

STORAGE OPERATORS and \forall -POSITIVE TYPES in *TTR* TYPE SYSTEM

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Abstract In 1990, J.L. Krivine introduced the notion of storage operator to simulate "call by value" in the "call by name" strategy. J.L. Krivine has shown that, using Gödel translation of classical into intuitionistic logic, we can find a simple type for the storage operators in *AF2* type system. This paper studies the \forall -positive types (the universal second order quantifier appears positively in these types), and the Gödel transformations (a generalization of classical Gödel translation) of *TTR* type system. We generalize, by using syntactical methods, the J.L. Krivine's Theorem about these types and for these transformations. We give a proof of this result in the case of the type of recursive integers.

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1 Introduction

The strategy of left reduction (iteration of head reduction denoted by \succ) has the following advantages :

- It has good mathematical properties stated by the normalisation Theorem : if a λ -term is normalizable, then we obtain the normal form by left reduction.
- It seems more economic since we compute a λ -term only when we need it.

Now, a drawback of the strategy of left reduction (call by name) is the fact that the argument of a function is computed as many times as it is used. The purpose of storage operators is precisely to correct this drawback.

Let F be a λ -term (a function), and \underline{N} the set of normal Church integers. During the computation, by left reduction, of $(F)\theta_n$ (where $\theta_n \simeq_{\beta} \underline{n}$), θ_n may be computed several times (as many times as F uses it). We would like to transform $(F)\theta_n$ to $(F)\underline{n}$. We also want this transformation depends only on θ_n (and not F). In other words we look for some closed λ -terms T with the following properties :

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- For every F , $n \in \mathbb{N}$, and $\theta_n \simeq_\beta \underline{n}$, we have $(T)\theta_n F \succ (F)\underline{n}$;
- The computation time of the head reduction $(T)\theta_n F \succ (F)\underline{n}$ depends only on θ_n .

Therefore the first definition : A closed λ -term T is called storage operator for \underline{N} if and only if for every $n \in \mathbb{N}$, and for every $\theta_n \simeq_\beta \underline{n}$, $(T)\theta_n f \succ (f)\underline{n}$ (where f is a new variable).

It is clear that a storage operator satisfies the required properties. Indeed,

- Since we have $(T)\theta_n f \succ (f)\underline{n}$, then the variable f never comes in head position during the reduction, and we may then replace f by any λ -term.
- The computation time of the head reduction $(T)\theta_n F \succ (F)\underline{n}$ depends only on θ_n .

We showed (see [12]) that it is not possible to get the normal form of θ_n . We then change the definition : A closed λ -term T is called storage operator for \underline{N} if and only if for every $n \in \mathbb{N}$, there is a closed λ -term $\tau_n \simeq_\beta \underline{n}$ (for example $\tau_n = (\underline{s})^n \underline{0}$, where \underline{s} is a λ -term for the successor), such that for every $\theta_n \simeq_\beta \underline{n}$, $(T)\theta_n f \succ (f)\tau_n$ (where f is a new variable).

If we take $T_1 = \lambda n((n)\lambda x \lambda y(x)\lambda z(y)(\underline{s})z)\lambda f(f)\underline{0}$, and $T_2 = \lambda n \lambda f(((n)\lambda x \lambda y(x)(\underline{s})y)f)\underline{0}$, then it is easy to check that : for every $\theta_n \simeq_\beta \underline{n}$, $(T_1)\theta_n f \succ (f)(\underline{s})^n \underline{0}$, and $(T_2)\theta_n f \succ (f)(\underline{s})^n \underline{0}$. Therefore T_1 and T_2 are storage operators for \underline{N} .

The $AF2$ type system is a way of interpreting the proof rules for the second order intuitionistic logic plus equational reasoning as construction rules for terms. In this system we have the possibility to define the data types, the representation in λ -calculus being automatically extracted from the logical definition of the data type. At the logical level the data type are defined by second order formulas expressing the usual iterative definition of the corresponding algebras of terms and the data receive the corresponding iterative definition in λ -calculus. For example, the type of integers is the formula : $N[x] = \forall X\{\forall y[X(y) \rightarrow X(sy)] \rightarrow [X(0) \rightarrow X(x)]\}$ (X is a unary predicate variable, 0 is a constant symbol for zero, and s is a unary function symbol for successor).

If we try to type a storage operator T in $AF2$ type system, we naturally find the type $\forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$ (where O is a particular 0-ary predicate symbol which represents an arbitrary type). Indeed, if $\vdash_{AF2} \tau_n : N[s^n(0)]$, and f is of type $N[s^n(0)] \rightarrow O$, then $f : N[s^n(0)] \rightarrow O \vdash_{AF2} (f)\tau_n : O$. It is natural to have $(T)\theta_n f$ of type O . If $\vdash_{AF2} \theta_n : N[s^n(0)]$, then the type for T must be $\forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$.

It is easy to check that $\vdash_{AF2} T_1, T_2 : \forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$.

The type $\forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$ does not characterize the storage operators. Indeed, if we take $T = \lambda n \lambda f(f)n$, we obtain :

- $n : N[x], f : N[x] \rightarrow O \vdash_{AF2} (f)n : O$, then, $\vdash_{AF2} T : \forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$.
- For every $\theta_n \simeq_\beta \underline{n}$, $(T)\theta_n f \succ (f)\theta_n$, therefore T is not a storage operator for \underline{N} .

This comes from the fact that the type $\forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$ does not take into account the independance of τ_n with θ_n . To solve this problem, we must prevent the use of the first $N[x]$ in $\forall x\{N[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$ as well as his subtypes to prove the second. Therefore, we will replace the first $N[x]$ by a new type $N^*[x]$ with the following properties :

- $\vdash_{AF2} \underline{n} : N^*[s^n(0)]$ (for example, take $N^*[x] = \forall X\{\forall y[F(X, y) \rightarrow F(X, sy)] \rightarrow [F(X, 0) \rightarrow F(X, x)]\}$) ;
- If $\nu : N^*[x], x_i : \forall y[F(G, y) \rightarrow F(G, sy)], y_j : F(H, a) \vdash_{AF2} t : N[s^n(0)]$, then $\vdash_{AF2} t' : N[s^n(0)]$, where t' is the normal form of t ;
- There is a closed λ -term T , such that $\vdash_{AF2} T : \forall x\{N^*[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$.

A simple solution for the second property is to take a formula $F(X, a)$ ending with a new constant symbol. Indeed, since $N[x]$ does not contain this symbol, we cannot use the variables ν, x_i, y_j in the typing of t' . We suggest the following proposition :

$$N^*[x] = \forall X\{\forall y[(X(y) \rightarrow O) \rightarrow (X(sy) \rightarrow O)] \rightarrow [(X(0) \rightarrow O) \rightarrow (X(x) \rightarrow O)]\}.$$

It is easy to chech that $\vdash_{AF2} T_1, T_2 : \forall x\{N^*[x] \rightarrow [(N[x] \rightarrow O) \rightarrow O]\}$ (see [6] and [12]).

For each formula F of $AF2$, we indicate by F^* the formula obtained by putting \neg in front of each atomic formulas of F (F^* is called the Gödel translation of F).

J.L. Krivine has shown that the type $\forall x\{N^*[x] \rightarrow \neg\neg N[x]\}$ characterize the storage operators for \underline{N} (see [6]). But the λ -term τ_n obtained may contain variables substituted by λ -terms u_1, \dots, u_m depending on θ_n . Since the λ -term τ_n is $\beta\eta$ -equivalent to \underline{n} , therefore, the left reduction of the $\tau_n[u_1/x_1, \dots, u_m/x_m]$ is equivalent to the left reduction of τ_n and the λ -terms u_1, \dots, u_m will therefore never be evaluated during the reduction.

Taking into account the above remarks, we modify again the definition : A closed λ -term T is called a storage operator for \underline{N} if and only if for every $n \in \mathbb{N}$, there is a λ -term $\tau_n \simeq_\beta \underline{n}$, such that for every $\theta_n \simeq_\beta \underline{n}$, there is a substitution σ , such that $(T)\theta_n f \succ (f)\sigma(\tau_n)$ (where f is a new variable).

The $AF2$ type system is satisfactory from an extensional point of view : one can construct programs for all the functions whose termination is provable in the second order Peano arithmetic. But from an intensional point of view the situation is very different : we cannot always obtain the simple (in term of time complexity, for instance) programs we need. For example we cannot find a λ -term of type $\forall x\forall y\{N[x], N[y] \rightarrow N[\min(x, y)]\}$ (\min is a binary function symbol defined by equations) in $AF2$ type system that computes the minimum of two Church integers in time $O(\min)$ ³

³R. David gives a λ -term of type $N, N \rightarrow N$ ($N = \forall X\{[X \rightarrow X] \rightarrow [X \rightarrow X]\}$) in F type system that computes the minimum of two Church integers in time $O(\min.\text{Log}(\min))$. The notion of storage operators plays an important tool in this construction (see [2]).

The *TTR* type system is an extension of *AF2* based on recursive definitions of types, which is intended to solve the basic problems of efficiency mentioned before. In *TTR* we have a logical operator μ of least fixed point. If A is a formula, C an n -ary predicate symbol which appears and occurs positively in A , x_1, \dots, x_n first order variables, and t_1, \dots, t_n terms, then $\mu C x_1 \dots x_n A < t_1, \dots, t_n >$ is a formula called the least fixed point of A in C calculated over the terms t_1, \dots, t_n . The intended logical meaning of the formula $\mu C x_1 \dots x_n A < t_1, \dots, t_n >$ is $K(t_1, \dots, t_n)$, where K is the least X , such that $X(x_1, \dots, x_n) \longleftrightarrow A$. *TTR* allows to define the multisorted term algebras as least fixed points. For example the type of recursive integers is the formula : $N^r[x] = \mu C z [\forall X \{ \forall y [C(y) \rightarrow X(sy)] \rightarrow [X(0) \rightarrow X(z)] \}] < x >$ (X is a unary predicate variable, 0 is a constant symbol for zero, and s is a unary function symbol for successor).

In this paper we study the types D of *TTR*, and the transformations $*$, for which we have the following result : if $\vdash_{TTR} T : D^* \rightarrow \neg \neg D$, then for every λ -term t with $\vdash_{TTR} t : D$, there are λ -terms τ_t and τ'_t such that $\tau_t \simeq_\beta \tau'_t$, $\vdash_{TTR} \tau'_t : D$, and for every $\theta_t \simeq_\beta t$, there is a substitution σ , such that $(T)\theta_t f \succ (f)\sigma(\tau_t)$ (where f is a new variable).

We prove ⁴ that, to obtain this result, it suffices to assume that :

- The universal second order quantifier appears positively in D (\forall -positive type) ⁵.
- The transformation $*$ satisfies the following properties :
 - If $A = C(t_1, \dots, t_n)$, then $A^* = A$;
 - If $A = X(t_1, \dots, t_n)$, then $A^* = F_X[t_1/x_1, \dots, t_n/x_n] < X_1, \dots, X_r >$ where F_X is a formula ending with \perp and having $x_1, \dots, x_n, X_1, \dots, X_r$ as free variables ;
 - $(A \rightarrow B)^* = A^* \rightarrow B^*$;
 - $(\forall x A)^* = \forall x A^*$.
 - $(\forall X A)^* = \forall X_1 \dots X_r A^*$.
 - $(\mu C x_1 \dots x_n A < t_1, \dots, t_n >)^* = \mu C x_1 \dots x_n A^* < t_1, \dots, t_n >$.

We give the proof of this result in the case of the type of recursive integers.

2 Basic notions of pure λ -calculus

Our notation is standard (see [1] and [5]).

We denote by Λ the set of terms of pure λ -calculus, also called λ -terms.

Let $t, u, u_1, \dots, u_n \in \Lambda$, the application of t to u is denoted by $(t)u$. In the same way we write

⁴J.L. Krivine and the author proved independently the same result for *AF2* type system (see [7] and [12]).

⁵This types were studied by some authors (in particular R. Labib-Sami), and have remarkable properties (see [8]).

$(t)u_1 \dots u_n$ instead of $(\dots((t)u_1)\dots)u_n$.

The β -reduction (resp. β -equivalence) is denoted by $t \rightarrow_\beta u$ (resp. $t \simeq_\beta u$).

The set of free variables of a λ -term t is denoted by $Fv(t)$.

The notation $t[u_1/x_1, \dots, u_n/x_n]$ represents the result of the simultaneous substitution of λ -terms u_1, \dots, u_n to the free variables x_1, \dots, x_n of t (after a suitable renaming of the bounded variables of t).

With each normal λ -term, we associate a set of λ -terms $STE(t)$ by induction :

if $t = \lambda x_1 \dots \lambda x_n (y) t_1 \dots t_m$, then $STE(t) = \{t\} \cup \bigcup_{1 \leq i \leq m} STE(t_i)$.

Let us recall that a λ -term t either has a head redex [i.e. $t = \lambda x_1 \dots \lambda x_n (\lambda x u) v v_1 \dots v_m$, the head redex being $(\lambda x u)v$], or is in head normal form [i.e. $t = \lambda x_1 \dots \lambda x_n (x) v v_1 \dots v_m$].

The notation $t \succ t'$ means that t' is obtained from t by some head reductions, and we denote by $n(t, t')$, the number of steps to go from t to t' .

A λ -term t is said to be solvable if and only if the head reduction of t terminates.

We define an equivalence relation \sim on Λ by : $u \sim v$ if and only if there is a t , such that $u \succ t$, and $v \succ t$. In particular, if v is in head normal form, then $u \sim v$ means that v is the head normal form of u .

Theorem 2.1 ([6]). *If $t \succ t'$, then for every $u_1, \dots, u_n \in \Lambda$:*

- 1) *there is a $v \in \Lambda$, such that $(t)u_1 \dots u_n \succ v$, $(t')u_1 \dots u_n \succ v$, and $n((t)u_1 \dots u_n, v) = n((t')u_1 \dots u_n, v) + n(t, t')$.*
- 2) *$t[u_1/x_1, \dots, u_n/x_n] \succ t'[u_1/x_1, \dots, u_n/x_n]$, and $n(t[u_1/x_1, \dots, u_n/x_n], t'[u_1/x_1, \dots, u_n/x_n]) = n(t, t')$.*

Remark. Theorem 2.1 shows that to make the head reduction of $(t)u_1 \dots u_n$ (resp. $t[u_1/x_1, \dots, u_n/x_n]$), it is equivalent (same result, and same number of steps) to make some steps in the head reduction of t , and then make the head reduction of $(t')u_1 \dots u_n$ (resp. $t'[u_1/x_1, \dots, u_n/x_n]$).

3 Basic notions of typed λ -calculus

3.1 The AF2 type system

The types will be formulas of second order predicate logic over a given language.

The logical symbols are \perp (for absurd), \rightarrow and \forall (and no other ones).

There are individual variables : x, y, \dots (also called first order variables) and n -ary predicate variables ($n = 0, 1, \dots$) : X, Y, \dots (also called second order variables).

The terms and the formulas are up in the usual way.

The formula $F_1 \rightarrow (F_2 \rightarrow (\dots \rightarrow (F_n \rightarrow G)\dots))$ is denoted by $F_1, F_2, \dots, F_n \rightarrow G$, and $F \rightarrow \perp$ is denoted by $\neg F$. The formula $\forall v_1 \dots \forall v_n F$ is denoted by $\forall \mathbf{v} F$, and the sentence " \mathbf{v} is not free in A " means that for all $1 \leq i \leq n$, v_i is not free in A .

If X is a unary predicate variable, t and t' two terms, then the formula $\forall X[Xt \rightarrow Xt']$ is denoted by $t = t'$, and is said to be equation. A particular case of $t = t'$ is a formula of the forme $t[u_1/x_1, \dots, u_n/x_n] = t'[u_1/x_1, \dots, u_n/x_n]$ or $t'[u_1/x_1, \dots, u_n/x_n] = t[u_1/x_1, \dots, u_n/x_n]$, u_1, \dots, u_n being terms of the language.

After, we denote by \mathbf{E} a system of function equations.

A context Γ is a set of the form $x_1 : A_1, \dots, x_n : A_n$ where x_1, \dots, x_n are distinct variables and A_1, \dots, A_n are formulas.

We are going to describe a system of typed λ -calculus called second order functional arithmetic (shortened in $AF2$ for Arithmétique Fonctionnelle du seconde ordre). The rules of typing are the following :

- (1) $\Gamma, x : A \vdash_{AF2} x : A$.
- (2) If $\Gamma, x : B \vdash_{AF2} t : C$, then $\Gamma \vdash_{AF2} \lambda xt : B \rightarrow C$.
- (3) If $\Gamma \vdash_{AF2} u : B \rightarrow C$, and $\Gamma \vdash_{AF2} v : B$, then $\Gamma \vdash_{AF2} (u)v : C$.
- (4) If $\Gamma \vdash_{AF2} t : A$, and x does not appear in Γ , then $\Gamma \vdash_{AF2} t : \forall xA$.
- (5) If $\Gamma \vdash_{AF2} t : \forall xA$, then, for every term u , $\Gamma \vdash_{AF2} t : A[u/x]$.
- (6) If $\Gamma \vdash_{AF2} t : A$, and X does not appear in Γ , then $\Gamma \vdash_{AF2} t : \forall XA$.
- (7) If $\Gamma \vdash_{AF2} t : \forall XA$, then, for every formula G , $\Gamma \vdash_{AF2} t : A[G/X(x_1, \dots, x_n)]$ (*)
- (8) If $\Gamma \vdash_{AF2} t : A[u/x]$, then $\Gamma \vdash_{AF2} t : A[v/x]$, $u = v$ being a particular case of an equation of \mathbf{E} .

(*) $A[G/X(x_1, \dots, x_n)]$ is obtained by replacing in A each atomic formula $X(t_1, \dots, t_n)$ by $G[t_1/x_1, \dots, t_n/x_n]$. To simplify, we write sometimes $A[G/X]$ instead of $A[G/X(x_1, \dots, x_n)]$.

Whenever we obtain the typing $\Gamma \vdash_{AF2} t : A$ by means of these rules, we say that "the λ -term t is of type A in the context Γ , with respect to the equation of \mathbf{E} ".

Theorem 3.1 ([5],[9]).

- 1) *Conservation Theorem:* If $\Gamma \vdash_{AF2} t : A$, and $t \rightarrow_{\beta} t'$, then $\Gamma \vdash_{AF2} t' : A$.
- 2) *Strong normalization:* If $\Gamma \vdash_{AF2} t : A$, then t is strongly normalizable.

3.2 The TTR type system

Let X be a predicate variable or predicate symbol, and A a type of $AF2$.

We define the notions " X is positive in A " and " X is negative in A " by induction :

- If X does not appears in A , then X is positive and negative in A ;

- If $A = X(t_1, \dots, t_n)$, then X is positive in A , and X is not negative in A ;
- If $A = B \rightarrow C$, then X is positive (resp. negative) in A if and only if X is negative (resp. positive) in B , and X is positive (resp. negative) in C ;
- If $A = \forall v B$, and $v \neq X$, then X is positive (resp. negative) in A if and only if X is positive (resp. negative) in B .

We add to the second order predicate calculus a new logic symbol μ , and we allow a new construction for formulas : if A is a formula, C an n -ary predicate symbol which appears positively in A , x_1, \dots, x_n first order variables, and t_1, \dots, t_n terms, then $\mu C x_1 \dots x_n A < t_1, \dots, t_n >$ is a formula called the least fixed point of A in C calculated over the terms t_1, \dots, t_n .

We extend the notions " X is positive in a type" and " X is negative in a type" by the following way : X is positive (resp. negative) in $\mu C x_1 \dots x_n A < t_1, \dots, t_n >$ if and only if X is positive (resp. negative) in A .

We extend the definition of the substitution by assuming that C, x_1, \dots, x_n are bounded in the formula $\mu C x_1 \dots x_n A < t_1, \dots, t_n >$.

We define on these formulas a binary relation \subseteq by : $A \subseteq B$ if and only if it is obtained by using the following rules :

$$\begin{array}{ll}
(ax) A \subseteq A & (\rightarrow) \frac{A \subseteq A' \quad B \subseteq B'}{A' \rightarrow B \subseteq A \rightarrow B'} \\
(\forall i_g) \frac{A[G/v] \subseteq B}{\forall v A \subseteq B} \quad (1) & (\forall i_d) \frac{A \subseteq B}{A \subseteq \forall v B} \quad (2) \\
(e) \frac{A \subseteq B[v/y]}{A \subseteq B[w/y]} \quad (3) & (tr) \frac{A \subseteq D \quad D \subseteq B}{A \subseteq B} \\
(\mu_d) D[\mu C x_1 \dots x_m D < z_1, \dots, z_m > / C(z_1, \dots, z_m)][t_1/x_1, \dots, t_m/x_m] \subseteq \mu C x_1 \dots x_m D < t_1, \dots, t_m > \\
(\mu'_g) \mu C x_1 \dots x_m D < t_1, \dots, t_m > \subseteq D[\mu C x_1 \dots x_m D < z_1, \dots, z_m > / C(z_1, \dots, z_m)][t_1/x_1, \dots, t_m/x_m] \\
(\mu_g) \frac{D[E/C(x_1, \dots, x_m)] \subseteq E}{\mu C x_1 \dots x_m D < t_1, \dots, t_m > \subseteq E[t_1/x_1, \dots, t_m/x_m]}
\end{array}$$

- (1) G is a formula if v is a second order variable, and a term if v is a first order variable.
- (2) v is not free in A .
- (3) $v = w$ is a particular case of an equation of **E** .

(μ_d) and (μ'_g) are the rules of factorisation and development of a fixed point.

(μ_g) expresses the fact that $\mu C x_1 \dots x_m D < t_1, \dots, t_m >$ is a least fixed point.

We are going to describe a system of typed λ -calculus called theory of recursive types (shortened in *TTR* for Théorie des Types Récurifs) where the types are formulas of language. The rules of typing are the following :

- The typing rules (1),..., (8) of *AF2* type system.
- $(\subseteq) \frac{\Gamma \vdash_{TTR} t : A \quad A \subseteq B}{\Gamma \vdash_{TTR} t : B}$
- $(Y) \frac{\Gamma \vdash_{TTR} t : \forall x_1 \dots \forall x_m [C(x_1, \dots, x_m) \rightarrow E] \rightarrow \forall x_1 \dots \forall x_m [D \rightarrow E]}{\Gamma \vdash_{TTR} (Y)t : \forall x_1 \dots \forall x_m [\mu C x_1 \dots x_m D < x_1, \dots, x_m > \rightarrow E]}$
 where C is not free in E and G , and Y is the Turing's fixed point.

The rule (Y) expresses also the fact that $\mu C x_1 \dots x_m D < t_1, \dots, t_m >$ is a least fixed point.

Theorem 3.2 ([12],[18]).

- 1) *Conservation Theorem* If $\Gamma \vdash_{TTR} t : A$, and $t \rightarrow_\beta t'$, then $\Gamma \vdash_{TTR} t' : A$.
- 2) *Strong normalization* If $\Gamma \vdash_{TTR} t : A$ without using the rule (Y) , then t is strongly normalizable.
- 3) *Weak normalization* If $\Gamma \vdash_{TTR} t : A$, and if all least fixed points of A are positives, then t is normalizable.

The TTR^\diamond type system is the subsystem of *TTR* where we only have propositional variables and constants (predicate variables or predicate symbols are of arity 0). So, first order variables, function symbols, and finite sets of equations are useless. With each predicate variable (resp. predicate symbol) X , we associate a predicate variable (resp. a predicate symbol) X^\diamond of TTR^\diamond type system. For every formula A of *TTR*, we define the formula A^\diamond of TTR^\diamond obtained by forgetting in A the first order part. If $\Gamma = x_1 : A_1, \dots, x_n : A_n$ is a context of *TTR*, then we denote by Γ^\diamond , the context $x_1 : A_1^\diamond, \dots, x_n : A_n^\diamond$ of TTR^\diamond . We write $\Gamma \vdash_{TTR^\diamond} t : A$ if t is tyable in TTR^\diamond of type A in the context Γ .

Theorem 3.3 If $\Gamma \vdash_{TTR} t : A$, then $\Gamma^\diamond \vdash_{TTR^\diamond} t : A^\diamond$.

Proof By induction on the length of the derivation $\Gamma \vdash_{TTR} t : A$. \square

Theorem 3.4

- 1) *Conservation Theorem* If $\Gamma \vdash_{TTR^\diamond} t : A$, and $t \rightarrow_\beta t'$, then $\Gamma \vdash_{TTR^\diamond} t' : A$.
- 2) *Strong normalization* If $\Gamma \vdash_{TTR^\diamond} t : A$ without using the rule (Y) , then t is strongly normalizable.
- 3) *Weak normalization* If $\Gamma \vdash_{TTR^\diamond} t : A$, and if all least fixed points of A are positives, then t is normalizable.

Proof We use Theorems 3.2 and 3.3. \square

Remark We cannot if the reverse of 2)-Theorem 3.2 is true, but the λ -term $t = \lambda x(\lambda y((x)(y)\lambda x x)(y)\lambda x \lambda y x)\lambda x(x)x$ (which is strongly normalizable, and untypable in $AF2$ type system (see [3])) is typable in TTR type system. Indeed, if we take $B = \mu C(\forall X X \rightarrow C)$, we check easily that $\vdash_{TTR^\circ} t : [B \rightarrow (B \rightarrow B)] \rightarrow B$.

4 Properties of TTR type system

4.1 Permutations Lemmas

Lemma 4.1 1) *The typing rules (5), (7), and (8) are admissible.*

2) *In the typing, we may replace the succession of n times (\subseteq) and m times (4) (resp. (6)), by the succession of m times (4) (resp. (6)) and n times (\subseteq).*

3) *If $\Gamma \vdash_{TTR} t : B$ is derived from $\Gamma \vdash_{TTR} t : A$, then we may assume that we begin by the applications of (4), (6), and next by (\subseteq).*

Proof Easy. \square

Lemma 4.2 1) *If $A \subseteq B$, then, for every sequence of terms and/or formulas \mathbf{G} , $A[\mathbf{G}/\mathbf{v}] \subseteq B[\mathbf{G}/\mathbf{v}]$, and we use the same proof rules.*

2) *If $\Gamma \vdash_{TTR} t : A$, then, for every sequence of terms and/or formulas \mathbf{G} , $\Gamma[\mathbf{G}/\mathbf{v}] \vdash_{TTR} t : A[\mathbf{G}/\mathbf{v}]$, and we use the same typing rules.*

Proof By induction on the length of the derivation $A \subseteq B$ (resp. $\Gamma \vdash_{TTR} t : A$). \square

Corollary 4.1 *If $\Gamma, x : A \vdash_{TTR} (x)u_1 \dots u_n : B$, then :*

$n = 0$, and there is \mathbf{v}_0 not free in A and Γ , such that $\forall \mathbf{v}_0 A \subseteq B$,

or

$n \geq 1$, and there are types C_i, B_i ($i = 1, \dots, n$) and \mathbf{v}_i ($i = 1, n$) not free in A and Γ , such that $\forall \mathbf{v}_0 A \subseteq C_1 \rightarrow B_1$, $\forall \mathbf{v}_i B_i \subseteq C_{i+1} \rightarrow B_{i+1}$ $1 \leq i \leq n-1$, $\forall \mathbf{v}_n B_n \subseteq B$, and $\Gamma, x : A \vdash_{TTR} u_i : C_i$ $1 \leq i \leq n$.

Proof By induction on n . \square

Lemma 4.3 1) *If X is positive (resp. negative) in D , and $A \subseteq B$, then $D[A/X] \subseteq D[B/X]$ (resp. $D[B/X] \subseteq D[A/X]$).*

2) *We may eliminate the rule (μ'_g).*

Proof 1) By induction on D .

2) By rule (μ_d), we have $A[\mu C x_1 \dots x_n A < y_1, \dots, y_n > / C(y_1, \dots, y_n)] \subseteq \mu C x_1 \dots x_n A < x_1, \dots, x_n >$,

then, by 1), $A[A[\mu Cx_1 \dots x_n A < y_1, \dots, y_n > /C(y_1, \dots, y_n)]/C(x_1, \dots, x_n)] \subseteq A[\mu Cx_1 \dots x_n A < x_1, \dots, x_n > /C(x_1, \dots, x_n)]$, and, by using the rule (μ_g) , we obtain $\mu Cx_1 \dots x_n A < t_1, \dots, t_n > \subseteq A[\mu Cx_1 \dots x_n A < y_1, \dots, y_n > /C(y_1, \dots, y_n)][t_1/x_1, \dots, t_n/x_n]$. \square

4.2 Without-arrow types and arrow types

Definitions

- 1) A type A is said to be without-arrow type if and only if A does not contain any arrow.
- 2) Each without-arrow type A contains a unique atomic formula $X(t_1, \dots, t_n)$. We denote X by $At(A)$. We distinguish between two kinds of without-arrow types :
 - A without-arrow type A is said to be of kind 1 if and only if $At(A)$ is free in A .
 - A without-arrow type A is said to be of kind 2 if and only if $At(A)$ is bounded in A .

Lemma 4.4 1) If A is a without-arrow type of kind 1, and $A \subseteq B$, then B is a without-arrow type of kind 1, and $At(A) = At(B)$.

2) If A is a without-arrow type of kind 2, then, for every type B , we have $A \subseteq B$.

Proof 1) By induction on the length of the derivation $A \subseteq B$.

2) Easy. \square

Definition A type A is said to be arrow type if and only if A contains at least an arrow.

Lemma 4.5 If A is an arrow type, and $A \subseteq B$, then B is an arrow type.

Proof By induction on the length of the derivation $A \subseteq B$. \square

Corollary 4.2 Let A be an atomic formula. If $\Gamma \vdash_{TTR} t : A$, then t does not begin by λ . Other words, if $\Gamma \vdash_{TTR} \lambda x u : B$, then B is an arrow type.

Proof If t begins by λ , then there are E, F , and \mathbf{v} , such that $\forall \mathbf{v}(E \rightarrow F) \subseteq A$, therefore, by Lemma 4.5, A is an arrow type. \square

Definition For every arrow type A , we define the type $Rep(A)$ as follows, by induction on A :

- $Rep(E \rightarrow F) = E \rightarrow F$;
- $Rep(\forall v B) = \forall v Rep(B)$;
- $Rep(\mu Cx_1 \dots x_n B < t_1, \dots, t_n >) =$
 $Rep(B)[\mu Cx_1 \dots x_n B < y_1, \dots, y_n > /C(y_1, \dots, y_n)][t_1/x_1, \dots, t_n/x_n]$.

Lemma 4.6 If A is an arrow type, then :

- 1) there are G, D and \mathbf{v} such that $Rep(A) = \forall \mathbf{v}(G \rightarrow D)$.
- 2) $A \subseteq Rep(A)$, and $Rep(A) \subseteq A$.

Proof By induction on A . \square

Remark. The Lemma 4.6 means that if A is an arrow type, then $Rep(A)$ is an "equivalent" type to A of the form $\forall \mathbf{v}(G \rightarrow D)$. In the rest of the paper, we denoted G by A_g and D by A_d .

Lemma 4.7 *Let A, B be two types, and X, X' two predicate variables or predicate symbols, such that X' is not free in A .*

- 1) *If X is positive in A , and X' is positive in B , then X' is positive in $A[B/X]$.*
- 2) *If X is positive in A , and X' is negative in B , then X' is negative in $A[B/X]$.*
- 3) *If X is negative in A , and X' is positive in B , then X' is negative in $A[B/X]$.*
- 4) *If X is negative in A , and X' is negative in B , then X' is positive in $A[B/X]$.*

Proof By induction on A . \square

Lemma 4.8 *Let A be an arrow type.*

- 1) *If X is positive (resp. negative) in A , then X is positive (resp. negative) in $Rep(A)$.*
- 2) *If \mathbf{G} is a sequence of terms and/or formulas, then $Rep(A[\mathbf{G}/\mathbf{v}]) = Rep(A)[\mathbf{G}/\mathbf{v}]$.*

Proof 1) We argue by induction on A . The only non-trivial case is the one where $A = \mu Cx_1 \dots x_n B < t_1, \dots, t_n >$. If X is positive (resp. negative) in A , then X is positive (resp. negative) in B . By the induction hypothesis, we have X is positive (resp. negative) in $Rep(B)$, therefore, by Lemma 4.7, X is positive (resp. negative) in $Rep(A)$.

2) By induction on A . \square

Theorem 4.1 *Let A, B be two arrow types, such that $Rep(A) = \forall \mathbf{v}(A_g \rightarrow A_d)$ and $Rep(B) = \forall \mathbf{v}'(B_g \rightarrow B_d)$. If $A \subseteq B$, then there is a sequence of terms and/or formulas \mathbf{G} , such that $B_g \subseteq A_g[\mathbf{G}/\mathbf{v}]$, and $A_d[\mathbf{G}/\mathbf{v}] \subseteq B_d$.*

Proof We argue by induction on the length of the derivation $A \subseteq B$. Let us look at the rule used in the last step. The only non-trivial cases are :

- (tr) : then $A \subseteq D$, and $D \subseteq B$. If $Rep(D) = \forall \mathbf{v}''(D_g \rightarrow D_d)$, by the induction hypothesis, there are sequences \mathbf{G} and \mathbf{G}'' such that $D_g \subseteq A_g[\mathbf{G}/\mathbf{v}]$, $A_d[\mathbf{G}/\mathbf{v}] \subseteq D_d$, $B_g \subseteq D_g[\mathbf{G}''/\mathbf{v}'']$, and $D_d[\mathbf{G}''/\mathbf{v}''] \subseteq B_d$. It is clear that we may assume that \mathbf{v}'' is not free in A_g and A_d , therefore, by Lemma 4.2, we have $B_g \subseteq A_g[\mathbf{G}/\mathbf{v}][\mathbf{G}''/\mathbf{v}'']$, and $A_d[\mathbf{G}/\mathbf{v}][\mathbf{G}''/\mathbf{v}''] \subseteq B_d$. Let $\mathbf{G}' = \mathbf{G}[\mathbf{G}''/\mathbf{v}'']$, then $B_g \subseteq A_g[\mathbf{G}'/\mathbf{v}]$, and $A_d[\mathbf{G}'/\mathbf{v}] \subseteq B_d$.

- (μ_d) : then $A = D[\mu Cx_1 \dots x_k D < y_1, \dots, y_k > / C(y_1, \dots, y_k)][t_1/x_1, \dots, t_k/x_k]$, and $B = \mu Cx_1 \dots x_k D < t_1, \dots, t_k >$. Therefore, by Lemma 4.8, $Rep(A) = Rep(B)$, $A_g = B_g$, and $B_d = A_d$, and so $B_g \subseteq A_g$, and $A_d \subseteq B_d$.

- (μ_g) : then $A = \mu C x_1 \dots x_k D < t_1, \dots, t_k >$, $B = E[t_1/x_1, \dots, t_k/x_k]$, and $D[E/C(x_1, \dots, x_k)] \subseteq E$. Therefore $Rep(D) = \forall \mathbf{v}(D_g \rightarrow D_d)$ with

$$D_g[\mu C x_1 \dots x_k D < y_1, \dots, y_k > / C(y_1, \dots, y_k)][t_1/x_1, \dots, t_k/x_k] = A_g,$$

$$D_d[\mu C x_1 \dots x_k D < y_1, \dots, y_k > / C(y_1, \dots, y_k)][t_1/x_1, \dots, t_k/x_k] = A_d, \text{ and}$$

$$Rep(E) = \forall \mathbf{v}'(E_g \rightarrow E_d) \text{ with } E_g[t_1/x_1, \dots, t_k/x_k] = B_g, E_d[t_1/x_1, \dots, t_k/x_k] = B_d.$$

By the induction hypothesis, there is a sequence \mathbf{G} , such that $E_g \subseteq D_g[E/C(x_1, \dots, x_k)][\mathbf{G}/\mathbf{v}]$, and $D_d[E/C(x_1, \dots, x_k)][\mathbf{G}/\mathbf{v}] \subseteq E_d$. C is positive in D , therefore, by Lemma 4.8, C is negative in D_g , and C is positive in D_d .

$D[E/C(x_1, \dots, x_k)] \subseteq E$, then $\mu C x_1 \dots x_k D < y_1, \dots, y_k > \subseteq E[y_1/x_1, \dots, y_k/x_k]$, and, by 1)-Lemma 4.3, $E_g \subseteq D_g[\mu C x_1 \dots x_k D < y_1, \dots, y_k > / C(y_1, \dots, y_k)][\mathbf{G}/\mathbf{v}]$, and

$$D_d[\mu C x_1 \dots x_k D < y_1, \dots, y_k > / C(y_1, \dots, y_k)][\mathbf{G}/\mathbf{v}] \subseteq E_d, \text{ and so, by Lemma 4.2,}$$

$$E_g[t_1/x_1, \dots, t_k/x_k] \subseteq D_g[\mu C x_1 \dots x_k D < y_1, \dots, y_k > / C(y_1, \dots, y_k)][\mathbf{G}/\mathbf{v}][t_1/x_1, \dots, t_k/x_k],$$

and $D_d[\mu C x_1 \dots x_k D < y_1, \dots, y_k > / C(y_1, \dots, y_k)][\mathbf{G}/\mathbf{v}][t_1/x_1, \dots, t_k/x_k] \subseteq E_d[t_1/x_1, \dots, t_k/x_k]$.

Let $\mathbf{G}' = \mathbf{G}[t_1/x_1, \dots, t_k/x_k]$, then $B_g \subseteq A_g[\mathbf{G}'/\mathbf{v}]$, and $A_d[\mathbf{G}'/\mathbf{v}] \subseteq B_d$. \square

Corollary 4.3 *Let B be an atomic formula. If $\Gamma, x : A \rightarrow B \vdash_{TTR} (x)u_1 \dots u_n : C$, then $n = 1$, and $\Gamma, x : A \rightarrow B \vdash_{TTR} u_1 : A$.*

Proof By Corollary 4.1, we have $\forall \mathbf{v}(A \rightarrow B) \subseteq F \rightarrow G$, $\Gamma, x : A \rightarrow B \vdash_{TTR} u_1 : F$, and \mathbf{v} is not free in Γ and $A \rightarrow B$. Therefore, by Theorem 4.1, $F \subseteq A$, and $B \subseteq G$, then $\Gamma, x : A \rightarrow B \vdash_{TTR} u_1 : A$. If $n > 1$, then $\forall \mathbf{v}'G \subseteq H \rightarrow J$, and \mathbf{v}' is not free in Γ and $A \rightarrow B$. Therefore $\forall \mathbf{v}'B \subseteq H \rightarrow J$, and $\forall \mathbf{v}'B$ is a without-arrow type of kind 1. A contradiction. \square

Lemma 4.9 *If $x_1 : A_1, \dots, x_n : A_n \vdash_{TTR} t : A$, $B_i \subseteq A_i$ $1 \leq i \leq n$, and $A \subseteq B$, then $x_1 : B_1, \dots, x_n : B_n \vdash_{TTR} t : B$.*

Proof We argue by induction on t . The only non-trivial cases are :

- If $t = \lambda x u$, then $x_1 : A_1, \dots, x_n : A_n, x : E \vdash_{TTR} u : F$, $\forall \mathbf{v}(E \rightarrow F) \subseteq A$, and \mathbf{v} is not free in E and A_j $1 \leq j \leq n$. We may assume that \mathbf{v} is not free in E and B_j $1 \leq j \leq n$. By the induction hypothesis, we have $x_1 : B_1, \dots, x_n : B_n, x : E \vdash_{TTR} u : F$, and so $x_1 : B_1, \dots, x_n : B_n \vdash_{TTR} t : B$.

- If $t = (Y)u$, then $\forall \mathbf{v} \forall y_1 \dots \forall y_m [\mu C y_1 \dots y_m E < y_1, \dots, y_m > \rightarrow D] \subseteq A$, $x_1 : A_1, \dots, x_n : A_n \vdash_{TTR} u : \forall y_1 \dots \forall y_m [C(y_1, \dots, y_m) \rightarrow D] \rightarrow \forall y_1 \dots \forall y_m [E \rightarrow D]$, C is positive in E , C is not free in D , and \mathbf{v} is not free in A_j $1 \leq j \leq n$. We may assume that \mathbf{v}, C are not free in B_j $1 \leq j \leq n$. By the induction hypothesis, we have $x_1 : B_1, \dots, x_n : B_n \vdash_{TTR} u : \forall y_1 \dots \forall y_m [C(y_1, \dots, y_m) \rightarrow D] \rightarrow \forall y_1 \dots \forall y_m [E \rightarrow D]$, and so $x_1 : B_1, \dots, x_n : B_n \vdash_{TTR} (Y)u : A$. \square

5 \forall -positive types

5.1 Properties of \forall -positive types

Definition We define two sets of types, the set Ω^+ of \forall -positive types, and the set Ω^- of \forall -negative types in the following way :

- If A is an atomic type, then $A \in \Omega^+$, and $A \in \Omega^-$;
- If $T^+ \in \Omega^+$, and $T^- \in \Omega^-$, then, $T^- \rightarrow T^+ \in \Omega^+$, and $T^+ \rightarrow T^- \in \Omega^-$;
- If $T^+ \in \Omega^+$, then $\forall x T^+ \in \Omega^+$;
- If $T^+ \in \Omega^+$, then $\forall X T^+ \in \Omega^+$;
- If $T^- \in \Omega^-$, then $\forall x T^- \in \Omega^-$;
- If $T^- \in \Omega^-$, and X is not free in T^- , then $\forall X T^- \in \Omega^-$;
- If $T^+ \in \Omega^+$, x_1, \dots, x_n first order variables, t_1, \dots, t_n terms, C an n -ary predicate symbol which appears and is positive in T^+ , then $\mu C x_1 \dots x_n T^+ < t_1, \dots, t_n > \in \Omega^+$.

Remarks

- 1) A least fixed point is not a \forall -negative type.
- 2) If $T^+ \in \Omega^+$, then all least fixed points of T^+ are positives. Therefore, by 3)-Theorem 3.2, if $\Gamma \vdash_{TTR} t : T^+$, then t is normalizable.

Lemma 5.1 Let $T^-, T'^- \in \Omega^-$, $T^+, T'^+ \in \Omega^+$, and X a predicate variable or predicate symbol.

- 1) If X is positive (resp. negative) in T^- , then $T^-[T'^-/X] \in \Omega^-$ (resp. $T^-[T'^+/X] \in \Omega^-$).
- 2) If X is positive (resp. negative) in T^+ , then $T^+[T'^+/X] \in \Omega^+$ (resp. $T^+[T'^-/X] \in \Omega^+$).
- 3) If $T[F/X] \in \Omega^+$ (resp. $T[F/X] \in \Omega^-$), then $T \in \Omega^+$ (resp. $T \in \Omega^-$).

Proof 1), 2) By induction on T^- and T^+ .

3) By induction on T . \square

Definition With each type T of TTR , we associate the set $Fv_2(T)$ of free predicate variables and free predicate symbols of T .

Theorem 5.1 Let $T^- \in \Omega^-$, and $T^+ \in \Omega^+$.

- 1) If $T^- \subseteq A$, then $A \in \Omega^-$, and $Fv_2(A) \subseteq Fv_2(T^-)$.
- 2) If $B \subseteq T^+$, then $B \in \Omega^+$, and $Fv_2(B) \subseteq Fv_2(T^+)$.

Proof We argue by induction on the length of the derivations $T^- \subseteq A$, and $B \subseteq T^+$. Let us look at the rule used in the last step.

1) The only non-trivial case is (μ_d) .

Then $T^- = T'[\mu C x_1 \dots x_n T' < y_1, \dots, y_n > / C(y_1, \dots, y_n)][t_1/x_1, \dots, t_n/x_n]$, and

$A = \mu C x_1 \dots x_n T' < t_1, \dots, t_n >$. Since $T^- \in \Omega^-$, then, by Lemma 5.1, $\mu C x_1 \dots x_n T' < y_1, \dots, y_n > \in \Omega^-$, which is impossible.

2) The only non-trivial cases are :

- (μ_d) : then $B = D[\mu C x_1 \dots x_n D < y_1, \dots, y_n > / C(y_1, \dots, y_n)][t_1/x_1, \dots, t_n/x_n]$, and $T^+ = \mu C x_1 \dots x_n D < t_1, \dots, t_n >$. Since $T \in \Omega^+$, then $D \in \Omega^+$, and so, by Lemma 5.1, $B \in \Omega^+$, and $Fv_2(B) = Fv_2(D) - \{C\} = Fv_2(T^+)$.

- (μ_g) : then $B = \mu C x_1 \dots x_n D < t_1, \dots, t_n >$, $T^+ = E[t_1/x_1, \dots, t_n/x_n]$, and $D[E/C(x_1, \dots, x_n)] \subseteq E$. Since $T^+ \in \Omega^+$, then $E \in \Omega^+$, and, by the induction hypothesis, $D[E/C(x_1, \dots, x_n)] \in \Omega^+$, and $Fv_2(D[E/C(x_1, \dots, x_n)]) \subseteq Fv_2(E)$. By Lemma 5.1, we have $D \in \Omega^+$, and $Fv_2(D) - \{C\} \subseteq Fv_2(D[E/C(x_1, \dots, x_n)]) \subseteq Fv_2(E)$, and so $B \in \Omega^+$, and $Fv_2(B) = Fv_2(D) - \{C\} \subseteq Fv_2(D[E/C(x_1, \dots, x_n)]) \subseteq Fv_2(E) = Fv_2(T^+)$. \square

5.2 The TTR_0 type system

We define on the types of TTR a binary relation \subseteq_0 by the following way :

$A \subseteq_0 B$ if and only if $A \subseteq B$, and in the proof we use only the weak version of $(\forall i_g)$:

$$(\forall i_{g_0}) \frac{A[G/v] \subseteq_0 B}{\forall v A \subseteq_0 B}$$

where G is a term if v is an individual variable, and G is a predicate variable or a predicate symbol having the same arity of v if v is a predicate variable.

Lemma 5.2 *If $A \subseteq_0 B$, then, for every sequence of terms and/or formulas \mathbf{G} , $A[\mathbf{G}/\mathbf{v}] \subseteq_0 B[\mathbf{G}/\mathbf{v}]$, and we use the same proof rules.*

Proof Same proof as 1)-Lemma 4.2. \square

Lemma 5.3 *Let A be an arrow type, and $Rep(A) = \forall \mathbf{v}(A_g \rightarrow A_d)$.*

1) *If $A \in \Omega^-$ (resp. $A \in \Omega^+$), then $A_g \in \Omega^+$, and $A_d \in \Omega^-$ (resp. $A_g \in \Omega^-$, and $A_d \in \Omega^+$).*

2) *$A \subseteq_0 Rep(A)$, and $Rep(A) \subseteq_0 A$.*

Proof By induction on A . \square

Lemma 5.4 *If $T^- \in \Omega^-$, $T^+ \in \Omega^+$, and $T^- \subseteq T^+$, then $T^- \subseteq_0 T^+$.*

Proof By induction on the length of the derivation $T^- \subseteq T^+$. \square

Definition We denote by TTR_0 , the TTR type system without the rules (5), (7), (8) and by replacing the rule (\subseteq) by :

$$(\subseteq_0) \frac{\Gamma \vdash_{TTR_0} t : A \quad A \subseteq_0 B}{\Gamma \vdash_{TTR_0} t : B}$$

Theorem 5.2 *Let $A_1, \dots, A_n \in \Omega^-$, $\Gamma = x_1 : A_1, \dots, x_n : A_n$, $A \in \Omega^+$, and t a normal λ -term. If $\Gamma \vdash_{TTR} t : A$, then $\Gamma \vdash_{TTR_0} t : A$, and in this typing each variable is assigned of a \forall -negative type, and each $u \in STE(t)$ is typable of a \forall -positive type.*

Proof We argue by induction on t .

- If $t = x_i$ $1 \leq i \leq n$, then $\forall \mathbf{v} A_i \subseteq A$, and \mathbf{v} is not free in Γ . Since $A_i \in \Omega^-$, then $\forall \mathbf{v} A_i \in \Omega^-$, and, by Lemma 5.4, $\forall \mathbf{v} A_i \subseteq_0 A$. Therefore $\Gamma \vdash_{TTR_0} t : A$.

- If $t = \lambda x u$, then $\Gamma, x : B \vdash_{TTR} u : C$, $\forall \mathbf{v} (B \rightarrow C) \subseteq A$, and \mathbf{v} is not free in Γ . Since $\forall \mathbf{v} (B \rightarrow C)$ is an arrow type, then, by Lemma 4.5, A is an arrow type. If $Rep(A) = \forall \mathbf{v}' (A_g \rightarrow A_d)$, then, by 1)-Lemma 5.3, $A_g \in \Omega^-$, and $A_d \in \Omega^+$. By Theorem 4.1, there is a sequence \mathbf{G} , such that $A_g \subseteq B[\mathbf{G}/\mathbf{v}]$, and $C[\mathbf{G}/\mathbf{v}] \subseteq A_d$. By 2)-Lemma 4.2, we have $\Gamma, x : B[\mathbf{G}/\mathbf{v}] \vdash_{TTR} u : C[\mathbf{G}/\mathbf{c}]$, and, by Lemma 4.9, $\Gamma, x : A_g \vdash_{TTR} u : A_d$. By the induction hypothesis, we have $\Gamma, x : A_g \vdash_{TTR_0} u : A_d$, and so, by 2)-Lemma 5.3, $\Gamma \vdash_{TTR_0} t : A$.

- If $t = (x_i)u_1 \dots u_k$ $1 \leq i \leq n$ and $k \neq 0$, then $\forall \mathbf{v}_0 A_i \subseteq C_1 \rightarrow B_1$, $\forall \mathbf{v}_j B_i \subseteq C_{j+1} \rightarrow B_{j+1}$ $1 \leq j \leq k-1$, $\forall \mathbf{v}_k B_k \subseteq A$ where $\mathbf{v}_0, \dots, \mathbf{v}_k$ are not free in Γ , and $\Gamma \vdash_{TTR} u_j : C_j$ $1 \leq j \leq k$. By Theorems 4.1, 5.1, and Lemmas 4.4, 5.4, we have

- $A_i = \forall \mathbf{v}'_0 A'_i$, $A'_i = C'_1 \rightarrow \forall \mathbf{v}'_1 B'_1$, $B_j = C'_{j+1} \rightarrow \forall \mathbf{v}'_{j+1} B'_{j+1}$ $1 \leq j \leq k-1$, $C'_j \in \Omega^+$, and $\forall \mathbf{v}'_j B'_j \in \Omega^-$ $1 \leq j \leq k$.
- $C_j \subseteq C'_j[\mathbf{G}_0/\mathbf{v}'_0] \dots [\mathbf{G}_{j-1}/\mathbf{v}'_{j-1}]$, $\forall \mathbf{c}'_j B'_j[\mathbf{G}_0/\mathbf{v}'_0] \dots [\mathbf{G}_{j-1}/\mathbf{v}'_{j-1}] \subseteq \mathbf{B}_j$ $1 \leq j \leq k$, and $\forall \mathbf{v}_k \forall \mathbf{v}'_k B'_k[\mathbf{G}_0/\mathbf{v}'_0] \dots [\mathbf{G}_{k-1}/\mathbf{v}'_{k-1}] \subseteq_0 A$.

Since $\Gamma \vdash_{TTR} u_j : C_j$ $1 \leq j \leq k$, then $\Gamma \vdash_{TTR} u_j : C'_j[\mathbf{G}_0/\mathbf{v}'_0] \dots [\mathbf{G}_{j-1}/\mathbf{v}'_{j-1}]$, and, by the induction hypothesis, $\Gamma \vdash_{TTR_0} u_j : C'_j[\mathbf{G}_0/\mathbf{v}'_0] \dots [\mathbf{G}_{j-1}/\mathbf{v}'_{j-1}]$. It is easy to check that $\Gamma \vdash_{TTR_0} t : B'_k[\mathbf{G}_0/\mathbf{v}'_0] \dots [\mathbf{G}_{k-1}/\mathbf{v}'_{k-1}]$, then $\Gamma \vdash_{TTR_0} t : \forall \mathbf{v}_k \forall \mathbf{v}'_k B'_k[\mathbf{G}_0/\mathbf{v}'_0] \dots [\mathbf{G}_{k-1}/\mathbf{v}'_{k-1}]$, and $\Gamma \vdash_{TTR_0} t : A$. \square

6 Gödel transformation

6.1 \perp -types of TTR

Definition Let A be a type of TTR . We say that A is an \perp -type if and only if A is obtained by the following rules :

- \perp is an \perp -type.

- If A is an \perp -type, then $B \rightarrow A$ is an \perp -type for every type B .
- If A is an \perp -type, then $\forall vA$ is an \perp -type for every variable v .
- If A is an \perp -type, C an n -ary predicate symbol which appears and is positive in A , x_1, \dots, x_n first order variables, and t_1, \dots, t_n terms, then $\mu Cx_1 \dots x_n A < t_1, \dots, t_n >$ is an \perp -type.

Lemma 6.1 *If A is an \perp -type, and $A \subseteq B$, then B is an \perp -type.*

Proof By induction on the length of the derivation $A \subseteq B$. \square

Lemma 6.2 *Let t be a normal λ -term, $A_1, \dots, A_n \in \Omega^-$, $A \in \Omega^+$, \perp does not appear in the types A_1, \dots, A_n, A , and B_1, \dots, B_m are \perp -types. If $\Gamma = x_1 : A_1, \dots, x_n : A_n, y_1 : B_1, \dots, y_m : B_m \vdash_{TTR} t : A$, then $x_1 : A_1, \dots, x_n : A_n \vdash_{TTR} t : A$.*

Proof We argue by induction on t .

- If t is a variable, then $t = x_i$ $1 \leq i \leq n$ or $t = y_i$ $1 \leq i \leq m$.
 - The case $t = x_i$ is trivial.
 - If $t = y_i$, then $\forall \mathbf{v} B_i \subseteq A$ and \mathbf{v} is not free in Γ . Since B_i is an \perp -type, then, by Lemma 6.1, A is an \perp -type, and \perp appears in A . A contradictoire.
- If $t = \lambda x_{n+1} t'$, then $\Gamma, x_{n+1} : A_{n+1} \vdash_{TTR} t' : D$, $\forall \mathbf{v} (A_{n+1} \rightarrow D) \subseteq A$, \mathbf{v} is not free in Γ . Since $A \in \Omega^+$, then, by Theorem 5.1, we have $A_{n+1} \in \Omega^-$, $D \in \Omega^+$, and $Fv_2(\forall \mathbf{v} (A_{n+1} \rightarrow D)) \subseteq Fv_2(A)$. Therefore \perp does not appear in A_{n+1} and D . By the induction hypothesis, we have $x_1 : A_1, \dots, x_n : A_n, x_{n+1} : A_{n+1} \vdash_{TTR} t : D$, and so $x_1 : A_1, \dots, x_n : A_n \vdash_{TTR} t : A$.
- If $t = (x)u_1 \dots u_k$ $k \geq 1$, then two case can be see :
 - If $x = y_i$ $1 \leq i \leq m$, then, by Corollary 4.1, we have $\forall \mathbf{v}_0 B_i \subseteq C_1 \rightarrow D_1, \forall \mathbf{v}_j D_j \subseteq C_{j+1} \rightarrow D_{j+1}$ $1 \leq i \leq k-1, \forall \mathbf{v}_k D_k \subseteq A$, where $\mathbf{v}_0, \dots, \mathbf{v}_k$ are not free in A and Γ , and $\Gamma \vdash_{TTR} u_j : C_j$ $1 \leq j \leq k$. Since B_i is an \perp -type, then, by Lemma 6.1, D_j $1 \leq j \leq k$ and A are \perp -types, and \perp appears in A . A contradictoire.
 - If $x = x_i$ $1 \leq i \leq n$, then, by Corollary 4.1, we have $\forall \mathbf{v}_0 A_i \subseteq C_1 \rightarrow D_1, \forall \mathbf{v}_j D_j \subseteq C_{j+1} \rightarrow D_{j+1}$ $1 \leq j \leq k-1, \forall \mathbf{v}_k D_k \subseteq A$, where $\mathbf{v}_0, \dots, \mathbf{v}_k$ are not free in A and Γ , and $\Gamma \vdash_{TTR} u_j : C_j$ $1 \leq j \leq k$. Since $A_i \in \Omega^-$, then, by Theorem 5.1, we have $C_j \in \Omega^+, D_i \in \Omega^-$ $1 \leq j \leq k$, and $Fv_2(C_j) \cup Fv_2(D_j) \subseteq Fv_2(A_i)$ $1 \leq j \leq k$. Therefore \perp does not appear in C_j $1 \leq j \leq k$. By the inductive hypothesis, we have $x_1 : A_1, \dots, x_n : A_n \vdash_{TTR} u_j : C_j$ $1 \leq j \leq k$, and so $x_1 : A_1, \dots, x_n : A_n \vdash_{TTR} t : A$. \square

6.2 Gödel transformations

Definition With each predicate variable X , we associate a finite no empty set of predicate variables $V_X = \{X_1, \dots, X_r\}$ having the same arity of X , such that : if $X \neq Y$, then $V_X \cap V_Y = \emptyset$. With each n -ary predicate variable X , and with each sequence of individual variables x_1, \dots, x_n , we associate a formula F_X such that :

- F_X is an \perp -type ;
- F_X does not contain any predicate symbol ;
- the free variables of F_X are among x_1, \dots, x_n and the elements of V_X .

For each formula A , we define the formula A^* by the following induction way :

- If $A = C(t_1, \dots, t_n)$, and C is a predicate symbol, then $A^* = A$.
- If $A = X(t_1, \dots, t_n)$, and X is a predicate variable, then $A^* = F_X[t_1/x_1, \dots, t_n/x_n]$.
- If $A = B \rightarrow C$, then $A^* = B^* \rightarrow C^*$.
- If $A = \forall x B$, then $A^* = \forall x B^*$.
- If $A = \forall X B$, then $A^* = \forall X_1 \dots \forall X_r B^*$, where $V_X = \{X_1, \dots, X_r\}$.
- If $A = \mu C x_1 \dots x_n D < t_1, \dots, t_n >$, then $A^* = \mu C x_1 \dots x_n D^* < t_1, \dots, t_n >$.

A^* is called the Gödel transformation of A .

Remark. In order to show that the above transformation is well defined, we need to prove the following Lemma :

Lemma 6.3 *Let C be a predicate variable or a predicate symbol, and A a type of TTR . If C is positive in A (resp. negative in A), then C is positive in A^* (resp. negative in A^*).*

Proof By induction on A . \square

Lemma 6.4 *1) If $A \subseteq_0 B$, then $A^* \subseteq_0 B^*$, and we use the same proof rules.*

2) If $\Gamma \vdash_{TTR_0} t : A$, then $\Gamma^ \vdash_{TTR_0} t : A^*$, and we use the same typing rules.*

Proof By induction on the length of the derivation $A \subseteq_0 B$ (resp. $\Gamma \vdash_{TTR_0} t : A$). \square

Corollary 6.1 *Let $D \in \Omega^+$, and t a normal λ -term. If $\vdash_{TTR} t : D$, then $\vdash_{TTR} t : D^*$.*

Proof By induction on the length of the derivation $\vdash_{TTR} t : D$, and we use Theorem 5.2 and Lemma 6.4. \square

7 Storage operators

7.1 Definition of storage operators

Definitions

- 1) Let T be a closed λ -term, and D, E two closed types of TTR (resp. TTR^\diamond). We say that T is a storage operator for the pair of types (D, E) if and only if for every λ -term t with $\vdash_{TTR} t : D$ (resp. $\vdash_{TTR^\diamond} t : D$), there are λ -terms τ_t and τ'_t such that $\tau_t \simeq_\beta \tau'_t$, $\vdash_{TTR} \tau'_t : E$ (resp. $\vdash_{TTR^\diamond} \tau'_t : E$), and for every $\theta_t \simeq_\beta t$, $(T)\theta_t f \succ (f)\tau_t[t_1/x_1, \dots, t_n/x_n]$, where $Fv(\tau_t) = \{f, x_1, \dots, x_n\}$ and t_1, \dots, t_n are λ -terms which depend on θ_t .
- 2) If $D = E$, we say that T is a storage operator for the type D .

Examples The type of recursive integers is the formula :

$$N^r[x] = \mu N x \Phi(N, x) < x >$$

where

$$\Phi(N, x) = \forall X \{ \forall y (Ny \rightarrow Xsy), X0 \rightarrow Xx \}$$

(s is a unary function symbol for successor and 0 is a constant symbol for zero).

For each integer n , we define the recursive integer \bar{n} by induction : $\bar{0} = \lambda f \lambda x x$ and $\overline{n+1} = \lambda f \lambda x (f)\bar{n}$. Let \bar{N} be the set of recursive integers.

We have $\bar{N} = \{t / t \text{ is a closed normal } \lambda\text{-term} / \vdash_{TTR} t : N^r[s^n(0)], n \geq 0\}$ (see [19]).

Let $\bar{s} = \lambda n \lambda f \lambda x (f)n$. It is easy to check that \bar{s} is a λ -term for successor, and $\vdash_{TTR} \bar{s} : \forall y (N^r[y] \rightarrow N^r[sy])$.

Define

$T_1 = (Y)H$ where $H = \lambda x \lambda y ((y)\lambda z (G)(x)z)\delta$, $G = \lambda x \lambda y (x)\lambda z (y)(\bar{s})z$, and $\delta = \lambda f (f)\bar{0}$;

$T_2 = \lambda \nu (\nu)\rho \tau \rho$ where $\tau = \lambda d \lambda f (f)\bar{0}$, and $\rho = \lambda y \lambda z (G)(y)z \tau z$,

then, for every $\theta_n \simeq_\beta \bar{n}$, $(T_i)\theta_n f \succ (f)(\bar{s})^n \bar{0}$ ($i = 1, 2$).

Therefore, for every $n \geq 0$, T_1 and T_2 are storage operators for $N^r[s^n(0)]$.

Typing of T_1

We use in the typing the Gödel transformation with $V_X = \{X\}$, and $F_X = \neg X(x_1, \dots, x_n)$ for every second order variable X of arity n .

- We have $\vdash_{TTR} \bar{0} : N^r[0]$, then $\vdash_{TTR} \delta : \neg \neg N^r[0]$.
- We have $\vdash_{TTR} \bar{s} : \forall y (N^r[y] \rightarrow N^r[sy])$, then
 $x : \neg \neg N^r[y], y : \neg N^r[sy], z : N^r[y] \vdash_{TTR} (y)(\bar{s})z : \perp$; hence :
 $x : \neg \neg N^r[y], y : \neg N^r[sy] \vdash_{TTR} (x)\lambda z (y)(\bar{s})z : \perp$; therefore :
 $\vdash_{TTR} G : \forall y (\neg \neg N^r[y] \rightarrow \neg \neg N^r[sy])$.

- We have $y : \Phi^*(N, x) \vdash_{TTR} y : \forall y(Ny \rightarrow \neg\neg N^r[sy]), \neg\neg N^r[0] \rightarrow \neg\neg N^r[x]$; thus :
 $x : \forall x(Nx \rightarrow \neg\neg N^r[x]), y : \Phi^*(N, x), z : Ny \vdash_{TTR} (G)(x)z : \neg\neg N^r[sy]$; therefore :
 $x : \forall x(Nx \rightarrow \neg\neg N^r[x]), y : \Phi^*(N, x) \vdash_{TTR} \lambda z(G)(x)z : \forall y(Ny \rightarrow \neg\neg N^r[sy])$; hence
 $x : \forall x(Nx \rightarrow \neg\neg N^r[x]) \vdash_{TTR} \lambda y((y)\lambda z(G)(x)z)\delta : \forall x(\Phi^*(N, x) \rightarrow \neg\neg N^r[x])$; therefore :
 $\vdash_{TTR} H : \forall x(Nx \rightarrow \neg\neg N^r[x]) \rightarrow \forall x(\Phi^*(N, x) \rightarrow \neg\neg N^r[x])$.

And finally $\vdash_{TTR} T_1 : \forall x\{N^{r*}[x] \rightarrow \neg\neg N^r[x]\}$.

Typing of T_2

We use in the typing the Gödel transformation with $V_X = \{X, X'\}$, and

$F_X = X(x_1, \dots, x_n), X'(x_1, \dots, x_n) \rightarrow \perp$ for every second order variable X of arity n .

Let $R = \forall X\forall y\{(X, X \rightarrow \neg\neg N^r[0], X \rightarrow \neg\neg N^r[y]), X \rightarrow \neg\neg N^r[sy]\}$, $D = R \rightarrow \neg\neg N^r[0]$, and $F[x] = R, D, R \rightarrow \neg\neg N^r[x]$.

- $\vdash_{TTR} \lambda f(f)\bar{0} : \neg\neg N^r[0]$; therefore : $\vdash_{TTR} \tau : X \rightarrow \neg\neg N^r[0]$, and $\vdash_{TTR} \tau : R \rightarrow \neg\neg N^r[0]$.
- By the previous typing, we have $\vdash_{TTR} G : \forall y(\neg\neg N^r[y] \rightarrow \neg\neg N^r[sy])$; hence :
 $y : X, X \rightarrow \neg\neg N^r[0], X \rightarrow \neg\neg N^r[y], z : X \vdash_{TTR} (G)(y)z\tau z : \neg\neg N^r[sy]$; therefore
 $\vdash_{TTR} \rho : R$.
- Check that $\Phi^*(\lambda xF[x]/N, x) \subseteq F[x]$.
 $\Phi^*(\lambda xF[x]/N, x) =$
 $\forall X\forall X'\{\forall y(F[y], Xsy, X'sy \rightarrow \perp), (X0, X'0 \rightarrow \perp) \rightarrow (Xx, X'x \rightarrow \perp)\}$;
therefore by specifying Xx by R , and $X'x$ by $\neg\neg N^r[x]$; we obtain :
 $\Phi^*(\lambda xF[x]/N, x) \subseteq \forall y(F[y], R, \neg\neg N^r[sy] \rightarrow \perp), (R, \neg\neg N^r[0] \rightarrow \perp) \rightarrow (R, \neg\neg N^r[x] \rightarrow \perp)$. We
need to check that $R \subseteq \forall y(F[y], R, \neg\neg N^r[sy] \rightarrow \perp)$, this is absolutely true.

Therefore $N^{r*}[x] \subseteq F[x]$ and $\nu : N^{r*}[x] \vdash_{TTR} \nu : R, D, R \rightarrow \neg\neg N^r[x]$; then :

$\nu : N^{r*}[x] \vdash_{TTR} (\nu)\rho\tau\rho : \neg\neg N^r[x]$; and finally $\vdash_{TTR} T_2 : \forall x\{N^{r*}[x] \rightarrow \neg\neg N^r[x]\}$.

7.2 General Theorem

Theorem 7.1 *Let D, E be two \forall -positive closed types of TTR , such that \perp does not appear in E . If $\vdash_{TTR} T : D^* \rightarrow \neg\neg E$, then T is a storage operator for the pair (D, E) .*

Proof It is a consequence from the following Theorem :

Theorem 7.2 *Let D, E be two \forall -positive closed types of TTR° , such that \perp does not appear in E . If $\vdash_{TTR} T : D^* \rightarrow \neg\neg E$, then T is a storage operator for the pair (D, E) .*

Indeed:

Lemma 7.1 1) If $T \in \Omega^+$ (resp. $T \in \Omega^-$) then $T^\diamond \in \Omega^+$ (resp. $T^\diamond \in \Omega^-$).

2) For each Gödel transformation $*$ of TTR , there is a Gödel transformation * of TTR^\diamond such that : for every type D of TTR , $D^{*\diamond} = D^\diamond^*$.

Proof 1) By induction on T .

2) * is the restriction of $*$ on the types of TTR^\diamond . \square

Let t be a normal λ -term, such that $\vdash_{TTR} t : D$. If $\vdash_{TTR} T : D^* \rightarrow \neg\neg E$, then, by Theorem 3.3, $\vdash_{TTR^\diamond} T : D^{*\diamond} \rightarrow \neg\neg E^\diamond$. By 2)-Lemma 7.1, there is a Gödel transformation * , such that $\vdash_{TTR^\diamond} T : D^{\diamond^*} \rightarrow \neg\neg E^\diamond$. Therefore, there are λ -terms τ_t and τ'_t , such that $\tau_t \simeq_\beta \tau'_t$, $\vdash_{TTR^\diamond} \tau'_t : E^\diamond$, and $(T)t f \succ (f)\tau_t[t_1/x_1, \dots, t_n/x_n]$. By 2)-Corollary 6.1, we have $\vdash_{TTR} t : D^*$, then $f : \neg E \vdash_{TTR} (T)t f : \perp$, and $f : \neg E \vdash_{TTR} (f)\tau_t[t_1/x_1, \dots, t_n/x_n] : \perp$. Therefore $f : \neg E \vdash_{TTR} (f)\tau'_t : \perp$, and, by Corollary 4.1, $\vdash_{TTR} \tau'_t : E$. \square

We give the proof of Theorem 7.2 in a particular case.

Let $N^r = \mu N[\forall X\{N \rightarrow X, X \rightarrow X\}]$, and $*$ the Gödel transformation with $V_X = \{X\}$, and $F_X = \neg X(x_1, \dots, x_n)$ for every second order variable X of arity n .

We will prove that : If $\vdash_{TTR^\diamond} T : N^{r*} \rightarrow \neg\neg N^r$, then T is a storage operator for N^r .

Because of : if t is a closed normal λ -term with $\vdash_{TTR^\diamond} t : N^r$, then $t = \bar{n}$ for a certain integer n , and it is suffices to prove that : If $\vdash_{TTR^\diamond} T : N^{r*} \rightarrow \neg\neg N^r$, then, for every $n \geq 0$, there is an $m \geq 0$ and $\tau \simeq_\beta \bar{m}$, such that, for every λ -term $\theta_n \simeq_\beta \bar{n}$, there is a substitution σ , such that $(T)\theta_n f \sim (f)\sigma(\tau)$.

Lemma 7.2 If $\Gamma' = \Gamma, x : N^r \stackrel{*}{\vdash}_{TTR^\diamond} (x)u_1 \dots u_n : \perp$, then $n = 3$, and there is a type G , such that $\Gamma' \vdash_{TTR^\diamond} u_1 : N^r \stackrel{*}{\rightarrow} \neg G$, $\Gamma' \vdash_{TTR^\diamond} u_2 : \neg G$, and $\Gamma' \vdash_{TTR^\diamond} u_3 : G$.

Proof By Corollary 4.1, we have $\forall \mathbf{v}_0 N^{r*} \subseteq A_1 \rightarrow B_1$, $\forall \mathbf{v}_i B_i \subseteq A_{i+1} \rightarrow B_{i+1}$ $1 \leq i \leq n-1$, $\forall \mathbf{v}_n B_n \subseteq \perp$, $\mathbf{v}_0, \dots, \mathbf{v}_n$ are not free in N^{r*} and Γ , and $\Gamma' \vdash_{TTR^\diamond} u_i : A_i$ $1 \leq i \leq n$. Since $\forall \mathbf{v}_0 N^{r*} \subseteq A_1 \rightarrow B_1$, then, by Theorem 4.1, there is a formula F , such that $A_1 \subseteq N^{r*} \rightarrow \neg F$ and $\neg F \rightarrow \neg F \subseteq B_1$. We have also $\forall \mathbf{v}_1 B_1 \subseteq A_2 \rightarrow B_2$, then $\forall \mathbf{v}_1 (\neg F \rightarrow \neg F) \subseteq A_2 \rightarrow B_2$, and, by Theorem 4.1, there is a sequence of formulas \mathbf{F}_1 , such that $A_2 \subseteq \neg F[\mathbf{F}_1/\mathbf{v}_1]$ and $\neg F[\mathbf{F}_1/\mathbf{v}_1] \subseteq B_2$. Now, since $\forall \mathbf{v}_2 B_2 \subseteq A_3 \rightarrow B_3$, we have $\forall \mathbf{v}_2 (\neg F[\mathbf{F}_1/\mathbf{v}_1]) \subseteq A_3 \rightarrow B_3$, and, by Theorem 4.1, there is a sequence of formulas \mathbf{F}_2 , such that $A_3 \subseteq F[\mathbf{F}_1/\mathbf{v}_1, \mathbf{F}_2/\mathbf{v}_2]$ and $\perp \subseteq B_3$. By Corollary 4.1, we have $n = 3$. Let $G = F[\mathbf{F}_1/\mathbf{v}_1, \mathbf{F}_2/\mathbf{v}_2]$. Since $\mathbf{v}_1, \mathbf{v}_2$ are not free in N^{r*} and Γ , we deduce $\Gamma' \vdash_{TTR^\diamond} u_1 : N^{r*} \rightarrow \neg G$, $\Gamma' \vdash_{TTR^\diamond} u_2 : \neg G$, and $\Gamma' \vdash_{TTR^\diamond} u_3 : G$. \square

Let $n \geq 0$.

Definition An n -special application θ is a function from $\{0, 1, \dots, n\}$ to Λ with the following properties : $\theta(0) \succ \bar{0}$ and $\theta(m+1) \succ \lambda f_m \lambda x_m (f_m) \theta(m)$ $0 \leq m \leq n-1$.

Lemma 7.3 For every $\theta_n \simeq_\beta \bar{n}$, there is an n -special application θ , such that $\theta(n) = \theta_n$.

Proof Easy. \square

Definitions

1) Let $0 \leq m \leq n$ and $\mathbf{u} = u_{m,1}, u_{m,2}, u_{m,3}, \dots, u_{n-1,1}, u_{n-1,2}, u_{n-1,3}$ a sequence of λ -terms. We denote by $x_{m,\mathbf{u}}$ a constant which does not appear in \mathbf{u} .

2) Let θ be an n -special application. The n -special substitution S_θ is the function on the set Λ defined by induction :

- If $u = x$, then $S_\theta(x) = x$;
- If $u = \lambda xv$, then $S_\theta(u) = \lambda y S_\theta(v[y/x])$ where $y \notin Fv(\theta(n))$;
- If $u = (v)w$, then $S_\theta(u) = (S_\theta(v))S_\theta(w)$;
- If $u = x_{m,\mathbf{u}}$, then
 $S_\theta(u) = \theta(m)[S_\theta(u_{m,1})/f_m, S_\theta(u_{m,2})/x_m, \dots, S_\theta(u_{n-1,1})/f_{n-1}, S_\theta(u_{n-1,2})/x_{n-1}]$.

An n -special substitution is the application S_θ associated to a some n -special application θ .

Lemma 7.4 Let $\{U_i \succ V_i\}_{1 \leq i \leq r}$ be a sequence of head reductions such that :

$V_i = (x_{m,\mathbf{u}})u_1 u_2 u_3$ $0 \leq m \leq n$, $[U_{i+1} = (u_1)x_{m-1,u_1,u_2,u_3,\mathbf{u}}u_3$ if $m \neq 0$, and $U_{i+1} = (u_2)u_3$ if $m = 0]$, and S_θ an n -special substitution. For every $1 \leq i \leq r$, $S_\theta(U_1) \sim S_\theta(V_i)$.

Proof We argue by induction on i .

The case $i = 0$ is a consequence of Theorem 2.1.

Assume that is true for i , and prove it for $i+1$.

If $V_i = (x_{m,\mathbf{u}})u_1 u_2 u_3$ $0 \leq m \leq n$, then

$$S_\theta(V_i) = (\theta(m)[S_\theta(u_{m,1})/f_m, S_\theta(u_{m,2})/x_m, \dots, S_\theta(u_{n-1,1})/f_{n-1}, S_\theta(u_{n-1,2})/x_{n-1}]) S_\theta(u_1) S_\theta(u_2) S_\theta(u_3).$$

- If $m \neq 0$, then $\theta(m) \succ \lambda f_{m-1} \lambda x_{m-1} (f_{m-1}) \theta(m-1)$,
and $S_\theta(V_i) \sim (S_\theta(u_1)) \theta(m-1) [S_\theta(u_{m-1,1})/f_{m-1}, S_\theta(u_{m-1,2})/x_{m-1}, \dots, S_\theta(u_{n-1,1})/f_{n-1}, S_\theta(u_{n-1,2})/x_{n-1}] S_\theta(u_3) = S_\theta(U_{i+1})$.
- If $m = 0$, then $\theta(m) \succ \lambda f \lambda x x$, and $S_\theta(V_i) \sim (\lambda f \lambda x x) S_\theta(u_1) S_\theta(u_2) S_\theta(u_3) \sim (S_\theta(u_2)) S_\theta(u_3) = S_\theta(U_{i+1})$.

By the induction hypothesis we have $S_\theta(U_1) \sim S_\theta(V_i)$, then $S_\theta(U_1) \sim S_\theta(U_{i+1})$, and, by Theorem 2.1, $S_\theta(U_1) \sim S_\theta(V_{i+1})$. \square

Definition A context $\Gamma = f : \neg N, x_n, \mathbf{u}_0 : N^{r*}, x_{m_1}, \mathbf{u}_1 : N^{r*}, \dots, x_{m_s}, \mathbf{u}_s : N^{r*}$ where $0 \leq m_j \leq n$, $1 \leq j \leq s$, is called n -good.

Lemma 7.5 *There is a sequence of head reductions $\{U_i \succ V_i\}_{1 \leq i \leq r}$ such that :*

- $U_1 = (T)x_n f$ and $V_r = (f)\tau$ where $\tau \simeq_\beta \bar{l}$ for some $l \geq 0$;
- $V_i = (x_{m,\mathbf{u}})u_1 u_2 u_3$ $0 \leq m \leq n$, and
 $U_{i+1} = (u_1)x_{m-1, u_1, u_2, u_3, \mathbf{u}} u_3$ if $m \neq 0$, and $U_{i+1} = (u_2)u_3$ if $m = 0$;
- For every $1 \leq i \leq r$, there is an n -good context Γ_i such that $\Gamma_i \vdash_{TTR^\circ} V_i : \perp$.

Proof Since $\vdash_{TTR^\circ} T : N^{r*} \rightarrow \neg \neg N^r$, then $x_n : N^{r*}, f : \neg N^r \vdash_{TTR^\circ} (T)x_n f : \perp$, and, by Corollary 4.3 and Lemma 7.2, we have $(T)x_n f \succ V_1$ where $V_1 = (f)\tau$ or $V_1 = (x_n)u_1 u_2 u_3$.

Assume that we have the head reduction $U_k \succ V_k$ and $V_k \neq (f)\tau$. Then $V_k = (x_{m,\mathbf{u}})u_1 u_2 u_3$ $0 \leq m \leq n$, and, by the induction hypothesis, there is an n -good context Γ_k such that $\Gamma_k \vdash_{TTR^\circ} (x_{m,\mathbf{u}})u_1 u_2 u_3 : \perp$. By Lemma 7.2, there is a type G , such that $\Gamma_k \vdash_{TTR^\circ} u_1 : N^{r*} \rightarrow \neg G$, $\Gamma_k \vdash_{TTR^\circ} u_2 : \neg G$, and $\Gamma_k \vdash_{TTR^\circ} u_3 : G$.

- If $m = 0$, let $U_{k+1} = (u_2)u_3$. Let $\Gamma_{k+1} = \Gamma_k$. We have $\Gamma_{k+1} \vdash_{TTR^\circ} U_{k+1} : \perp$.
- If $m \neq 0$, let $U_{k+1} = (u_1)x_{m-1, u_1, u_2, u_3, \mathbf{u}} u_3$. The variable $x_{m-1, u_1, u_2, u_3, \mathbf{u}}$ is not used before. Indeed, if it is, by Lemma 7.4, the λ -term $(T)\bar{n}f$ is not solvable. That is impossible because $f : \neg N^r \vdash_{TTR^\circ} (T)\bar{n}f : \perp$. Therefore $\Gamma_{k+1} = \Gamma_k, x_{m-1, u_1, u_2, u_3, \mathbf{u}} : N^{r*}$ is an n -good context and $\Gamma_{k+1} \vdash_{TTR^\circ} U_{k+1} : \perp$.

By Corollary 4.3 and Lemma 7.2, we have $U_{k+1} \succ V_{k+1}$ where $V_{k+1} = (f)\tau$ or $V_{k+1} = (x_{s,\mathbf{v}})v_1 v_2 v_3$ $0 \leq s \leq n$.

This construction always terminates. Indeed, if not, by Lemma 7.4, the λ -term $(T)\bar{n}f$ is not solvable. That is impossible because $f : \neg N^r \vdash_{TTR^\circ} (T)\bar{n}f : \perp$.

Therefore there is $r \geq 0$ and an n -good context Γ_r such that $V_r = (f)\tau$ and $\Gamma_r \vdash_{TTR^\circ} V_r : \perp$. By Lemma 6.2, we have $\tau \simeq_\beta \bar{l}$ for some $l \geq 0$. \square

Let θ_n be a λ -term such that $\theta_n \simeq_\beta \bar{n}$. By Lemma 7.3, let θ be an n -special application such that $\theta(n) = \theta_n$. Let S_θ the n -special substitution associated to θ . By Lemma 7.4, we have for every $1 \leq i \leq r$, $(T)\theta_n f \sim S_\theta(V_i)$. In particular, for $i = n$, $(T)\theta_n f \sim S_\theta((f)\tau) = (f)S_\theta(\tau)$. Then T is a storage operator for N^r . \square

References

- [1] H. BARENDREGT. *The lambda calculus: Its Syntax and Semantics*. North Holland, 1984.
- [2] R. DAVID. *The Inf function in system F*. Theoretical Computer Science, 135 (423-431), 1994.
- [3] P. GIANNINI and S. RONCHI. *Characterization of typing in polymorphic type discipline*. LICS, Edinbourg (61-70), 1988.
- [4] J.L. KRIVINE. *Lambda calcul, évaluation paresseuse et mise en mémoire*. Informatique Théorique et Applications, Vol. 25,1, p. 67-84, 1991.
- [5] J.L. KRIVINE. *Lambda calcul, types et modèle*. Masson, Paris, 1990.
- [6] J.L. KRIVINE. *Opérateurs de mise en mémoire et traduction de Gödel*. Archive. Math. Logic 30. (241-267), 1990.
- [7] J.L. KRIVINE. *Mise en mémoire (preuve générale)*. Manuscript, 1991.
- [8] R. LABIB-SAMI. *Typer avec (ou sans) types auxiliaires*. Manuscript, 1986.
- [9] D. LEIVANT. *Reasonning about functional programs and complexity classes associated with type disciplines*. In 24th Annual Symposium on Foundations of Computer Science, volum 44 (460-469), 1983.
- [10] D. LEIVANT. *Typing and computation properties of lambda expressions*. Theoretical Computer Science, 44 (51-68), 1986.
- [11] J. MITCHELL. *Polomorphic type*. Information and Computation, 76 (2/3), (211-249), 1988.
- [12] K. NOUR. *Opérateurs de mise en mémoire en lambda-calcul pur et typé*. Thèse de doctorat, Université de Savoie, 1993.
- [13] K. NOUR. *Strong storage operators and data types*. Archive. Math. Logic 34. (65-78), 1995.
- [14] K. NOUR. *Opérateurs propres de mise en mémoire*. C.R.A.S. Paris, t. 317, Série I, p. 1-6, 1993.

- [15] K. NOUR. *Preuve syntaxique d'un théorème de J.L. Krivine sur les opérateurs de mise en mémoire.*
C.R.A.S. Paris, t. 318, Série I, p. 201-204, 1994.
- [16] K. NOUR. *Opérateurs de mise en mémoire et types \forall -positifs.*
Submitted to publication in Theoretical Informatics and Applications, 1993.
- [17] K. NOUR and R. DAVID. *Storage operators and directed lambda-calculus.*
Journal of Symbolic Logic (to appear).
- [18] M. PARIGOT. *Programming with proofs : a second order type theory.*
ESOP'88, LNCS 300, (145-159), 1988.
- [19] M. PARIGOT. *On representation of data in lambda calculus.*
To appear in LNCS.
- [20] M. PARIGOT. *Recursive programming with proofs.*
Theoretical Computer Science, 94 (335-356), 1992.