Decomposing planar graphs into graphs with degree restrictions

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Abstract

Given a graph G, a decomposition of G is a partition of its edges. A graph is (d,h)-decomposable if its edge set can be partitioned into a d-degenerate graph and a graph with maximum degree at most h. For $d \leq 4$, we are interested in the minimum integer h_d such that every planar graph is (d,h_d)-decomposable. It was known that $h_3 \leq 4$ and $h_2 \leq 8$ and $h_1 = \infty$. This paper proves that $h_4 = 1, h_3 = 2$ and $4 \leq h_2 \leq 6$.

1 Introduction

We consider only finite simple graphs. Given a graph G, a *decomposition* of G is a collection of spanning subgraphs $H_1,\ldots,H_t$ such that each edge of G is an edge of H_i for exactly one $i \in \{1,\ldots,t\}$. In other words, $E(H_1),\ldots,E(H_t)$ is a partition of $E(G)$.

A graph is *d-degenerate* if every subgraph has a vertex of degree at most d. Given non-negative integers d and h, a *(d,h)-decomposition* of a graph G is a decomposition H_1,H_2 of G such that H_1 is d-degenerate and H_2 has maximum degree at most h. We say G is *(d,h)-decomposable* if there exists a (d,h)-decomposition of G. This paper studies (d,h)-decomposability of planar graphs.

Decomposing a graph into subgraphs with simpler structure is a fundamental problem in graph theory. The classical Nash-Williams Arboricity Theorem [12] (see also [11, 13]) gives a necessary and sufficient condition under which a graph can be decomposed into k forests. The Nine-Dragon Tree Conjecture [10], confirmed by Jiang and Yang [8], gives a sharp density condition under which a graph can be decomposed into k forests with one of them having bounded maximum degree. The page number of a graph G is the minimum k such that G can

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be decomposed into k planar graphs. A proper edge colouring of G is a decomposition of G into matchings. The problem of decomposing a graph G into star forests, linear forests, graphs of bounded maximum degree, etc., are studied extensively in the literature.

The concept of (d, h)-decomposition has not been defined formally in the literature (as to our knowledge). However, such decompositions raise naturally in the study of many problems. For example, it was proved in [6] that if a graph G is $(1, h)$-decomposable, then G has game chromatic number $\chi_g(G)$ at most $4 + h$. It was shown in [6] that outerplanar graphs are $(1, 3)$-decomposable, and hence have game chromatic number at most 7. A result in [16] implies that planar graphs are $(2, 8)$-decomposable, and such a decomposition (with some more structure constraints) was used to show that planar graphs have game chromatic number at most 19 (the currently best known upper bound for the game chromatic number of planar graphs is 17 [17]). A similar decomposition were used to derive upper bound on the game chromatic number of graphs G embeddable on an orientable surface of genus $g \geq 1$, namely, $\chi_g(G) \leq \lfloor \frac{1}{2}(3\sqrt{1 + 48g}) + 23) \rfloor$. It is known that if G decomposes into $H_1, H_2, \ldots, H_k$, then the spectral radius of G is bounded by the summation of the spectral radius of H_i, i.e., $\rho(G) \leq \rho(H_1) + \rho(H_2) + \cdots + \rho(H_k)$ [2,15] . The currently best known upper bounds on the spectral radius of planar graphs (namely, $\rho(G) \leq \sqrt{8\Delta - 16} + 3.47$) was obtained by Dvořák and Mohar [2] by applying the result that every planar graph G decomposes into H_1, H_2, with H_1 has an orientation of maximum out-degree 2, and H_2 has maximum degree at most 4.

In this paper, we are interested in the minimum integer h_d such that every planar graph is (d, h_d)-decomposable. Since every planar graph is 5-degenerate, the problem is interesting only for $d \leq 4$. As observed above, a result in [16] implies that every planar graph is $(2, 8)$-decomposable, i.e., $h_2 \leq 8$. A result in [4] implies that every planar graph is $(3, 4)$-decomposable, i.e., $h_3 \leq 4$. In this paper, we prove the following results:

Theorem 1.1. *Every planar graph is* $(4, 1)$*-decomposable.*

Theorem 1.2. *Every planar graph is* $(3, 2)$*-decomposablee.*

Since planar graphs of minimum degree 5 is neither $(3, 1)$-decomposble nor $(4, 0)$-decomposable, we conclude that $h_4 = 1$ and $h_3 = 2$.

Theorem 1.3. *Every planar graph is* $(2, 6)$*-decomposable.*

Proposition 1.4. *Not all planar graphs are* $(2, 3)$*-decomposable.*

As a consequence of Theorem 1.3 and Proposition 1.4, we have $4 \leq h_2 \leq 6$. The exact value of h_2 remains an open problem.

Note that for every integer h, the complete bipartite graph with two vertices in one part and $2h + 2$ vertices in the other part is not $(1, h)$-decomposable. Thus $h_1 = \infty$.

A graph G is *h-defective k-choosable* if for any k-list assignment L of G, there is an L-colouring of G in which each vertex v has at most h-neighbours coloured the same colour as v. The concept of *h-defective k-paintable* is an online version of h-defective k-choosable, defined through a two-person game (see [7] for its definition), and *h-defective k-DP-colourable* is a generalization of h-defective k-choosable (see [9] for its definition). We remark that (d, h)-decomposable graphs are easily seen to be h-defective $(d+1)$-choosable, h-defective $(d+1)$-paintable, as well as h-defective $(d+1)$-DP-colourable. On the other hand, (d, h)-decomposable seems to be considerably stronger than h-defective $(d+1)$-choosability and h-defective $(d+1)$-paintability. Cushing and Kierstead [1] proved that every planar graph is 1-defective 4-choosable. This result was strengthened recently by Grytczuk and Zhu [5] who proved that every planar graph is 1-defective 4-paintable. As observed above, planar graphs with minimum degree 5 are not $(3, 1)$-decomposable. Eaton and Hull [3], and independently Škrekovski [14] proved that every planar graph is 2-defective 3-choosable. Gutowski, Han, Krawcyzk and Zhu [**?**] [Defective 3-paintability of planar graphs, Electronic Journal of Combinatorics, Volume 25, Issue 2 (2018)] showed that there are planar graphs that are not 2-defective 3-paintable, but every planar graph is 3-defective 3-paintable. We show in this paper that not every planar graph is $(2, 3)$-decomposable.

The proof of Theorem 1.1 (given in Section 4) uses standard discharging method. Theorem 1.3 and Theorem 1.2 are obtained by proving stronger and more technical statements in Section 2 and Section 3, respectively. The technical statement used to derive Theorem 1.3 is more intriguing and the proof is also more complicated.

We end this section with some definitions and notation. A vertex ordering σ of G is d-*degenerate* if every vertex has at most d earlier neighbors in the ordering σ. Note that a graph G is d-degenerate if and only if it has a d-degenerate ordering. For $S \subseteq V(G)$ and a vertex ordering σ of G, let $\sigma - S$ denote a subordering of σ obtained by deleting the vertices in S. We also note that a graph G is d-degenerate if and only if it has an acyclic orientation whose maximum out-degree is at most d. Therefore, when we prove Theorems 1.2 and 1.3, we find a pair (D, H), where H is a subgraph of G with $\Delta(H) \leq h$ and D is an acyclic orientation of $G - E(H)$ with $\Delta^+(D) \leq d$.

Let G be a plane graph. A *plane subgraph* of G is a subgraph of G whose plane embedding is inherited. We say G is a *near triangulation* if G is a 2-connected plane graph and every face of G except the outer face is a triangle. Note that the outer face of a near plane triangulation G is a cycle since G is 2-connected. A *boundary vertex* and *boundary edge* of G are a vertex and an edge, respectively, on the boundary cycle of G. For a boundary edge uv, v is called a *boundary neighbor* of u.

An arc, which is a directed edge, is represented by an ordered pair of vertices; namely, uv is an (undirected) edge whereas (u, v) is an arc from u to v. For a graph G and a set E of unordered pairs on $V(G)$, let $G+E$ (resp. $G-E$) denote the graph obtained from G by adding (resp. deleting) the elements of E to (resp. from) the edge set of G. If $|E| = 1$, say $E = \{ww'\}$, then denote $G+E$ (resp. $G-E$) by $G+ww'$ (resp. $G-ww'$). For a digraph D and a set A of ordered pairs on $V(D)$, define $D+A$, $D-A$, $D+(w, w')$, and $D-(w, w')$ similarly. Moreover, for a digraph D and vertices $x, y \in V(D)$, let $D-xy$ denote the subdigraph $D-\{(x, y), (y, x)\}$. We often drop the parentheses to improve the readability. For instance, for a digraph D and sets A_1, A_2, A_3 of ordered pairs on $V(D)$, both $D - A_1 + A_2 + A_3$ and $D - A_1 + (A_2 + A_3)$ denote $((D - A_1) + A_2) + A_3$.

For two (di)graphs G_1 and G_2, let $G_1 \cup G_2$ be the (di)graph such that $V(G_1 \cup G_2) =$

$V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

2 Proof of (2,6)-decomposability

Assume G is a near triangulation, xy is a boundary edge of G, and w is a boundary vertex of G. We denote by $b_{G,xy}(w)$ the number of vertices in $\{x, y\}$ that are boundary neighbors of w. Recall that w is a boundary neighbor of x if xw is a boundary edge of G. If there is no confusion, then we use $b(w)$ to denote $b_{G,xy}(w)$. Instead of proving Theorem 1.3 directly, we prove the following more technical result, which is easily seen to imply Theorem 1.3.

Theorem 2.1. *Let G be a near triangulation, xy be a boundary edge of G, and z be a boundary vertex of G other than x and y. Then there exist a subgraph H and an acyclic orientation D of $G - E(H)$ satisfying the following:*

- (i) *For every interior vertex w, $\deg^+_D(w) \leq 2$ and $\deg_H(w) \leq 6$.*
- (ii) *For every boundary vertex w, $\deg^+_D(w) \leq 1$ and $\deg_H(w) \leq 5 - b(w)$.*
- (iii) $\deg^+_D(y) = \deg_H(y) = 0$, $N^+_D(x) = \{y\}$, *and* $\deg_H(x) \leq 1$. *If* $\deg_H(x) = 1$, *then the neighbor s of x in H is a boundary vertex and $s \in N_G(x) \cap N_G(y)$.*
- (iv) $\deg_H(z) \leq 4 - b(z)$. *If equality holds, then* $\deg_H(w) \leq 4 - b(w)$ *for every boundary neighbor w of z.*
- (v) *For the boundary neighbors z' and z'' of z,* $\deg_H(z) + \deg_H(z') + \deg_H(z'') \leq 12 - b(z') - b(z'')$.

Let us call such (D, H) a $(2, 6)$-decomposition of G with respect to (x, y, z).

Lemma 2.2. *Let G be a near triangulation, xy be a boundary edge of G, and z be a boundary vertex of G other than x and y. If (D, H) is a $(2, 6)$-decomposition of G with respect to (x, y, z), then there is a $(2, 6)$-decomposition of G with respect to (y, x, z).*

Proof. Let (D, H) be a $(2, 6)$-decomposition of G with respect to (x, y, z). If $\deg_H(x) = 0$, then let $D' = D - (x, y) + (y, x)$ and $H' = H$. If $\deg_H(x) = 1$, then let w be the neighbor of x in H. Then w is a boundary vertex and (w, y) is an arc of D. Let $D' = D - (x, y) - (w, y) + (w, x) + (y, x)$ and $H' = (H + wy) - wx$. Then (D', H') is a $(2, 6)$-decomposition of G with respect to (y, x, z). □

Proof of Theorem 2.1. We use induction on $|V(G)|$. If $|V(G)| = 3$, then $G = K_3$. Let D be a digraph with two arcs (x, y) and (z, y), and H be a graph with one edge xz. Then (D, H) is a $(2, 6)$-decomposition of G with respect to (x, y, z). Suppose $|V(G)| \geq 4$. Let C be the boundary cycle of G, and let z' and z'' be the boundary neighbors of z. For simplicity, we denote $b_{G,xy}(w)$ by $b(w)$.

Case 1 $C = (x, y, z)$ is a triangle.

Let $G' = G - z$. Since G contains at least four vertices, G' is a near triangulation. Let C' be the boundary cycle of G', and let w be a boundary vertex of G' other than x and y. See Figure 1. By the induction hypothesis, there is a $(2,6)$-decomposition (D', H') of G' with respect to (x, y, w).

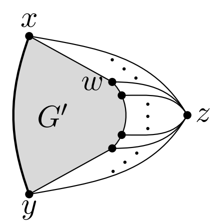

Figure 1: An illustration for **Case 1**

If $\deg_{H'}(x) = 0$, then let $D = D' + \{(u,z) \mid u \in V(C') \setminus \{x,y\}\} + (z,y)$ and $H = H' + xz$. If $\deg_{H'}(x) = 1$, then for the vertex s with $sx \in E(H')$, s belongs to $N_{G'}(x) \cap N_{G'}(y) \cap V(C')$ by Condition (iii), so let $D = D' + \{(s,x),(z,y)\} + \{(u,z) \mid u \in V(C') \setminus \{x,y,s\}\}$ and $H = (H' - sx) + \{sz, xz\}$.

In both cases, we can easily check Conditions (i)-(iii). Since $b(z) = 2$ and $\deg_H(z) \le 2$, Condition (iv) holds. Since $b(z') + b(z'') = 2$, we have $\deg_H(z) + \deg_H(z') + \deg_H(z'') = \deg_H(z) + \deg_H(x) + \deg_H(y) \le 2 + 1 = 3 \le 12 - 2$, so Condition (v) holds. Thus (D, H) is a $(2,6)$-decomposition of G with respect to (x, y, z).

Case 2 C has a chord uv that either separates xy and z or is incident with one of x, y, z.

Let G_1 and G_2 be the plane subgraphs of G separated by uv. Namely, $G_1 = G[V_1], G_2 = G[V_2]$, where $V_1 \cup V_2 = V(G)$ and $V_1 \cap V_2 = \{u, v\}$. Then each G_i is a near triangulation. Let C_i be the boundary cycle of G_i. Without loss of generality, assume $x, y \in V(G_1)$. We divide the proof into three subcases: (1) $z \notin V(G_1)$, (2) $z \in \{u,v\}$, and (3) $z \in V(G_1) \setminus \{u,v\}$. In each case, we will find a $(2,6)$-decomposition of G_1 with respect to (x, y, w') for some $w' \in \{z, u, v\}$, and a $(2,6)$-decomposition (D_2, H_2) of G_2 with respect to (u, v, w^*) or (v, u, w^*) for some vertex w^*. Let $D = D_1 \cup (D_2 - uv)$ and $H = H_1 \cup H_2$. It is clear that D is acyclic.

For simplicity, denote $b_{G_1,xy}(w)$ and $b_{G_2,uv}(w)$ by $b_1(w)$ and $b_2(w)$, respectively. If $w \in V(G_i) \setminus \{u,v\}$, then $\deg^+_D(w) = \deg^+_{D_i}(w)$, $\deg_H(w) = \deg_{H_i}(w)$. If w is a boundary vertex of G, then $b_i(w) \ge b(w)$. If $w \in \{u,v\}$, then $\deg^+_D(w) = \deg^+_{D_1}(w)$, $\deg_H(w) = \deg_{H_1}(w) + \deg_{H_2}(w)$, and $b_1(w) \ge b(w)$. Hence, Condition (i) immediately holds, and Conditions (ii)-(v) hold except those regarding the degrees in H involving u or v. From now on, we will prove that Condition (ii) holds when $w \in \{u,v\}$, Condition (iii) holds when x or y is in $\{u,v\}$, and Conditions (iv) and (v) hold when z, z', or z'' is in $\{u,v\}$.

Case 2-1 $z \notin V(G_1)$.

We may assume $v \notin \{x,y\}$. See the first figure of Figure 2. By Lemma 2.2, we may assume $x \ne u$. Note that $z \notin \{u,v\}$ and $z', z'' \in V(C_2)$. Let (D_1, H_1) be a (2,6)-decomposition of G_1 with respect to (x, y, v). If $\deg_{H_1}(u) \le 4 - b_1(u)$ and $y \ne u$, then let (D_2, H_2) be a $(2,6)$-

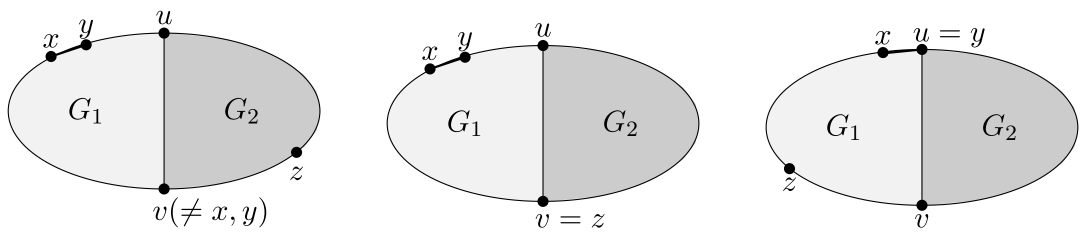

Figure 2: Illustrations for **Case 2**

decomposition of G_2 with respect to (u,v,z). Otherwise, let (D_2,H_2) be a $(2,6)$-decomposition of G_2 with respect to (v,u,z). Recall that it is enough to check Conditions (ii)-(v) for the case where $\deg_H(u)$ or $\deg_H(v)$ is involved.

Condition (ii) holds, since

$$\deg_H(v)\le\deg_{H_1}(v)+\deg_{H_2}(v)\le(4-b_1(v))+1\le 5-b(v)$$
$$\deg_H(u)\le\deg_{H_1}(u)+\deg_{H_2}(u)\le\begin{cases}(4-b_1(u))+1\le 5-b(u) & \text{if } \deg_{H_1}(u)\le 4-b_1(u)\\ (5-b_1(u))+0\le 5-b(u) & \text{otherwise.}\end{cases}$$

If $u=y$, then $\deg_H(y)=\deg_{H_1}(y)+\deg_{H_2}(y)=0+0=0$, which implies Condition (iii). To check Condition (iv), suppose that $\deg_H(z)=4-b(z)$. Since $\deg_H(z)=\deg_{H_2}(z)\le 4-b_2(z)\le 4-b(z)$, we conclude that $b_2(z)=b(z)$ and $\deg_{H_2}(z)=4-b_2(z)$. Since $b_2(z)=b(z)$, either $b(z)=b_2(z)=0$ or $u=y$ and y is a boundary neighbor of z. For the first case, $\{z',z''\}\cap\{u,v\}=\emptyset$, so u and v are not involved. For the second case, since $b_1(v)=b(v)+1$, we have the following:

$$\deg_H(z')\le\deg_{H_1}(z')+\deg_{H_2}(z')\le\begin{cases}0+0\le 4-b(z') & \text{if } z'=u\\ 4-b_1(z')+1\le 4-b(z') & \text{if } z'=v.\end{cases}$$

Thus Condition (iv) holds.

By Condition (ii), $\deg_H(z')\le 5-b(z')$ and $\deg_H(z'')\le 5-b(z'')$. If $\{z',z''\}=\{u,v\}$, then $\deg_{H_2}(z)\le 4-b_2(z)\le 2$, so $\deg_H(z)+\deg_H(z')+\deg_H(z'')\le 2+5-b(z')+5-b(z'')=12-b(z')-b(z'')$. If $z'\in\{u,v\}$ and $z''\notin\{u,v\}$, then $\deg_{H_2}(z)\le 4-b_2(z)\le 3$, and so

$$\deg_H(z)+\deg_H(z')+\deg_H(z'')\le 3+5-b(z')+\deg_{H_2}(z'')\le 12-b(z')-b(z''),$$

where the last inequality is from Condition (iv) for (D_2,H_2) stating that $\deg_{H_2}(z)=3$ implies $\deg_{H_2}(z'')\le 4-b_2(z'')\le 4-b(z'')$. Therefore Condition (v) holds.

Case 2-2 $z\in\{u,v\}$.

We may assume $v=z$. Let $z'\in V(G_1)$ and $z''\in V(G_2)$. See the second figure of Figure 2. If $u\in\{x,y\}$, then we may assume $u=y$ by Lemma 2.2. Now let (D_1,H_1) be a $(2,6)$-decomposition of G_1 with respect to (x,y,z). If $\deg_{H_1}(u)\le 4-b_1(u)$ and $u\neq y$, then let

(D_2, H_2) be a (2,6)-decomposition of G_2 with respect to (u, v, z''). Otherwise, let (D_2, H_2) be a $(2,6)$-decomposition of G_2 with respect to (v, u, z''). Conditions (ii) and (iii) hold by the same reasoning as in Case 2-1. Moreover, since $\deg_H(z'') \leq 4 - b_2(z'') = 4 - b(z'')$, Condition (v) immediately follows from Condition (iv). Hence, it is enough to show that one of the following holds:

- $\deg_H(z) \leq 3 - b(z)$.
- $\deg_H(z) = 4 - b(z)$ and $\deg_H(z') \leq 4 - b(z')$.

If $\deg_{H_1}(z) = 4 - b_1(z)$, then by Condition (iv) for (D_1, H_1), we know $\deg_{H_1}(u) \leq 4 - b_1(u)$. Hence by definition, (D_2, H_2) is a $(2,6)$-decomposition of G_2 with respect to (u, v, z''). Therefore $\deg_{H_2}(z) = 0$ and $\deg_H(z) = \deg_{H_1}(z) = 4 - b_1(z) + 0 \leq 4 - b(z)$. Moreover, if $\deg_H(z) = 4 - b(z)$, then $\deg_{H_1}(z) = 4 - b_1(z)$, and hence by Condition (iv) for (D_1, H_1), we have $\deg_{H_1}(z') \leq 4 - b_1(z')$, and hence $\deg_H(z') = \deg_{H_1}(z') \leq 4 - b(z')$.

Assume $\deg_{H_1}(z) \leq 3 - b_1(z)$. If $y = u$, then $b(z) = b_1(z) - 1$ and $\deg_H(z) \leq \deg_{H_1}(z) + 1$. Hence $\deg_H(z) \leq 3 - b(z)$, and we are done.

If $y \neq u$, then $b(z) = b_1(z)$ and $\deg_H(z) \leq \deg_{H_1}(z) + 1$. Hence $\deg_H(z) \leq 4 - b(z)$ holds. Suppose to the contrary that none of two above conditions previously mentioned holds, i.e., $\deg_H(z) = 4 - b(z)$ and $\deg_H(z') = 5 - b(z')$. Then $\deg_{H_1}(z) = 3 - b_1(z)$ and $\deg_{H_2}(z) = 1$. So (D_2, H_2) is a $(2,6)$-decomposition with respect to (v, u, z''). By the definition of (D_2, H_2), this implies that $\deg_{H_1}(u) = 5 - b_1(u)$. Moreover, $\deg_{H_1}(z') = 5 - b_1(z')$ and $z' \neq x$. Since $z' \neq x$ and $y \neq u$, this implies $b_1(z) = 0$. However,

$$\deg_{H_1}(z) + \deg_{H_1}(z') + \deg_{H_1}(u) = (3 - b_1(z)) + (5 - b_1(z')) + (5 - b_1(u)) = 13 - b_1(z') - b_1(u).$$

This is a contradiction to the assumption that (D_1, H_1) is a $(2,6)$-decomposition of G_1 with respect to (x, y, z), as Condition (v) for (D_1, H_1) is not satisfied.

Case 2-3 $z \in V(G_1) \setminus \{u, v\}$.

By the case assumption, the chord uv is incident with either x or y. By Lemma 2.2, we may assume $y = u$. See the last figure of Figure 2. Note that $z', z'' \in V(C_1)$. Let (D_1, H_1) be a $(2,6)$-decomposition of G_1 with respect to (x, y, z), and let (D_2, H_2) be a $(2,6)$-decomposition with respect to (v, u, z^*), where $z^* \in V(C_2) \setminus \{u, v\}$. Note that Condition (iii) clearly holds by definition. Conditions (ii), (iv), (v) hold since $b_1(v) = b(v) + 1$ and $\deg_H(v) \leq \deg_{H_1}(v) + 1$.

Case 3 Neither Case 1 nor Case 2 applies, in other words, C has at least four vertices and for every chord uv of C, the vertices x, y, z lie in the same component of $G - \{u, v\}$.

Case 3-1 z is a boundary neighbor of either x or y.

By Lemma 2.2, we may assume $yz \in E(C)$. Since C has at least four vertices, we may assume $z' \notin \{x, y\}$. See the first figure of Figure 3. Let $G' = G - z$, and let P be the boundary path of G' from y to z' not containing x. Let (D', H') be a $(2,6)$-decomposition of G' with respect to (x, y, z'). For simplicity, let $X = V(P) \setminus \{y, z'\}$.

If $\deg_{H'}(x) = 0$, then let $D = D' + (z, y) + \{(u, z) \mid u \in X\}$ and $H = H' + zz'$. It is easy to observe that (D, H) is a $(2,6)$-decomposition of G with respect to (x, y, z). Suppose

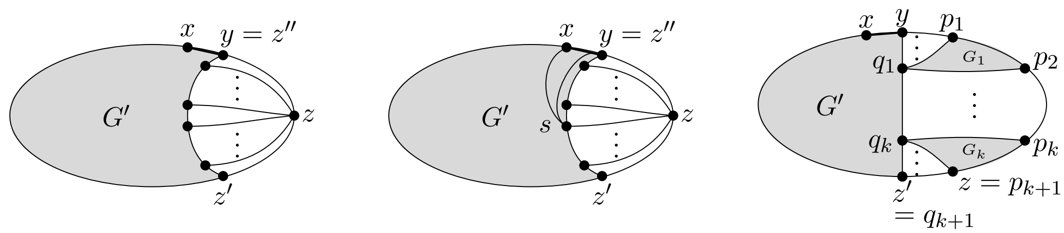

Figure 3: Illustrations for **Case 3**

$\deg_{H'}(x) = 1$. Then by Condition (iii) for (D', H'), for the vertex s with $xs \in E(H')$, s belongs to $N_{G'}(x) \cap N_{G'}(y) \cap V(C')$. See the second figure of Figure 3. Since G has no chord incident with either x or y, $s \in X$. Let $D = D' + (s,x) + (z,y) + \{(u,z) \mid u \in V(P) \setminus \{y, z', s\}\}$ and $H = (H' - sx) + \{zz', sz\}$. Then (D,H) is a $(2,6)$-decomposition of G with respect to (x,y,z).

Case 3-2 Neither x nor y is a boundary neighbor of z.

Then z', z'' are different from x, y. By Lemma 2.2, we may assume x, y, z'', z, z' is the clockwise ordering on C. See the last figure of Figure 3. Let p_1 be the boundary neighbor of y other than x. Let P be the clockwise subpath of C joining p_1 and z. Note that by our case assumption, $|V(P)| \geq 2$. Let G' be the block of $G - V(P)$ containing x, y, z', and let C' be the boundary cycle of G'. Let Q be the clockwise subpath of C' joining y and z'.

Claim 2.3. *Every two adjacent vertices q and q' on Q have a common neighbor in $V(P)$.*

Proof. Since G is a near triangulation, q and q' have a common neighbor w in $V(G) \setminus V(G')$. If w is not on P, then qq' cannot be a boundary edge of $G - V(P)$, which is a contradiction. □

Since there is no chord incident with y, $|V(Q)| \geq 3$. By Claim 2.3, every vertex of Q has a neighbor in $V(P)$. If every vertex of Q has exactly one neighbor in $V(P)$, then $V(P) = \{z\}$, which is a contradiction. Let $q_0 = y$, $q_1, q_2, \ldots, q_k$ $(k \geq 1)$ be the vertices of Q in the order from y to z' that are adjacent to at least two vertices in $V(P)$ and let $q_{k+1} = z'$. By Claim 2.3, for $i \in \{1, 2, \ldots, k+1\}$, let $p_i \in V(P)$ be the vertex adjacent to q_{i-1} and q_i. Note that P is a path from p_1 to $p_{k+1} = z$, and $yp_1 \in E(G)$.

For $j \in \{0, 1, \ldots, k\}$, let Q_j be the subpath of Q from q_j to q_{j+1}. For $i \in \{1, \ldots, k\}$, let P_i be the subpath of P from p_i to p_{i+1}. Let C_i be the cycle consisting of P_i and vertex q_i, and let G_i be the maximal plane subgraph of G with boundary cycle C_i. Let (D_i, H_i) be a $(2,6)$-decomposition of G_i with respect to (p_{i+1}, q_i, p_i). Then, clearly $(p_{i+1}, q_i), (p_i, q_i)$ are arcs of D_i. Modify D_i and H_i by reversing the orientation of (p_i, q_i) in D_i, removing (p_{i+1}, q_i) from D_i, and then adding $p_{i+1}q_i$ to H_i. Then for $i \in \{1, \ldots, k\}$, D_i is still acyclic and

$$\deg^+_{D_i}(q_i) = \deg_{H_i}(q_i) = 1, \deg^+_{D_i}(p_i) = \deg^+_{D_i}(p_{i+1}) = 0, \deg_{H_i}(p_i) \leq 3, \text{ and } \deg_{H_i}(p_{i+1}) \leq 2.$$

Let (D', H') be a $(2,6)$-decomposition of G' with respect to (x, y, z'). Let

$$\begin{aligned} D &= D' \cup \left(\bigcup_{i=1}^{k} D_i \right) + \{(p_1, y)\} + \{(q, p_{i+1}) \mid q \in V(Q_i) \setminus \{q_i, q_{i+1}\}, i \in \{0, 1, \dots, k\}\}, \\ H &= H' \cup \left(\bigcup_{i=1}^{k} H_i \right) + zz'. \end{aligned}$$

Suppose $\deg_{H'}(x) = 1$. Then the vertex s such that $sx \in E(H')$ belongs to $V(Q) \setminus \{y, z'\}$ by the case assumption. Delete sx from H and then add arc (s, x) to D. For the smallest index i such that $s \in V(Q_i)$, if $i \geq 1$, then modify D by reversing the orientation of (s, p_{i+1}) in D, and if $i = 0$, then modify D and H by deleting arc (s, p_{i+1}) from D and then adding the edge sp_{i+1} to H. Clearly, D is acyclic. From the definition of (D, H), $N_D^+(x) = \{y\}$ and $\deg_D^+(y) = \deg_H(x) = \deg_H(y) = 0$, so Condition (iii) holds.

For the vertex s (if it exists), $\deg_D^+(s) \leq 2$ and $\deg_H(s) \leq 6$. For $w \in V(Q) \setminus \{y, z', s\}$,

$$\deg_D^+(w) = \deg_{D'}^+(w) + 1 \leq 2 \quad \text{and} \quad \deg_H(w) \leq \deg_{H'}^+(w) + 1 \leq 6,$$

and therefore Condition (i) holds. It is easy to check that

$$\begin{aligned} &\deg_D^+(p_i) \leq 1, \text{ for } i \in \{1, 2, \dots, k\}, \\ &\deg_H(p_1) \leq 1 + 3 = 5 - b(p_1), \quad \deg_H(p_i) \leq 3 + 2 = 5 - b(p_i) \text{ for } i \in \{2, 3, \dots, k\}, \\ &\deg_D^+(z) \leq 1, \quad \deg_H(z) \leq 1 + \deg_{H_k}(z) \leq 1 + 2 = 3 < 4 = 4 - b(z) \\ &\deg_D^+(z') \leq 1, \quad \deg_H(z') = \deg_{H'}(z') + 1 \leq 5 - b(z'), \text{ and} \\ &\deg_H(z'') = \deg_{H_k}(z'') \leq 5 - b_{G_k, p_{k+1}q_k}(z'') = 4 \leq 4 - b(z'') \text{ if } z'' \neq p_k. \end{aligned}$$

If $z'' = p_k$, then the boundary cycle of G_k is a triangle. Thus, $\deg_{H_k}(p_k) \leq 4 - b_{G_k, p_{k+1}q_k}(p_k) = 2$, which implies that

$$\deg_H(z'') = \deg_H(p_k) \leq \begin{cases} 2 + 2 = 4 = 4 - b(z'') & \text{if } z'' = p_k, k > 1, \\ 1 + 2 = 3 = 4 - b(z'') & \text{if } z'' = p_1, \end{cases}$$

so Conditions (ii) and (iv) hold.

It remains to check Condition (v). As shown above, whether z'' is p_k or not, we have $\deg_H(z'') \leq 4 - b(z'')$. Therefore

$$\deg_H(z) + \deg_H(z') + \deg_H(z'') \leq 3 + (5 - b(z')) + (4 - b(z'')) = 12 - b(z') - b(z'').$$

□

We finish this section by proving the following, which implies Proposition 1.4.

Proposition 2.4. *Let G be a plane triangulation on at least 11 vertices. If G' is the plane graph obtained from G by adding a new vertex v_f to every face f of G and adding all edges between v_f and the vertices of f, then G' is not $(2,3)$-decomposable.*

Proof. Let $n = |V(G)|$. Since G is a triangulation, G has $2n-4$ faces and $3n-6$ edges. Suppose to the contrary that G' is $(2,3)$-decomposable. Let (D,H) be a $(2,3)$-decomposition of G' that maximizes $|E(H) \cap (E(G') \setminus E(G))|$. Let σ be a 2-degenerate ordering of D.

We claim that for every face f of G, $\deg_H(v_f) \geq 1$. Suppose $\deg_H(v_f) = 0$ for some face f of G. Let v_1, v_2, v_3 be the vertices of G incident with f. Then $\deg_D(v_f) = 3$, so some v_j comes later than v_f in σ. We may assume v_1 is the last in σ among $\{v_f, v_1, v_2, v_3\}$. Since σ is a 2-degenerate ordering, either v_1v_2 or v_1v_3 is in H, say $v_1v_2 \in E(H)$. Let $D' = D - v_fv_1 + v_1v_2$ and $H' = H - v_1v_2 + v_fv_1$. Then (D', H') is a $(2,3)$-decomposition of G', which is a contradiction to the maximality of $|E(H) \cap (E(G') \setminus E(G))|$. Therefore $\deg_H(v_f) \geq 1$ for every face f of G, and thus $|E(H) \setminus E(G)| \geq |F(G)| = 2n-4$.

In $\sum_{v \in V(G)} \deg_H(v)$, an edge in $E(H) \cap E(G)$ is counted twice and an edge in $E(H) \setminus E(G)$ is counted once. Hence, together with the fact that $\Delta(H) \leq 3$,

$$3n \geq \sum_{v \in V(G)} \deg_H(v) \geq 2|E(H) \cap E(G)| + |E(H) \setminus E(G)| \geq 2|E(H) \cap E(G)| + (2n-4).$$

From the fact that $|E(D) \cap E(G)| \leq 2n-3$, we have $|E(H) \cap E(G)| \geq (3n-6)-(2n-3) = n-3$, so $3n \geq 2(n-3) + 2n - 4 = 4n - 10$, which is a contradiction since $n \geq 11$. $\square$

3 Proof of (3,2)-decomposability

Note that for a near triangulation and a boundary edge xy, there always exists a boundary vertex z distinct from x, y that is not incident with a chord of the boundary cycle. Instead of proving Theorem 1.2 directly, we prove the following more technical result.

Theorem 3.1. *Let G be a near triangulation, xy be a boundary edge of G, and z be a boundary vertex other than x, y that is not incident with a chord of the boundary cycle. When neither x nor y is a boundary neighbor of z, let z' be a boundary neighbor of z. Then there exist a subgraph H and an acyclic orientation D of $G - E(H)$ satisfying the following:*

(i) *For every interior vertex w, $\deg^+_D(w) \leq 3$ and $\deg_H(w) \leq 2$.*

(ii) *For every boundary vertex w, $\deg^+_D(w) \leq 2$ and $\deg_H(w) \leq 2$. Moreover, if $w \neq z'$, then $\deg^+_D(w) + \deg_H(w) \leq 3$.*

(iii) $\deg^+_D(y) = \deg_H(x) = \deg_H(y) = 0$, $N^+_D(x) = \{y\}$, *and* $\deg^+_D(z) + \deg_H(z) \leq 2$.

Let us call such (D,H) a $(3,2)$-decomposition of G with respect to (x,y,z) or (x,y,z,z').

Proof. We use induction on $|V(G)|$. If $|V(G)| = 3$, then $G = K_3$. Let D be a digraph with arcs (x,y), (z,x) and (z,y), and H be the empty graph. Then (D,H) is a $(3,2)$-decomposition of G with respect to (x,y,z). Suppose $|V(G)| \geq 4$. Let C be the boundary cycle of G.

Case 1 $C = (x,y,z)$ is a triangle.

Let $G' = G - z$. Note that G' is a near triangulation and let C' be the boundary cycle of G'. Let $w \in N_G(z) \setminus \{x,y\}$ such that w is not incident with a chord of C'. By the induction hypothesis, there is a $(3,2)$-decomposition (D', H') of G' with respect to (x, y, w, w') or (x, y, w) depending on the existence of w'. Then (D, H), where $D = D' + \{(z, y), (z, x)\} + \{(u, z) \mid u \in V(C') \setminus \{x, y\}\}$ and $H = H'$, satisfies Conditions (i)-(iii).

Case 2 C has a chord uv.

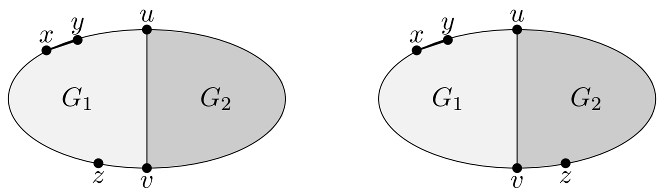

Figure 4: Illustrations for **Case 2**

Case 2-1 There is a chord uv of C such that $x, y, z \in V(G_i)$ for some $i \in \{1, 2\}$, where G_1 and G_2 are the plane subgraphs of G separated by uv.

Let C_i be the boundary cycle of G_i. Without loss of generality, assume $x, y, z \in V(G_1)$. See the first figure of Figure 4. Choose the chord uv so that G_2 is minimum, so C_2 has no chord. Note that $z \notin \{u, v\}$, since z is not incident with a chord of C. Therefore, $z' \in V(G_1)$ if neither x nor y is a boundary neighbor of z in G. By the induction hypothesis, there is a $(3,2)$-decomposition (D_1, H_1) of G_1 with respect to (x, y, z, z') or (x, y, z) depending on the existence of z'. Let z'' be a boundary neighbor of v in G_2 other than u. By the induction hypothesis, there is a $(3,2)$-decomposition (D_2, H_2) of G_2 with respect to (u, v, z''). Note that z'' is not incident with a chord of C_2 since it has no chord. Let $D = D_1 + (D_2 - uv)$ and $H = H_1 + (H_2 - uv)$. Since $N^+_{D_2}(u) = \{v\}$, $\deg^+_{D_2}(v) = \deg_{H_2}(u) = \deg_{H_2}(v) = 0$ by Condition (iii) for (D_2, H_2), it follows that D is acyclic and Conditions (i)-(iii) are easily verified.

Case 2-2 For every chord uv of C, $x, y \in V(G_1)$ and $z \in V(G_2) \setminus V(G_1)$, where G_1 and G_2 are the plane subgraphs of G separated by uv. See the second figure of Figure 4.

Let C_i be the boundary cycle of G_i. Choose the chord uv so that G_1 is minimum, so C_1 has no chord. By the induction hypothesis, there is a $(3,2)$-decomposition (D_1, H_1) of G_1 with respect to (x, y, w) where w is a boundary vertex of G_1 so that wx is a boundary edge of G_1. Note that w is not incident with a chord of C_1.

If either zu or zv is a boundary edge of G, then there exists a $(3,2)$-decomposition (D_2, H_2) of G_2 with respect to (u, v, z) by the induction hypothesis. If z is neither adjacent to u nor v, then $z' \in V(G_2) \setminus \{u, v\}$, so let (D_2, H_2) be a $(3,2)$-decomposition of G_2 with respect to (u, v, z, z'). Let $D = D_1 + (D_2 - uv)$ and $H = H_1 + (H_2 - uv)$. Since $N^+_{D_2}(u) = \{v\}$, $\deg^+_{D_2}(v) = \deg_{H_2}(u) = \deg_{H_2}(v) = 0$ by Condition (iii) for (D_2, H_2), it follows that D is acyclic and Conditions (i)-(iii) are also easily verified.

Case 3 C is not a triangle and has no chord.

Let zw be the boundary edge of G where $w \notin \{x, y, z'\}$, and let w^* be the other boundary neighbor of z in G. Note that $w^* \in \{x, y, z'\}$. For simplicity, let $U = N_G(z) \setminus \{w, w^*\}$. Let $G' = G - z$. Note that G' is a near triangulation, and let C' be the boundary cycle of G'. Let

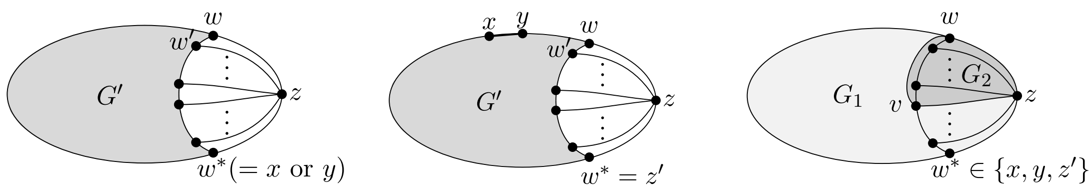

Figure 5: Illustrations for **Case 3**

w' be the interior vertex of G which is a boundary neighbor of w in G'. (Such w' exists, since G has no chord and so $\deg_G(z) \geq 3$.)

Case 3-1 C' has no chord at the vertex w.

We find a (3,2)-decomposition (D', H') of G' with respect to (x, y, w, w') (if y or x is a boundary neighbor of w in G, then we do not consider w') by the induction hypothesis. Note that $\deg^+_{D'}(w) + \deg_{H'}(w) \leq 2$. Let $\tilde{D} = D' + \{(u, z) \mid u \in U\}$ for simplicity.

Suppose $w^* \in \{x, y\}$. See the first figure of Figure 5. If $\deg_{H'}(w) \leq 1$, then let $D = \tilde{D} + (z, w^*)$ and $H = H' + zw$. If $\deg_{H'}(w) = 2$, then let $D = \tilde{D} + \{(w, z), (z, w^*)\}$ and $H = H'$. Since $\deg^+_{D'}(y) = 0$ and $N^+_{D'}(x) = \{y\}$, it follows that D is acyclic. Moreover, $\deg_{H'}(w) = 2$ implies $\deg^+_{D'}(w) = 0$, so Conditions (i)-(iii) are verified.

Suppose $w^* = z'$. See the second figure of Figure 5. We divide into four cases according to $\deg_{H'}(w)$ and $\deg^+_{D'}(z')$. Note that $\deg_{H'}(w) = 2$ implies $\deg^+_{D'}(w) = 0$.

- If $\deg_{H'}(w) \leq 1$ and $\deg^+_{D'}(z') \leq 1$, then let $D = \tilde{D} + \{(z', z)\}$ and $H = H' + zw$.
- If $\deg_{H'}(w) = 2$ and $\deg^+_{D'}(z') \leq 1$, then let $D = \tilde{D} + \{(z', z), (w, z)\}$ and $H = H'$.
- If $\deg_{H'}(w) \leq 1$ and $\deg^+_{D'}(z') = 2$, then let $D = \tilde{D}$ and $H = H' + \{zw, zz'\}$.
- If $\deg_{H'}(w) = 2$ and $\deg^+_{D'}(z') = 2$, then let $D = \tilde{D} + \{(w, z)\}$ and $H = H' + zz'$.

Clearly, the resulting digraph D is acyclic. It is also easy to check Conditions (i) and (iii). By Condition (ii) for (D', H'), we have $\deg^+_{D'}(z') + \deg_{H'}(z') \leq 3$, so Condition (ii) is also satisfied.

Case 3-2 C' has a chord wv.

Since there is no chord of C by the case assumption, $v \in N_G(z) \setminus \{w^*, w'\}$. Then $G - \{z, w, v\}$ has two components V_1 and V_2. Let $G_i = G[V_i \cup \{z, w, v\}]$ for each i, and assume $x, y \in V(G_1)$. Note that each G_i is a near triangulation. See the last figure of Figure 5. By the induction hypothesis, there is a $(3, 2)$-decomposition (D_1, H_1) of G_1 with respect to (x, y, z, z') (if either zy or zx is a boundary edge of G_1 (or G), then we do not consider z'). By the induction hypothesis, there is a $(3, 2)$-decomposition (D_2, H_2) of G_2 with respect to (w, v, z). Let $D = D_1 + (D_2 - \{zw, vz, vw\})$ and $H = H_1 + H_2$.

By Condition (iii) for (D_2, H_2), (s, t) is an arc of D_2, for every edge st of G joining an interior vertex s of G_2 and a boundary vertex t of G_2. Hence, D is acyclic and Conditions (i)-(iii) are easily verified. □

4 Proof of $(4,1)$-decomposability

A *d-vertex*, a *d^+-vertex*, and a *d^--vertex* are a vertex of degree d, at least d, and at most d, respectively. A *d-neighbor* is a neighbor that is a d-vertex. A *d^+-neighbor* and a *d^--neighbor* are defined analogously. Note that even though a matching is a collection of edges, we sometimes refer to it as a subgraph with maximum degree one.

Let G be a minimum counterexample to Theorem 1.1 with respect to the number of vertices. We may assume that G is a triangulation, and fix an embedding of G. The following lemma reveals some reducible configurations of G.

Lemma 4.1. *The following structures cannot appear in G:*

(i) *A 4^--vertex.*

(ii) *Two adjacent 5-vertices.*

(iii) *A 5-vertex with three consecutive 6^--neighbors.*

(iv) *A 5-vertex with two 7-neighbors and three 6^--neighbors.*

(v) *A 7-vertex with three consecutive 6^--neighbors where two of them are 5-vertices.*

Proof. In all cases, we will obtain a $(4,1)$-decomposition of G, which is a contradiction.

(i) Suppose to the contrary that there is a 4^--vertex v. By the minimality of G, $G-v$ has a $(4,1)$-decomposition (D', M') with a 4-degenerate ordering σ' of D'. Let $M = M'$, and let D be the graph from D' by adding all edges incident to v. Clearly, M is a matching and the ordering obtained by appending v to σ' is a 4-degenerate ordering of D, so D is 4-degenerate.

(ii) Suppose to the contrary that there are two adjacent 5-vertices u and v. By the minimality of G, $G - \{u, v\}$ has a $(4,1)$-decomposition (D', M') with a 4-degenerate ordering σ' of D'. Let $M = M' \cup \{uv\}$ and let $D = G - M$. Clearly, M is a matching and the ordering obtained by appending v, u to σ' is a 4-degenerate ordering of D, so D is 4-degenerate.

(iii) Suppose to the contrary that there is a 5-vertex v with three 6^--neighbors u_1, u_2, u_3, and $u_1u_2, u_2u_3 \in E(G)$. By the minimality of G, $G - \{v, u_1, u_2, u_3\}$ has a $(4,1)$-decomposition (D', M') with a 4-degenerate ordering σ' of D'. Let $M = M' \cup \{vu_1, u_2u_3\}$ and let $D = G - M$. Clearly, M is a matching, and the ordering obtained by appending u_3, u_1, u_2, v to σ' is a 4-degenerate ordering of D, so D is 4-degenerate.

(iv) Suppose to the contrary that there is a 5-vertex v with three 6^--neighbors and two 7-neighbors. Let $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$ where $u_1u_5 \in E(G)$ and $u_iu_{i+1} \in E(G)$ for $i \in \{1, 2, 3, 4\}$.

By (ii) and (iii), we may assume that u_1, u_2, u_4 are the 6-vertices, and u_3 and u_5 are the 7-vertices. By the minimality of G, $G - N_G[v]$ has a $(4,1)$-decomposition (D', M') with a 4-degenerate ordering σ' of D'. Let $M = M' \cup \{vu_5, u_1u_2, u_3u_4\}$ and let $D = G - M$. Clearly, M is a matching, and the ordering obtained by appending $u_5, u_1, u_3, u_2, u_4, v$ to σ' is a 4-degenerate ordering of D, so D is 4-degenerate.

(v) Suppose to the contrary that there is a 7-vertex v with three consecutive neighbors u_1, u_2, u_3 where two of them are 5-vertices. By (ii), u_1, u_3 are 5-vertices and u_2 is a 6-vertex.

By the minimality of G, $G - \{v, u_1, u_2, u_3\}$ has a $(4,1)$-decomposition (D', M') with a 4-degenerate ordering σ' of D'. Let $M = M' \cup \{vu_1, u_2u_3\}$ and let $D = G - M$. Clearly, M is a matching, and the ordering obtained by appending v, u_2, u_3, u_1 to σ' is a 4-degenerate ordering of D, so D is 4-degenerate. $\square$

We use the discharging method to reach the final contradiction, to conclude that the minimum counterexample G could not have existed. By Euler's formula, recall that

$$\sum_{v \in V(G)} (\deg_G(v) - 6) + \sum_{f \in F(G)} (2\deg_G(f) - 6) = -12.$$

Since $\deg_G(f) \geq 3$ for every face $f \in F(G)$, we know

$$\sum_{v \in V(G)} (\deg_G(v) - 6) \leq -12.$$

Let the initial charge of each vertex v be $\deg_G(v) - 6$, and note that the initial charge sum is negative. We will reach a contradiction by showing that the final charge at each vertex is non-negative after the discharging rules, which preserves the charge sum. The following is our one discharging rule:

[R] Each 6^+-vertex v sends charge $(\deg_G(v) - 6)/d_5(v)$ to each of its 5-neighbors, where $d_5(v)$ is the number of 5-neighbors of v.

By Lemma 4.1 (ii), $d_5(v) \leq \left\lfloor \frac{\deg_G(v)}{2} \right\rfloor$. Thus, an 8-vertex and a 7-vertex send charge at least $\frac{1}{2}$ and at least $\frac{1}{3}$, respectively, to each 5-neighbor.

By the rule [**R**], the final charge of a 6^+-vertex is non-negative. By Lemma 4.1 (i), it remains to check 5-vertices. Take a 5-vertex v, and let $N_G(v) = \{u_1, u_2, u_3, u_4, u_5\}$ where $u_1u_5 \in E(G)$ and $u_iu_{i+1} \in E(G)$ for $i \in \{1, 2, 3, 4\}$. If v has at least two 8^+-neighbors, then the final charge of v is non-negative. If v has no 8^+-neighbors, then by Lemma 4.1 (iii) and (iv), it has at least three 7-neighbors, and the final charge of v is non-negative.

Assume v has exactly one 8^+-neighbor u_5. If v has at least two 7^+-neighbors other than u_5, then it has non-negative final charge. If v has no 7^+-neighbor other than u_5, then this is a contradiction to Lemma 4.1 (iii). Thus, v has exactly one 7-neighbor, so it has three 6^--neighbors. By Lemma 4.1 (iii), we may assume that u_3 is the 7-neighbor and u_1, u_2, u_4 are the 6^--neighbors. By Lemma 4.1 (ii) and (v), the 7-vertex u_3 has at most two 5-neighbors. Thus u_3 sends charge at least $\frac{1}{2}$ to v by the rule [**R**]. Since u_5 sends charge at least $\frac{1}{2}$ to v by the rule [**R**], the final charge of v is non-negative.

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References

[1] William Cushing and H. A. Kierstead. Planar graphs are 1-relaxed, 4-choosable. *European J. Combin.*, 31(5):1385–1397, 2010.

[2] Zdeněk Dvořák and Bojan Mohar. Spectral radius of finite and infinite planar graphs and of graphs of bounded genus. *Journal of Combinatorial Theory, Series B*, 100(6):729–739, 2010.

[3] Nancy Eaton and Thomas Hull. Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, 25:79–87, 1999.

[4] Daniel Gonçalves. Covering planar graphs with forests, one having bounded maximum degree. *Journal of Combinatorial Theory, Series B*, 99(2):314–322, 2009.

[5] Jarosaw Grytczuk and Xuding Zhu. The alon-tarsi number of a planar graph minus a matching. *Journal of Combinatorial Theory, Series B*, 2020.

[6] DJ Guan and Xuding Zhu. Game chromatic number of outerplanar graphs. *Journal of Graph Theory*, 30(1):67–70, 1999.

[7] Grzegorz Gutowski, Ming Han, Tomasz Krawczyk, and Xuding Zhu. Defective 3-paintability of planar graphs. *The Electronic Journal of Combinatorics*, 25(2):2–34, 2018.

[8] Hongbi Jiang and Daqing Yang. Decomposing a graph into forests: the nine dragon tree conjecture is true. *Combinatorica*, 37(6):1125–1137, 2017.

[9] Yifan Jing, Alexandr Kostochka, Fuhong Ma, Pongpat Sittitrai, and Jingwei Xu. Defective dp-colorings of sparse multigraphs. *arXiv preprint arXiv:1912.03421*, 2019.

[10] Mickael Montassier, Patrice Ossona de Mendez, André Raspaud, and Xuding Zhu. Decomposing a graph into forests. *Journal of Combinatorial Theory, Series B*, 102(1):38–52, 2012.

[11] C St JA Nash-Williams. Edge-disjoint spanning trees of finite graphs. *Journal of the London Mathematical Society*, 1(1):445–450, 1961.

[12] C St JA Nash-Williams. Decomposition of finite graphs into forests. *Journal of the London Mathematical Society*, 1(1):12–12, 1964.

[13] William Thomas Tutte. On the problem of decomposing a graph into n connected factors. *Journal of the London Mathematical Society*, 1(1):221–230, 1961.

[14] R. Škrekovski. List improper colourings of planar graphs. *Combin. Probab. Comput.*, 8(3):293–299, 1999.

[15] Hermann Weyl. Das asymptotische verteilungsgesetz der eigenwerte linearer partieller differentialgleichungen (mit einer anwendung auf die theorie der hohlraumstrahlung). *Mathematische Annalen*, 71(4):441–479, 1912.

[16] Xuding Zhu. The game coloring number of pseudo partial k-trees. *Discrete Mathematics*, 215(1):245 – 262, 2000.

[17] Xuding Zhu. Refined activation strategy for the marking game. *Journal of Combinatorial Theory, Series B*, 98(1):1–18, 2008.