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A KURATOWSKI THEOREM FOR THE PROJECTIVE PLANE

The Ohio State University

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A KURATOWSKI THEOREM FOR THE
PROJECTIVE PLANE

DISSERTATION

Presented in Partial Fulfillment of the Requirements for
the Degree Doctor of Philosophy in the Graduate
School of The Ohio State University

By

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* * * * *

The Ohio State University

1980


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Chapter 1

SOME BASIC DEFINITIONS AND THE STATEMENT OF THE MAIN RESULT

§1.1 The Statement of the Main Result

We shall assume the reader is familiar with the basic terms and definitions and notation of graph theory. An embedding of a graph G into a surface M , $G \subset M$, is a realization of a homeomorphic image of G as a subspace of M . A graph G is irreducible for a surface M provided there does not exist an embedding of G in M , denoted $G \not\subset M$ but for any proper subgraph $H \subset G$, $H \subset M$. Irreducible graphs are the smallest (with respect to inclusion) graphs which fail to embed on a given surface. Let $I(M)$ denote the set of homeomorphy classes of irreducible graphs for the surface M .

The real projective plane, P , is defined as the orbit space of the antipodal involution on the two-sphere. P can also be described as the nonorientable surface of genus 1. The main result of this paper will be to list the set of all irreducible graphs for the projective plane.

Theorem 1.1. $I(P)$ is the set of 103 graphs listed in the appendix.

The proof of this theorem is in §1.4. This theorem is similar in nature to Kuratowski's theorem [7] which states

$$I(\mathbb{R}^2) = \{K_{3,3}, K_5\}$$

where \mathbb{R}^2 denotes the real euclidean plane.

Glover, Huneke, and Wang [5] have shown that the 103 graphs in the appendix are distinct, irreducible graphs for the real projective plane. Thus to prove theorem 1.1 it suffices to show this list is complete, i.e., it contains all irreducible graphs for P .

Glover and Huneke have shown [3] that $|I(P)|$ is finite but their bound is rather large compared to the number 103 of theorem 1.1. Another precursor to theorem 1.1 is the determination of the cubic graphs in $I(P)$, [4],[8].

The reader should note the graphs of appendix A are individually named using a letter between A and G and a numerical subscript. The graphs named with letter A all have Betti number 12, the graphs with letter B have Betti number 11, and so forth. Within each letter class the graphs are ordered by the numerical subscripts in a manner consistent with the decreasing lexicographic ordering on the vertex valency sequences.

§1.2 The Topology of the Projective Plane

A simple cycle C in P is called essential (denoted $\neq *$) if the topological complement of C in P (denoted P/C) is connected, and is called null (denoted $\approx *$) otherwise. The fundamental group of P is \mathbb{Z}_2 . Essential cycles correspond to the nonzero element in \mathbb{Z}_2 , while null cycles correspond to 0. The following is a well known theorem of topology.

Lemma 1.2. Any two essential cycles in P must intersect each other.

Proof. See ([5], lemma 2.2).

□

The star of a vertex, $st(v)$, is the vertex v together with the interiors of all edges incident with v . The closed star of v , $\overline{st(v)}$, is the topological closure of $st(v)$ in G , i.e., $\overline{st(v)} = st(v) \cup$ all vertices of distance 1 from v . The preceding two terms shall also be applied to arbitrary subgraphs of G .

A subgraph K of a graph G will be called a k-graph if there exist a graph L of G , $K \subset L \subset G$.

Such that

- 1) $K \approx K_4$ or $K \approx K_{2,3}$ (\approx denotes homeomorphism),
- 2) $L \setminus st(K)$ is connected, and
- 3) the quotient space $\frac{L}{L \setminus st(K)} \approx K_{3,3}$ or $\frac{L}{L \setminus st(K)} \approx K_5$.

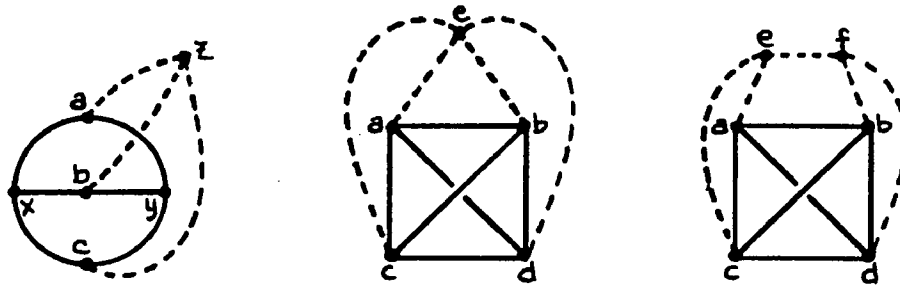


Figure 1.1

Figure 1.1 illustrates the homeomorphism types of k -graphs. The solid lines are K , while the dotted lines represent minimal representations of $L \setminus K$. A k -graph homeomorphic to a $K_{2,3}$ with bipartition sets $\{a,b,c\}, \{x,y\}$ will be called a $\underline{k}_{2,3}$ and denoted $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$. Similarly a k -graph homeomorphic to a K_4 on vertices $\{a,b,c,d\}$ will be called a \underline{k}_4 and denoted $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. A $K_{3,3}$ graph with vertex set $\{a,b,c\} \cup \{x,y,z\}$ will be denoted $\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}$.

Lemma 1.3. If K is a k -graph of G and $i: G \subset P$ is an embedding of G then there exists a cycle C of K such that $i(C)$ is essential.

Proof: See (GHW [5], lemma 2.5).

□

§1.3 Two Partial Orderings for $I(P)$

Let v be a vertex of a graph G , and label the n distinct vertices adjacent to v by $1, 2, 3, \dots, j, j+1, \dots, n$. From G we can construct a new graph, $S_v(G)$, by a process called splitting a vertex. $S_v(G)$ is formed by removing $st(v)$ from G , adding two new vertices v, v' and an edge e joining them, and adding in edges joining $1, 2, \dots, j$ to v and $j+1, \dots, n$ to v' . More specifically we shall denote the resulting graph $S_{v:(1, \dots, j)}(G)$. The reverse of this operation, contracting an edge, will be denoted by $\frac{G}{e}$. We note contracting an edge is the same as topologically identifying the edge with a point. As an example we note $K_{3,3}$ may be constructed from K_5 by splitting a vertex and deleting two edges.

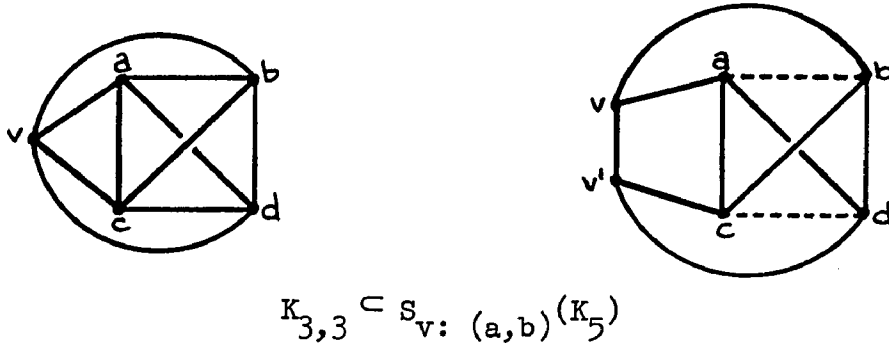


Figure 1.2

Lemma 1.4. If $G \notin M$ then $S_v(G) \notin M$.

Proof. See GHW ([5], lemma 0.2). □

Thus $S_v(G)$ must contain some $G' \in I(M)$ if $G \in I(M)$. We shall say G' is an elementary derivation of G , and denote this by $G \geq G'$. Note G' is constructed from G by splitting a vertex and deleting a set (possibly empty) of edges. We shall consider the reflexive, transitive relation, also denoted \geq , on $I(M)$ generated by elementary derivations.

Lemma 1.5. $(I(M), \geq)$ is a partially ordered set.

Proof. By definition it is reflexive and transitive. It remains to show the relation is antisymmetric. Consider the function σ assigning to each graph its valency sequence. Partially order the valency sequences lexicographically, and also denote this ordering by \geq . The function σ is order preserving, i.e., $G > G' \Rightarrow \sigma(G) > \sigma(G')$. Since the range is antisymmetric the domain is also. □

For a given surface Σ let $I^M(\Sigma)$ denote the set of maximal elements in $(I(\Sigma), \geq)$.

Theorem 7.1. $I^M(P) = \{A_1, A_2, B_1, B_3, D_9\}$.

Proof. See chapter 7.

□

Maximal graphs are of interest because by a sequence of vertex splittings and edge deletions we can generate large numbers of irreducible graphs from $I^M(P)$. However it should be noted that this process is extremely tedious. Finding each of the possibly many irreducible graphs contained in any particular $S_v(G)$ may involve the deletion of several edges. Also for a given $G \in I(P)$ many different vertex splittings are possible. Glover, Huneke, and Wang have announced [6] that the 103 graphs of appendix A are the entire set of graphs derivable from the 5 maximal graphs. To avoid the difficulty of this check we shall use a courser partial ordering (subordering) in which the type of edge deletions allowed are explicitly stated. For this we need the following lemma.

Lemma 1.6. Let M be a surface and let G be a graph. Let v be a cubic vertex in G adjacent to vertices a, b . If $G \subset M$ then $G \cup (a, b) \subset M$.

Proof. (From [5]) $G \subset M$ with a local neighborhood of v as in figure 1.3. As shown in that figure we may extend $G \subset M$ to $G \cup (a, b) \subset M$. □

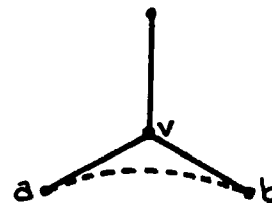


Figure 1.3

Let $G, G' \in I(P)$, $G' \subset S_v(G)$ for some vertex v . We shall say G' is an elementary $*$ -derivation of G , $G \xrightarrow{*} G'$, provided:

1) both G, G' do not contain disjoint k -graphs, and

$e \in E(S_v(G) \setminus G') \Rightarrow e$ is in a 3-cycle opposite one of the two new vertices created in the splitting, said vertex being valency 3 in $S_v(G)$;

or 2) both G, G' contain disjoint k -graphs, K_1 and K_2 , with $G' = S_v(G)$, and either:

a) v is disjoint from $(K_1 \cup K_2)$,

b) $v \in K_1$ and the bipartition of edges incident with v in the splitting is $\{\text{the edges of } K_1\}, \{\text{edges not in } K_1\}$,

or c) $v \in K_1 = k_{2,3}$ with v one of the valency 2 vertices.

The reader is referred to figure 1.4 for some illustrations of elementary $*$ -derivations.

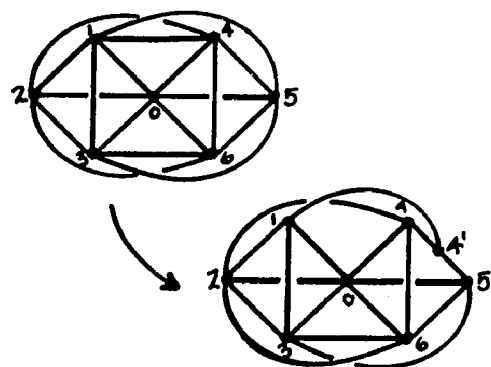
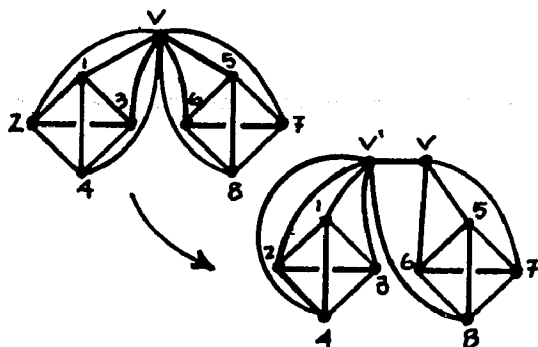
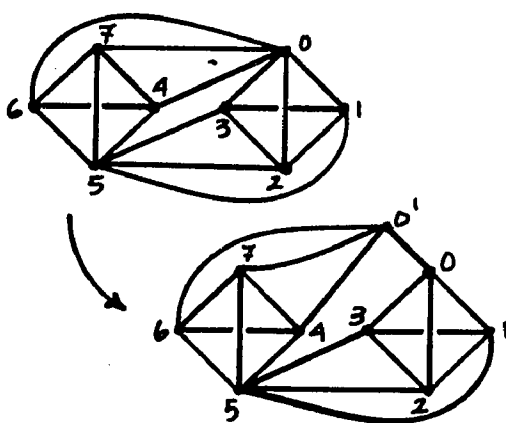
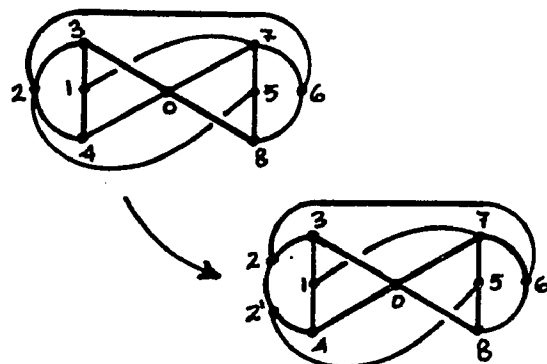
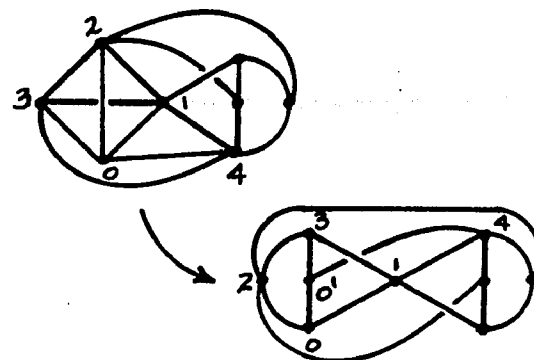
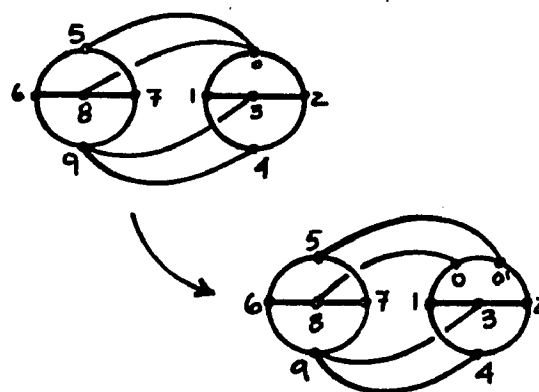
Type 1) $S_4:(0,2,6) - (1,5)$ Type 2a) $S_v:(5,6,7)$ Type 2b) $S_0:(1,2,3)$ Type 1) $S_2:(3,6)$ Type 1) $S_0:(1,2) - \{(1,2), (3,4)\}$ Type 2c) $S_0:(1,8)$

Figure 1.4

We shall consider the reflexive transitive relation, also denoted $\overset{>}{*}$, generated by elementary $*$ -derivations. Note $G \overset{>}{*} G'$ implies $G \geq G'$ so it follows from lemma 1.5 that $\overset{>}{*}$ is a partial ordering. Let $I_*^M(P)$ denote the set of maximal elements in $(I(P), \overset{>}{*})$.

Let $G, G' \in I(P)$, $G' \subset S_V(G)$. Define an elementary S-derivation, $G \overset{>}{s} G'$, if $G' = S_V(G)$. Again let $\overset{>}{s}$ also denote the reflexive transitive partial ordering generated by elementary S-derivations. Maximal elements in $(I(P), \overset{>}{s})$ will be called sources. Sources are often called minimal minors in the literature. Minimal elements in $(I(P), \overset{>}{s})$ will be called sinks. Finally consider the partially ordered set $(I(P), \overset{>}{s*})$ where $\overset{>}{s*} = \overset{>}{*} \cap \overset{>}{s}$. Maximal elements in this set will be called *-sources, minimal elements *-sinks. Observe $\overset{>}{s*}$ agrees with $\overset{>}{*}$ if G, G' contain disjoint k -graphs, and $\overset{>}{s*}$ agrees with $\overset{>}{s}$ if G, G' do not contain disjoint k -graphs.

§1.4 An Outline of the Proof of the Main Result

Recall

Theorem 1.1. $I(P)$ is the set of 103 graphs listed in the appendix.

Proof. The 103 graphs of the appendix are each in $I(P)$ by [3]. Theorem 1.7 identifies a set of graphs containing $I_{*}^M(P)$. Theorem 6.1 identifies the 103 graphs as all those in $I(P)$ below a graph in the set identified by theorem 1.7. Finally we note that $G \not\geq G' \Rightarrow |v(G)| < |v(G')|$ shows each graph in $I(P)$ is \leq_{*} to a maximal graph, i.e., $(I(P), \leq_{*})$ does not contain an infinite chain.

Theorem 1.7 depends on the results in chapters 2, 3, 4, and 5, which are independent of chapter 1. Theorem 6.1, and all the material of chapter 6, is proved independent of the preceeding chapters.

□

We point out we do not find all relations in $(I(P), \leq)$, or even all relations in $(I(P), \leq_{*})$. We examine only enough relations to guarantee reaching each graph in $I(P)$.

Let $G \in I(P)$ and let H_1, H_2 be k -graphs of G . Let $\underline{H_1 \vee H_2}$ denote a one point union of H_1 and H_2 with the property each H_i contains a cycle disjoint from H_{3-i} . Likewise write

$\underline{H_1 \vee H_2}$ if H_2 contains a cycle disjoint from H_1 . Similarly $\underline{H_1 \vee H_2}$ means H_1 contains a cycle disjoint from H_2 . Any such one point union will be called a wedge product, or more simply a wedge.

Theorem 1.7. $I_*^M(P) \subseteq \{A_1, A_2, A_5, B_1, B_3, C_1, C_2, C_7, C_{11}, D_1, D_4, D_5, D_9, D_{12}, D_{17}, E_1, E_3, E_6, E_8, E_9, E_{11}, E_{18}, E_{19}, E_{20}, E_{22}, E_{26}, E_{27}, E_{42}, F_2, F_4, F_6, G\}$.

Proof. Theorem 2.1 states if $G \in I_*^M(P)$ and G contains disjoint k -graphs then $G \in \{A_1, A_5, B_3, C_1, C_2, C_7, C_{11}, D_1, D_4, D_5, D_9, D_{12}, D_{17}, E_1, E_6, E_8, E_9, E_{11}, E_{19}, E_{20}, E_{26}, E_{27}, E_{42}, F_2, F_4, F_6, G\}$. Theorem 3.1 states that if $G \in I_*^M(P)$ and G does not contain disjoint k -graphs but contains a wedge of k -graphs then $G \in \{A_2, B_1, E_{22}\}$. Theorem 4.1 states that there does not exist $G \in I_*^M(P)$ such that G contains a cycle disjoint from a k -graph but G does not contain either disjoint or a wedge of k -graphs. Theorem 5.1 states that if $G \in I_*^M(P)$ and G does not contain a cycle disjoint from a k -graph then $G \in \{E_3, E_{18}\}$. The proofs of these theorems will complete the proof of theorem 1.7. It should be noted these theorems give an exhaustive list of candidates for maximal graphs. The actual proof of maximality follows from theorem 6.1 which shows no two of these graphs are related by \leq_x .

□

§1.5 Some Definitions

A graph G will be called projective if $G \subset P$, and nonprojective if $G \not\subset P$. An edge e of a nonprojective graph G is reducible if $G \setminus e$ is nonprojective, and irreducible otherwise. A θ -graph is any graph homeomorphic to the greek letter θ , i.e., the union of two cycles along a common arc.

Given a graph G and a subgraph H , a (G,H) -bridge, B , is defined as the topological closure of a path component of $G \setminus H$. If (G,H) is clear we shall refer to a bridge. A bridge may consist of a single edge, or of several vertices and edges. $B \cap H$ is a set of vertices called vertices of attachment, abbreviated voa . An n -bridge is a bridge with $|\text{voa}(B)| = n$. Two bridges are equivalent if their voa are equal. Given $\varphi: H \subset P$ a region D is a component of $P \setminus \varphi(H)$. A bridge is (φ, D) -admissible if there exists an embedding $\bar{\varphi}: H \cup B \subset P$ with $\bar{\varphi}|_H = \varphi$ and $\varphi(B) \subset \bar{D}$. A bridge is φ -admissible if it is (φ, D) -admissible for some region D , and φ -inadmissible otherwise. A bridge is φ -transferable if it is (φ, D) -admissible for more than one region D . The prefix φ - shall be dropped if the embedding φ is understood from context. A region D is φ -dead if there does not exist a (φ, D) -admissible bridge. The cycle bounding D is φ -dead if D is φ -dead. A subgraph $L \subset H$ is dead if there does not exist a (G,H) -bridge with a voa in L . Let C be a simple cycle bounding a planer region D . Two bridges B_1, B_2 are C -skew if there exists $\text{voa}(B_1) u_1, v_1$, and $\text{voa}(B_2) u_2, v_2$, such that u_1, u_2, v_1, v_2 is the cyclic order on C .

Lemma 1.8. Let C be a simple cycle, D a region of $\varphi: C \subset \mathbb{R}^2$.
 Let B_1, B_2 be two (φ, D) -admissible bridges. Then $B_1 \cup B_2$ is
 $\varphi(D)$ -admissible if and only if B_1 and B_2 are neither C -skew
 nor equivalent 3-bridges.

Proof. See [1].

□

Figure 1.5 shows $K_{3,3} \subset P$, where the dotted circle is understood
 to be identified $x = -x$. Likewise in the right hand picture the
 dotted circle is understood.

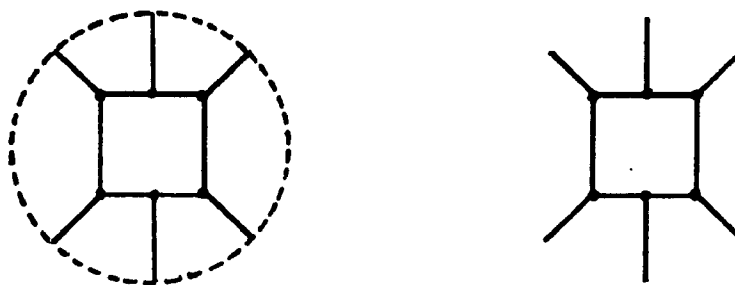
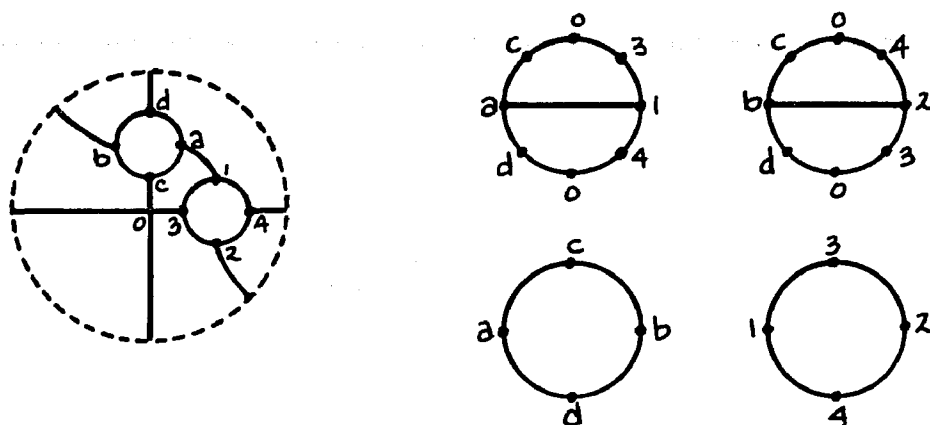


Figure 1.5

An embedding $G \subset P$ will also be called an unlabeled embedding. If G is a labeled graph, i.e., we name the vertices, $G \subset P$ will be called a labeled embedding. If G possesses symmetry there will be more labeled embeddings that can be distinguished than unlabeled embeddings that can be distinguished. In describing a labeled embedding we will often give a picture of its regions, as in figure 1.6.



Two descriptions of the same labeled embedding.

Figure 1.6

Let (a,b) be an edge of $H \subset G$. If $v \in V(G)$, $v \in (a,b)$ we shall denote v by ab . Likewise a vertex of G in edge (a,b) of H will be denoted a^2b . Here (a,b) denotes the interior of an edge, $[a,b]$ will denote $\overline{(a,b)}$, i.e., including the endpoints. Similarly $[a,b)$ denotes $(a,b) \cup \{a\}$.

Chapter 2

DISJOINT k -GRAPHS

§2.1 The Disjoint k -graph Theorem

Theorem 2.1. Let $G \in I_*^M(P)$ contain disjoint k -graphs. Then
 $G \in \{A_1, A_5, B_3, C_1, C_2, C_7, C_{11}, D_1, D_4, D_5, D_9, D_{12}, D_{17}, E_1, E_6, E_8, E_9, E_{11}, E_{19},$
 $E_{20}, E_{26}, E_{27}, E_{42}, F_2, F_4, F_6, G\}.$

Proof. Lemma 2.4 says that if G is not connected then
 $G \in \{A_5, C_{11}, E_{42}\}.$ Lemma 2.5 says that if G has a cut point then
 $G \in \{A_1, C_1, E_1\}.$ Lemma 2.8 says that if G is two-connected and
contains disjoint k_4 's then $G \in \{B_3, C_7, D_{17}\}.$ Lemma 2.11 says
that if G is two-connected and contains disjoint $k_{2,3}$'s then
 $G \in \{D_1, D_9, E_6, E_8, E_9, E_{11}, E_{26}, E_{27}, F_2, F_4, F_6, G\}.$ Lemma 2.13 says that
if G is two-connected and contains a k_4 disjoint from a $k_{2,3}$
then $G \in \{C_2, D_4, D_5, D_{12}, E_{19}, E_{20}\}.$ The proofs of these lemmas will
complete the proof of this theorem.

□

Lemma 2.2. Let $G \in I_*^M(P),$ and let $e = (x, y)$ be an edge of
 G with endpoints x and $y.$ If

- 1) G does not contain disjoint k -graphs,
 or 2) G contains disjoint k -graphs K_1 and K_2 and either
 a) e is entirely contained in one open arc connecting cubic
 vertices of $k_{2,3} = K_i$, $i = 1$ or 2
 or b) either $[x,y)$ or $(x,y]$ is disjoint from $(K_1 \cup K_2)$.

Then $\frac{G}{e} \subset P$.

Proof. Let $G' = \frac{G}{e}$, and by way of contradiction suppose $G' \not\subset P$.
 $G' \setminus e' \subset P$ for all $e' \in E(G')$ by removing the edge corresponding
 to e' in G , embedding, and applying the contrapositive of lemma
 1.4 to contract e . Thus G' is irreducible, but $G' \supsetneq G$ contradicts
 G maximal with respect to \supsetneq .

□

Lemma 2.3. If a graph G contains two disjoint k -graphs then
 G is nonprojective.

Proof. By lemma 1.3 if $G \subset P$ then each k -graph must contain
 an essential cycle. The k -graphs being disjoint contradicts
 lemma 1.2.

□

Lemma 2.4. Suppose $G \in I_*^M(P)$ is not connected. Then
 $G \in \{A_5, C_{11}, E_{42}\}$.

Proof. We first observe each component must be nonplanar or else we could embed G by placing the planar component C in a region of $G \setminus C \subset P$. Each component must be a Kuratowski graph by lemma 2.3. Hence the result follows.

□

Lemma 2.5. Suppose $G \in I_*^M(P)$ has a cut point. Then $G \in \{A_1, C_1, E_1\}$.

Proof. Let v be a cut point, and C_1, C_2 be two components (including v) of $G \setminus v$. By a method similar to lemma 2.4 we see each C_i must contain a Kuratowski graph. Observe $C_i \setminus \text{st}(v)$ contains a k -graph, hence by lemma 2.3 G is the wedge product of two Kuratowski graphs. If the wedge point v is in an arc (a, b) of one of these Kuratowski graphs, then $\frac{G}{[a, v]}$ still contains disjoint k -graphs, contradicting lemma 2.2.

□

§2.2 Disjoint k_4 's

Lemma 2.6. Suppose $G \in I_*^M(P)$ contains a k_4 disjoint from a k -graph. Then the k_4 is on 4 vertices, i.e., an edge of the k_4 is an edge of G .

Proof. Let v be a vertex on the edge joining a and b in the k_4 . G consists of disjoint k -graphs together with whatever edges are needed to complete them to Kuratowski graphs. Observe $\frac{G}{[a,v]}$ still contains disjoint k -graphs, contradicting lemma 2.2.

□

Lemma 2.7. Let $G \in I_*^M(P)$ be two-connected and contain disjoint k_4 's. Then $|v(G)| = 8$.

Proof. By way of contradiction suppose $|v(G)| = 9$. By lemma 2.6 there exists a disjoint union $v \parallel \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \parallel \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. There exists two paths from v to one k_4 , without loss of generality (v,a) and (v,b) , or else vertex v is not used in the completion of the k -graphs. If v,c,d all connect to the same component of G

$G \setminus st \left(\begin{array}{c} v \\ \diagup \quad \diagdown \\ a \quad b \\ \diagdown \quad \diagup \\ c \quad d \end{array} \right)$ then $\begin{pmatrix} a & b \\ v & c & d \end{pmatrix}$ is a k -graph, showing

(c,d) is reducible by lemma 2.3. Hence without loss of generality we have (v,c) . Suppose v and d both connect to $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. If vertex v connects to only one vertex of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, say 1, then $\frac{G}{[1,v]}$ still contains disjoint k -graphs. By symmetry d must also

connect to two vertices of $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. If $(v,1), (d,1)$ both occur then one of them is reducible. Hence given the assumption v and d connect to $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ we conclude $(v,1), (v,2), (d,3), (d,4)$, a contradiction since $G \setminus (1,2) \supset \begin{pmatrix} a & b \\ c & d \end{pmatrix} \parallel \begin{pmatrix} 1 & 2 \\ v & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Hence v, d cannot both connect to $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, which implies v connects to d . By symmetry no two vertices v, a, b, c, d can connect to $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, hence G is not two-connected, a contradiction.

□

Lemma 2.8. Let $G \in \mathcal{I}_*^M(P)$ be two-connected and contain disjoint k_4 's. Then $G \in \{B_3, C_7, D_{17}\}$.

Proof. By lemma 2.7 $|v(G)| = 8$. Call the k_4 's $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ respectively. Note G may not contain a subgraph homeomorphic to figure 2.1 or else $G \setminus (a,1)$ still contains disjoint k -graphs.

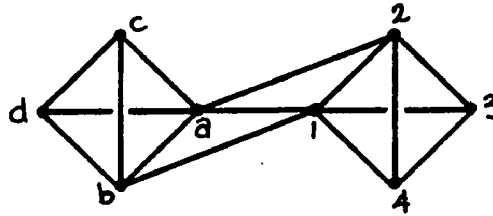


Figure 2.1

If all 4 vertices $1, 2, 3, 4$ are adjacent to vertex a , then wherever b connects gives a subgraph as in figure 2.1. If vertices $1, 2, 3$ are adjacent to a then b, c, d must all be adjacent to 4 ,

giving B_3 . If vertices 1,2 are adjacent to a , then vertices b,c,d must be adjacent to 3,4. The only choice is $(b,3)(c,3),(d,4)$ giving C_7 . If no two 1,2,3,4 are adjacent to the same vertex we get $(a,1),(b,2),(c,3),(d,4)$ which is graph D_{17} .

□

§2.3 Disjoint $k_{2,3}$'s

Lemma 2.9. Let $G \in I_*^M(P)$ contain a $k_{2,3}$ disjoint from a k -graph. Then the $k_{2,3}$ is on 5 vertices.

Proof. Let G contain $\begin{pmatrix} 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}$ disjoint from a k -graph, and suppose $G \supset v \in (1,4)$. Then $\frac{G}{[1,v]}$ still contains disjoint k -graphs, contradicting lemma 2.2.

□

Lemma 2.10. Let $G \in I_*^M(P)$ be two-connected and contain disjoint $k_{2,3}$'s. Then $|v(G)| = 10$.

Proof. There are at least 10 vertices needed in the $k_{2,3}$'s. By way of contradiction suppose G contained an eleventh vertex, v . By lemma 2.9 G contains a disjoint union $v \parallel \begin{pmatrix} d & e \\ a & b & c \end{pmatrix} \parallel \begin{pmatrix} 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}$. Without loss of generality, we have $(v,1), (v,2)$ or else v is not needed in the completion of either k -graph. Suppose $(v,3)$ is an edge. By lemma 2.9 $\begin{pmatrix} v & 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}$ is on 6 vertices. It remains to identify the 3 edges from a, b, c respectively. If two of the 3 connect to adjacent vertices then the edge joining these vertices is reducible. By two-connected the only two possibilities are $(a,1), (b,1), (c,2)$ or $(a,1), (b,2), (c,3)$. In the former graph we can contract $(c,2)$ and still have $\begin{pmatrix} a & b \\ 1 & d & e \end{pmatrix} \parallel \begin{pmatrix} 2 & 3 \\ 4 & 5 & v \end{pmatrix}$ and in the latter graph we can remove $st(5)$ and still have the nonprojective graph G of appendix A. Thus $(v,3)$ is not an edge of G .

Since $(v,3)$ is not an edge we have that v and 3 must both connect to $\begin{pmatrix} d & e \\ a & b & c \end{pmatrix}$. If v connects to only one vertex v' then (v,v') is contractible. If v connects to two vertices and one is d (e respectively) then (v,d) ((v,e) respectively) is reducible. Thus without loss of generality we have $(v,a)(v,b)(3,a)(3,c)$, a contradiction since (v,a) is reducible.

□

Lemma 2.11. Let $G \in I_*^M(P)$ contain disjoint $k_{2,3}$'s and be two-connected. Then $G \in \{D_1, D_9, E_6, E_8, E_9, E_{11}, E_{26}, E_{27}, F_2, F_4, F_6, G\}$.

Proof. Let G contain $\begin{pmatrix} d & e \\ a & b & c \end{pmatrix} \parallel \begin{pmatrix} 4 & 5 \\ 1 & 2 & 3 \end{pmatrix}$. By lemma 2.10 we know $|v(G)| = 10$. If $(a,1)(b,2)(c,3)$ are all edges then we have graph G of the appendix. If $(a,1), (b,2)$ are edges we have three possible graphs: $(c,1)(a,3)$ gives F_6 , $(c,1)(d,3)$ gives F_2 , and $(c,4)(d,3)$ gives F_4 . Next suppose $(a,1)$ is an edge and we have no other edges of this type. If $(2,a)(3,a)$ then we have two graphs: $(b,4)(c,4)$ gives E_6 , $(b,4)(c,5)$ gives E_9 . If we have $(2,a)$ then we have two graphs: $(3,d)(b,4)(c,5)$ gives E_{27} , $(3,d)(b,4)(c,4)$ gives a graph with a reducible edge $G \setminus (3,4) = F_1$. If both 1 and a are dead we have three possible graphs: $(b,4)(c,4)(2,d)(3,d)$ gives E_8 , $(b,4)(c,4)(2,d)(3,e)$ gives E_{11} , and $(b,4)(c,5)(2,d)(3,e)$ gives E_{26} . This exhausts the graphs with an edge $(a,1)$.

We now have that vertices a, b, c must be adjacent to either vertices 4 or 5, and likewise that vertices 1, 2, 3 must be adjacent to either d or e . We have three possible graphs:

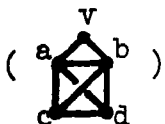
$(a,4)(b,4)(c,4)(1,d)(2,d)(3,d)$ gives D_1 , $(a,4)(b,4)(c,4)(1,d)(2,d)(3,d)$ gives graph E_4 together with a reducible edge $(3,4)$, and $(a,4)(b,4)(c,5)(1,d)(2,d)(3,e)$ gives D_9 .

□

§2.4 A Disjoint k_4 and $k_{2,3}$

Lemma 2.12. Let $G \in \mathcal{I}_*^M(P)$ be two connected and contain a $k_{2,3}$ disjoint from a k_4 . Then $|v(G)| = 9$.

Proof. G contains at least 9 vertices. By way of contradiction suppose $|v(G)| \geq 10$. By lemmas 2.6 and 2.9, G contains a disjoint union $((\begin{smallmatrix} 2 & 4 \\ 1 & 3 & 5 \end{smallmatrix}) \parallel (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})) \parallel v$. We shall examine how many edges (edge disjoint paths) join v to $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$. If $(v,a)(v,b)(v,c)(v,d)$ are in G and G were two-connected then there exists two edges from $(\begin{smallmatrix} 2 & 4 \\ 1 & 3 & 5 \end{smallmatrix})$ connecting to (without loss of generality) a and b , $G \setminus (a,b) \supset (\begin{smallmatrix} 2 & 4 \\ 1 & 3 & 5 \end{smallmatrix}) \parallel (\begin{smallmatrix} a & v \\ d & c \end{smallmatrix})$ shows (a,b) is reducible. If $(v,a), (v,b), (v,c)$ are in G then by assuming not the previous case, v and d both connect to $(\begin{smallmatrix} 2 & 4 \\ 1 & 3 & 5 \end{smallmatrix})$. If v connects only to a vertex v' then (v,v') is contractible. If v connects to vertices 1,2 then $G \setminus (v,2)$ still contains disjoint k -graphs, hence $(v,2)$ is reducible. Thus without loss of generality we have $(v,1), (v,3), (d,1), (d,3)$, which contains reducible edge $(d,1)$. If $(v,a)(v,b)$ are in G then they must be used to complete the k -graph $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix})$ into a Kuratowski graph or else one of the two would be reducible. Thus vertices v,c,d must all connect to the same component of

$G \setminus st$  $\cdot G \setminus (a,b) \supset (\begin{smallmatrix} 2 & 4 \\ 1 & 3 & 5 \end{smallmatrix}) \parallel (\begin{smallmatrix} a & b \\ v & c & d \end{smallmatrix})$ implies

edge (a,b) is reducible. If (v,a) alone is in G then $\frac{G}{[v,a]}$ still contains disjoint k -graphs contradicting lemma 2.2. Finally if

v does not connect to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then v must be used to complete the k -graph $\begin{pmatrix} 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$, implying we have edges $(v,1), (v,3), (v,5)$. The symmetry between $v,1,2,3,4,5$ shows G is not two-connected.

□

Lemma 2.13. Let $G \in \mathcal{I}_*(P)^M$ be two-connected and contain a disjoint $k_{2,3}$ and k_4 . Then $G \in \{C_2, D_4, D_5, D_{12}, E_{19}, E_{20}\}$.

Proof. Let G contain $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \parallel \begin{pmatrix} 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$. By lemma 2.2 $|v(G)| = 9$. If $G \supset (1,a)(1,b)(1,c)(1,d)$ then regardless of where 3 connects, say $(3,a)$, the edge $(1,a)$ is reducible. If $G \supset (1,a)(1,b)(1,c)$ then we must have $(3,d)(5,d)$ giving D_4 . If $G \supset (1,a)(1,b)$ then we have two possible graphs: $(3,c)(5,d)$ giving E_{19} , or $(3,c)(5,c)(d,2)$ giving F_1 together with reducible edges $(a,b)(c,d)$. Thus no more than one of $\{a,b,c,d\}$ may be adjacent to either 1,3,5. If $(1,a)(3,a)(5,a) \in E(G)$ then we have two possibilities: $(b,2)(c,2)(d,2)$ giving C_2 , or $(b,2)(c,2)(d,4)$ which gives E_4 together with reducible edges $(a,d)(b,c)$. If $(1,a)(3,a) \in E(G)$ then we have two possibilities: $(5,b)(2,c)(2,d)$ giving D_5 , or $(5,b)(2,c)(4,d)$ giving D_{12} . Finally we have $(1,a)(3,b)(5,c)(2,d)$, giving E_{20} .

□

§2.5 Some Useful Corollaries

Lemma 2.14. Suppose $G \in I(P)$ is not three-connected. Then G contains disjoint k -graphs.

Proof. See [2]. We note this lemma holds for any $G \in I(P)$, although we will only use it for $G \in I_*^M(P)$.

□

Lemma 2.15. Let L_1 be a two-connected subgraph of G , L_2 a component of $G \setminus \text{st}(L_1)$, and e an edge not in $L_1 \cup \text{st}(L_2)$. If $G \not\subset P$ but $G \setminus e \subset P$ with L_1 null, then there exists a k -graph disjoint from L_2 .

Proof. See [2].

□

Lemma 2.16. Let $G \in I(P)$ contain a θ -graph disjoint from a k -graph. Then G contains disjoint k -graphs.

Proof. See [2].

□

Lemma 2.17. Let $G \in I_*^M(P)$ contain a cubic (valency three) vertex v with $\text{st}(v)$ disjoint from a k -graph. Then G contains disjoint k -graphs.

Proof. Let a, b, c denote the vertices adjacent to v . Define a graph $G' = G \cup \{(a, b), (b, c), (c, a)\} \setminus \text{st}(v)$ (see figure 2.2).

Case 1. $G' \not\subset P$. Then G' contains some irreducible subgraph.

If we delete an edge not in the cycle (a, b, c) we can embed $(G' \cup \text{st}(v)) \setminus e$ (and hence embed $G' \setminus e$) by first embedding $G \setminus e$ and then applying lemma 1.6 three times. If we delete two of the three edges in (a, b, c) , say (a, b) and (b, c) , we get a homeomorph of $G \setminus (b, v)$, hence it embeds. Thus either G' or $G' \setminus (\text{one edge of } \{(a, b), (b, c), (c, a)\})$, say $G' \setminus (a, b)$, must be irreducible.

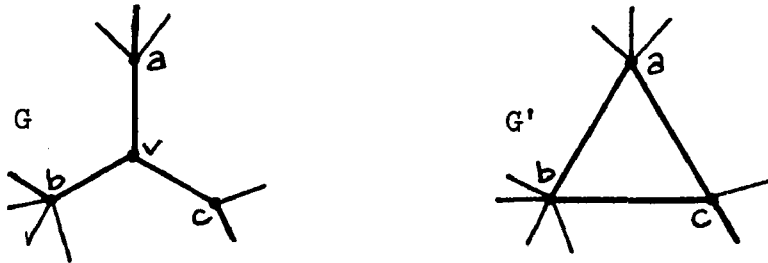


Figure 2.2

But $G \approx S_{a:(b,c)}(G') \setminus (b,c) \approx S_{a:(a,b)}(G' \setminus (a,b))$ contradicting G maximal w.r.t. $\frac{\gamma}{\kappa}$.

Case 2. $G' \subset P$. $\overline{St(v)}$ is disjoint from a k -graph, hence for any embedding of G' cycle (a,b,c) must be null by lemmas 1.2 and 1.3. Consider the planer region bounded by (a,b,c) . If we could embed $st(v)$ in this region we would have an embedding of G and a contradiction. Thus there exists a blocking bridge in this region. If G is not three connected then G contains disjoint k -graphs (lemma 2.14), hence there exists some vertex v' adjacent to $\{a,b,c\}$, so that G contains a k -graph disjoint from $\begin{pmatrix} v & v' \\ a & b & c \end{pmatrix}$.

□

Lemma 2.18. Let $G \in I_{\kappa}^M(P)$ contain a subgraph homeomorphic to that of figure 2.3 with the $K_{2,3}$ a $k_{2,3}$ k -graph. Moreover suppose (a,b) is an edge with vertices a,b cubic. Then G contains disjoint k -graphs.

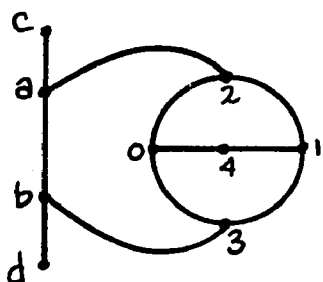


Figure 2.3

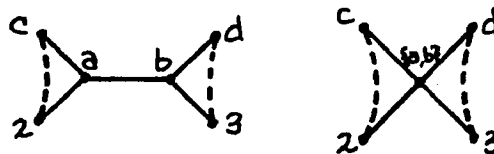


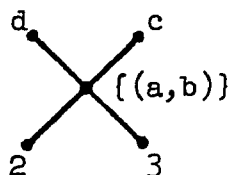
Figure 2.4

Proof. By way of contradiction, suppose G does not contain disjoint k -graphs. Note G is three-connected by lemma 2.14.

Define $G' = \frac{G}{[a,b]} \cup \{(c,2), (d,3)\}$ (figure 2.4).

Case 1. $G' \not\subset P$. G' must contain some irreducible subgraph. If we delete an edge not in $st(a) \cup st(b)$ and not $(c,2), (d,3)$, we can embed by first removing the corresponding edge in G , embedding, applying lemma 1.6 twice, and then applying the contrapositive of lemma 1.4; i.e., adding in edges $(c,2), (d,3)$ and then contracting edge (a,b) . If we delete an edge of the type $(\{(a,b)\}, 2)$ we can embed a homeomorph by embedding $\frac{G}{(c,a)}$ (lemma 2.2) and filling in edge $(d,3)$ by lemma 1.6. Thus the only possible reducible edges in G' are $(c,2), (d,3)$. Regardless of whether these edges are reducible G' contains an irreducible graph $\stackrel{>}{*} G$, contradicting G maximal.

Case 2. $G \subset P$. Note $G \subset P$ with a local neighborhood of $\{a,b\}$



or else we can split vertex $\{a,b\}$ and embed G .

Adding m edges $(d,3), (c,2)$ must give both cycles $(\{(a,b)\}, d, 3)$ and $(\{(a,b)\}, c, 2)$ essential or else $(\{(a,b)\}, d, 3)$ (for instance) separates G and edge $(c,2)$ must intersect $(\{(a,b)\}, d, 3)$, a contradiction. Cycle $(0, 2, 1, 4)$ is disjoint from $(\{a,b\}, d, 3)$ hence

cycle $(0,2,1,4)$ must be null for all embeddings. Likewise cycle $(0,3,1,4)$ must be null for all embeddings. This gives $\begin{pmatrix} 0 & 1 \\ 2 & 3 & 4 \end{pmatrix}$ null contradicting lemma 1.3.

□

Lemma 2.19. Let $G \in \mathcal{I}_*^M(P)$ contain either a cycle disjoint from a K_5 or an n -cycle disjoint from a $K_{3,3}$, $n \geq 4$. Then G contains disjoint k -graphs.

Proof. If v is a cubic vertex on the cycle then regardless of where its third adjacent vertex is we have $\overline{\text{st}(v)}$ disjoint from a k -graph, hence the result by lemma 2.17. If v is not cubic then it must connect to the star of at least two vertices of the Kuratowski graph (i.e., there exist paths from v to two vertices). By the pidgeonhole principle in either case two vertices on the cycle must connect to the star of some vertex in the Kuratowski graph, giving θ -graph disjoint from a k -graph and the conclusion by lemma 2.16. Note if the Kuratowski graph is a $K_{3,3}$ (6 vertices) and the cycle is a 3-cycle we cannot force two vertices on the cycle to connect to the open star of some vertex.

□

Chapter 3

THE WEDGE OF k -GRAPHS

§3.1 Statement of the Result and Standing Assumptions

The result of chapter 2 classified the subset of $I_*^M(P)$ whose graphs contained disjoint k -graphs. In this chapter we make the standing assumption $G \in I_*^M(P)$ does not contain disjoint k -graphs. For ease of reference, let H3 refer to this standing assumption.

Theorem 3.1. Let $G \in I_*^M(P)$ contain a wedge of k -graphs, one containing a cycle disjoint from the other, but not disjoint k -graphs. Then $G \in \{A_2, B_1, E_{22}\}$.

Proof. Considering the two types of k -graphs and possible one point unions the theorem naturally breaks into 5 cases (figure 3.1). Each of the 5 cases will be covered in the appropriate proposition. The proofs of the propositions will complete the proof of the theorem. We observe that a unioning of a k_4 with another subgraph along the interior of an edge is the same as the unioning with a $k_{2,3}$. Also the condition one k -graph must contain a cycle disjoint from the other eliminates one possible wedge of $k_{2,3}$'s.

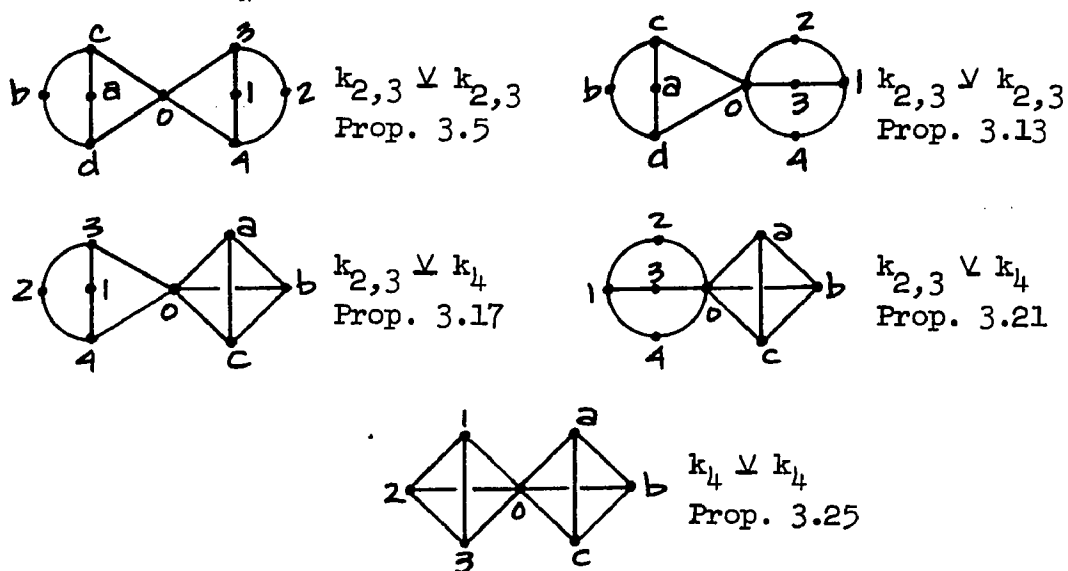


Figure 3.1

For convenience we state the propositions, with the standing assumption H3, G does not contain disjoint, k -graphs.

Proposition 3.5. Let $G \in \mathcal{I}_*^M(P)$ satisfy H3 and let $G \supset G \supset k_{2,3} \vee k_{2,3}$. Then $G = E_{22}$.

Proposition 3.13. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H3 such that G contains a wedge $k_{2,3} \vee k_{2,3}$.

Proposition 3.17. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H3 such that G contains a wedge $k_{2,3} \vee k_4$.

Proposition 3.21. There does not exist a $G \in I_*^M(P)$ satisfying H3 such that G contains a wedge $k_{2,3} \vee k_4$.

Proposition 3.25. Let $G \in I_*^M(P)$ satisfy H3 and let $G \supset k_4 \vee k_4$. Then $G \in \{A_2, B_1\}$.

□

Throughout all of chapter 3 we shall maintain the standing assumption H2, G does not contain disjoint k -graphs. From the lemmas of §2.5 we gain the following standing assumptions:

- 1) G is 3-connected
- 2) G does not contain a θ -graph disjoint from a k -graph
- 3) G does not contain cubic vertex v with $\overline{st(v)}$ disjoint from a k -graph
- 4) $G \not\supset$ a subgraph homeomorphic to figure 2.3 with $[a,b]$ dead
- 5) G does not contain a cycle disjoint from a K_5
- 6) G does not contain a cycle with 4 or more vertices disjoint from a $K_{3,3}$

The proofs in chapter 3 will be by contradiction, assuming a subgraph as in figure 3.1 and considering possible augmentations of that subgraph. The particular contradiction reached will often be referred to by name (described as in the above list) rather than by the number of the lemma.

Definition. An S_v -independent argument is a proof where each implication follows regardless of whether we consider G or $S_v(G)$.
An S-independent argument is one which is S_v -independent for all S_v .

Lemma 3.2. If there exists an S_v -independent argument showing there does not exist a $G \in I_*^M(P)$, G containing H then there exists an S_v -independent argument showing there does not exist a $G \in I_*^M(P)$, G containing $S_v(H)$.

Proof. Each implication follows independent of the splitting.
Thus the original proof suffices to show the second statement.

□

Lemma 3.3. The following implications may be used in an S_v -independent argument:

- 1) Each of lemmas 2.15 to 2.19
- 2) A contradiction identifying disjoint k -graphs in G
- 3) A contradiction identifying G with a known non-maximal graph
- 4) Lemma 1.6 (there does not exist a cubic vertex in a 3-cycle);
provided S_v preserves the 3-cycle.

Proof.

- 1) Each lemma describes assumptions which must still hold when considering $S_v(G)$ instead of G . Hence the implications still follow,

- 2) G contains disjoint k -graphs implies $S_v(G)$ contains disjoint k -graphs,
- 3) G contains $S_v(G')$, $G' \in I(P)$ implies $G \notin I_*^M(P)$,
- 4) Obvious

□

On occasion we shall make S_v -independent proofs by considering G contains H , G contains $S_v(H)$, as two separate cases. Also we shall add new "tools" to our list of S_v -independent implications of the previous lemma. As an example the reader should note proposition 3.10 is an S -independent version of proposition 3.5.

§3.2 The Wedge of $k_{2,3}$'s Each With a Cycle Disjoint From the Other

Lemma 3.4. Let G contain $k_{2,3} \vee k_{2,3}$ as in figure 3.2. Then there is exactly one unlabeled and four labeled embeddings of this wedge which may extend to an embedding of G .

Proof. Cycles $(1,3,2,4)$ and (a,c,b,d) are both disjoint from k -graphs, hence they must embed null. Cycles $(0,3,2,4), (0,3,1,4), (0,c,b,d), (0,c,a,b)$ must all embed essentially. This implies the unique unlabeled embedding of figure 3.2. Observe regions III, IV in figure 3.2 are dead, hence we need only depict regions I, II. The four labeled embeddings (arising from symmetries) are shown in figure 3.3. Note the natural bijection between embeddings of the wedge, H , and embeddings of $S_0: (3,c)^{(H)}$, shown in figure 3.4. Also embeddings of $S_0: (3,d)^{(H)}$ are shown in figure 3.5.

□

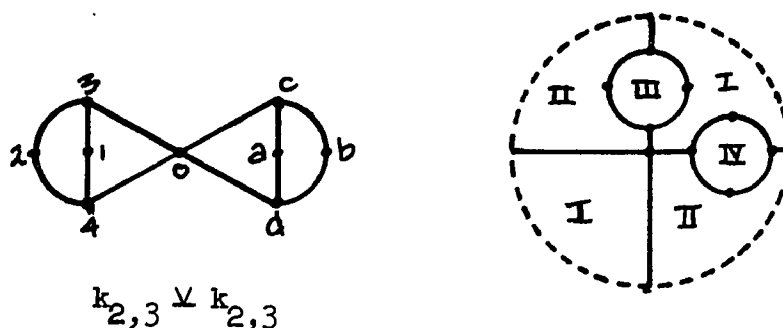
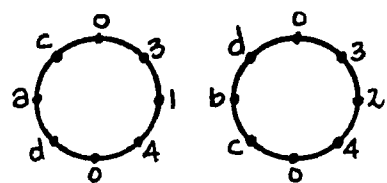
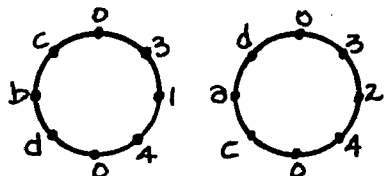


Figure 3.2



embedding 1



embedding 2

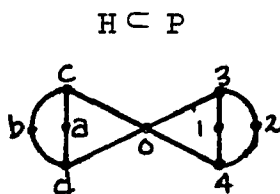
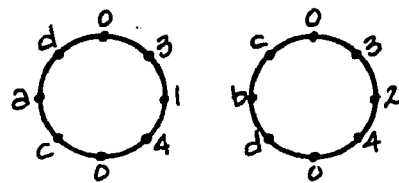
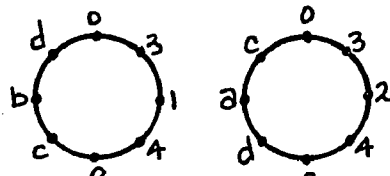


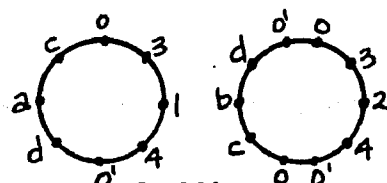
Figure 3.3



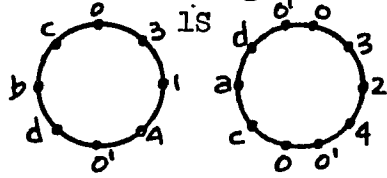
embedding 3



embedding 4



embedding 1S



embedding 2S

$$S_0: (3, c)(H) \subset P$$

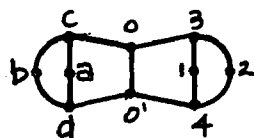
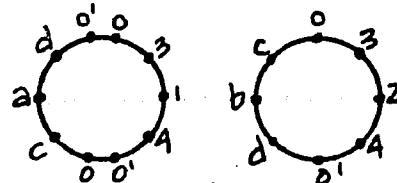
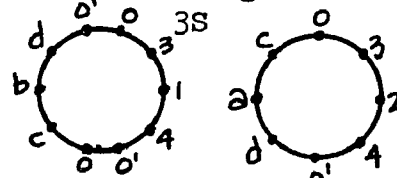


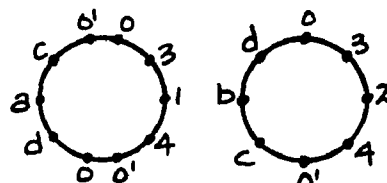
Figure 3.4



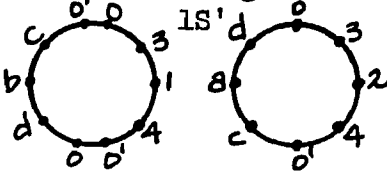
embedding 3S



embedding 4S



embedding 1S'



embedding 2S'

$$S_0: (3, d)(H) \subset P$$

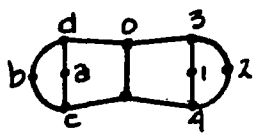
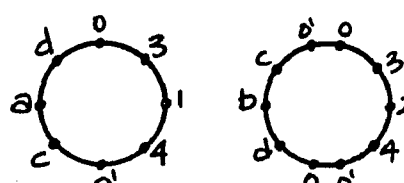
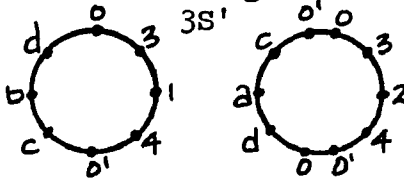


Figure 3.5



embedding 3S'



embedding 4S'

Proposition 3.5. Let $G \in \mathcal{I}_*^M(P)$ satisfy H3 and let $G \supset k_{2,3} \vee k_{2,3}$. Then $G = E_{22}$.

Proof. We consider how we may nonhomeomorphically complete the given k -graphs to Kuratowski graphs. If the vertex missing from the $K_{3,3}$ containing the $k_{2,3} \begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ does not lie in $\begin{pmatrix} c & d \\ 0 & a & b \end{pmatrix}$ then we have a θ -graph disjoint from $\begin{pmatrix} c & d \\ 0 & a & b \end{pmatrix}$, a contradiction by lemma 2.16. Thus each of the k -graphs complete to Kuratowski graphs by bridges to the other k -graph.

If $(1,a), (2,b)$ are both arcs in G then we shall apply lemma 3.9 (see figure 3.11).

If $(1,a)$ but not $(2,b)$ is an arc of G then G contains a graph homeomorphic to that of figure 3.10, or G contains a graph homeomorphic to a splitting of the graph in figure 3.10. In either case lemma 3.8 is applicable.

If neither $(1,a)$ nor $(2,b)$ are arcs of G then let $(1,c)$ be an arc of G ($(1,0c)$ is a splitting of this possibility, but we will use (S_c) -independent arguments). If $(2,c)$ is an arc of G then we have two possibilities; $(a,3), (b,3)$ or $(a,3), (b,4)$. The former graph is E_5 , which is not maximal, and the latter graph is covered in lemma 3.7. Finally if $(2,c)$ is not an arc of G then by symmetry we have $(2,d), (a,3), (b,4)$ giving graph E_{22} .

□

Lemma 3.6. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H3 and containing a homeomorph of either H (the graph of figure 3.6), or a splitting thereof where $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} c & d \\ 0 & a & b \end{pmatrix}$ are k-graphs of G.

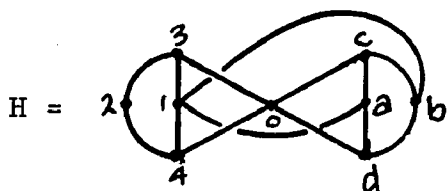


Figure 3.6

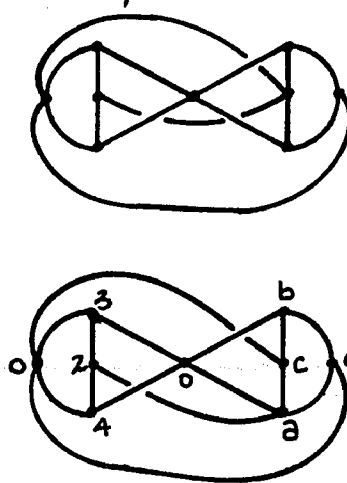


Figure 3.7

Proof. To complete $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ to a k-graph vertex 2 must connect somewhere. Figure 3.7 shows $(2,a), (2,c)$ both give F_1 , which is not maximal. Observe this is an S-independent argument, hence there does not exist a G containing a splitting of H.

□

Lemma 3.7. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H3 and containing a subgraph homeomorphic to either the graph of figure 3.8 or a splitting thereof.

Proof. $\text{St}(d)$ is disjoint from $\binom{1 \ 2}{c \ 3 \ 4}$ so by the standing assumption G does not contain disjoint k -graphs and by lemma 2.17 vertex d is not cubic. Where may it connect? If $(d,1)$ or $(d,2)$ then G contains E_{22} . (See figure 3.9.).

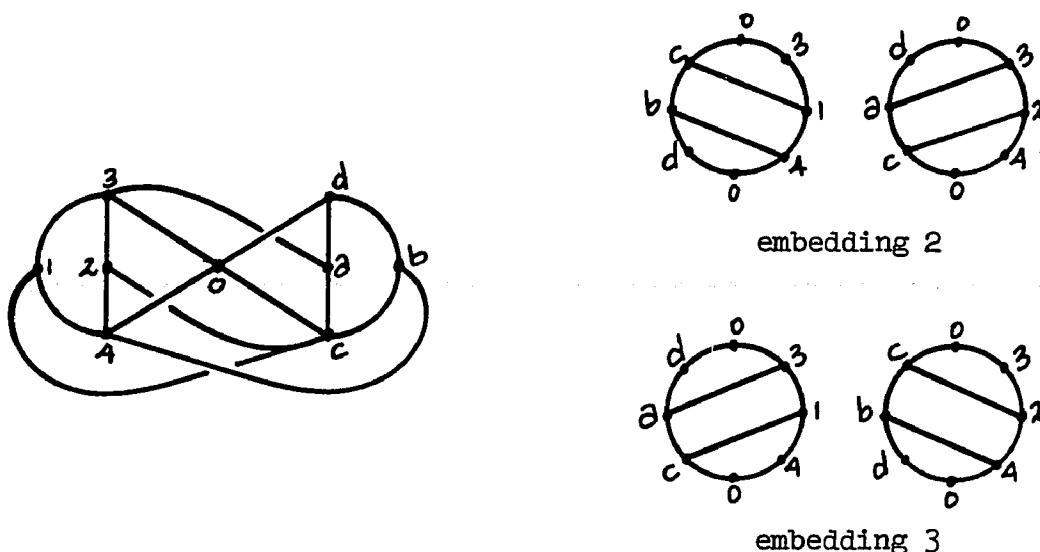


Figure 3.8

Thus we have either $(d,3)$ or $(d,4)$, without loss of generality assume $(d,3)$. Cycle $(d,a,3,0)$ is disjoint from $\binom{4 \ c}{1 \ 2 \ b}$, hence the addition of this edge creates a θ -graph disjoint from a k -graph, and by lemma 2.16 G contains disjoint k -graphs, a contradiction. Observe this is an S -independent argument, hence there does not exist a $G \in I_{*}^M(P)$ containing either H or a splitting thereof.

□

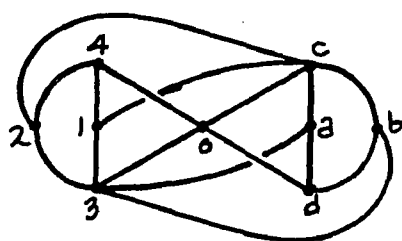
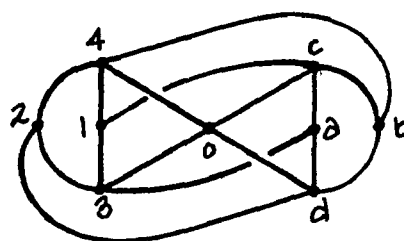
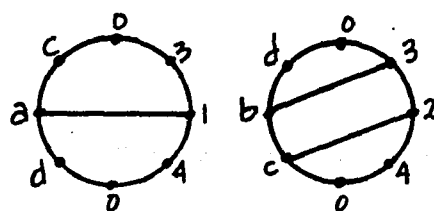
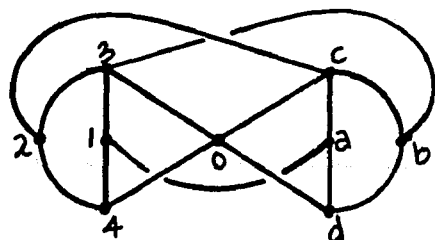
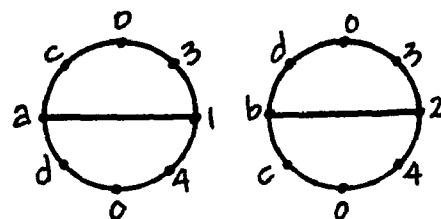
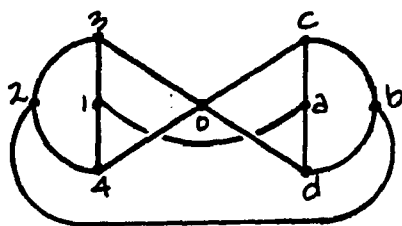
 E_5  E_{22}

Figure 3.9

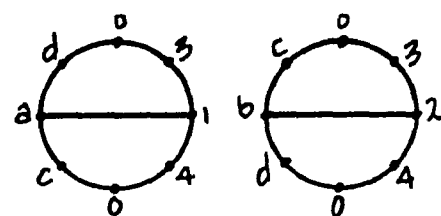


embedding 1

Figure 3.10



embedding 1



embedding 3

Figure 3.11

Lemma 3.8. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H_3 and containing a subgraph homeomorphic to either that of figure 3.10 or to a splitting thereof.

Proof. Observe by lemma 3.3 there exists a unique embedding of this subgraph. By lemma 2.18 we see that neither $[1,4]$ nor $[a,d]$ is dead. We shall assume there exists such a G containing figure 3.10 and use S -independent arguments to establish the results.

Suppose vertex 4 connects somewhere. Edge $(4,b)$ gives a nonprojective graph with $[a,d]$ dead, contradicting lemma 2.18. Likewise an edge $(4, \text{st}(c))$ gives either a nonprojective graph, or else the edge embeds in region $(2,c,0,4)$. In the latter case lemma 2.15 gives us a k -graph on $(2,c,0,4)$, and since the region admits no transferable bridges the resulting graph is nonprojective. In either case $[a,d]$ contradicts lemma 2.18. Thus we see we have either edge $(4,0d), (4,d)(4,ad)$ or $(4,a)$. By lemma 2.15 there exists a k -graph on cycle $(a,1,4,0,d)$. We note the arc $(4, _)$ is an edge of G , as 4 connecting to two places gives a θ -graph disjoint from $(\begin{smallmatrix} 3 & c \\ 0 & 2 & b \end{smallmatrix})$. The k -graph on cycle $(a,1,4,0,d)$ consists of the cycle, the edge $(4, _)$, and a bridge B blocking the extension of an embedding of $G \setminus (4, _)$ to an embedding of G .

We shall analyze the possibilities for B . If 0 is not a v.o.a. of B then we have a θ -graph disjoint from $(\begin{smallmatrix} 3 & c \\ 0 & 2 & b \end{smallmatrix})$. Note that if G contains $S_0: (3,c)$ then the same holds. Hence either G contains exactly the subgraph of figure 3.10 or $S_0: (3,d)$. In either case bridge B must be an edge out of 0 . If B does not

transfer to cycle $(a,1,3,0,c)$ then $B = (0,14)$ or $(0,ad)$, and we have either edge $[1,14]$ or $[a,ad]$ contradicting lemma 2.18. Hence $B = (0,1)$. To block this edge from transferring to $(a,1,3,0,c)$ we have either $(3,c), (3,a)$, or we have $S_0: (3,d)$ with $(1,0)$ and either $(0,c)$ or $(0,a)$. The first case has a θ -graph disjoint from a k -graph on cycle $(a,1,4,0,d)$; the second has a θ -graph disjoint from $\begin{pmatrix} 3 & 5 \\ a & b & 2 \end{pmatrix}$ if $(4,d)$ and reducible edge $(1,a)$ if $(4,a)$. In the $S_0: (3,d)$ case, $(0,c)$ gives a θ -graph disjoint from $\begin{pmatrix} c & d \\ 0 & a & b \end{pmatrix}$, hence we have $(0,a)$. Edge $(4,d)$ a θ -graph disjoint from $\begin{pmatrix} 0' & a \\ 0 & 1 & c \end{pmatrix}$ and edge $(a,0)$ in cycle $(4,a,d,0)$ gives a θ -graph disjoint from a k -graph. We have exhausted vertex 4 connecting anywhere using S -independent arguments.

To avoid $[1,4], [a,d]$ contradicting lemma 2.18 we have vertices 1, a non-cubic. Note edges $(1,b)$ contradicts lemma 3.4, and $st(d)$ is dead. Edge $(1,0)$ creates 4 cubic in a 3-cycle or vertex d cubic disjoint from $\begin{pmatrix} 0 & 2 \\ 3 & c & 1 \end{pmatrix}$ unless we have $(1,0)$ and $S_0: (3,d)$. In this case examining the unique embedding $1S'$ of figure 3.5 and embedding $(1,0)$ in region $(a,1,4,0',0,d)$ we get edge $(a,0')$. Embedding $(1,0)$ in region $(a,1,3,0,0',c)$ implies vertex 3 connects somewhere; edge $(3,c)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ 0' & 1 & d \end{pmatrix}$ hence we must have $(3,a), (1,c)$ and G contains E_5 . Thus $(1,0)$ cannot be an edge. We have forced into the case $(1,c), (3,a)$ and again, G contains E_5 . The above are S -independent arguments, hence the result is established.

□

Lemma 3.9. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying $H3$ and containing a subgraph homeomorphic to that of figure 3.11.

Proof. By way of contradiction let G be such a graph, and note figure 3.11 also gives the two unique embeddings of the subgraph. We shall first prove there does not exist a vertex v disjoint from figure 3.11.

If $a_1 \in V(G)$ then $(a_1, 0)$, or else we have a θ -graph disjoint from a k -graph; note $\overline{st(1)}, \overline{st(2)}, \overline{st(a)}, \overline{st(b)}$ are all disjoint from k -graphs. Connecting a and 1 somewhere without contradicting lemma 3.8 forces either $(a, 1)$ or $(a, 0)$. In either case vertex a_1 is cubic in a 3-cycle, forcing by lemma 1.6 a vertex v in $(a, 1), (a, 0)$, respectively, giving a k -graph $\begin{pmatrix} a_1 & v \\ 0 & a & 1 \end{pmatrix}$. Examining the embeddings and deleting $st(v)$ forces a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ a_1 & v & 4 \end{pmatrix}$. Thus there is not a vertex a_1 .

Let v be a vertex disjoint from the subgraph of figure 3.11, B the bridge containing v . Without loss of generality two v.o.a. must be 0 and c . If B had 4 or more v.o.a., G contains a θ -graph disjoint from a k -graph; hence B consists of a single cubic vertex v and there exists a vertex $0c$. Let v' be the third vertex adjacent to c . If $(0c, 1)$ then G contains a splitting of the graph in figure 3.10, say H , by examining $\begin{pmatrix} 0 & c & d \\ v & 0c & \end{pmatrix} \cup \begin{pmatrix} 1 & 2 \\ 3 & 4 & d \end{pmatrix}$. Thus $(0c, 3)$, and above union of k -graphs implies $v' = 4$ to avoid G containing H . By symmetry $(04, d)$, and there is a θ -graph disjoint from $\begin{pmatrix} 4 & d \\ 04 & 1 & 2 \end{pmatrix}$. Thus there is not a vertex disjoint from the wedge $k_{2,3} \vee k_{2,3}$.

Next we will show $(a,0)$ is not an edge of G , by contradiction.

We have three cases:

- 1) $(c,1)(d,1)$, nonprojective with $[2,b]$ disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & c & a \end{pmatrix}$ violating lemma 2.18.
- 2) $(c,3)(c,4)$, projective, embedding 3 of figure 3.11, is unique. $St(2), st(b)$ are both disjoint from k -graphs, hence they are non-cubic. Examining embedding 3 we get $(b,3)(2,c)$ and contradict lemma 3.9..
- 3) $(c,1)(d,4)$, projective embedding 3 unique. $\overline{St(b)}$ is disjoint from $\begin{pmatrix} 0 & 1 \\ a & 3 & 4 \end{pmatrix}$, hence b is not cubic. The inadmissible bridge from b is $(b,04)$, giving a θ -graph disjoint from $\begin{pmatrix} b & 4 \\ 2 & d & 04 \end{pmatrix}$. Avoiding a contradiction of lemma 3.8, there must be skew edges in either region $(b,c,0,3,2)$ or region $(b,2,4,d)$. If $(0,b),(c,2)$ or $(0,b),(c,3)$ then the wedge $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix} \vee \begin{pmatrix} a & b \\ c & d & 0 \end{pmatrix}$ gives a contradiction of lemma 3.8, $(0,2),(b,3)$ gives a nonprojective graph contradicting lemma 1.6. Thus $(a,0)$ is not an edge.

To finish the lemma suppose vertex a connects somewhere. If $(a,13)$ or $(a,3)$ then we may always add this edge in either embedding, unless 1 connects somewhere. Any such connection gives G containing a splitting of the graph in figure 3.10. If $(a,03)$ then b is cubic disjoint from $\begin{pmatrix} 03 & 1 \\ 3 & 4 & a \end{pmatrix}$. Edge $(b,03)$ gives 4-cycle $(1,3,2,4)$ disjoint from $\begin{pmatrix} 0 & a & b \\ c & d & 03 \end{pmatrix}$, contradicting lemma 2.19, hence $(b,04)$. $\overline{St(1)}$ is disjoint from $\begin{pmatrix} 0 & b \\ 04 & c & d \end{pmatrix}$ and any edge out of 1 contradicts

lemma 3.8. Hence there does not exist an edge out of a , and by disjointness arguments there does not exist an edge out of $st(1)$.

The only possible crossing edges remaining are $(c, 03), (3, 0c)$; the resulting graph viewed as a wedge $\begin{pmatrix} 0c & 03 \\ 0 & c & 3 \end{pmatrix} \vee \begin{pmatrix} d & 4 \\ 0 & a & 2 \end{pmatrix}$ contradicts lemma 3.6.

□

Lemma 3.10. There does not exist a $G \in I_*^M(P)$ satisfying H3 and containing a subgraph homeomorphic to that of figure 3.12.

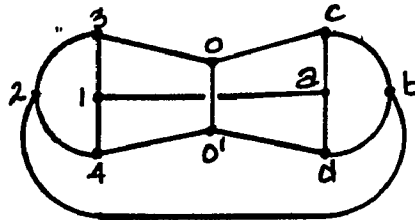


Figure 3.12

Proof. Because of the bijection between embeddings of this subgraph H and $\frac{H}{(0,0')}$ we see $[0,0']$ must not be dead. If there is a vertex $00'$ then $\overline{st(0)}$ is disjoint from $\begin{pmatrix} 4 & d \\ a & 2 & 0' \end{pmatrix}$, hence WLOG we have either $(0,13)$ or $(0,1)$, since $(0,4)$ is lemma 3.9.

Suppose $(0,13)$ is an arc of G . If there exists $v \in (0,13)$ then v connects somewhere on cycle (a,c,b,d) , else G contains a θ -graph disjoint from $\begin{pmatrix} a & b \\ c & d & 4 \end{pmatrix}$. An earlier lemma, depending on where v connects, applies if we write G as $\begin{pmatrix} 0 & 13 \\ 3 & v & 1 \end{pmatrix} \cup \begin{pmatrix} a & b \\ (1,4) & c & d & 4 \end{pmatrix}$.

Thus $(0,13)$ is an edge, not an arc of G . Next delete $(0,13)$ and embed G , using either embedding 1S or 3S of figure 3.4. There exists an edge out of 3 which makes $(0,13)$ inadmissible, by S_v independent arguments we need only consider edges between existing vertices. Again, $(3,0), (3,0')$ give a θ -graph disjoint from $\begin{pmatrix} a & b \\ c & d & 4 \end{pmatrix}$. Edge $(3,c)$ gives a θ -graph disjoint from $\begin{pmatrix} 4 & d \\ a & 2 & 0' \end{pmatrix}$, and $(3,a)$ is covered by an earlier lemma if we examine $\begin{pmatrix} 13 & 4 \\ 1 & 2 & 0 \end{pmatrix} \cup \begin{pmatrix} c & d \\ 0' & a & b \end{pmatrix}$. We need not consider $(3,b)$, as this edge does not make $(0,13)$ inadmissible, hence we must have $(3,d)$, with embedding 1s unique. $\overline{st(1)}$ disjoint from $\begin{pmatrix} 0 & d \\ 3 & 0' & c \end{pmatrix}$, $\overline{st(4)}$ disjoint from $\begin{pmatrix} c & d \\ a & b & 3 \end{pmatrix}$ hence we must have a k -graph on cycle $(a,1,4,0',d)$ giving a nonprojective graph with vertex 13 cubic in a triangle, a contradiction. Hence $(13,0)$ is not an edge of G , so $(1,0)$ must be.

Note $\overline{st(2)}$ is disjoint from $\begin{pmatrix} 0 & a \\ c & 0' & 1 \end{pmatrix}$, so we will break into cases depending on where vertex 2 connects.

If $(2,bc)$ then $\overline{st(bc)}$ is disjoint from $\begin{pmatrix} 0' & 1 \\ 0 & 4 & d \end{pmatrix}$, so bc must not be cubic. Avoiding a contradiction of lemmas 3.6, 3.7, and 3.8, bc may only connect to $0, 00'$, or $0'$. The first is equivalent to edge $(13,0)$, which case was just handled, the second contradicts lemma 3.8 if we examine $\begin{pmatrix} 3 & 4 \\ 1 & 2 & 0' \end{pmatrix} \cup \begin{pmatrix} bc & d \\ 0' & c & b \end{pmatrix}$, and the third contradicts lemma 3.9. Hence 2 does not connect to the $st(b)$.

If $(2,c)$ is an arc of G , first consider the existence of a vertex $2c$. Any edge out of $2c$ except $(2c,0)$ gives either a

θ -graph disjoint from a k -graph or an earlier case of completing the k -graphs to Kuratowski graphs. If $(2c,0)$ is an edge, the avoiding $2c$ cubic in a 3-cycle we get a vertex $0c$. Edge $(0c,2)$ gives a θ -graph disjoint from $(\begin{smallmatrix} 0' & 1 \\ 0 & 4 & d \end{smallmatrix})$, $(0c,1)$ gives a θ -graph disjoint from $(\begin{smallmatrix} 0 & 1 \\ 0c & 3 & 4 \end{smallmatrix})$, $(0c,4)$ gives the wedge $(\begin{smallmatrix} 2 & 0c \\ 4 & v & b \end{smallmatrix}) \vee (\begin{smallmatrix} 0' & 1 \\ 0 & 4 & d \end{smallmatrix})$. By S_v -independent arguments $0c$ cannot connect anywhere, hence $(2,c)$ is an edge, not an arc, of G . Next delete $(2,c)$ and embed, using either embeddings 1S or 3S of figure 3.4. Vertex b connects somewhere or we may always extend $G \setminus (2,c) \subset P$ to $G \subset P$. Avoiding a contradiction of lemmas 3.6, 3.7, and 3.8 gives that b connects to $[0,0']$. Edges $(b,0), (b,00')$ give embedding 1S unique, avoiding vertex 3 cubic in a 3 cycle and $st(d)$ disjoint from a k -graph implies $(3,d)$, a contradiction since $\overline{st(a)}$ is disjoint from $(\begin{smallmatrix} 0 & 2 \\ b & 3 & 4 \end{smallmatrix})$. Hence we get $(b,0')$ is in G . Deleting this edge and embedding gives either $(d,4)$ or $(d,3)$. The former graph contains a θ -graph disjoint from $(\begin{smallmatrix} c & 1 \\ 0 & a & 2 \end{smallmatrix})$. The latter graph has embedding 3 unique, $st(4)$ disjoint from $(\begin{smallmatrix} c & d \\ a & b & 3 \end{smallmatrix})$ and $st(a)$ disjoint from $(\begin{smallmatrix} 0' & 2 \\ 0 & 4 & b \end{smallmatrix})$, hence inadmissible bridge $(a,4)$. The graph is B_{11} , which is not maximal.

If $(2,d)$ is an edge of G , then vertex b must connect somewhere. Edge $(b,0')$ gives θ -graph disjoint from $(\begin{smallmatrix} 2 & 0' \\ 4 & d & b \end{smallmatrix})$ hence we have $(b,0)$. Deleting this edge and embedding implies there exists a bridge out of c which prevents the embedding from extending to an embedding of G , yet any such bridge gives a nonprojective graph with $st(a)$ disjoint from $(\begin{smallmatrix} 0 & 2 \\ 3 & 4 & b \end{smallmatrix})$. Hence $(2,d)$ is not an edge of G .

If $(2,0'd)$ is an edge of G then $\overline{st(d)}$ is disjoint from $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$, so d is not cubic. There is a unique embedding even after contracting $(0'd,0')$. One of the vertices a,b , will be cubic disjoint from a k -graph.

If $(2,0c)$ is an edge vertex c is not cubic. If $(c,3)$ then G is nonprojective, but so is $\frac{G}{[0c,0]}$. Avoiding a θ -graph disjoint from $\begin{pmatrix} 1 & 0' \\ 4 & 0 & a \end{pmatrix}$ implies either $(c,4)$ or $(c,14)$. The former contains a wedge $\begin{pmatrix} c & 2 \\ 0c & 4 & b \end{pmatrix} \vee \begin{pmatrix} 1 & 0' \\ 4 & 0 & a \end{pmatrix}$ and the latter is covered in lemma 3.8.

If $(2,00')$ then we have a θ -graph disjoint from $\begin{pmatrix} 2 & 0' \\ 00' & 4 & b \end{pmatrix}$.

If $(2,0)$ then vertex 3 connecting anywhere gives a wedge $\begin{pmatrix} 1 & 2 \\ 0 & 3 & 4 \end{pmatrix} \vee \begin{pmatrix} c & d \\ 0 & a & b \end{pmatrix}$, contradicting proposition 3.5.

If $(2,0')$ we have vertices 3 and 4 cubic in a triangle. Examining $\begin{pmatrix} c & d \\ a & b & 0 \end{pmatrix} \cup \begin{pmatrix} 1 & 2 \\ 3 & 4 & 0' \end{pmatrix}$ we are forced to $(3,a)(4,b)$; with a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ 1 & c & 3 \end{pmatrix}$. Having exhausted the places 2 may connect, the lemma is proved.

□

Corollary 3.11. Let $G \in I_*^M(P)$ satisfy H3. If G contains $k_{2,3} \vee k_{2,3}$ or a splitting thereof then $G = E_{22}$.

Proof. The proofs of lemmas 3.6-3.9 completed the proof of proposition 3.5, which covers G containing a wedge $k_{2,3} \vee k_{2,3}$. The proofs of lemmas 3.6, 3.7, and 3.8 are S -independent, and lemma 3.10 is an S -independent version of lemma 3.9.

□

§3.3 The Wedge of $k_{2,3}$'s, One Containing a Cycle Disjoint From the Other

Lemma 3.12. Let G contain $H = k_{2,3} \vee k_{2,3}$ as in figure 3.13. Then there exist exactly one unlabeled and 12 labeled embeddings of H into P which may extend to an embedding of G .

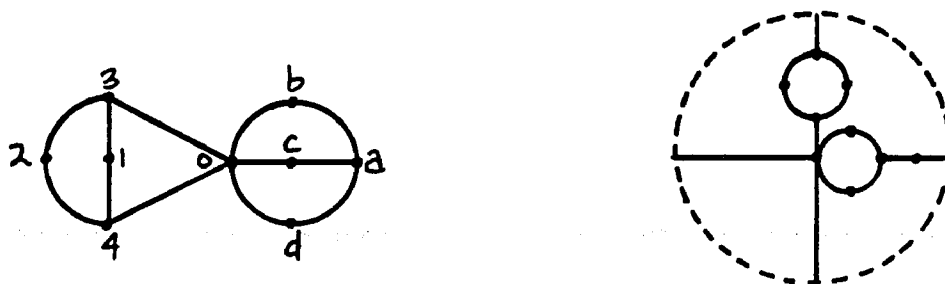


Figure 3.13

Proof. Cycle $(1,3,2,4)$ is disjoint from k -graph hence this cycle must embed null. Exactly one of the cycles $(0,b,a,c)$, $(0,b,a,d)$, $(0,c,a,d)$ must embed null. Thus the unique unlabeled embedding is shown in figure 3.13. The 12 labeled embeddings arise from the symmetries involved, and are shown in figure 3.14. Note region $(1,3,2,4)$ is dead.

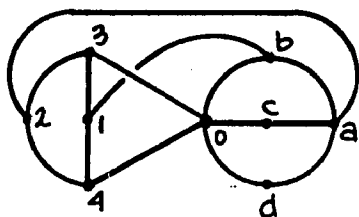
□

Proposition 3.13. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H3, G containing a wedge $k_{2,3} \vee k_{2,3}$.

Proof. Let G contain a wedge $k_{2,3} \vee k_{2,3}$ labeled as in figure 3.13. We shall break into cases depending on how the $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ k -graph completes to a Kuratowski graph.

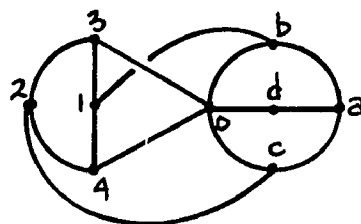
The missing vertex of the Kuratowski graph associated with $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ must lie on $\begin{pmatrix} a & 0 \\ b & c & d \end{pmatrix} \setminus \{0\}$, else G contains a θ -graph disjoint from $\begin{pmatrix} a & 0 \\ b & c & d \end{pmatrix}$. If vertices 1,2 both connect to the interior of the same arc of the $k_{2,3} \begin{pmatrix} a & 0 \\ b & c & d \end{pmatrix}$ then we can apply corollary 3.11. The remaining three possibilities are $(1,b)(2,a)$, $(1,b)(2,c)$, or $(1,a)(2,a)$. These three cases are shown in figures 3.15-3.17. The nonexistence of G containing these first two subgraphs is proven in lemmas 3.14-3.15 respectively for the first two cases. We note the symmetry $(0,a)(1,3)(2,4)$ shows any completion of $\begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$ in the third case reduces it to one of the first two cases. Hence the proof of these two lemmas shall complete the proof of the proposition.

□



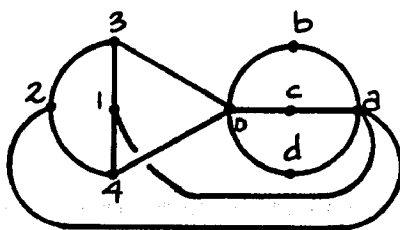
lemma 3.14

Figure 3.15



lemma 3.15

Figure 3.16



lemma 3.16

Figure 3.17

Lemma 3.14. There does not exist a $G \in I_*^M(P)$ satisfying H_3 , G containing H homeomorphic to the graph of figure 3.15, where $\begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$ is a $k_{2,3}$.

Proof. By way of contradiction we shall suppose such a G and break into cases depending on how c, d connect to complete $\begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$ to a $K_{3,3}$. The reader is referred to figure 3.18.

Case 1. $(c,13)$. We examine where d connects to complete the k -graph. If $(d,1), (d,13)$ then G is a splitting of E_4 . If $(d,23)$ then the wedge $\begin{pmatrix} 0 & 23 \\ 4 & 3 & d \end{pmatrix} \vee \begin{pmatrix} b & c \\ a & 1 & 0 \end{pmatrix}$ contradicts corollary 3.11.

If $(d,2)$ then observe $(a,2)$ is an edge, since any edge out of a_2 creates a θ -graph disjoint from $\begin{pmatrix} 0 & 2 \\ 3 & 4 & d \end{pmatrix}$ or a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$. Vertex d is cubic in a 3-cycle, so d connects somewhere else. By the symmetry $(c\ 3)(b\ 4)(2\ a)$ (the symmetry is described by the permutation on $V(G)$) we may assume $(d,3)$.

Examining the 12 embeddings we see embedding 8 in figure 3.14 is the unique embedding of our subgraph. $\overline{St(1)}, \overline{St(13)}$ are both disjoint from k -graphs, hence they are not cubic. The skew bridges in region $(c,13,1,b,a)$ give us a nonprojective graph with d cubic in a 3-cycle, a contradiction.

Case 2. $(c,1)$. Again we examine where vertex d connects. As in case 1 we may again force $(d,3)$.

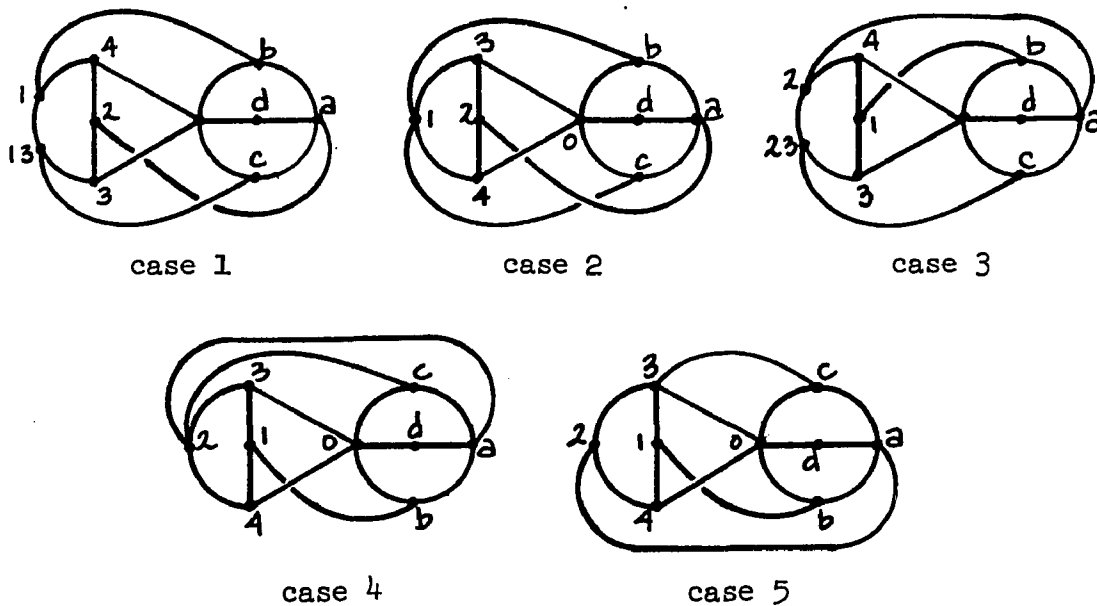


Figure 3.18

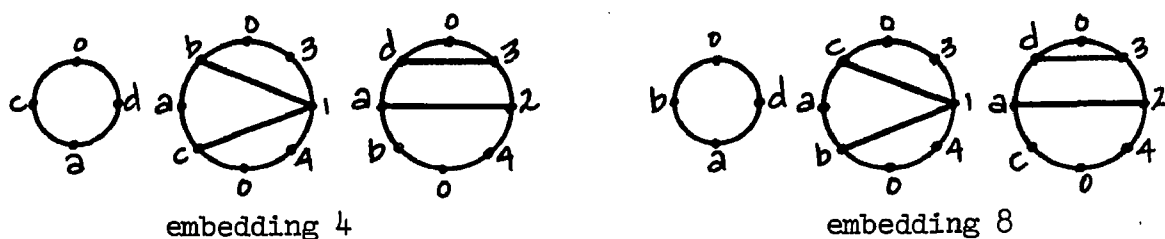


Figure 3.19

Vertex d is cubic in a 3-cycle, we shall examine where else d may connect. If d connects to $st(0)$ or $st(3)$ then we have a θ -graph disjoint from $(\begin{smallmatrix} a & 1 \\ b & 2 & c \end{smallmatrix})$. If d connects to $st(a), st(2), st(3)$ then we have a θ -graph disjoint from $(\begin{smallmatrix} 0 & 1 \\ b & c & 4 \end{smallmatrix})$. Edge $(d, 4)$ gives a wedge $(\begin{smallmatrix} 3 & 4 \\ 1 & 2 & d \end{smallmatrix}) \vee (\begin{smallmatrix} 0 & a \\ b & c & d \end{smallmatrix})$ and we apply corollary 3.11. Thus if d connects anywhere we must have (d, b) , with embedding 8 unique. We examine why this embedding does not extend to an embedding of the whole graph. If there existed an inadmissible bridge, it would have to be embedding 4 admissible else edge (b, d) is reducible. The only candidates for such a bridge are $(c, 14)$ and $(b, 3)$; the first graph being case 1 of this lemma and the latter graph containing a splitting of D_3 . Since any set of equivalent 3-bridges creates a θ -graph disjoint from k -graph, embedding 8 must not extend because of a pair of skew bridges. The only live regions are $(a, 2, 4, 0, c), (c, 1, 3, 0)$, and $(b, 1, 4, 0)$; the rest are disjoint from a k -graph. Note the skew edges must be embedding 4 admissible.

In region $(a,2,4,0,c)$ the bridges must involve vertex c , or they are not embedding 4 admissible. Edge $(c,2)$ gives a θ -graph disjoint from $\begin{pmatrix} 1 & 2 \\ 3 & 4 & c \end{pmatrix}$, hence we have $(c,4)$. Edge $(a,0)$ gives a θ -graph disjoint from $\begin{pmatrix} d & b \\ a & 0 \end{pmatrix}$ and edge $(2,0)$ contains a splitting of B_1 . In the remaining regions $(0,14)$ is equivalent to $(0,24)$, and $(0,13)$ is equivalent to $(d,23)$ instead of $(d,3)$. Thus the only choice is $(1,0), (b,4)$ and $(c,3)$, which has (b,d) reducible. We conclude vertex d is cubic, and $st(d)$ does not connect anywhere.

By d being cubic we know there exists a vertex 03 . Vertex 03 adjacent to either $a, d, 3, 1, 2$ all give a θ -graph disjoint from a k -graph. Hence $(03, b)$, giving a wedge $\begin{pmatrix} 3 & a \\ 2 & d & b \end{pmatrix} \vee \begin{pmatrix} 0 & 1 \\ b & 4 & c \end{pmatrix}$ and contradicting corollary 3.11.

Case 3. $(c,23)$. Again we examine where vertex d connects.

Cases 1 and 2 rule out $st(1)$, and $(d,2)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & 4 & b \end{pmatrix}$. Edge $(d,3)$ gives a wedge $\begin{pmatrix} 0 & 1 \\ 3 & b & 4 \end{pmatrix} \vee \begin{pmatrix} a & 23 \\ c & 2 & 3 \end{pmatrix}$, contradicting corollary 3.11. Hence by S -independent arguments we have $(d,4)$, with $\overline{st(23)}$ disjoint from $\begin{pmatrix} b & d \\ 0 & a & 4 \end{pmatrix}$. If 23 connects to a, c , or d then we have a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & b & 4 \end{pmatrix}$; $(23, b)$ gives a wedge $\begin{pmatrix} 23 & 1 \\ 3 & 4 & b \end{pmatrix} \vee \begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$; and $(23, 0)$ gives a θ -graph disjoint from $\begin{pmatrix} b & 4 \\ 1 & 2 & d \end{pmatrix}$. These cover the possibilities, hence $(c,23)$ is not in G .

Case 4. $(c,2)$. Edge $(d,2)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & 4 & b \end{pmatrix}$, so by earlier cases of this lemma we must have $(d,3)$ or

$(d,03)$. We shall assume $(d,3)$ and use S_V -independent arguments.

Note there are 4 embeddings of this graph, shown in figure 3.20.

We examine vertex c cubic in cycle $(c,a,2)$.

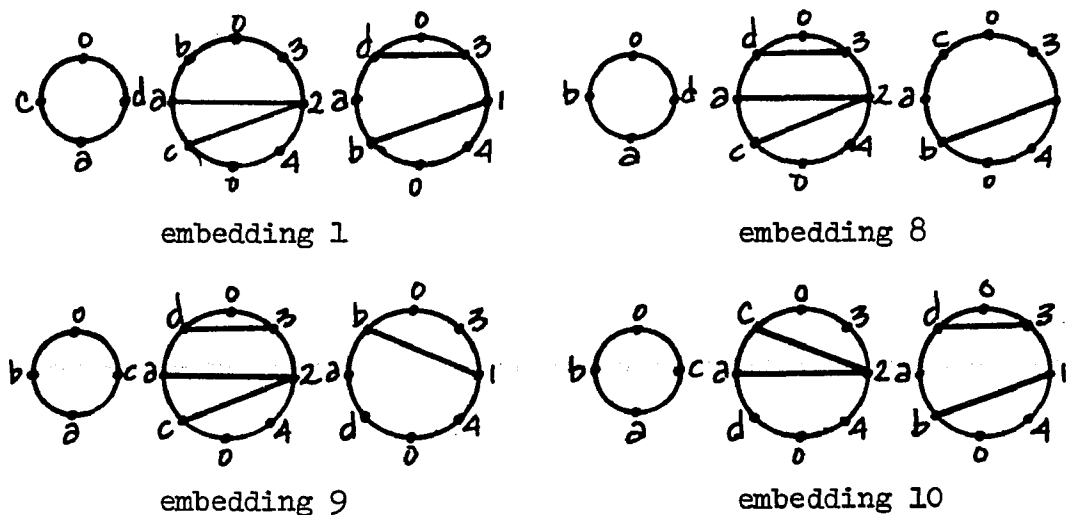


Figure 3.20

If there were a vertex a_2 any edge out of a_2 creates a θ -graph disjoint from k -graph except $(a_2,b), (a_2,0)$. The first graph contains a wedge $\begin{pmatrix} 3 & 4 \\ b & 1 & 2 \end{pmatrix} \vee \begin{pmatrix} a & 0 \\ b & c & d \end{pmatrix}$ contradicting corollary 3.11, the second graph is an earlier case of the lemma. Thus $(a,2)$ is an edge, not just an arc, of G . Deleting this edge and embedding we get either $(c,3)$ or $(c,4)$.

If $(c,3)$ is a bridge then we have either embedding 8 or 10. Deleting $(c,2)$ gives the same embeddings, hence we have either $(a,0)$ or $(a,4)$. Regardless the graph is nonprojective with vertex d cubic in a 3-cycle.

If $(c,4)$ is a bridge we have either embedding 1, 8, or 9, with vertex d cubic in a 3-cycle. Edge $(d,4)$ makes vertex 3 symmetric to vertex 4, hence we can apply the preceding paragraph. Edge (d,b) gives a θ -graph disjoint from $\begin{pmatrix} d & b \\ 0 & 3 \end{pmatrix}$, (d,c) gives a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & 4 & b \end{pmatrix}$, $(d,3)$ gives a θ -graph disjoint from $\begin{pmatrix} a & 4 \\ 1 & 2 & c \end{pmatrix}$. By S-independent arguments $st(d)$ is dead. Hence there exists a vertex 03. If 03 connects to either c or d then the preceding arguments apply, also note $(03,a)$ gives a wedge $\begin{pmatrix} 2 & 1 \\ 4 & 3 & a \end{pmatrix} \vee \begin{pmatrix} a & 0 \\ d & c & 03 \end{pmatrix}$, case 2. Thus we have $(03,b)$ and unique embeddings 8 and 9. Deleting $(a,2)$ implies a bridge $(c,3)$, a contradiction.

Case 5. Vertices c,d can only connect to 3,4,03,04.

If (c,d) both connect to $[3,0)$ there are three possibilities. Edges $(c,03),(d,03)$ give a θ -graph disjoint from $\begin{pmatrix} 1 & 2 \\ 3 & 4 & b \end{pmatrix}$. Edges $(c,3),(d,03)$ give two embeddings, 8 and 10; unique even under contracting $(3,03)$. This implies $(3,0d)$ giving a wedge $\begin{pmatrix} 03 & 0d \\ 0 & 3 & d \end{pmatrix} \vee \begin{pmatrix} 1 & 2 \\ 4 & 3 & b \end{pmatrix}$ contradicting corollary 3.11. Hence we have $(c,3)$ and $(d,3)$. If there is a vertex 03 then we have a wedge $\begin{pmatrix} 4 & b \\ 1 & a & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 3 \\ c & 03 & d \end{pmatrix}$ is case 2 of this lemma. Edge $(c,4)$ gives a nonprojective graph with d cubic in a 3-cycle. Hence we have (c,d) which gives nonmaximal graph D_6 .

We have $(c,3)$ and $(d,4)$, with both vertices c,d cubic in a 3-cycle. If $(c,03)$ then G contains a θ -graph disjoint from $\begin{pmatrix} a & 4 \\ 2 & b & d \end{pmatrix}$. The only place c and d can connect is to each other.

If this is not the case, then c, d are dead and there exist vertices $03, 04$. If $03, 04$ connect to the same place we have either case 2 of this lemma or corollary 3.11. Hence $(03, ab)(04, a)$ or $(03, ab)(04, b)$. The former graph contains a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ 04 & c & d \end{pmatrix}$ and the latter a θ -graph disjoint from $\begin{pmatrix} 3 & 4 \\ 1 & 2 & a \end{pmatrix}$.

Thus $(c, d) \in E(G)$ and vertices c, d are dead. All remaining bridges may connect only to a or $st(b)$. Note $(a, 03)$ gives a θ -graph disjoint from $\begin{pmatrix} a & 3 \\ 03 & 2 & b \end{pmatrix}$ and $(a, 1)$ is case 2 by examining a wedge $\begin{pmatrix} 1 & 2 \\ 3 & 4 & a \end{pmatrix} \vee \begin{pmatrix} a & 0 \\ b & c & d \end{pmatrix}$. Also observe $(a, 3)$ forces a vertex $a3$ or else c is cubic on a 3-cycle, yet $a3$ can connect nowhere. We conclude all bridges have a v.o.a. in $st(b)$. Edge $(b, 3)$ gives a nonprojective graph with vertex 1 cubic in a 3-cycle. Any splitting of this edge creates θ -graph disjoint from k -graph. Edge $(b, 1)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ c & d & 2 \end{pmatrix}$, hence $st(b)$ is dead. The only bridge addition which does not create a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$ is $(1, 0)$, which gives a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & b \end{pmatrix}$. This completes case 5, which completes the proof of the lemma.

□

Lemma 3.15. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H3, G containing H homeomorphic to the graph of figure 3.16 where $\begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$ is a $k_{2,3}$.

Proof. By way of contradiction we shall suppose such a G and break into cases depending on where d connects to complete the $k_{2,3}$ $\begin{pmatrix} 0 & a \\ b & c & d \end{pmatrix}$ to a Kuratowski graph.

Case 1. $(d,1)$. We shall use S_d -independent arguments to also cover the case $(d,13)$. Note the symmetry $(3\ b)(4\ d)(2\ a)$. Vertex a is cubic disjoint from $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$, by symmetry 2 also cannot be cubic. Edges $(2,a), (2,1)$ contradict lemma 3.14, edges $(2,d)(2,b)$ contradict corollary 3.11. By S -independent arguments we have $(2,0)$. This graph has 4 embeddings, generated from embedding 1 of figure 3.21 by the symmetries $b \sim d, 3 \sim 4$. Thus upon removing $(2,0)$ we may assume the graph embeds as an extension of embedding 1. We get either 1) $(c,4), (c,3)$ with a θ -graph disjoint from $\begin{pmatrix} d & b \\ 1 & 0 & a \end{pmatrix}$, 2) $(c,4), (3,a)$ with a θ -graph disjoint from $\begin{pmatrix} 1 & a \\ 3 & d & b \end{pmatrix}$, or 3) $(c,4), (b,3)$ with a θ -graph disjoint from $\begin{pmatrix} 2 & 0 \\ 4 & c \end{pmatrix}$.

Case 2. $(d,3)$. $\overline{st(a)}$ is disjoint from $\begin{pmatrix} 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$, so a is not cubic. We examine possible places where a may connect.

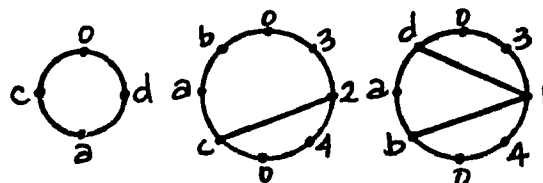
If $(a,03)$ then $\overline{st(2)}$ is disjoint from a k -graph. Since cycle $(0,c,2,4)$ is disjoint from $\begin{pmatrix} 3 & a \\ d & b & 03 \end{pmatrix}$ we must have $(2,b)$, with the wedge $\begin{pmatrix} 4 & b \\ 0 & 2 & 1 \end{pmatrix} \vee \begin{pmatrix} 03 & d \\ 0 & 3 & a \end{pmatrix}$ contradicting corollary 3.11. If $(a,04)$ then embeddings 9,10 are unique even under contracting $(04,4)$. This implies either $(4,d)$ or $(04,c), (4,a)$. The former graph contradicts corollary 3.11 and the latter contains $st(1)$ disjoint from $\begin{pmatrix} 0 & a \\ 04 & c & d \end{pmatrix}$.

If $(a,3)$ is a bridge then that bridge is reducible unless vertex d is not cubic. By the previous case and corollary 3.11 we have (d,b) and $st(d)$ is dead. Consider $\frac{G}{[2,c]} \cup \{(3,4), (a,0)\}$. Since G is maximal this graph embeds and we know cycles $(\{2,c\}, 3, 4)$, $(\{2,c\}, a, 0)$ are essential and $(0,b,d), (a,b,d), (b,d,3,1)$ are null. This implies k -graph $(\begin{smallmatrix} b & d \\ 0 & a & 3 \end{smallmatrix})$ is null, hence $[2,c]$ cannot be dead. If vertex 2 connects anywhere but (b,d,a) then there is a θ -graph disjoint from either $(\begin{smallmatrix} 0 & a \\ b & c & d \end{smallmatrix})$ or $(\begin{smallmatrix} 3 & b \\ 1 & d & a \end{smallmatrix})$. Edge $(2,a)$ gives case 1 of this lemma, any splitting of this is lemma 3.14. Edge $(2,b)$ is a contradiction of corollary 3.11, hence 2 and, by S_v -independent arguments $st(2)$, are dead; we conclude vertex c is not cubic. Edge $(c,3)$ creates vertex 2 cubic in a 3-cycle, yet any other connection of c gives a θ -graph disjoint from either $(\begin{smallmatrix} 3 & 4 \\ 1 & 2 & 0 \end{smallmatrix})$ or $(\begin{smallmatrix} 3 & b \\ 1 & d & a \end{smallmatrix})$. Hence c is dead and we conclude $(a,3)$ is not in G .

If $(a,4)$ is a bridge we examine where vertex d can connect. Edges $(d,4), (d,c), (d,03), (3,0d)$ all give a contradiction of corollary 3.11, hence d is cubic. Avoiding a 3-cycle implies 03 , without loss of generality $(03,b)$ and a contradiction as G is nonprojective with 03 cubic in a 3-cycle.

Having exhausted the possible connections of a we conclude case 2 does not hold.

□



embedding 1

Figure 3.21

§3.4 A $k_{2,3}$ Wedge a k_4 , Each Containing a Cycle Disjoint from the Other

Lemma 3.16. Let G contain $H = k_{2,3} \vee k_4$ as shown in figure 3.22. Then there is exactly one unlabeled and 12 labeled embeddings of H which may extend to an embedding of G .

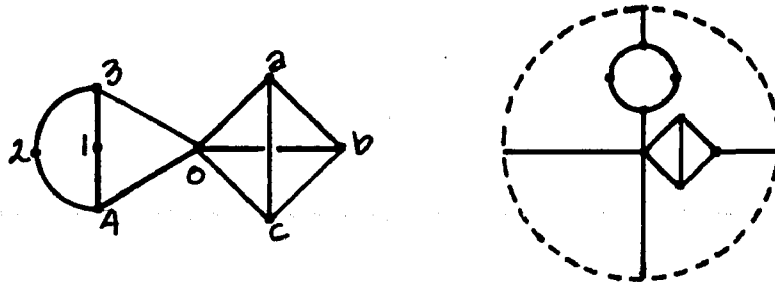


Figure 3.22

Proof. Cycle $(1,3,2,4)$ is disjoint from a k -graph hence it must embed null. One of the cycles $(0,a,b), (0,a,c), (0,b,c)$ must be null. The unique unlabeled embedding is shown in figure 3.22. For a particular choice, say $(0,a,b)$, embedding null, there are 4 labeled embeddings based on the symmetries $3 \sim 4, 1 \sim 2$. The 12 resulting labeled embeddings are shown in figure 3.23.

□

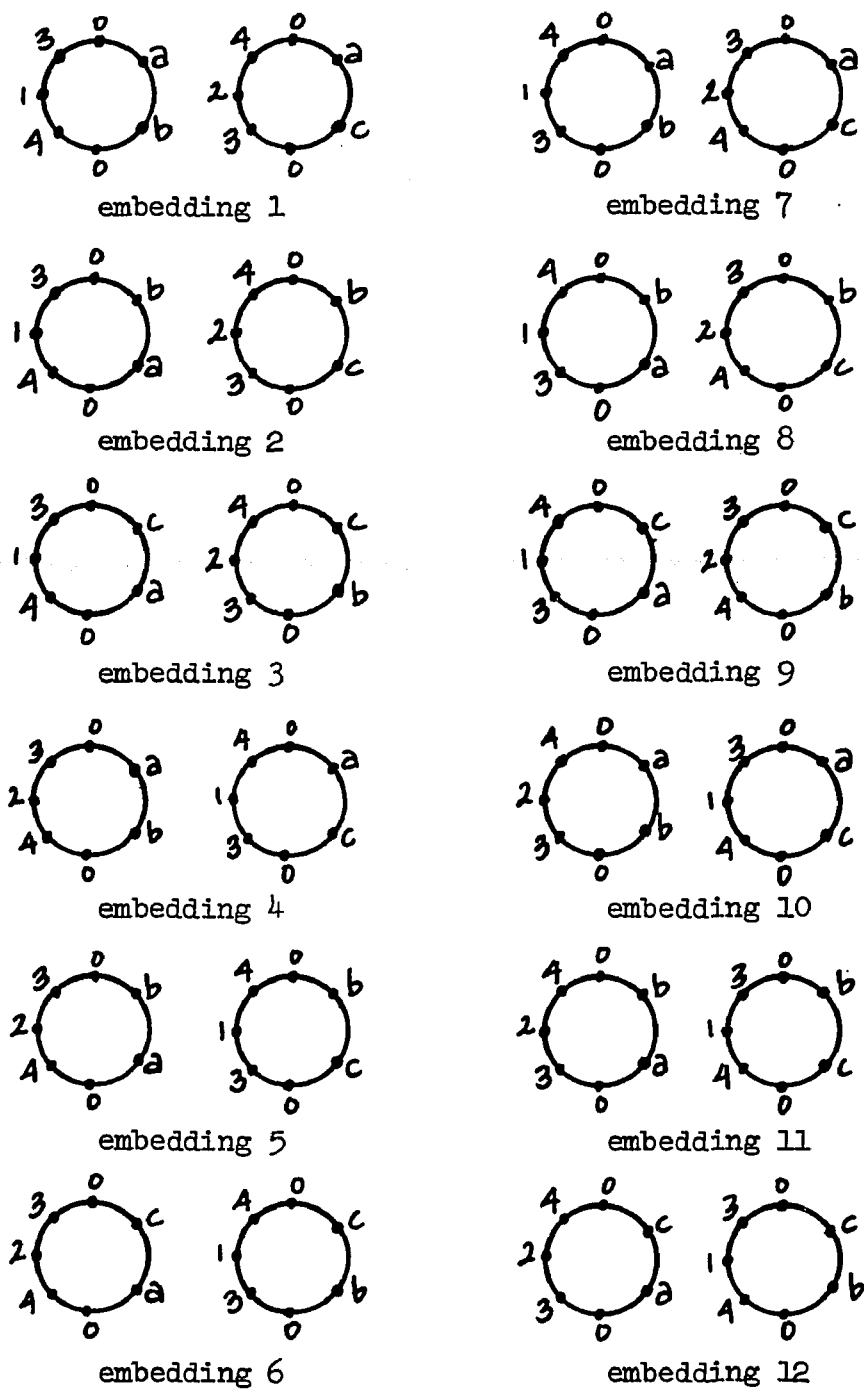


Figure 3.23

Proposition 3.17. There does not exist a $G \in \mathcal{L}_*^M(P)$ satisfying H_3 and containing $k_{2,3} \vee k_4$.

Proof. We label $k_{2,3} \vee k_4$ as in figure 3.22. By propositions 3.5 and 3.13 we know G does not contain a wedge of $k_{2,3}$'s, hence $(0,a), (0,b), (0,c), (a,b), (a,c), (b,c)$ are all edges, not arcs, of G . Moreover if there existed a vertex v disjoint from this wedge then the bridge containing v must connect twice to one of the k -graphs. We cannot avoid either a wedge $k_{2,3} \vee k_{2,3}$ or a θ -graph disjoint from a k -graph. The proof of the proposition will be broken into cases depending on how the $k_{2,3}$ completes to a $K_{3,3}$. The two choices are shown in figure 3.24 and 3.25 and the proof that there does not exist a graph G containing this subgraph follows in lemma 3.18, 3.19, respectively.

□

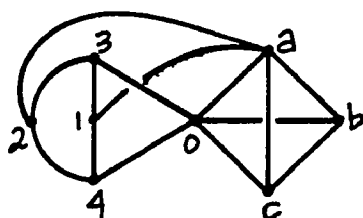


Figure 3.24

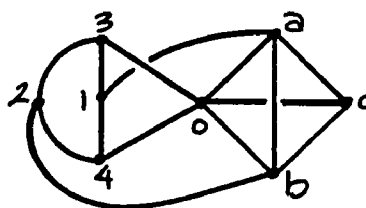


Figure 3.25

Lemma 3.18. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H_3 and containing a subgraph homeomorphic to that of figure 3.24 where $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ is a k -graph.

Proof. By way of contradiction suppose G were such a graph. Note the only embeddings which admit both $(1,a)$ and $(2,a)$ are embeddings 1,4,7,10 of figure 3.23. Also observe the symmetry $(0\ a)(1\ 3)(2\ 4)$. From this we see vertices 1,2,3,4 are all vertex transitive. Also $(3,0)(4,0)$ must be edges of G or else there exists a vertex disjoint from this wedge.

If b, c connect to the same vertex then G is D_3 , which is not maximal. Next suppose $(b,13)$ is in G , note embedding 7 is unique. Without loss of generality we have either $(c,2)$ or $(c,24)$. $\overline{St(1)}$ and $\overline{st(13)}$ are each disjoint from k -graphs, hence we have $(1,b), (13,a)$. Regardless of where c connects we have $\overline{st(3)}$ is disjoint from $\begin{pmatrix} 1 & c \\ a & b \end{pmatrix}_4$, a contradiction. Thus b and c can connect only to 1,2,3 or 4.

Suppose $(b,1), (c,3)$ are in G . If there exists a ninth vertex then it must be either 13,14,23,24. By the preceeding paragraph such a vertex may only connect to a . Supposing $(14,a)$ gives 14 cubic in a 3-cycle. Supposing $(24,a), (23,a)$ gives $(2,0)$ or $(2,c)$ respectively and vertex 24,23 respectively cubic in a 3-cycle. Finally supposing $(13,a)$ and avoiding a cubic vertex in a 3-cycle implies $v \in (1,13)$ and $(v,0)$, which case was considered in the preceeding paragraph. Hence there does not exist a ninth vertex.

Examining the unique embedding for skew bridges gives $(2,0), (4,a)$ or $(3,a), (1,0), (2,c), (4,b)$. The first graph is B_5 , while the second has edge $(c,3)$ reducible. Hence G does not contain edges of the type $(b,1)$ with $(c,3)$.

Next suppose $(b,3), (c,4)$ are in G . Note embeddings 7,10 are unique. Again if there were a ninth vertex without loss of generality it is 13. Having previously eliminated $(13,b)$ or $(13,c)$ we have either $(13,0)$ or $(13,a)$. If $(13,0)$ then avoiding vertex 13 cubic in a 3-cycle implies either $(a,03)$ or $(13,a), (1,0)$. The former graph contains $\overline{st(3)}$ disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ and the latter contains a wedge $\begin{pmatrix} b & 2 \\ c & 3 \end{pmatrix} \vee \begin{pmatrix} a & 1 \\ 13 & 0 \end{pmatrix}$ as covered by the preceeding paragraph. If $(13,a)$ then 13 is still cubic in a 3-cycle. $St(13)$ as before can connect nowhere except a , hence we get a θ -graph disjoint from $\begin{pmatrix} 3 & 4 \\ 2 & 0 \end{pmatrix}$. We conclude there are exactly 8 vertices in G . We have already checked the cases where any bridges are incident with vertices b or c . The only possible edge additions are $(a,3), (a,4)$. Adding in both still gives a projective graph.

□

Lemma 3.19. There does not exist a $G \in \Gamma_*^M(P)$ satisfying $H3$ and containing a subgraph homeomorphic to that of figure 3.25 where $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ is a k -graph.

Proof. By way of contradiction we shall suppose such a G and break into cases depending on where c connects in the completion of $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ to a Kuratowski graph. Note the subgraph of figure 3.25 embeds as an extension of embeddings 2,3,4,8,9,10 of figure 3.23.

Case 1. $(c,03)$. Note $\overline{st(3)}$ is disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$; 03 is cubic in a 3-cycle, cycle $(03,0,c)$ is disjoint from $\begin{pmatrix} 1 & 2 \\ 3 & 4 & a \end{pmatrix}$ hence it embeds null, and embeddings 8 and 10 do not admit $(c,03)$. Without loss of generality vertex 03 connects to a , and vertex 3 connects to either a,b or c . If $(3,a)$ then observe in each embedding cycle $(03,3,1,a)$ is null, hence $(03,1)$ and we have a θ -graph disjoint from k -graph. Edge $(3,b)$ gives a θ -graph disjoint from $\begin{pmatrix} 03 & a \\ 0 & c \end{pmatrix}$. Edge $(3,c)$ always embeds in region $(c,0,03,3)$, so by lemma 2.15 there exists a k -graph on this cycle. This gives either a wedge of $k_{2,3}$'s or a wedge $\begin{pmatrix} 4 & b \\ 0 & 1 & 2 \end{pmatrix} \vee \begin{pmatrix} 3 & 0 \\ 03 & c \end{pmatrix}$.

Case 2. $(c,1)$. Note this union of Kuratowski graphs has embeddings 3,4,9 and 10 of figure 3.23 as its only embeddings. We shall first eliminate the existence of a ninth vertex.

If 13 is a vertex then note $(13,a)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & 2 \\ b & 3 & 4 \end{pmatrix}$, and $(13,b)$ was eliminated in lemma 3.18. Hence $(13,0)$, and vertices $13,3$ are cubic in a 3-cycle. There does not exist a vertex in $(13,0)$ or $(3,0)$ because this would create a vertex disjoint from subgraph of figure 3.22, by elimination we cannot avoid 13 cubic in a 3-cycle.

If 03 is a vertex note $(03,a)$ is case 1, hence $(03,b)$.

Again 03 is cubic in a 3-cycle, the opposite edge $(0,b)$ is indeed an edge, not an arc, of G , and $st(03)$ can connect only to b .

A cubic vertex in a 3-cycle is unavoidable.

If 23 is a vertex note $(23,a)$ was eliminated in lemma 3.18.

Edge $(23,0)$ is the same graph as we get from adding $(13,0)$, which was previously covered. Hence $(23,b)$, with vertices $2,23$ cubic in a 3-cycle. By elimination $[2,23]$ can only connect to b , and a cubic vertex in a 3-cycle is unavoidable.

We conclude $|v(G)| = 8$, and examine possible graphs. If $(2,0)$ is an edge then delete it and embed. The 4 embeddings are generated by $3 \sim 4$, $a \sim c$ so without loss of generality, suppose $G \setminus (2,c)$ embeds in P as an extension of embedding 3. We get $(3,b)$ and either $(4,b)$ or $(4,c)$. The first graph is B_4 and the second contains a θ -graph disjoint from $\begin{pmatrix} 2 & 0 \\ 3 & b \end{pmatrix}$. Hence $(2,0)$ is not an edge, and avoiding a contradiction from lemma 3.18 implies vertex 2 is dead. Vertex 1 can only connect to 0, yet this makes $\overline{st(2)}$ disjoint from $\begin{pmatrix} 0 & 1 \\ a & c \end{pmatrix}$. Thus the only possible edges are $(3,a), (3,c), (4,a)$, and $(4,c)$ since $(3,b)$ and $(4,b)$ make vertex 2 cubic in a 3-cycle. These edges are symmetric, as are the 4 embeddings, so it is easy to check adding only 3 gives a projective graph and adding in all 4 a nonprojective graph. This graph without edge $(0,3)$ is B_2 .

Case 3. $(c,13)$. Note embeddings 3 and 4 of figure 3.23 are the only possibilities. As in lemma 2.18 $\frac{G}{[1,13]} \cup \{(c,3), (4,a)\}$ should embed with cycles $(\{1,13\}, c, 3)$ and $(\{1,13\}, 4, a)$ essential. Cycle (a, b, c) is disjoint from $(\begin{smallmatrix} 3 & 4 \\ 0 & 1 & 2 \end{smallmatrix})$, cycle $(0, a, b)$ is disjoint from cycle $(\{1,13\}, c, 3)$, and cycle $(0, b, c)$ is disjoint from cycle $(\{1,13\}, 4, a)$ gives a null k -graph $(\begin{smallmatrix} 0 & a \\ b & c \end{smallmatrix})$. Hence $[1,13]$ is not dead. Cycle $(1, a, c, 13)$ is disjoint from $(\begin{smallmatrix} 2 & 0 \\ 3 & 4 & b \end{smallmatrix})$ hence 1 cannot connect to a or c . Edge $(1, b)$ was lemma 3.18. By symmetry and S_v -independent arguments we have $(v, 0)$ for v either 1 or a new vertex in $(1, 13)$. The former graph has edge $(4, a)$ by deleting $(v, 0)$ and examining embeddings, and $\overline{st(2)}$ is now disjoint from $(\begin{smallmatrix} 1 & c \\ 13 & 0 & a \end{smallmatrix})$. The latter graph is D_2 .

Case 4. $(c, 3)$. Note the only embeddings are 2, 3, 4 and 9 of figure 2.23. We shall first eliminate the existence of a ninth vertex.

If 13 is a vertex, note $(13, b)$ contradicts lemma 3.18, $(13, c)$ is case 3, and $(13, 0)$ is the same graph as adding $(c, 23)$, an earlier case. Hence $(13, a)$, and vertices 1 and 13 are cubic in a 3-cycle. Having eliminated any other connection of 13, we cannot avoid a cubic vertex in a 3-cycle.

If 14 is a vertex then we have $(14, 0)$. Since either region $(1, 4, 0, a)$ or $(1, 4, 0, c, a)$ is null by deleting $(14, 0)$ and embedding we get either edge $(4, a)$ or $(4, c)$. The former graph is nonprojective with 14 cubic in a 3-cycle, and the latter graph is case 3 by considering the wedge $(\begin{smallmatrix} 14 & 3 \\ 0 & 1 & 4 \end{smallmatrix}) \vee (\begin{smallmatrix} 0 & a \\ b & c \end{smallmatrix})$.

If 03 is a vertex then we have either $(03,a)$ or $(03,b)$, only one gives cubic vertex in a 3-cycle, so we must have both. $\overline{St(4)}$ is disjoint from $\begin{pmatrix} 03 & c \\ a & b & 3 \end{pmatrix}$. Edge $(4,a)$ gives a θ -graph disjoint from $\begin{pmatrix} 3 & b \\ c & 2 & 03 \end{pmatrix}$, edge $(4,c)$ gives a wedge $\begin{pmatrix} 3 & b \\ c & 2 & 03 \end{pmatrix} \vee \begin{pmatrix} 4 & a \\ c & 1 & 0 \end{pmatrix}$, and edge $(4,b)$ gives θ -graph disjoint from $\begin{pmatrix} a & 3 \\ 1 & 03 & c \end{pmatrix}$, eliminating possibilities and giving a contradiction.

If 04 is a vertex note $(04,c)$ was a previous case. Avoiding a cubic vertex in a 3-cycle implies $(04,a)$ and $(04,b)$. $\overline{St(4)}$ is disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$, so we have edge $(4,c)$ or $(4,a)$. The former graph contains a θ -graph disjoint from $\begin{pmatrix} 3 & 4 \\ 1 & 2 & c \end{pmatrix}$ and the latter θ -graph disjoint from $\begin{pmatrix} 3 & b \\ 0 & 2 & c \end{pmatrix}$.

We conclude G does not contain a ninth vertex, and examine possible graphs on 8 vertices. By elimination the possible edge additions are $(1,0), (2,0), (3,a), (3,b), (4,a), (4,b), (4,c)$. If $(4,a)$ is an edge then avoiding cubic in a 3-cycle implies $(1,0)$. We have only embeddings 2 and 9, and symmetry $(4,b)(1,c)$. Examining the embeddings gives either $(2,0), (4,b)$ or $(3,a)$. The former graph is case 2 upon examining the wedge $\begin{pmatrix} 2 & c \\ 0 & 3 & b \end{pmatrix} \vee \begin{pmatrix} 0 & 1 \\ a & 4 \end{pmatrix}$ and the latter graph is B_2 . Hence $(4,a)$ is not an edge, by symmetry neither is $(4,b)$. If $(4,c)$ then the symmetry $(0,c)$ implies not $(1,0)$ or $(2,0)$ by case 2. The symmetry $(4,3)$ shows no possible edge additions, yet the graph is projective. Hence vertex 4 is dead, which implies vertices 1,2 are dead. The addition of the remaining edges $(3,a)(3,b)$ gives a projective graph.

□

§3.5 A $k_{2,3}$ Wedge a k_4 , The k_4 Containing a Cycle Disjoint

From the $k_{2,3}$

Lemma 3.20. Let G contain a subgraph H , $H = k_{2,3} \vee k_4$, as shown in figure 3.26. Then there is exactly one unlabeled and 36 labeled embeddings of this subgraph which may extend to an embedding of G .

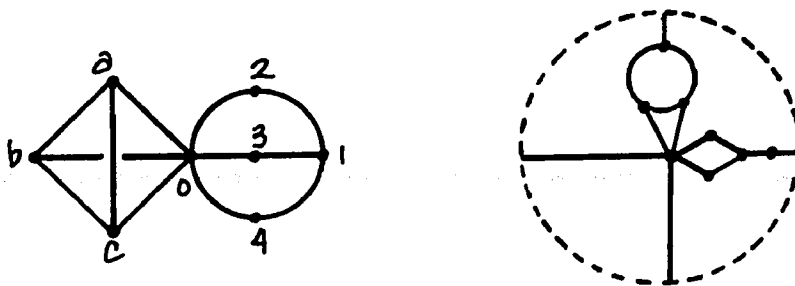


Figure 3.26

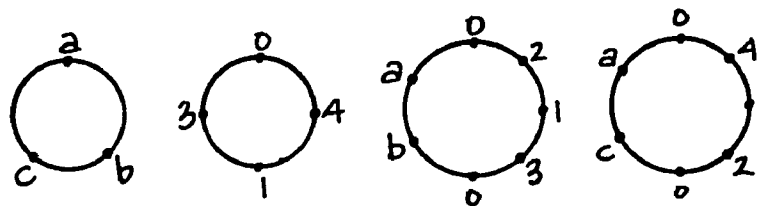


Figure 3.27

Proof. Each embedding is homeomorphic to the unique unlabeled embedding of figure 3.26. We label the vertices and depict a typical labeled embedding in figure 3.27. Any other labeled embedding arises from a symmetry, and hence may be described as the product of two permutations, one on the set $\{a, b, c\}$ and the other on the set $\{2, 3, 4\}$. There are 36 such symmetries, each giving rise to a

different labeled embedding. We shall refer to these embeddings by the permutations describing the symmetry. As examples the 12 embeddings arising from permutations on the stabilizer of a are shown in figure 3.28.

□

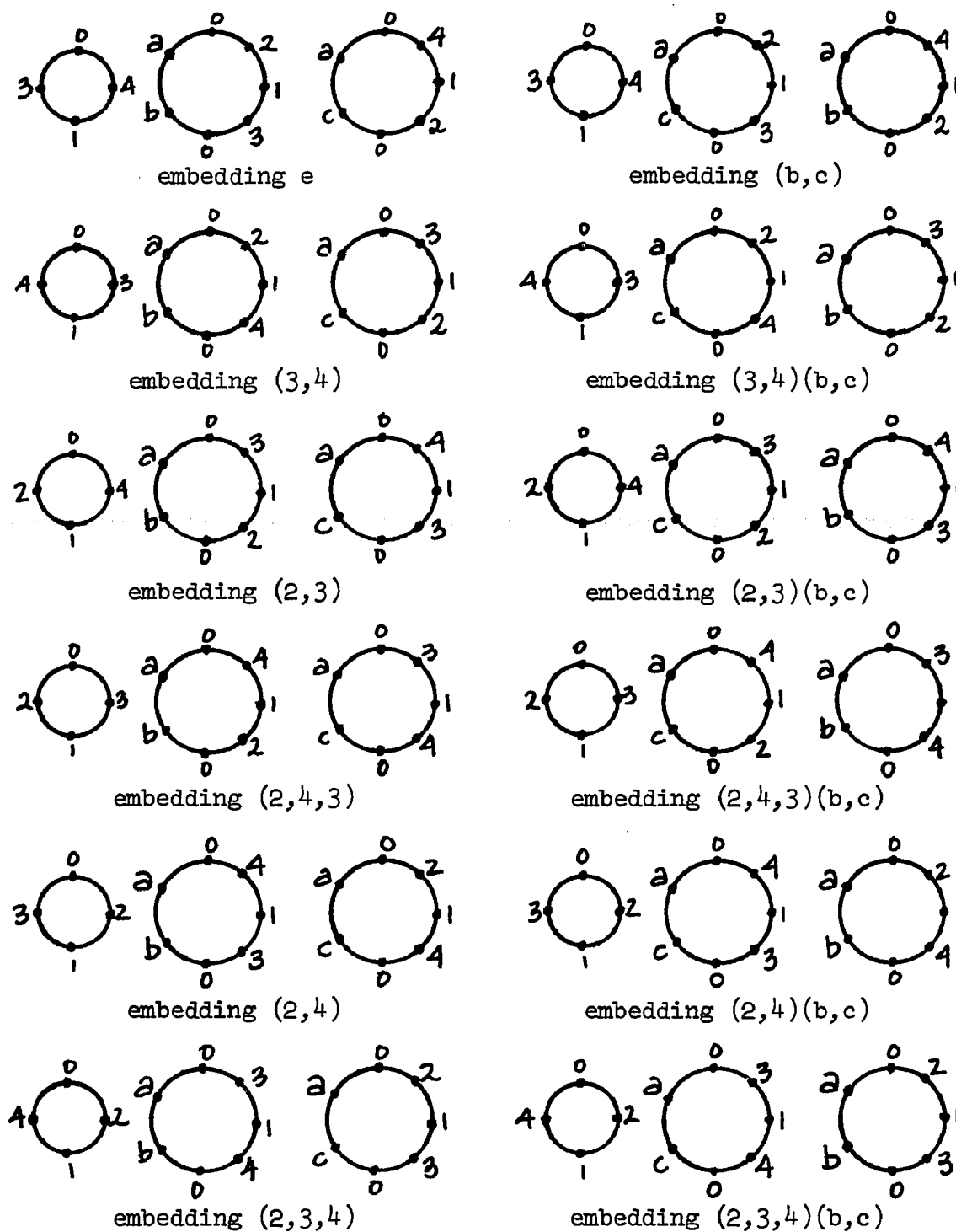


Figure 3.28

Proposition 3.21. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H_3 and containing $k_{2,3} \vee k_4$.

Proof. We break the proof into cases depending on how the $k_{2,3}$ completes to a Kuratowski graph. Label the $k_{2,3} \vee k_4$ as in figure 3.26, i.e., $\begin{pmatrix} 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \vee \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. Since G cannot contain a $k_{2,3} \vee k_{2,3}$ by proposition 3.13 vertices $2,3,4$ may connect only to $a, 0a, b, 0b, c, 0c$. We will use S_v -independent arguments hence we will assume $2,3,4$ may connect only to a, b, c . The proof of the proposition naturally falls into three cases: 1) $(2,a), (3,a), (4,a)$; 2) $(2,a), (3,a), (4,b)$; 3) $(2,a), (3,b), (4,c)$. The proofs that there does not exist a G containing these three subgraphs are lemma 3.22, 3.23, 3.24 following. The proofs of these lemmas completes the proof of the proposition.

□

Lemma 3.22. Let H denote any graph homeomorphic to the graph of figure 3.29, where $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ is a k -graph. Then there does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H_3 and containing either H or $S_a(H)$.

Proof. By way of contradiction, suppose such a G exists. We shall break into cases depending on where vertices b, c connect in the completion of $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. Case 1 is $(b,2)(c,2)$; Case 2 is $(b,1)(c,2)$; and Case 3 is $(b,2)(c,3)$.

Case 1. $(b,2)(c,2)$. Cycle $(0,3,1,4)$ is disjoint from $\begin{pmatrix} a & b \\ c & 2 \end{pmatrix}$, cycle $(a,3,1,4)$ is disjoint from $\begin{pmatrix} b & 2 \\ 0 & c \end{pmatrix}$, hence the k -graph $\begin{pmatrix} 3 & 4 \\ 0 & 1 & a \end{pmatrix}$ would be null if G embedded. There G is nonprojective with $\overline{st(1)}$ disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. Note this is an S -independent argument.

Case 2. $(b,1)(c,2)$. First observe $(c,02)$ contains a θ -graph disjoint from $\begin{pmatrix} a & 1 \\ 2 & 3 & 4 \end{pmatrix}$ and $(c,12)$ contains a wedge $\begin{pmatrix} a & 1 \\ 3 & 4 & b \end{pmatrix} \vee \begin{pmatrix} c & 2 \\ a & 0 & 12 \end{pmatrix}$ contradicting proposition 3.13. Hence we have $(c,2)$ and not a splitting thereof. Secondly observe cycle $(0,b,c)$ is disjoint from $\begin{pmatrix} 1 & a \\ 2 & 3 & 4 \end{pmatrix}$ hence we must have one of the 12 embeddings of figure 3.28. Examining these shows there are exactly two embeddings of our graph (figure 3.30) based on the symmetry $3 \sim 4$.

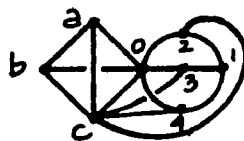


Figure 3.29

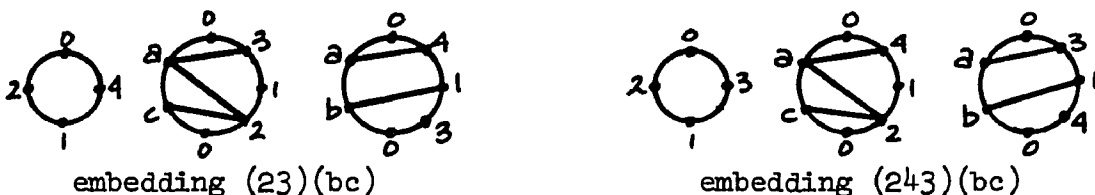


Figure 3.30

We shall now prove there does not exist a $G \in I_*^M(P)$ containing $S_a(H)$. If $(2,0a), (3,a), (4,a)$ then G is nonprojective with $0a$ cubic in a 3-cycle. If $(2,a), (3,0a), (4,0a)$ then G contains a θ -graph disjoint from $(0a \begin{smallmatrix} 3 & 4 \\ 0 & 1 \end{smallmatrix})$. If $(2,0a), (3,0a), (4,a)$ then G contains a wedge $(0a \begin{smallmatrix} c & \\ 0 & 2 \end{smallmatrix} \begin{smallmatrix} a \\ a \end{smallmatrix}) \vee (3 \begin{smallmatrix} 0 & 1 \\ 4 & b \end{smallmatrix})$, contradicting proposition 3.13. Finally if $(2,a), (3,a), (4,0a)$ we still have the two embeddings of figure 3.30, we will examine $\frac{G}{[a,0a]}$. If an inadmissible bridge becomes admissible we have either $(a,04)$ or $(2,0a)$. The former graph contains a wedge $(0 \begin{smallmatrix} 2 & b \\ 0 & c \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix}) \vee (04 \begin{smallmatrix} 0 & a \\ 0a & 3 \end{smallmatrix})$ and the latter graph was just covered. By contracting $(a,0a)$ we must have "unskewed" skew bridges. We have already eliminated $(0a,2)$, and observing $(0a,1)$ gives a θ -graph disjoint from $(0a \begin{smallmatrix} 0 & 1 \\ 3 & 4 \end{smallmatrix})$ shows the skew bridges must be on region $(a,0a,03)$ or $(0a,0,4)$. We must get a k -graph on these regions by lemma 2.15, but this k -graph wedge $(0 \begin{smallmatrix} 2 & b \\ 0 & c \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix})$ must contradict either corollary 3.11, proposition 3.13 or proposition 3.17. Hence we have exactly H and not $S_a(H)$.

Vertices 3 and 4 are cubic in a 3-cycle. If there was a vertex $0a$ then $(0a,1)$ (we have just eliminated any other choice) giving a θ -graph disjoint from $(3 \begin{smallmatrix} 0 & 1 \\ 4 & 0a \end{smallmatrix})$. If there is a vertex $a3$ then the bridge from $a3$ must embed, else vertex 4 remains cubic in a 3-cycle. Region $(a,0,3)$ is dead, because as above a bridge on this region gives k -graph wedge $(0 \begin{smallmatrix} 2 & b \\ 0 & c \end{smallmatrix} \begin{smallmatrix} 1 \\ 1 \end{smallmatrix})$. Edges $(a3,4)$ or $(a3,2)$ give a contradiction by proposition 3.17. Thus $(a3,1)$ and avoiding cubic in a triangle implies a vertex 13,

resulting in a contradiction by proposition 3.17. We see a_3 is not a vertex, by the symmetry $(2\ b)(a\ 0)$ neither is $a_4, 0_3, 0_4$.

Edge $(3, c)$ gives a nonprojective graph with 4 cubic in a 3-cycle. Edge $(3, 4)$ gives a wedge $\begin{pmatrix} 0 & 3 \\ a & 4 \end{pmatrix} \vee \begin{pmatrix} b & 2 \\ 0 & c \end{pmatrix}$, proposition 3.17. Edges $(3, 2)$ and $(4, 2)$ together give a θ -graph disjoint from k -graph, by symmetry we do not have edges $(3, b), (4, b)$ together. Hence $(3, 2), (4, b)$, deleting one of them gives either $(0, 1)$ or $(a, 1)$, by symmetry $(a, 1)$ and a θ -graph disjoint from $\begin{pmatrix} 4 & b \\ a & 1 \end{pmatrix}$.

Case 3. $(2, b)(3, c)$. Note $\overline{St(1)}$ is disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. Edge $(1, b)$ or $(1, c)$ gives case 2, edge $(1, 0_b)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 0_b & 4 \end{pmatrix}$, and edge $(1, 0_3)$ a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 4 & 2 \end{pmatrix}$. Edge $(1, 0_4)$ implies either $(4, b)$ or $(4, c)$, regardless 0_4 is cubic in a 3-cycle. Edge $(1, 0_a)$ gives either a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 0_a & 3 \end{pmatrix}$ or edges $(3, 0_a), (2, 0_a)$ and a θ -graph disjoint from $\begin{pmatrix} a & 0 \\ 4 & b \end{pmatrix}$. Finally $(0, 1)$ gives either $(4, 3)$ or $(2, 3)$. The former graph contains a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$ and in the latter graph avoiding vertex 4 cubic in a 3-cycle implies 0_1 or 4 connect someplace, any such connection gives a θ -graph disjoint from a k -graph.

□

Lemma 3.23. Let H denote the graph of figure 3.31, where $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ is a k -graph. Then there does not exist a $G \in \Gamma_*^M(P)$ satisfying H_3 and containing either H , $S_a(H)$ or $S_b(H)$.

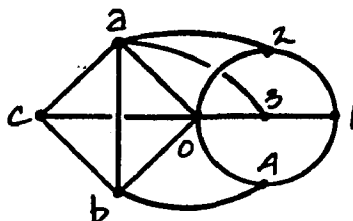


Figure 3.31

Proof. By way of contradiction, suppose G were such a graph. Note a splitting of H of the type described which does not create $k_{2,3} \vee k_{2,3}$ is the equivalent of $(4,0b)$ replacing $(4,b), (2,0a)$ replacing $(2,a)$, and/or $(3,0a)$ replacing $(3,a)$. We shall break into cases depending on where vertex c connects to complete the k -graph $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. Case 1 is $(c,2)$ or a splitting thereof, case 2 is $(c,1)$. Note $(c,4)$ gives a wedge $\begin{pmatrix} 2 & 3 \\ 1 & a & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & c \\ b & 4 \end{pmatrix}$.

Case 1. $(c,2)$. First we shall show the cycle $(0,4,b)$ must be a 3-cycle.

If there exists a vertex 04 then edge $(04,a)$ contradicts the previous lemma, edge $(04,c)$ gives a wedge $\begin{pmatrix} 2 & 3 \\ 0 & 1 & a \end{pmatrix} \vee \begin{pmatrix} 04 & b \\ c & 4 & 0 \end{pmatrix}$, edge $(04,2)$ gives a wedge $\begin{pmatrix} 04 & 1 \\ 2 & 4 & 0 \end{pmatrix} \vee \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, edge $(04,3)$ gives a wedge $\begin{pmatrix} 04 & 1 \\ 3 & 4 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$, so 04 may connect only to 1 or to b . If $(04,1)$ then $\overline{\text{st}(4)}$ is disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$, 4 cannot connect to a or c by previous cases, and 4 cannot connect to $\begin{pmatrix} 0 & 1 \\ 2 & 3 & 4 \end{pmatrix} \setminus \{0\}$ without creating a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. Hence G contains edge $(4,0)$, and avoiding 04 cubic in a 3-cycle

implies $(04, b)$ giving a wedge $\begin{pmatrix} 4 & 0 \\ b & 04 \end{pmatrix} \vee \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} a$, contradicting proposition 3.17. Thus 04 may only be adjacent to b . Avoiding 04 cubic in a 3-cycle implies $(4, 0b)$, a contradiction since G contains a wedge $\begin{pmatrix} 0 & 2 \\ a & c \end{pmatrix} \vee \begin{pmatrix} 04 & 0b \\ 0 & b \end{pmatrix} 4$. Thus 04 is not a vertex.

If there exists a vertex $0b$ which connects anywhere but 4 we get a wedge $\begin{pmatrix} 0 & b \\ c & 4 \end{pmatrix} 0b \vee \begin{pmatrix} 2 & 3 \\ 0 & 1 \end{pmatrix} a$. Edge $(0b, 4)$ gives $0b$ cubic in a 3-cycle. Hence there is not a vertex $0b$.

If there exists a vertex $b4$ then we have both $(b4, 0)$ and $(b4, 1)$ or else $b4$ is cubic in a 3-cycle. Avoiding 4 cubic in a 3-cycle implies $(4, 3)$ or $(4, 2)$. Regardless G contains a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$.

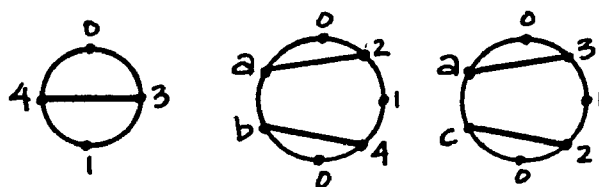
Thus cycle $(0, b, 4)$ is indeed a 3-cycle of G . Vertex 4 must connect elsewhere, the only candidates are $(4, 2), (4, 12), (4, 3)$ or $(4, 13)$. By S_v -independent arguments we will consider $(4, 2)$ or $(4, 3)$. Before considering these two subcases we shall show that neither $(1, b)$ nor $(1, 0)$ can be in G .

If $(1, b)$ then avoiding a θ -graph disjoint from $\begin{pmatrix} 0 & 2 \\ a & c \end{pmatrix}$ implies $(4, 2)$. If cycle $(0, 3, a)$ is not a 3-cycle of G then we are considering $S_a: (0, 2)$ and a wedge $\begin{pmatrix} 0 & b \\ 1 & 4 \end{pmatrix} \vee \begin{pmatrix} c & 0a \\ 0 & a \end{pmatrix} 2$. Thus vertex 3 must connect somewhere. The three possibilities, $(3, c), (3, 0a)$ and $(3, b)$ contain a θ -graph disjoint from $\begin{pmatrix} 3 & c \\ 0 & a \end{pmatrix}$, θ -graph disjoint from $\begin{pmatrix} b & 2 \\ c & 1 \end{pmatrix} 4$, and a wedge $\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} b \vee \begin{pmatrix} 0 & a \\ 2 & c \end{pmatrix}$ respectively. Thus $(1, b)$ is not in G .

If $(1, 0)$ then $(4, 3)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$, hence $(4, 2)$. Vertex 3 must connect somewhere else, since $(1, 0)$

is an edge, not an arc of G . The k -graphs $\begin{pmatrix} 0 & 2 \\ 1 & 4 \end{pmatrix} \vee \begin{pmatrix} 0 & c \\ a & b \end{pmatrix}$ not involving vertex 3 show any additional connection yields θ -graph disjoint from a k -graph.

Having eliminated the possible edges $(1,b)$ and $(1,0)$ we proceed by breaking into the two subcases, $(4,3)$ or $(4,2)$.

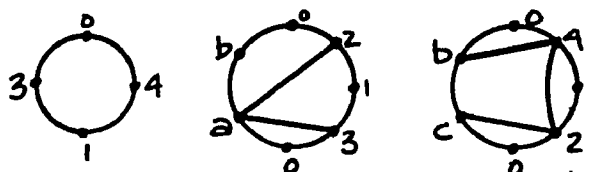


embedding $(3,4)$

Figure 3.32

Suppose $(4,3)$ is an edge of G and consider embedding $(3,4)$ as shown in figure 3.32. This embedding does not extend to an embedding of G . The three inadmissible bridges are $(0,1), (b,3), (c,4)$. The first bridge was just covered two paragraphs ago, and the latter pair of cases both contain wedges $k_{2,3} \vee k_4$. Because any pair of equivalent 3-bridges creates a θ -graph disjoint from a k -graph, we conclude there exists a pair of skew bridges for this embedding. Regions $(2,0,c)$ wedges $\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix}_a$, region $(a,0,3)$ wedges $\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}_c$, and region $(b,0,4)$ wedges $\begin{pmatrix} 2 & 3 \\ 0 & a \end{pmatrix}_1$ so any pair of skew bridges on these regions create a wedge of k -graphs as has previously been eliminated. We have shown $(b,1)$ is not in G , and $(b,2)$ gives a

θ -graph disjoint from $\begin{pmatrix} a & b \\ c & 2 \end{pmatrix}$, hence $(a,3,1,2,c)$ does not admit skew bridges. Skew bridges on $(0,3,4)$ give a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. If there are skew bridges on $(a,0,2)$ they are $(a,0,2), (2,0a)$, and G contains either a θ -graph disjoint from a k -graph or a wedge $k_{2,3} \vee k_4$ depending on whether G contains $(1,c)$ or $(1,a)$. Finally observe skew bridges on region $(a,3,1,2,c)$ cannot contain (a,v) for $v \in [1,2]$ as this bridge is transferable and $(a,1,3)$ gives a wedge $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \vee \begin{pmatrix} 1 & 3 \\ 13 & 4 & 0 \end{pmatrix}$. Thus $(4,3)$ is not an edge of G .



embedding (ab)

Figure 3.33

We conclude $(4,2)$ is an edge of G and we consider embedding $(a\ b)$ as shown in figure 3.33. Note skew bridges on cycles $(0,3,1,4), (4,1,2)$ and $(a,2,1,3)$ all give a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$. Skew bridges on $(b,4,2,c)$ gives a wedge $\begin{pmatrix} c & b \\ 0 & 4 \end{pmatrix} \vee \begin{pmatrix} 2 & 3 \\ 0 & 1 & a \end{pmatrix}$, and also as before regions $(b,0,4), (c,0,2), (a,0,3)$ cannot contain skew bridges or we have a previous wedge of k -graphs. Skew bridges on $(a,b,c,2)$ must be $(a,0,2), (b,2)$ and $(1,a)$ gives a wedge

$k_{2,3} \vee \begin{pmatrix} a & b \\ c & 2 \end{pmatrix}$, and edge $(1,c)$ gives a wedge $k_{2,3} \vee \begin{pmatrix} a & b \\ c & 2 \end{pmatrix}$. Hence this embedding must have an inadmissible bridge, either $(a,4), (c,3)$, or $(c,1)$. The first contradicts lemma 3.22, the second creates the symmetry $2 \sim 3$ and hence was covered in the preceding subcase, hence we conclude $(c,1)$.

Having established $(4,2), (c,1)$ are in G we consider vertex 3 which is cubic in a 3-cycle. If $(3,0,a)$ is not a 3-cycle then we must have $S_a: (0,2)$ which gives a wedge $\begin{pmatrix} 0 & a \\ b & 0a \end{pmatrix} \vee \begin{pmatrix} c & 4 \\ 1 & 2 \end{pmatrix} \vee b$. Thus vertex 3 connects somewhere. Edge $(3,b)$ gives a wedge $\begin{pmatrix} 3 & 4 \\ 0 & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & 2 \\ a & c \end{pmatrix}$. Edge $(3,c)$ gives $2 \sim 3$, having eliminated $(4,3)$ previously leads to a contradiction. Thus 3 connects to $st(a)$, giving a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ 2 & c \end{pmatrix}$.

Case 2. $(c,1)$. Observe cycle $(0,b,4)$ is disjoint from $\begin{pmatrix} a & 1 \\ 2 & 3 \end{pmatrix} \vee c$, hence that cycle is dead. Again we cannot have $(4,c)$ or $(4,a)$, by previous casework. A vertex Ob gives a wedge $\begin{pmatrix} 2 & 3 \\ 0 & a \end{pmatrix} \vee \begin{pmatrix} 0 & b \\ c & 4 \end{pmatrix} \vee Ob$. Hence without loss of generality we have $(4,3)$. We shall work on vertex 2 being cubic in a 3-cycle.

If G contains $S_a: (0,3)$ then the new vertex, called Oa in keeping with convention, is cubic in a 3-cycle. Edges $(Oa,1)$ or $(Oa,2)$ give us case 1 considering the wedge $\begin{pmatrix} 0 & a \\ 0a & c \end{pmatrix} \vee \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}$. Edge $(Oa,03)$ gives a θ -graph disjoint from $\begin{pmatrix} 1 & a \\ 2 & b \end{pmatrix} \vee c$ and edge $(Oa,4)$ gives the previous lemma. Thus G does not contain this splitting.

If there is a bridge on cycle $(a,0,2)$ then delete that bridge and embed. If cycle $(a,0,2)$ is null then we get an earlier case of a k -graph wedge $\begin{pmatrix} 0 & c & 4 \\ & b & 1 \end{pmatrix}$, by lemma 2.15. Examining the possible embeddings we see the only one with $(a,0,2)$ essential is the one of figure 3.34. The bridge on cycle $(a,0,2)$ must be $(a,02)$, and the bridge blocking an extension of this embedding is $(2,b)$. The resulting graph contains a θ -graph disjoint from $\begin{pmatrix} 1 & b \\ 2 & 4 & c \end{pmatrix}$, hence there is no bridge on $(a,0,2)$.

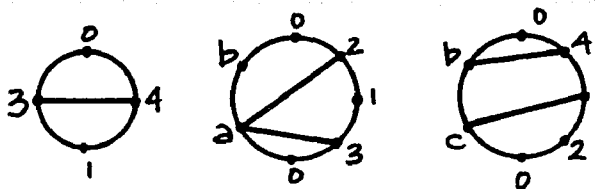


Figure 3.34

We have vertex 2 is in fact cubic in a 3-cycle and there is no bridge on $(a,0,2)$. If $a2$ is a vertex then avoiding a θ -graph disjoint from either $\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$ or $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$ implies $(a2,0)$, and a θ -graph disjoint from $\begin{pmatrix} a & 1 \\ 2 & 3 & a0 \end{pmatrix}$. A vertex 02 has several possible connections. Edge $(02,0b)$ contradicts the previous lemma using the wedge $\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} \vee \begin{pmatrix} 0 & b \\ a & 0b & c \end{pmatrix}$. Edge $(0b,0c)$ is case 1 using the wedge $\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix} \vee \begin{pmatrix} 0 & c \\ a & b & 0c \end{pmatrix}$. Edge $(02,b)$ gives 02 cubic in a

3-cycle, yet we have exhausted the places where it may connect.

Thus $(a,0,2)$ is a 3-cycle.

Edges $(2,0c), (2,0b)$ are ruled out by the same k -graphs which ruled out $(02,0c), (02,0b)$ respectively, thus $(2,b)$ is in G .

Vertices $2,c$ are dead, $st(4)$ is dead, by symmetry so is $st(3)$.

The only edges left are $(1,0a), (1,0b)$ with θ -graphs disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & 4 & 0a \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 3 & 4 & 0b \end{pmatrix}$ respectively.

□

Lemma 3.24. Let H denote the graph of figure 3.35. Then there does not exist a $G \in \Gamma_{\times}^M(P)$ satisfying $H3$ and either $H, S_a(H), S_b(H)$ or $S_c(H)$.

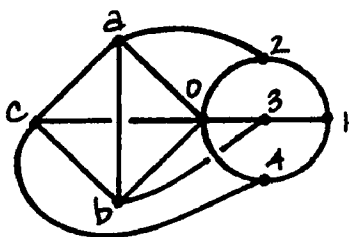


Figure 3.35

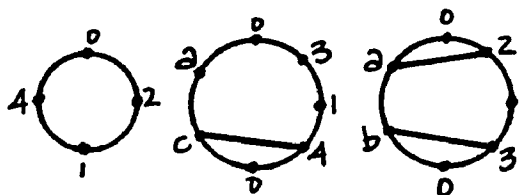


Figure 3.36

Proof. By way of contradiction suppose G were such a graph. Note if $(1,0)$ is in G , avoiding cubic vertices in 3-cycles we have $(2,3)$. Any connection from 4 gives a θ -graph disjoint from a k -graph. Hence $(1,0)$ is not in G .

Next consider the embedding of figure 3.36, this does not extend to an embedding of G . If there were skew bridges on cycle $(a,0,2)$ then G contains a k -graph wedge $\begin{pmatrix} 3 & c \\ 0 & b & 4 \end{pmatrix}$ is a previous proposition. By symmetry there are not skew bridges on $(b,0,3)$ or $(c,0,4)$. Since $(0,1)$ is not in G there are not skew bridges on $(a,2,1,4)$. By the previous lemma $(a,3), (a,4), (b,2), (b,4), (c,2), (c,3)$ are not edges of G , so the only possibilities for skew bridges on $(a,2,1,3)$ or $(a,0,3,1,4,c)$ are $(2,3), (1,a)$ or $(3,4), (1,0a)$. The latter graph contains a wedge $\begin{pmatrix} 0a & 2 \\ 0 & a & 1 \end{pmatrix} \vee \begin{pmatrix} b & 4 \\ 0 & c & 3 \end{pmatrix}$. In the former graph vertex 4 is cubic in a 3-cycle, avoiding the previous lemma implies $(4,c)$ giving a θ -graph disjoint from $\begin{pmatrix} 1 & 2 \\ 3 & a \end{pmatrix}$. Thus this embedding must not extend by reason of an inadmissible bridge. Bridges $(c,2), (b,4)$ give graphs covered in the previous lemma, hence we have $(1,0b)$. Avoiding $2,4$ cubic in a 3-cycle implies $(2,4)$ which is symmetric to a previous case.

□

§3.6 A Wedge of k_4 's

Proposition 3.25. Let $G \in \Gamma_*^M(P)$ satisfy H3 and contain $k_4 \vee k_4$ as shown in figure 3.37. Then $G \in \{A_2, B_1\}$.

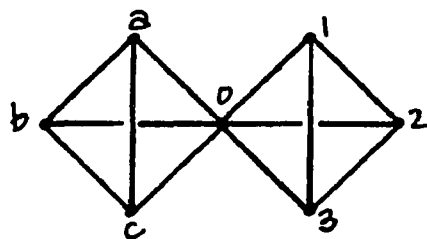


Figure 3.37

Proof. First we note $|v(G)| = 7$. Any vertex v disjoint from this subgraph connects to (WLOG) $0,1$ giving a wedge $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \vee \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}_v$. Likewise if there was a vertex $12 \in (1,2)$ then G contains a wedge $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix} \vee \begin{pmatrix} 1 & 2 \\ 12 & 3 \end{pmatrix}_0$. We shall break into cases depending on $\deg(1)$, the valency of vertex 1.

Case 1. $\deg(1) = 6$. We have edges $(1,a), (1,b), (1,c)$. Edges $(2,a), (3,a)$ give graph A_2 . Avoiding this, the largest graph possible is $(2,a)(2,b)(3,c)$ which is projective.

Case 2. $\deg(1) = 5$. We have edges $(1,a)(1,b)$. If $(2,a)(2,b)$ then the largest graph with only one valency 5 vertex is $(3,c)$, which is projective. If $(2,a)(2,c)$ then avoiding the previous sentence implies $(3,b), (3,c)$ giving B_1 . If both vertices 2,3 have valency ≤ 4 then the resulting graph is projective.

Case 3. $\deg(1) = 4$. Avoiding cases 1 and 2 the largest graph possible is $(1,a)(2,b)(3,c)$ which is projective.

□

Chapter 4

A CYCLE DISJOINT FROM A k -GRAPH

§4.1 Statement of the Result and Standing Assumptions

Chapter 2 characterizes graphs in $I_*^M(P)$ which contain disjoint k -graphs. Chapter 3 characterizes graphs in $I_*^M(P)$ which contain a wedge, V , of k -graphs. In chapter 4 we make the standing assumption, H_4 , that $G \in I_*^M(P)$ contains neither disjoint k -graphs nor a wedge, V , of k -graphs.

Theorem 4.1. There does not exist a $G \in I_*^M(P)$ which contains a cycle disjoint from a k -graph but which does not contain either:

- 1) disjoint k -graphs, or
- 2) a one point union of k -graphs, at least one containing a cycle disjoint from the other.

Proof. The condition G contains a cycle disjoint from a k -graph is exhaustively covered by the four propositions listed below. Their proofs will complete the proof of this theorem.

Proposition 4.6. There does not exist a $G \in I_*^M(P)$ satisfying H_4 and containing an n -cycle disjoint from a $k_{2,3}$, where $n \geq 4$.

Proposition 4.12. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing an n -cycle disjoint from a k_4 , where $n \geq 4$.

Proposition 4.18. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing a 3-cycle disjoint from a $k_{2,3}$.

Proposition 4.22. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing a 3-cycle disjoint from a k_4 .

□

We note standing assumption H^4 includes H^3 , the standing assumption of chapter 3. The reader is referred to the list of standing assumptions in §3.1.

§4.2 A 4-cycle Disjoint From a $k_{2,3}$

The goal of this section is the proof of proposition 4.6, concerning G containing an n -cycle $\parallel k_{2,3}$ for $n \geq 4$. We first shall prove a partial result where G contains a $K_{3,3}$, $\begin{pmatrix} a & b & c \\ \alpha & x & y \end{pmatrix}$, and an n -cycle disjoint from $\begin{pmatrix} a & b & c \\ \alpha & x & y \end{pmatrix} \setminus (\alpha, c)$ for $n \geq 4$.

Lemma 4.2. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H^4 and containing a subgraph, H , homeomorphic to the graph of figure 4.1, where H has at least 3 vertices $1, 2, 3$ as indicated.

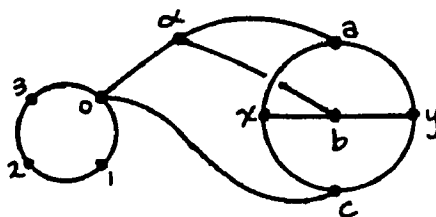


Figure 4.1

Proof. By way of contradiction suppose G is such a graph. If vertex 2 connects to either $st(\alpha)$ or $st(c)$ then G contains a θ -graph disjoint from a k -graph. If $(2, b)$ then G contains a wedge $\begin{pmatrix} o & 2 \\ 1 & 3 & b \end{pmatrix} \vee \begin{pmatrix} x & y \\ b & a & c \end{pmatrix}$ and if $(2, x)$ then G contains a wedge $\begin{pmatrix} o & 2 \\ 1 & 3 & x \end{pmatrix} \vee \begin{pmatrix} a & b \\ x & y & \alpha \end{pmatrix}$. By S_v -independent arguments (corollary 3.11) vertex 2 is dead, a contradiction.

□

Lemma 4.3. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing a subgraph, H , homeomorphic to the graph of figure 4.2 where H has vertices 2,3 as indicated.

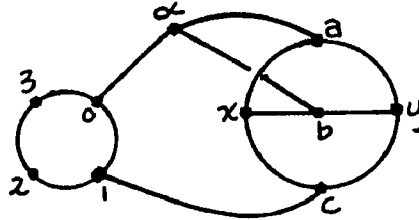


Figure 4.2

Proof. By way of contradiction suppose G were such a graph. If vertex 2 connects to α, c then G contains a θ -graph disjoint from a k -graph. If edge $(2, a)$ is in G then G contains a wedge $\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b \end{pmatrix}$. By symmetry and S_V -independent arguments 2 may connect only to x or y , likewise 3 may only connect to a or b . Assume G contains $(2, x)$ and $(3, a)$.

Next observe $\overline{st(0)}$ is disjoint from $\begin{pmatrix} x & y \\ a & b \end{pmatrix}$ which implies 0, and by symmetry 1, must connect elsewhere. Edge $(0, c)$ gives a θ -graph disjoint from $\begin{pmatrix} a & b \\ x & y \end{pmatrix}$, edge $(0, x)$ gives a wedge $\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix} \vee \begin{pmatrix} a & b \\ x & y \end{pmatrix}$, and edge $(0, y)$ gives $\overline{st(2)}$ disjoint from $\begin{pmatrix} \alpha & y \\ a & b \end{pmatrix}$ which implies edge $(2, y)$ giving a wedge $\begin{pmatrix} 0 & a \\ 3 & \alpha \end{pmatrix} \vee \begin{pmatrix} 2 & c \\ y & 1 \end{pmatrix}$. If $(0, a)$ and $(0, b)$ are both in G then $\overline{st(2)}$ disjoint from $\begin{pmatrix} a & b \\ 0 & \alpha \end{pmatrix}$ implies $(2, y)$ and a θ -graph disjoint from $\begin{pmatrix} x & y \\ 2 & b \end{pmatrix}$. If $(0, a\alpha)$ then $\overline{st(\alpha)}$ disjoint from $\begin{pmatrix} a & c \\ x & y \end{pmatrix}$

implies α connects somewhere. Edge (α, c) is lemma 2.19, (α, b) contains θ -graph disjoint from $\begin{pmatrix} a & c \\ x & y & 2 \end{pmatrix}$, (α, x) contains a wedge $\begin{pmatrix} 0 & 2 \\ 1 & 3 & x \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ and (α, ax) respectively (α, ay) is equivalent to the case where $(3, ay)$ respectively $(3, ax)$ is an edge of G . Hence (α, y) and $a\alpha$ is cubic in a triangle so that $a\alpha$ connects somewhere, yet any such connection yields either a θ -graph disjoint from a k -graph or $\overline{st(2)}$ disjoint from a k -graph. Thus $(0, a\alpha)$ is not an edge of G . If $(0, a)$ then vertex 3 cubic in a 3-cycle implies $(3, b)$ is an edge of G and G contains a wedge $\begin{pmatrix} a & b \\ 3 & y & \alpha \end{pmatrix} \vee \begin{pmatrix} 1 & x \\ 2 & c & a \end{pmatrix}$. Thus by S_v -independent arguments we may assume G contains $(0, b)$ and by symmetry $(1, y)$.

Vertex α is cubic in a 3-cycle, and (α, v) for v not in cycle (x, b, y, c) gives θ -graph disjoint from a k -graph. Edge (α, c) contains a 4-cycle disjoint from $\begin{pmatrix} a & b & c \\ \alpha & x & y \end{pmatrix}$ contradicting lemma 2.19, (α, x) gives a wedge $\begin{pmatrix} 0 & 2 \\ x & 1 & 3 \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$, (a, y) gives a wedge $\begin{pmatrix} 0 & 2 \\ y & 1 & 3 \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$, and (α, b) gives a θ -graph disjoint from $\begin{pmatrix} a & c \\ 2 & x & y \end{pmatrix}$. By S_v -independent arguments these exhaust the possibilities.

□

Lemma 4.4. There does not exist a $G \in \Gamma_*^M(P)$ satisfying H_4 and containing a subgraph, H , homeomorphic to the graph of figure 4.3 where H has vertices 1, 3 as indicated.

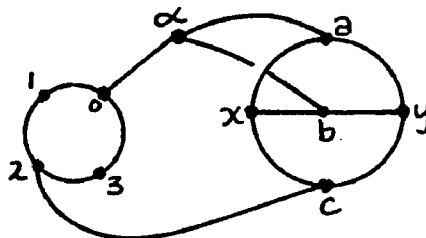


Figure 4.3

Proof. By way of contradiction suppose G were such a graph. Note there cannot exist vertices $01, 12, 23, 03$ else we can apply lemma 4.3. Also note the symmetries $(1\ 3), (a\ b), (x\ y)$, and $(a\ x)(b\ y)(\alpha\ c)(0\ 2)$. We examine where vertices $1, 3$ may connect.

Edges $(1, \alpha)$ or $(1, c)$ create a θ -graph disjoint from a k -graph, hence they both may connect only to cycle (a, x, b, y) . Edges $(1, x)(3, x)$ together give a wedge $\begin{pmatrix} 1 & 3 \\ 0 & 2 & x \end{pmatrix} \vee \begin{pmatrix} a & b \\ x & y & \alpha \end{pmatrix}$. Edges $(3, by), (1, ax)$ together give a subgraph $\begin{pmatrix} \alpha & ax \\ b & a & 0 \end{pmatrix} \cup \begin{pmatrix} by & c \\ b & y & 2 \end{pmatrix}$, by S_x -independent arguments and symmetry one of the pair $1, 3$ must connect to an existing vertex, without loss of generality suppose $(1, x)$. If $(3, y)$ then we have a cycle disjoint from $\begin{pmatrix} x & \alpha \\ a & b & 0 \end{pmatrix}$. $\overline{St(2)}$ is disjoint from $\begin{pmatrix} a & b \\ x & y & \alpha \end{pmatrix}$ yet anywhere 2 connects creates a θ -graph disjoint from a k -graph. By S_y -independent arguments we conclude $(3, b)$, giving the graph of figure 4.4.

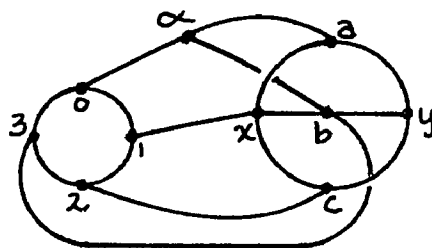
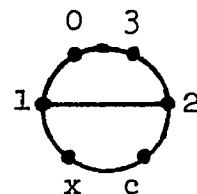


Figure 4.4

As was observed before, $\overline{\text{st}(2)}$ is disjoint from $\begin{pmatrix} a & b \\ x & y & \alpha \end{pmatrix}$. Avoiding a contradiction we have 2 connecting only to x, y, cx, cy . Likewise 0 may connect only to $a, b, \alpha a, b\alpha$. We examine where vertex 0 may connect.

If $(0, \alpha b)$ is in G note the cycle $(0, \alpha, \alpha b)$ is disjoint from $\begin{pmatrix} a & b & c \\ x & y & 3 \end{pmatrix} \setminus (3, a)$. If this cycle is a 4-cycle lemma 4.3 0α , and

avoiding k -graph disjoint from the θ -graph



implies $(0\alpha, a)$, yet this gives θ -graph disjoint from $\begin{pmatrix} x & 2 \\ 1 & b & c \end{pmatrix}$. Hence $(0, \alpha, \alpha b)$ is a 3-cycle and both $\alpha, \alpha b$ are not cubic. If either connects outside of cycle (a, x, b, y) then we get a θ -graph disjoint from a k -graph or a contradiction of lemma 2.19. Edges $(\alpha, b), (\alpha b, x)$ give a wedge $\begin{pmatrix} 0 & \alpha \\ \alpha & b & x \end{pmatrix} \vee \begin{pmatrix} b & c \\ x & y & 2 \end{pmatrix}$, any other connections give a wedge $k_4 \vee \begin{pmatrix} x & 2 \\ 1 & b & c \end{pmatrix}$. Thus $(0, \alpha b)$ is not an edge of G . Observe $(0, \alpha a)$ is not an edge of G since renaming $\alpha a, \alpha$ by $\alpha, \alpha b$ respectively gives the subgraph we just considered.

If $(0,b)$ is an edge of G then avoiding vertex 3 cubic in a triangle implies $(3,a)$. Since this creates a graph with a symmetric to b we may assume $(0,a)$ is an edge of G and by symmetry $(2,y)$ is also. Note (α,x) builds $\begin{pmatrix} \alpha & 0 \\ a & x \end{pmatrix}$ disjoint from a θ -graph. By S_x -independent arguments we have (α,y) and (c,a) giving $\begin{pmatrix} x & 2 \\ 1 & b & c \end{pmatrix}$ disjoint from a θ -graph.

□

Corollary 4.5. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing a $K_{3,3} \begin{pmatrix} a & b & c \\ \alpha & x & y \end{pmatrix}$ and an n -cycle disjoint from $\begin{pmatrix} a & b & c \\ \alpha & x & y \end{pmatrix} \setminus (\alpha,c)$ for $n \geq 4$.

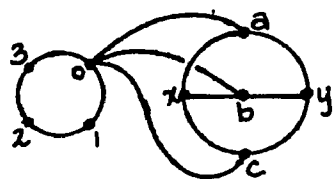
Proof. Depending on how (α,c) is added in we apply lemma 2.19 or one of the preceeding three lemmas. Observe this is an S -independent argument.

□

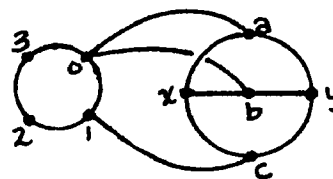
Proposition 4.6. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing an n -cycle disjoint from a $k_{2,3}$ for $n \geq 4$.

Proof. The vertex in the $K_{3,3}$ missing from the $k_{2,3}$ must lie on the n -cycle by corollary 4.5. The proposition naturally breaks into four cases, illustrated in figure 4.5. Each case is covered in a separate lemma. The proofs of these lemmas will complete the proof of this proposition.

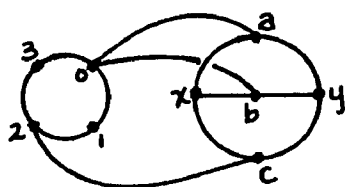
□



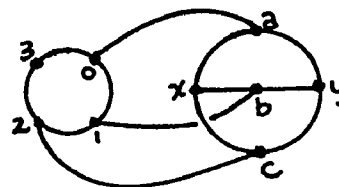
lemma 4.7



lemma 4.8



lemma 4.10



lemma 4.11

Figure 4.5

Lemma 4.7. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H_4 and containing a subgraph, H , homeomorphic to the graph of figure 4.6 where H contains vertices 1,2,3 as shown.

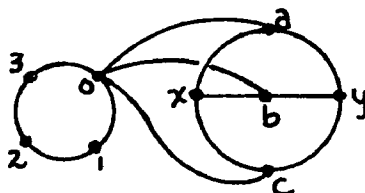


Figure 4.6

Proof. By way of contradiction suppose G were such a graph.

We first examine where vertex 2 may connect.

Suppose $(2,a)$ is an arc of G . Avoiding a wedge $\begin{pmatrix} 0 & 2 \\ 1 & 3 & a \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ implies edge $(1,a)$ and $st(1)$ is dead. Avoiding vertex 1 cubic in a 3-cycle implies vertices $0a$ and $2a$. Edge $(0a,x)$ gives cycle $(0,1,2,3)$ disjoint from $K_{3,3}$ contradicting corollary 4.5. If $0a$ connects anywhere else we get a similar contradiction or a θ -graph disjoint from a k -graph. Thus edge $(2,a)$ is not in G .

Suppose $(2,ax)$ is an arc of G . Avoiding a subgraph $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix} \cup \begin{pmatrix} 0 & 2 \\ 1 & 3 & ax \end{pmatrix}$ implies vertex 1 can only connect to $[a,ax]$. If $(1,a)$ is in G and there exists a vertex $0a$ we can repeat the above argument. Thus edge $(1,ax)$ and G contains a 4-cycle disjoint from $\begin{pmatrix} a & b & c \\ x & y & 0 \end{pmatrix} \setminus (a,x)$ contradicting corollary 4.5.

Thus G contains $(2,x)$, and since $\overline{st(2)}$ is disjoint from $\begin{pmatrix} 0 & y \\ a & b & c \end{pmatrix}$, $(2,y)$ is also in G . Next we shall examine where vertex 1 connects.

Suppose $(1,a)$ is in G . Edge $(3,a)$ gives a contradiction of corollary 3.11 and edge $(3,x)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & y \\ a & b & c \end{pmatrix}$. Edges $(3,by), (3,y)$ give a graph containing a wedge $\begin{pmatrix} 0 & by \\ 3 & b & y \end{pmatrix} \vee \begin{pmatrix} 2 & a \\ x & y & 1 \end{pmatrix}$ and a wedge $\begin{pmatrix} 0 & y \\ b & c & 3 \end{pmatrix} \vee \begin{pmatrix} 2 & a \\ x & y & 1 \end{pmatrix}$ respectively. Thus we have $(3,b)$ is in G . If there is a vertex $0b$ then G contains a wedge $\begin{pmatrix} 0 & b \\ 0b & 3 & x \end{pmatrix} \vee \begin{pmatrix} 2 & a \\ x & y & 1 \end{pmatrix}$, hence $(0,b)$ is an edge.

Avoiding vertex 3 cubic in a triangle implies edge $(3,c)$ giving a wedge $\begin{pmatrix} b & c \\ 0 & 3 \end{pmatrix} \vee \begin{pmatrix} 2 & a \\ 1 & x \end{pmatrix}$. These arguments are S_a -independent, hence $(1,ax)$ is not an arc of G .

We now have that vertices 1,3 may only connect to x or y . If $(1,x)$ and $(3,x)$ occur together then G contains a wedge $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b \end{pmatrix}$, hence $(1,x)$ and $(3,y)$ with $st(1), st(3)$ dead. Since $st(2)$ is dead G must contain a cubic vertex in a 3-cycle.

□

Lemma 4.8. There does not exist a $G \in \Gamma_*^M(P)$ satisfying H_4 and containing a subgraph, H , homeomorphic to the graph of figure 4.7 where H contains vertices 2,3 as shown.

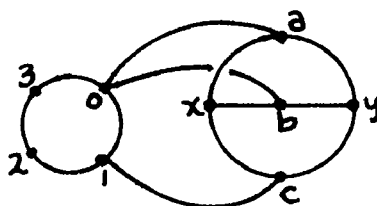


Figure 4.7

Proof. By way of contradiction let G be such a graph. We shall first examine where vertex 2 may connect.

If $(2,a)$ is in G then avoiding a wedge $\begin{pmatrix} 0 & 2 \\ 1 & 3 & a \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ implies edge $(3,a)$ with vertex 3 dead. Avoiding a cubic vertex in a 3-cycle there exist $0a, 2a$. Any connection of vertex $0a$ creates a 4-cycle disjoint from $K_{3,3} \setminus e$ contradicting corollary 4.5. If $(2,ax)$ is in G then again (corollary 3.11 is S_a -independent) 3 can only connect to $[a,ax]$. We get either a 4-cycle disjoint from $K_{3,3} \setminus e$ directly or a vertex as above from whence any connection gives the same contradiction.

Thus vertex 2 may only connect to $[x,c] \cup [c,y]$. Observe $(2,c)$ or $(2,cx)$ give $\overline{st(2)}$ disjoint from $\begin{pmatrix} a & b \\ x & y & 0 \end{pmatrix}$. If 2 connects to $st(c)$ again then G contains a θ -graph disjoint from a k -graph. Thus without loss of generality edge $(2,x)$ is in G .

We shall examine where vertex 3 may connect. First consider the case $(3,y)$. Using lemma 2.18 with k -graph $\begin{pmatrix} a & b \\ x & y & 0 \end{pmatrix}$ we conclude $[1,2]$ is not dead. Edges $(2,a), (2,b)$ contradict corollary 3.11, $(2,c)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & y \\ 3 & a & b \end{pmatrix}$, and $(2,y)$ gives a wedge $\begin{pmatrix} 2 & c \\ 1 & x & y \end{pmatrix} \vee \begin{pmatrix} 0 & y \\ a & b & 3 \end{pmatrix}$. By S -independent arguments $st(2)$ is dead, by the symmetry $(0x)(3c)(12)$ $st(1)$ is dead, a contradiction. By S_y -independent arguments edges $(3,ay), (3,by)$ are also eliminated.

If edge $(3,ax)$ is in G then again $[1,2]$ cannot be dead. If 2 connects somewhere then for the same reasons as in the previous paragraph we have either $(2,y)$ or $(2,cy)$. Using the symmetry $(23)(ax1)(0x)(ca)$ this is exactly the case of the preceeding

paragraph. Thus vertex 1 is not dead, we examine where it may connect. Edge $(1,ax)$ or $(1,a)$ contradict corollary 3.11, $(1,b)$ gives a 4-cycle disjoint from $\begin{pmatrix} 1 & x & y \\ b & 2 & c \end{pmatrix} \setminus (2,y)$, so 1 may connect only to x, cx, y . If $(1,x)$ then since $st(2)$ is dead there exists $1x$, either edge $(1x, cx)$ or $(1x, y)$ gives 4-cycle disjoint from $\begin{pmatrix} x & 3 & a \\ 2 & ax & 0 \end{pmatrix} \setminus (a, 2)$. If $(1, cx)$ then $(1, c, cx)$ is a 3-cycle by the same reasoning, and $(1, y)$ gives a θ -graph disjoint from $\begin{pmatrix} x & 3 \\ 0 & 2 & ax \end{pmatrix}$. Thus $(1, y)$ is an arc of G and vertex c connects somewhere. Any choice gives either a θ -graph disjoint from $\begin{pmatrix} 0 & y \\ 1 & b & a \end{pmatrix}$, 4-cycle disjoint from a $K_{3,3} \setminus e$, or lemma 4.7. Thus 3 does not connect to ax .

If $(3, x)$ then consider the 3-cycle $(2, 3, x)$. If there exists $2x$ we get either a 4-cycle disjoint from $\begin{pmatrix} a & b & 3 \\ 0 & x & y \end{pmatrix} \setminus (3, y)$ or a θ -graph disjoint from $\begin{pmatrix} x & y \\ 2x & a & b \end{pmatrix}$. Thus we must also have $(3, a)$. Now note a vertex $3x$ gives a θ -graph disjoint from a k -graph unless $(3x, b)$, in which case replacing (b, x) with $(b, 3x)(3x, x)$ shows we are in an earlier case. Thus vertex 2 connects somewhere else. Edge $(2, y)$ gives a wedge $\begin{pmatrix} c & 2 \\ 1 & x & y \end{pmatrix} \vee \begin{pmatrix} x & 0 \\ b & a & 3 \end{pmatrix}$, edge $(2, cy)$ gives a wedge $\begin{pmatrix} 2 & c \\ x & 1 & cy \end{pmatrix} \vee \begin{pmatrix} a & b \\ x & y & 0 \end{pmatrix}$, $(2, x)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & y \\ a & b & c \end{pmatrix}$ hence $(2, c)$, and vertex 2 is dead. Now vertex 1 cannot be cubic. Edges $(1, c)$ or $(1, y)$ give a θ -graph disjoint from $\begin{pmatrix} 1 & x \\ 2 & c \end{pmatrix}$, edge $(1, a)$ gives a wedge $\vee k_{2,3}$'s hence $(1, b)$ and vertex 1 is dead. Deleting $(3, x)$ and noting cycles $(0, 1, 2, 3), (0, 3, a), (0, 1, b)$, and $(1, 2, c)$ all embed null implies $(3, x)$ is reducible.

Thus we must have $(3,a)$, and avoiding a cubic vertex in a 3-cycle we also have $(3,b)$. $\overline{St(1)}$ is disjoint from $\begin{pmatrix} a & b \\ x & y & 3 \end{pmatrix}$ hence 1 connects somewhere. Anywhere but $(1,y)$ gives a θ -graph disjoint from a k -graph, hence $(1,y)$. Examining cycle $(1,2,x,c)$ disjoint from $\begin{pmatrix} a & b \\ 0 & 3 & y \end{pmatrix}$ shows we must have edge $(c,0)$ in G which contradicts lemma 4.7.

□

Lemma 4.9. Let G contain H , a $k_{2,3}$ disjoint from a cycle, with the vertex of the $K_{3,3}$ missing from the $k_{2,3}$ lying on the cycle. Then there are exactly 1 labeled and 6 unlabeled embeddings of H which can allow an extension to an embedding of G .

Proof. If this embedding extends then the $k_{2,3}$ contains an essential cycle, there are 3 labeled choices. For each choice the cycle containing the missing vertex must embed in the non-null region, either clockwise or counterclockwise. The possibilities are shown in figure 4.8.

□

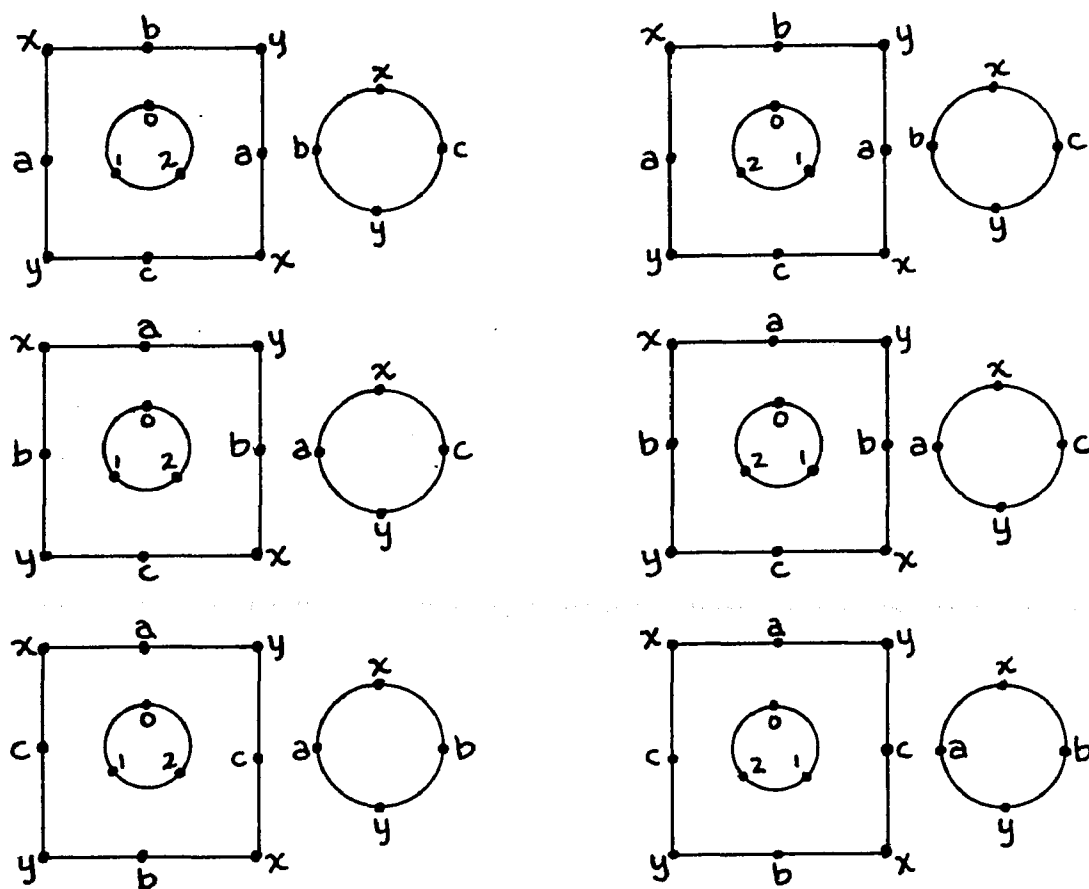


Figure 4.8

Lemma 4.10. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing a subgraph, H , homeomorphic to the graph of figure 4.9 where H contains vertices 1 and 3 as shown.

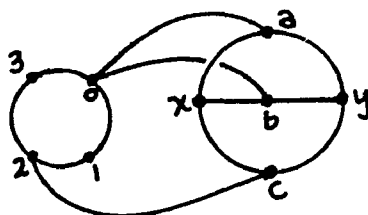


Figure 4.9

Proof. By way of contradiction suppose G were such a graph. Before proceeding note $(0,1), (1,2), (2,3)$ and $(3,4)$ are all dead by lemma 4.8. Also note $\overline{\text{st}(2)}$ is disjoint from $\begin{pmatrix} a & b \\ 0 & x & y \end{pmatrix}$ and finally observe vertices 1 and 3 may only connect to cycle (a,x,b,y) . We shall break into cases based on where 1,3 can connect. If $(1,ax)$ is in G then $(3,a)$ gives a contradiction of corollary 3.11. If $(3,y)$ then $(2,c), (2,x)$ give a θ -graph disjoint from $\begin{pmatrix} 0 & y \\ a & b & 3 \end{pmatrix}$ and $(2,y)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & ax \\ 1 & b & a \end{pmatrix}$. Thus we have either $(3,b), (3,bx)$ or $(3,x)$, cases 1-3 respectively. If there is no edge of type $(1,ax)$ we have either $(1,a), (3,x)$; $(1,a)(3,b)$; or $(1,x), (3,y)$ cases 4-6 of figure 4.10 respectively.

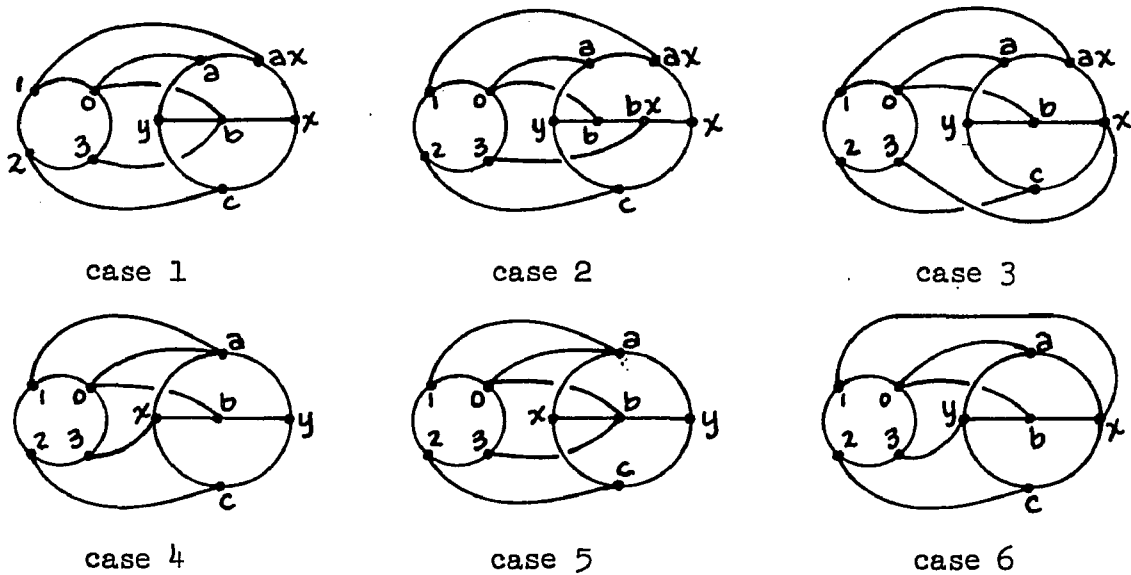


Figure 4.10

Case 1. We observe vertex 3 is cubic in a 3-cycle. If Ob then avoiding corollary 4.5 we can have only (Ob,x) or (Ob,y) . If Ob is cubic then there exists a vertex ax, ay respectively and wherever they connect G contains a θ -graph disjoint from a k -graph. If Ob is not cubic then G contains a θ -graph disjoint from $\begin{pmatrix} x & y \\ b & Ob & c \end{pmatrix}$. Thus vertex 3 is not cubic. If $(3,b)$ then G contains a θ -graph disjoint from $\begin{pmatrix} a & c \\ x & y & 2 \end{pmatrix}$, $(3,y)$ gives lemma 4.8 by considering $(3,2,y,c)$ disjoint from $\begin{pmatrix} ax & 0 \\ a & b & 1 \end{pmatrix}$. Thus G must contain $(3,x)$.

Next we note $\overline{st(2)}$ is disjoint from $\begin{pmatrix} a & b \\ x & y & 0 \end{pmatrix}$ implies vertex 2 is not cubic, 2 may connect only to x, y, cx, cy . Before considering the choices, by examining lemma 4.9 we see our subgraph embeds in exactly 2 ways, shown in figure 4.11.

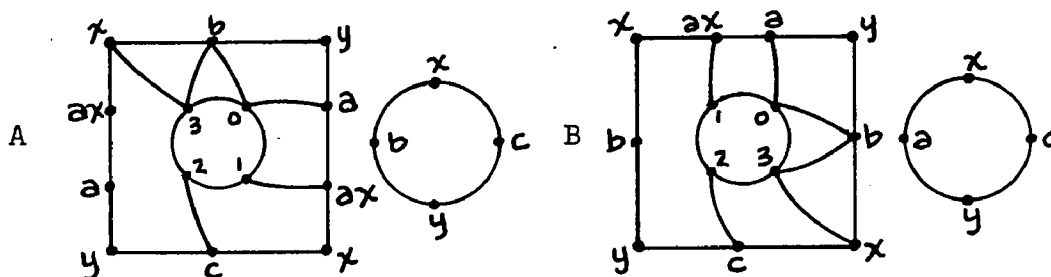


Figure 4.11

We note cycles $(1,2,c,x,ax)$, $(2,3,x,c)$ are disjoint from θ -graphs, hence they cannot contain k -graphs. If $(2,x)$ or $(2,cx)$ then upon deleting this edge we get a k -graph on one of these cycles by lemma 2.15. If $(2,y)$ the deleting implies (c,b) , (c,a) or (c,ax) . The first contains a θ -graph disjoint from a k -graph, the second is the previous lemma with $k_{2,3}$ disjoint from $(ax,a,0,1)$, and the third bridge transfers to cycle $(1,2,c,x,ax)$, a contradiction.

Case 2. Using lemma 4.9 we see our subgraph only embeds two ways, shown in figure 4.12. We examine where vertex 2 can connect, either

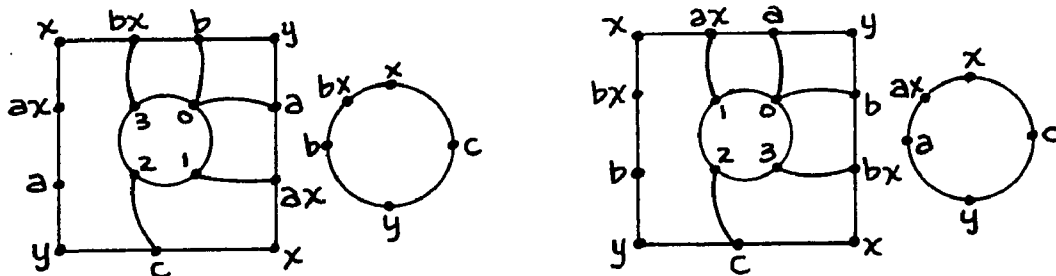


Figure 4.12

x , cx , cy , or y , keeping in mind the symmetries $(1\ 3)(a\ b)(ax\ bx)$ and $(1\ a\ b\ 3)(2\ ax\ y\ bx)(c\ x)$. Cycles $(1,2,c,x,ax)$, $(3,2,c,x,bx)$ are both disjoint from θ -graphs, hence by lemma 2.15 we cannot have $(2,x)$ or $(2,cx)$. Thus $(2,y)$ and by symmetry (ax,bx) . Deleting $(2,y)$ and embedding gives either (c,a) or (c,ax) . The former

graph contains a subgraph $(\begin{smallmatrix} 0 & 2 \\ 1 & 3 \end{smallmatrix} \ y) \cup (\begin{smallmatrix} a & x \\ b & ax \end{smallmatrix} \ c)$ and the latter contains a θ -graph disjoint from a k -graph.

Case 3. We note the symmetry $(1\ a)(2\ y)(3\ b)$ and the three embeddings shown in figure 4.13. Again we examine where vertex 2

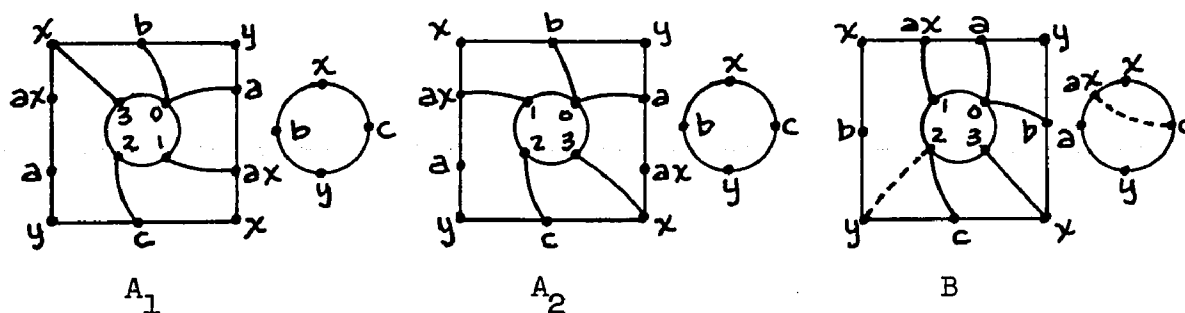


Figure 4.13

may connect. If $(2, x)$ then vertex 3 is cubic in a 3-cycle, yet any other connection gives an earlier case of this lemma. If $(2, cx)$ then deleting that bridge and embedding in one of the above embeddings implies $(c, 1), (c, 3)$ or (c, ax) . The first two are lemma 4.8 and the last graph contains a 4-cycle disjoint from $(\begin{smallmatrix} ax & cx \\ 2 & x \end{smallmatrix} \ 3) \setminus (3, c)$. Edge $(2, cy)$ is the same as $(2, cx)$, hence $(2, y)$. Deleting this bridge and embedding implies $(c, a), (c, b)$ and (c, ax) . Again (by symmetry) the first two are lemma 4.8 hence (c, ax) with embedding B unique. Every region except $(1, 2, y, b, x, ax)$ is disjoint from a θ -graph, hence any admissible bridge embeds in this region. Vertex

2 is dead, $(1,x)$ gives a wedge $\begin{pmatrix} y & ax \\ a & c & x \end{pmatrix} \vee \begin{pmatrix} 1 & 3 \\ x & 0 & 2 \end{pmatrix}$, $(1,b)$ is a splitting of this so vertex 1 cannot be a v.o.a. of an admissible bridge. Thus admissible bridges are $(a,bx), (y,x)$ and $\overline{st(3)}$ is disjoint from $\begin{pmatrix} ax & y \\ b & c & a \end{pmatrix}$. We conclude there exists an inadmissible bridge for embedding B. The only candidate for such a bridge which avoids a cubic vertex in a 3-cycle is (a,bx) , which gives cycle $(0,1,2,3)$ disjoint from $\begin{pmatrix} bx & y & ax \\ b & a & x \end{pmatrix} \setminus (ax,b)$.

Case 4. Vertex 1 is cubic in a 3-cycle. If there is a vertex Oa then either (Oa,x) or (Oa,y) , since $(Oa, st(a))$ is equivalent to $(1,ax)$ which has been ruled out. Both (Oa,x) and (Oa,y) give a wedge $\begin{pmatrix} x & y \\ Oa & b & c \end{pmatrix} \vee \begin{pmatrix} 1 & 3 \\ 0 & 2 & x \end{pmatrix}$ hence we can have only one of them. If (Oa,x) then there exists ax , as a result we get either the previous case with cycle (Oa,a,ax,x) disjoint from $\begin{pmatrix} 0 & 2 \\ 1 & 3 & y \end{pmatrix}$ or a θ -graph disjoint from $\begin{pmatrix} x & y \\ ax & b & c \end{pmatrix}$. If (Oa,y) then there exists ay . Edges $(ay,1), (ay,3), (ay,0)$ are earlier cases, $(ay,2)$ contradicts corollary 3.11, edge (ay,x) gives a θ -graph disjoint from $\begin{pmatrix} x & y \\ ay & b & c \end{pmatrix}$, hence ay may connect only to b or c . If both edges occur then G contains a θ -graph disjoint from $\begin{pmatrix} b & c \\ x & y & ay \end{pmatrix}$, if just (ay,b) (edge (ay,c) respectively) then there exists a vertex by (vertex cy respectively) and any connection gives a θ -graph disjoint from a k -graph. Thus Oa is not a vertex and 1 is not cubic.

If $(1,b)$ then lemma 4.8 applies, hence edge $(1,y)$. $\overline{st(2)}$ is disjoint from $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$, $(2,y)$ gives a θ -graph disjoint from $\begin{pmatrix} 0 & x \\ a & b & 3 \end{pmatrix}$ and anything else gives a θ -graph disjoint from $\begin{pmatrix} 0 & y \\ a & b & 1 \end{pmatrix}$.

Case 5. If either vertex 1,3 connect elsewhere then we have an earlier case. Thus there exist vertices O_a, O_b which can connect only to x, y . If (O_a, x) with (O_b, x) then G contains a θ -graph disjoint from $\begin{pmatrix} a & x \\ a & O_b & a_x \end{pmatrix}$, hence (O_a, x) and (O_b, y) . Vertices a_x, b_y must connect to c or we get a θ -graph disjoint from $\begin{pmatrix} a & x \\ a_x & O_a & 2 \end{pmatrix}$, yet this gives an earlier case with cycle $(0, 1, 2, 3)$ disjoint from $\begin{pmatrix} a_x & b_y \\ a & b & c \end{pmatrix}$.

Case 6. Note vertices 1,3 are now dead. If $(2, x)$ then G contains a θ -graph disjoint from $\begin{pmatrix} 0 & y \\ a & b & 3 \end{pmatrix}$. By symmetry 2 is cubic, contradicting lemma 2.17.

□

Lemma 4.11. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing a subgraph, H , homeomorphic to the graph of figure 4.14 where H contains a vertex 3 as shown.

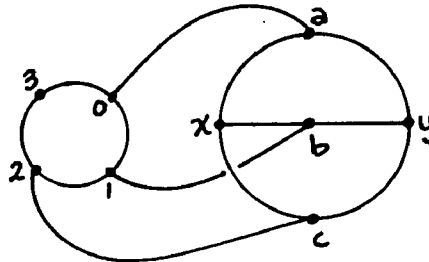


Figure 4.14

Proof. By way of contradiction suppose G were such a graph. Up to symmetry vertex 3 may connect only to ay, a or y forming cases 1, 2 and 3 respectively.

Case 1. $(3, ay)$. Note the symmetry $(1\ a)(2\ ay)(b\ x)(c\ y)$. If $(0, 3)$ is not an edge we get either a θ -graph disjoint from a k -graph or a contradiction of corollary 3.11. Likewise neither vertices 0 nor 3 can connect anywhere else without an earlier case, so $[0, 3]$ is dead.

Next, if there were a vertex 23 any connection gives a contradiction of corollary 4.5. By lemma 2.18 vertex 2 is not cubic. We can only have edge $(2, x)$ or $(2, y)$, or else we get either lemma 4.8 or we contradict corollary 4.5. By the above symmetry and $(a\ y)(a\ ay)(0\ 3)(1\ 2)(b\ c)$ vertices $1, a, ay$ also cannot be cubic. Examining $\frac{G}{(0, 3)} \cup \{(1, a), (2, ay)\} \subset P$ as shown in figure 4.15 we have (without loss of generality) $(2, y), (c, a), (1, x)$ and (b, ay) . Since vertices $0, 3, 1, 2, a, ay$ are all dead any bridge additions occur on cycle (b, x, c, y) , the graph is projective with (b, x, c, y) null and disjoint from a θ -graph, by lemma 4.5 we must get a θ -graph disjoint from a k -graph.

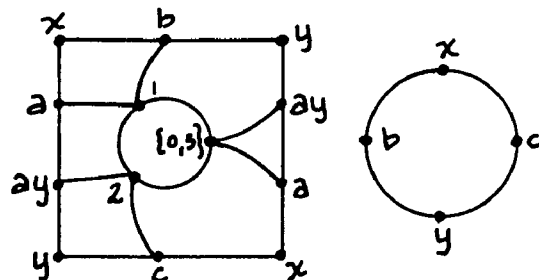


Figure 4.15

Case 2. $(3,a)$. If cycle $(0,a,3)$ is not a 3-cycle then we have the previous lemma. Thus vertices $0,3$ connect somewhere, avoiding the previous case implies they can only connect to x or y . If $(a,x),(3,x)$ then G contains a wedge $\begin{pmatrix} x & a \\ 0 & 3 \end{pmatrix} \vee \begin{pmatrix} b & c \\ a & 1 \end{pmatrix} \vee \begin{pmatrix} c & y \\ 1 & 2 \end{pmatrix}$, hence edges $(0,x),(3,y)$. Examining extensions of the embeddings of lemma 4.9 we get exactly 3 embeddings, shown in figure 4.16.

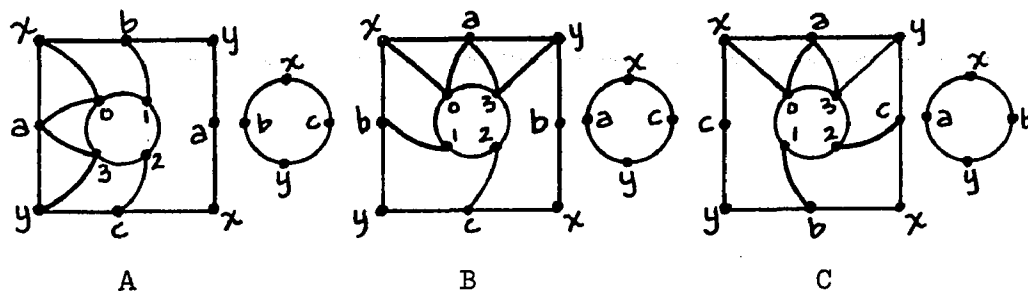


Figure 4.16

If there exists a vertex 12 then $(12,a)$, gives a wedge of $k_{2,3}$'s, $(12,x)$ or $(12,y)$ give a wedge $\vee k_{2,3}$'s and $(12,bx)$ gives a θ -graph disjoint from $\begin{pmatrix} 1 & bx \\ 12 & 0 \end{pmatrix} \vee \begin{pmatrix} 1 & bx \\ 12 & 0 \end{pmatrix}$. By lemma 2.18 either 1 or 2 is not cubic, without loss of generality, suppose 1 is not cubic. Either edge $(1,a)$ or $(1,c)$ give the previous lemma, edge $(1,y)$ gives $\begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b \end{pmatrix}$. Thus we have either edge $(1,x)$ or $(1,bx)$. Deleting this bridge and embedding with either an extension of A or B gives a k -graph disjoint from a θ -graph

(k-graph on cycle $(0,1,b,x)$) hence we must have an extension of C , yet this implies (b,c) and a θ -graph disjoint from $\begin{pmatrix} 1 & x \\ b & c \end{pmatrix}$.

Case 3. $(3,y)$. By elimination vertex 3 may only connect to x . Suppose $(3,x)$ is in G . $\overline{St(1)}$ is disjoint from $\begin{pmatrix} x & y \\ a & c \end{pmatrix}_3$ implies $(1,bx)$ or else we get an earlier lemma. Cycle $(1,b,bx)$ is a 3-cycle since it is disjoint from a $K_{3,3} \setminus e$, vertices (b,bx) may only connect to (x,a,y,c) . Edge (b,a) gives a θ -graph disjoint from $\begin{pmatrix} c & 3 \\ 2 & x \end{pmatrix}_y$ hence $(b,x), (bx,y)$ and G contains a wedge $\begin{pmatrix} 3 & c \\ 2 & x \end{pmatrix}_y \vee \begin{pmatrix} y & 1 \\ b & bx \end{pmatrix}_a$. Thus vertex 3 is cubic.

If vertex 0 is not cubic note $(0,y)$ implies a wedge $\begin{pmatrix} 0 & y \\ 0y & a \end{pmatrix}_3 \vee \begin{pmatrix} b & c \\ x & y \end{pmatrix}_z$, so 0 may connect only to ax, x . If $(0,ax)$ then $(0,a,ax)$ disjoint from a $K_{3,3} \setminus e$ implies it is a 3-cycle; so a must connect elsewhere. Edge (a,x) gives a θ -graph disjoint from $\begin{pmatrix} 2 & y \\ 3 & c \end{pmatrix}_1$, a connecting to cycle $(0,1,2,3)$ gives an earlier lemma, (a,b) is case 1 if we examine $\begin{pmatrix} a & x \\ b & ax \end{pmatrix}_y$, and (a,c) is case 1 by examining $\begin{pmatrix} a & x \\ c & ax \end{pmatrix}_y$. Hence under the supposition 0 is not cubic so we get $(0,x)$.

If $(0,x)$ is in G note $(2,x)$ gives a wedge of k -graphs, $(2,y)$ makes the dead vertex 3 cubic in a 3-cycle, and $(2,cx)$ is symmetric to $(0,ax)$ which was just ruled out. Vertices 2,3 cubic implies $\frac{G}{(2,3)} \cup \{(0,y), (1,c)\}$ is projective, the unique embedding with cycles $(\{2,3\}, 0,y), (\{2,3\}, 1,c)$ essential is shown in figure 4.17.

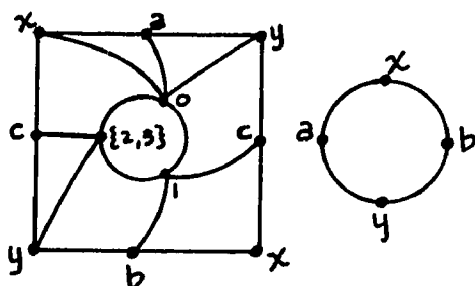


Figure 4.17

Vertex a is cubic in a 3-cycle (if cycle $(a,x,0)$ contains a fourth vertex we apply the previous lemma), by figure 4.17 we have either (a,b) or (a,by) . The latter gives case 1 if we use the k -graph $\begin{pmatrix} by & x \\ a & b & c \end{pmatrix}$, thus (a,b) , and we get the symmetry $(0,y)(c,1)(b,x)$. Finally note $(1,y)$ gives a wedge $\begin{pmatrix} 2 & y \\ 3 & c & 1 \end{pmatrix} \vee \begin{pmatrix} 0 & b \\ 1 & a & x \end{pmatrix}$, $(1,cx)$ gives cycle $(2,3,y,c)$ disjoint from $\begin{pmatrix} 1 & x & a \\ 0 & b & cx \end{pmatrix} \setminus (a,cx), (1,bx)$ is equivalent to $(0,ax)$ if we use $\begin{pmatrix} 0 & bx \\ 1 & x & a \end{pmatrix}$, hence 1 may only connect to x . If 1 and c are both non-cubic we get $\overline{St(3)}$ disjoint from $\begin{pmatrix} b & x \\ a & c & 1 \end{pmatrix}$ hence without loss of generality 1 is not cubic. $[1,2]$ dead contradicts lemma 2.18.

We conclude $(0,x)$ is not in G , vertex 0 is dead, by symmetry so is vertex 2 . We shall complete the lemma by showing vertex 1 is cubic and hence contradicting lemma 2.18. Observe

$$\frac{G}{(2,3)} \cup \{(1,c), (0,y)\} \text{ embeds in } P_1 \text{ as does } \frac{G}{(3,0)} \cup \{(2,y), (1,a)\}.$$

The unique embeddings are shown in figure 4.18. We see vertex 1 may only connect to x or bx , regardless vertex b connects elsewhere.

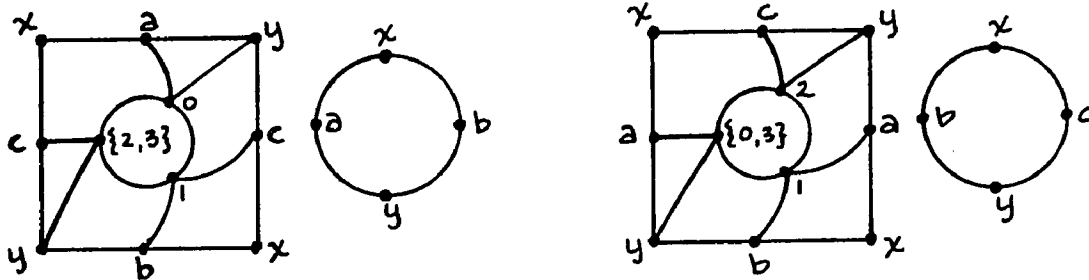


Figure 4.18

The only two possibilities, $st(a)$ or $st(c)$ do not extend one of the above embeddings. Thus l is cubic, and the proof of the lemma is complete.

□

§4.3 A 4-cycle Disjoint From a k_4

Proposition 4.12. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H_4 and containing an n -cycle $\perp k_4$ for $n \geq 4$.

Proof. By way of contradiction suppose G were such a graph. Label the cycle and the k_4 as in figure 4.19, call this graph H . First we will show there is not a vertex v disjoint from H .

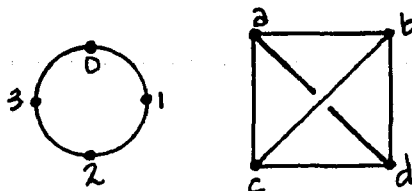


Figure 4.19

Let v be such a vertex. Note v may not connect twice to the cycle. If (v,a) and (v,b) then v must be in the same component of $G \setminus \text{st}(k_4)$ as the "missing" vertex of the k_4 , else a and b form a cut set separating v and this vertex, contradicting lemma 2.14. Cycle $(0,1,2,3)$ is disjoint from $\begin{pmatrix} a & b \\ c & d & v \end{pmatrix}$ contradicting proposition 4.6. If $(v,a), (v,b)$, and (v,c) are in G , note v must be the "missing" vertex of the k_4 , else $\begin{pmatrix} v & d \\ a & b & c \end{pmatrix}$ is a $k_{2,3}$ disjoint from a 4-cycle. The remaining arc (v,d) must intersect cycle $(0,1,2,3)$ by lemma 2.19. Also note no bridge hits the cycle $(0,1,2,3)$ and the interior of an edge of the k_4 , else the k_4 is

a $k_{2,3}$. If $(v,0)$ and $(d,0)$ then without loss of generality $(1,a), (2,a), (3,b)$. Avoiding 1 cubic in a 3-cycle implies $(1,b)$ giving a wedge $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \vee \begin{pmatrix} 1 & 3 \\ b & 0 \end{pmatrix}$. Thus $(v,0)$ and $(d,0)$ are not both in G . Note each of the vertices $0,1,2,3$ must connect to the k_4 , either because they are degree 2 or are cubic vertices disjoint from a k_4 . Also note not all of a,b,c connect to the cycle or $\begin{pmatrix} v & d \\ a & b \end{pmatrix}$ is a k -graph, without loss of generality suppose c does not connect to the cycle. Whichever pair out of $0,1,2,3$ which are not adjacent to $\{v,d\}$ must (by avoiding a cubic vertex in a 3-cycle) be adjacent to without loss of generality a , creating a wedge of k -graphs. Thus there is not a vertex disjoint from H .

We summarize our knowledge of G : 1) there does not exist a vertex v disjoint from H , 2) an edge of the k_4 is an edge of G , 3) all bridges are edges joining the cycle to $\{a,b,c,d\}$. With this information the proof shall proceed based on the valency of vertex 0 .

If $(0,a), (0,b), (0,c), (0,d)$ then without loss of generality G contains edge $(2,a)$. $\begin{pmatrix} 0 & 2 \\ 1 & 3 \end{pmatrix}$ must be a k -graph, because if vertex 1 only connects to a it is cubic in a 3-cycle.

If $(0,a), (0,b), (0,c)$ then again edge $(2,a)$ and k -graph $\begin{pmatrix} 0 & 2 \\ a & 1 \end{pmatrix}$, as $(2,d)$ has $\overline{\text{st}(2)}$ disjoint from $\begin{pmatrix} 0 & a \\ b & c \end{pmatrix}$.

If $(0,a), (0,b)$ then suppose $(2,c), (2,d)$. Vertices 1 and 3 must connect somewhere twice. Edges $(1,a)(1,b)$ imply $(3,c)$ giving a θ -graph disjoint from $\begin{pmatrix} 0 & 1 \\ a & b \end{pmatrix}$. Edges $(1,a)(1,c)$ imply

$(3,b), (3,d)$. Any other edge addition yields a wedge $k_{2,3} \vee k_4$, yet this graph is projective. We conclude vertex 2 is cubic, say $(2,c)$. Both vertices 1 and 3 cannot connect to d, say $(1,d)$ note c is dead else 2 is cubic in a 3-cycle. Thus $(3,a), (3,b)$ are the only possible edge additions, yet the resulting graph is projective.

We have G contains edges $(0,a), (1,b), (2,c), (3,d)$ and there exists a vertex 0_1 . This vertex can only connect to a or b , or else we get a wedge $k_{2,3} \vee k_4$, suppose $(0_1,a)$. Now 0_1 is dead and cubic, hence there exists $v \in (0_1,a)$ and (v,a) giving θ -graph disjoint from $\begin{pmatrix} a & b \\ c & 2 \end{pmatrix}$.

□

For easy reference we summarize the results of proposition 4.6 and proposition 4.12 in the following corollary.

Corollary 4.13. There does not exist a $G \in I_{*}^M(P)$ satisfying H_4 and containing an n -cycle disjoint from a k -graph for $n \geq 4$.

§4.4 A 3-cycle Disjoint From a $k_{2,3}$

The goal of this section is to prove proposition 4.18, concerning G contains a cycle disjoint from a $k_{2,3}$. We shall first prove the partial result where G contains a $K_{3,3}$, $\begin{pmatrix} a & b & c \\ 0 & x & y \end{pmatrix}$, and a 3-cycle $(1,2,3)$ disjoint from $\begin{pmatrix} a & b & c \\ 0 & x & y \end{pmatrix} \setminus (0,c)$.

Lemma 4.14. There does not exist a $G \in I_*^M(P)$ satisfying H_4 and containing a subgraph homeomorphic to the graph of figure 4.20.

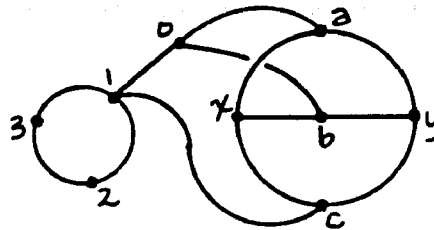


Figure 4.20

Proof. By way of contradiction let G be such a graph. Note $(1,2,3)$ is a 3-cycle of G by corollary 4.13, so 2 and 3 are of valency at least 4. Observe neither 2 or 3 may connect to $st(0), st(c)$ or we get a θ -graph disjoint from a k -graph. Also, if 2 and 3 are both adjacent to $st(a)$ we get a 4-cycle disjoint from a k -graph. Note the symmetries $(x y), (a b), (2 3)$, and $(c 0)(x a)(b y)$. Finally each embedding of $(1,2,3) \parallel \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ given by lemma 4.9 extends in 2 ways to an embedding of $((1,2,3) \parallel \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}) \cup st(0)$, the 12 embeddings are given in figure 4.21.

Case 1. $(2, ax)$. Vertex 3 connects twice to $[b, y]$. The two unique embeddings are shown in figure 4.22. The remaining bridge with

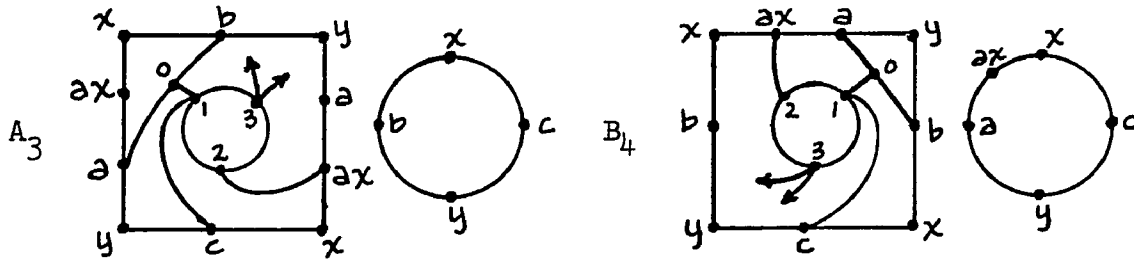


Figure 4.22

v.o.a. 2 must connect to either a or x . From the embeddings $(2, 0)$ implies $(ax, 0), (ax, 1), (ax, 3)$ or (ax, y) . The first 3 contain cycle (x, b, y, c) disjoint from a k -graph, contradicting corollary 4.13, and the fourth contains cycle $(2, a, 0, 1)$ disjoint from $\begin{pmatrix} x & y \\ ax & b & c \end{pmatrix}$. Bridge $(2, x)$ implies either $(ax, b), (ax, c)$ or $(ax, 1)$. The first contains a θ -graph disjoint from $\begin{pmatrix} 1 & y \\ c & 0 & 3 \end{pmatrix}$, the second a θ -graph disjoint from $\begin{pmatrix} 2 & x \\ ax & c \end{pmatrix}$, and the third a θ -graph disjoint from $\begin{pmatrix} 2 & x \\ ax & 1 \end{pmatrix}$. Case 1 is done.

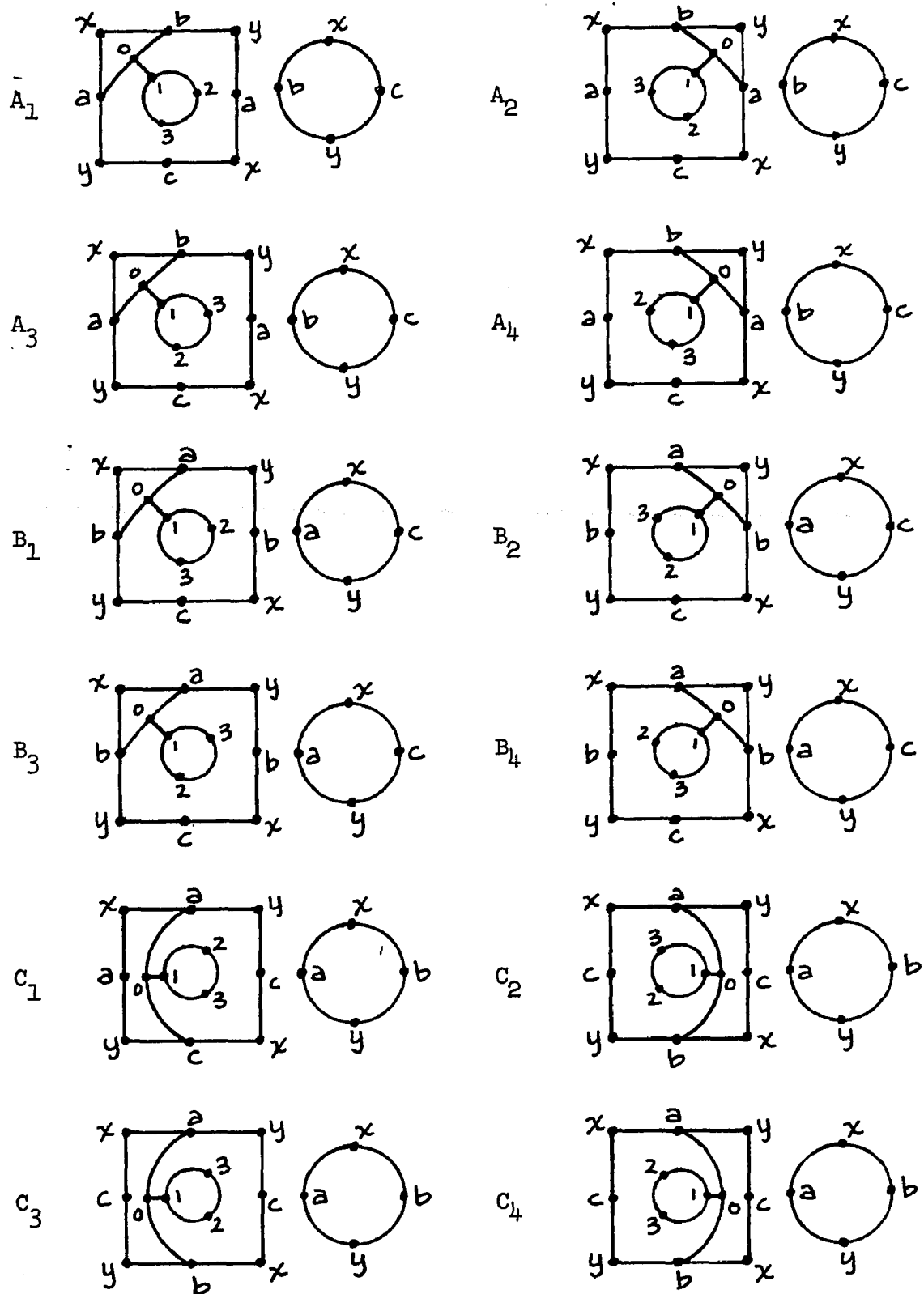


Figure 4.21

Case 2. Since 2 and 3 both connect only to cycle (a,x,b,y) and they must connect to existing vertices by case 1, we have either $(2,x),(2,y),(3,a),(3,b)$ or $(2,x),(2,c),(3,y),(3,b)$. The former graph contains cycle $(x,c,y,2)$ disjoint from $\begin{pmatrix} 0 & 3 \\ 1 & a & b \end{pmatrix}$ hence the latter. Note vertices 2 and 3 are dead, and we have the unique embeddings A_3, B_4 based on the symmetry $(x\ y)(2\ 3)(0\ c)$.

Suppose vertex 0 is not cubic. If $(0,c)$ then either $(1,b),(1,x),(1,a)$, or $(1,y)$. All are symmetric and $(0,c)(1,b)$ give a θ -graph disjoint from $\begin{pmatrix} a & c \\ x & y & 0 \end{pmatrix}$. If edge $(0,ax)$ then B_2 is a unique embedding with ax connecting somewhere. Any such connection gives a θ -graph disjoint from a k -graph. If $(0,x)$ then we have either $(1,b),(c,b)$, or (a,b) . The first graph contains a θ -graph disjoint from $\begin{pmatrix} 1 & y \\ 3 & b & c \end{pmatrix}$, the second a θ -graph disjoint from $\begin{pmatrix} c & 3 \\ 1 & b & y \end{pmatrix}$, and the third a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ x & b \end{pmatrix}$. By S_0 -independent arguments $st(0)$ and by symmetry $st(c)$ are dead.

We know $1,2,3,0,c$ and star thereof are all dead. Edge (a,b) implies a vertex ab , without loss of generality (ab,y) and cycle $(c,y,3,1)$ is disjoint from $\begin{pmatrix} a & b \\ ab & x & 0 \end{pmatrix}$. If (a,bx) then G contains a θ -graph disjoint from $\begin{pmatrix} a & b \\ bx & 0 & y \end{pmatrix}$. This exhausts the possibilities.

□

Lemma 4.15. There does not exist a $G \in I_*^M(P)$ satisfying H^4 and containing a subgraph homeomorphic to the graph of figure 4.23.

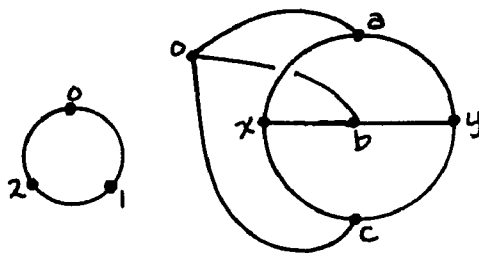


Figure 4.23

Proof. To avoid a θ -graph disjoint from a k -graph each vertex $\{0,1,2\}$ must connect to adjacent vertices on $\begin{pmatrix} 0 & x & y \\ a & b & c \end{pmatrix}$. Without loss of generality we assume G contains edges $(1,0)$ and $(1,c)$ and apply lemma 4.14.

□

Lemma 4.16. There does not exist a $G \in I_{*}^M(P)$ satisfying H^4 and containing a subgraph homeomorphic to the graph of figure 4.24.

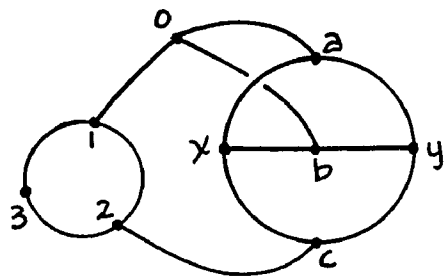


Figure 4.24

Proof. By way of contradiction let G be such a graph. The vertex 3 must connect twice to cycle (a, x, b, y) or else G contains a θ -graph disjoint from a k -graph. Note the symmetry $(1\ 2)(c\ 0)(a\ x)(b\ y)$. We break the proof into three cases; $(3, ax)$ case 1, $(3, x)(3, y)$ case 2, and $(3, a)(3, x)$ case 3.

Case 1. $(3, ax)$. Again there are exactly 12 embeddings of the subgraph given in the lemma, the 5 admitting $(3, ax)$ are given in figure 4.25. We examine where vertex 2 may connect. Edge $(2, b)$

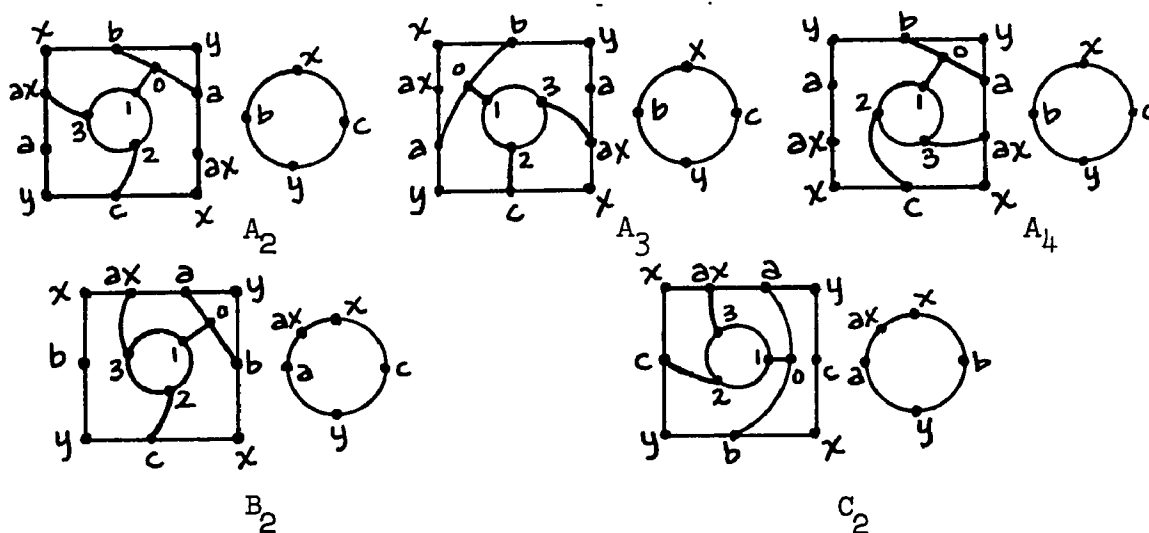


Figure 4.25

gives a 4-cycle disjoint from $\begin{pmatrix} b & c \\ x & y & 2 \end{pmatrix}$. Edges $(2,a), (2,ax)$ imply 3 connects again to $[a,ax]$ or else G contains a k -graph disjoint from cycle (y,b,x,c) , yet such a connection creates a θ -graph disjoint from $\begin{pmatrix} b & c \\ x & y & 1 \end{pmatrix}$. Hence 2 connects only $[x,c) \cup (c,y]$, by symmetry 1 connects only to $[a,0) \cup (0,b]$. Regardless of where 2,1 connect $c,0$ respectively must connect to cycle (a,y,b,x,ax) .

Suppose $(2,x)$ we examine where c connects. Edge (c,ax) gives cycle $(0,a,y,b)$ disjoint from $\begin{pmatrix} 2 & ax \\ c & x & 3 \end{pmatrix}$, and (c,bx) gives either cycle (c,bx,b,y) disjoint from $\begin{pmatrix} 1 & ax \\ a & 3 & 2 \end{pmatrix}$ or a bridge from 3 to $[ax,x]$ and a wedge $\begin{pmatrix} x & 3 \\ 2 & ax \end{pmatrix} \vee \begin{pmatrix} 0 & y \\ a & b & 2 \end{pmatrix}$. If (c,a) then this pair of edges does not eliminate any of the 5 embeddings, hence deleting $(2,x)$ implies either $(c,1), (c,0)$, or $(c,3)$, all of which give contradictions. If (c,b) then again this pair of edges does not eliminate any of the embeddings, c must connect elsewhere, yet the possibilities are exhausted. We conclude we cannot have $(2,x)$, hence $(2,y)$ and $(1,b)$.

Vertex 0 now connects to the cycle (a,y,b,x,ax) . If edge $(0,y)$ then G contains a θ -graph disjoint from $\begin{pmatrix} 0 & ax \\ 1 & a & b \end{pmatrix}$, hence either edges $(c,a), (c,b)$ or (c,ax) . Edge (c,b) is transferable in each embedding, since $st(1)$ and $st(2)$ are now dead. By examining the embeddings we get (x,y) or $(x,0)$. The former graph contains a θ -graph disjoint from $\begin{pmatrix} b & c \\ x & y \end{pmatrix}$ and the latter graph a θ -graph disjoint from $\begin{pmatrix} 0 & ax \\ a & x & 3 \end{pmatrix}$. Edge (c,ax) is transferable in every embedding, hence we get either edges $(x,0), (x,y)$ or (x,a) . The

first gives a θ -graph disjoint from $\begin{pmatrix} 2 & y \\ ax & c \end{pmatrix}$, the second a θ -graph disjoint from $\begin{pmatrix} c & ax \\ x & y \end{pmatrix}$, and the third a θ -graph disjoint from $\begin{pmatrix} c & ax \\ x & a \end{pmatrix}$. Hence G contains edge (c,a) and by symmetry edge $(0,x)$, and cycle $(b,x,0,1)$ is disjoint from $\begin{pmatrix} a & 2 \\ 3 & c & y \end{pmatrix}$ contradicting corollary 4.13.

Case 2. $(3,x),(3,y)$. Checking extensions of the embeddings in figure 4.21 we see there are exactly 4 embeddings of our subgraph, shown in figure 4.26.

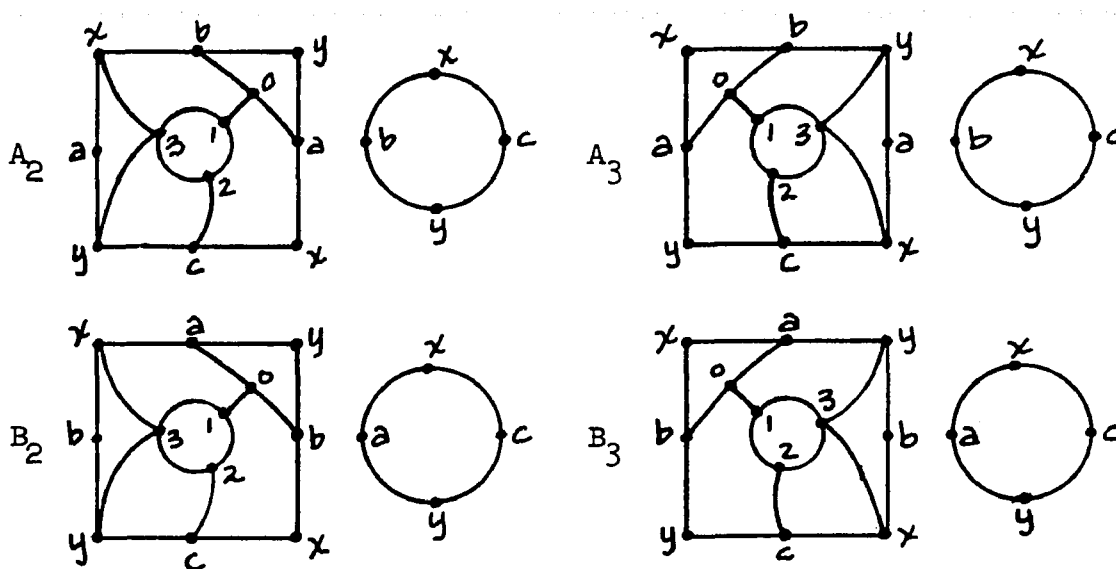


Figure 4.26

By lemma 4.14 we do not have edge $(1,c)$ in G . If $(1,x)$ then cycle $(3,2,c,y)$ is disjoint from $\begin{pmatrix} 0 & x \\ a & b & 1 \end{pmatrix}$, so by S_x -independence we have $(1,a)$. If $1a$ then $(1a,b)$ giving a θ -graph disjoint from $\begin{pmatrix} a & b \\ x & 0 & 1a \end{pmatrix}$, hence $(1,a)$ is an edge, not an arc, of G . Next, note examining cycle $(0,1,a)$ disjoint from

$\begin{pmatrix} 3 & b & c \\ 2 & x & y \end{pmatrix} \setminus (2,b)$ shows 2 cannot connect to a or b, hence without loss of generality we have edge $(2,x)$. Now, deleting $(1,a)$ and embedding implies either edge $(0,x), (0,ax), (0,ay), (0,y)$ or $(0,cy)$. The first contains a wedge $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$, the next two contain a θ -graph disjoint from $\begin{pmatrix} x & y \\ b & c & 3 \end{pmatrix}$, $(0,cy)$ gives the previous lemma, hence G contains edge $(0,y)$. Deleting $(0,y)$ implies either $(1,b), (c,b), (2,b)$ or $(b,3)$, we need only consider embeddings A_3, B_3 or else G contains a θ -graph disjoint from k -graph on cycle $(a,y,b,0)$. The first gives a θ -graph disjoint from $\begin{pmatrix} 3 & c \\ 2 & x & y \end{pmatrix}$, the second a θ -graph disjoint from $\begin{pmatrix} 0 & a \\ 1 & y \end{pmatrix}$, the third a θ -graph disjoint from $\begin{pmatrix} 2 & x \\ 3 & b & c \end{pmatrix}$, hence $(b,3)$. By the symmetry $(0\ 2)(a\ 3)(x\ y)(b\ c)$ we have (a,c) giving a θ -graph disjoint from $\begin{pmatrix} 2 & a \\ 1 & c & x \end{pmatrix}$.

Case 3. $(3,a), (3,x)$. The graph embeds in exactly 6 ways, shown in figure 4.27. Vertex 3 is dead or else either case 1 or case 2 applies. Note the symmetry $(1\ 2)(0\ c)(a\ x)(b\ y)$.

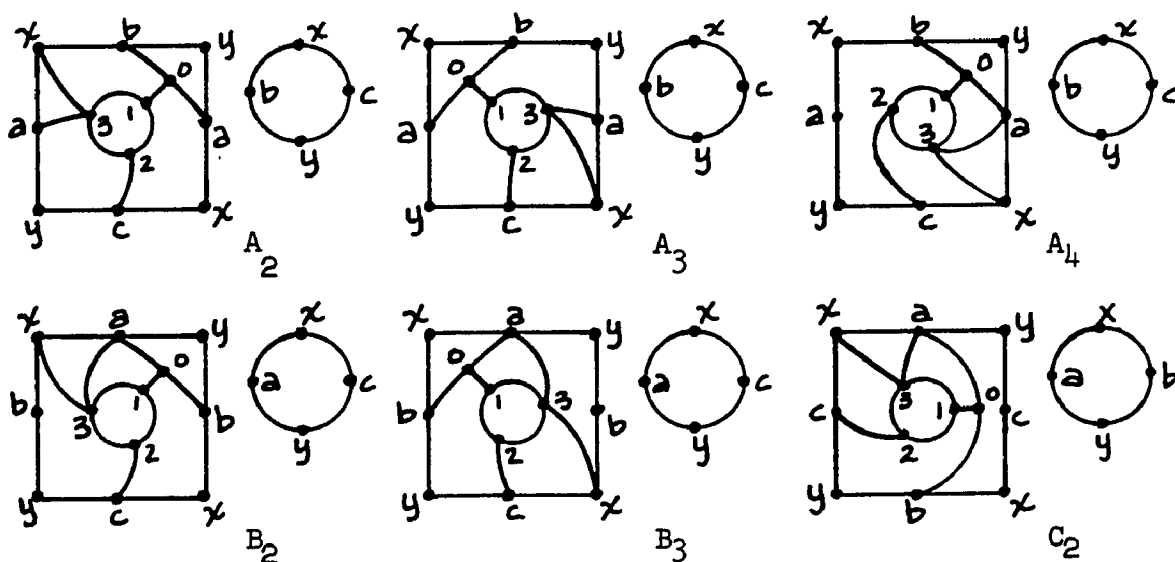


Figure 4.27

If $(1,y)$ is in G then $\begin{pmatrix} 0 & y \\ a & b \end{pmatrix}_1$ disjoint from cycle $(3,2,c,x)$. Edge $(1,c)$ gives lemma 4.14, edges $(1,2)$ or $(1,3)$ give a θ -graph disjoint from a k_4 -graph. Edge $(1,x)$ creates a 4-cycle (a,y,b,c) disjoint from a K_4 , in order that the K_4 is not a k_4 k -graph G must have edge $(2,x)$. Vertex c is cubic in a 3-cycle, and any connection creates $\begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$, a k_4 . Thus we have $(1,v)$ for $v \in [a,0) \cup (0,b]$, by symmetry $(2,v')$ for $v' \in [x,c) \cup (c,y]$.

If $(1,b)$ note $(1,0,b)$ disjoint from $\begin{pmatrix} 3 & c \\ 2 & x \end{pmatrix}_y$ implies 0 is not cubic. Edge $(0,c)$ is lemma 4.14, edges $(0,2)$ or $(0,3)$ give a θ -graph disjoint from a k -graph. Edge $(0,ax)$ is case 1, viewing cycle $(1,2,3)$ disjoint from $\begin{pmatrix} ax & b & c \\ 0 & a & x \end{pmatrix} \setminus (c,0)$; likewise $(0,ay)$ is case 1 upon a similar symmetry. If $(0,by)$ then considering $\begin{pmatrix} a & by \\ x & y \end{pmatrix}_0$ in place of $\begin{pmatrix} a & b \\ x & y \end{pmatrix}_0$ shows that it is the equivalent to adding $(1,bx)$, which was previously eliminated; likewise $(0,x)$ is the same case as edge $(1,by)$. Note 0 may only connect to x or y . If $(0,x)$ then $(2,x)$ gives a wedge $\begin{pmatrix} x & y \\ a & b \end{pmatrix}_c \vee \begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix}$, hence let us assume $(2,y)$. Vertex c connects to either a or b , the former giving a θ -graph disjoint from $\begin{pmatrix} 2 & c \\ y & a \end{pmatrix}$ and the latter a θ -graph disjoint from $\begin{pmatrix} 0 & 3 \\ 1 & a \end{pmatrix}_x$. We conclude $(0,y)$, and examine where 2 connects. If $(2,x)$ then c connects to either a or b , yet (c,a) contains a cycle $(1,0,y,b)$ disjoint from $\begin{pmatrix} 2 & a \\ c & x \end{pmatrix}_3$. Also note edge $(2,y)$ is symmetric to edge $(1,b)$, hence we also conclude (c,b) . Regardless of $(2,x)$ or $(2,y)$, deleting (c,b) and examining the embeddings

implies either (x,y) or $(x,1)$. The former graph contains a θ -graph disjoint from $\begin{pmatrix} x & y \\ b & c \end{pmatrix}$ and the latter has been previously considered. We conclude $(1,b)$ is not in G .

In case 3 we have now forced $(1,a)$, and by symmetry $(2,x)$. Deleting $(2,x)$ and examining the embeddings implies either $(c,3), (c,0), (c,b)$ or (c,a) . The first two have been previously considered. The third edge is transferable, deleting it and embedding gives $(x,y), (y,2)$ or $(y,3)$, giving a θ -graph disjoint from $\begin{pmatrix} b & c \\ x & y \end{pmatrix}$ and two previous cases respectively. Hence we have (c,a) , by symmetry $(x,0)$ and G contains a wedge $\begin{pmatrix} x & 1 \\ 2 & 3 \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$. This completes case 3 and the proof of the lemma.

□

Corollary 4.17. There does not exist a $G \in I_*^M(P)$ satisfying H_4 containing a $K_{3,3} \begin{pmatrix} a & b & c \\ 0 & x & y \end{pmatrix}$ and a cycle disjoint from $\begin{pmatrix} a & b & c \\ 0 & x & y \end{pmatrix} \setminus (0,c)$.

Proof. The result follows immediately from corollary 4.5, lemma 4.14, lemma 4.15, and lemma 4.16.

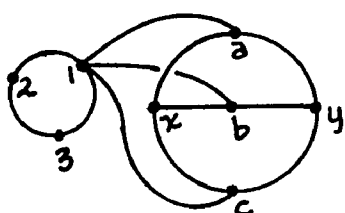
□

Proposition 4.18. There does not exist a $G \in I_*^M(P)$ satisfying H_4 containing a 3-cycle disjoint from a $K_{2,3}$.

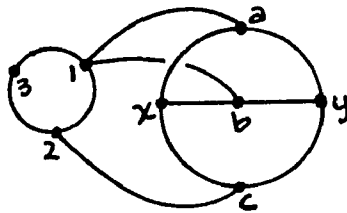
Proof. The vertex in the $K_{3,3}$ missing from the $K_{2,3}$ must be on the 3-cycle by corollary 4.17. The proposition naturally breaks

into three cases, illustrated in figure 4.28. Each case shall be covered in a separate lemma. The proofs of these lemmas will complete the proof of this proposition.

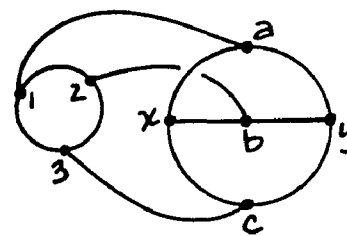
□



lemma 4.19



lemma 4.20



lemma 4.21

Figure 4.28

Lemma 4.19. There does not exist a $G \in \Gamma_*^M(P)$ satisfying H^4 and containing a subgraph homeomorphic to the graph of figure 4.29.

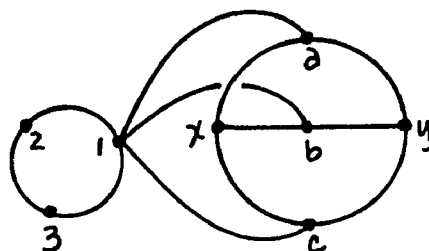


Figure 4.29

Proof. By way of contradiction suppose G was such a graph. Note there exists vertices $2,3$ as shown and by proposition 4.6 $(1,2,3)$ is a 3-cycle. We shall break into cases depending on where vertices $2,3$ connect. If $(2,ax)$ then we cannot have $(3,ax)$ by corollary 4.17, $(3,a)$ and $(3,x)$ give proposition 4.6. Hence we have either $(3,by)$ (case 1) or $(3,b)$ and $(3,x)$ (case 2) since $(3,b)$ and $(3,c)$ give cycle $(1,2,ax,a)$ disjoint from $(x^b y^c 3)$. If edges $(2,a)$ and $(2,b)$ then G must have edges $(3,c), (3,y)$ (case 3) since $(3,x), (3,y)$ gives $E_{18} \cup (1,2)$. Edges $(2,x), (2,a)$ give either $(3,c), (3,y)$ (case 4) or $(3,x), (3,y)$ (case 5). If G does not contain $(2,a), (2,x)$ then G contains $(2,x), (2,y), (3,x), (3,y)$ giving $E_3 \cup (2,3)$. The five cases are illustrated in figure 4.30.

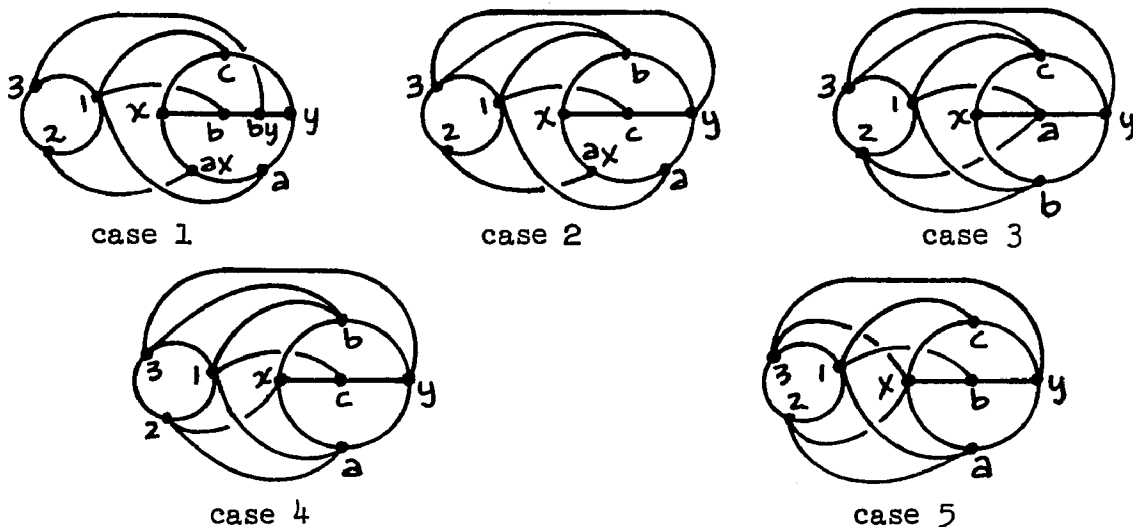


Figure 4.30

Case 1. $(2,ax), (3,by)$. Vertex 3 may only connect to $[b,y]$. If $(3,b)$ then cycle $(3,b,y)$ is disjoint from $(\begin{smallmatrix} a & c \\ 1 & y \end{smallmatrix} \begin{smallmatrix} 2 \\ ax \end{smallmatrix}) \setminus (2,y)$, hence edge $(3,y)$ and by symmetry $(2,x)$. We note by examining lemma 4.9 there are exactly 3 embeddings, shown in figure 4.31.

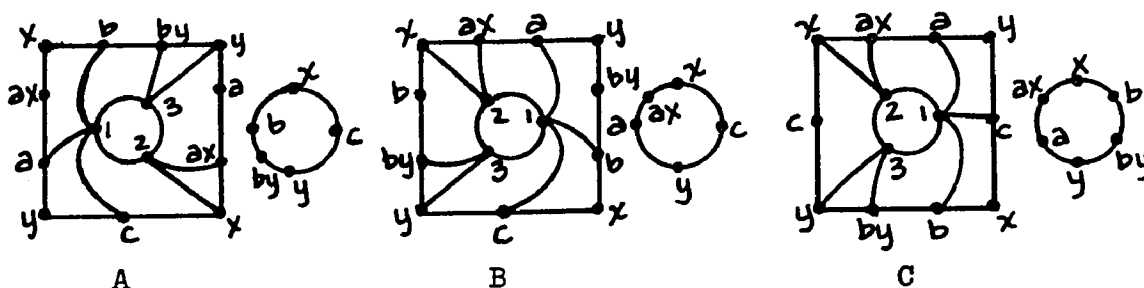


Figure 4.31

Deleting $(2,x)$ and checking the embeddings implies (ax,c) or (ax,b) . The former gives a 4-cycle disjoint from $(\begin{smallmatrix} 1 & y \\ b & a \end{smallmatrix} \begin{smallmatrix} 3 \end{smallmatrix})$ and the latter a 4-cycle disjoint from $(\begin{smallmatrix} 1 & y \\ c & a \end{smallmatrix} \begin{smallmatrix} 3 \end{smallmatrix})$.

Case 2. $(2,ax), (3,b), (3,y)$. We observe cycle $(3,b,y)$ is disjoint from $(\begin{smallmatrix} 1 & ax \\ 2 & a \end{smallmatrix} \begin{smallmatrix} x \end{smallmatrix})$ hence it must embed null. Thus there are exactly 3 embeddings corresponding to the 3 embeddings of figure 4.31. Again 2 cannot connect to b, ax, a, x or cy by similar arguments, hence $(2,c)$ and embedding C is unique.

Next note $\overline{st(ax)}$ is disjoint from $(\begin{smallmatrix} 1 & y \\ b & c \end{smallmatrix} \begin{smallmatrix} 3 \end{smallmatrix})$, so ax is not cubic. It cannot be a v.o.a. for an admissible bridge since the cycles bounding ax , $(ax,2,x,c), (ax,2,1,a)$ and (ax,a,y,b,x) , are

all disjoint from a 4-cycle, by lemma 2.15 we would get a contradiction of corollary 4.13. Avoiding cubic vertex in a 3-cycle implies either edge $(ax, cy), (ax, lc)$ or (ax, lb) . The first and third contradict corollary 4.17 and the second contains a θ -graph disjoint from $\begin{pmatrix} x & ax \\ lc & x & y \end{pmatrix}$.

Case 3. $(2, a), (2, b), (3, c), (3, y)$. We note there are exactly two embeddings, shown in figure 4.32, the two embeddings based on the symmetry $(a \ b)$.

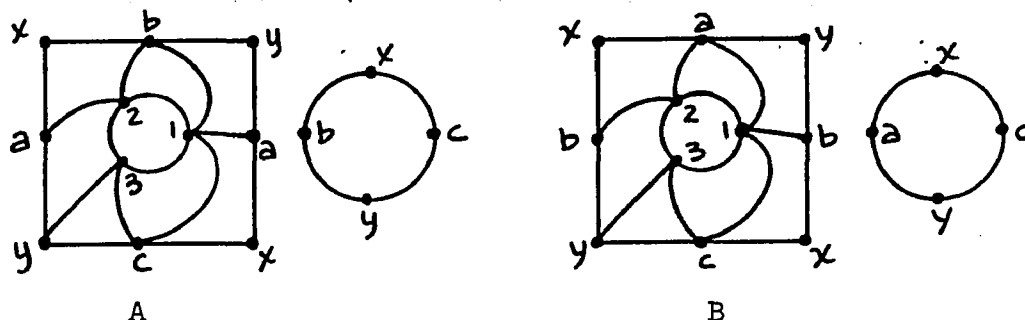


Figure 4.32

First we shall rule out the possibility of an admissible bridge. Note regions $(a, x, b, 2), (1, 2, 3), (a, y, b, 1), (1, 3, c), (3, c, y)$ occur in both embeddings and are either disjoint from 4-cycles or θ -graphs. Using lemma 2.15 an admissible bridge contradicts either corollary 4.13 or lemma 2.16. An admissible bridge on $(1, 2, b)$ creates a θ -graph disjoint from a k -graph. An admissible bridge on $(2, 3, y, a)$ which is not admissible on $(2, 3, y, b)$ gives $\begin{pmatrix} 1 & 2 \\ 3 & a \end{pmatrix}$ disjoint from cycle (c, x, b, y) , a bridge admissible on both gives $(a, 3)$ and G contains a wedge $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix} \vee \begin{pmatrix} a & 1 \\ 2 & 3 \end{pmatrix}$. A similar argument holds

on cycle $(1,a,x,c)$ except for the bridge $(1a,x)$, which gives a cubic vertex in a 3-cycle and $(b,c),(x,y)$ with a θ -graph disjoint from a k -graph. We conclude there is not an admissible bridge.

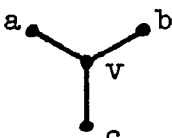
If a bridge inadmissible in either embedding has 3 or more v.o.a. then no two v.o.a. are on the same region, else there is a k -graph on that region. Recalling vertex 3 is dead implies 2 is also dead. If 1 is a v.o.a. then so is cy , and G contains a 4-cycle disjoint from $\begin{pmatrix} a & b \\ 1 & 2 & x \end{pmatrix}$. Since there does not exist a vertex cy , any 3 v.o.a. on $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ give 2 v.o.a. on a common region.

If a 2-bridge is inadmissible in either embedding involves a ninth vertex, by symmetry it is either ax, ay, cx or cy . If it is a vertex ax then $\overline{st(x)}$ is disjoint from $\begin{pmatrix} 1 & 2 \\ 3 & a \end{pmatrix}$, if cy then G contains a 4-cycle disjoint from $\begin{pmatrix} a & b \\ 1 & 2 & x \end{pmatrix}$. If it is a vertex ay then edge (ay,bx) gives a 4-cycle disjoint from $\begin{pmatrix} 1 & y \\ c & b & 3 \end{pmatrix}$ and if it is a vertex cx then $(cx,2)$ gives a 4-cycle disjoint from $\begin{pmatrix} 1 & 2 \\ 3 & c \end{pmatrix}$. Thus the 2-bridge is between existing vertices, hence it is $(3,x)$ giving $E_{18} \cup \{(1,2), (3,c)\}$.

Case 4. $(2,a),(2,x),(3,b),(3,y)$. Note cycle $(3,b,y)$ is disjoint from $\begin{pmatrix} 1 & x \\ c & 2 & a \end{pmatrix}$, hence it embeds null as does $(2,x,a)$. This graph has 3 embeddings corresponding to the embeddings of figure 4.31. Also note $st(3)$ is dead implies $st(2)$ is dead. If edge $(1,x)$ then G contains a wedge $\begin{pmatrix} 1 & x \\ 2 & a \end{pmatrix} \vee \begin{pmatrix} 1 & 3 \\ y & b \end{pmatrix}$, edge $(1,by)$

gives a 4-cycle disjoint from $\begin{pmatrix} 1 & x \\ c & 2 & a \end{pmatrix}$. Thus $st(1)$ is dead as well.

If there is a ninth vertex disjoint from our graph then it must connect to $[c,x] \cup [x,a]$, avoiding a cubic vertex in a triangle

implies  giving a wedge $\begin{pmatrix} 1 & 2 \\ 3 & b \end{pmatrix} \vee \begin{pmatrix} v & y \\ a & b & c \end{pmatrix}$. If the

ninth vertex is on the k-graph $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ then it is either cx or bx . If it is cx then edge (cx,v) for either $v \in st(x)$ or $v \in st(y)$. Regardless replacing the k-graph $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ with $\begin{pmatrix} v & y \\ a & b & c \end{pmatrix}$ gives an earlier case. Similarly the ninth vertex is not bx .

We conclude G has exactly 8 vertices and $(a,b), (a,c), (b,c)$ are the only possible bridge additions. $G \cup \{\text{these bridges}\}$ is still projective.

Case 5. $(2,c), (2,x), (3,x), (3,y)$. Note $st(3)$ is dead. Edge $(2,y)$ has been ruled out so $st(2)$ is also dead. Edge $(1,x)$ gives a wedge $\begin{pmatrix} 1 & y \\ a & b & c \end{pmatrix} \vee \begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix}$. Edge $(1,y)$ is equivalent under the symmetry $(1\ x)$ to (x,y) , which gives either (c,b) and a θ -graph disjoint from $\begin{pmatrix} x & y \\ b & c \end{pmatrix}$ or (c,a) which is case 4 under the symmetry $(3\ a)$. Thus $st(1)$ is also dead. By the aforementioned symmetries $st(c)$ and $st(x)$ are dead, so the only live vertices are $[b,y] \cup [y,c]$. Without loss of generality G contains edge (b,cy) which is a previous case if we replace the k-graph $\begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ with $\begin{pmatrix} x & cy \\ a & b & c \end{pmatrix}$. This completes the proof of case 5 and lemma 4.19.

□

Lemma 4.20. There does not exist a $G \in \mathcal{I}_*^M(P)$ satisfying H^4 and containing a subgraph homeomorphic to the graph of figure 4.33.

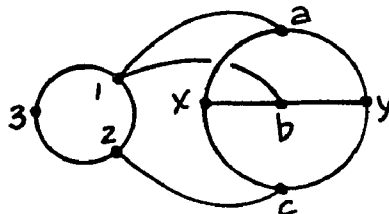


Figure 4.33

Proof. By way of contradiction suppose G were such a graph. Note there exists a vertex 3 as in figure 4.33. We shall break into cases depending on where vertices 2,3 connect.

Suppose $(3,c)$ is in G . If $(2,ax)$ then 3 may not connect to vertex a (cycle (x,b,y,c) is disjoint from $\begin{pmatrix} 1 & 2 \\ 3 & a \end{pmatrix}$), to vertex c (θ -graph disjoint from $\begin{pmatrix} a & b \\ x & y & 1 \end{pmatrix}$) or to vertex b (cycle $(1,2,ax,a)$ is disjoint from $\begin{pmatrix} b & c \\ x & y & 3 \end{pmatrix}$). Under the symmetry $(1\ c)(x\ a)(b\ y)$ this exhausts the possibilities, hence edge $(2,ax)$ is not in G . If $(2,a)$ is an edge of G then we have either $(3,b)$ (case 1) or $(3,x)$ (case 2).

Suppose $(3,c)$ is not in G . If $(2,ax)$ then the symmetry $(1\ 2)(a\ ax)(x\ y)(b\ c)$ implies 3 may only connect to $[x,a] \cup [a,ax] \cup [ax,y]$. If $(3,x)$ and $(3,y)$ then cycle $(1,2,ax,a)$ is disjoint from $\begin{pmatrix} x & y \\ b & c & 3 \end{pmatrix}$, so 3 must connect to $[a,ax]$. Avoiding cycle (x,b,y,c) disjoint from a k -graph implies $(3,a), (3,ax)$, and 3 is dead. Embedding $G \setminus (ax,2)$ gives cycle $(3,a,ax)$ null

as is $(3,1,2)$ (disjoint from k -graphs $\begin{pmatrix} b & c \\ 1 & x & y \end{pmatrix}, \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ respectively), hence we have a local embedding around 3 as in figure 4.34. This embedding extends to include $(2,ax)$, a contradiction. Hence G does not contain edge $(2,ax)$. If $(2,a)$ then again we do not have $(3,c), (3,b)$. If $(3,a)$ then $\begin{pmatrix} 1 & 2 \\ 3 & a \end{pmatrix}$ is disjoint from cycle (x,b,y,c) , hence $(3,x)$ and $(3,y)$ (case 3).

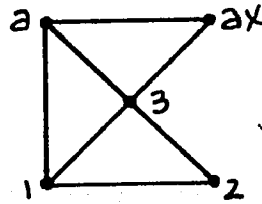


Figure 4.34

Under the supposition $(3,c)$ is not in G we know 2 may only connect to x or cx . If $(3,y)$ then cycle $(2,x,c)$ being disjoint from $\begin{pmatrix} 1 & y \\ a & b & 3 \end{pmatrix}$ gives an earlier case. If $(3,a)$ and $(3,b)$ then cycle $(2,x,c)$ disjoint from $\begin{pmatrix} a & b \\ 1 & 3 & y \end{pmatrix}$ shows we have in an earlier case. If $(3,x), (3,ax)$ then cycle $(3,x,ax)$ disjoint from $\begin{pmatrix} 1 & y \\ a & b & c \end{pmatrix}$ implies $(1,ax)$. Thus we have $(3,x), (3,a)$ (case 4). Note in case 4 we have $(2,x)$, as $(2,cx)$ gives cycle $(2,c,cx)$ disjoint from $\begin{pmatrix} a & b & 3 \\ 1 & x & y \end{pmatrix} \setminus (3,y)$. The 4 cases are illustrated in figure 4.35.

Note by lemma 4.9 we get 12 embeddings of the graph in figure 4.33, these embeddings are given in figure 4.36.

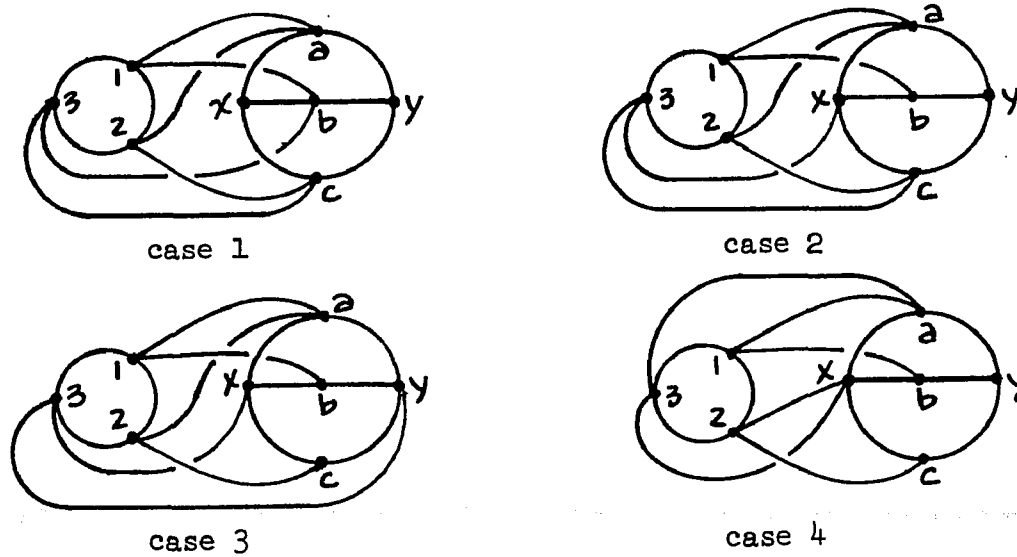


Figure 4.35

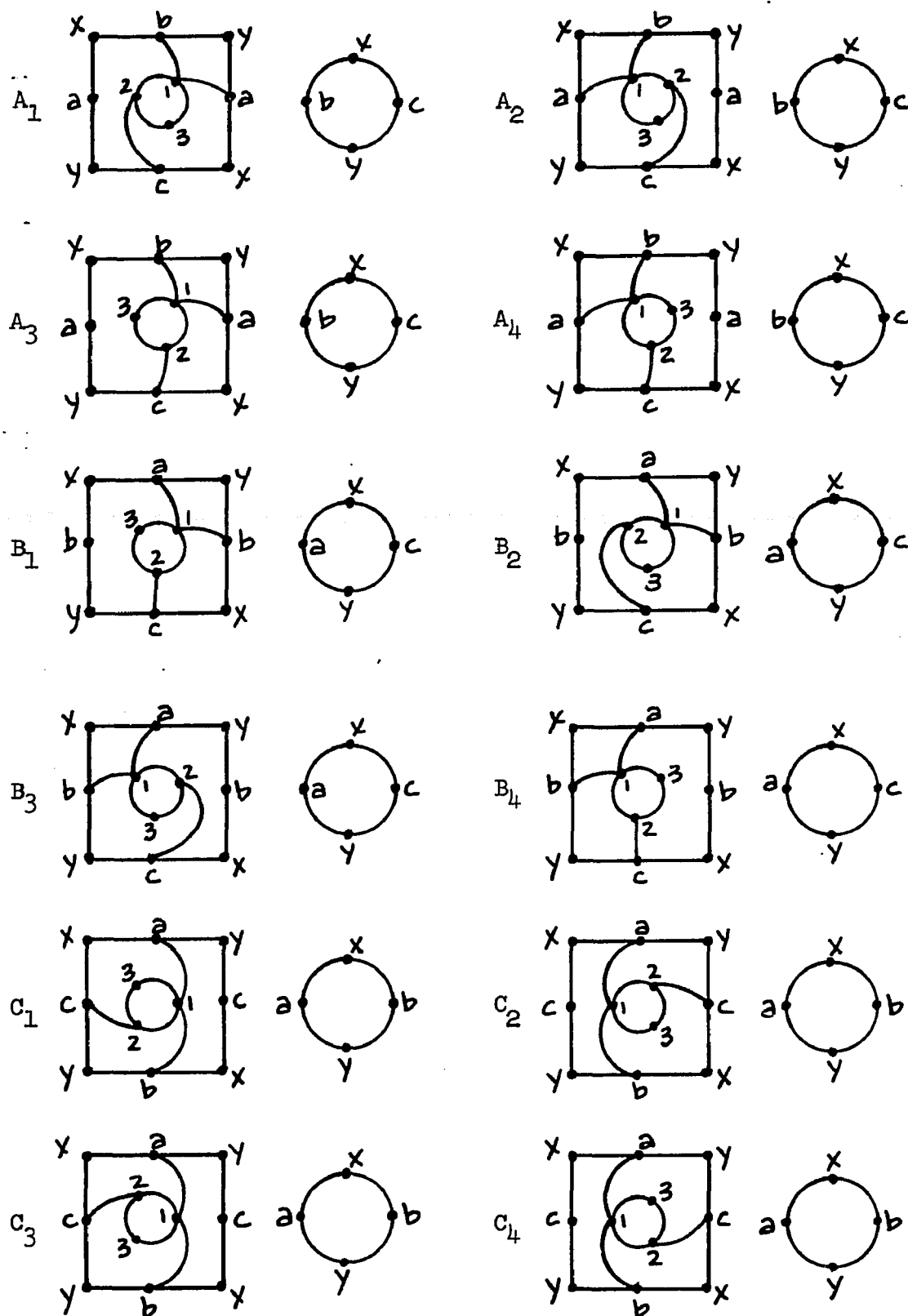


Figure 4.36

Case 1. $(2,a),(3,b),(3,c)$. Note cycles $(1,2,3),(1,2,a),(1,3,b),(2,3,c)$ are all disjoint from k -graphs, hence they must be 3-cycles.

Suppose there exists v disjoint from this subgraph. If $(v,2)$ then $(v,x),(v,y)$ giving cycle $(3,1,a,2)$ disjoint from $(\begin{smallmatrix} x & y \\ b & c \end{smallmatrix} v)$. If $(v,x),(v,y)$ and without loss of generality (v,a) then $(\begin{smallmatrix} x & y \\ b & c \end{smallmatrix} v)$ is disjoint from cycle $(a,1,3,2)$. If $(v,a),(v,b),(v,c)$ then cycle $(1,2,c,3)$ is disjoint from $(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix} v)$, hence v connects only to $[a,x] \cup [x,b]$. If v connects entirely in $[a,x]$ then there exists a vertex ax and without loss of generality edge (ax,c) . This gives the equivalent of vertex v connecting to $[a,x]$ and $[x,b]$ upon considering the k -graph $(\begin{smallmatrix} y & ax \\ a & b \end{smallmatrix} c)$. Thus without loss of generality there exists both ax, bx connecting somewhere. Edge (ax,b) gives $(\begin{smallmatrix} a & b \\ ax & v \end{smallmatrix} y)$ disjoint from cycle $(1,2,c,3)$, therefore G has edges (ax,c) and (ax,b) . G now contains cycle (c,ax,x,bx) disjoint from $(\begin{smallmatrix} a & b \\ v & x \end{smallmatrix} 1)$; hence we conclude there is not a vertex v disjoint from this subgraph.

If there is a ninth vertex on this subgraph then it must be, without loss of generality, cx . Examining cycle $(c,2,3)$ disjoint from $(\begin{smallmatrix} a & b \\ x & y \end{smallmatrix} 1)$ corollary 4.17 implies either (cx,a) or (cx,b) . If (cx,a) then either edge (ax,c) with cycle $(2,1,b,3)$ disjoint from $(\begin{smallmatrix} cx & ax \\ a & x \end{smallmatrix} c)$ or edge (ax,b) with vertex x disjoint from $(\begin{smallmatrix} a & b \\ ax & y \end{smallmatrix} 1)$ contradicting the previous paragraph. Hence (cx,a) and (cx,b) and again vertex x is disjoint from $(\begin{smallmatrix} y & cx \\ a & b \end{smallmatrix} c)$ contradicts the previous paragraph. Hence $|v(G)| = 8$.

If G contains edge (a,b) then vertices x and y are both cubic in a 3-cycle. Edge (x,y) gives cycle $(1,2,c,3)$ disjoint from $\begin{pmatrix} a & b \\ x & y \end{pmatrix}$. If edge $(x,1)$ then $(y,1)$, or else a θ -graph disjoint from $\begin{pmatrix} 1 & x \\ a & b \end{pmatrix}$. Deleting edge (a,b) and embedding implies edge (x,y) , a contradiction. If edges $(x,2)$ and $(y,2)$ then considering cycle $(2,3,c)$ disjoint from $\begin{pmatrix} a & b \\ 1 & x & y \end{pmatrix}$ gives lemma 4.19. Thus $(x,2), (y,3)$ and x, y are dead. Every edge addition has been ruled out, yet our graph is still projective.

If there is no edge of the type (a,b) then note again given edges $(2,x), (2,y)$ we can apply lemma 4.19 as above. If $(1,x), (2,x), (3,x)$ then G contains a wedge $\begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b & c \end{pmatrix}$ hence we have at most $(1,x), (2,x), (3,y)$ which gives a projective graph.

Case 2. $(2,a), (3,c), (3,x)$. Note 3 is dead by exhaustion of cases. By the symmetry $(c,1)(2,3)(a,x)(b,y)$ vertex 2 is also dead. We will show G contains exactly 8 vertices.

If cx is a vertex note (cx,y) gives a θ -graph disjoint from $\begin{pmatrix} x & y \\ a & b & cx \end{pmatrix}$. Edges $(cx,a), (cx,b)$ respectively are equivalent to edges $(3,bx)(3,ax)$ which was contradicted. By S -independent arguments there is no vertex cx . If cy is a vertex then again (cy,x) gives a θ -graph disjoint from $\begin{pmatrix} x & y \\ cy & a & b \end{pmatrix}$. Edge (cy,b) has cycle (cy,y,b) disjoint from $\begin{pmatrix} x & c \\ 2 & 3 \end{pmatrix}$ implies (cy,a) and cycle $(c,2,1,3)$ is disjoint from $\begin{pmatrix} a & b \\ x & y & cy \end{pmatrix}$. Edge (cy,a) is

equivalent, using $\begin{pmatrix} x & cy \\ a & b & c \end{pmatrix}$. Thus there is no new vertex in $st(c)$ and by symmetry $st(1)$.

If ax is a vertex then (ax,y) gives a θ -graph disjoint from a k -graph, edge (ax,b) or (ax,c) are equivalent to edge $(3,cx), (3,bx)$ respectively, and $(ax,1)$ is equivalent to $(2,ay)$, all of which have been eliminated. If ay is a vertex we have either edges (ay,b) or (ay,c) . The first is equivalent to the existence of cy , using $\begin{pmatrix} x & ay \\ a & b & c \end{pmatrix}$, and the latter gives a 3-cycle with y dead.

If bx is a vertex then $(bx,a), (bx,c)$ give similar contradictions. Edge (bx,y) is equivalent to a vertex $1b$. If by is a vertex note (by,x) is equivalent to $1b$ using $\begin{pmatrix} x & y \\ by & a & c \end{pmatrix}$, $(by,1)$ contains $(by,y,1)$ disjoint from $\begin{pmatrix} a & 3 & b \\ 1 & 2 & x \end{pmatrix} \setminus (b,2)$ and edge (by,a) or (by,c) is equivalent to vertices cy, ay respectively.

If the ninth vertex, v , is disjoint from our subgraph then edge $(v,1)$ implies edges $(v,x), (v,y)$ giving cycle $(x,c,2,3)$ disjoint from $\begin{pmatrix} 1 & y \\ a & b & v \end{pmatrix}$. All 3 of the vertices a,b,c cannot be v.o.a., avoiding cubic vertices in a 3-cycle imply edges $(v,a), (v,b), (v,x), (v,y)$ giving cycle $(a,1,3,2)$ disjoint from $\begin{pmatrix} x & y \\ v & b & c \end{pmatrix}$.

Given G contains exactly 8 vertices, note (x,y) implies (b,a) or (b,c) and a θ -graph disjoint from $\begin{pmatrix} x & y \\ a & b \end{pmatrix}$, θ -graph disjoint from $\begin{pmatrix} x & y \\ b & c \end{pmatrix}$ respectively. The four possible remaining edges are $(1,x), (c,a), (1,y), (c,b)$, addition of all four yields a projective graph.

Case 3. $(2,a),(3,x),(3,y)$. We note the symmetries $(1\ 2), (a\ 3)$. Since $st(3)$ is dead by exhaustion of cases so is $st(a)$. If 1 is not dead we have $(1,x)$ and cycle $(1,b,x)$ is disjoint from $(\begin{smallmatrix} 2 & y \\ a & c \end{smallmatrix} \ 3)$ is lemma 4.19, hence $st(1)$ and by symmetry $st(2)$ are dead.

If bx is a vertex note (bx,y) is equivalent to lb , using $(\begin{smallmatrix} x & y \\ a & c \end{smallmatrix} \ bx)$ in place of $(\begin{smallmatrix} x & y \\ a & b \end{smallmatrix} \ c)$. Any other connection creates a 3-cycle disjoint from a $K_{3,3} \setminus e$. By symmetry there does not exist a ninth vertex.

The only possible edge additions are $(b,c)(x,y)$, adding in both gives a projective graph.

Case 4. $(2,x),(3,a),(3,x)$. By exhaustion of the casework $st(3)$ is dead. If 2 connects to cx then both c, cx connect elsewhere. Edge (cx,a) or (cx,b) is an earlier case by considering $(\begin{smallmatrix} cx & y \\ a & b \end{smallmatrix} \ c)$. Edge (cx,y) is equivalent to edge $(2,y)$ (deleting (c,cx)), using cycle $(1,a,3)$ disjoint from $(\begin{smallmatrix} x & y \\ 2 & b \end{smallmatrix} \ c)$ we get either case 1 or case 2. Thus $st(2)$ is dead.

If there exists a vertex cx then the only connection not giving a contradiction is (cx,y) . Cycle (cx,y,c) is a 3-cycle (disjoint from a k -graph) yet cx is cubic, a contradiction. We conclude $(2,x,c)$ is a 3-cycle and c is not cubic.

If (c,v) for $v \in \{ax, bx\}$ then examining $(\begin{smallmatrix} v & y \\ a & b \end{smallmatrix} \ c)$ we get an earlier case. If (c,b) then y is not cubic, (y,x) gives a θ -graph disjoint from $(\begin{smallmatrix} x & y \\ b & c \end{smallmatrix})$. Thus edge $(y,1)$ and symmetry

$(1\ x)(y\ c)(2\ a)$. By the symmetry $st(a)$ is dead, also note $(1,x)$ gives a wedge $\begin{pmatrix} 1 & 2 \\ 3 & x \end{pmatrix} \vee \begin{pmatrix} x & y \\ a & b \\ c \end{pmatrix}$. Thus all bridges lie on cycle (y,b,c) , yet since there exists an embedding with this cycle null we get a k -graph disjoint from a θ -graph. By exhaustion on where c connects we have (c,a) . Again $(y,1)$ is forced, as is either (b,ay) or (b,cy) . Regardless, there exists a 4-cycle disjoint from $\begin{pmatrix} 1 & x \\ 2 & 3 \\ b \end{pmatrix}$. This completes the proof of case 4 and lemma 4.20.

□

Lemma 4.21. There does not exist a $G \in I_*^M(P)$ satisfying H_4 and containing a subgraph homeomorphic to the graph of figure 4.37.

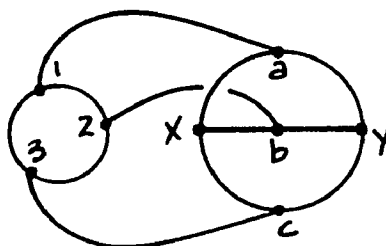


Figure 4.37

Proof. Vertices $(1,2,3)$ non-cubic imply without loss of generality edges $(1,x), (2,x), (3,y)$. Since $st(3)$ is dead c connects somewhere, without loss of generality (c,a) . Considering cycle $(2,x,b)$ disjoint from $\begin{pmatrix} a & 3 \\ 1 & c \\ y \end{pmatrix}$ shows we can apply lemma 4.20.

□

§4.5 A 3-cycle Disjoint From a k_4

Proposition 4.22. There does not exist a $G \in I_*^M(P)$ satisfying H_4 containing a 3-cycle disjoint from a k_4 .

Proof. By way of contradiction let G be such a graph. Label the cycle and the k_4 as shown in figure 4.38.

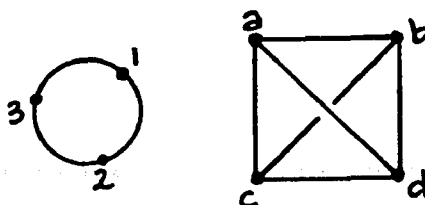


Figure 4.38

If v is a vertex disjoint from this subgraph then v may not connect twice to the cycle. If (v,a) and (v,b) then v is in the same component of $G \setminus \text{st}(k_4)$ as the missing vertex of the k_4 , else $\{a,b\}$ is a cut set, contradicting lemma 2.14. Now the cycle is disjoint from $\begin{pmatrix} a & b \\ v & c & d \end{pmatrix}$. If $(v,a), (v,b), (v,c)$ then v is the missing vertex of the k_4 , by lemma 2.19 the arc (v,d) intersects cycle $(1,2,3)$. Note neither v nor d may connect twice to the cycle, if all 3 of $\{a,b,c\}$ connect to the cycle then $\begin{pmatrix} v & d \\ a & b & c \end{pmatrix}$ is a $k_{2,3}$, without loss of generality, assume c does not. If $(v,1)(d,1)$ then $(2,a), (2,b), (3,a), (3,b)$ and cycle $(2,3,a)$ is disjoint from a $k_{2,3}$. If edges $(v,1), (d,2)$ then $(3,a), (3,b)$ and 1 may only connect to a or b , regardless G contains a wedge of k_4 's.

Since there does not exist a disjoint vertex all edges connect from the cycle to the k_4 . If there exists a vertex ab then

$\begin{pmatrix} a & b \\ ab & c & d \end{pmatrix}$ is disjoint from cycle $(1,2,3)$, thus $|v(G)| = 7$.

If $(1,a)(1,b)(1,c)(1,d)$ then WLOG $(2,a),(2,b),(3,c)$ and $(3,d)$. The graph is projective and any edge addition gives a wedge $k_4 \vee k_4$. If edges $(1,a),(1,b),(1,c)$ are in G then without loss of generality assume $(2,d),(2,c)$. Vertex 3 may connect only to a,b,d . If all 3 are present then the graph is projective and any edge addition gives a wedge, $k_4 \vee k_4$. If $(3,a),(3,d)$ and $(2,b)$ the graph is symmetric. If none of the vertices $1,2,3$ can have valency ≥ 5 there are only 2 possible graphs, both are projective.

□

Chapter 5

NO CYCLE DISJOINT FROM A k -GRAPH

§5.1 Statement of the Result

Theorem 5.1. Let $G \in I_*^M(P)$ and suppose G does not contain a cycle disjoint from a k -graph. Then $G \in \{E_3, E_{18}\}$.

Proof. Since G is nonplanar G contains either a $K_{3,3}$ or a K_5 . If G contains a K_5 with no vertex disjoint, then any bridge addition creates a $K_{3,3}$. If G does contain a vertex disjoint and G is 3-connected then the bridge containing the vertex creates either a $K_{3,3}$ or a cycle disjoint from a k -graph. Finally if G is not 3-connected then by lemma 2.14 G contains disjoint k -graphs and hence a cycle disjoint from a k -graph. Therefore we may assume G contains a $K_{3,3}$, say G contains $\begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$.

Lemma 5.8 states $G \supset K_{3,3}$ then the $K_{3,3}$ is on 6 vertices, i.e., an edge of the $K_{3,3}$ is an edge of G . Hence G contains a vertex v disjoint from $\begin{pmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{pmatrix}$. By lemma 2.14 the bridge containing v is adjacent to at least 3 vertices of the $K_{3,3}$. If two adjacent vertices are v.o.a. then we may replace the edge with an arc containing v , contradicting lemma 5.8. Hence without loss of generality v is adjacent to 0,2,4, giving $K_{4,3}$.

The existing graph is projective, hence there exists an eighth vertex v' disjoint from this $K_{4,3}$. By an argument similar to that above, v' connects to 3 vertices in one of the bipartition sets of the $K_{4,3}$. One choice gives $G = K_{3,5} = E_3$, and the other choice gives $G = K_{4,4} \setminus K_2 = E_{18}$.

□

The remainder of chapter 5 shall be devoted to the lemmas we use to prove lemma 5.8 which is in turn used to prove theorem 5.1. Note that if G contains disjoint k -graphs then G contains a cycle disjoint from a k -graph. Hence we shall use G does not contain disjoint k -graphs for the remainder of the chapter. Observe we may also use G is 3-connected, G does not contain a θ -graph disjoint from a k -graph, etc. using the various lemmas of §2.5.

§5.2 Case 1

Section 5.1 shall concern $G \in I_*^M(P)$ containing a subgraph homeomorphic to that of figure 5.1. Observe said subgraph is a $K_{3,3}$ with an arc connecting the interiors of opposite edges.

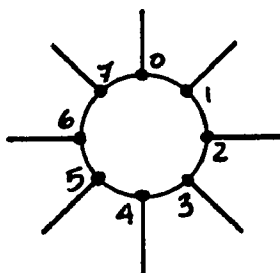


Figure 5.1

Lemma 5.2. Let $G \in I_*^M(P)$ contain a vertex v disjoint from a subgraph homeomorphic to figure 5.1. Then G contains a cycle disjoint from a k -graph.

Proof. By way of contradiction suppose G does not contain a cycle disjoint from a k -graph. We examine the bridge containing the vertex v . It must have at least 3 v.o.a. We will break into cases depending on where these v.o.a. are.

If 01 is a v.o.a. then another v.o.a. cannot be in $st(1), st(2), st(0), st(7)$, else G contains a cycle disjoint from a k -graph. Likewise if the bridge connects to $st(4)$ then cycle $(2,3,7,6)$ is disjoint from $\begin{pmatrix} 01 & 4 \\ 0 & 1 \end{pmatrix}_v$ or a splitting thereof. Hence by symmetry the other two v.o.a. must be 3 and 6; giving cycle $(0,01,1,5,4)$ disjoint from $\begin{pmatrix} 3 & 6 \\ 7 & 2 \end{pmatrix}_v$. Thus 01 must not be a v.o.a.

If vertex 0_4 is a v.o.a. then $st(0), st(1)$ cannot contain v.o.a, or else there is a cycle disjoint from $(\begin{smallmatrix} 3 & 6 \\ 4 & 7 & 2 \end{smallmatrix})$. By symmetry the only possible v.o.a. are vertices 6,2, giving case 1 of figure 5.2.

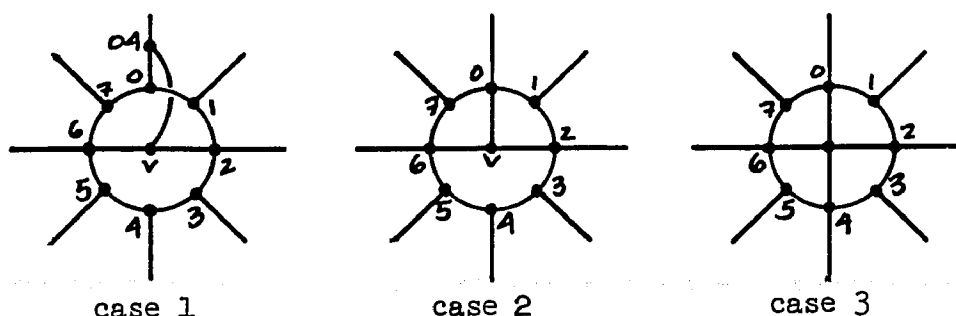


Figure 5.2

We have reduced to cases where the only v.o.a. are existing vertices 0 through 7. Without loss of generality let 0 be a v.o.a. If 1 is a v.o.a. then cycle $(v, 0, 1)$ is disjoint from $(\begin{smallmatrix} 3 & 6 \\ 2 & 4 & 7 \end{smallmatrix})$. If 3 is a v.o.a. then cycle $(1, 2, 6, 5)$ is disjoint from $(\begin{smallmatrix} 0 & 3 \\ 7 & 4 & v \end{smallmatrix})$. Hence the only possible v.o.a. sets are $\{0, 2, 6\}$ or $\{0, 2, 4, 6\}$. Finally we note that v.o.a. set $\{0, 2, 4, 6\}$ must give case 3 of figure 5.2 as any splitting contains a cycle disjoint from a k-graph.

Case 1. Vertex v is cubic, $(v, 2), (v, 6)$ are edges, so by lemma 1.6 there exists a vertex 26. Vertex 26 is disjoint from a subgraph homeomorphic to 5.1 so by previous arguments 26 connects to $st(4)$, creating a cycle disjoint from $(\begin{smallmatrix} 0 & 6 \\ v & 7 & 1 \end{smallmatrix})$.

Case 2. As in case 1, we have a vertex 26, which connects to either $st(0)$ or $st(4)$. If $(26, st(0))$ then cycle $(7, 6, 5, 4, 3)$ is disjoint from $(\begin{smallmatrix} 0 & 2 \\ 1 & v & 26 \end{smallmatrix})$, and $(26, st(4))$ gives cycle $(4, 3, 2, 26)$ disjoint from $(\begin{smallmatrix} 0 & 6 \\ 1 & 7 & v \end{smallmatrix})$.

Case 3. First we shall show $|v(G)| = 9$. If there exists \bar{v} disjoint from our subgraph then the v.o.a. are either $\{0, 2, 4, 6\}$ or $\{1, 3, 5, 7\}$. The former contains cycle $(3, 4, 5, 6, 7)$ disjoint from $(\begin{smallmatrix} 0 & 2 \\ v & \bar{v} & 1 \end{smallmatrix})$ and the latter contains cycle $(7, \bar{v}, 3)$ disjoint from $(\begin{smallmatrix} v & 5 \\ 6 & 4 & 1 \end{smallmatrix})$. If there exists a vertex 15 then it cannot connect to $st(1), st(0)$ or $st(7)$ without a cycle disjoint from $(\begin{smallmatrix} 2 & 4 \\ v & 3 & 5 \end{smallmatrix})$. Symmetry exhausts the possibilities, hence there does not exist a vertex 15. If there exists a vertex 01 then it cannot connect to $st(1), st(0)$, or $st(7)$ without a cycle disjoint from $(\begin{smallmatrix} 2 & 4 \\ v & 3 & 5 \end{smallmatrix})$. Likewise a connection to $st(2)$ or $st(3)$ gives a cycle disjoint from $(\begin{smallmatrix} 0 & 6 \\ v & 7 & 5 \end{smallmatrix})$. A connection to $st(4)$ or $st(5)$ gives a cycle disjoint from $(\begin{smallmatrix} v & 7 \\ 0 & 6 & 3 \end{smallmatrix})$. Since by the previous construction it does not connect to $st(v)$, we have exhausted the possibilities. Finally suppose there exists a vertex 04. If 04 connects to $st(0), st(1)$ or $st(2)$ then there exists a cycle disjoint from $(\begin{smallmatrix} 4 & 6 \\ v & 3 & 5 \end{smallmatrix})$, by symmetry these are the only choices. Thus we conclude $|v(G)| = 9$.

The only possible edge additions are of the type $(i, i^{\pm 3}) \bmod 8$, as $(i, i^{\pm 2})$ gives a cycle disjoint from a k -graph. Therefore without loss of generality $(0, 3)$ is an edge. Vertex 7 is cubic in

a 3-cycle, hence we have edges $(7,2)$ or $(7,4)$. The former contains cycle $(4,5,6,v)$ disjoint from $(\begin{smallmatrix} 0 & 2 \\ 1 & 3 & 7 \end{smallmatrix})$ and the latter contains cycle $(4,5,6,7)$ disjoint from $(\begin{smallmatrix} 0 & 2 \\ 1 & 3 & v \end{smallmatrix})$.

□

Lemma 5.3. Let $G \in I_*^M(P)$ contain a subgraph homeomorphic to that of figure 5.1, and suppose $|v(G)| \geq 9$. Then G contains a cycle disjoint from a k -graph.

Proof. By way of contradiction let G be a graph as described. By lemma 5.2 the ninth vertex must be (without loss of generality) either 04 or 01 .

If there exists a vertex 04 then a connection to $st(0)$ or $st(1)$ creates a cycle disjoint from $(\begin{smallmatrix} 6 & 3 \\ 7 & 2 & 4 \end{smallmatrix})$. By symmetry the only possibilities are either $(04,26)$ (case 1 of figure 5.3) or $(04,2)$ (case 2 of figure 5.3).

If there exists a vertex 01 then a connection to $st(1)$ or $st(2)$ creates a cycle disjoint from $(\begin{smallmatrix} 4 & 7 \\ 0 & 3 & 5 \end{smallmatrix})$, and $(01, st(3))$ gives cycle $(1,2,6,5)$ disjoint from $(\begin{smallmatrix} 0 & 3 \\ 01 & 4 & 7 \end{smallmatrix})$. By symmetry we have either $(01,4)$ or $(01,45)$, cases 3 and 4 of figure 5.3 respectively.

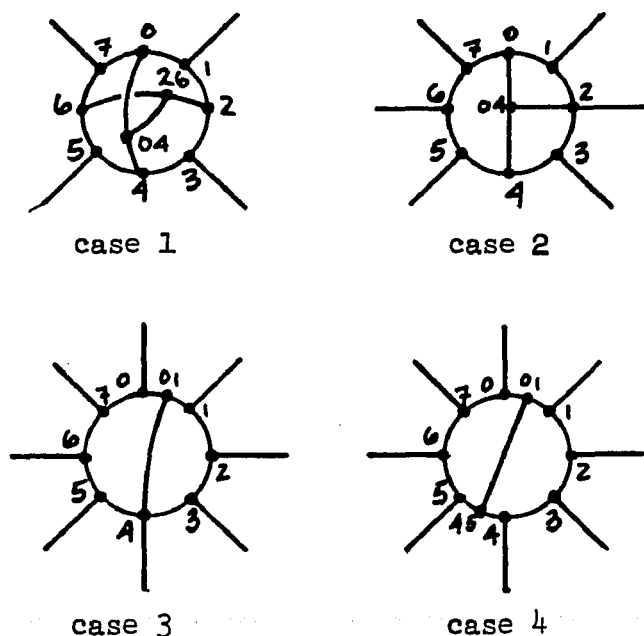
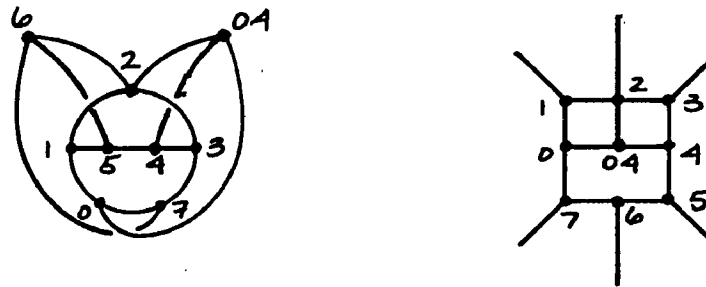


Figure 5.3

Case 1. We note the graph is both vertex and edge transitive. If there existed an additional vertex then without loss of generality $v \in (04, 26)$, contradicting lemma 5.2. Since the graph is projective there exists an additional edge out of, without loss of generality, 04 , by the previous argument we have edge $(04, 2)$, giving a cycle disjoint from $(\begin{smallmatrix} 5 & 7 \\ 1 & 3 & 6 \end{smallmatrix})$.

Case 2. First we shall eliminate the possibility of a tenth vertex. The edges fall into 5 symmetry classes, represented by $(0, 7), (3, 7), (2, 3), (2, 04), (0, 04)$. By lemma 5.2 the possible tenth vertex lies in the interior of one of these edges.



Symmetries of case 2

Figure 5.4

If there exist a vertex 07 then by previous arguments it connects to either 3, 34, or 4. The former two graphs contain a cycle disjoint from $\begin{pmatrix} 1 & 04 \\ 0 & 2 & 4 \end{pmatrix}$ and the latter graph contains cycle $(0, 04, 2, 1)$ disjoint from $\begin{pmatrix} 4 & 7 \\ 07 & 5 & 3 \end{pmatrix}$. If there exist a vertex 37 then it must connect to either vertex 1, with cycle $(37, 7, 0, 1)$ disjoint from $\begin{pmatrix} 2 & 4 \\ 04 & 3 & 6 \end{pmatrix}$, or to vertex 5, with cycle $(37, 3, 4, 5)$ disjoint from $\begin{pmatrix} 0 & 2 \\ 04 & 1 & 6 \end{pmatrix}$. If there exists a vertex 23 it connects to either 6, 67, 7, yielding a cycle disjoint from $\begin{pmatrix} 1 & 04 \\ 0 & 2 & 4 \end{pmatrix}$. Lemma 5.2 shows that there does not exist a vertex in edge $(2, 04)$. Finally if there exists a vertex $v \in (0, 04)$ then we must have either edge $(v, 2)$ or $(v, 6)$. The former graph contains cycle $(v, 04, 2)$ disjoint from $\begin{pmatrix} 5 & 7 \\ 0 & 3 & 6 \end{pmatrix}$ and the latter graph contains cycle $(v, 0, 7, 6)$ disjoint from $\begin{pmatrix} 2 & 4 \\ 04 & 3 & 5 \end{pmatrix}$. Having exhausted the possibilities we conclude $|v(G)| = 9$.

If $(04,6)$ is an edge of G then vertex 4 is dead, and by symmetry so are 2 and 6. Vertex 0 can connect only to 3,2 or 4 (up to symmetry). The first has cycle $(3,0,7)$ disjoint from $(\begin{smallmatrix} 04 & 5 \\ 6 & 4 & 1 \end{smallmatrix})$, the second has cycle $(0,1,2)$ disjoint from $(\begin{smallmatrix} 4 & 6 \\ 04 & 5 & 3 \end{smallmatrix})$, and the last was handled in lemma 5.2. Hence by symmetry both 0 and 4 are dead. Edge $(1,7)$ gives cycle $(0,1,7)$ disjoint from $(\begin{smallmatrix} 2 & 4 \\ 04 & 3 & 6 \end{smallmatrix})$, so the only two edges which may be added are $(1,3)$ and $(5,7)$. Adding in both edges gives cycle $(2,04,6)$ disjoint from $(\begin{smallmatrix} 1 & 3 \\ 5 & 7 \end{smallmatrix})$, and adding in only one edge gives a projective graph. Thus we see $(04,6)$ is not an edge, and vertices 04,6 are dead.

Vertex 2 cannot be adjacent to 0,4,5 or 7, since such an edge makes either vertex 04 or 6 cubic in a 3-cycle. Hence vertex 2 is dead also. Also note $(1,7)$ is not an edge because the resulting graph contains cycle $(0,1,7)$ disjoint from $(\begin{smallmatrix} 2 & 4 \\ 04 & 3 & 6 \end{smallmatrix})$. We are left with 5 possible edge additions, $(0,3), (1,4), (0,5), (1,3)$ and $(4,7)$. If $(1,4)$ is an edge then avoiding a cubic vertex in a 3-cycle implies edge $(0,5)$, giving cycle $(5,6,7,0)$ disjoint from $(\begin{smallmatrix} 2 & 4 \\ 1 & 3 & 04 \end{smallmatrix})$. By symmetry $(0,3)$ is not an edge. Of the 3 remaining edges adding in any two still gives a projective graph, and adding all 3 gives cycle $(1,2,3)$ disjoint from $(\begin{smallmatrix} 5 & 7 \\ 0 & 4 & 6 \end{smallmatrix})$.

Case 3. Consider the specific embedding shown in figure 5.1. This embedding does not extend to an embedding of G . Lemma 5.2 rules out the existence of equivalent 3-bridges, and cases 1, 2 rule out inadmissible bridges. Each of the regions $(i, i+1, i+5, i+4)$

mod 8 are disjoint from a similar "complementary" region, so skew bridges on such a region must create a cycle disjoint from a k -graph. Hence there must exist skew bridges on region $(0,1,2,3,4,5,6,7)$ and bridge $(01,4)$ can fit in region $(0,1,5,4)$.

By previous arguments there cannot exist edges $(i, i+\frac{3}{2}), (i, i+\frac{5}{2}) \pmod{8}$. Also edges $(i, i+3), (i, i+\frac{7}{2}) \pmod{8}$ are all outer region admissible, where outer region refers to one of the type $(i, i+1, i+5, i+4)$. Hence the skew bridges must be of the form $(i, i+2)$, pick i s.t. the skew bridges are $(i-1, i+1)(i, i+2)$. We observe avoiding a cubic vertex in a 3-cycle implies vertex 01 must connect to both vertices 4,5. Symmetry shows we need only consider the cases $i = 0, 1, 2, 3$ or 4, where i determines the location of the skew bridges as described previously. If $i = 0$ then cycle $(01, 4, 5)$ is disjoint from $(\begin{smallmatrix} 2 & 7 \\ 0 & 3 & 6 \end{smallmatrix})$, if $i = 1$ then cycle $(01, 4, 5)$ is disjoint from $(\begin{smallmatrix} 2 & 7 \\ 0 & 3 & 6 \end{smallmatrix})$, if $i = 2$ then cycle $(1, 2, 3)$ is disjoint from $(\begin{smallmatrix} 0 & 5 \\ 01 & 4 & 7 \end{smallmatrix})$, if $i = 3$ then cycle $(01, 0, 4)$ is disjoint from $(\begin{smallmatrix} 3 & 6 \\ 2 & 5 & 7 \end{smallmatrix})$, and if $i = 4$ then cycle $(01, 4, 5)$ is disjoint from $(\begin{smallmatrix} 4 & 7 \\ 0 & 3 & 6 \end{smallmatrix})$.

Case 4. By cases 1-3 we know vertex 01 is dead, hence by symmetry all vertices are dead. The only possible bridge additions are of the type $(12, 56)$. These bridges may be added repeatedly and the graph still embeds by an extension of the embedding shown in figure 5.1.

□

Lemma 5.4. Let $G \in I_*^M(P)$ contain a subgraph homeomorphic to that of figure 5.1, and suppose $|v(G)| = 8$. Then G contains a cycle disjoint from a k -graph.

Proof. Observe all edges to be added are of the type (i, i^{+2}) and (i, i^{+3}) .

If $(0,3)$ is an edge then $4,7$ are both cubic in a 3-cycle. If $(4,2)$ then cycle $(0,3,7)$ is disjoint from $\begin{pmatrix} 2 & 5 \\ 1 & 4 & 6 \end{pmatrix}$, and if $(4,7)$ then cycle $(1,2,5,6)$ is disjoint from $\begin{pmatrix} 0 & 3 \\ 4 & 7 \end{pmatrix}$. If $(4,1)$ is an edge then we must have (by symmetry) either $(5,7)$ or $(2,7)$. The former graph contains cycle $(5,6,7)$ disjoint from $\begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$ and the latter has either edge $(5,2)$ and hence $(6,1)$ with cycle $(0,3,4)$ disjoint from $\begin{pmatrix} 2 & 6 \\ 1 & 5 & 7 \end{pmatrix}$ or edge $(5,0)$ with cycle $(2,6,7)$ disjoint from $\begin{pmatrix} 0 & 4 \\ 1 & 3 & 5 \end{pmatrix}$. Thus we must have $(4,6)$ and by symmetry $(5,7)$. Observe avoiding earlier cases implies 5 may only connect to 3. The only possible edge additions are $\{(5,3), (6,0), (3,1), (0,2)\}$. Edge $(1,3)$ implies (since 2 is cubic in a 3-cycle) $(0,2)$ giving cycle $(4,5,6)$ disjoint from $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$. Thus there are only two possible edge additions, and G together with these edges is projective. Hence we see $(0,3)$ is not an edge, and there cannot exist edges of the type (i, i^{+3}) .

Without loss of generality let $(0,2), (1,3)$ be edges. Note edge $(4,6)$ creates cycle $(4,5,6)$ disjoint from $\begin{pmatrix} 0 & 3 \\ 1 & 2 & 7 \end{pmatrix}$, hence $(4,6), (5,7)$ are not in G . Also note if $(3,5)$ is an edge of G vertex 4 is cubic in a 3-cycle, hence $(4,2)$ is an edge. By

symmetry and the previous argument our graph may not be augmented further, yet it is still projective. We have only two possible edge additions remaining, $(1,7)$ and $(2,4)$. Adding on both gives a projective graph.

□

Corollary 5.5. Let $G \in I_*^M(P)$ contain a $K_{3,3}$ and a bridge adjacent to two points not vertices of the $K_{3,3}$. Then G contains a cycle disjoint from a k -graph.

Proof. The two vertices lie in the topological interior of edges of the $K_{3,3}$. If they both lie in the same edge, or in edges incident with a common vertex, then there exists a cycle disjoint from a k -graph. If they lie in opposite edges then we apply either lemma 5.3 or lemma 5.4.

□

§5.3 Case 2

Section 5.3 shall concern $G \in I_*^M(P)$ containing a subgraph homeomorphic to that of figure 5.5. Observe this graph is a $K_{3,3}$ with an added edge connecting a vertex and the interior of a non-incident edge.

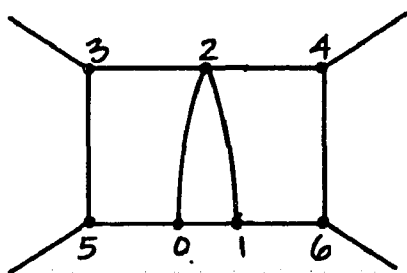


Figure 5.5

Lemma 5.6. Let $G \in I_*^M(P)$ contain a vertex disjoint from a subgraph homeomorphic to figure 5.5. Then G contains a cycle disjoint from a k -graph.

Proof. By way of contradiction let G be such a graph without a cycle disjoint from a k -graph. We examine possible vertices of attachment for the bridge containing v . By corollary 5.5 there are not v.o.a. in the interior of edges. Observe vertices 0,1 may both be viewed as in the interior of an edge.

If 46 is a v.o.a. then two others must be chosen from the set $\{2,3,5\}$. If two adjacent vertices are chosen, say 3 and 5, then replacing the edge $(3,5)$ with the arc through v gives a contradiction of corollary 5.5. Hence the v.o.a. are 46,2,5 and G

contains cycle $(2,0,1)$ disjoint from $(\begin{smallmatrix} 46 \\ v \end{smallmatrix} \begin{smallmatrix} 5 \\ 4 \end{smallmatrix} \begin{smallmatrix} 6 \\ 6 \end{smallmatrix})$. Thus 46 is not a v.o.a.; by symmetry, neither is 35,45,46.

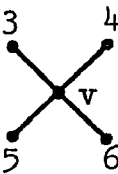
If 24 is a v.o.a. then two others must be chosen from $\{3,5,6\}$. Again choosing nonadjacent vertices gives v.o.a. $\{24,5,6\}$, and G contains cycle $(2,0,1)$ disjoint from $(\begin{smallmatrix} 5 \ 6 \\ 3 \ 4 \end{smallmatrix} \begin{smallmatrix} v \\ v \end{smallmatrix})$. Hence 24, and by symmetry 23, are not v.o.a. A similar argument shows a v.o.a. may not lie in the interior of any edge, i.e., the v.o.a. must lie in the set $\{0, \dots, 6\}$.

If 1 is a v.o.a. then the other two cannot be adjacent. Choosing 3,4 gives cycle $(2,0,1)$ disjoint from $\parallel (\begin{smallmatrix} 3 \ 4 \\ 5 \ 6 \end{smallmatrix} \begin{smallmatrix} v \\ v \end{smallmatrix})$ hence the v.o.a. are 1,2 and 5. Avoiding v cubic in a 3 cycle gives a vertex 12. If 12 is adjacent to 1,4 or 6 then G contains a cycle disjoint from $(\begin{smallmatrix} 2 \ 5 \\ v \ 3 \end{smallmatrix} \begin{smallmatrix} 0 \\ 0 \end{smallmatrix})$, if adjacent to 2 or 3 then G contains a cycle disjoint from $(\begin{smallmatrix} 1 \ 5 \\ v \ 0 \end{smallmatrix} \begin{smallmatrix} 4 \\ 4 \end{smallmatrix})$, if adjacent to 0 or v then G contains a cycle disjoint from $(\begin{smallmatrix} 3 \ 4 \\ 2 \ 5 \end{smallmatrix} \begin{smallmatrix} 6 \\ 6 \end{smallmatrix})$, and if adjacent to 5 then G contains a cycle $(3,2,4,6)$ disjoint from $(\begin{smallmatrix} 1 \ 5 \\ v \ 0 \end{smallmatrix} \begin{smallmatrix} 12 \\ 12 \end{smallmatrix})$. Hence vertex 1, and by symmetry 0, are not a v.o.a.

If 2 is a v.o.a., then without loss of generality so are 3 and 5, any other v.o.a. gives cycle $(2,0,1)$ disjoint from a k -graph. The bridge must consist of the single cubic vertex v , and avoiding cubic in a 3-cycle forces the existence of vertex 35, which may connect only to 2,4 or 6. The first gives a graph containing cycle $(2,0,1)$ disjoint from $(\begin{smallmatrix} 3 \ 5 \\ 6 \ v \end{smallmatrix} \begin{smallmatrix} 35 \\ 35 \end{smallmatrix})$, the second contradicts corollary 5.5 using the $K_{3,3}$ $(\begin{smallmatrix} v \ 35 \ 6 \\ 3 \ 4 \end{smallmatrix} \begin{smallmatrix} 5 \\ 5 \end{smallmatrix})$, and the third contains

cycle $(2,3,v)$ disjoint from $(\begin{smallmatrix} 5 & 6 \\ 0 & 4 \\ & 35 \end{smallmatrix})$. Hence 2 is not a v.o.a., leaving only the set $\{3,4,5,6\}$.

Without loss of generality let 3,5 be v.o.a. If 35 is a vertex then we must have either $(35,4)$ or $(35,6)$, because 35 is disjoint from a subgraph like figure 5.5, giving a k -graph disjoint from cycle $(2,0,1)$. Hence $\{3,4,5,6\}$ is exactly the v.o.a. set, and the

bridge is , or else cycle $(2,0,1)$ is disjoint

from a k -graph. Vertices 0,1 are still cubic in a 3-cycle, without loss of generality $st(1)$ connects to a vertex not 0 or 2. If $(st(1),5)$ then cycle $(5,1,0)$ is disjoint from $(\begin{smallmatrix} 3 & 4 \\ 2 & 6 \\ & v \end{smallmatrix})$. By symmetry we have $(st(1),4)$ which gives cycle $(3,v,6)$ disjoint from $(\begin{smallmatrix} 0 & 4 \\ 1 & 2 \\ & 5 \end{smallmatrix})$.

□

Lemma 5.7. Let $G \in \mathcal{I}_*^M(P)$ contain a vertex disjoint from a $K_{3,3}$ on 7 or more vertices. Then G contains a cycle disjoint from a k -graph.

Proof. By way of contradiction let G be as described without a cycle disjoint from a k -graph. Denote the $K_{3,3}$ by $(\begin{smallmatrix} 2 & 6 & 5 \\ 0 & 3 & 4 \end{smallmatrix})$ and let the seventh vertex, 1, lie in edge $(0,6)$. Avoiding a cycle disjoint from a k -graph and using corollary 5.5 we have the graph of figure 5.5. A vertex disjoint from the original $K_{3,3}$

but not from this subgraph must be 12. Corollary 5.5 implies 12 connects only to vertices, vertices 1,2,0 or 6 all give a cycle disjoint from a k -graph. If 12 connects to two adjacent vertices we may again apply corollary 5.5, hence we have $(12,5)$ and $st(12)$ is dead. Since vertices 3,4 and 0 are disjoint from $K_{3,3}$'s $\begin{pmatrix} 12 & 0 & 4 \\ 1 & 5 & 2 \end{pmatrix}$, $\begin{pmatrix} 1 & 2 & 5 \\ 0 & 12 & 3 \end{pmatrix}$ and $\begin{pmatrix} 12 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}$ respectively, we know $st(3)$, $st(4)$ and $st(0)$ are all dead. The only live edge for a ninth vertex is $(1,6)$. By corollary 5.5 and the above arguments 16 may connect only to vertex 2 (symmetrically 5) giving cycle $(3,5,4,6)$ disjoint from $\begin{pmatrix} 1 & 2 \\ 12 & 0 & 16 \end{pmatrix}$. Hence the graph G contains exactly 8 vertices, the only live ones being 1,2,5 and 6. Note edge $(1,2)$, $(1,5)$ make vertices 12,0 respectively cubic vertices in a 3-cycle, yet they are dead, a contradiction. Hence vertex 1 is dead. Adding in the three remaining possible edges gives a projective graph.

□

Lemma 5.8. Let $G \in \mathcal{I}_*^M(P)$ contain a $K_{3,3}$ on 7 or more vertices. Then G contains a cycle disjoint from a k -graph.

Proof. By way of contradiction let G be as described, G not containing a cycle disjoint from a k -graph. As in lemma 5.7 G contains a subgraph homeomorphic to that of figure 5.5. We observe vertices 0,1 are both cubic in a 3-cycle. By lemma 5.7 $(0,2)$, $(1,2)$ must both be edges. We will first show there does not exist a vertex 01, then examine where 0 and 1 connect.

Suppose 01 is a vertex. By repeated use of the fact that there does not exist a cubic vertex in a 3-cycle we may assume 01 connects somewhere other than vertex 2. If $(01,6)$ then vertex 1 is disjoint from $\begin{pmatrix} 0 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}$ contradicting lemma 5.7. By symmetry we may assume $(01,4)$. Observe $G = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 01 & 2 & 5 & 6 \end{pmatrix} \setminus \{(1,6), (01,3), (5,0)\}$ and by symmetry a ninth vertex, if it exists, lies on either $(1,2), (2,4), (01,1)$. If 12 is a vertex then G contains a cycle disjoint from a k -graph by lemma 5.7. If 24 is a vertex it must connect to either 1, $v \in (0,1)$, or 46. The former pair are corollary 5.5, while the latter graph contains a cycle disjoint from $\begin{pmatrix} 0 & 3 \\ 2 & 5 & 6 \end{pmatrix}$. A vertex $v \in (1,01)$ may only connect to 2 and/or 4. Thus the only possible additional vertices lie on edges $(6,3), (3,5), (5,1), (1,01), (01,0), (0,6)$ and must connect to 2 and/or 4.

Next we examine possible edge addition between existing vertices. By symmetry these fall into three classes, represented by edges $(01,3), (0,1), (01,2)$. The first type was previously eliminated, and the second has cycle $(0,01,1)$ disjoint from $\begin{pmatrix} 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}$. Thus the only edge additions are $(0,4), (1,4), (3,4), (01,2), (5,2), (6,2)$. Thus under the supposition of a vertex 01 we have characterized all edge additions, whether between existing vertices or with one endpoint a new vertex. Adding these edges still gives a projective graph, in particular embedding as an extension of the embedding shown in figure 5.6. Thus there does not exist 01 , and $(0,1,2)$ is a 3-cycle of G .

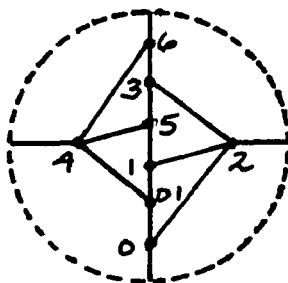


Figure 5.6

Next we examine where $0,1$ may connect. First suppose $(0,6)$ is an edge of G , observe vertex 1 is disjoint from $\begin{pmatrix} 0 & 3 & 4 \\ 2 & 5 & 6 \end{pmatrix}$. If there exists an eighth vertex then by lemma 5.7 and the above argument it must be 16 . If 16 connects to $0,2$ we get a cycle disjoint from a k -graph. If 16 connects to 3 or 4 then avoiding 16 cubic in a 3 -cycle (opposite edge must be an edge or we apply lemma 5.7) 16 must connect somewhere else. Edge $(16,5)$ gives cycle $(0,1,2)$ disjoint from $\begin{pmatrix} 5 & 6 \\ 16 & 3 & 4 \end{pmatrix}$, and 16 connecting to both 3 and 4 gives $\begin{pmatrix} 3 & 4 \\ 16 & 5 & 6 \end{pmatrix}$ disjoint from cycle $(0,1,2)$. Hence 16 is not a vertex and $|v(G)| = 7$. We shall break into cases depending on valency (1) . If valency $(1) = 6$ consider the extension of the embedding shown in figure 5.7. This does not extend to an embedding of G , and all edge additions are admissible. Hence there must be skew edges in one of the regions bounded by a 4 -cycle, yet any choice gives a cycle through vertex 1 which is disjoint from a k -graph. If $\text{val}(1) = 5$ suppose $(1,3)$ is not an edge. The above

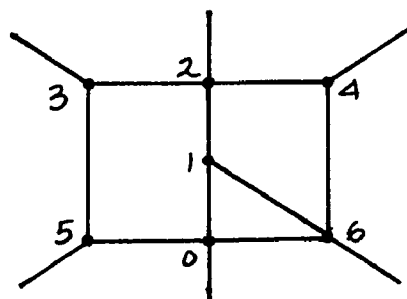


Figure 5.7

argument still holds except for skew edges $(4,0)(2,5)$, this graph contains cycle $(2,3,5)$ disjoint from $\begin{pmatrix} 0 & 1 \\ 4 & 6 \end{pmatrix}$. If vertex 3 is cubic then G embeds, since K_6 triangulates P . Hence vertex 1 is of valence 4. Let $(1,5)$ be the additional edge and note we have $\begin{pmatrix} 2 & 5 & 6 \\ 0 & 1 & 3 & 4 \end{pmatrix} \cup (0,1)$, so by symmetry 0 is dead. Vertex 3 must connect to 4, so 3 and 4 are dead. Any edge of triangle $(2,5,6)$ gives a cycle disjoint from a k -graph. Thus $(1,4)$ must be the additional edge and vertex 1 is dead. Vertices 3,5 cannot be cubic, so without loss of generality $(3,0), (5,2)$ giving cycle $(1,4,6)$ disjoint from $\begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}$. Thus $|v(G)| = 7$ implies G contains a cycle disjoint from a k -graph, and we conclude $(0,6)$ is not an edge of G , and hence we now have vertices 0,1 may only be adjacent to vertices 3,4.

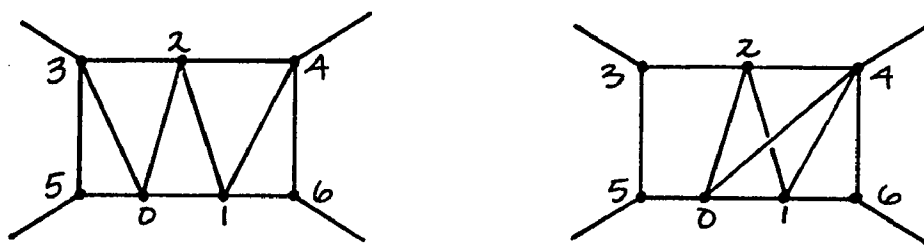


Figure 5.8

If 0,1 connect to different vertices we have the graph on the left of figure 5.8. It was previously argued edges $(0,1), (0,2), (1,2)$ are edges, not arcs, of G ; the same arguments show $(0,3), (0,5), (3,5), (1,4), (4,6), (1,6)$ are all edges. If 23 is a vertex then a connection to an edge or to vertices 0,1,5,6 violate corollary 5.5, hence $(23,4)$, and G contains a cycle $(0,1,2)$ disjoint from $\begin{pmatrix} 3 & 4 \\ 23 & 5 & 6 \end{pmatrix}$. If 36 is a vertex observe edge $(36,2)$ and G contains cycle $(0,3,5)$ disjoint from $\begin{pmatrix} 2 & 6 \\ 1 & 4 & 36 \end{pmatrix}$, hence $(36,4)$. Using vertex $5 \in (0,4)$ in $\begin{pmatrix} 0 & 2 & 6 \\ 1 & 3 & 4 \end{pmatrix}$ we get either $(5,2)$ or $(5,6)$. The former graph contains cycle $(1,4,6)$ disjoint from $\begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}$ and the latter cycle $(0,2,3)$ disjoint from $\begin{pmatrix} 4 & 6 \\ 1 & 5 & 36 \end{pmatrix}$. Thus $|v(G)| = 7$. Vertex 5 is cubic in a 3-cycle, $(5,2)$ gives cycle $(1,4,6)$ disjoint from $\begin{pmatrix} 0 & 2 \\ 3 & 5 \end{pmatrix}$, hence $(5,6)$ and vertices 5,6 are dead. Likewise edge $(1,3)$ gives cycle $(4,6,5)$ disjoint from $\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$ hence vertices 0,1 are dead. The only remaining edge addition is $(3,4)$, and the resulting graph is still projective.

Finally we have that 0 and 1 must connect to the same vertex, giving the right hand graph of figure 5.8. Moreover vertices 0 and

1 are dead. Using vertex $5 \in (0,4) \subset \begin{pmatrix} 0 & 2 & 6 \\ 4 & 1 & 3 \end{pmatrix}$ we get that $(0,5)$ and $(4,5)$ are edges, not arcs of G , $(5,2)$ is an edge of G , and vertices 4 and 5 are dead. Vertex $3 \in (6,5) \subset \begin{pmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \end{pmatrix}$ implies edge $(3,4)$, a contradiction.

□

Chapter 6

COMPLETION OF THE RESULT

§6.1 Derivation of the 103 Graphs

Let the set of specific graphs listed in theorem 1.7 be denoted by \mathcal{G} , i.e., $\mathcal{G} = \{A_1, A_2, A_5, B_1, B_3, C_1, C_2, C_7, C_{11}, D_1, D_4, D_5, D_9, D_{12}, D_{17}, E_1, E_3, E_6, E_8, E_9, E_{11}, E_{18}, E_{19}, E_{20}, E_{22}, E_{26}, E_{27}, E_{42}, F_2, F_4, F_6, G\}$. The goal of this chapter will be theorem 6.1 which identifies $\{G' \in I(P) \mid G' \leq_* G, G \text{ an arbitrary graph in } \mathcal{G}\}$. The reader should recall that theorem 1.7 (using the results of chapters 2,3,4 and 5) showed that $\mathcal{G} \supset I_*^M(P)$, the set of maximal graphs with respect to \leq_* . Theorem 6.1 identifies the 103 graphs in the appendix and completes the proof of theorem 1.1, the main result. However it should be noted this chapter does not depend on results in previous chapters. We examine possible splittings and deletions of graphs, independent of $\mathcal{G} \supset I_*^M(P)$. As a corollary we note the fact that no distinct elements G_1, G_2 in \mathcal{G} are comparable. This, together with theorem 1.7, establishes the graphs actually are maximal, and hence $\mathcal{G} = I_*^M(P)$.

Theorem 6.1. The set $\{G \in I(P) \mid G' \leq_* G, G \in \mathcal{G}\}$ consists of precisely 103 graphs, listed in the appendix.

Proof. We partition \mathcal{G} into four subsets, $H_1 = \{A_1, A_5, B_3, C_7, D_{17}\}$,
 $H_2 = \{C_1, C_2, C_{11}, D_4, D_5, D_{12}, E_{19}, E_{20}\}$, $H_3 = \{D_1, D_9, E_1, E_6, E_8, E_9, E_{11}, E_{26},$
 $E_{27}, E_{42}, F_2, F_4, F_6, G\}$ and $H_4 = \{A_2, B_1, E_3, E_{18}, E_{22}\}$. Lemmas 6.2-6.4
exhaustively search for $G' \leq^* G$, G in H_1, H_2 and H_3 respectively.
This completes the listing for $G' \in I(P)$, G' contains disjoint
 k -graphs. $H_4 = \{G \in \mathcal{G} \mid G \text{ does not contain disjoint } k\text{-graphs}\}$.
Lemma 6.7 finds $H_5 = \{G' \in I(P) \mid G' \leq^* G \in H_4, G' \text{ a } * \text{-source}\}$.
Lemmas 6.9-6.16 exhaustively search for graphs \leq_s a graph in H_5 .
The complete set of graphs thus found are listed in the appendix.

□

§6.2 Disjoint k-graphs

Recall \mathcal{L} as defined in §6.1. The goal of this section is to find all $G' \in I(P)$, $G' \leq_{\times} G$ for some $G \in \mathcal{L}$, where G contains disjoint k-graphs. Two graphs are comparable under \leq_{\times} if and only if either they both contain disjoint k-graphs or they both do not contain disjoint k-graphs. Thus, equivalently, the goal of this section is to find all $G' \in I(P)$, G' contains disjoint k-graphs, $G' \leq_{\times} G$ for some $G \in \mathcal{L}$.

The reader is asked to recall the definition of an elementary \ast -derivation for G, G' containing disjoint k-graphs, K_1 disjoint K_2 .

$G \geq_{\times} G'$ provided $G' = S_v(G)$ and either

a) $v \parallel (K_1 \cup K_2)$,

b) $v \in K_1$ and the bipartition of edges incident with v in the splitting is $\{\text{the edges of } K_1\} \cup \{\text{edges not in } K_1\}$,

or c) $v \in K_1 = K_{2,3}$ with v one of the valency two vertices in K_1 .

We shall call these type a, type b, and type c splittings respectively. The reader is referred to figure 1.4 for illustrations.

No edge deletions are allowed in these splittings. Thus, for example, any splitting which creates a cubic vertex in a 3-cycle is not allowed by lemma 1.6.

We cite some restrictions for type a, type b and type c splittings may occur. For type a splittings, if v is a cut point of G then $S_v(G)$ may not be 2-connected, otherwise the newly created edge (v, v') would be reducible. Similarly if the partition

on edges incident with v created by the splitting is the same as the partition created by v being a cut point then again (v, v') is reducible. For type b splittings we need at least 2 edges incident with v which are not part of the k -graph to which v belongs. For any splitting we need valency $(v) \geq 4$. These criteria shall be applied in upcoming lemmas to greatly decrease the number of splittings considered.

Finally before proceeding, $\{G \in \mathcal{S} \mid G \supset \coprod k\text{-graphs}\}$ breaks naturally into the following three subsets:

$$H_1 = \{G \in \mathcal{S} \mid G \supset k_4 \coprod k_4\} = \{A_1, A_5, B_3, C_7, D_{17}\},$$

$$H_2 = \{G \in \mathcal{S} \mid G \supset k_4 \coprod k_{2,3}\} = \{C_1, C_2, C_{11}, D_4, D_5, D_{12}, E_{19}, E_{20}\},$$

$$H_3 = \{G \in \mathcal{S} \mid G \supset k_{2,3} \coprod k_{2,3}\} = \{D_1, D_9, E_1, E_6, E_8, E_9, E_{11}, E_{26}, E_{27}, E_{42}, F_2, F_4, F_6, G\}.$$

Lemma 6.2. Let $H_1 = \{A_1, A_5, B_3, C_7, D_{17}\}$. Then $\{G' \in I(P) \mid G' \leq_{\ast} G, G \in H_1\} = H_1 \cup \{A_3, A_4, B_8, B_{10}\}$.

Proof. We refer the reader to figure 6.1. Graphs A_5, C_7, D_{17} are all \ast -sinks.

If $G = B_3$ there are no type a or type c splittings. Up to isomorphism there is a unique type b splitting, creating B_8 . Any type a splitting on B_8 creates a cubic vertex in a 3-cycle. The unique type b splitting gives B_{10} . B_{10} is a \ast -sink, so we have all $G' \leq_{\ast} B_3$.

If $G = A_1$, avoiding a cubic vertex in a 3-cycle gives a unique type a splitting. The resulting graph is A_3 . Again there is a unique (up to symmetry) type a splitting yielding A_4 , a $*$ -sink. Thus we have all $G' \leq^* A_1$, which completes the proof of the lemma.

□

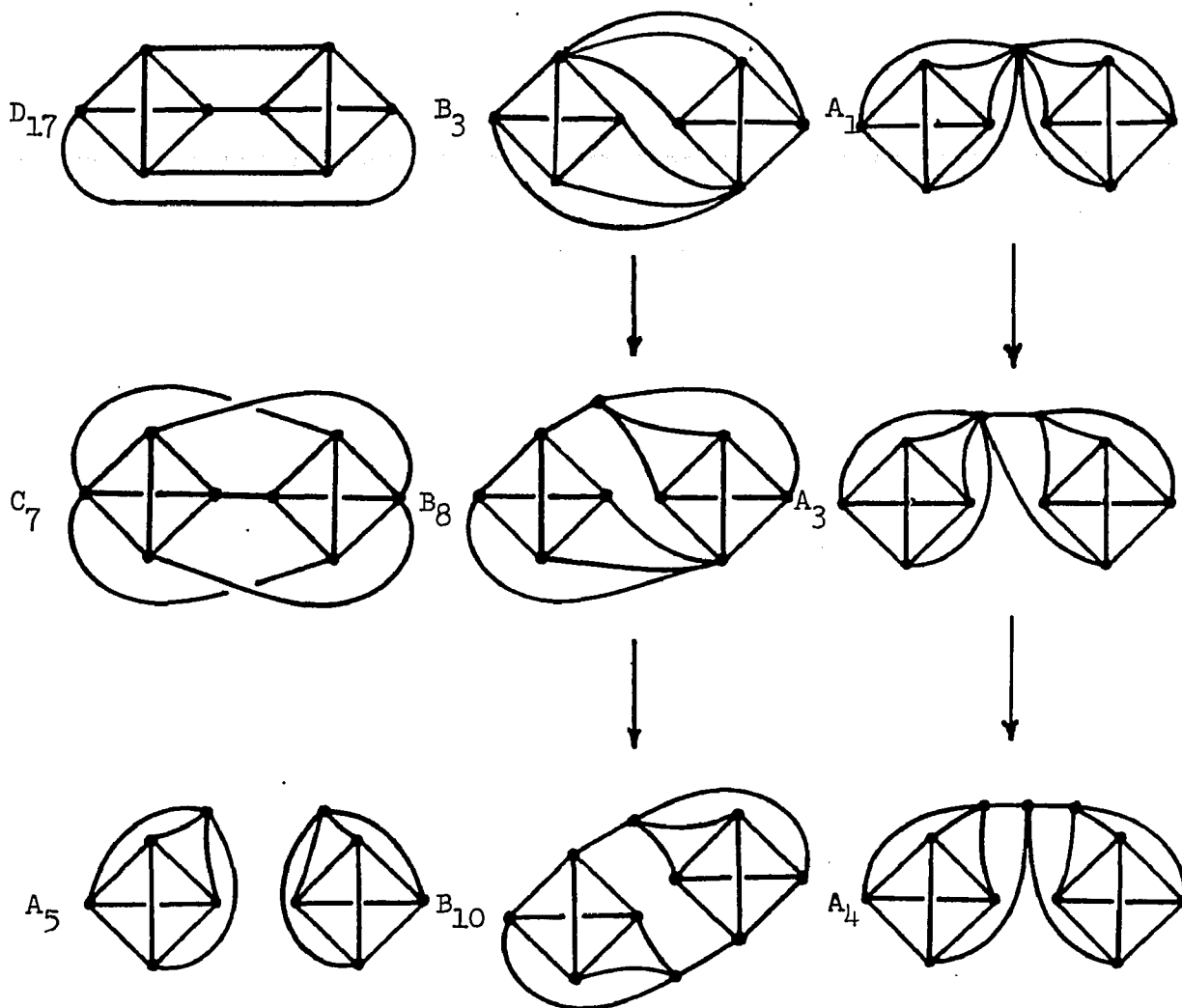


Figure 6.1

Lemma 6.3. Let $H_2 = \{C_1, C_2, C_{11}, D_4, D_5, D_{12}, E_{19}, E_{20}\}$. Then
 $\{G' \in I(P) \mid G' \leq^* G, G \in H_2\} = H_2 \cup \{C_5, C_6, C_8, C_9, C_{10}, D_{13}, D_{18}, E_{23}\}$.

Proof. We refer the reader to figure 6.2. Graphs C_{11} and E_{20} are both $*$ -sinks.

In graph $D_5, S_1: (2,3)$ has edge $(4,5)$ reducible. The only other type b splitting creates a cubic vertex in a 3-cycle, hence D_5 is a $*$ -sink. In graph $D_{12}, S_1: (2,3)$ has edge $(4,5)$ reducible so D_{12} is also a $*$ -sink.

In graph E_{19} there are no type a splittings and any type b splitting creates a cubic vertex in a 3-cycle. Both type c splitting gives E_{23} , a $*$ -sink.

In graph C_2 we note the symmetry $(1\ 2)$ using the labeling of figure 6.2. The type b splitting is unique up to symmetry yielding C_5 . In C_5 there again is a unique type b splitting giving C_9 . There are no type b or type c splittings in C_9 . The unique up to symmetry type a splitting which avoids a cubic vertex in a 3-cycle is $S_1: (3,4)$, this splitting has $(5,6)$ reducible, so C_9 is a $*$ -sink and we have all $G' \leq^* C_2$.

In the graph C_1 there are only type a splittings. Recall we must avoid the creation of a 2-connected graph. The two splittings which do not create a cubic vertex in a 3-cycle give graphs C_6 and C_8 . Doing both splittings gives C_{10} , a $*$ -sink.

In the graph D_4 we note the symmetry $(1\ 2)(3\ 4)(5\ 6)$. This symmetry shows there is a unique type b splitting, which gives D_{13} .

In D_{13} there again is a unique type b splitting. This splitting creates D_{18} , a $*$ -sink.

□

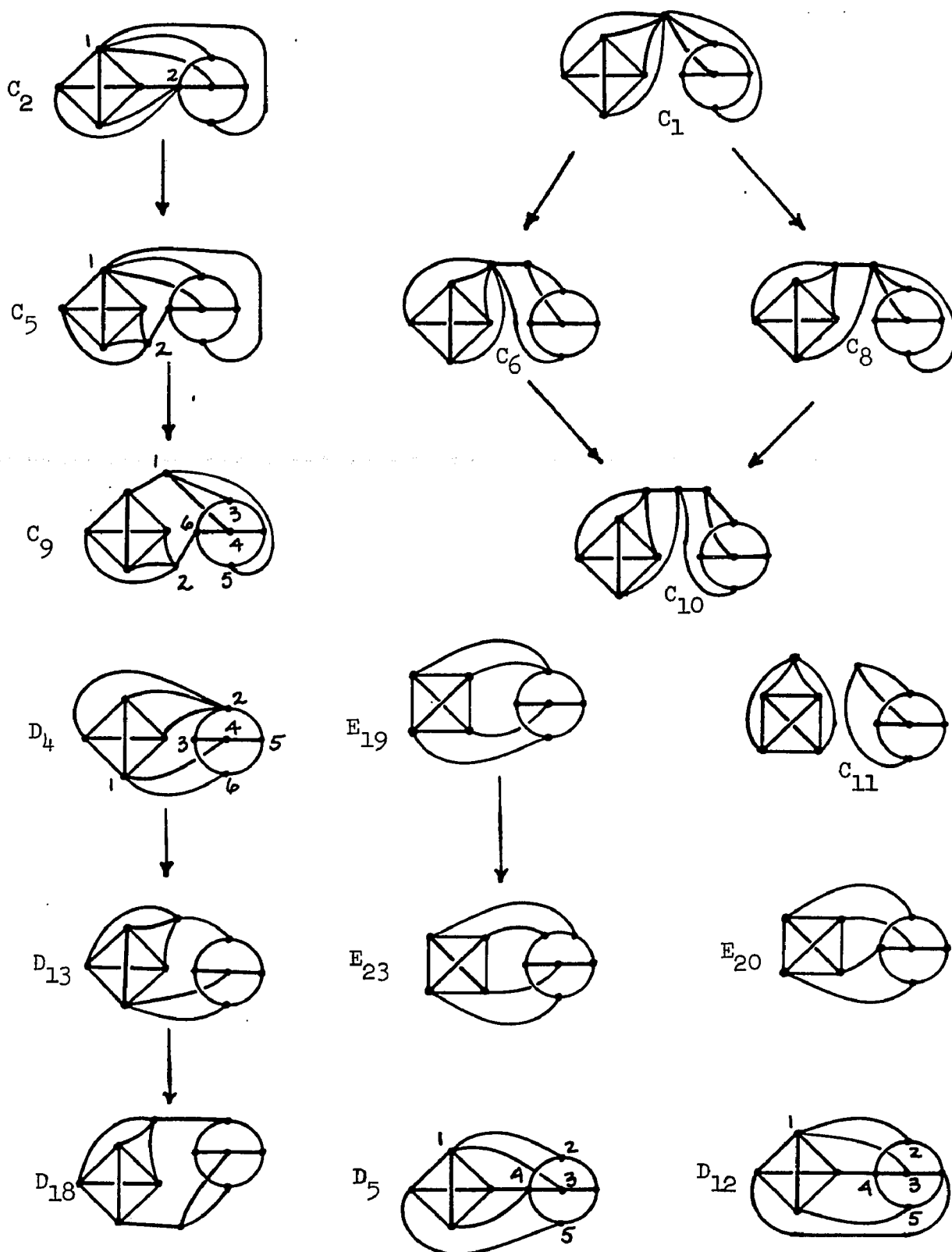


Figure 6.2

Lemma 6.4. Let $H_3 = \{D_1, D_9, E_1, E_6, E_8, E_9, E_{11}, E_{26}, E_{27}, E_{42}, F_2, F_4, F_6, G\}$. Then $\{G' \in I(P) \mid G' \leq^* G, G \in H_3\} = H_3 \cup \{E_{13}, E_{16}, E_{29}, E_{30}, E_{36}, E_{37}, E_{40}, F_7, F_8, F_{10}, F_{11}, F_{12}\}$.

Proof. Examining figure 6.3 it is clear E_{26}, E_{42}, F_4 and G are all $*$ -sinks.

Graph D_1 admits a unique up to symmetry type b splitting on vertex 1; however, the new edge $(1, 1')$ is reducible giving E_1 . The graph D_9 admits a unique up to symmetry type b splitting on vertex 1, but $S_1: (2, 3)^{(D_9)} \setminus \text{st}(4) = F_2$. In a similar manner $S_1: (2, 3)^{(E_8)} \setminus (4, 5) = F_2$ and $S_1: (2, 3)^{(E_{11})} \setminus (4, 5) = F_2$. These exhaust the possible type b splittings. There are no type a or type c splittings, hence D_1, D_9, E_8 and E_{11} are $*$ -sinks.

Graph E_{27} admits a unique up to symmetry type c splitting yielding E_{30} , a $*$ -sink. The unique type b splitting on E_{27} is $S_1: (2, 3)^{(E_{27})} \setminus \text{st}(4) = G$. We have all $G' \leq^* E_{27}$.

For the remaining graph splittings the reader is referred to figure 6.4.

Graph E_1 admits a unique up to symmetry type a splitting which gives E_{16} . Applying the same process again gives E_{40} , a $*$ -sink. Since neither E_1 nor E_{16} have any type b or type c splittings we have all $G' \leq^* E_1$.

Given graph E_6 we note the symmetry $(1\ 4)(2\ 5)(3\ 6)$. This symmetry shows the two type b splittings both give E_{13} . A type c splitting on E_6 is by symmetry equivalent to a splitting of vertex 1.

Any such splitting has a reducible edge $(4,i)$ for $i = 7, 8$ or 9 ; e.g., $S_1: (7,2)$ has $(4,7)$ reducible. In E_{13} the unique type b splitting gives E_{36} . Any type c splitting has the new edge (v,v') reducible. In E_{36} the only splitting possible is a type a splitting on vertex 4, or symmetrically vertex 1. $S_1: (2,3)$ has edge $(3,4)$ reducible. Thus E_{36} is a $*$ -sink and we have found all $G' \leq_* E_6$.

Given graph E_9 the unique up to symmetry type c splitting gives E_{29} . The unique up to symmetry type c splitting on E_{29} gives E_{37} , a $*$ -sink. In either E_9 or E_{29} the unique type b splitting gives a reducible edge $(1,2)$. Thus we have all $G' \leq_* E_9$.

Given graph F_2 the unique type b splitting is $S_1: (2,3)$ which has edge $(4,5)$ reducible. The unique type c splitting gives F_{10} , a $*$ -sink.

Given graph F_6 the unique up to symmetry type b splitting gives F_7 and the unique up to symmetry type c splitting gives F_8 . In F_8 the unique type b splitting $S_1: (2,3)$ has edge $(4,5)$ reducible. Similarly, the new edge in any type c splitting of F_7 is reducible. A type b splitting on F_7 gives F_{11} , each type c splitting on F_8 gives F_{12} . Both graphs are $*$ -sinks so we have found all $G' \leq_* F_6$.

□

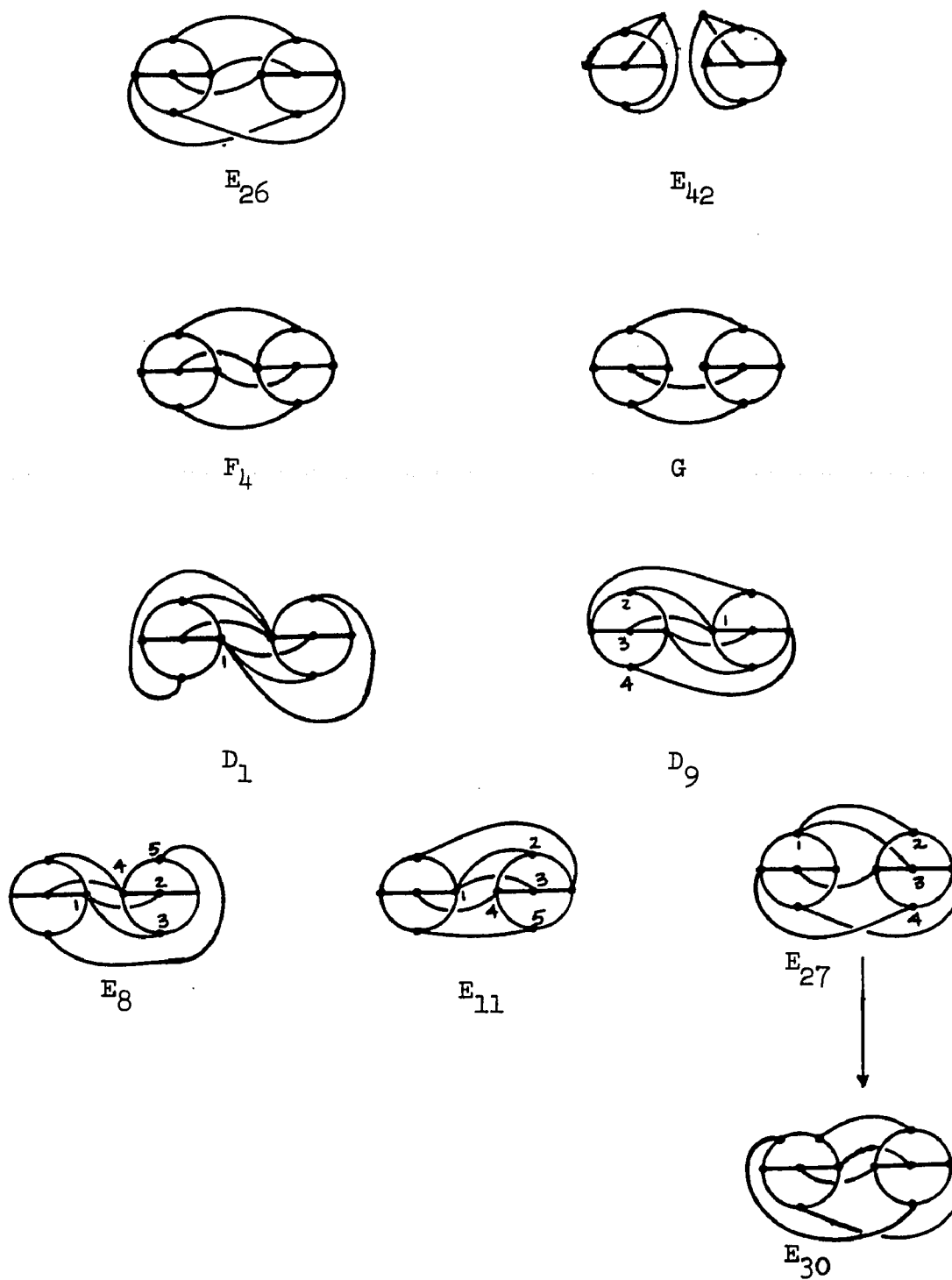


Figure 6.3

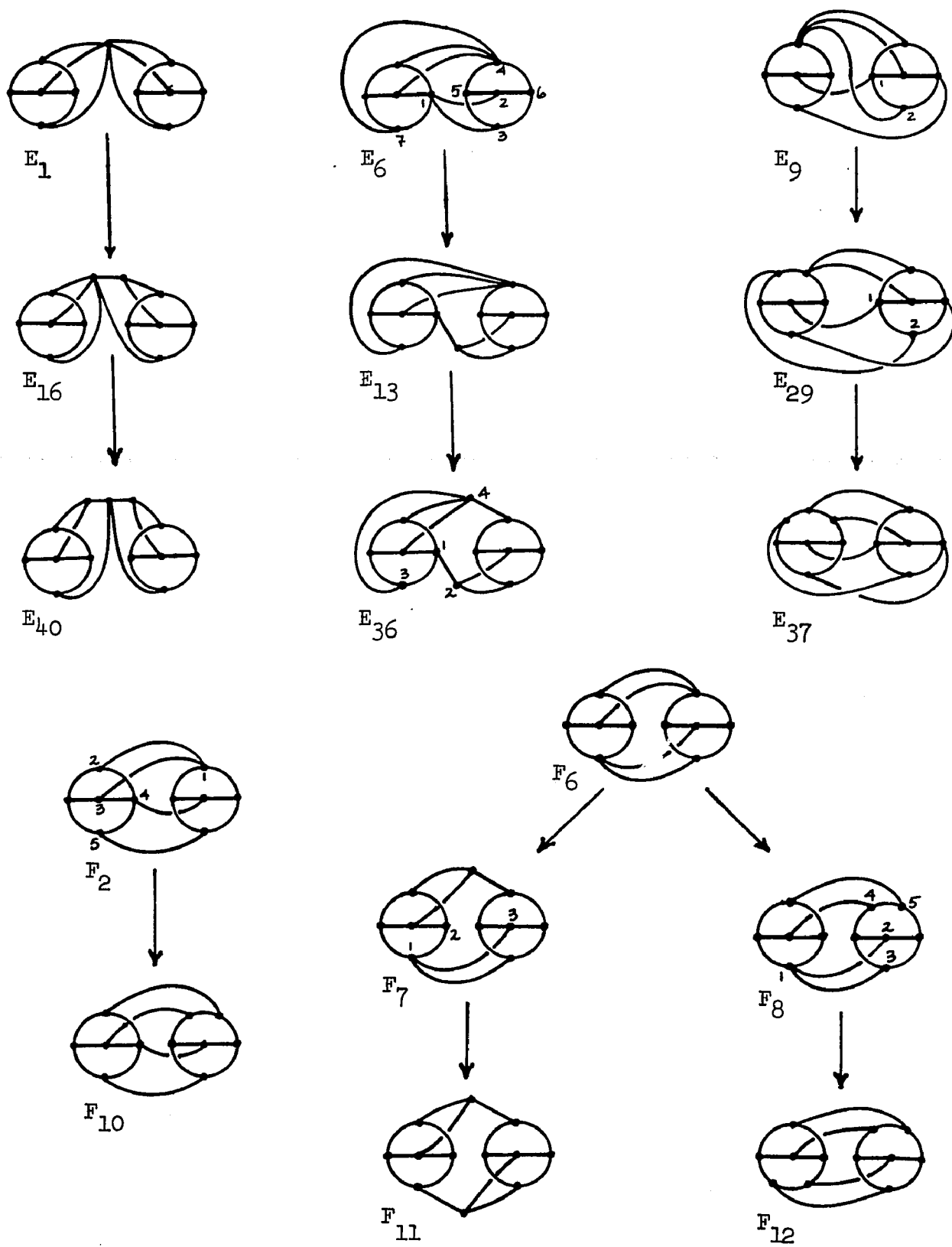


Figure 6.4

§6.3 No Disjoint k -graphs

Recall \mathcal{L} and the partition H_i ($i = 1, 2, 3, 4$) as defined in §6.1. Also recall \leq_x respects the dichotomy of $I(P)$ determined by the existence or nonexistence of disjoint k -graphs. In §6.2 all graphs in $I(P)$ which contain disjoint k -graphs were determined. In this section only those graphs in $I(P)$ which do not contain disjoint k -graphs will be considered.

$H_4 = \{A_2, B_1, E_3, E_{18}, E_{22}\} = \{G \in \mathcal{L} \mid G \not\parallel k\text{-graphs}\}$. In this section we shall find all $G' \in I(P)$ such that $G' \leq_x G$, $G \in H_4$.

Note any splitting which creates a graph containing disjoint k -graphs is not a relation in \leq_x , and hence need not be considered towards this goal. We shall first find all $*$ -sources (for a definition see §1.3), then examine all \geq_s derivations on the $*$ -sources.

Lemma 6.5. Let H_2 be a $*$ -source in $I(P)$. If H_2 is an elementary $*$ -derivative of G_2 then G_2 is a $*$ -source.

Proof. By way of contradiction suppose $G_2 \leq_s G_1$, where G_2 is derived from G_1 by a single elementary splitting operation. Let e_1 denote the new edge created, so that $\frac{G_2}{e_1} = G_1$. Note e_1 is not in a 3-cycle of G_2 , hence e_1 is also in H_2 . Define $H_1 = \frac{H_2}{e_1}$. We will show $H_1 \in I(P)$.

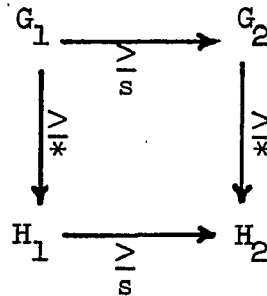


Figure 6.5

Suppose H_1 is projective, and let φ be an embedding. Let v be a cubic vertex created in $S(G_2)$ which forces an edge deletion in creating H_2 . If $v \in V(H_1)$ is not cubic then e_1 must be incident with v , yet this implies e_1 is in a 3-cycle, a contradiction. Thus using lemma 1.6 we may extend φ to $\varphi': S(G_1) \subset P$. The embedding φ' contradicts lemma 1.4, hence H_1 is nonprojective.

Let e be an arbitrary edge of H_1 . By embedding $H_2 \setminus e \subset P$ and applying the contrapositive of lemma 1.4 we get $H_1 \setminus e$ is projective.

The two preceding paragraphs show $H_1 \in I(P)$. Since $H_1 = \frac{H_2}{e_1}$, $H_1 \geq_s H_2$ which contradicts H_2 is a $*$ -source.

□

Lemma 6.5 says in order to find all $*$ -sources we need only consider splittings of $*$ -sources. Let G be a $*$ -source in $I(P)$, (a,b,c) a 3-cycle in G . We observe $S_a: (b,c)(G) \setminus (b,c) =$

$$S_b: (a,c)(G) \setminus (a,c) = S_c: (a,b)(G) \setminus (a,b) = (G \setminus (a,b,c)) \cup \left(\begin{array}{c} a \quad b \\ \quad \diagdown \quad \diagup \\ \quad v \\ \quad \diagup \quad \diagdown \\ \quad c \end{array} \right).$$

Call this operation a triangle replacement. If $(a,b,c), (a,0,1)$ are both 3 cycles with only vertex a in common, and valency $(a) = 4$, then we may consider a double triangle replacement. This corresponds to splitting vertex a and deleting two edges, $(0,1)$ and (b,c) .



Figure 6.6

Rephrasing lemma 6.5, all $*$ -sources are generated from $*$ -sources by triangle replacements or double triangle replacements. Note given a candidate for a double triangle replacement, a single triangle replacement creates a cubic vertex in a 3-cycle.

Lemma 6.6. Let $G, G' \in I(P)$, with G' either a triangle replacement or a double triangle replacement of G . Suppose $S_v(G)$ contains a reducible edge e , e not in a triangle which is replaced in the creation of G' . Then e is reducible in $S_v(G')$.

Proof. Suppose G, G', e and v are as given in the hypothesis, assume e is not reducible in $S_v(G')$ and embed $S_v(G') \setminus e \subset P$. Locally the vertices created in the triangle replacement look as shown in figure 6.7. In either case we may extend the embedding to include the dotted edges, giving $S_v(G) \setminus e \subset P$. This contradicts e reducible for $S_v(G)$. We note the condition e is not in a triangle replaced in the creation of G' is used only to be sure a corresponding edge exists in G' .

□



Figure 6.7

Lemma 6.7. Let $H_4 = \{A_2, B_1, E_3, E_{18}, E_{22}\}$. Then $\{G' \in I(P) \mid G' \leq^* G, G \in H_4; G' \text{ a } * \text{-source}\} = H_4 \cup \{B_7, C_3, C_4, D_2, D_3, E_2, E_5, F_1\}$.

Proof. We shall examine all possible triangle replacements. We refer to the graphs as labeled in figure 6.8.

Graph $A_2 = K_7 \setminus (\frac{11}{3} K_2)$, so up to symmetry there are only two types of triangles, represented by $(1,4,5)$ and $(0,1,2)$. Replacing

triangle $(1,4,5)$ gives B_7 , where the new vertex is labeled "7". Replacing triangle $(0,1,2)$ gives $D_3 \cup \{(1,5), (2,6)\}$, hence it does not generate an irreducible graph. Thus B_7 is the only \ast -source which is an elementary \ast -derivative of A_1 .

Graph B_7 has the symmetries $(2\ 3)(4\ 5)$ and $(1\ 5)(2\ 6)$. Using these symmetries the triangles fall into 4 equivalence classes, represented by $(3,5,6), (2,3,6), (0,1,2)$ and $(0,2,3)$. Replacing triangle $(3,5,6)$ gives C_3 , where the new vertex is labeled "8", and replacing $(2,3,6)$ gives C_4 . Using lemma 6.6 we see replacing triangle $(0,1,2)$ gives a graph with $(2,6)$ reducible and replacing $(0,2,3)$ gives $(2,4)$ reducible.

Graph C_3 has the symmetries $(1\ 4)(3\ 6)$ and $(1\ 3)(4\ 6)(7\ 8)$. The triangles fall into 3 classes represented by $(2,4,6), (0,1,2)$ and $(0,4,6)$. Replacing a triangle in the first class gives D_2 , replacing a triangle in the second class, say $(0,1,2)$, gives a graph with $(2,6)$ reducible by lemma 6.6, and replacing a triangle in the class represented by $(0,4,6)$ gives $D_{10} \leq_s D_7$.

Graph D_2 contains the symmetry $(1\ 2\ 3)(7\ 9\ 8)(4\ 6\ 5)$. The triangles fall into symmetry classes represented by $(1,2,3)$ and $(0,1,2)$. Replacing the former gives E_2 and replacing the latter gives $F_5 \cup (0,4)$.

Graph E_2 does not contain a triangle.

Graph C_4 is a 6 wheel with vertex 7 attaching to alternating rim vertices 1,4 and 5 and vertex 8 attaching to alternating rim vertices 2,3 and 6. Thus all triangles are symmetric, replacing $(0,1,2)$ gives $D_{15} \leq_{\ast} D_7$.

We have found all \ast -sources G' with $G' \leq_{\ast} A_2$.

Graph B_1 is $K_7 \setminus \text{cycle } (1,4,3,6)$. The triangles fall into 2 classes, the first represented by $(0,2,5)$ and the second a double triangle replacement represented by $\{(0,4,6)(2,5,6)\}$. Replacing $(0,2,5)$ gives $E_3 \cup \{(1,3), (4,6)\}$ and the double triangle replacement gives D_3 . Note in the latter replacement valency $(6) = 2$, hence 6 is not labeled as a vertex of D_3 .

In D_3 we have the symmetries $(2\ 5)$ and $(1\ 3)$. Triangle $(0,1,2)$ shares a valency 4 vertex with $(1,3,5)$, performing a double triangle replacement gives F_1 . By symmetry we have considered all triangles containing either vertex 2 or vertex 5. Replacing triangle $(0,1,3)$ gives E_5 .

Since E_5 and F_1 do not contain any triangles we have found all \ast -sources G' , $G' \leq_{\ast} B_1$.

Observing E_3, E_{18} and E_{22} are triangle-free completes the proof of the lemma.

□

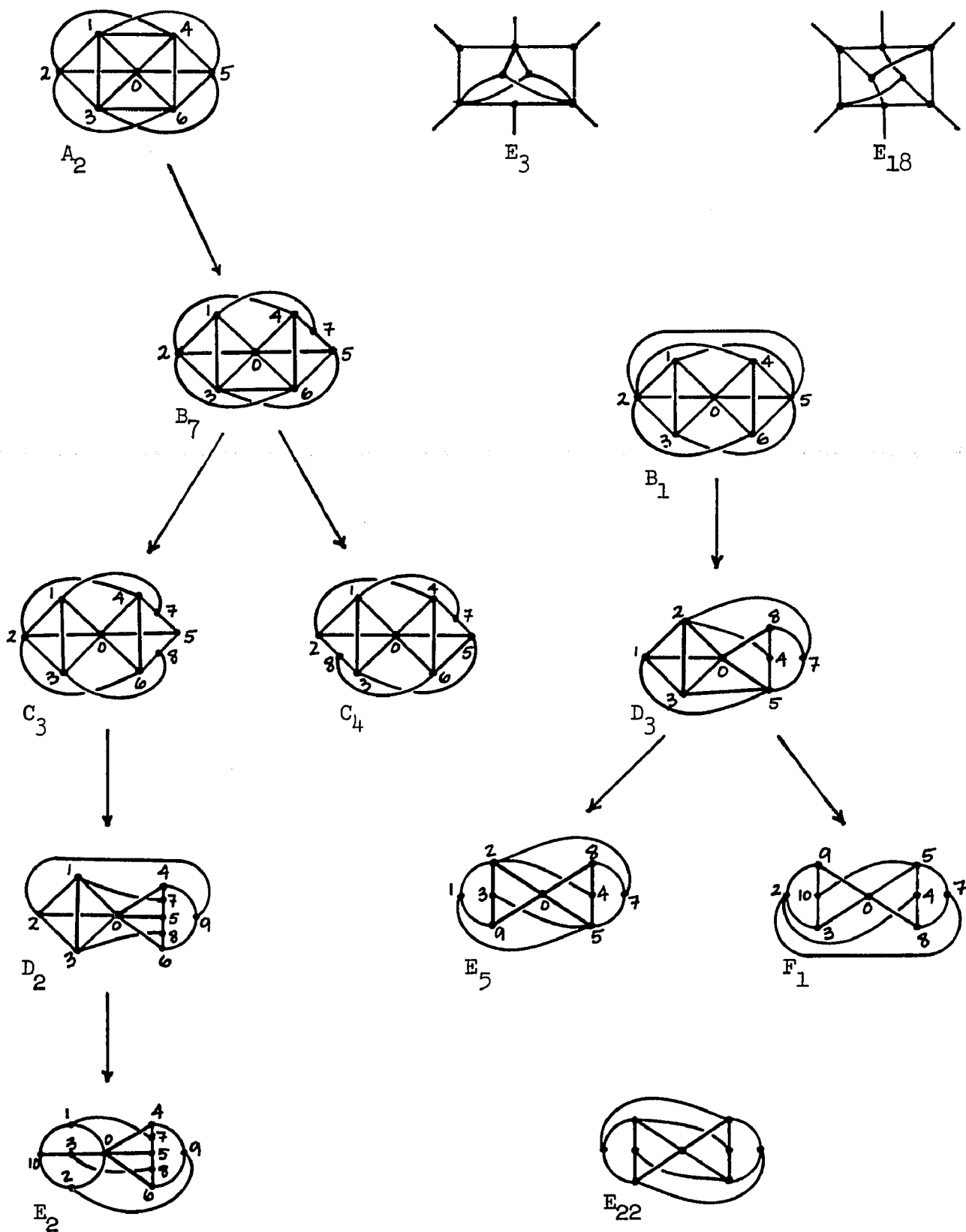


Figure 6.8

Lemma 6.8. Let $G, G' \in I(P)$, $G' = S_{v_1}(G)$. Suppose $S_{v_2}(G)$ contains a reducible edge e_1 , where $v_2 \neq v_1$. Then $S_{v_2}(G')$ also has e_1 reducible.

Proof. By way of contradiction suppose $S_{v_2}(G') \setminus e_1 \subset P$. Applying the contrapositive of lemma 1.4 we get $S_{v_2}(G) \setminus e_1 \subset P$, contradicting e_1 reducible. Note we need $v_1 \neq v_2$ to ensure $S_{v_2}(G')$ is well defined.

□

Lemma 6.9. A_2, B_7, C_3, C_4 and D_2 are all $*$ -sinks. Also $\{G \in I(P) \mid G \leq_s E_2\} = \{E_2, E_{17}, E_{38}\}$.

Proof. We label the graphs as shown in figure 6.8.

Recall A_2 is $K_7 \setminus \{(1,6), (2,5), (3,4)\}$. $S_1: (4,5) = B_7 \cup \{(4,5)\}$, $S_1: (3,4) = D_3 \cup \{(0,2), (1,5), (2,6)\}$, $S_0: (4,5,6) = D_{17} \cup \{(1,4), (2,6), (3,5)\}$, $S_0: (4,5) = D_3 \cup \{(2,4), (3,5), (4,5)\}$, $S_0: (3,4) = E_3 \cup \{(1,2), (2,6), (6,5), (5,1)\}$, and $S_0: (3,4,5) = E_{18} \cup \{(1,2), (2,6), (3,5), (4,5)\}$. By symmetry this exhausts the possibilities. Since every splitting contains reducible edges A_2 is a $*$ -sink.

B_7 contains the symmetries $(2\ 3)(4\ 5)$ and $(1\ 5)(2\ 6)$. By lemma 6.6 any splitting of vertex 2 contains a reducible edge, by symmetry so does any splitting of 3 and 6. Any splitting of 1

creates a cubic vertex in a 3-cycle. By lemma 6.6 any splitting of 0 contains a reducible edge. Thus B_7 is also a $*$ -sink.

In C_3 any splitting of 1 creates a reducible edge by lemma 6.6. By symmetry we need not consider splittings of 3,4 or 6. By lemma 6.6 any splitting of 0 or 2 contains a reducible edge. Thus C_3 is a $*$ -sink.

As was observed in lemma 6.7 all valency 4 vertices of C_4 are similar. By lemma 6.6 we need only consider $S_1: (2,3) = D_{15} \cup \{(0,5)\}$. Any splitting of 0 contains a reducible edge by lemma 6.6, hence C_4 is a $*$ -sink.

In D_2 note vertex 1 is similar to 2 and 3, any splitting of these vertices gives a cubic vertex in a 3-cycle. Any splitting of 0 gives a reducible edge by lemma 6.6.

In E_2 , 0 is the only non-cubic vertex. Lemma 6.6 does not apply because the reducible edges in $S_0(D_2)$ is in cycle (1,2,3). We note $S_0: (4,5)$ and $S_0: (4,5,6)$ both contain θ -graph \parallel k -graph. $S_0: (2,5) = E_{17}$, $S_0: (1,2,5) = E_{38}$, and $S_0: (2,4,6) = F_9 \cup \{(0,5)\}$. By the symmetry shown in the right hand side of figure 6.9 these exhaust the possible splittings. Any additional splittings of E_{17} and E_{38} give a splitting of $S_0: (4,5)$ and thus contain a θ -graph disjoint from a k -graph, except $S_{11}: (1,6)^{(E_{17})} = F_9 \cup \{(2,6)\}$. Thus we have found all $G' \in I(P)$, $G' \leq_s E_2$.

□

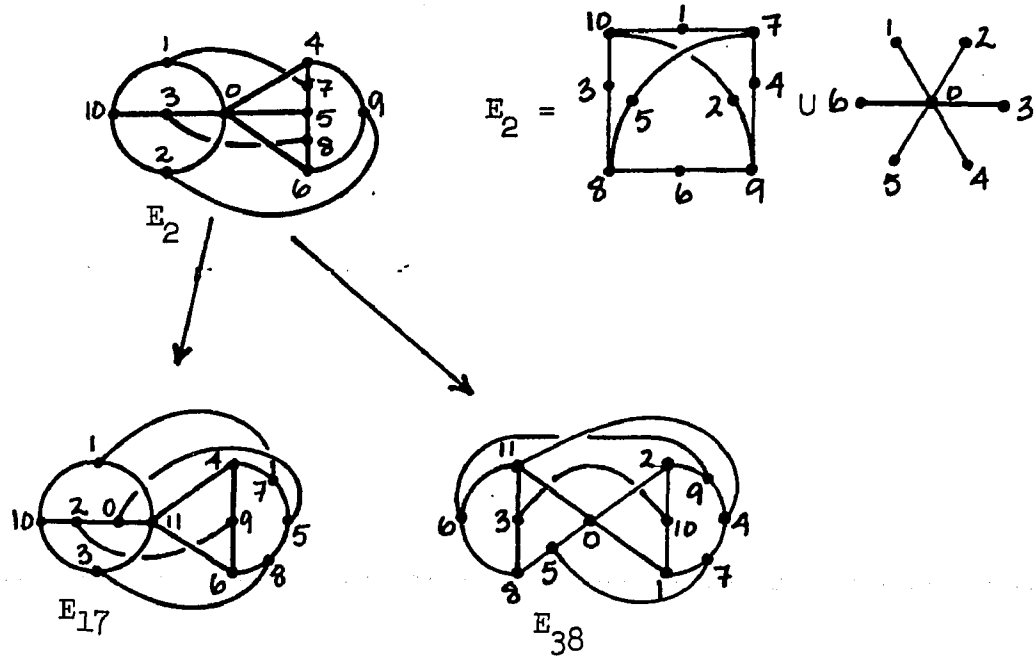


Figure 6.9

Lemma 6.10. $\{G \in \mathcal{I}(P) \mid G \leq_s B_1\} = \{B_1, B_2, B_4, B_5, B_6, B_9, B_{11}\}.$

Proof. We shall refer to the graphs as labeled in figure 6.10.

Note $B_1 = K_7 \setminus \text{cycle } (1, 4, 3, 6)$. Any splitting of vertex 1 creates a reducible edge so by lemma 6.8 and symmetry we need never consider splitting vertices 1, 3, 4 or 6. $S_5: (1, 4) = B_2$, $S_5: (1, 3, 4) = B_4$, and $S_5: (1, 2, 4) = B_5$. We observe $S_5: (0, 2) = E_3 \cup \{(0, 2), (1, 3), (4, 6)\}$ and $S_5: (1, 2)$ has $(1, 2)$ reducible, thus we need never consider these splittings. These splittings exhaust the possibilities in B_1 .

In B_4 if we split 5 again we get B_6 . If we split a second vertex in B_6 we get a graph containing C_4 . In B_4 we cannot split 7 without a forced deletion. If we split another vertex, say

2, then we must get B_9 , else the graph contains C_4 . Splitting all 3 vertices gives B_{11} .

In B_2 if we split 7 we must get B_6 , else the graph contains C_3 . Since $B_6 \leq_s B_4$ we have already found all graphs less than B_6 . If we split a second vertex we get a graph containing C_4 .

Finally in graph B_5 we note $\frac{B_5}{(4,5)} = B_1$, hence since we need not consider S_4 we need not consider S_7 . By symmetry we cannot split vertex 5. If we split another vertex we get either a graph below B_2 or B_4 or we get a graph containing C_4 .

□

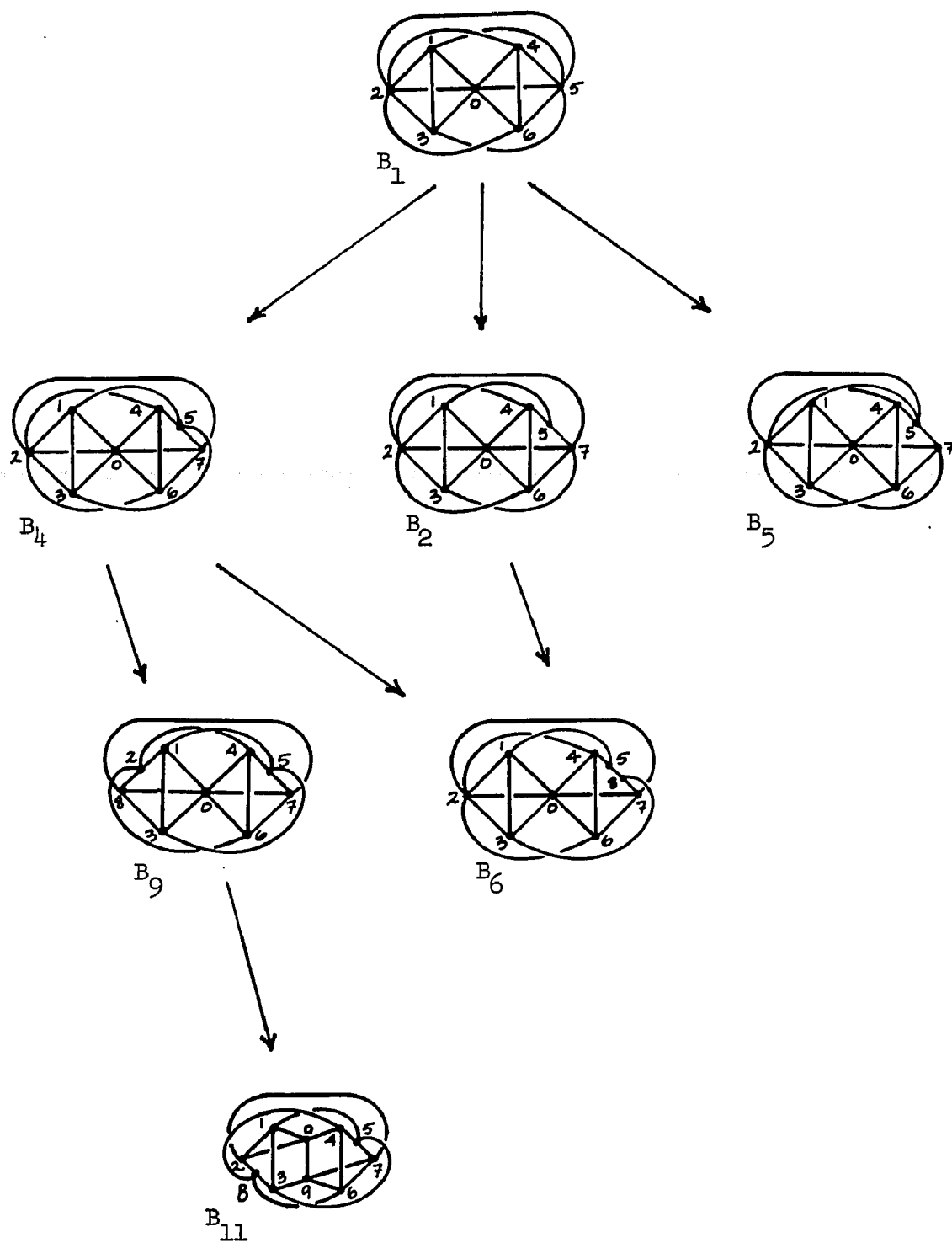


Figure 6.10

Lemma 6.11. $\{G \in I(P) \mid G \leq_s D_3\} = \{D_3, D_6, D_7, D_8, D_{10}, D_{11}, D_{14}, D_{15}, D_{16}, D_{19}\}.$

Proof. We shall refer to the graphs as labeled in figure 6.11. We observe any splitting of either vertex 2 or vertex 3 creates a cubic vertex in a 3-cycle. By lemma 6.8 we will need only consider splitting vertices 0, 2, 5. Note the symmetries $(2\ 5), (4\ 7)$ and $(1\ 3).$

Case 1. $G' \leq_s S_0(D_3).$

Without loss of generality in splitting 0 we get 2 vertices adjacent to 0. If the two are among the set $\{1, 2, 3, 5\}$ then we get a cubic vertex in a 3-cycle. Thus assume 8 is adjacent to 0.

$S_0: (8, 5)$ contains θ -graph disjoint from a k -graph, thus

$S_0: (8, 1) = D_8$ is unique.

In D_8 note the symmetry $(1\ 9)$, this says we need not consider splitting 9. If we split a second vertex then avoiding a cubic vertex in a 3-cycle or a θ -graph disjoint from a k -graph we have

$S_2: (3, 4) = E_{28} \cup \{(5, 7)\}$ or $S_2: (4, 9) = D_{14}.$

In D_{14} splitting vertex 10 gives D_{16} . In D_8 vertex 2, and by symmetry vertex 7, had a unique splitting. Applying this splitting to vertex 7 in D_{14} gives D_{19} . The only possible graph below D_{16} and D_{19} is $S_5: (1, 4)(D_{16}) = F_9 \cup \{(1, 3), (5, 12)\}.$ We have found all $G' \leq_s S_0(D_3).$

Case 2. $G' \leq_s D_3$ which do not split vertex 0.

Without loss of generality in D_3 we consider splitting vertex 5.

Avoiding a θ -graph disjoint from a k -graph implies either

$S_5: (0,4) = D_6$ or $S_5: (1,4) = D_7$. In D_6 we have the symmetry

$(0\ 2)(5\ 7)(4\ 8)$ shows we need not consider splitting vertex 2, or

else we are in case 1. If we split vertex 9 again it is equivalent

to splitting vertex 9 in D_7 , from which it is seen avoiding a

θ -graph disjoint from a k -graph implies D_{11} , or D_{10} . Thus we

have found all $G' \leq_s D_6$ and all $G' \leq D_3$ which involve splitting

either vertex 2 or 5 into 3 cubic vertices. The only remaining

possibility is to split vertex 2 in D_7 in a manner similar to

$D_3 \geq_s D_7$, giving D_{15} .

□

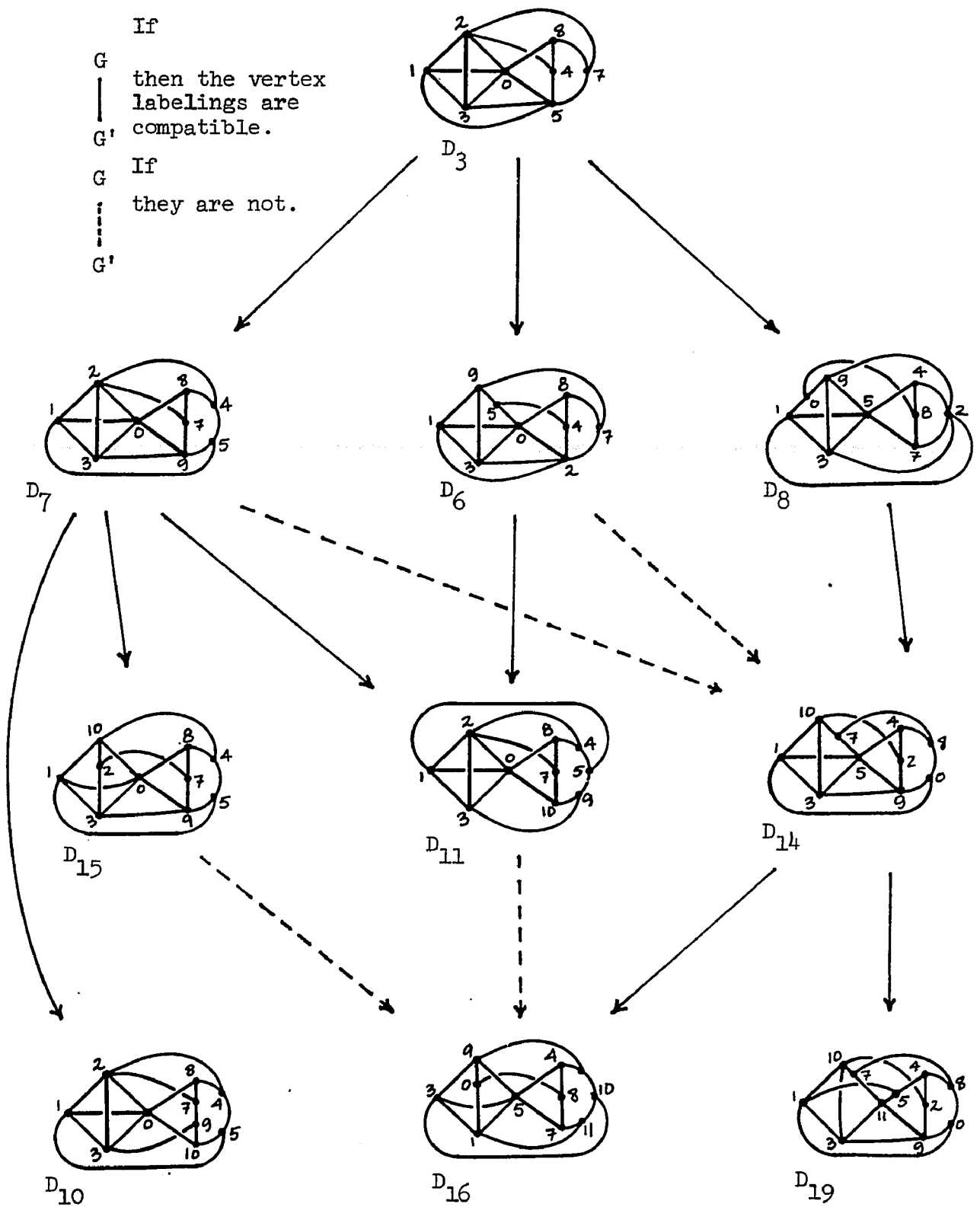


Figure 6.11

Lemma 6.12. $\{G \in I(P) \mid G \leq_s E_{18}\} = \{E_{18}, E_{21}, E_{24}, E_{28}, E_{31}, E_{32}, E_{38}\}.$

Proof. We shall refer to the graphs as labeled in figure 6.12.

Note $E_{18} = K_{4,4} \setminus (0,4)$, with bipartition sets $\{0,1,2,3\}, \{4,5,6,7\}.$

Up to symmetry the only possible splitting is $S_1: (4,5)^{(E_{18})} = E_{21}.$

In E_{21} we consider continuing to split vertices in the bipartition set $\{0,1,2,3\}.$ $S_3: (6,7)^{(E_{21})}$ contains a θ -graph disjoint from a k -graph. Since 6 is symmetric to 7 we have without loss of generality $S_3: (4,6)^{(E_{21})} = E_{24}.$ In E_{24} $S_2: (4,6)$ and $S_2: (4,5)$ both contain \parallel k -graphs, hence $S_2: (5,6)^{(E_{24})} = E_{31}$ is unique. We have exhausted graphs which involve splittings in only one bipartition set.

In E_{31} vertices 5,6,7 are all symmetric. $S_5: (0,2) = F_9 \cup \{(8,7)\}$ and $S_5: (0,1) = F_9 \cup \{(3,6)\}$ show any splitting contains a reducible edge. Thus we have found all graphs which involve splitting all 3 valency 4 vertices of a bipartition set.

In E_{24} observe vertex 5 is symmetric with 6. $S_7: (0,2) = F_5 \cup \{(3,6)\}$ and $S_7: (2,9) = F_9 \cup \{(2,5)\}.$ By symmetry we need not consider splitting vertex 7. $S_5: (0,2) = F_9 \cup \{(7,8)\}$ and $S_5: (0,1) = F_5 \cup \{(3,6)\}.$ Thus the only two splittings are $S_5: (1,2)$ and $S_6: (2,9).$ Doing one gives $E_{32},$ doing both gives $E_{38}.$ We have exhausted the possible graphs where two vertices in the same bipartition set are split.

In E_{21} we need only consider a single splitting of a vertex in the set $\{4,5,6,7\}.$ $S_5: (0,2)^{(E_{21})} = E_{28},$ $S_5: (0,1)^{(E_{21})} = F_5 \cup \{(3,6)\},$

$S_6: (0,2) = F_5 \cup \{(3,6)\}$ and $S_6: (0,8) = E_{28}$. These exhaust the possibilities by symmetry, and complete the proof of the lemma.

□

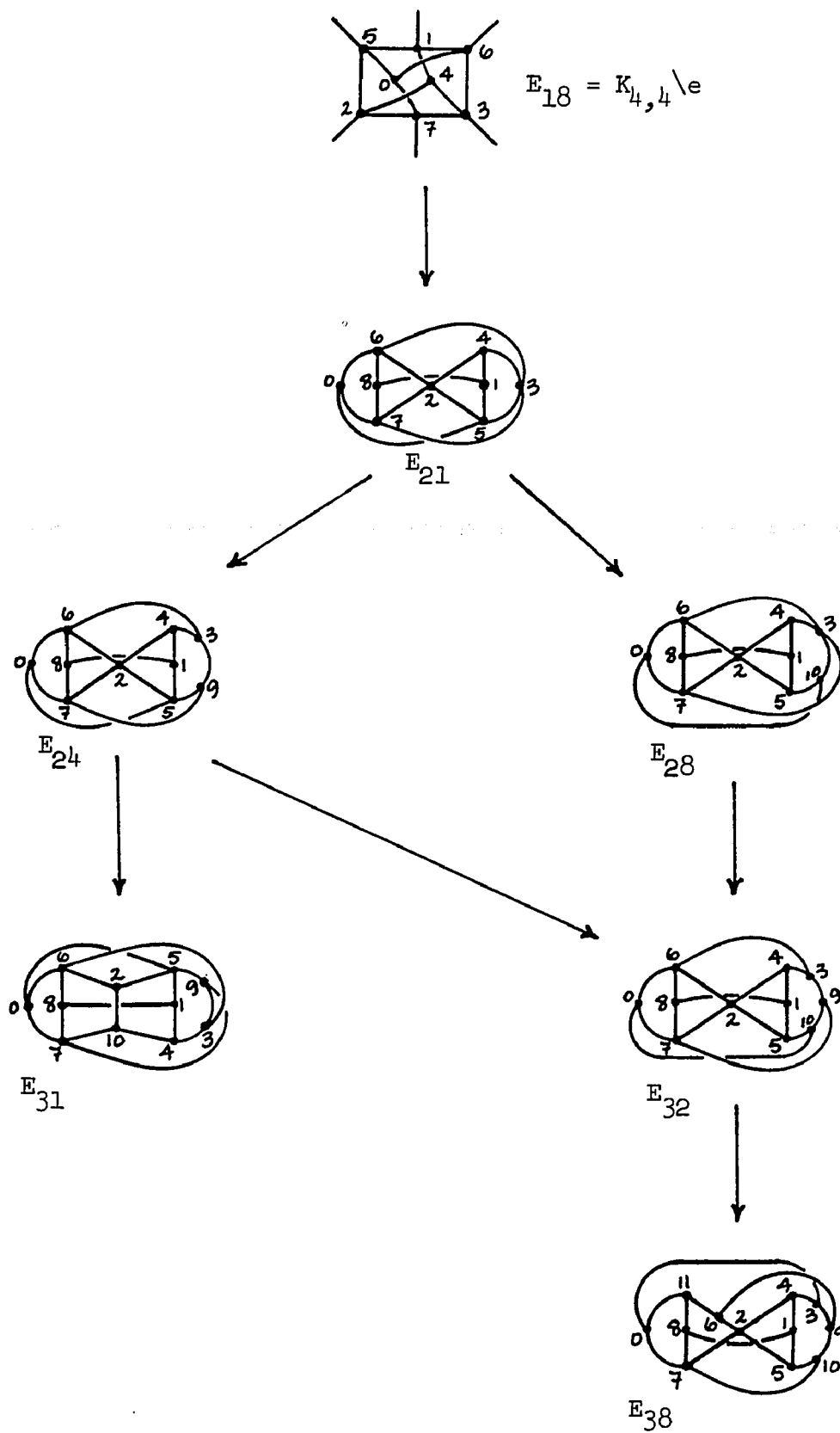


Figure 6.12

Lemma 6.13. $\{G \in I(P) \mid G \leq_s E_3\} = \{E_3, E_4, E_9, E_{10}, E_{14}, E_{17}, E_{31}\}$.

Proof. The reader is referred to figure 6.13. Note $E_3 = K_{3,5}$ with 0,1,2 the valency 5 vertices. Without loss of generality $S_0: (3,4)^{(E_3)} = E_4$. We shall first consider $G \leq_s E_3$ where one valency 5 vertex is split into 3 cubic vertices.

In E_4 splitting vertex 8, up to symmetry we need only examine $S_8: (0,5)^{(E_4)} = E_7$. In E_7 symmetry shows the only two splittings are $S_2: (3,5)^{(E_7)} = F_3 \cup \{(1,4)\}$ and $S_2: (3,6)^{(E_7)} = E_{14}$. $S_{10}: (2,4)^{(E_{14})} = F_3 \cup \{(1,6)\}$ and $S_{10}: (2,5)^{(E_{14})} = E_{17}$. Since $E_{17} \leq_s E_2$ we know by lemma 6.9 E_{17} is a $*$ -sink. If we split the valency 5 vertex in E_{14} we have either $S_1: (4,5) = F_5 \cup \{(7,9)\}$, $S_1: (4,7) = F_5 \cup \{(5,8)\}$, $S_1: (3,4)$ contains a θ -graph disjoint from a k -graph or $S_1: (3,5) = F_9 \cup \{(1,4)\}$. Thus we have all $G \leq_s E_3$ which involve splitting a vertex twice.

In E_4 if we split vertex 2 we have by symmetry $S_2: (3,6) = E_{10}$. From the above we need only consider splitting vertex 1. $S_1: (3,4)^{(E_{10})}$ and $S_1: (3,6)$ both contain θ -graph disjoint from a k -graph. $S_1: (3,5)^{(E_{10})} = E_{31}$, which is a $*$ -sink by lemma 6.12 $S_1: (4,6)$ contains a θ -graph disjoint from a k -graph, $S_1: (4,5) = F_5 \cup \{(7,8)\}$, $S_1: (5,6) = F_5 \cup \{(0,7)\}$ and $S_1: (5,7) = (1,4) \cup F_3$. Thus we have all graphs $G' \leq_s E_{10}$.

□

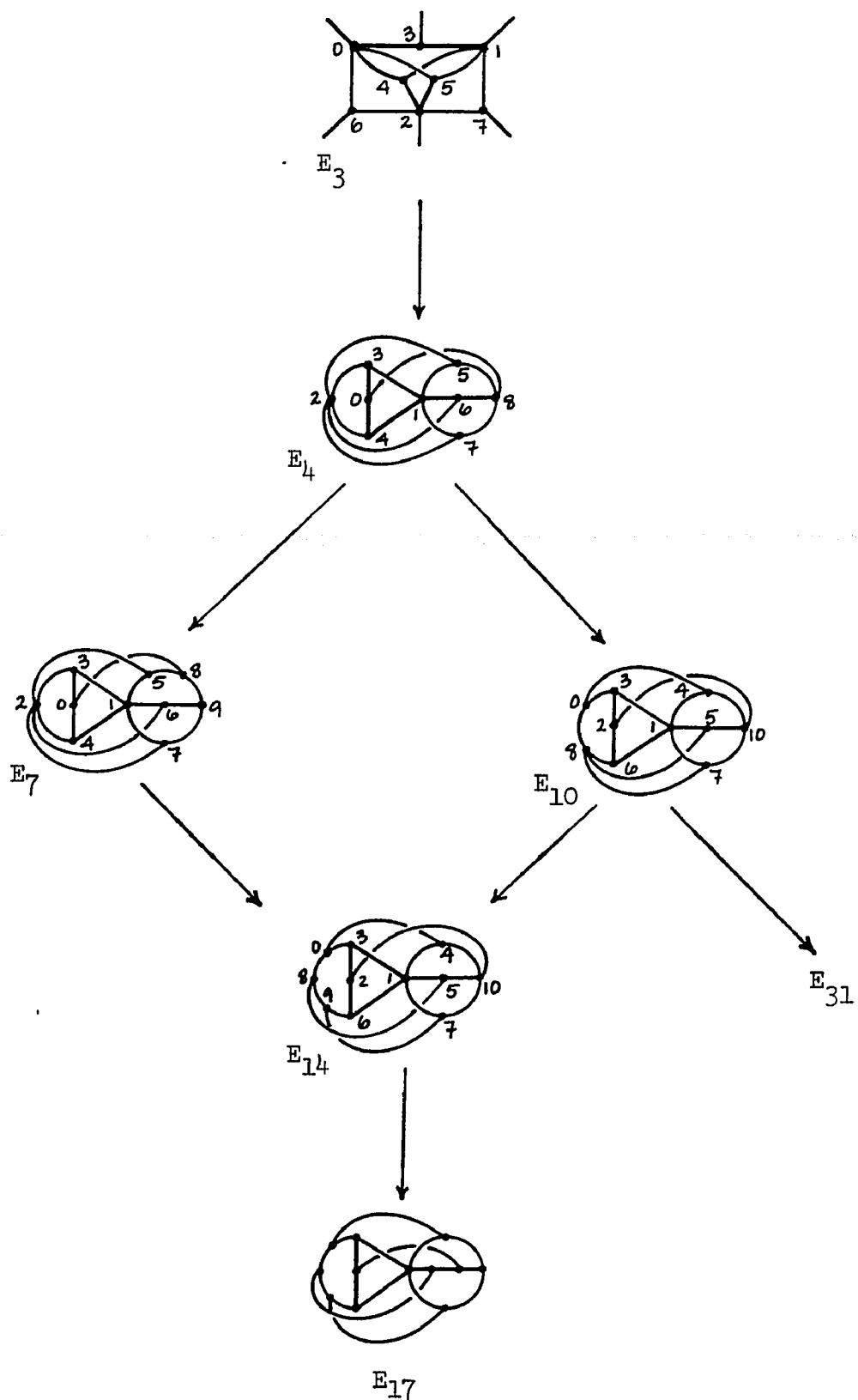


Figure 6.13

Lemma 6.14. $\{G \in I(P) \mid G \leq_s E_5\} = \{E_5, E_7, E_{12}, E_{14}, E_{15}, E_{17}, E_{33}, E_{34}, E_{39}, E_{41}\}.$

Proof. We label the graphs as in figure 6.14. Viewing $\begin{pmatrix} 3 & 4 \\ 2 & 8 & 1 \end{pmatrix} \vee \begin{pmatrix} 1 & 7 \\ 0 & 5 & 6 \end{pmatrix}$ we see $S_0: (7,8)^{(E_5)} = E_7$. From lemma 6.13 we know $\{G \in I(P) \mid G \leq_s E_7\} = \{E_7, E_{14}, E_{17}\}$. Thus we need not consider this splitting of vertex 0. Since $S_0: (1,8)^{(E_5)}$ and $S_0: (2,8)^{(E_5)}$ both contain θ -graph disjoint from a k -graph we will assume throughout this lemma 0 is not split. In E_5 $S_1: (3,4) \supset \parallel k$ -graphs and $S_1: (0,5) = F_1 \cup \{(0,5)\}$. Thus without loss of generality we have $S_1: (4,5)^{(E_5)} = E_{12}$.

We shall first consider graphs in which a valency 5 vertex of E_5 is split into three cubic vertices. In E_{12} this corresponds to S_9 , $S_9: (0,1) = E_{15}$ is the only choice by lemma 6.8. Clearly splitting vertex 2 once gives E_{39} , splitting twice gives E_{41} . Thus we have all graphs where a valency 5 vertex is split twice.

It only remains to check $S_2(E_{12})$. $S_2: (3,6)^{(E_{12})} = E_{34}$ and $S_2: (4,5)^{(E_{12})} = E_{33}$. Any further splitting gives $G' \leq_s E_{15}$ so we are done.

□

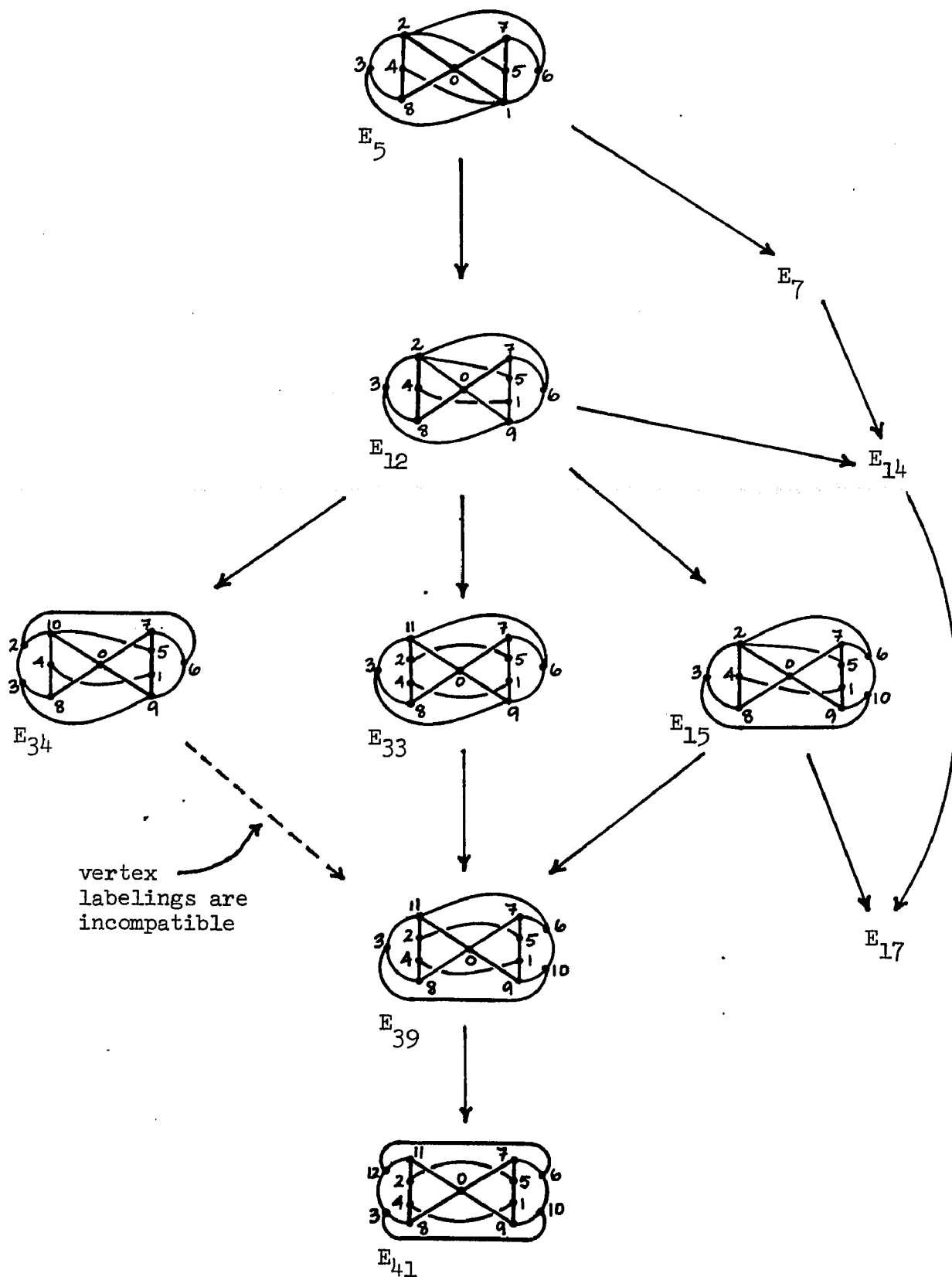


Figure 6.14

Lemma 6.15. $\{G \in I(P) \mid G \leq_s E_{22}\} = \{E_{22}, E_{25}, E_{33}, E_{34}, E_{35}, E_{39}, E_{41}\}.$

Proof. We label the graphs as in figure 6.15. $E_{22} = K_{5,4} \setminus \{(1,6), (2,7), (3,5), (4,8)\}$ where the bipartition sets are $\{0,1,2,3,4\}, \{5,6,7,8\}$. Observe $S_0: (5,8)$ contains disjoint k -graphs, by symmetry we may assume throughout this lemma vertex 0 is not split. Also by symmetry there is a unique splitting of E_{22} ,

$$S_5: (1,4) = E_{25}.$$

In E_{25} the valency 4 vertices are in two symmetry classes, $\{7\}$ and $\{6,8\}$. Note $S_7: (0,1)^{(E_{25})} = E_{34}$, $S_7: (0,3)^{(E_{25})} = E_{33}$ and $S_7: (0,4)^{(E_{25})} = F_5 \cup \{(3,8)\}$. By lemma 6.14 we know $\{G \in I(P) \mid G \leq_s E_{33} \text{ or } G \leq_s E_{34}\} = \{E_{33}, E_{34}, E_{39}, E_{41}\}$, so we need not consider splitting vertex 7. Since $S_6: (0,2)^{(E_{25})} = F_5 \cup \{(4,7)\}$ and $S_6: (0,3)^{(E_{25})} = F_5 \cup \{(1,7)\}$, there is a unique splitting on vertex 6, by symmetry there also is a unique splitting on vertex 8. Splitting one of these vertices gives E_{35} , splitting both gives E_{39} . Since $E_{39} \leq_s E_{33}$ the proof of the lemma is completed.

□

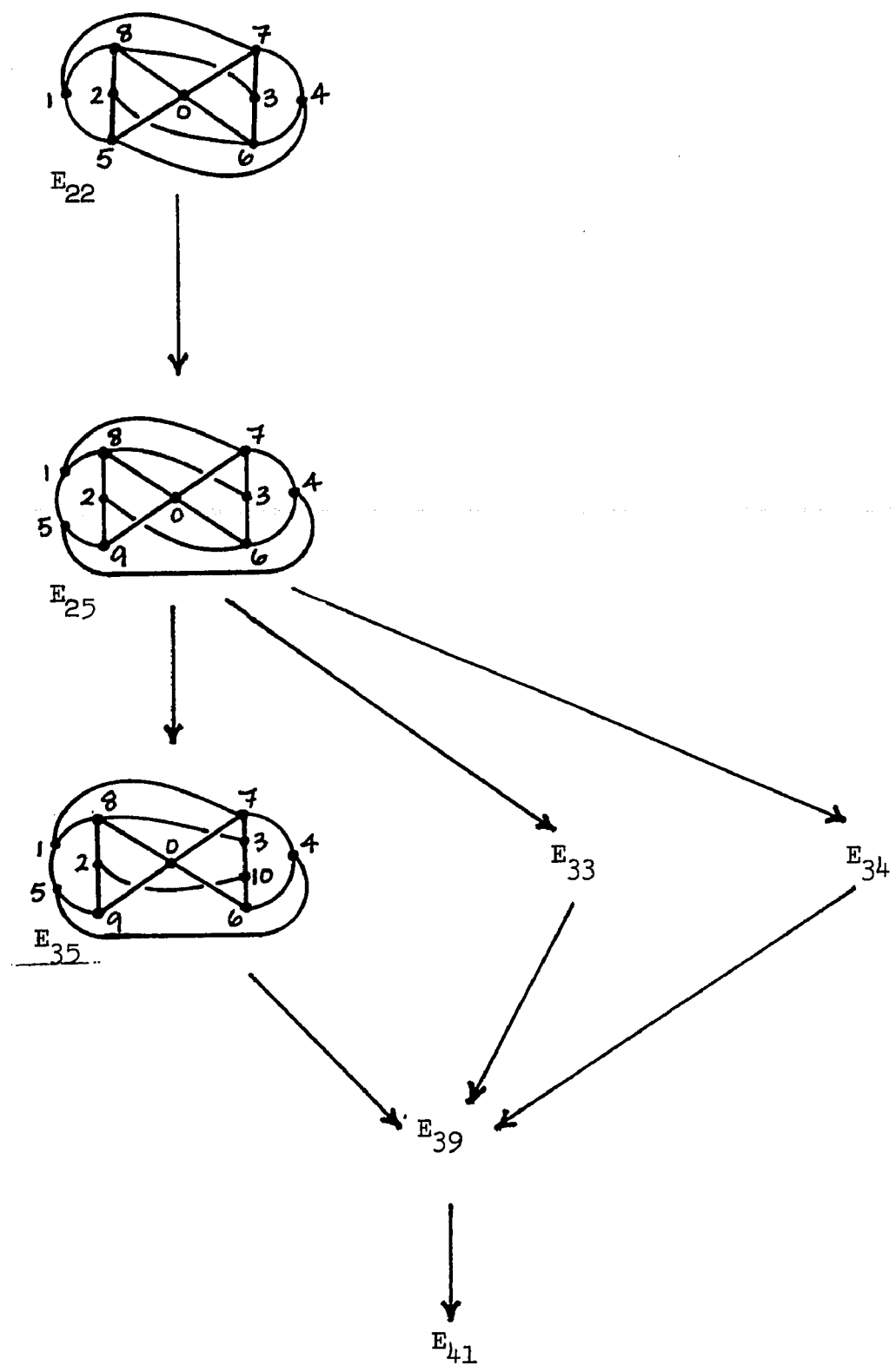


Figure 6.15

Lemma 6.16. $\{G \in I(P) \mid G \leq_s F_1\} = \{F_1, F_3, F_5, F_9, F_{13}, F_{14}\}.$

Proof. Label the graphs as shown in figure 6.16. We shall first find $\{G \in I(P) \mid G \leq_s S_0(F_1)\}$. Note $S_0: (6,8)^{(F_1)}$ contains a θ -graph disjoint from a k -graph hence up to symmetry there exists a unique splitting of 0, $S_0: (3,8)^{(F_1)} = F_5$.

In F_5 observe $S_2: (1,4)$ contains a θ -graph disjoint from $\begin{pmatrix} 1 & 4 \\ 2 & 3 & 5 \end{pmatrix}$. Up to symmetry there is a unique splitting, $S_2: (1,6)^{(F_5)} = F_9$. Vertex 1 is symmetric to 2 in F_5 , but the two possible splittings of vertex 1 are not symmetric in F_9 .

$S_1: (2,3)^{(F_9)} = F_{14}$ and $S_1: (2,5)^{(F_9)} = F_{13}$. Since the graphs are cubic we have found all $G \leq_s S_0(F_1)$.

In F_1 we now need only consider splitting vertices 1 and 2. $S_2: (1,4)$ contains a θ -graph disjoint from $\begin{pmatrix} 1 & 4 \\ 2 & 3 & 5 \end{pmatrix}$, so up to symmetry $S_2: (1,6)^{(F_1)} = F_3$ is unique. Both $S_1(F_5)$ give F_9 , since $F_9 \leq_s S_0(F_1)$ the proof of the lemma is completed.

□

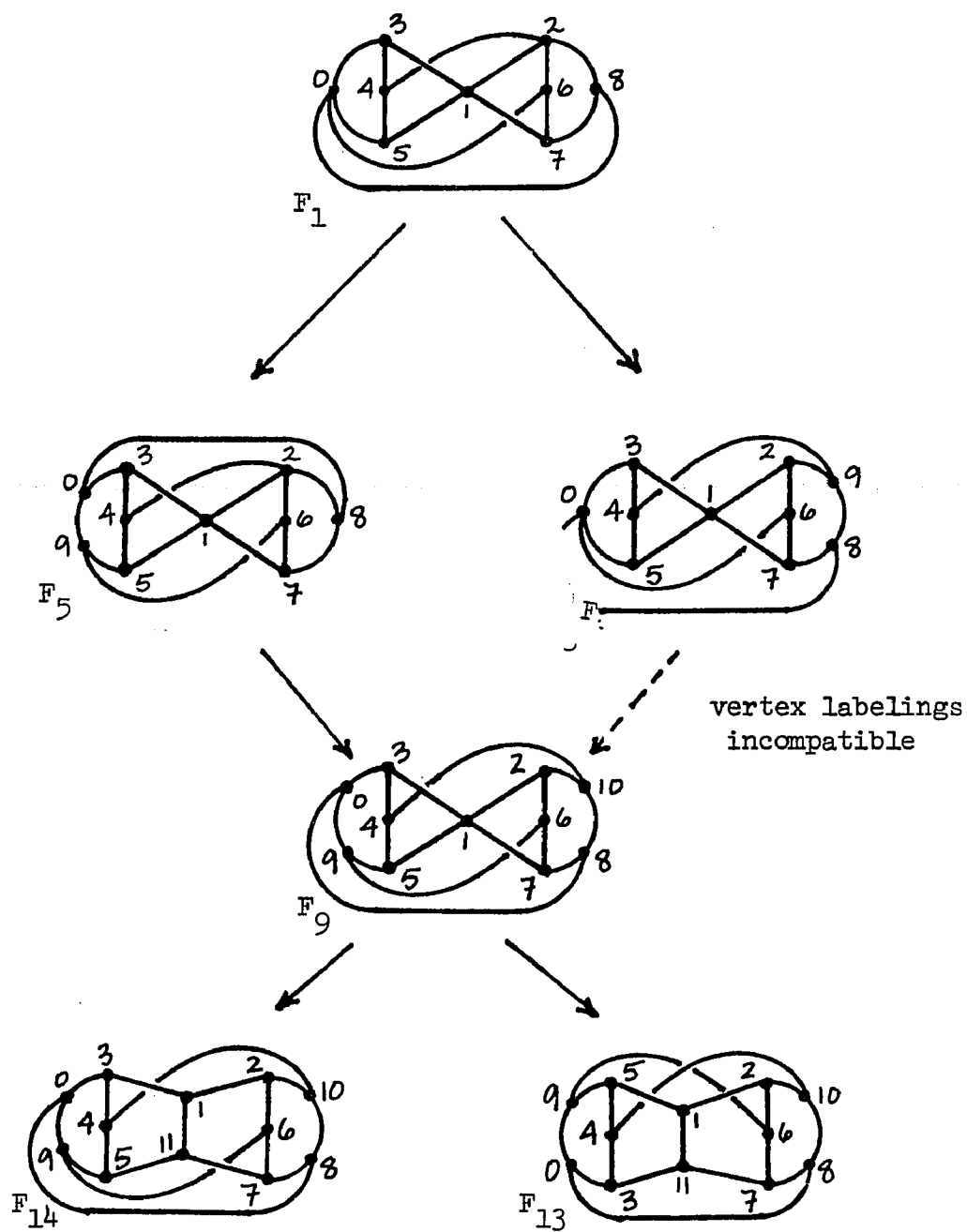


Figure 6.16

Chapter 7

CONCLUSIONS

In this section we offer several theorems which follow from our main result. In §7.2 we investigate some possible research directions and give some concluding remarks.

§7.1 Further Results

Theorem 7.1. $I^M(P) = \{A_1, A_2, B_1, B_3, D_9\}$.

Proof. Recall \mathcal{D} as defined in §6.1. Following from theorem 6.1 and theorem 1.7 $\mathcal{D} = I_*^M(P)$. Since \leq_x is a course ordering we know $I^M(P) \subset I_*^M(P)$. The proof of the theorem breaks into two parts; first finding \leq derivations for $G \in I_*^M(P) \setminus I^M(P)$, secondly showing $\{A_1, A_2, B_1, B_3, D_9\}$ are incomparable under \leq .

For the first part of the proof we refer the reader to figures 7.1, 7.2 and 7.3. In these figures we have identified each graph in $I_*^M(P)$ (excepting those claimed in $I^M(P)$) as an elementary derivation of some other graph. Thus we conclude $I^M(P) \subset \{A_1, A_2, B_1, B_3, D_9\}$.

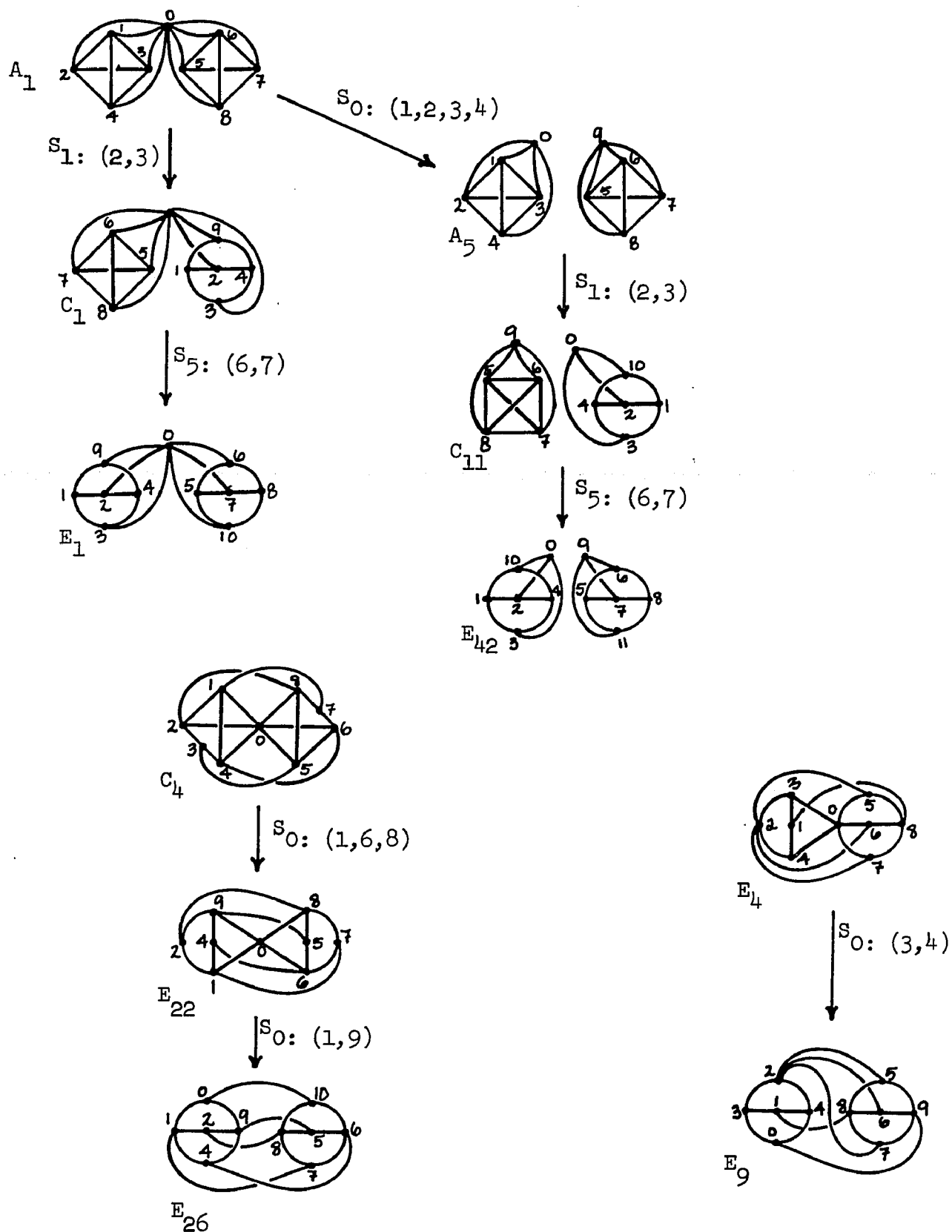


Figure 7.1

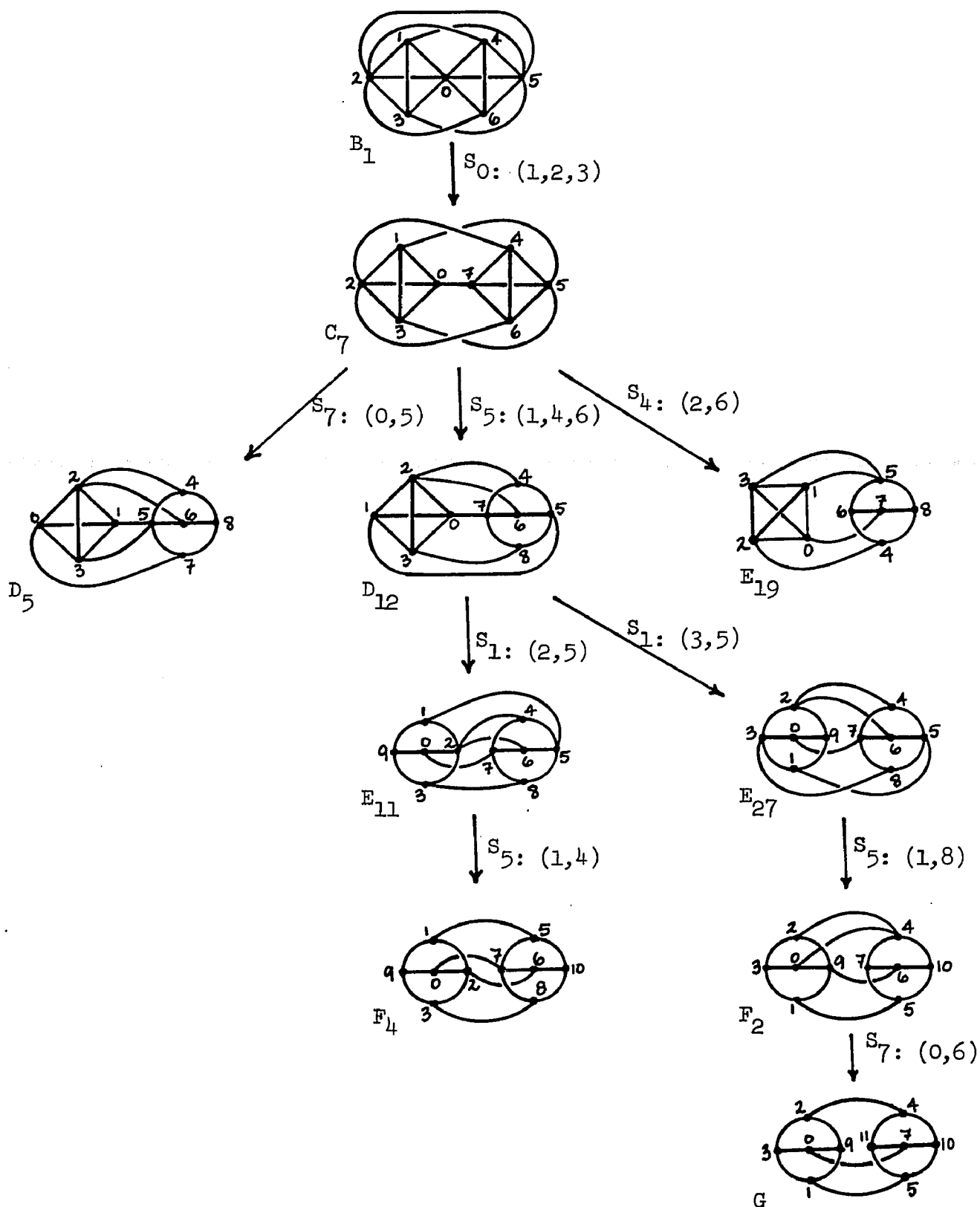


Figure 7.2

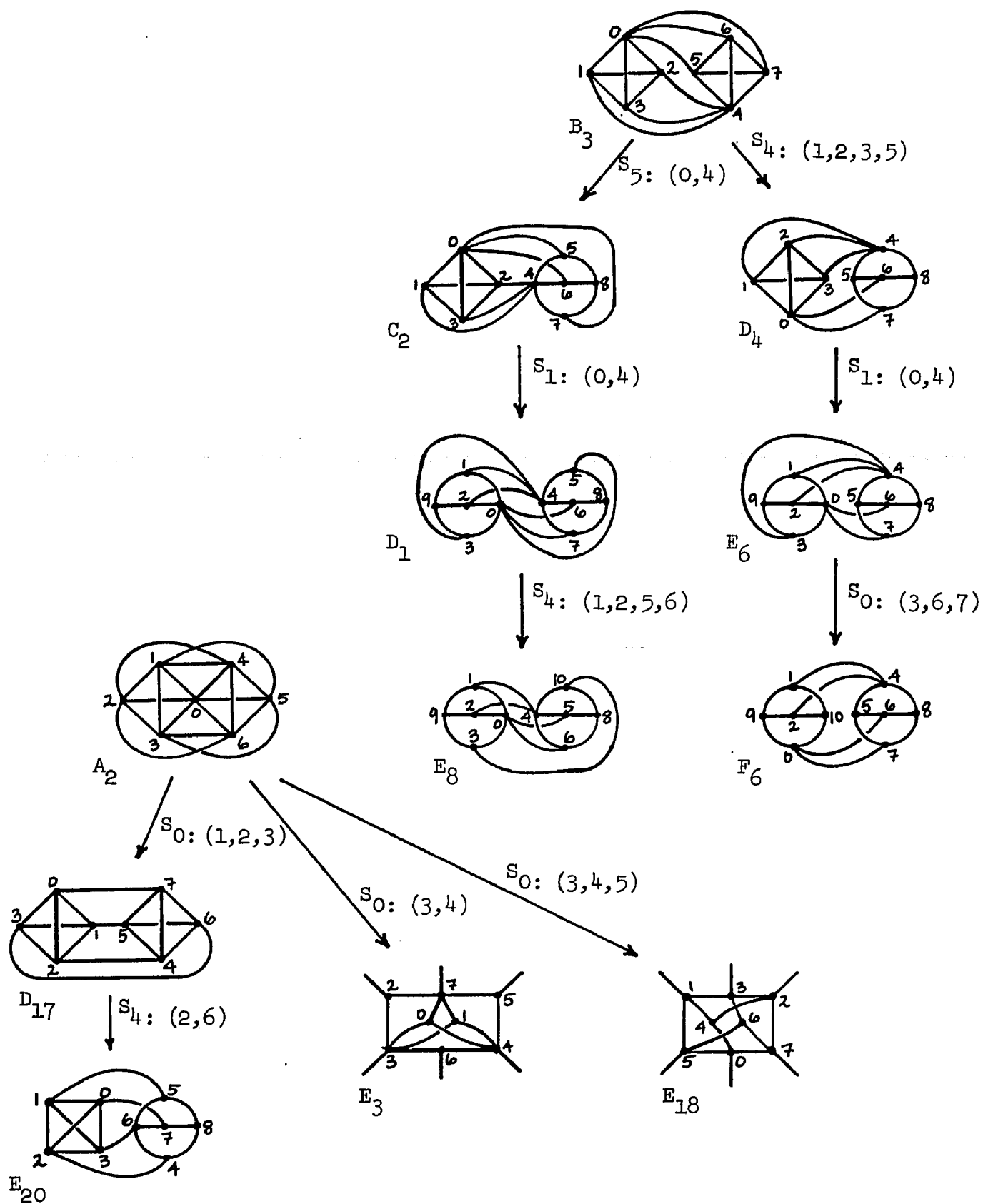


Figure 7.3

For the second part of the proof we need to show each of the 5 graphs are maximal. It suffices to show the 5 are pairwise incomparable under \leq . As in the proof of lemma 1.5 we shall use the function σ which assigns to each graph its valency sequence. We offer the following obvious facts:

- 1) $G > G' \Rightarrow \sigma(G) > \sigma(G')$,
- 2) $G > G' \Rightarrow \beta(G) \geq \beta(G')$ where $\beta(G)$ denotes the betti number of G .

From the appendix we have $\sigma(A_1) = (8, 4^8) > \sigma(B_1) = (6^3, 4^4) > \sigma(B_3) = (6^2, 4^6) > \sigma(A_2) = (6, 5^6) > \sigma(D_9) = (5^2, 4^2, 3^6)$. From the contrapositive of 1) we immediately conclude $A_1 \in I^M(P)$. For graph B_1 we need only check if it derives from A_1 . Since no splitting of a single valency 8 vertex gives 3 valency 6 vertices we conclude $B_1 \in I^M(P)$. Likewise we know $B_3 \not\leq A_1$, since we checked all splittings of B_1 (lemma 6.10) we conclude $B_3 \in I^M(P)$. Graph $A_2 \not\leq A_1$ (A_2 contains too many vertices of valency > 4), 1) and 2) combine to show A_2 is not comparable to either B_1 or B_3 . We note $D_9 \not\leq A_1$ since the only way to get two valency 5 vertices is to split the valency 8 vertex. The valency 5 vertices are adjacent in $S(A_1)$ but not in D_9 . Observe B_1 contains 4 valency 4 vertices while D_9 contains only 2. Any splitting of a valency 4 vertex, or edge deletion of an edge incident to a valency 4 vertex, was examined in lemma 6.7. Such an operation decreases the β number by 2, thus $D_9 \not\leq B_1$. Finally in graph B_3 splitting a valency 6 vertex causes 2 edge deletions. Splitting both valency

6 vertices gives a graph G' with $\beta(G') < 9$. By 2) we conclude $D_9 \not\in B_3$. Thus $D_9 \in I^M(P)$ and the proof of the theorem is complete.

□

Define the Kuratowski cover number of G , $K(G)$, as the least k s.t. $G = \bigcup_{i=1}^k H_i$, where $H_i \in I(\mathbb{R}^2)$.

Theorem 7.2. $G \in I(P)$ implies $K(G) = 2$.

Proof. It is clear $K(G) \geq 2$. To show $K(G) \leq 2$ we check the 103 cases, writing each graph as the union of two Kuratowski graphs. We note in many cases the union is easy, e.g., G contains disjoint k -graphs. The graphs as shown in the appendix emphasize the k -graphs of G , in each case there is a completion of the k -graphs to Kuratowski graphs which gives all of G .

□

§7.2 Some Related Problems

The real projective plane is probably the last surface Σ for which an explicit listing seems realistic. For example, it is easy to verify the number of one-connected graphs in $I(\tilde{\Sigma}_2)$ is over 4000. However, many interesting questions about $I(\Sigma)$ remain open. The following are but two well known unsolved problems:

Conjecture. $|I(\Sigma)| < \infty$ for all surfaces Σ

Conjecture. 1) $G \in I(\Sigma_n) \Rightarrow K(G) = 2n + 1,$

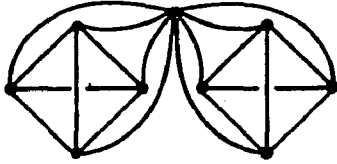
2) $G \in I(\tilde{\Sigma}_n) \Rightarrow K(G) = n + 1.$

We note theorem 7.2 shows the second conjecture (part 2) is true for $n = 1$.

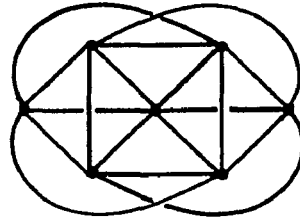
Of special interest in $I(\Sigma)$ are the maximal graphs and the minimal graphs. Observe bounding $|I^M(\Sigma)|$ also bounds $|I(\Sigma)|$. On the other hand, given an arbitrary graph we can check its genus by $|I^M(\Sigma)|$ also bounds $|I(\Sigma)|$. On splitting (in all possible ways) to a set of cubic graphs and looking for minimal irreducible subgraphs. Characterizing any of these sets remains an open question.

Finally, given $G, G' \in I(\Sigma)$, $G' \leq G$, what does $S(G) \setminus G'$ look like? Which edges are reducible in $S(G)$? The set $I(P)$ provides a place to investigate these problems.

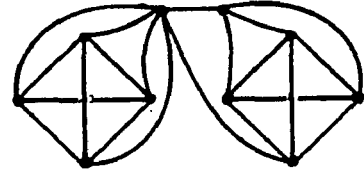
APPENDIX



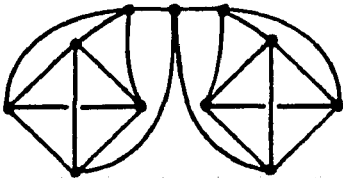
A_1 $9(8,4^8)$ maximal



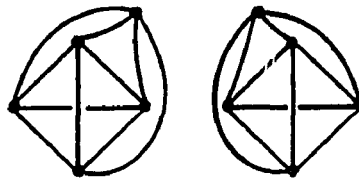
A_2 $7(6,5^6)$ maximal



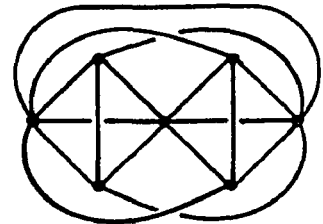
A_3 $10(6,4^9)$ ($\langle A_1 \rangle$)



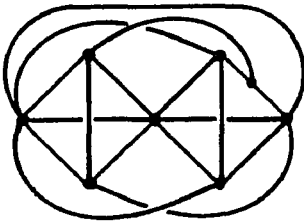
A_4 $11(4^{11})$ ($\langle A_3 \rangle$)



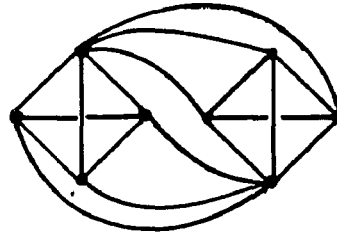
A_5 $10(4^{10})$ ($\langle A_1 \rangle$)



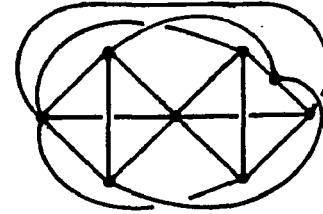
B_1 $7(6^3,4^4)$ maximal



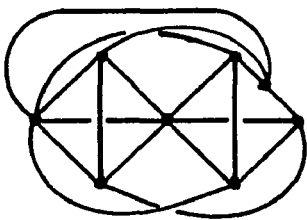
B_2 $8(6^2,5,4^4,3)$ ($\langle B_1 \rangle$)



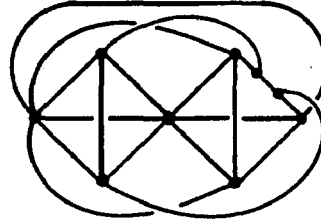
B_3 $8(6^2,4^6)$ maximal



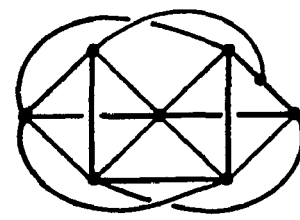
B_4 $8(6^2,2^6)$ ($\langle B_1 \rangle$)



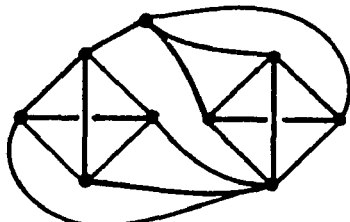
B_5 $8(6^2,4^6)$ ($\langle B_1 \rangle$)



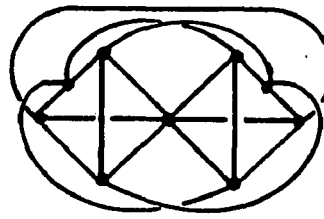
B_6 $9(6^2,4^5,3^2)$ ($\langle B_4 \rangle$)



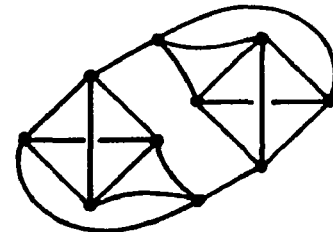
B_7 $8(6,5^3,4^3,3)$ ($\langle A_2 \rangle$)



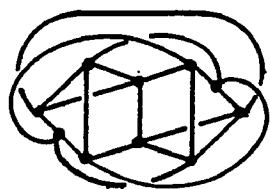
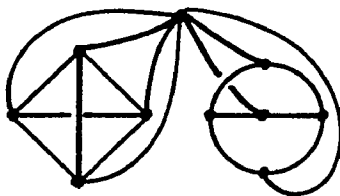
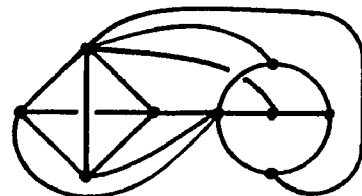
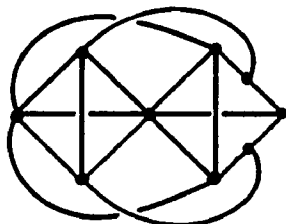
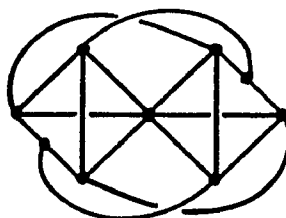
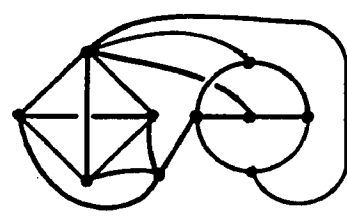
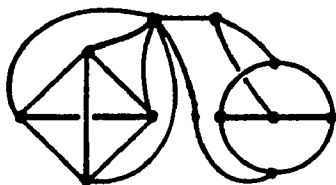
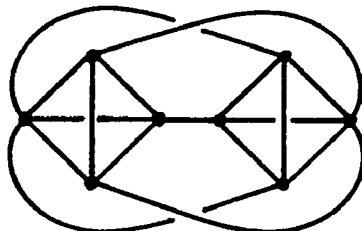
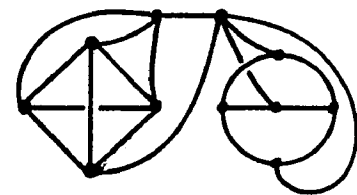
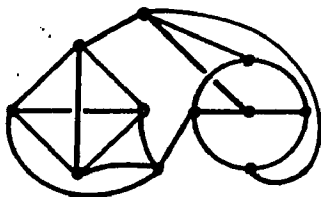
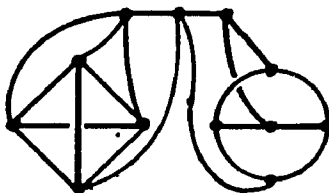
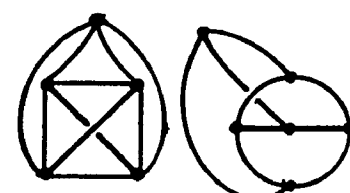
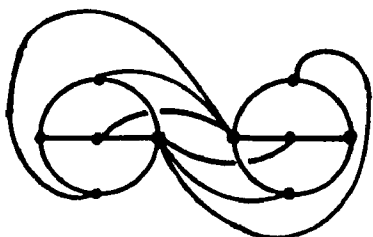
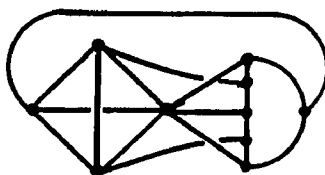
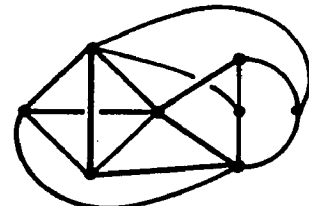
B_8 $9(6,4^8)$ ($\langle B_3 \rangle$)

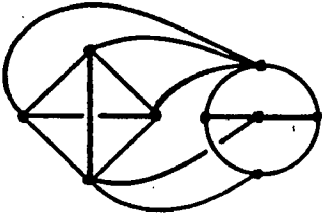
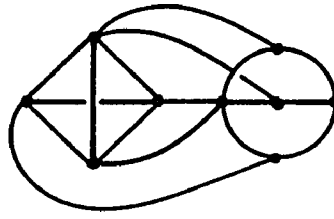
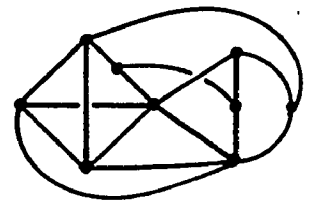
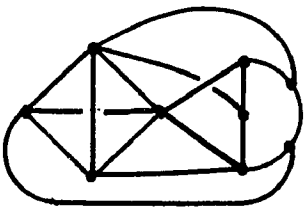
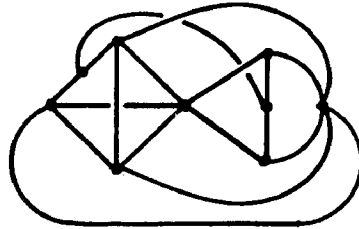
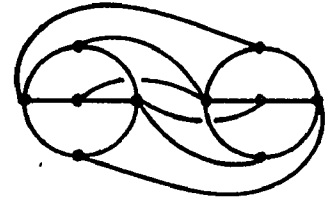
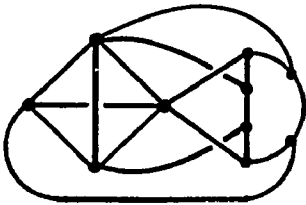
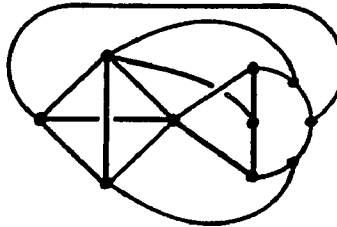
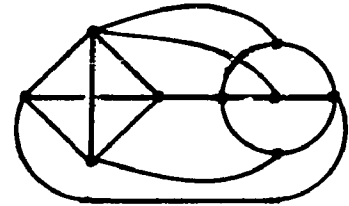
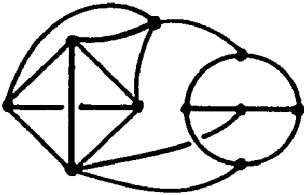
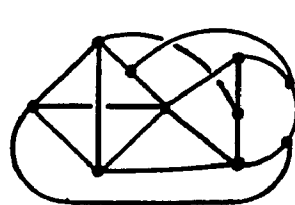
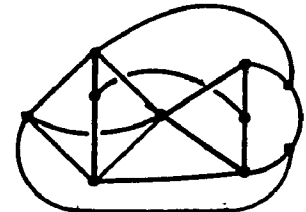
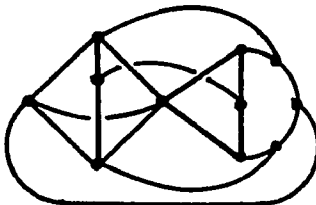
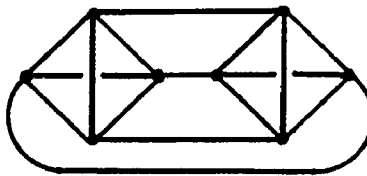
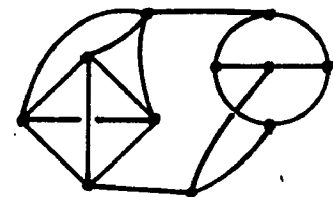


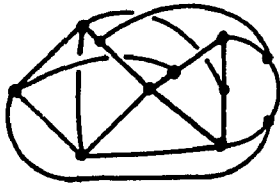
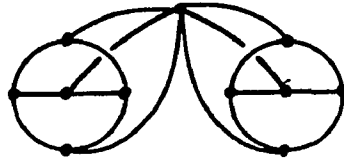
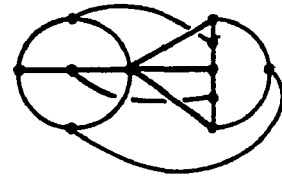
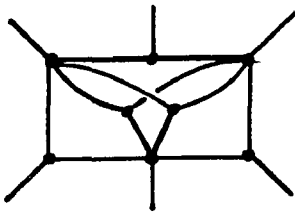
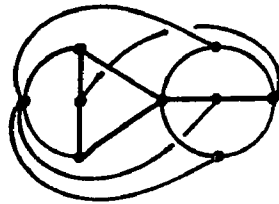
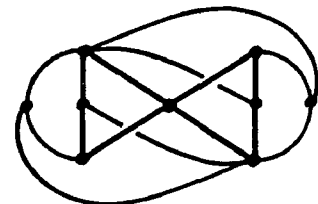
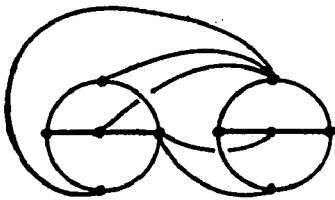
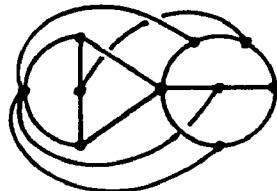
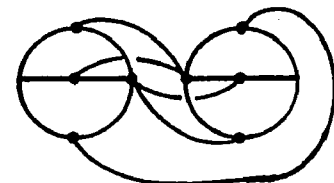
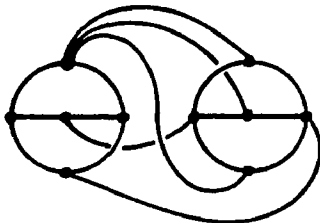
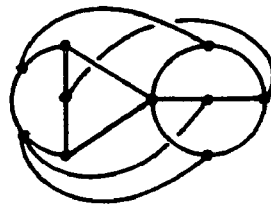
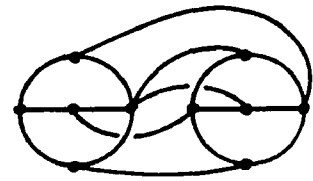
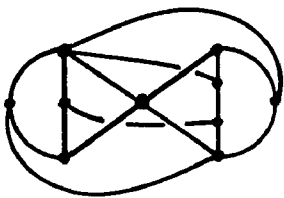
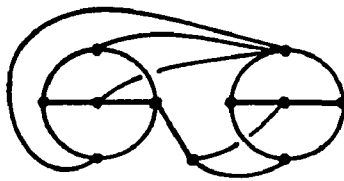
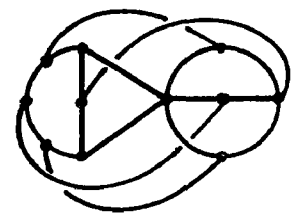
B_9 $9(6,4^8)$ ($\langle B_4 \rangle$)

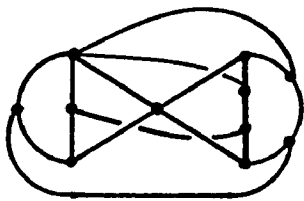
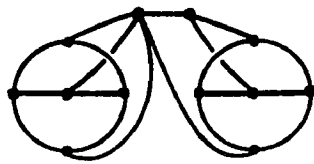
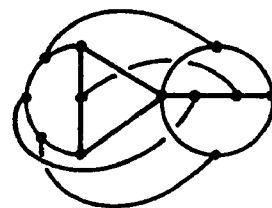
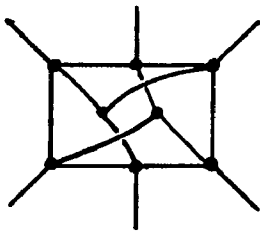
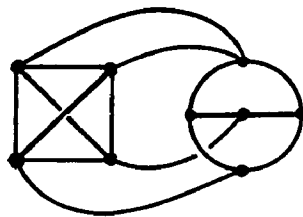
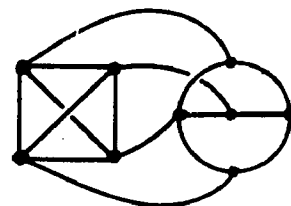
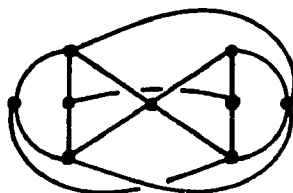
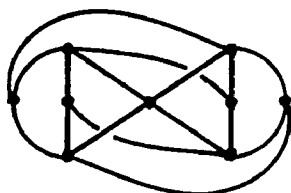
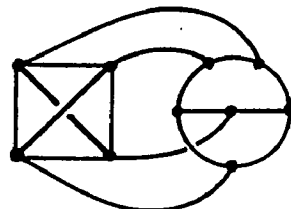
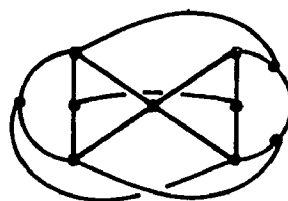
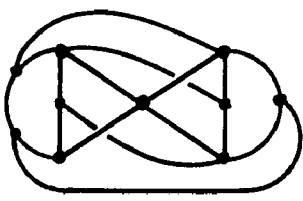
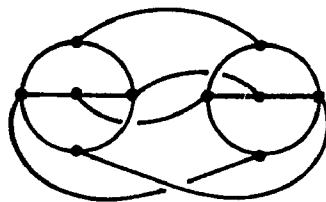
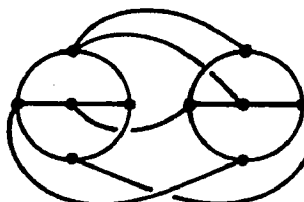
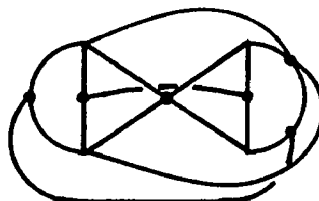
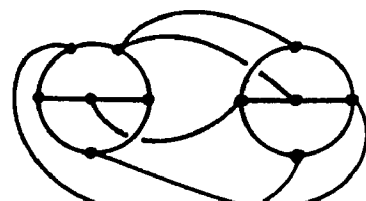


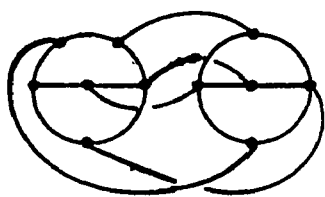
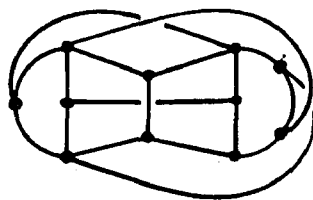
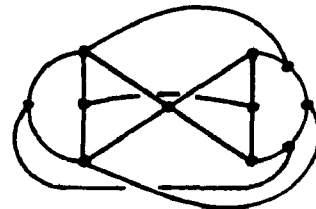
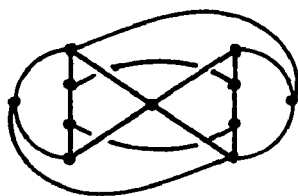
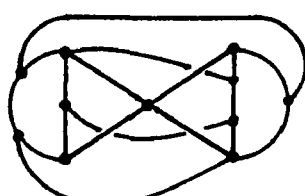
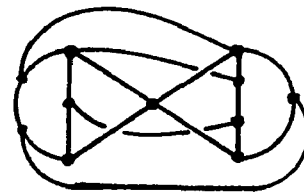
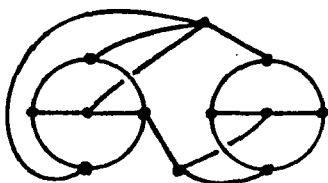
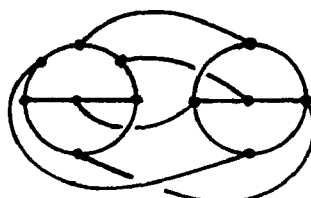
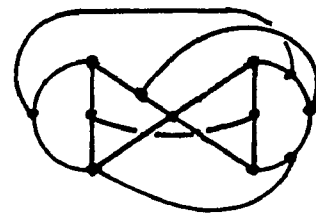
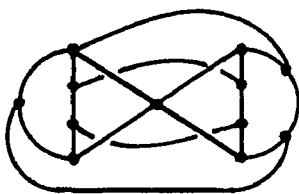
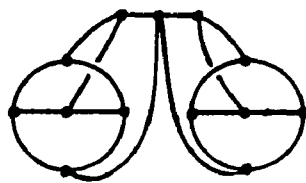
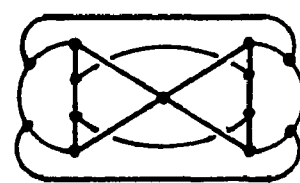
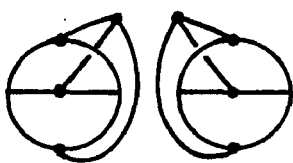
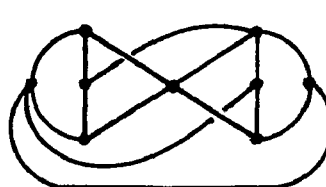
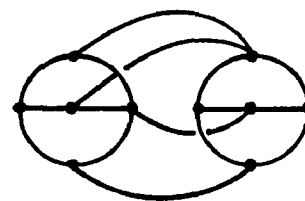
B_{10} $10(4^{10})$ ($\langle B_8 \rangle$)

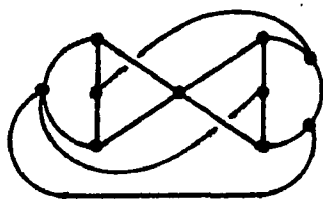
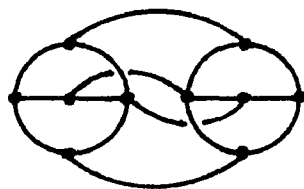
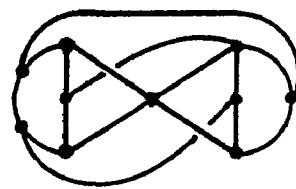
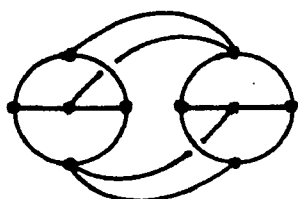
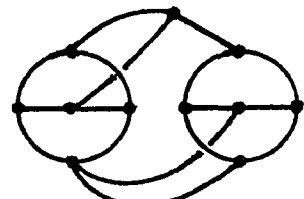
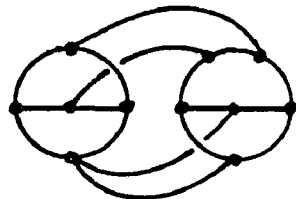
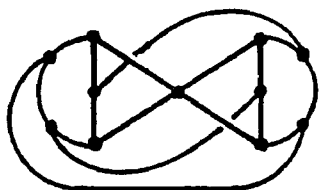
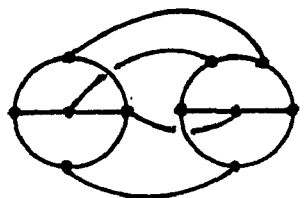
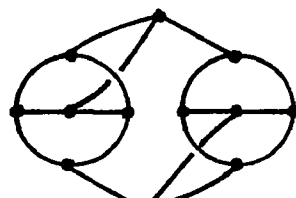
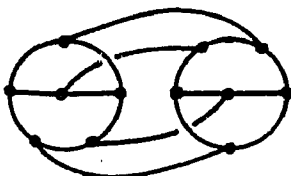
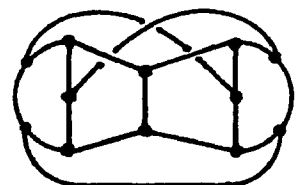
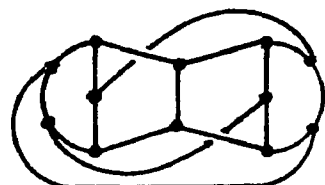
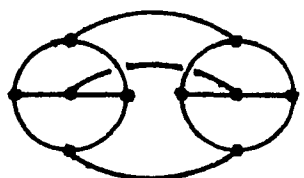

 $B_{11} \ 10(4^{10}) \ (\langle B_9 \rangle)$

 $C_1 \ 10(7, 4^4, 3^5) \ (\langle A_1 \rangle)$

 $C_2 \ 9(6^2, 4^3, 3^4) \ (\langle B_3 \rangle)$

 $C_3 \ 9(6, 5, 4^4, 3^3) \ (\langle B_7 \rangle)$

 $C_4 \ 9(6, 4^6, 3^2) \ (\langle B_7 \rangle)$

 $C_5 \ 10(6, 4^5, 3^4) \ (\langle C_2 \rangle)$

 $C_6 \ 11(6, 4^4, 3^6) \ (\langle C_1 \rangle)$

 $C_7 \ 8(5^2, 4^6) \ (\langle B_1 \rangle)$

 $C_8 \ 11(5, 4^5, 3^5) \ (\langle C_1 \rangle)$

 $C_9 \ 11(4^7, 3^4) \ (\langle C_5 \rangle)$

 $C_{10} \ 12(4^6, 3^6) \ (\langle C_6 \rangle)$

 $C_{11} \ 11(4^5, 3^6) \ (\langle C_1 \rangle)$

 $D_1 \ 10(6^2, 3^8) \ (\langle C_2 \rangle)$

 $D_2 \ 10(6, 4^3, 3^6) \ (\langle C_3 \rangle)$

 $D_3 \ 8(5^3, 4^2, 3^3) \ (\langle B_1 \rangle)$


 $D_4 \ 9(5^2, 4^3, 3^4) \ (\triangleleft B_3)$

 $D_5 \ 10(6, 4^3, 3^6) \ (\triangleleft C_3)$

 $D_6 \ 9(5^2, 4^3, 3^4) \ (\triangleleft D_3)$

 $D_7 \ 9(5^2, 4^3, 3^4) \ (\triangleleft D_3)$

 $D_8 \ 9(5^2, 4^3, 3^4) \ (\triangleleft D_3)$

 $D_9 \ 10(5^2, 4^2, 3^6) \text{ maximal}$

 $D_{10} \ 10(5^2, 4^2, 3^6) \ (\triangleleft D_7)$

 $D_{11} \ 10(5^2, 4^2, 3^6) \ (\triangleleft D_7)$

 $D_{12} \ 9(5, 4^5, 3^3) \ (\triangleleft C_7)$

 $D_{13} \ 10(5, 4^4, 3^5) \ (\triangleleft D_4)$

 $D_{14} \ 10(5, 4^4, 3^5) \ (\triangleleft D_7)$

 $D_{15} \ 10(5, 4^4, 3^5) \ (\triangleleft D_7)$

 $D_{16} \ 11(5, 4^3, 3^7) \ (\triangleleft D_{15})$

 $D_{17} \ 8(4^8) \ (\triangleleft A_2)$

 $D_{18} \ 11(4^5, 3^6) \ (\triangleleft D_{13})$


 $D_{19} \ 11(4^5, 3^6) \ (\triangleleft D_{14})$

 $E_1 \ 11(6, 3^{10}) \ (\triangleleft C_1)$

 $E_2 \ 11(6, 3^{10}) \ (\triangleleft D_2)$

 $E_3 = K_{3,5} \ 8(5^3, 3^5) \ (\triangleleft A_1)$

 $E_4 \ 9(5^2, 4, 3^6) \ (\triangleleft E_3)$

 $E_5 \ 9(5^2, 4, 3^6) \ (\triangleleft D_3)$

 $E_6 \ 10(5^2, 3^8) \ (\triangleleft D_4)$

 $E_7 \ 10(5^2, 3^8) \ (\triangleleft E_4)$

 $E_8 \ 10(5^2, 3^8) \ (\triangleleft E_5)$

 $E_9 \ 10(5, 4^2, 3^7) \ (\triangleleft E_4)$

 $E_{10} \ 10(5, 4^2, 3^7) \ (\triangleleft E_4)$

 $E_{11} \ 10(5, 4^2, 3^7) \ (\triangleleft D_{12})$

 $E_{12} \ 10(5, 4^2, 3^7) \ (\triangleleft E_5)$

 $E_{13} \ 11(5, 4, 3^9) \ (\triangleleft E_6)$

 $E_{14} \ 11(5, 4, 3^9) \ (\triangleleft E_{10})$


 $E_{15} \ 11(5,4,3^9) \ (\triangleleft E_{12})$

 $E_{16} \ 12(5,3^{11}) \ (\triangleleft E_1)$

 $E_{17} \ 12(5,3^{11}) \ (\triangleleft E_{14})$

 $E_{18} = K_{4,4} \setminus e \ 8(4^6,3^2) \ (\triangleleft B_9)$

 $E_{19} \ 9(4^5,3^4) \ (\triangleleft D_3)$

 $E_{20} \ 9(4^5,3^4) \ (\triangleleft D_{17})$

 $E_{21} \ 9(4^5,3^4) \ (\triangleleft E_{18})$

 $E_{22} \ 9(4^5,3^4) \ (\triangleleft C_4)$

 $E_{23} \ 10(4^4,3^6) \ (\triangleleft E_{19})$

 $E_{24} \ 10(4^4,3^6) \ (\triangleleft E_{21})$

 $E_{25} \ 10(4^4,3^6) \ (\triangleleft E_{22})$

 $E_{26} \ 10(4^4,3^6) \ (\triangleleft E_{22})$

 $E_{27} \ 10(4^4,3^6) \ (\triangleleft D_{12})$

 $E_{28} \ 10(4^4,3^6) \ (\triangleleft E_{21})$

 $E_{29} \ 11(4^3,3^8) \ (\triangleleft E_{27})$


 $E_{30} \ 11(4_3, 3^8) \ (\triangleleft E_{26})$

 $E_{31} \ 11(4^3, 3^8) \ (\triangleleft E_{24})$

 $E_{32} \ 11(4^3, 3^8) \ (\triangleleft E_{24})$

 $E_{33} \ 11(4^3, 3^8) \ (\triangleleft E_{25})$

 $E_{34} \ 11(4^3, 3^8) \ (\triangleleft E_{25})$

 $E_{35} \ 11(4^3, 3^8) \ (\triangleleft E_{25})$

 $E_{36} \ 12(4^2, 3^{10}) \ (\triangleleft E_{13})$

 $E_{37} \ 12(4^2, 3^{10}) \ (\triangleleft E_{29})$

 $E_{38} \ 12(4^2, 3^{10}) \ (\triangleleft E_{32})$

 $E_{39} \ 12(4^2, 3^{10}) \ (\triangleleft E_{33})$

 $E_{40} \ 13(4, 3^{12}) \ (\triangleleft E_{16})$

 $E_{41} \ 13(4, 3^{12}) \ (\triangleleft E_{39})$

 $E_{42} \ 12(3^{12}) \ (\triangleleft E_{40})$
 minimal

 $F_1 \ 9(4^3, 3^6) \ (\triangleleft D_3)$

 $F_2 \ 10(4^2, 3^8) \ (\triangleleft F_1)$


 $F_3 \ 10(4^2, 3^8) \ (\langle F_1 \rangle)$

 $F_4 \ 10(4^2, 3^8) \ (\langle E_{25} \rangle)$

 $F_5 \ 10(4^2, 3^8) \ (\langle F_1 \rangle)$

 $F_6 \ 10(4^2, 3^8) \ (\langle E_6 \rangle)$

 $F_7 \ 11(4, 3^{10}) \ (\langle F_6 \rangle)$

 $F_8 \ 11(4, 3^{10}) \ (\langle F_6 \rangle)$

 $F_9 \ 11(4, 3^{10}) \ (\langle F_5 \rangle)$

 $F_{10} \ 11(4, 3^{10}) \ (\langle F_4 \rangle)$

 $F_{11} \ 12(3^{12}) \ (\langle F_7 \rangle)$

 $F_{12} \ 12(3^{12}) \ (F_{10})$
minimal

 $F_{13} \ 12(3^{12}) \ (\langle F_9 \rangle)$

 $F_{14} \ 12(3_{12}) \ (\langle F_9 \rangle)$

 $G \ 10(3^{10}) \ (\langle F_3 \rangle)$

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