On a Spector ultrapower of the Solovay model^{*}

Vladimir Kanovei^{\dagger} Michiel van Lambalgen^{\ddagger}

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Abstract

We prove that a Spector–like ultrapower extension \mathfrak{N} of a countable Solovay model \mathfrak{M} (where all sets of reals are Lebesgue measurable) is equal to the set of all sets constructible from reals in a generic extension $\mathfrak{M}[\alpha]$ where α is a random real over \mathfrak{M} . The proof involves an almost everywhere uniformization theorem in the Solovay model.

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[†]Moscow Transport Engineering Institute, kanovei@sci.math.msu.su

[‡]University of Amsterdam, michiell@fwi.uva.nl

Introduction

Let \mathcal{U} be an ultrafilter in a transitive model \mathfrak{M} of \mathbb{ZF} . Assume that an ultrapower of \mathfrak{M} via \mathcal{U} is to be defined. The first problem we meet is that \mathcal{U} may not be an ultrafilter in the universe because not all subsets of the index set belong to \mathfrak{M} .

We can, of course, extend \mathcal{U} to a true ultrafilter, say \mathcal{U}' , but this may cause additional trouble. Indeed, if \mathcal{U} is a special ultrafilter in \mathfrak{M} certain properties of which were expected to be exploit, then most probably these properties do not transfer to \mathcal{U}' ; assume for instance that \mathcal{U} is countably complete in \mathfrak{M} and \mathfrak{M} itself is countable. Therefore, it is better to keep \mathcal{U} rather than any of its extensions in the universe, as the ultrafilter.

If \mathfrak{M} models **ZFC**, the problem can be solved by taking the *inner* ultrapower. In other words, we consider only those functions $f: I \longrightarrow \mathfrak{M}$ (where $I \in \mathfrak{M}$ is the carrier of \mathfrak{U}) which belong to \mathfrak{M} rather than *all* functions $f \in \mathfrak{M}^{I}$, to define the ultrapower. This version, however, depends on the axiom of choice in \mathfrak{M} ; otherwise the proofs of the basic facts about ultrapowers (e. g. Loś' theorem) will not work.

The "choiceless" case can be handled by a sophisticated construction of SPEC-TOR [1991], which is based on ideas from both forcing and the ultrapower technique. As presented in KANOVEI and VAN LAMBALGEN [1994], this construction proceeds as follows. One has to add to the family of functions $\mathcal{F}_0 = \mathfrak{M}^I \cap \mathfrak{M}$ a number of new functions $f \in \mathfrak{M}^I$, $f \notin \mathfrak{M}$, which are intended to be choice functions whenever we need such in the ultrapower construction.

In this paper, we consider a very interesting choiceless case: \mathfrak{M} is a Solovay model of **ZF** plus the principle of dependent choice, in which all sets of reals are Lebesque measurable, and the ultrafilter \mathcal{L} on the set I of Vitali degrees of reals in \mathfrak{M} , generated by sets of positive measure.

1 On a.e. uniformization in the Solovay model

In this section, we recall the uniformization properties in a Solovay model. Thus let \mathfrak{M} be a countable transitive Solovay model for Dependent Choices plus "all sets are Lebesgue measurable", as it is defined in SOLOVAY [1970], – the ground model. The following known properties of such a model will be of particular interest below.

Property 1 [True in \mathfrak{M}] $\mathbb{V} = \mathbb{L}(\text{reals})$; in particular every set is real-ordinal-definable.

To state the second property, we need to introduce some notation.

Let $\mathcal{N} = \omega^{\omega}$ denote the Baire space, the elements of which will be referred to as *real numbers* or *reals*.

Let P be a set of pairs such that dom $P \subseteq \mathcal{N}$ (for instance, $P \subseteq \mathcal{N}^2$). We say that a function f defined on \mathcal{N} uniformizes P a.e. (almost everywhere) iff the set

$$\{\alpha \in \operatorname{dom} P : \langle \alpha, f(\alpha) \rangle \notin P\}$$

has null measure. For example if the projection dom P is a set of null measure in \mathcal{N} then any f uniformizes a.e. P, but this case is not interesting. The interesting case is the case when dom P is a set of full measure, and then f a.e. uniformizes P iff for almost all α , $\langle \alpha, f(\alpha) \rangle \in P_{\alpha}$.

Property 2 [True in \mathfrak{M}]

Any set $P \in \mathfrak{M}$, $P \subseteq \mathbb{N}^2$, can be uniformized a.e. by a Borel function. (This implies the Lebesgue measurability of all sets of reals, which is known to be true in \mathfrak{M} independently.)

This property can be expanded (with the loss of the condition that f is Borel) on the sets P which do not necessarily satisfy dom $P \subseteq \mathcal{N}$.

Theorem 3 In \mathfrak{M} , any set P with dom $P \subseteq \mathfrak{M}$ admits an a.e. uniformisation.

Proof Let *P* be an arbitrary set of pairs such that dom $P \subseteq \mathcal{N}$ in \mathfrak{M} . Property 1 implies the existence of a function $D : (\operatorname{Ord} \cap \mathfrak{M}) \times (\mathcal{N} \cap \mathfrak{M})$ onto \mathfrak{M} which is \in -definable in \mathfrak{M} .

We argue in \mathfrak{M} . Let, for $\alpha \in \mathcal{N}$, $\xi(\alpha)$ denote the least ordinal ξ such that

$$\exists \gamma \in \mathcal{N} \left[\left\langle \alpha, D(\xi, \gamma) \right\rangle \in P \right].$$

(It follows from the choice of D that $\xi(\alpha)$ is well defined for all $\alpha \in \mathbb{N}$.) It remains to apply Property 2 to the set $P' = \{ \langle \alpha, \gamma \rangle \in \mathbb{N}^2 : \langle \alpha, D(\xi(\alpha), \gamma) \rangle \in P \}$. \Box

2 The functions to get the Spector ultrapower

We use a certain ultrafilter over the set of Vitali degrees of reals in \mathfrak{M} , the initial Solovay model, to define the ultrapower.

Let, for $\alpha, \alpha' \in \mathbb{N}$, α vit α' if and only if $\exists m \ \forall k \geq m \ (\alpha(k) = \alpha'(k))$, (the *Vitali equivalence*).

- For $\alpha \in \mathbb{N}$, we set $\underline{\alpha} = \{\alpha' : \alpha' \text{ vit } \alpha\}$, the Vitali degree of α .
- $\underline{\mathbb{N}} = \{\underline{\alpha} : \alpha \in \mathbb{N}\}; i, j \text{ denote elements of } \underline{\mathbb{N}}.$

As a rule, we shall use *underlined* characters \underline{f} , \underline{F} , ... to denote functions defined on $\underline{\mathcal{N}}$, while functions defined on \mathcal{N} itself will be denoted in the usual manner.

Define, in \mathfrak{M} , an ultrafilter \mathcal{L} over $\underline{\mathcal{N}}$ by: $\underline{X} \subseteq \underline{\mathcal{N}}$ belongs to \mathcal{L} iff the set $X = \{\alpha \in \mathcal{N} : \underline{\alpha} \in \underline{X}\}$ has full Lebesgue measure. It is known (see e.g. VAN LAMBALGEN [1992], Theorem 2.3) that the measurability hypothesis implies that \mathcal{L} is κ -complete in \mathfrak{M} for all cardinals κ in \mathfrak{M} .

One cannot hope to define a good \mathcal{L} -ultrapower of \mathfrak{M} using only functions from $\mathcal{F}_0 = \{\underline{f} \in \mathfrak{M} : \operatorname{dom} \underline{f} = \underline{N}\}$ as the base for the ultrapower. Indeed consider the identity function $\mathfrak{i} \in \mathfrak{M}$ defined by $\mathfrak{i}(i) = i$ for all $i \in \underline{N}$. Then $\mathfrak{i}(i)$ is nonempty for all $i \in \underline{N}$ in \mathfrak{M} , therefore to keep the usual properties of ultrapowers we need a function $\underline{f} \in \mathcal{F}_0$ such that $\underline{f}(i) \in i$ for almost all $i \in \underline{N}$, but Vitali showed that such a choice function yields a nonmeasurable set.

Thus at least we have to add to \mathcal{F}_0 a new function \underline{f} , not an element of \mathfrak{M} , which satisfies $\underline{f}(i) \in i$ for almost all $i \in \underline{N}$. Actually it seems likely that we have to add a lot of new functions, to handle similar situations, including those functions the existence of which is somehow implied by the already added functions. A general way how to do this, extracted from the exposition in SPECTOR [1991], was presented in KANOVEI and VAN LAMBALGEN [1994]. However in the case of the Solovay model the a.e. uniformization theorem (Theorem 3) allows to add essentially a single new function, corresponding to the i-case considered above.

The generic choice function for the identity

Here we introduce a function \mathfrak{r} defined on $\mathcal{N} \cap \mathfrak{M}$ and satisfying $\mathfrak{r}(i) \in i$ for all $i \in \mathcal{N} \cap \mathfrak{M}$. \mathfrak{r} will be generic over \mathfrak{M} for a suitable notion of forcing.

The notion of forcing is introduced as follows. In \mathfrak{M} , let \mathbb{P} be the set of all functions p defined on $\underline{\mathbb{N}}$ and satisfying $p(i) \subseteq i$ and $p(i) \neq \emptyset$ for all i.¹ (For example $i \in \mathbb{P}$.) We order \mathbb{P} so that p is stronger than q iff $p(i) \subseteq q(i)$ for all i. If $G \subseteq \mathbb{P}$ is \mathbb{P} -generic over \mathfrak{M} , G defines a function \mathfrak{r} by

 $\mathfrak{r}(i)$ = the single element of $\bigcap_{p \in G} p(i)$

¹Or, equivalently, the collection of all sets $X \subseteq \mathcal{N}$ which have a nonempty intersection with every Vitali degree. Perhaps this forcing is of separate interest.

for all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$. Functions \mathfrak{r} defined this way will be called \mathbb{P} -generic over \mathfrak{M} . Let us fix such a function \mathfrak{r} for the remainder of this paper.

The set of functions used to define the ultrapower

We let \mathcal{F} be the set of all superpositions $f \circ \mathfrak{r}$ where² \mathfrak{r} is the generic function fixed above while $f \in \mathfrak{M}$ is an arbitrary function defined on $\mathcal{N} \cap \mathfrak{M}$. Notice that in particular any function $f \in \mathfrak{M}$ defined on $\underline{\mathcal{N}} \cap \mathfrak{M}$ is in \mathcal{F} : take $f(\alpha) = f(\underline{\alpha})$.

To see that \mathcal{F} can be used successfully as the base of an ultrapower of \mathfrak{M} , we have to check three fundamental conditions formulated in KANOVEI and VAN LAMBALGEN [1994].

Proposition 4 [Measurability] Assume that $E \in \mathfrak{M}$ and $f_1, ..., f_n \in \mathcal{F}$. Then the set $\{i \in \underline{\mathbb{N}} \cap \mathfrak{M} : E(f_1(i), ..., f_n(i))\}$ belongs to \mathfrak{M} .

Proof By the definition of \mathcal{F} , it suffices to prove that $\{i : \mathfrak{r}(i) \in E\} \in \mathfrak{M}$ for any set $E \in \mathfrak{M}$, $E \subseteq \mathcal{N}$. By the genericity of \mathfrak{r} , it remains then to prove the following in \mathfrak{M} : for any $p \in \mathbb{P}$ and any set $E \subseteq \mathcal{N}$, there exists a stronger condition q such that, for any i, either $q(i) \subseteq E$ or $q(i) \cap E = \emptyset$. But this is obvious. \Box

Corollary 5 Assume that $V \in \mathfrak{M}$, $V \subseteq \mathfrak{N}$ is a set of null measure in \mathfrak{M} . Then, for \mathcal{L} -almost all i, we have $\mathfrak{r}(i) \notin V$.

Proof By the proposition, the set $I = \{i : \mathfrak{r}(i) \in V\}$ belongs to \mathfrak{M} . Suppose that, on the contrary, $I \in \mathcal{L}$. Then $A = \{\alpha : \underline{\alpha} \in I\}$ is a set of full measure. On the other hand, since $\mathfrak{r}(i) \in i$, we have $A \subseteq \bigcup_{\beta \in V} \underline{\beta}$, where the right-hand side is a set of null measure because V is such a set, contradiction.

Proposition 6 [Choice] Let $f_1, ..., f_n \in \mathcal{F}$ and $W \in \mathfrak{M}$. There exists a function $\underline{f} \in \mathcal{F}$ such that, for \mathcal{L} -almost all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$, it is true in \mathfrak{M} that

$$\exists x \ W(f_1(i), ..., f_n(i), x) \longrightarrow W(f_1(i), ..., f_n(i), f(i))$$

Proof This can be reduced to the following: given $W \in \mathfrak{M}$, there exists a function $\underline{f} \in \mathcal{F}$ such that, for \mathcal{L} -almost all $i \in \underline{\mathcal{N}} \cap \mathfrak{M}$,

$$\exists x \ W(\mathbf{r}(i), x) \longrightarrow W(\mathbf{r}(i), f(i)) \tag{(*)}$$

in M.

²To make things clear, $f \circ \mathfrak{r}(i) = f(\mathfrak{r}(i))$ for all i.

We argue in \mathfrak{M} . Choose $p \in \mathbb{P}$. and let $p'(i) = \{\beta \in p(i) : \exists x W(\beta, x)\}$, and $X = \{i : p'(i) \neq \emptyset\}$. If $X \notin \mathcal{L}$ then an arbitrary <u>f</u> defined on <u>N</u> will satisfy (*), therefore it is assumed that $X \in \mathcal{L}$. Let

$$q(i) = \begin{cases} p'(i) & \text{iff} \quad i \in X \\ p(i) & \text{otherwise} \end{cases}$$

for all $i \in \underline{N}$; then $q \in \mathbb{P}$ is stronger than p. Therefore, since \mathfrak{r} is generic, one may assume that $\mathfrak{r}(i) \in q(i)$ for all i.

Furthermore, DC in the Solovay model \mathfrak{M} implies that for every $i \in X$ the following is true: there exists a function ϕ defined on q(i) and such that $W(\beta, \phi(\beta))$ for every $\beta \in q(i)$. Theorem 3 provides a function Φ such that for almost all α the following is true: the value $\Phi(\alpha, \beta)$ is defined and satisfies $W(\beta, \Phi(\alpha, \beta))$ for all $\beta \in q(\underline{\alpha})$. Then, by Corollary 5, we have

for all
$$\beta \in q(\mathfrak{r}(i))$$
, $W(\beta, \Phi(\mathfrak{r}(i), \beta))$

for almost all *i*. However, $\underline{\mathbf{r}}(i) = i$ for all *i*. Applying the assumption that $\mathbf{r}(i) \in q(i)$ for all *i*, we obtain $W(\overline{\mathbf{r}}(i), \Phi(\mathbf{r}(i), \mathbf{r}(i)))$ for almost all *i*. Finally the function $f(i) = \Phi(\mathbf{r}(i), \mathbf{r}(i))$ is in \mathcal{F} by definition. \Box

Proposition 7 [Regularity] For any $\underline{f} \in \mathcal{F}$ there exists an ordinal $\xi \in \mathfrak{M}$ such that for \mathcal{L} -almost all i, if f(i) is an ordinal then $f(i) = \xi$.

Proof To prove this statement, assume that $\underline{f} = f \circ \mathfrak{r}$ where $f \in \mathfrak{M}$ is a function defined on \mathcal{N} in \mathfrak{M} .

We argue in \mathfrak{M} . Consider an arbitrary $p \in \mathbb{P}$. We define a stronger condition p' as follows. Let $i \in \underline{\mathcal{N}}$. If there does not exist $\beta \in p(i)$ such that $f(\beta)$ is an ordinal, we put p'(i) = p(i) and $\xi(i) = 0$. Otherwise, let $\xi(i) = \xi$ be the least ordinal ξ such that $f(\beta) = \xi$ for some $\beta \in p(i)$. We set $p'(i) = \{\beta \in p(i) : f(\beta) = \xi(i)\}$.

Notice that $\xi(i)$ is an ordinal for all $i \in \underline{\mathbb{N}}$. Therefore, since the ultrafilter \mathcal{L} is κ -complete in \mathfrak{M} for all κ , there exists a single ordinal $\xi \in \mathfrak{M}$ such that $\xi(i) = \xi$ for almost all i.

By genericity, we may assume that actually $\mathfrak{r}(i) \in p'(i)$ for all $i \in \underline{\mathbb{N}} \cap \mathfrak{M}$. Then ξ is as required.

The ultrapower

Let $\mathfrak{N} = \text{Ult}_{\mathcal{L}} \mathfrak{F}$ be the ultrapower. Thus we define:

- $f \approx g$ iff $\{i : f(i) = g(i)\} \in \mathcal{L}$ for $f, g \in \mathcal{F}$;
- $[f] = \{g : g \approx f\}$ (the *L*-degree of f);

- $[f] \in^* [g]$ iff $\{i : f(i) \in g(i)\} \in \mathcal{L}$;
- $\mathfrak{N} = \{ [f] : f \in \mathfrak{F} \}, \text{ equipped with the above defined membership } \in^*$.

Theorem 8 \mathfrak{N} is an elementary extension of \mathfrak{M} via the embedding which associates $x^* = [\underline{\mathbb{N}} \times \{x\}]$ with any $x \in \mathfrak{M}$. Moreover \mathfrak{N} is wellfounded and the ordinals in \mathfrak{M} are isomorphic to the \mathfrak{M} -ordinals via the mentioned embedding.

Proof See KANOVEI and VAN LAMBALGEN [1994].

Comment. Propositions 4 and 6 are used to prove the Loś theorem and the property of elementary embedding. Proposition 7 is used to prove the wellfoundedness part of the theorem.

3 The nature of the ultrapower

Theorem 8 allows to collapse \mathfrak{N} down to a transitive model $\widehat{\mathfrak{N}}$; actually $\widehat{\mathfrak{N}} = \{\widehat{X} : X \in \mathfrak{N}\}$ where

$$\widehat{X} = \{\widehat{Y} : Y \in \mathfrak{N} \text{ and } Y \in X\}.$$

The content of this section will be to investigate the relations between \mathfrak{M} , the initial model, and $\widehat{\mathfrak{N}}$, the (transitive form of its) Spector ultrapower. In particular it is interesting how the superposition of the "asterisk" and "hat" transforms embeds \mathfrak{M} into $\widehat{\mathfrak{N}}$.

Lemma 9 $x \mapsto \widehat{x^*}$ is an elementary embedding \mathfrak{M} into $\widehat{\mathfrak{N}}$, equal to identity on ordinals and sets of ordinals (in particular on reals).

Proof Follows from what is said above.

Thus $\widehat{\mathfrak{N}}$ contains all reals in \mathfrak{M} . We now show that $\widehat{\mathfrak{N}}$ also contains some new reals. We recall that $\mathfrak{r} \in \mathfrak{F}$ is a function satisfying $\mathfrak{r}(i) \in i$ for all $i \in \underline{\mathfrak{N}} \cap \mathfrak{M}$.

Let $\mathbf{a} = [\widehat{\mathbf{r}}]$. Notice that by Loś $[\mathbf{r}]$ is a real in \mathfrak{N} , therefore \mathbf{a} is a real in $\widehat{\mathfrak{N}}$.

Lemma 10 a is random over \mathfrak{M} .

Proof Let $B \subseteq \mathbb{N}$ be a Borel set of null measure coded in \mathfrak{M} ; we prove that $\mathbf{a} \notin B$. Being of measure 0 is an absolute notion for Borel sets, therefore $B \cap \mathfrak{M}$ is a null set in \mathfrak{M} as well. Corollary 5 implies that for \mathcal{L} -almost all i, we have $\mathfrak{r}(i) \notin B$. By Loś, $\neg ([\mathfrak{r}] \in B^*)$ in \mathfrak{N} . Then $\mathbf{a} \notin \widehat{B^*}$ in $\widehat{\mathfrak{N}}$. However, by the absoluteness of the Borel coding, $\widehat{B^*} = B \cap \widehat{\mathfrak{N}}$, as required. \Box

Thus $\widehat{\mathfrak{N}}$ contains a new real number **a**. It so happens that this **a** generates all reals in $\widehat{\mathfrak{N}}$.

Lemma 11 The reals of $\widehat{\mathfrak{N}}$ are exactly the reals of $\mathfrak{M}[\mathbf{a}]$.

Proof It follows from the known properties of random extensions that every real in $\mathfrak{M}[\mathbf{a}]$ can be obtained as $F(\mathbf{a})$ where F is a Borel function coded in \mathfrak{M} . Since \mathbf{a} and all reals in \mathfrak{M} belong to $\widehat{\mathfrak{N}}$, we have the inclusion \supseteq in the lemma.

To prove the opposite inclusion let $\beta \in \widehat{\mathfrak{N}} \cap \mathcal{N}$. Then by definition $\beta = [F]$, where $F \in \mathcal{F}$. In turn $F = f \circ \mathfrak{r}$, where $f \in \mathfrak{M}$ is a function defined on $\mathcal{N} \cap \mathfrak{M}$. We may assume that in \mathfrak{M} f maps reals into reals. Then, first, by Property 2, f is a.e. equal in \mathfrak{M} to a Borel function $g = B_{\gamma}$ where $\gamma \in \mathcal{N} \cap \mathfrak{M}$ and B_{γ} denotes, in the usual manner, the Borel subset (of \mathcal{N}^2 in this case) coded by γ . Corollary 5 shows that we have $F(i) = B_{\gamma}(\mathfrak{r}(i))$ for \mathcal{L} -almost all i. In other words, $F(i) = B_{\gamma^*(i)}(\mathfrak{r}(i))$ for \mathcal{L} -almost all i. By Łoś, this implies $[F] = B_{[\gamma^*]}([\mathfrak{r}])$ in \mathfrak{N} , therefore $\beta = B_{\gamma}(\mathfrak{a})$ in $\widehat{\mathfrak{N}}$. By the absoluteness of Borel coding, we have $\beta \in \mathbb{L}[\gamma, \mathfrak{a}]$, therefore $\beta \in \mathfrak{M}[\mathfrak{a}]$. \Box

We finally can state and prove the principal result.

Theorem 12 $\widehat{\mathfrak{N}} \subseteq \mathfrak{M}[\mathbf{a}]$ and $\widehat{\mathfrak{N}}$ coincides with $\mathbb{L}^{\mathfrak{M}[\mathbf{a}]}(\text{reals})$, the smallest subclass of $\mathfrak{M}[\mathbf{a}]$ containing all ordinals and all reals of $\mathfrak{M}[\mathbf{a}]$ and satisfying all the axioms of \mathbf{ZF} .

Proof Very elementary. Since $\mathbb{V} = \mathbb{L}(\text{reals})$ is true in \mathfrak{M} , the initial Solovay model, this must be true in $\widehat{\mathfrak{N}}$ as well. The previous lemma completes the proof. \Box

Corollary 13 The set $\mathcal{N} \cap \mathfrak{M}$ of all "old" reals does not belong to $\widehat{\mathfrak{N}}$.

Proof The set in question is known to be non-measurable in the random extension $\mathfrak{M}[\mathbf{a}]$; thus it would be non-measurable in $\widehat{\mathfrak{N}}$ as well. However $\widehat{\mathfrak{N}}$ is an elementary extension of \mathfrak{M} , hence it is true in $\widehat{\mathfrak{N}}$ that all sets are measurable.

References

- 1. V. KANOVEI and M. VAN LAMBALGEN [1994] Another construction of choiceless ultrapower. University of Amsterdam, Preprint X-94-02, May 1994.
- M. VAN LAMBALGEN [1994] Independence, randomness, and the axiom of choice. J. Symbolic Logic, 1992, 57, 1274 – 1304.
- 3. R. M. SOLOVAY [1970] A model of set theory in which every set of reals is Lebesgue measurable. Ann. of Math., 1970, 92, 1 – 56.
- 4. M. SPECTOR [1991] Extended ultrapowers and the Vopenka Hrbáček theorem without choice. J. Symbolic Logic, 1991, 56, 592 607.