') - |B(G') - B| - \delta(Y) + 1 = \tau(G') - |B(G')| + |B| - |B - B(G')| - \delta(Y) + 1.$$

Plugging this into the definition of $\partial(G)$, we obtain

$$\partial(G) \leq \partial(G') + \frac{3|Y|}{4} + \frac{|B - B(G')| + \delta(Y) - 1}{4}. \qquad \square$$

Lemma 4.4. *Every vertex $v \in V - A$ satisfies $d(v) \geq 4$.*

Proof. Suppose that $d(v) \leq 3$. Apply Lemma 4.3 with $Y = \{v\}$. Let $G' = G - Y$. Note that if v is a boundary vertex, then $\delta(Y) = 0$. So $|B - B(G')| + \delta(Y) \leq 1$. Therefore

$$\partial(G) \leq \partial(G') + \frac{3}{4} + \frac{1}{4}.$$

By the minimality of $(G; A)$, $f(G'; A) \geq \partial(G')$. Therefore $f(G; A) = f(G'; A) + 1 \geq \partial(G)$, a contradiction. $\square$

Lemma 4.5. *There are no disjoint nonempty subsets X, Y of $V(G)$ such that Y is a set of $4|X|$ interior vertices of G, $G[X \cup Y]$ is connected, and Y is collectable in $G - X$.*

Proof. Suppose that there exist disjoint nonempty sets $X, Y \subseteq V(G)$ such that Y is a subset of $4|X|$ interior vertices of G, $G[X \cup Y]$ is connected, and Y is collectable in $G - X$. Let $G' = G - (X \cup Y)$. We apply Lemma 4.3. Since $|B - B(G')| + \delta(X \cup Y) \leq |X|$, we have $\partial(G) \leq \partial(G') + \frac{3}{4}(|X| + |Y|) + \frac{1}{4}|X| = \partial(G') + 4|X|$. As G is extreme, $f(G'; A) \geq \partial(G')$. Hence $f(G; A) \geq f(G'; A) + |Y| = f(G'; A) + 4|X| \geq \partial(G') + 4|X| \geq \partial(G)$, a contradiction. $\square$

Lemma 4.6. *For any two distinct special cycles C_1, C_2 of G, $Y_{C_1} \cap Y_{C_2} = \emptyset$.*

Proof. Assume to the contrary that C_1, C_2 are two special cycles of G with $Y_{C_1} \cap Y_{C_2} \neq \emptyset$. Observe that for each $i = 1, 2$, $V(T_{C_i})$ has two vertices of degree 4 and one vertex of degree 4, 5, or 6 in G.

If T_{C_1} and T_{C_2} share an edge, say $T_{C_1} = xyz$ and $T_{C_2} = xyz'$, then one of x, y, say x, has degree 4. Since G is simple, $z \neq z'$. Let v be the other neighbor of x. By inspecting all graphs in $\mathcal{Q}$, we deduce that each of z, z' is either adjacent to v or has degree at most 5 in G. So in $G - v$, the set $\{x, y, z, z'\}$ is collectable, contrary to Lemma 4.5.

Assume T_{C_1} and T_{C_2} have a common vertex, say $T_{C_1} = xyz$ and $T_{C_2} = xy'z'$. If none of y, z, y', z' have degree 6, then we can delete x and collect y, z, y', z', contrary to Lemma 4.5. So we may assume that $d_G(y) = 6$ and hence $d_G(x) = d_G(z) = 4$ and all the faces incident to x are triangles because G_{C_1} is isomorphic to Q_4^{++}. Thus we may assume $yy', zz' \in E(G)$. By deleting y, we can collect x, z, z', and y', again contrary to Lemma 4.5. (We collect y' ahead of z' if $d_G(z') = 6$ and collect z' ahead of y' otherwise.) Thus $V(T_{C_1}) \cap V(T_{C_2}) = \emptyset$.

If $X_{C_1} \cap V(T_{C_2}) \neq \emptyset$, then for a vertex v of maximum degree in $V(T_{C_2})$, after deleting v, we can collect the other two vertices of T_{C_2} and two vertices of T_{C_1}, contrary to Lemma 4.5. So $X_{C_1} \cap V(T_{C_2}) = \emptyset$ and by symmetry, $X_{C_2} \cap V(T_{C_1}) = \emptyset$.

If $X_{C_1} \cap X_{C_2}$ contains a vertex v, then by deleting v, we can collect two vertices from each of T_{C_1} and T_{C_2}, again contrary to Lemma 4.5 because $V(T_{C_1}) \cap V(T_{C_2}) = \emptyset$. $\square$

Lemma 4.7. *If C is a special cycle of G, then there is a vertex $u \in X_C$ such that $V(T_C)$ is collectable in $G - u$ and $G' = G - (V(T_C) \cup \{u\})$ has no type-c special cycle.*

Proof. Suppose the lemma fails for some special cycle C of G with $|E(C)| = k$. Then G_C is isomorphic to a graph $Q \in \mathcal{Q}$. We may assume $G_C = Q$. Then $V(T_C)$ is collectable in $G - v_1$. Put $Y = V(T_C) \cup \{v_1\}$ and $G' = G - Y$. Since G' has a type-c special cycle C', $G'_{C'}$ has a facial cycle C'' with $Y = V(\mathrm{int}_G(C''))$.

Then C'' consists of the subpath $C - v_1$ from v_2 to v_k of length $k - 2$ and a path P from v_k to v_2 in G'. As $G'_{C'}$ is special, $3 \leq |E(C'')| \leq 5$. So $|E(P)| \leq 5 - (k - 2) \leq 4$. Now $N_G(v_1) \subseteq V(P) \cup \{y, z\}$, so $d_G(v_1) \leq |E(P)| + 3 \leq 10 - k \leq 7$. If $d_G(v_1) \leq 6$, then after deleting v_2 we can collect Y: use the order x, y, v_1, z if $d_G(v_1) \leq 5$; else $d_G(v_1) = 6$ and $k \leq 4$, so use the order x, y, z, v_1. This contradicts Lemma 4.5. Thus $d_G(v_1) = 7$. So $k = 3$, $|E(P)| = 4$, $|E(C'')| = 5$, $G_C = Q_1$, and v_1 is adjacent to all vertices of P.

Setting $u = v_3$, and using symmetry between v_1 and v_3, we see that v_3 is also an interior vertex with $d_G(v_3) = 7$.

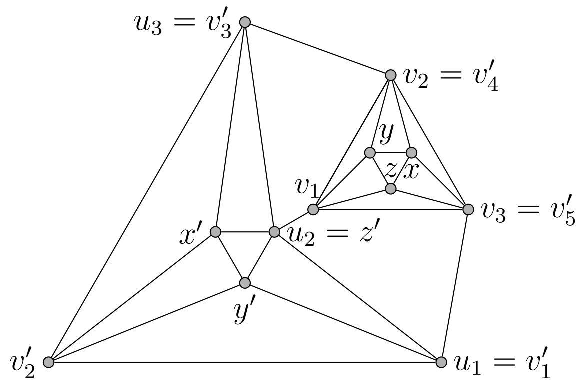


Figure 3: The graph $\mathrm{int}_G(C'')$ in the last part of the proof of Lemma 4.7 when $z' = u_2$. Note that $d_G(v_3) = 7$ and v_3 is an interior vertex.

Let $P = v_3u_1u_2u_3v_2$. Now $G'_{C'}$ is isomorphic to Q_4 since C'' is a facial 5-cycle. Assume $u \mapsto u'$ is an isomorphism from Q_4 to $G'_{C'}$. Then $C'' = z'v_3'v_4'v_5'v_1'z'$.

If there is $w \in \{v_2, v_3\} \cap \{v_1', v_3'\}$ then after deleting w we can collect $\{x, y, z, x', y', z'\}$, contrary to Lemma 4.5. Else $\{v_1', v_3'\} = \{u_1, u_3\}$ and therefore $z' = u_2$, see Figure 3. After deleting $\{v_2, u_1\}$, we can collect $\{x, y, z, v_3, v_1, z', x', y'\}$, contrary to Lemma 4.5, as both v_1 and v_3 are interior vertices. $\square$

Lemma 4.8. *G has no special cycle.*

Proof. Assume to the contrary that C is a special cycle of G. By Lemma 4.7, there is a vertex $u \in X_C$ such that $V(T_C)$ is collectable in $G - u$ and $G' = G - (V(T_C) \cup \{u\})$ has no type-c special cycle. Observe that $f(G; A) \geq f(G'; A) + 3$. So it suffices to show that $\partial(G) \leq \partial(G') + 3$. Since G' has no type-c special cycles, every exposed special cycle of G' is a special cycle of G.

If $u \notin B$, then $B(G') = B$ and so $B - B(G') = \emptyset$. As $\delta(V(T_C) \cup \{u\}) = 0$, we deduce from Lemma 4.3 that $\partial(G) \leq \partial(G') + \frac{3}{4} \cdot 4$.

Thus we may assume that $u \in B$ and so $|B - B(G')| = 1$. If C is exposed in G, then by Lemmas 4.3 and 4.6, $\partial(G) \leq \partial(G') + \frac{3}{4} \cdot 4 + \frac{1-1}{4}$.

If C is not exposed in G, then X_C has some interior vertex v. Since v is adjacent to a vertex of T_C, v is exposed in G'. By Lemma 4.6, $v \notin X_{C'}$ for every exposed special cycle C' of G', because C' is a special cycle of G. Therefore, in an optimal special cycle packing of G', at most $|B(G') - B| - 1$ of the cycles are not exposed in G. So,

$$\tau(G) \geq \tau(G') - (|B(G') - B| - 1) = \tau(G') - (|B(G')| - |B| + 1) + 1.$$

Thus

$$\begin{aligned}\partial(G) &= \frac{3}{4}|V(G)| + \frac{1}{4}(|B| - \tau(G)) \\ &\leq \frac{3}{4}(|V(G')| + 4) + \frac{1}{4}(|B| + (-\tau(G') + |B(G')| - |B|)) = \partial(G') + 3. \qquad \square\end{aligned}$$

Lemma 4.9. *Let s be an integer. Let X and Y be disjoint subsets of $V(G)$ such that Y is collectable in $G - X$. If $|B(G - (X \cup Y))| \geq |B(G)| + s$, $G[X \cup Y]$ is connected, and $(X \cup Y) \cap B(G) \neq \emptyset$, then $s + |Y| < 3|X|$.*

Proof. Let $G' = G - (X \cup Y)$. Since $(X \cup Y) \cap B(G) \neq \emptyset$ and $G[X \cup Y]$ is connected, any special cycle of G' is also a special cycle of G. So $\tau(G') = 0$ and $\partial(G) \leq \partial(G') + \frac{3}{4}(|X \cup Y|) - \frac{s}{4}$. As $X \cup Y \neq \emptyset$ and $(G; A)$ is extreme, $f(G'; A) \geq \partial(G')$. Thus

$$f(G'; A) + |Y| \leq f(G; A) < \partial(G) \leq \partial(G') + \frac{3}{4}|X \cup Y| - \frac{s}{4} \leq f(G'; A) + \frac{3}{4}|X \cup Y| - \frac{s}{4}.$$

This implies that $s + |Y| < 3|X|$. □

Lemma 4.10. *G is 2-connected and $|A| = 2$.*

Proof. Suppose G is not 2-connected. If $|V(G)| \leq 3$, then G is $(3, A)$-degenerate, so $f(G; A) = \partial(G)$ and we are done. Else $|V(G)| > 3$. As G is connected, it has a cut-vertex x. Let G_1, G_2 be subgraphs of G such that $G = G_1 \cup G_2$, $V(G_1) \cap V(G_2) = \{x\}$, and $|V(G_1) \cap A| \leq |V(G_2) \cap A|$. Observe that if $x \notin A$, then $A \cap V(G_1) = \emptyset$ by the choice of G_1 because A is usable in G.

Let $A_1 = V(G_1) \cap A$ if $x \in A$ and $A_1 = \{x\}$ otherwise. Let $A_2 = V(G_2) \cap A$. Note that for each $i = 1, 2$, A_i is usable in G_i. For $i = 1, 2$, let X_i be a maximum A_i-good set in G_i.

Let $X := (X_1 \cup X_2 - \{x\}) \cup (X_1 \cap X_2)$. We claim that X is A-good in G. If $x \in A$, then collect $X_1 - A$, $X_2 - A$, $A \cap X$. If $x \notin A$ and $x \in X_1 \cap X_2$, then collect $X_1 - \{x\}$, X_2. If $x \notin A$ and $x \notin X_1 \cap X_2$, then collect $X_1 - \{x\}$, $X_2 - \{x\}$. This proves the claim that X is A-good in G.

As $(G; A)$ is extreme, $f(G_i; A_i) \geq \partial(G_i)$ for $i = 1, 2$.

If $x \in B$ then $B(G_i) = B(G) \cap V(G_i)$ for $i = 1, 2$. Note that any special cycle of G_i is a special cycle of G and so $\tau(G_i) = 0$ for $i = 1, 2$ by Lemma 4.8 and hence $\partial(G) = \partial(G_1) + \partial(G_2) - 1$.

If $x \notin B$, then we may assume $V(G_1) \cap B(G) = \emptyset$. Hence $B(G) = B(G_2)$. Since only one inner face of G_2 contains vertices of G_1, $\tau(G_2) \leq 1$ by Lemma 4.8. Note that

$$\partial(G) = \partial(G_1) + \partial(G_2) - \frac{3}{4} + \frac{1}{4}\tau(G_2) - \frac{1}{4}(|B(G_1)| - \tau(G_1)).$$

Since $\tau(G_1) \leq |B(G_1)| - 2$, we have $\partial(G) \leq \partial(G_1) + \partial(G_2) - 1$.

In both cases, we have the contradiction:

$$f(G; A) \geq |X_1| + |X_2| - 1 = f(G_1; A_1) + f(G_2; A_2) - 1 \geq \partial(G_1) + \partial(G_2) - 1 \geq \partial(G).$$

Thus G is 2-connected, and hence $|A| \leq 2$. As $(G; A)$ is extreme, we have $|A| = 2$. □

In the following, set $A = \{a, a'\}$.

Lemma 4.11. *The boundary cycle $\mathbf{B}$ has no chord.*

Proof. Assume $\mathbf{B}$ has a chord $e := xy$. Let P_1, P_2 be the two paths from x to y in $\mathbf{B}$ such that $A \subseteq V(P_1)$. Since e is a chord, both P_1 and P_2 have length at least two.

Set $G_1 = \mathrm{int}[P_1 + e]$ and $G_2 = \mathrm{int}[P_2 + e]$. As $\tau(G) = 0$ by Lemma 4.8, we know that $\tau(G_1) = \tau(G_2) = 0$. Hence $\partial(G) = \partial(G_1) + \partial(G_2) - 2$. We may assume that $A \subseteq V(G_2)$. Let $A_1 = \{x, y\}$ and $A_2 = A$.

For $i = 1, 2$, let X_i be a maximum A_i-good set in G_i. Then $X = (X_1 \cup X_2 - \{x, y\}) \cup (X_1 \cap X_2)$ is an A-good set in G: collect $X_1 - \{x, y\}$, $(X_2 - \{x, y\}) \cup (X_1 \cap X_2)$. Thus

$$f(G; A) \geq f(G_1; A_1) + f(G_2; A_2) - 2 \geq \partial(G_1) + \partial(G_2) - 2 = \partial(G),$$

contrary to the choice of G. $\square$

Lemma 4.12. *G is a near plane triangulation.*

Proof. By Lemma 4.10, every face boundary of G is a cycle of G. Assume to the contrary that G has an interior face F which is not a triangle. Then $V(F)$ has a pair of vertices non-adjacent in G because G is a plane graph. Let $e \notin E(G)$ be an edge drawn on F joining them. Then $G' = G+e$ is a plane graph with $B(G') = B(G)$. As G is extreme, G' is not a counterexample. As $f(G'; A) \leq f(G; A)$, we conclude $\tau(G') > \tau(G)$, and hence G' has an exposed special cycle C and e is an edge of G'_C. By (d) of Observation 3.2, there is a vertex $v \in X_C$ such that after deleting v, we can collect all the three vertices of T_C. In $G - (V(T_C) \cup \{v\})$, all vertices in $(V_C \cup V(F)) - (V(T_C) \cup \{v\})$ are exposed. By Lemma 4.9, none of these vertices can be an interior vertex of G, because otherwise $|B(G - (V(T_C) \cup \{v\})| \geq |B(G)|$. So all these vertices are boundary vertices of G. By Lemmas 4.10 and 4.11, G is 2-connected, $|A| = 2$, and $B(G)$ has no chord, so G has no other vertices and $\mathrm{int}(B(G)) = T_C$, as $v \in X_C$ is also a boundary vertex of G. By the definition of usable sets, the two vertices in A are adjacent.

By Lemma 4.4, $\|u, V(T_C)\| \geq 2$ for every vertex $u \in B(G) - A$, and $\|w, B(G)\| \geq 2$ for every vertex $w \in V(T_C)$. On the other hand, the number of vertices $u \in B(G)$ with $\|u, V(T_C)\| \geq 2$ is at most 3. So $|B(G)| \leq 3 + |A| = 5$.

If $|B(G)| = 3$, then G is triangulated. Suppose $|B(G)| = 4$. If $\|u, V(T_C)\| \geq 2$ for three vertices $u \in B(G)$, then G is isomorphic to Q_2; else G is isomorphic to Q_3. Both are contradictions. If $|B(G)| = 5$, then G is isomorphic to Q_4 or Q_4^+, again a contradiction. $\square$

5 Properties of separating cycles

In a plane graph G, a cycle C is called *separating* if both $V(\mathrm{int}(C))$ and $V(\mathrm{ext}(C))$ are nonempty. In this section we will discuss properties of separating cycles in G.

Lemma 5.1. *Suppose T is a separating triangle of G and let $I = \mathrm{int}(T)$. Then*
(a) $\|V(T), V(I)\| \geq 6$,
(b) $|I| \geq 3$,
(c) $\|x, V(I)\| \geq 1$ for all $x \in V(T)$, and

(d) for all distinct x, y in $V(T)$, $|N(\{x,y\}) \cap V(I)| \geq 2$.

Proof. If $|I| \leq 2$, then I contains a vertex v with $d_G(v) \leq 3$, contrary to Lemma 4.4. Thus $|I| \geq 3$ and (b) holds. Moreover, $I^+ := \text{int}[T]$ is triangulated and therefore $\|I^+\| = 3|I^+| - 6$ and $\|I\| \leq 3|I| - 6$. Thus

$$\|V(T), V(I)\| = \|I^+\| - \|T\| - \|I\| \geq 3(3 + |I|) - 6 - 3 - (3|I| - 6) = 6.$$

Thus (a) holds. As I^+ is triangulated and T is separating, every edge of T is contained in a triangle of I^+ other than T; so (c) holds.

If $|(N(x) \cup N(y)) \cap V(I)| \leq 1$, then $|I| = 1$ because G is a near plane triangulation. This contradicts (b). So (d) holds. □

Lemma 5.2. *Let C be a separating cycle in G such that $V(C) \cap A = \emptyset$. Assume X, Y are disjoint subsets of G such that $X \cup Y \neq \emptyset$, Y is collectable in $G - X$, and $G[X \cup Y]$ is connected. Let $G_1 = \text{int}[C] - (X \cup Y)$, $G_2 = \text{ext}(C) - (X \cup Y)$, $B_1 = B(G_1)$, $B_2 = B(G_2)$, $G_2' = \text{ext}[C] - (X \cup Y)$, $A' = V(C) - (X \cup Y)$. If A' is usable in G_1 and collectable in G_2', then*

$$|Y| + |B_1| + |B_2| < 3|X| + |B| + \tau(G_2) \leq 3|X| + |B| + 1.$$

In particular,

$$|Y| < \begin{cases} 3|X| + |B| - |B_1| - |B_2| & \text{if } (X \cup Y) \cap B \neq \emptyset, \\ 3|X| + \tau(G_2) - |B_1| & \text{otherwise.} \end{cases}$$

Proof. Since A' is usable, $(X \cup Y) \cap V(C) \neq \emptyset$ and so $X \cup Y$ lies in the infinite face of G_1. Thus any special cycle of G_1 is also a special cycle of G. Thus by Lemma 4.8, $\tau(G) = \tau(G_1) = 0$. By Lemma 4.2, in an optimal special cycle packing of G_2, at most one cycle is type-c and there are no type-a or type-b cycles. Therefore $\tau(G_2) \leq 1$.

As A' is collectable in G_2', we have

$$f(G; A) \geq f(G_1; A') + f(G_2; A) + |Y|.$$

On the other hand,

$$\partial(G) = \partial(G_1) + \partial(G_2) + \frac{3}{4}(|X| + |Y|) - \frac{1}{4}(|B_1| + |B_2| - |B| - \tau(G_2)).$$

As $f(G_1; A') \geq \partial(G_1)$ and $f(G_2; A) \geq \partial(G_2)$, we have

$$\partial(G) - \frac{3}{4}(|X| + |Y|) + \frac{1}{4}(|B_1| + |B_2| - |B| - \tau(G_2)) \leq f(G_1; A') + f(G_2; A) \leq f(G; A) - |Y|.$$

As $f(G; A) < \partial(G)$, it follows that

$$|Y| + |B_1| + |B_2| < 3|X| + |B| + \tau(G_2) \leq 3|X| + |B| + 1.$$

Note that if $(X \cup Y) \cap B \neq \emptyset$, then $\tau(G_2) = 0$. In this case, we have

$$|Y| + |B_1| + |B_2| < 3|X| + |B|.$$

If $(X \cup Y) \cap B = \emptyset$, then $B_2 = B$. In this case, we have $|Y| + |B_1| < 3|X| + \tau(G_2)$. □

Lemma 5.3. *Let C be a separating triangle of G. If C has no vertex in $B(G)$, then either $\|v, V(\mathrm{ext}(C))\| \geq 3$ for all vertices $v \in V(C)$ or $\|v, V(\mathrm{ext}(C))\| \geq 4$ for two vertices $v \in V(C)$.*

Proof. Suppose not. Let $C = xyzx$ be a counterexample with the minimal area. We may assume that $\|x, V(\mathrm{ext}(C))\| \leq 2$ and $\|y, V(\mathrm{ext}(C))\| \leq 3$. By Lemma 5.1(c), z has a neighbour w in $I := \mathrm{int}(C)$. If w is the only neighbour of z in I, then by Lemma 5.1(b), $C' := xwyx$ is a separating triangle. However, w has only 1 neighbour in $\mathrm{ext}(C')$ and x has at most 3 neighbours in $\mathrm{ext}(C')$, contradicting the choice of C.

Thus $\|z, V(I)\| \geq 2$.

We apply Lemma 5.2 with C, $X = \{z\}$ and $Y = \emptyset$. Then $A' := \{x, y\}$ is usable in $G_1 := \mathrm{int}[C] - z$, A' is collectable in $G_2' := \mathrm{ext}[C] - z$ and $B_1 := B(G_1) \supseteq \{x, y\} \cupdot N_I(z)$. So $|B_1| \geq 4$, and this contradicts Lemma 5.2. $\square$

Lemma 5.4. *Let C be a separating induced cycle of length 4 in G having no vertex in $B(G)$. Then exactly one of the following holds.*

(a) $|B(\mathrm{int}(C))| \geq 4$.

(b) $|V(\mathrm{int}(C))| \leq 2$ *and every vertex in* $\mathrm{int}(C)$ *has degree* 4 *in* G.

Proof. Suppose that $|B(\mathrm{int}(C))| \leq 3$. By Euler's formula, we have

$$\|\mathrm{int}[C]\| = 3|V(\mathrm{int}[C])| - 7 = 3|V(\mathrm{int}(C))| + 5$$

as G is a near plane triangulation. Then since C is induced, by Lemma 4.4,

$$\begin{aligned} 0 &\leq \sum_{v \in V(\mathrm{int}(C))} (d(v) - 4) \\ &= \|\mathrm{int}[C]\| - \|C\| + \|\mathrm{int}(C)\| - 4|V(\mathrm{int}(C))| \\ &= (3|V(\mathrm{int}(C))| + 5) - 4 + \|\mathrm{int}(C)\| - 4|V(\mathrm{int}(C))| \\ &= 1 - |V(\mathrm{int}(C))| + \|\mathrm{int}(C)\|. \end{aligned} \tag{5.1}$$

Suppose that $\mathrm{int}(C)$ has a cycle. Since $|B(\mathrm{int}(C))| \leq 3$, we deduce that $B(\mathrm{int}(C)) = xyzx$ is a triangle. By Euler's formula applied on $G[V(C) \cup B(\mathrm{int}(C))]$, we have

$$\|V(C), B(\mathrm{int}(C))\| = (3 \cdot 7 - 7) - 3 - 4 = 7,$$

hence $\mathbf{B}(\mathrm{int}(C))$ is a facial triangle by Lemma 5.3. Therefore, x, y, z have degree 4, 4, 5 in G by (5.1) and Lemma 4.4. Let $w, w' \in V(C)$ be consecutive neighbours of x in $V(C)$. From G, we can delete w and collect x, y, z. Let $G' = G - \{w, x, y, z\}$. If G' has an exposed special cycle, then the face of G' containing w has length at most 5, implying that $\|w, V(\mathrm{ext}(C))\| \leq 2$ because $C - w$ is a subpath of an exposed special cycle of G', as C is induced. Then we can delete w' and collect x, y, z, w, contradicting Lemma 4.5. Therefore G' has no exposed special cycles. Then $\partial(G) = \partial(G') + 3$ and $f(G; A) \geq f(G'; A) + 3 \geq \partial(G') + 3 = \partial(G)$, a contradiction.

Therefore $\operatorname{int}(C)$ has no cycles. Then $\|\operatorname{int}(C)\| \leq |V(\operatorname{int}(C))| - 1$, and so in (5.1) the equality must hold. This means $\operatorname{int}(C)$ is a tree and every vertex in $\operatorname{int}(C)$ has degree 4 in G by Lemma 4.4. If $\operatorname{int}(C)$ has at least 3 vertices, then let w be a vertex in $V(C)$ adjacent to some vertex in $\operatorname{int}(C)$. By deleting w, we can collect all the vertices in $\operatorname{int}(C)$. Similarly we can choose w so that $G' = G - w - V(\operatorname{int}(C))$ contains no special cycle, and that leads to the same contradiction. Thus we deduce (b). $\square$

6 Degrees of boundary vertices

Lemma 6.1. *Each vertex in B has degree at most* 5.

Proof. Assume to the contrary that $x \in B$ has $d(x) \geq 6$. Then deleting x exposes at least 4 interior vertices. Apply Lemma 4.9 with $X = \{x\}, Y = \emptyset$ and $s = 3$, we obtain a contradiction. $\square$

Recall that $A = \{a, a'\}$.

Lemma 6.2. *Each vertex in $B - A$ has degree* 5.

Proof. Suppose that there is a vertex $x \in B - A$ with $d(x) < 5$. By Lemma 4.4, $d(x) = 4$. By Lemma 4.11, exactly two of the neighbors of x are in B. Consider two cases.

Case 1: x has a neighbour $y \in B - A$. As $|A| = 2$, we have $|B| \geq 4$. As G is a near plane triangulation, there is a vertex $z \in N(x) \cap N(y)$ such that $xyzx$ is a facial triangle. As $\mathbf{B}$ has no chords by Lemma 4.11, $(N(x) \cap N(y)) \cap B(G) = \emptyset$.

Suppose there is $z' \in N(x) \cap N(y) - \{z\}$. Since $d(x) = 4$ and G is a near plane triangulation, $xzz'x$ is a facial triangle. Since $d(z) \geq 4$ by Lemma 4.4, $T := yzz'y$ is a separating triangle. As $d(y) \leq 5$ by Lemma 6.1, y has a unique neighbour $y' \in V(\operatorname{int}(T))$ and therefore both $yy'zy$ and $yy'z'y$ are facial triangles. By Lemma 5.1(b), $\operatorname{int}(T)$ contains at least three vertices and so $T' := zz'y'z$ is a separating triangle with $\|z, V(\operatorname{ext}(T'))\| = 2$ and $\|y', V(\operatorname{ext}(T'))\| = 1$, contrary to Lemma 5.3. So $N(x) \cap N(y) = \{z\}$.

If $d(y) = 5$, then deleting z and collecting x and y exposes three vertices in $(N^\circ(x) \cup N^\circ(y)) - \{z\}$, the resulting graph $G' = G - \{x, y, z\}$ has $|B(G')| \geq |B| + 1$. Apply Lemma 4.9 with $X = \{z\}, Y = \{x, y\}$, and $s = 1$, we obtain a contradiction.

Hence $d(y) = 4$. By repeating the same argument, we deduce that for all edges $vv' \in \mathbf{B} - A$, we have (i) $d(v) = 4 = d(v')$ and (ii) $|N(v) \cap N(v')| = 1$.

Let x', y' be vertices such that $N^\circ(x) = \{x', z\}$ and $N^\circ(y) = \{y', z\}$. As G is a near plane triangulation and $\mathbf{B}$ is chordless, $G - B$ is connected. Let $J = \{x', z, y'\}$. If $V - B \neq J$, then there exist $b \in J$ and $t \in (V - B) - J$ such that b and t are adjacent. Then deleting b and collecting x, y exposes all vertices in $(J - \{b\}) \cup \{t\}$. Let $G' = G - \{x, y, b\}$. Then $|B(G')| \geq |B| + 1$. With $X = \{b\}$, $Y = \{x, y\}$, and $s = 1$, this contradicts Lemma 4.9. Hence $V - B = J$.

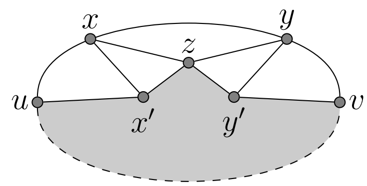


Figure 4: Case 1 in the proof of Lemma 6.2. The dashed line may have other vertices and the gray region has other edges but no interior vertices.

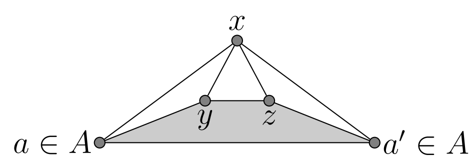


Figure 5: Case 2 in the proof of Lemma 6.2. The gray region may have other vertices.

Let u, v be vertices in B so that $uxyv$ is a path in $\mathbf{B}$. Since G is a near plane triangulation, x' is adjacent to u and z, and y' is adjacent to v and z, see Figure 4. Then $ux'zy, xzy'v$ are paths in G. If $A = \{u, v\}$, then $\mathbf{B}$ is a 4-cycle and as $d(x'), d(y') \geq 4$, we must have $x'y' \in E(G)$, which implies that G is isomorphic to Q_2^+ and $\mathbf{B}$ is a special cycle, contrary to Lemma 4.8. Therefore $A \neq \{u, v\}$ and since $y \notin A$, we deduce that $v \notin A$. This implies $d(v) = 4$. Then v has another neighbour in J, and by the observation that y and v have only one common neighbour y', we deduce that v is non-adjacent to z. Thus v is adjacent to x', and x' is adjacent to y'.

Furthermore every vertex in $B - \{u, x, y, v\}$ has degree at most 3, because $\mathbf{B}$ has no chords and x' is the only possible interior neighbor. By Lemma 4.4, every vertex in $B - \{u, x, y, v\}$ is in A. Then G is isomorphic to Q_4^{++} and $\mathbf{B}$ is a special cycle, contrary to Lemma 4.8.

Case 2: $N_G(x) \cap B \subseteq A$. Then $\mathbf{B} = xaa'x$. Since G is a near plane triangulation and $d(x) = 4$, the neighbours of x form a path of length 3 from a to a', say $ayza'$ where a, y, z, a' are the neighbours of x. (See Figure 5.)

If $|N^\circ(y)| \geq 3$, then deleting y and collecting x exposes at least three vertices in $N^\circ(y)$. Let $G' = G - \{x, y\}$. Then $|B(G')| \geq |B| + 2$. With $X = \{y\}, Y = \{x\}$, and $s = 2$, this contradicts Lemma 4.9.

Thus $|N^\circ(y)| \leq 2$ and so $d(y) \leq 5$. (Note that y may be adjacent to a'.) By symmetry, $|N^\circ(z)| \leq 2$ and $d(z) \leq 5$.

If y is adjacent to a', then z is non-adjacent to a and so $d(z) = 4$ by Lemma 4.4. Then $T := yza'y$ is a separating triangle, as $\mathrm{int}(T)$ contains a neighbour of z. Since $d(y) \leq 5$ and $d(z) = 4$, we have $|N(\{y, z\}) \cap V(\mathrm{int}(T))| = 1$, contrary to Lemma 5.1(d).

So y is non-adjacent to a'. By symmetry, z is non-adjacent to a. As $|N^\circ(y)|, |N^\circ(z)| \leq 2$ and $d(y), d(z) \geq 4$, y and z have a unique common neighbour w and $d(y) = d(z) = 4$. Since G is a near plane triangulation, w is adjacent to both a and a'.

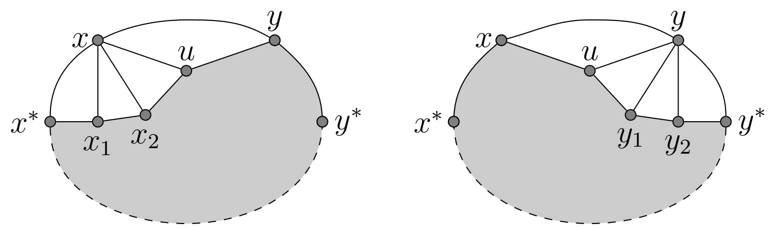


Figure 6: The situation in the proof of Lemma 7.1(b).

If $d(w) > 4$, then deleting w and collecting y, z, x exposes at least one vertex and so $|B(G - \{x, y, z, w\})| \geq |B|$. With $X = \{w\}, Y = \{x, y, z\}$, and $s = 0$, this contradicts Lemma 4.9. This implies $d(w) = 4$, hence $B(G)$ is a special cycle, contrary to Lemma 4.8. □

7 The boundary is a triangle

In this section we prove that $|B| = 3$.

Lemma 7.1. *If $xy \in E(\mathbf{B} - A)$, then the following hold:*

(a) There are $S := \{x_1, x_2, u, y_1, y_2\} \subseteq V - B$ and $x^*, y^* \in B$ such that $x^*x_1x_2uy$ is a path in $G[N(x)]$ and $xuy_1y_2y^*$ is a path in $G[N(y)]$.
(b) $d(x_2), d(u), d(y_1) \geq 5$.
(c) The vertices x_1, x_2, u, y_1, y_2 are all distinct.
(d) $|N^\circ(\{x_2, u\}) - S| \leq 2$ and $|N^\circ(\{y_1, u\}) - S| \leq 2$.
(e) $x_2y_1, x_2y_2, x_1y_1, ux_1, uy_2 \notin E$.
(f) There is $w_1 \in (N(\{x_2, u, y_1\}) \cap B) - \{x, y\}$; in particular $G[S]$ is an induced path.
(g) $x_2, u \notin N(x^*)$ and $y_1, u \notin N(y^*)$.
(h) Neither x^* nor y^* is equal to the vertex w_1 from (f).

Proof. (a) By Lemma 6.2, $d(x) = 5 = d(y)$. By Lemmas 4.10 and 4.11, there are $x^*, y^* \in B$ with $N(x) \cap B = \{x^*, y\}$ and $N(y) \cap B = \{x, y^*\}$. As G is a near plane triangulation, there is $u \in N(x) \cap N(y)$. So (a) holds.

(b) (See Figure 6.) As $d(u) \geq 4$ by Lemma 4.4, $x_2 \neq y_1$. Assume $d(x_2) = 4$. If x_2 is adjacent to y, then $x_2 = y_2$, implying that $d(x_2) > 4$, contradicting the assumption. Thus x_2 is non-adjacent to y and deleting u and collecting x_2, x, y exposes y_1, y_2 (note that it is possible that $x_1 \in \{y_1, y_2\}$, so we do not count it as exposed). We have $|B(G - \{u, x_2, x, y\})| \geq |B|$. With $X = \{u\}, Y = \{x_2, x, y\}$, and $s = 0$, this contradicts Lemma 4.9. Thus $d(x_2) \geq 5$ by Lemma 4.4. By symmetry, $d(y_1) \geq 5$. If $d(u) = 4$, then we can delete x_2, collect u, x, y, and expose y_1, y_2. This contradicts Lemma 4.9 applied with $X = \{x_2\}, Y = \{u, x, y\}$, and $s = 0$. So (b) holds.

(c) Since $d(u) \geq 5$, we deduce $x_2 \neq y_1$, and if $x_1 = y_1$, then $T := x_1x_2ux_1$ is a separating triangle (see Figure 7), since $d(x_2) \geq 5$. As $\|x_2, V(\text{ext}(T))\| = 1$ and $\|u, V(\text{ext}(T))\| = 2$, this contradicts Lemma 5.3. So $x_1 \neq y_1$. By symmetry, $x_2 \neq y_2$.

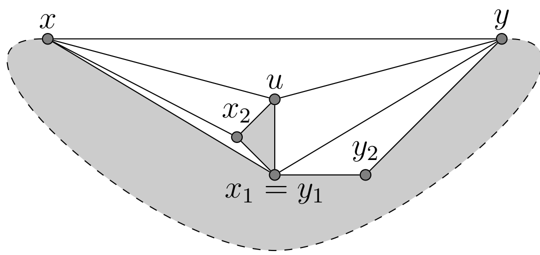


Figure 7: When $x_1 = y_1$ in the proof of Lemma 7.1(c). Gray regions may have other vertices.

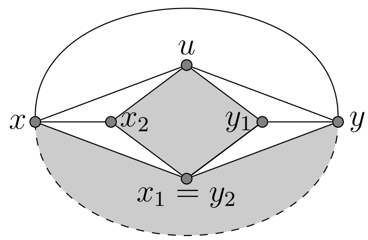


Figure 8: When $x_1 = y_2$ in Lemma 7.1(c). Gray regions may have other vertices.

It remains to show that $x_1 \neq y_2$. Suppose not. By (b), $d(x_2) \geq 5$, so $C := x_1x_2uy_1x_1$ is a separating 4-cycle (see Figure 8). We first prove the following.

$$\text{For all } u' \in V(C) - \{u\},\ |N(\{u, u'\}) \cap V(\text{int}(C))| \leq 3. \tag{7.1}$$

Suppose not. Then deleting u, u' and collecting x, y exposes two vertices in $V(C) - \{u, u'\}$ and at least 4 vertices in $\text{int}(C)$. So $|B(G - \{u, u', x, y\})| \geq |B| - 2 + 2 + 4$. This contradicts Lemma 4.9 with $X = \{u, u'\}$, $Y = \{x, y\}$, and $s = 4$. This proves (7.1).

If u is adjacent to x_1, then $C_1 := x_1x_2ux_1$ and $C_2 := x_1uy_1x_1$ are both separating triangles by (b). Then $|N(\{u, x_1\}) \cap V(\text{int}(C_i))| \geq 2$ for each $i \in \{1, 2\}$ by Lemma 5.1(d). Thus $|N(\{u, x_1\}) \cap V(\text{int}(C))| \geq 4$, contrary to (7.1). So u is non-adjacent to x_1.

If x_2 is adjacent to y_1, then $C_3 := ux_2y_1u$ is a separating triangle by (b). Then $|N(\{u, x_2\}) \cap V(\text{int}(C_3))| \geq 2$ by Lemma 5.1(d). As $\|u, V(\text{ext}(C_3))\| = 2$, Lemma 5.3 implies that $\|x_2, V(\text{ext}(C_3))\| \geq 4$, hence $|N(\{u, x_2\}) \cap V(\text{int}(x_1x_2y_1x_1))| \geq 2$. Thus $|N(\{u, x_2\}) \cap V(\text{int}(C))| \geq 4$, contrary to (7.1). So C has no chord.

By (b), C is a separating induced cycle of length 4 in G. By Lemma 5.4, either $|B(\text{int}(C))| \geq 4$ or $|V(\text{int}(C))| \leq 2$ and every vertex in $\text{int}(C)$ has degree 4 in G.

By (7.1), $d(x_2), d(y_1) \leq 6$. If $|B(\text{int}(C))| \geq 4$, then deleting u, x_1 and collecting x, y, x_2, y_1 exposes at least 4 vertices and therefore $|B(G - \{u, x_1, x, y, x_2, y_1\})| \geq |B| + 2$. This contradicts Lemma 4.9 applied with $X = \{u, x_1\}$, $Y = \{x, y, x_2, y_1\}$, and $s = 2$.

Therefore we may assume $1 \leq |V(\text{int}(C))| \leq 2$ and every vertex in $\text{int}(C)$ has degree 4 in G. As x_2 is non-adjacent to y_1, x_1 has at least one neighbour in $\text{int}(C)$ and therefore after deleting x_1, we can collect all vertices in $V(\text{int}(C))$ and then collect x_2, y_1 and u, this contradicts Lemma 4.5. So (c) holds.

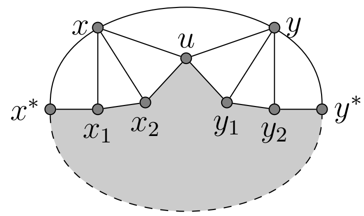


Figure 9: The situation in the proof of Lemma 7.1(d); x_1, x_2, u, y_1, y_2 are all distinct. The gray region has other vertices.

(d) (See Figure 9.) If $|N^\circ(\{x_2, u\}) - S| \geq 3$, then deleting x_2, u and collecting x, y exposes x_1, y_1, y_2, and three other vertices and so $|B(G - \{x_2, u, x, y\})| \geq |B| - 2 + 6$. By applying Lemma 4.9 with $X = \{x_2, u\}$, $Y = \{x, y\}$, and $s = 4$, we obtain a contradiction. So we deduce that $|N^\circ(\{x_2, u\}) - S| \leq 2$. By symmetry, $|N^\circ(\{y_1, u\}) - S| \leq 2$.

(e) Suppose x_1 is adjacent to u. By (b) and (d), $d(x_2) = 5$. Thus $T := ux_1x_2u$ is a separating triangle. Let w_1, w_2 be the two neighbours of x_2 other than x_1, x, u so that $x_1w_1w_2u$ is a path in G. Such a choice exists because G is a near plane triangulation. As $d(w_2) \geq 4$ by Lemma 4.4, and u has no neighbours in $\text{int}(ux_1w_1w_2u)$ by (d), x_1 is adjacent to w_2. As $d(w_1) \geq 4$, $T' := x_1w_1w_2x_1$ is a separating triangle. Note that $x_2x_1w_1x_2$, $x_2w_1w_2x_2$, $x_2w_2ux_2$, and ux_1w_2u are facial triangles. Thus $\|w_1, V(\text{ext}(T'))\| = 1$ and $\|w_2, V(\text{ext}(T'))\| = 2$, contrary to Lemma 5.3. So x_1 is non-adjacent to u. By symmetry, y_2 is non-adjacent to u.

Suppose that x_2 is adjacent to y_1. Let $T'' := ux_2y_1u$. By (b), $d(u) \geq 5$, so T'' is a separating triangle. By (d), $\|z, V(\text{int}(T''))\| \leq 2$ for all $z \in V(T'')$. By Lemma 5.1,

$$\sum_{z \in V(T'')} \|z, V(\text{int}(T''))\| = \|V(T''), V(\text{int}(T''))\| \geq 6$$

and therefore $\|z, V(\text{int}(T''))\| = 2$ for all $z \in V(T'')$. By (d), $N(u) \cap V(\text{int}(T'')) = N(x_2) \cap V(\text{int}(T'')) = N(y_1) \cap V(\text{int}(T''))$. Then u, x_2, y_1, and their neighbours in $\text{int}(T'')$ induce a K_5 subgraph, contradicting our assumption on G. Thus x_2 is non-adjacent to y_1.

Suppose that x_2 is adjacent to y_2. Since x_2 is non-adjacent to y_1, (b) and (d) imply that $d(y_1) = 5$. Let w_1, w_2 be the two neighbours of y_1 other than u, y, y_2 such that $uw_1w_2y_2$ is a path in G. By (d), $N^\circ(u) - S \subseteq \{w_1, w_2\}$. If u is adjacent to both w_1 and w_2, then uw_1w_2u, uy_1w_1u, $y_1w_1w_2y_1$ are facial triangles, implying that w_1 has degree 3, contradicting Lemma 4.4. Thus, as $d(u) \geq 5$ by (b), we deduce that $d(u) = 5$. Since G is a near plane triangulation, x_2 is adjacent to w_1 and ux_2w_1u, uw_1y_1u are facial triangles. If $x_2w_1w_2y_2x_2$ is a separating cycle, then deleting w_1, w_2 and collecting y_1, u, y, x exposes at least 4 vertices and so $|B(G - \{w_1, w_2, y_1, u, y, x\})| \geq |B| - 2 + 4$. By applying Lemma 4.9 with $X = \{w_1, w_2\}$, $Y = \{y_1, u, y, x\}$, and $s = 2$, we obtain a contradiction. So $x_2w_1w_2y_2x_2$ is not a separating cycle. By Lemma 4.4, $d(w_2) \geq 4$ and therefore w_2 is adjacent to x_2 and $d(w_1) = 4 = d(w_2)$. Then, deleting y_1 and collecting

w_1, w_2, u, y, x exposes 3 vertices and $|B(G - \{y_1, w_1, w_2, u, y, x\})| = |B| - 2 + 3$. By applying Lemma 4.9 with $X = \{y_1\}$, $Y = \{w_1, w_2, u, y, x\}$, and $s = 1$, we obtain a contradiction. So x_2 is non-adjacent to y_2. By symmetry, x_1 is non-adjacent to y_1.

(f) Suppose that none of x_2, u, y_1 has neighbours in $B - \{x, y\}$. By (b), (d), and (e), $d(x_2) = 5 = d(y_1)$. If

$$|N^\circ(\{x_2, y_1\}) - \{u\}| \geq 5$$

then deleting u, x_2 and collecting x, y, y_1 exposes all vertices in $N^\circ(\{x_2, y_1\}) - \{u\}$ and so $|B(G - \{u, x_2, x, y, y_1\})| \geq |B| - 2 + 5$. By applying Lemma 4.9 with $X = \{u, x_2\}$, $Y = \{x, y, y_1\}$, and $s = 3$, we obtain a contradiction. Thus $|N^\circ(\{x_2, y_1\}) - \{u\}| \leq 4$ and therefore x_2, y_1 have the same set of neighbours in $V(G) - (B \cup S)$ by (c) and (e). Let w, w' be the neighbours of x_2 (and also of y_1) such that $w \in V(\text{int}(uy_1w'x_2u))$. Then w is the unique common neighbour of x_2, u, and y_1. By (d) and Lemma 4.4, w is adjacent to w'. Thus $d(w) = 4$. Deleting u and collecting w, x_2, y_1, x, y exposes at least 3 vertices including w' and so $|B(G - \{u, w, x_2, y_1, x, y\})| \geq |B| - 2 + 3$. This contradicts Lemma 4.9 applied with $X = \{u\}$, $Y = \{w, x_2, y_1, x, y\}$, and $s = 1$.

Thus at least one vertex of x_2, u, and y_1 is adjacent to a vertex in $B - \{x, y\}$. Then x_1 is non-adjacent to y_2. By (e), $G[S]$ is an induced path and (f) holds.

(g) Suppose that x^* is adjacent to x_2. As $d(x_1) \geq 4$ by Lemma 4.4, $T := x^*x_1x_2x^*$ is a separating triangle. Since $d(x^*) \leq 5$ by Lemma 6.1, x^* has a unique neighbour $w \in V(\text{int}(T))$. So w is adjacent to both x_1 and x_2. As $d(w) \geq 4$ by Lemma 4.4, $T' := wx_1x_2w$ is a separating triangle with $\|w, V(\text{ext}(T'))\| = 1$ and $\|x_1, V(\text{ext}(T'))\| = 2$, contrary to Lemma 5.3. So x^* is non-adjacent to x_2. By symmetry, y^* is non-adjacent to y_1.

Suppose u is adjacent to x^*. As $d(x_1) \geq 4$ and $d(x^*) \leq 5$ by Lemmas 4.4 and 6.1, x^* has a unique neighbour $w \in V(\text{int}(x^*x_1x_2ux^*))$ adjacent to both x_1 and u. By (b) and (d), w is adjacent to x_2. If uwx_2u is a separating triangle, then by Lemma 5.1(d), $|N(\{x_2, u\}) \cap V(\text{int}(uwx_2u))| \geq 2$, hence $|N^\circ(\{x_2, u\}) - S| \geq 3$, contrary to (d). So uwx_2u is facial. As $d(x^*) \leq 5$, wx^*x_1w and wx^*uw are facial triangles. As $d(x_2) \geq 5$ by (d), $T' := wx_1x_2w$ is a separating triangle. So $\|x_1, V(\text{ext}(T'))\| = 2$ and $\|x_2, V(\text{ext}(T'))\| = 2$, contrary to Lemma 5.3. Thus u is non-adjacent to x^*. By symmetry, u is non-adjacent to y^*. So (g) holds.

(h) Suppose that $w_1 = y^*$. By (g), y^* is adjacent to x_2. Let $C := y^*x_2uy_1y_2y^*$ and C' be the cycle formed by the path from x^* to y^* in $\mathbf{B}(G) - x - y$ together with the path $y^*x_2x_1x^*$. Since G is a near plane triangulation and $d(y_2) \geq 4$, by (f) there is $w \in N(y^*) \cap N(y_2) \cap V(\text{int}(C))$. By Lemma 6.1, $d(y^*) = 5$, and therefore x_2 is adjacent to w and $x_2wy^*x_2$ is a facial triangle. Let $y^{**} \in B$ be the neighbour of y^* other than y. Then $x_2y^{**}y^*x_2$ is also a facial triangle in G. Because x_2 is non-adjacent to x^* by (g), $y^{**} \neq x^*$. By (f) applied to yy^*, we have $y^* \in A$ because $uy_1y_2wx_2$ is not an induced path in G. Thus $x^* \notin A$ because $|A| = 2$. By Lemma 4.11, $\mathbf{B}(G)$ is chordless. Therefore by Lemma 6.2, $d(x^*) = 5$ and so $\|x^*, V(\text{int}(C'))\| = 2$. By (b), (d), and (e), we have $|N^\circ(y_1) - S| = 2$. Deleting x_1, u and collecting x, x^*, y, y_1 exposes at least 6

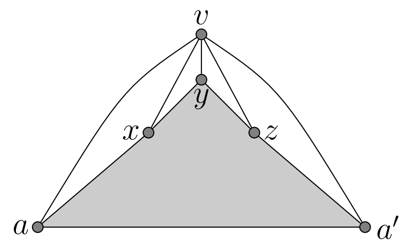


Figure 10: The situation of Lemma 8.1.

vertices, including two neighbours of x^* in $\text{int}(C')$ and two neighbours of y_1 in $\text{int}(C)$. So $|B(G-\{x_1, u, x, x^*, y, y_1\})| \geq |B|-3+6$. By applying Lemma 4.9 with $X = \{x_1, u\}$, $Y = \{x, x^*, y, y_1\}$, and $s = 3$, we obtain a contradiction. So $w_1 \neq y^*$. By symmetry, $w_1 \neq x^*$. Thus (h) holds. □

Lemma 7.2. $|B| = 3$.

Proof. For an edge $e = xy \in E(\mathbf{B} - A)$, let $x^*, x_1, x_2, u, y_1, y_2, y^*$ be as in Lemma 7.1. Suppose that $|B| \geq 4$. Then $x^* \neq y^*$. Lemma 7.1(h) implies that B has a vertex other than x, y, x^*, and y^*. So, $|B| \geq 5$.

We claim that $N^\circ(u) = \{x_2, y_1\}$. Suppose not. By Lemma 4.10, $|A| = 2$, so at least one vertex of $\{x^*, y^*\}$, say, y^* is not in A. By Lemma 7.1(f) applied to $yy*$, we deduce that u is non-adjacent to vertices in $N^\circ(y^*)$. Thus deleting u, y_2 and collecting y, x, y^* exposes at least 6 vertices and so $|B(G - \{u, y_2, y, x, y^*\})| \geq |B| - 3 + 6$. By applying Lemma 4.9 with $X = \{u, y_2\}$, $Y = \{y, x, y^*\}$, and $s = 3$, we obtain a contradiction. So $N^\circ(u) = \{x_2, y_1\}$.

Since $d(u) \geq 5$ by Lemma 7.1(b), u has at least one boundary neighbour $z \neq x, y$. Let $\mathbf{B}(x, z)$ be the boundary path from x to z not containing y, and $\mathbf{B}(y, z)$ be the boundary path from y to z not containing x. So $\mathbf{B}(x, z)$ and $\mathbf{B}(y, z)$ have only one vertex in common, namely z. One of $\mathbf{B}(x, z)$, $\mathbf{B}(y, z)$ has no internal vertex in A. We denote this path by $P(e, z)$. We choose $e = xy$ and z so that $P(e, z)$ is shortest. Assume $P(e, z) = \mathbf{B}(y, z)$. Let $e' = yy^*$. Then $e' \in E(\mathbf{B} - A)$. Let y_2 be the common neighbour of y and y^* and let $z' \neq y, y^*$ be a boundary neighbour of y_2. Then $P(e', z')$ is a proper subpath of $P(e, z)$, and hence is shorter. This contradicts our choice of e and z. □

8 The final contradiction

In this section we complete the proof of Theorem 3.2. First we prove a lemma.

Lemma 8.1. *If $B = \{a, a', v\}$ and $axyza'$ is a path in $G[N(v)]$ (see Figure 10), then the following hold.*

(a) x is non-adjacent to z.

(b) y is adjacent to neither a nor a'.

(c) z is non-adjacent to a and x is non-adjacent to a'.

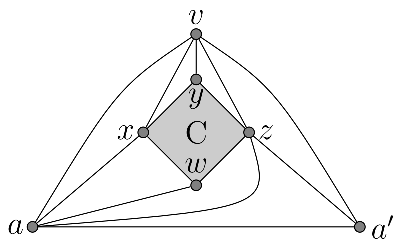


Figure 11: An illustration of the proof of Lemma 8.1(c).

(d) $d(x), d(y), d(z) \geq 5$.
(e) $|N^{\circ}(\{x, y, z\})| \leq 4$.
(f) *x and z have a common neighbour $w \notin \{y, v\}$.*
(g) $N(x) \cap N(z) = \{v, w, y\}$.

Proof. (a) Suppose x is adjacent to z. As $K_5 \not\subseteq G$, x is non-adjacent to a' or z is non-adjacent to a; by symmetry, assume x is non-adjacent to a'. Since $d(y) \geq 4$, $T := xyzx$ is a separating triangle. By Lemma 5.1, $|V(\text{int}(T))| \geq 3$. Since $\|y, V(\text{ext}(T))\| = 1$, Lemma 5.3 implies $\|x, V(\text{ext}(T))\| \geq 4$, and so $|N^{\circ}(x) \cap V(\text{ext}(T))| \geq 2$.

If $d(y) \leq 6$, then deleting x, z and collecting v, y exposes at least 5 vertices from $B(\text{int}(T))$ and $N^{\circ}(x) \cap V(\text{ext}(T))$ and so $|B(G - \{x, z, v, y\})| \geq |B| - 1 + 5$. By applying Lemma 4.9 with $X = \{x, z\}$, $Y = \{v, y\}$, and $s = 4$, we obtain a contradiction. Therefore, $d(y) \geq 7$. Then $|N^{\circ}(y) \cap V(\text{int}(T))| \geq 4$ and so deleting x, y and collecting v exposes at least 7 vertices, and $|B(G - \{x, y, v\})| \geq |B| - 1 + 7$. By applying Lemma 4.9 with $X = \{x, y\}$, $Y = \{v\}$, and $s = 6$, we obtain a contradiction. So (a) holds.

(b) Suppose y is adjacent to a. Then $T := axya$ is a separating triangle, because $d(x) \geq 4$ and the other triangles incident with x are facial. As $d(a) \leq 5$ by Lemma 6.1, a has a unique neighbour w in $\text{int}(T)$. As $d(w) \geq 4$, $T' := xwyx$ is a separating triangle. Now $\|w, V(\text{ext}(T'))\| = 1$, and $\|x, V(\text{ext}(T'))\| = 2$, contrary to Lemma 5.3. Thus y is non-adjacent to a. By symmetry, y is non-adjacent to a'. So (b) holds.

(c) Suppose that z is adjacent to a. By (a), z is non-adjacent to x. As $d(x) \geq 4$ and $d(a) \leq 5$ by Lemmas 4.4 and 6.1, there is $w \in (N(a) \cap N(x) \cap N(z)) - \{v\}$, and $xawx$, $wazw$, $aza'a$ are all facial triangles. (See Figure 11.) By (b), $y \neq w$. Since $d(y) \geq 4$ by Lemma 4.4, $C := xyzwx$ is a separating cycle of length 4. Let $I = \text{int}(C)$. Then $V = B \cup V(C) \cup V(I)$, (i) $\|x, V(\text{ext}(C))\| = 2$, (ii) $\|y, V(\text{ext}(C))\| = 1$, (iii) $\|z, V(\text{ext}(C))\| = 3$, and (iv) $\|w, V(\text{ext}(C))\| = 1$.

If w is adjacent to y, then we apply Lemma 5.2 with C, $X = \{w\}$, and $Y = \emptyset$. As y is adjacent to w, $A' := \{x, y, z\}$ is usable in $G_1 := \text{int}[C] - w$, and by (i–iii), A' is collectable in $G_2' := \text{ext}[C] - w$. As $|V(G_2)| = |B| = 3$, $\tau(G_2) = 0$. This contradicts Lemma 5.2.

So using (a), C is chordless and x has at least one neighbour in $\text{int}(C)$.

By Lemma 5.4, either $|B(I)| \geq 4$ or $|V(I)| \leq 2$ and every vertex in I has degree 4 in G. If $|V(I)| \leq 2$ and every vertex in I has degree 4 in G, then $V - \{x\}$ is A-good as we can collect $V(I), y, w, z, v, a', a$. Then $f(G; A) \geq |V(G)| - 1 \geq \partial(G)$, a contradiction. Therefore $|B(I)| \geq 4$.

If there is an edge $uu' \in E(C)$ with $|N(\{u, u'\}) \cap V(I)| \geq 4$, then we apply Lemma 5.2 with C, $X = \{u, u'\}$ and $Y = \emptyset$. Now $A' := V(C) - \{u, u'\}$ is usable in $G_1 := \text{int}[C] - \{u, u'\}$, A' is collectable in $G_2' := \text{ext}[C] - \{u, u'\}$, $|B_1| \geq 6$, and $B_2 = B$. As $G_2 = \mathbf{B}$, $\tau(G_2) = 0$. This contradicts Lemma 5.2. So $|N(\{u, u'\}) \cap V(I)| \leq 3$ for all edges $uu' \in E(C)$ and in particular, $\|u, V(I)\| \leq 3$ for all $u \in V(C)$. This implies $d(y) \leq 6$.

If $|N(\{x, y, z\}) \cap V(I)| \geq 4$, then we apply Lemma 5.2 with C, $X = \{x, z\}$ and $Y = \{v, y\}$. Then Y is collectable in $G - X$, $A' := \{w\}$ is usable in $G_1 := \text{int}[C] - \{x, y, z\}$, A' is collectable in $G_2' := \text{ext}[C] - \{x, y, z\}$, $|B_1| \geq 5$, and $B_2 = B - \{v\}$. As $(X \cup Y) \cap B \neq \emptyset$, this contradicts Lemma 5.2.

Therefore $|N(\{x, y, z\}) \cap V(I)| \leq 3$. Since $|B(I)| \geq 4$, there exists a vertex u in $B(I) - N(\{x, y, z\})$. Then w is the only neighbour of u in C.

Because G is a plane triangulation and $d(u) \geq 4$, w is adjacent to u. Since u is non-adjacent to x, y, z, we deduce that $B(I) \cap N(w)$ contains u and at least two of the neighbours of u. Since $\|w, V(I)\| \leq 3$, we deduce that $\|w, V(I)\| = 3$. Since $|N(\{x, w\}) \cap V(I)| \leq 3$, all neighbours of x in I are adjacent to w. Similarly all neighbours of z in I are adjacent to w. Since $|B(I)| \geq 4$, there is a vertex t in $B(I)$ non-adjacent to w. Then t is non-adjacent to x and z. Therefore t is adjacent to y. By the same argument, $\|y, V(I)\| = 3$ and every neighbour of x or z in I is adjacent to y. Thus, every vertex in $N(\{x, z\}) \cap V(I)$ is adjacent to both y and w.

If $\|x, V(I)\| \geq 2$, then x, y, w, and their common neighbours in I together with a are the branch vertices of a $K_{3,3}$-subdivision, using the path avy. So G is nonplanar, a contradiction. Thus, $\|x, V(I)\| \leq 1$ and similarly $\|z, V(I)\| \leq 1$. This means that $d(x) \leq 5$ and $B(\text{int}[C] - \{x, y, w\}) = B(I) \cup \{z\}$.

We apply Lemma 5.2 with C, $X = \{w, y\}$ and $Y = \{x\}$. Then Y is collectable in $G - X$ and $A' = \{z\}$ is usable in $G_1 := \text{int}[C] - \{w, x, y\}$, A' is collectable in $G_2' := \text{ext}[C] - \{w, x, y\}$, $|B_1| = |B(I) \cup \{z\}| \geq 5$, $B_2 = B$, and $G_2 = \mathbf{B}$. Thus $\tau(G_2) = 0$ and this contradicts Lemma 5.2. Hence z is non-adjacent to a. By symmetry, x is non-adjacent to a'. Thus (c) holds.

(d) Suppose $d(u) \leq 4$ for some $u \in \{x, y, z\}$. By Lemma 4.4, $d(u) = 4$. Let $u' := y$ if $u \neq y$, $u' := x$ otherwise. Then, deleting u' and collecting u, v exposes at least 2 vertices in $N^\circ(\{u, u'\})$ by (a) and (c) and so $|B(G - \{u, u', v\})| \geq |B| - 1 + 2$. By applying Lemma 4.9 with $X = \{u'\}$, $Y = \{u, v\}$, and $s = 1$, we obtain a contradiction. So (d) holds.

(e) Suppose $|N^\circ(\{x, y, z\})| \geq 5$. If $d(y) \leq 6$, then deleting x, z and collecting v, y exposes at least 5 vertices and so $|B(G - \{x, z, v, y\})| \geq |B| - 1 + 5$. By applying Lemma 4.9 with $X = \{x, z\}$, $Y = \{v, y\}$, and $s = 4$, we obtain a contradiction. Thus $d(y) \geq 7$. Then either $|N^\circ(\{x, y\}) - \{z\}| \geq 5$ or $|N^\circ(\{z, y\}) - \{x\}| \geq 5$. We may assume by symmetry that $|N^\circ(\{x, y\}) - \{z\}| \geq 5$. Then deleting x, y and collecting v exposes at least 6 vertices and so $|B(G - \{x, y, v\})| \geq |B| - 1 + 6$. By applying Lemma 4.9 with

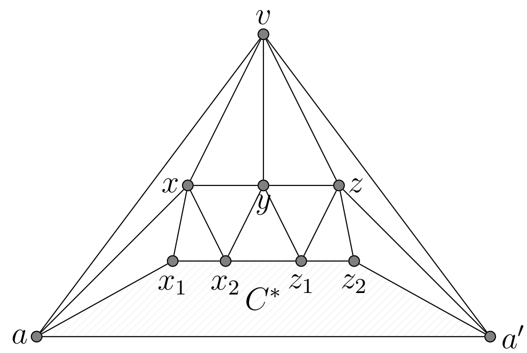


Figure 12: Proof of Lemma 8.1(f). There are no vertices in $\mathrm{int}(C^*)$.

$X = \{x, y\}$, $Y = \{v\}$, and $s = 5$, we obtain a contradiction. So (e) holds.

(f) Suppose $N(x) \cap N(z) = \{y, v\}$. By (d), $d(x), d(z) \geq 5$. By (e), $|N^\circ(\{x, z\}) - \{y\}| \leq 4$. By (c), z is non-adjacent to a and x is non-adjacent to a' and by (a), x is non-adjacent to z. So each of x and z have exactly two neighbours in $\mathrm{int}(axyza'a)$ and $d(x) = d(z) = 5$. Let x_1, x_2 be those neighbours of x and z_1, z_2 be those two neighbours of z. We may assume that $x_1x_2yz_1z_2$ is a path in G by swapping labels of x_1 and x_2 and swapping labels of z_1 and z_2 if necessary. By (e), we have $N^\circ(y) - \{x, z\} \subseteq \{x_1, x_2, z_1, z_2\}$. As $d(x_2) \geq 4$, y is not adjacent to x_1 because otherwise $x_1x_2yx_1$ is a separating triangle, that will make a new interior neighbour of y by Lemma 5.1(c), contrary to (e). By symmetry, y is not adjacent to z_2. So x_2 is adjacent to z_1 as G is a plane triangulation. Therefore $d(y) = 5$.

Let $C^* := ax_1x_2z_1z_2a'a$. Suppose that $w \in N(\{x_1, x_2, z_1, z_2\}) \cap V(\mathrm{int}(C^*))$. Then by symmetry, we may assume w is adjacent to x_1 or x_2. Deleting x_1, x_2 and collecting x, y, v, z exposes w, z_1, z_2 and so $|B(G - \{x_1, x_2, x, y, v, z\})| \geq |B| - 1 + 3$. By applying Lemma 4.9 with $X = \{x_1, x_2\}$, $Y = \{x, y, v, z\}$, and $s = 2$, we obtain a contradiction. Thus $N(\{x_1, x_2, z_1, z_2\}) \cap V(\mathrm{int}(C^*)) = \emptyset$ and therefore $|G| = 10$. See Figure 12.

By Observation 3.1 applied to $\mathrm{int}[C^*]$, there is a vertex $w \in \{x_1, x_2, y_1, y_2\}$ having degree at most 2 in $\mathrm{int}[C^*]$. By symmetry, we may assume that $w = x_i$ for some $i \in \{1, 2\}$. Since $d(x_i) \leq 4$, after deleting x_{3-i}, we can collect x_i, x, y, v, z, resulting in an outerplanar graph, which can be collected by Observation 3.1. So, $f(G; A) \geq 9 \geq \partial(G)$, a contradiction. So (f) holds.

(g) Suppose there is $w' \in N(x) \cap N(z) - \{v, w, y\}$. Let $C := xyzwx$. We may assume that w is chosen to maximize $|V(\mathrm{int}(C))|$. So w' is in $V(\mathrm{int}(C))$ and together with (a), we deduce that C is an induced cycle.

We claim that y is non-adjacent to w'. Suppose not. As $d(y) \geq 5$ by (d), $xw'yx$ or $zw'yz$ is a separating triangle. By symmetry, we may assume $xw'yx$ is a separating triangle. Thus $|N(\{x, y\}) \cap V(\mathrm{int}(xw'yx))| \geq 2$ by Lemma 5.1(d). Because G is a plane triangulation, by (e), w is adjacent to w' and $xww'x$, $zww'z$, and $yzw'y$ are facial triangles. Thus $\|y, V(\mathrm{ext}(xw'yx))\| = \|w', V(\mathrm{ext}(xw'yx))\| = 2$, contrary to Lemma 5.3. This proves the claim that y is non-adjacent to w'.

Therefore $\|y, V(\mathrm{int}(xyzw'x))\| = 2$ by (d) and (e). Let y_1, y_2 be two neighbours of y in $\mathrm{int}(xyzw'x)$ such that xy_1y_2z is a path in G. Because G is a plane triangulation, by (e), w' is adjacent to both y_1 and y_2 and $\mathrm{int}(xw'zwx)$ has no vertex. Then C is a separating induced cycle of length 4 and $|B(\mathrm{int}(C))| = 3$, contrary to Lemma 5.4. So (g) holds. □

Proof of Theorem 3.2. Let $(G; A)$ be an extreme counterexample. Then G is a near plane triangulation. Let $B = B(G)$ and $\mathbf{B} = \mathbf{B}(G)$. By Lemmas 4.10 and 7.2, $|B| = 3$ and $|A| = 2$. Let $A = \{a, a'\}$ and $v \in B - A$. By Lemma 6.2, $d(v) = 5$. As G is a plane triangulation, the neighbours of v form a path $axyza'$. By Lemma 8.1(g), x and z have exactly one common neighbour w in $G - v - y$. Then $C := xyzwx$ is a cycle of length 4. By symmetry and Lemma 8.1(d), we may assume that $d(x) \geq d(z) \geq 5$. By Lemma 8.1(e),

$$(d(x) - 3) + (d(z) - 3) - 1 \leq |N^\circ(\{x, y, z\})| \leq 4.$$

Therefore $d(z) = 5$ and $d(x) = 5$ or 6.

We claim that y is non-adjacent to w. Suppose that y is adjacent to w. By Lemma 8.1(d), $d(y) \geq 5$ and therefore at least one of $xywx$ and $yzwy$ is a separating triangle. If both of them are separating triangles, then $|N(\{x, y\}) \cap V(\mathrm{int}(xywx))| \geq 2$ and $|N(\{y, z\}) \cap V(\mathrm{int}(yzwy))| \geq 2$, by Lemma 5.1(d). Therefore $|N^\circ(\{x, y, z\})| \geq 2 + 2 + 1 = 5$, contrary to Lemma 8.1(e). This means that exactly one of $xywx$ and $yzwy$ is a separating triangle.

Suppose $yzwy$ is a separating triangle. Then $xywx$ is a facial triangle, and z has a neighbour in $\mathrm{int}(yzwy)$. As $d(z) = 5$, z has no neighbour in $\mathrm{int}(axwza'a)$. Therefore, w is adjacent to a', and $wza'w$ is a facial triangle. Thus $\|y, V(\mathrm{ext}(yzwy))\| = \|z, V(\mathrm{ext}(yzwy))\| = 2$, contrary to Lemma 5.3. So $yzwy$ is not a separating triangle.

Therefore $xywx$ is a separating triangle. By Lemma 5.1(d), $\mathrm{int}(xywx)$ has at least two vertices in $N^\circ(\{x, y, z\})$. By Lemma 8.1(d), z has a neighbour in $\mathrm{int}(axwza'a)$. Then already we found four vertices in $N^\circ(\{x, y, z\})$. This means that x has no neighbours in $\mathrm{int}(axwza'a)$ by Lemma 8.1(f). Hence $\|y, V(\mathrm{ext}(xywx))\| = \|x, V(\mathrm{ext}(xywx))\| = 2$, contrary to Lemma 5.3. This completes the proof of the claim that y is non-adjacent to w.

Therefore C is chordless by Lemma 8.1(a). By Lemma 8.1(d), $d(y) \geq 5$. Thus C is a separating induced cycle of length 4. By Lemma 5.4, either $|B(\mathrm{int}(C))| \geq 4$ or both $|V(\mathrm{int}(C))| \leq 2$ and every vertex in $\mathrm{int}(C)$ has degree 4 in G.

If $|B(\mathrm{int}(C))| \geq 4$, then deleting w, y and collecting z, v, x exposes at least 4 vertices and so $|B(G - \{w, y, z, v, x\})| = |B| - 1 + 4$. By applying Lemma 4.9 with $X = \{w, y\}$, $Y = \{z, v, x\}$, and $s = 3$, we obtain a contradiction.

Therefore $1 \leq |V(\mathrm{int}(C))| \leq 2$ and every vertex in $\mathrm{int}(C)$ has degree 4 in G. Deleting y and collecting all vertices in $\mathrm{int}(C)$ and z, v, x exposes w and so $B(G - (\{y, z, v, x\} \cup V(\mathrm{int}(C)))) \geq |B| - 1 + 1$. By applying Lemma 4.9 with $X = \{y\}$, $Y = V(\mathrm{int}(C)) \cup \{z, v, x\}$, and $s = 0$, we obtain a contradiction. □

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