Realization of digraphs in Abelian groups and its consequences

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Abstract

Let $\overrightarrow{G}$ be a directed graph with no component of order less than 3, and let Γ be a finite Abelian group such that $|\Gamma| \geq 4|V(\overrightarrow{G})|$ or if $|V(\overrightarrow{G})|$ is large enough with respect to an arbitrarily fixed $\varepsilon > 0$ then $|\Gamma| \geq (1+\varepsilon)|V(\overrightarrow{G})|$. We show that there exists an injective mapping φ from $V(\overrightarrow{G})$ to the group Γ such that $\sum_{x\in V(C)} \varphi(x) = 0$ for every connected component C of $\overrightarrow{G}$, where 0 is the identity element of Γ. Moreover we show some applications of this result to group distance magic labelings.

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1 Introduction

Let $\overrightarrow{G} = (V, A)$ be a directed graph. An arc $\overrightarrow{xy}$ is considered to be directed from x to y, moreover y is called the *head* and x is called the *tail* of the arc. For a vertex x, the set of head endpoints adjacent to x is denoted by $N^+(x)$, and the set of tail endpoints adjacent to x is denoted by $N^-(x)$.

Assume Γ is an Abelian group of order n with the operation denoted by $+$. For convenience we will write ka to denote $a + a + \ldots + a$ where the element a appears k times, $-a$ to denote the inverse of a, and we will use $a - b$ instead of $a + (-b)$. Moreover, the notation $\sum_{a \in S} a$ will be used as a short form for $a_1 + a_2 + a_3 + \ldots$, where $a_1, a_2, a_3, \ldots$ are all elements of the set S. The identity element of Γ will be denoted by 0. Recall that any group element $\iota \in \Gamma$ of order 2 (i.e., $\iota \neq 0$ and $2\iota = 0$) is called an *involution*.

Suppose that there exists a mapping ψ from the arc set $E(\overrightarrow{G})$ of $\overrightarrow{G}$ to an Abelian group Γ such that if we define a mapping φ from the vertex set $V(\overrightarrow{G})$ of G to Γ by

$$\varphi_\psi(x) = \sum_{y \in N^+(x)} \psi(yx) - \sum_{y \in N^-(x)} \psi(xy), \quad (x \in V(G)),$$

then φ_ψ is injective. In this situation, we say that $\overrightarrow{G}$ is *realizable* in Γ, and that the mapping ψ is Γ-*irregular*.

The corresponding problem in the case of simple graphs was considered in [2, 3, 4]. For $\Gamma = (\mathbb{Z}_2)^m$ the problem was raised in [13]. We easily see that if $\overrightarrow{G}$ is realizable in $(\mathbb{Z}_2)^m$, then every component of $\overrightarrow{G}$ has order at least 3 (recall that we are assuming $\overrightarrow{G}$ has no isolated vertex). The following results have been shown:

Theorem 1.1 ([6]). *Let $\overrightarrow{G}$ be a directed graph with no component of order less than 3. Then $\overrightarrow{G}$ is realizable in $(\mathbb{Z}_2)^m$ if and only if $|V(\overrightarrow{G})| \leq 2^m$ and $|V(\overrightarrow{G})| \neq 2^m - 2$.*

Theorem 1.2 ([9]). *Let p be an odd prime and let $m \geq 1$ be an integer. If $\overrightarrow{G}$ is a directed graph without isolated vertices such that $|V(\overrightarrow{G})| \leq p^m$, then $\overrightarrow{G}$ is realizable in $(\mathbb{Z}_p)^m$.*

In this paper we will prove that a directed graph $\overrightarrow{G}$ with no component of order less than 3 is realizable in any Γ of order at least $|V(\overrightarrow{G})|$ such that either Γ is of an odd order or Γ contains exactly three involutions. Moreover we will show that a directed graph $\overrightarrow{G}$ with no component of order less than 3 is realizable in any Γ such that $|\Gamma| \geq 4|V(\overrightarrow{G})|$. Further, the coefficient 4 will be improved substantially for $|V(\overrightarrow{G})|$ large enough. In the last section we will show some applications of this result.

2 Characterizations and sufficient conditions

A subset S of Γ is called a zero-sum subset if $\sum_{a\in S} a = 0$. It turns out that a realization of $\overrightarrow{G}$ in an Abelian group Γ is strongly connected with a zero-sum partition of Γ [1, 9]. Using exactly the same arguments as in [9] for elementary Abelian groups we show the following for general Abelian groups.

Theorem 2.1. *A directed graph $\overrightarrow{G}$ with no isolated vertices is realizable in Γ if and only if there exists an injective mapping φ from $V(G)$ to Γ such that $\sum_{x\in V\langle C\rangle} \varphi(x) = 0$ for every component C of G.*

Proof. The necessity is obvious. To prove the sufficiency, let φ be an injective mapping from $V(\overrightarrow{G})$ to Γ such that $\sum_{x\in V\langle C\rangle} \varphi(x) = 0$ for every connected component C of $\overrightarrow{G}$.

Let C be a connected component of $\overrightarrow{G}$. It suffices to show that there exists a mapping ψ from $E(C)$ to Γ satisfying:

$$\varphi(x) = \sum_{y\in N^+(x)} \psi(yx) - \sum_{y\in N^-(x)} \psi(xy), \quad (x \in V(G)).$$

Now we will construct a spanning tree of C. Let $V(C) = \{x_1, \ldots, x_k\}$ ($k = |V(C)|$) so that for each $2 \leq i < k$, there exists exactly one arc e_i between $\{x_l, \ldots, x_{i-1}\}$ and x_i. For each $2 \leq i < k$, define a subdigraph C_i of C by setting $V(C_i) = V(C)$ and $E(C_i) = \{e_{i+l}, \ldots, e_k\}$. Let $\psi(e_k) = \varphi(x_k)$. We define ψ backward inductively by

$$\psi(e_i) = \begin{cases} \varphi(x_i) - \sum_{y\in N^+_{C_i}(x_i)} \psi(x_iy) + \sum_{y\in N^-_{C_i}(x_i)} \psi(yx_i), & \text{if } x_i \text{ is the tail of } e_i, \\ -\varphi(x_i) + \sum_{y\in N^+_{C_i}(x_i)} \psi(x_iy) - \sum_{y\in N^-_{C_i}(x_i)} \psi(yx_i), & \text{if } x_i \text{ is the head of } e_i. \end{cases}$$

Finally, let $\psi(e) = 0$ for all $e \in E(C) \setminus \{e_2, \ldots, e_m\}$. Then the resulting mapping ψ has the desired property. $\square$

The following result is known.

Theorem 2.2 ([10, 14]). *Let Γ have order n. For every partition $n-1 = r_1 + r_2 + \ldots + r_t$ of $n-1$ with $r_i \geq 2$ for $1 \leq i \leq t$ and for any possible positive integer t, there is a partition of $\Gamma - \{0\}$ into pairwise disjoint subsets $A_1, A_2, \ldots, A_t$ such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = 0$ for all $1 \leq i \leq t$ if and only if either Γ is of an odd order or Γ contains exactly three involutions.*

By Theorems 2.1 and 2.2 we obtain the following immediately.

Theorem 2.3. *A directed graph $\overrightarrow{G}$ with no component of order less than 3 is realizable in any Γ of order at least $|V(\overrightarrow{G})|$ such that either Γ is of an odd order or Γ contains exactly three involutions.*

Before we proceed to groups having more than three involutions, we need some lemmas. For the sake of simplicity, for any element $a \in \Gamma$, we are going to use the notation $a/2$ for an arbitrarily chosen element $b \in \Gamma$ satisfying $2b = a$. Let $S_{a/2} = \{b \in \Gamma : 2b = a\}$.

Observation 2.4. *If Γ is an Abelian group of even order n, then $|S_{a/2}| \leq n/2$ for any element $a \in \Gamma$, $a \neq 0$.*

Proof. If for some $a \neq 0$ there exist $g_1, g_2 \in \Gamma$, $g_1 \neq g_2$ such that $2g_1 = a$ and $2g_2 = a$ then it follows that $2(g_1 - g_2) = 0$ and consequently $g_1 - g_2$ is an involution. Since $2g_1 = a \neq 0$, the number of involutions in Γ is less than $|\Gamma|/2$. $\square$

Lemma 2.5. *Let $\overrightarrow{G}$ be a directed graph with no component of order less than 3, and let Γ be a finite Abelian group such that $|\Gamma| \geq 4|V(\overrightarrow{G})|$. There exists a Γ-irregular labeling ψ of $\overrightarrow{G}$ such that $\psi(e) \neq 0$ for every $e \in E(\overrightarrow{G})$, and $\varphi_\psi(x) \neq 0$ for every $x \in V(\overrightarrow{G})$.*

Proof. The proof follows by induction on the number of arcs.

Suppose first that $\overrightarrow{G}$ is a path $\overrightarrow{P}_3$ with vertices, say, u, v and w and arcs e_1 and e_2. With no loss of generality we can assume that $e_1 \cap e_2 = v$. Let Γ be an arbitrary Abelian group of order at least 12. Set an element $a \neq 0$ in such a way that $\varphi_\psi(u) = a$ (namely, $\psi(vu) = a$ and $\psi(uv) = -a$). Now, choose any $b \notin \{0, a, -a, -2a\}$ and $b \notin S_{-a/2}$. The number of forbidden values is at most $4 + |\Gamma|/2 < |\Gamma|$. Set now the element b in such a way that $\varphi_\psi(v) = -a - b$. Both arc labels are different from 0, and so are the vertex weighted degrees, since $\varphi_\psi(u) = a$, $\varphi_\psi(v) = -a - b$ and $\varphi_\psi(w) = b$. It is also obvious that the weighted degrees are three distinct elements of Γ.

Now let $\overrightarrow{G}$ be arbitrary directed graph of order n with at least 3 edges, having no component of order less than 3, and let Γ be any Abelian group of order at least $4n$. In the induction step we can assume that for every proper subgraph $\overrightarrow{H}$ of $\overrightarrow{G}$ having no component of order less than 3 and for every Abelian group Γ' of order at least $4|\overrightarrow{H}|$, there is a Γ'-irregular labeling ψ_H of $\overrightarrow{H}$ in which no edge has label 0 and $\varphi_{\psi_H}(x) \neq 0$ for every $x \in V(\overrightarrow{H})$. In particular, there is such labeling of $\overrightarrow{H}$ with $\Gamma' = \Gamma$, since $|\Gamma| \geq 4n \geq 4|V(\overrightarrow{H})|$. We will extend ψ_H to the labeling ψ of $\overrightarrow{G}$, having the same properties.

We choose $\overrightarrow{H}$ in one of the following ways. If there is a component $C \cong \overrightarrow{P}_3$ of $\overrightarrow{G}$, then $\overrightarrow{H} = \overrightarrow{G} - C$. Otherwise, if there is a component C and an edge $e \in E(C)$ not being a bridge in C, then $\overrightarrow{H} = \overrightarrow{G} - e$. Finally, if $\overrightarrow{G}$ is a forest with each component of order at least 4, then choose any leaf edge e of any component and let $\overrightarrow{H} = \overrightarrow{G} - e$.

Let us consider the first case. Assume that $\overrightarrow{G} = \overrightarrow{H} \cup \overrightarrow{H}'$, where $V(H') = \{u, v, w\}$ and $E(H') = \{e_1, e_2\}$ such that $e_1 \cap e_2 = u$. Let ψ_H be a Γ-irregular labeling of $\overrightarrow{H}$ fulfilling the desired non-zero properties, existing by the induction hypothesis. Let $\psi(e) = \psi_H(e)$ for $e \in E(\overrightarrow{H})$. Now choose any element of $a \in \Gamma$ such that $a \neq 0$ and $a \neq \varphi_{\psi_H}(x)$ for $x \in V(\overrightarrow{H})$ and set the label a on the edge e_1 such that $\varphi_\psi(v) = a$. Such a can be chosen, as only $n-3$ vertex weighted degrees have been assigned so far and $|\Gamma| > n-2$. Now choose $b \in \Gamma$ such that $b \notin \{0, a, -a, -2a\}$, $b \notin S_{-a/2}$, $b \notin \{\varphi_{\psi_H}(x), -w(x) + a\}$ for $x \in V(H)$ and set b on the arc e_2 such that $\varphi_\psi(w) = b$. The number of forbidden elements is at most $4 + |\Gamma|/2 + 2(n-3) = 2n - 2 + |\Gamma|/2 < |\Gamma|$, so we can choose such b. Obviously, the two new edge labels are not 0 and neither are the three new weighted degrees $\varphi_\psi(v) = a$, $\varphi_\psi(w) = b$ and $\varphi_\psi(u) = -a - b$. Also, the three new weighted degrees are pairwise distinct and not equal to any $\varphi_\psi(x)$, where $x \in V(\overrightarrow{H})$.

In the second case, let $\overrightarrow{H} = \overrightarrow{G} - e$ and let ψ_H be a Γ-irregular labeling of $\overrightarrow{H}$ fulfilling the desired non-zero properties, existing by the induction hypothesis. Now let $\psi(y) = \psi_H(y)$ for $y \in E(\overrightarrow{H})$. Let us denote the tail of e by u and the head by v. Choose an element $a \in \Gamma$ such that $a \notin \{0, \varphi_{\psi_H}(u), -\varphi_{\psi_H}(v)\}$, $a \neq \varphi_{\psi_H}(u) - \varphi_{\psi_H}(x)$ for $x \in V(\overrightarrow{G}) \setminus \{u, v\}$ and $a \neq \varphi_{\psi_H}(x) - \varphi_{\psi_H}(v)$ for $x \in V(\overrightarrow{G}) \setminus \{u, v\}$, and $a \notin S_{(\varphi_{\psi_H}(u) - \varphi_{\psi_H}(v))/2}$. Set $\psi(uv) = a$. The number of forbidden values is at most $3 + 2(n-2) + |\Gamma|/2 < |\Gamma|$, so we can always choose such a. Note that two adjusted weighted degrees remain distinct and because of the way that a was chosen, they are different from any weighted degree $\varphi_\psi(x)$ for $x \in V(\overrightarrow{G}) \setminus \{u, v\}$. This means that ψ has the desired property.

Finally, consider the third case. Assume that the ends of e are u and v, where u is the pendant vertex. Having a Γ-irregular labeling ψ_H of $\overrightarrow{H}$ fulfilling the desired non-zero properties, we set $\psi(y) = \psi_H(y)$ for $y \in E(\overrightarrow{H})$. Then we choose $a \in \Gamma$ such that $a \notin \{0, -\varphi_{\psi_H}(v)\}$, $a \neq \varphi_{\psi_H}(x)$ for $x \in V(\overrightarrow{G}) \setminus \{u, v\}$ and $a \neq \varphi_{\psi_H}(v) - \varphi_{\psi_H}(x)$ for $x \in V(\overrightarrow{G}) \setminus \{u, v\}$, and $a \notin S_{\varphi_{\psi_H}(u)/2}$. There are at most $2 + 2(n-2) + |\Gamma|/2 < |\Gamma|$ forbidden values, so we can choose such a. Put label a on the edge e such that $\varphi_\psi(u) = a$. The adjusted weighted degree $\varphi_\psi(v)$ and the new weighted degree $\varphi_\psi(u)$ are distinct and different from any of the weighted degrees $\varphi_\psi(x)$ for $x \in V(G) \setminus \{u, v\}$, so also in this case $\overrightarrow{G}$ has the labeling ψ with the desired property. This completes the proof. $\square$

The above lemma implies the following.

Theorem 2.6. *A directed graph $\overrightarrow{G}$ with no component of order less than 3 is realizable in any Γ such that $|\Gamma| \geq 4|V(\overrightarrow{G})|$.*

3 Asymptotic result

Our goal here is to prove that if $|\Gamma|$ gets large, then it is possible to strengthen Theorem 2.6 considerably, by replacing the multiplicative constant 4 in the condition $|\Gamma| \geq 4|V(\overrightarrow{G})|$ with $(1 + o(1))$, and also omitting the assumption that Γ has more than one involution. In the proof we shall apply the following corollary of Theorem 1.1 from [7].

Lemma 3.1. *For every fixed $\varepsilon > 0$ there exists an $n_0 = n_0(\varepsilon)$ with the following properties. If $\mathcal{H}$ is a 3-uniform regular hypergraph with $n > n_0$ vertices such that the degree of regularity is at least $n/3$, and each vertex pair is contained in at most two hyperedges, then $\mathcal{H}$ contains at least $(1/3 - \varepsilon/3)n$*

pairwise disjoint hyperedges. Moreover if $\mathcal{H}$ is a 4-uniform hypergraph with $n > n_0$ vertices, such that the vertex degrees are nearly equal and at least $n/4$, and each vertex pair is contained in at most three hyperedges, then $\mathcal{H}$ contains at least $(1/4 - \varepsilon/4)n$ pairwise disjoint hyperedges.

In fact the degree condition $n/3$ (and also $n/4$ in the 4-uniform case) can be replaced with cn with any constant $c > 0$, but we shall not need this stronger version of the lemma to allow very small degrees.

As another tool, we will use the following corollary of Theorem 1.1.

Corollary 3.2 ([6]). *Let $p \geq 2$ be an integer, and let q_3, q_4, q_5 be nonnegative integers such that $3q_3 + 4q_4 + 5q_5 \leq 2^p$ and $3q_3 + 4q_4 + 5q_5 \neq 2^p - 2$. Then there exists a family $Z = \{S_1, \ldots, S_{q_3+q_4+q_5}\}$ of $q_3+q_4+q_5$ mutually disjoint zero-sum subsets of $(\mathbb{Z}_2)^p$ such that $|S_i| = 3$ for all $1 \leq i \leq q_3$, $|S_i| = 4$ for all $q_3 + 1 \leq i \leq q_3 + q_4$, and $|S_i| = 5$ for all $q_3 + q_4 + 1 \leq i \leq q_3 + q_4 + q_5$.*

The main result of this section is the following.

Theorem 3.3. *Let $\varepsilon > 0$ be fixed and assume that n is sufficiently large with respect to ε. Let $\Gamma \neq (\mathbb{Z}_2)^m$ be of order n, and consider any integers $r_1, r_2, \ldots, r_t$ with $n > (1+\varepsilon)(r_1 + r_2 + \ldots + r_t)$ and $r_i \geq 2$ for all $1 \leq i \leq t$. If $r_i = 2$ holds for at most $n/4$ terms r_i, then there exist pairwise disjoint subsets $A_1, A_2, \ldots, A_t$ in $\Gamma - \{0\}$ such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = 0$ for $1 \leq i \leq t$.*

Proof. In order to make the structure of the argument more transparent, we split it into several parts.

1° Assume $n > (2/\varepsilon) \cdot n_0(\varepsilon/2)$, where the function n_0 is from Lemma 3.1. We denote by I the set of involutions in Γ, and write R for the set of the other nonzero elements, i.e. $R = \Gamma \setminus (I \cup \{0\})$. We shall distinguish between the elements $\iota_1, \iota_2, \ldots$ of I with subscripts, and wrtite $a, b, c, \ldots$ for the elements of R. A generic element may simply be denoted by $\iota \in I$ or $a \in R$.

Since $\Gamma \neq (\mathbb{Z}_2)^m$, we have $|R| \geq n/2$. If $|I| \leq \varepsilon n/2$, then we omit I, and continue work in R alone. (This simplification also involves the elimination of the involution in case if Γ has just one; then of course the elements of R sum up to 0.) Otherwise we have both $|I|$ and $|R|$ larger than $n_0(\varepsilon/2)$. Below we describe the procedure for this more general case.

2° If there are terms r_i larger than 6, we modify the sequence by splitting each large term into a combination of terms 3 and 4. Once the new sequence admits suitable zero-sum subsets, a solution for the original sequence follows immediately. Note that this step does not create any new $r_i = 2$, i.e. the condition on the number of terms 2 does not get violated. Consequently we may assume $r_i \in \{2, 3, 4, 5\}$ for all $1 \le i \le t$.

In order to make further simplification, we state and prove the theorem in the following stronger form:

($\star$) The required disjoint zero-sum subsets A_i exist also under the weaker assumption that the number of $r_i = 2$ terms is at most $|R|/2$.

Note that the bound $|R|/2$ is absolutely tight for every Γ because a zero-sum pair necessarily is of the type $(a, -a)$.

Now, if there is an $r_i = 5$, we may split it into $2 + 3$, unless there are exactly $|R|/2$ terms $r_i = 2$. Similarly, if there is an $r_i = 4$, we may split it into $2+2$, unless the number of $r_i = 2$ terms is $|R|/2$ or $|R|/2-1$. In this way the family of sequences $r_1, \dots, r_t$ to be studied is reduced to the following two cases:

1. there are exactly $|R|/2$ or $|R|/2 - 1$ terms $r_i = 2$, and all the other terms are 3, 4, or 5; or
2. we have $r_i \in \{2, 3\}$ for all $1 \le i \le t$.

3° In case of 1., we can obtain the following much stronger result:

($\star\star$) If the number of $r_i = 2$ terms is $|R|/2$ or $|R|/2 - 1$, then the required disjoint zero-sum subsets A_i exist whenever $|\Gamma| \ge r_1 + \cdots + r_t + 5$.

Indeed, if $|I| = 1$ then we lose at most two elements from R and the only one element of I. Otherwise Γ has at least three involutions and we may apply Corollary 3.2. Namely, the 3-, 4-, and 5-terms can surely be assigned to suitable subsets A_i of I if their sum is at most $|I| - 2$; and creating the $(a, -a)$ pairs for $a \in R$ we lose at most two elements from R (and the 0-element of Γ).

4° In order to handle the case 2., we create an auxiliary set R^* whose elements represent the inverse pairs of R, i.e. each $a^* \in R^*$ stands for $(a, -a)$; hence $|R^*| = |R|/2$. Note that each inverse-free subset of R defines a unique subset of R^* with the same cardinality, in the natural way, while a k-element subset of R^* may arise from 2^k distinct subsets of R. One should be warned, however, that R^* does not inherit the group structure of R. Indeed, if $a+b=c$ holds inside R, then $a+(-b) \neq (-c)$; that is, $b^* = (-b)^*$, but $(a+b)^* \neq (a-b)^*$.

5° Using Corollary 3.2 again, inside I we define a large family T_I of pairwise disjoint triples which together nearly cover I, such that the sum of the three elements in each triple equals 0. If $|I|$ is a multiple of 3, we can partition I into 3-element zero-sum subsets. Otherwise, if $|I| = 2^p - 1 = 3q+1$, we find $q-1$ triples and one quadruple, each of whose elements sum up to zero. Thus T_I covers all but at most four elements of I.

6° An analogous set T_R of pairwise disjoint triples which nearly cover R is more complicated to construct, because we put *two* requirements instead of just one: if a triple $\{a, b, c\}$ is in T_R, then

- $a+b+c=0$; and
- the inverse triple $\{-a, -b, -c\}$ also belongs to T_R.

We first construct an edge-labeled complete graph whose vertex set is R, and each edge $ab \in \binom{R}{2}$ gets the label $\lambda(a,b) := (-a)+(-b)$. Hence $\lambda(-a,a) = 0$ for all a by definition, we shall disregard these edges. At each $a \in R$ precisely $|I|$ edges are labeled from I, and consequently $|R|-|I|-2$ edges are labeled from R. The strategy depends on whether the former or the latter is larger.

If $|I| < |R|/2$, or equivalently $|R| \geq 2|I|+2$, we keep the edges labeled from R. It means that for all such edges we have $T_{a,b} := (a, b, \lambda(a,b)) \in \binom{R}{3}$, moreover every $T_{a,b}$ is a zero-sum triple. This gives rise to the 3-uniform hypergraph, say H_R, whose hyperedges are the triples $T_{a,b}$, each pair of vertices belonging to either 0 or 1 hyperedge, and therefore each vertex being incident with exactly $(|R|-|I|-2)/2$ hyperedges, hence the vertex degrees satisfy the inequality $\frac{(|R|-|I|-2)/2}{|R|} \geq 1/2 - \frac{|I|+2}{4|I|+4} \geq 1/2 - 5/16 = 3/16$.

Note that if $T_{a,b} \in H_R$ then also $T_{-a,-b} \in H_R$ (and of course vice versa), and they yield the same triple $T_{(a,b)^*} = T_{(-a,-b)^*}$ in the corresponding system H_{R^*} over R^*. We observe that inside the 6-tuple $T_{a,b} \cup T_{-a,-b}$ there do not exist any further zero-sum triples. Indeed, a third such triple should contain at least one element from each of $T_{a,b}$ and $T_{-a,-b}$, hence it would be of the form $T_{a,-b}$ or alike. But then we would have $b-a \in \{a+b, -a-b\}$, from where $a+a=0$ or $b+b=0$ would follow, contrary to the assumption $a, b \notin I$. (This is in agreement with the comment above that R^* does not inherit the group structure of R.) It follows that the number of triples incident with a vertex in H_{R^*} is exactly the same as that in H_R. In particular, H_{R^*} is regular of a degree at least $\frac{3}{8}|R^*|$. Moreover the maximum number of triples containing a pair a^*, b^* increases from 1 to 2, but not more. Therefore Lemma 3.1 can be applied and we obtain $(1/3 - \varepsilon/6) \cdot |R^*|$ pairwise disjoint triples in R^*. Each of those triples (a^*, b^*, c^*) originates from a triple (a, b, c) with $a + b + c = 0$, hence it generates (a, b, c) and $(-a, -b, -c)$ inside R. We denote this collection of disjoint zero-sum triples by T_R. They together cover $(1 - \varepsilon/2) \cdot |R|$ elements of R in the case of $|I| < |R|/2$.

Otherwise, if $|I| \geq |R|/2$, note first that $|R| = |I| + 1 = |\Gamma|/2$ must hold, because $|R|$ is divisible by $|I| + 1$ in every Γ. Indeed, $a + \iota \in R$ holds for all $a \in R$ and $\iota \in I$. Moreover, if $b - a = \iota_1$ and $c - b = \iota_2$ then $c - a = \iota_1 + \iota_2 \in I$ hence the reflexive closure of the relation '$b - a \in I$' is an equivalence relation over R and each of its equivalence classes contains exactly $|I| + 1$ elements.

Consequently, the set $\{a + \iota \mid \iota \in I\}$ is the same as $R \setminus \{a\}$, therefore we have $-a = a + \iota_0$ for some $\iota_0 \in I$ (and of course $a = (-a) + \iota_0$ also). We observe that $-b = b + \iota_0$ holds for all $b \in R$. Indeed, if $b = a + \iota_1$, then $-b = \iota_1 - a = \iota_1 + (-a) = \iota_1 + a + \iota_0 = b + \iota_0$. (In case of larger R, this property would be guaranteed inside each equivalence class only.)

Let us put $x := \lceil |R|/6 \rceil$, and take x triples from T_I constructed above, such that none of them contains $\iota_0 \in I$. This selection can be done because $|T_I| \geq (|I| - 4)/3 = (|R| - 5)/3 \geq |R|/6 + 1$ whenever Γ is not too small. Recall that each member of T_I is of the form $(\iota_1, \iota_2, \iota_3)$ with $\iota_1 + \iota_2 + \iota_3 = 0$. We represent the x selected triples with new *vertices* $u_1, u_2, \ldots, u_x$, and define a nearly regular 4-uniform hypergraph on the vertex set $Q^* := R^* \cup \{u_1, u_2, \ldots, u_x\}$. The hyperedges are of the form (a^*, b^*, c^*, u_j), where a is any element, $b = a + \iota_1$, $c = b + \iota_2$ (hence $a = c + \iota_3$). We do this for all $a \in R$ and all permutations of $\iota_1, \iota_2, \iota_3$ in all of the first x triples.

It is important to note that a^*, b^*, c^* are three distinct elements because the triple of T_I containing ι_0 has not been selected. For the same reason, for any fixed permutation $(\iota_1, \iota_2, \iota_3)$ of any selected triple, the elements a and $-a$ yield the same quadruple; that is, disregarding the representing new vertex u_j, we have $\{a^*, (a + \iota_1)^*, (a + \iota_1 + \iota_2)^*\} = \{(-a)^*, (-a + \iota_1)^*, (-a + \iota_1 + \iota_2)^*\}$. Moreover, since any two of $\iota_1, \iota_2, \iota_3$ sum to the third, the quadruples of the form $\{a^*, (a + \iota_1)^*, (a + \iota_2)^*, (a + \iota_3)^*\}$ partition R^*; this holds for each u_j. Inside each such quadruple, three of the four 3-element subsets contain a^*. It follows that there are exactly $|R^*| = |R|/2$ hyperedges incident with any u_j, and the degree of an a^* equals $3x \approx |R|/2$. Thus, the conditions of

Lemma 3.1 are satisfied, and there is a large packing of 4-element hyperedges (a^*, b^*, c^*, u_j) covering all but at most $(\varepsilon/2) \cdot |Q^*|$ vertices of Q^*.

For every (a^*, b^*, c^*, u_j) and its corresponding $(\iota_1, \iota_2, \iota_3)$ we create the triples[1] $(a, -b, \iota_1)$, $(b, -c, \iota_2)$, $(c, -a, \iota_3)$. All these three are zero-sum triples, and they partition the 9-tuple $\{a, b, c, -a, -b, -c, \iota_1, \iota_2, \iota_3\}$. Observe further that no triangle from T_I is used more than once in the construction; this is ensured by the presence of vertices u_j. In this way we obtain T_R if $|I| \geq |R|/2$.

7° In case 2., we have $r_i \in \{2, 3\}$ for all $1 \leq i \leq t$. Let m_k denote the number of terms $r_i = k$ for $k = 2, 3$.

If $m_3 \leq |T_I|$, we simply take any m_3 triples from T_I, and choose $m_2 \leq |R|/2$ pairs $(a, -a)$ inside R. Otherwise two different situations may occur, depending on whether $|I| < |R|/2$ or not. In both cases we assume that $m_3 > |T_I|$ holds.

If $|I| < |R|/2$, we take the triples of T_I, moreover $2 \cdot \lceil (m_3 - |T_I|)/2 \rceil$ triples from T_R in such a way that if a triple (a, b, c) is selected, then we also select $(-a, -b, -c)$. This may yield one more triple than what we need, which we shall forget at the very end; but currently it is kept, in order to ensure that the rest of R consist of inverse pairs.

If $|I| \geq |R|/2$, we again start with the triples of T_I, but then replace $\lceil (m_3 - |T_I|)/2 \rceil$ of them with three triples from T_R each. This can be done by choosing $\lceil (m_3 - |T_I|)/2 \rceil$ from the first x members of T_I, and replacing them with the triples covering $\{a, b, c, -a, -b, -c, \iota_1, \iota_2, \iota_3\}$ as constructed above.

In either case, since I is covered with the exception of at most four elements, there remains enough room for selecting the m_2 pairs $(a, -a)$ in the part of R which is not covered by the selected triples. This completes the proof of the theorem. □

Corollary 3.4. *There exists a function $h_0 : \mathbb{N} \to \mathbb{N}$ with the following properties: $h_0(n) = o(n)$, and if $h \geq h_0$ is any integer function, then every Γ of order n admits a zero-sum set $A_0 \subset \Gamma$ such that $|A_0| = h(n)$ and $\Gamma \setminus A_0$ is partitionable into pairwise disjoint zero-sum subsets $A_1, A_2, \ldots, A_t$ with $|A_i| = r_i$ whenever $r_1 + r_2 + \ldots + r_t = n - |A_0|$ and $r_i \geq 3$ for all $1 \leq i \leq t$.*

Due to the possible strengthening indicated after Lemma 3.1, the above proof shows that only an overwhelming presence of values $r_i = 3$ can be

[1]Their inverse triples $(-a, b, \iota_1)$, $(-b, c, \iota_2)$, $(-c, a, \iota_3)$ would be equally fine.

responsible for the error term εn. For this reason, on slightly restricted sequences of the r_i we can obtain an almost optimal result.

Corollary 3.5. *If the number of $r_i = 3$ is at most $(1/3-c)\cdot n$ for a fixed $c > 0$, and the number of $r_i = 2$ does not exceed $n/4$, then for sufficiently large $n > n_c$ every sequence $r_1, \dots, r_t$ admits disjoint zero-sum subsets $A_1, \dots, A_t$ with $|A_i| = r_i$ in every Γ of order $n \geq r_1 + \ldots + r_t + 5$.*

As in the preceding proof, $n/4$ can be replaced with $|R|/2$ also here. On the one hand this condition depends on the actual Γ, while the restricted version given in the corollary is universally valid. On the other hand the modified condition $|R|/2$ is best possible for every Γ.

Remark 3.6. *The bound on the number of pairs $r_i = 2$ in Theorem 3.3 is tight, because $|R| = n/2$ may occur, and then only $n/4$ zero-sum pairs exist in Γ.*

The following conjecture was raised recently.

Conjecture 3.7 ([5])**.** *Let Γ of order n have more than one involution. For every partition $n-1 = r_1 + r_2 + \ldots + r_t$ of $n-1$ with $r_i \geq 3$ for $1 \leq i \leq t$ and for any possible positive integer t, there is a partition of $\Gamma - \{0\}$ into pairwise disjoint subsets $A_1, A_2, \dots, A_t$ such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = 0$ for $1 \leq i \leq t$.*

Note that the conjecture is true for $\Gamma \cong (\mathbb{Z}_2)^m$ as Egawa proved in [6]. Moreover since for every Γ having more than one involution we have $\sum_{g \in \Gamma} g = 0$, the following two observations are valid by Theorem 2.6 and Corollary 3.5, respectively.

Observation 3.8. *Let Γ of order n have more than one involution. For every partition $n-1 = r_1 + r_2 + \ldots + r_t$ of $n-1$, with $r_i \geq 3$ for $1 \leq i \leq t$ and $r_t \geq 3n/4$, for any possible positive integer t, there is a partition of $\Gamma - \{0\}$ into pairwise disjoint subsets $A_1, A_2, \dots, A_t$, such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = 0$ for $1 \leq i \leq t$.*

Observation 3.9. *Let Γ of large enough order n have more than one involution. For every partition $n-1 = r_1 + r_2 + \ldots + r_t$ of $n-1$, with $r_i \geq 4$ for $1 \leq i \leq t$, for any possible positive integer t, there is a partition of $\Gamma - \{0\}$ into pairwise disjoint subsets $A_1, A_2, \dots, A_t$, such that $|A_i| = r_i$ and $\sum_{a \in A_i} a = 0$ for $1 \leq i \leq t$.*

Proof. If $n_t \geq 5$, we apply Corollary 3.5 for $n_1, \ldots, n_{t-1}$ to create the first $t-1$ sets. The remaining n_t elements of Γ automatically sum up to zero, serving for the largest set. Otherwise, if all $n_1 = \ldots = n_t = 4$, using the notation in the proof of Theorem 3.3 we partition $I \cup \{0\} \cong (\mathbb{Z}_2)^m$ into zero-sum quadruples by Egawa's theorem, and partition R into quadruples of the type $(a, b, -a, -b)$. □

4 Some applications

Consider a simple graph $G = (V, E)$ whose order we denote by $n = |V|$. The *open neighborhood* $N(x)$ of a vertex x is the set of vertices adjacent to x, and the degree $d(x)$ of x is $|N(x)|$, the order of the neighborhood of x. In this paper we also investigate group distance magic labelings, which belong to a large family of magic-type labelings. Generally speaking, a magic-type labeling of $G = (V, E)$ is a mapping from V, E, or $V \cup E$ to a set of labels which most often is a set of integers or group elements. The magic labeling (in the classical point of view) with labels being the elements of an Abelian group has been studied for a long time (see papers by Stanley [11, 12]). Froncek in [8] defined the notion of group distance magic graphs, which are the graphs allowing a bijective labeling of vertices with elements of an Abelian group resulting in a constant sum of neighbor labels.

A Γ-*distance magic labeling* of a graph $G = (V, E)$ with $|V| = n$ is a bijection ℓ from V to an Abelian group Γ of order n such that the weight $w(x) = \sum_{y \in N(x)} \ell(y)$ of every vertex $x \in V$ is equal to the same element $\mu \in \Gamma$, called the *magic constant*.

Notice that the constant sum partitions of a group Γ lead to complete multipartite Γ-distance magic labeled graphs. For instance, the partition $\{0\}$, $\{1, 2, 4\}$, $\{3, 5, 6\}$ of the group $\mathbb{Z}_7$ with constant sum 0 leads to a $\mathbb{Z}_7$-distance magic labeling of the complete tripartite graph $K_{1,3,3}$ (see [5]). Using Theorem 3.8 we are able to prove the following.

Observation 4.1. *Let $G = K_{n_1, n_2, \ldots, n_t}$ be a complete t-partite graph such that $3 \leq n_1 \leq n_2 \leq \ldots \leq n_t$ and $n = n_1 + n_2 + \ldots + n_t$. Let Γ be an Abelian group of order n having more than three involutions. The graph G is Γ-distance magic whenever $n_t \geq 3n/4 - 1$.*

Proof. There exists a zero-sum partition $A'_1, A'_2, \ldots, A'_t$ of the set $\Gamma - \{0\}$ such that $|A'_t| = n_t - 1$ and $|A'_i| = n_i$ for every $1 \leq i \leq t-1$ by Theorem 3.8.

Set $A_t = A'_t \cup \{0\}$ and $A_i = A'_i$ for every $1 \leq i \leq t-1$. Label now the vertices from V_i, where V_i is the vertex class of cardinality n_i, using the elements from the set A_i for $i \in \{1, 2, \ldots, t\}$. $\square$

Analogously, for n large enough, by Observation 3.9 we can obtain:

Observation 4.2. *Let Γ be an Abelian group of large enough order n having more than one involution. If $G = K_{n_1,n_2,\ldots,n_t}$ is a complete t-partite graph such that $4 \leq n_1 \leq n_2 \leq \ldots \leq n_t$ and $n = n_1 + n_2 + \ldots + n_t$, then G is a Γ-distance magic graph.*

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