On topological properties of ultraproducts of finite sets

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Abstract

Motivated by the model theory of higher order logics, in [2] a certain kind of topological spaces had been introduced on ultraproducts. These spaces are called ultratopologies. Ultratopologies provide a natural extra topological structure for ultraproducts and using this extra structure in [2] some preservation and characterization theorems had been obtained for higher order logics.

The purely topological properties of ultratopologies seem interesting on their own right. We started to study these properties in [3], where some questions remained open. Here we present the solutions of two such problems. More concretely we show that

- (1) there are sequences of finite sets of pairwise different cardinalities such that in their certain ultraproducts there are homeomorphic ultratopologies and
- (2) if A is an infinite ultraproduct of finite sets then every ultratopology on A contains a dense subset D such that |D| < |A|.

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1 Introduction

In first order model theory the ultraproduct construction can be applied rather often. this is because ultraproducts preserve the validity of first order formulas. It is also natural to ask, what connections can be proved between certain higher order formulas and ultraproducts of models of them.

In [2] we answer related questions in terms of topological spaces which can be

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naturally associated to ultraproducts. These spaces are called ultratopologies and their definition can be found in [2] and also at the beginning of [3].

Although ultratopologies were introduced from logical (model theoretical) reasons, these spaces can be interesting on their own right. In [3] a systematic investigation about these topological properties has been started. However, in [3] some problems remained open. In the present note we are dealing with two such problems.

In Section 2 we give a positive answer for Problem 5.2 of [3]: there are sequences of finite sets of pairwise different cardinalities such that their certain ultraproducts are still homeomorphic with respect to some carefully chosen ultratopologies. In fact, in Theorem 2.1 below we show that if U is a good ultrafilter and A is any infinite ultraproduct of finite sets modulo U then there is an ultratopology on A in which the family of closed sets consists just the finite sebsets of A and the whole A. From this the affirmative answer for Problem 5.2 of [3] can be immediately deduced. In Section 3 we investigate the possible cardinalities of dense sets in ultratopologies, again on ultraproducts of finite sets. In Corollary 3.3 we show that if C is any ultratopology on an infinite ultraproduct A then the density of C is smaller then |A|, that is, one can always find a dense set whose cardinality is less than |A|.

Throughout we use the following conventions. I is a set and for every $i \in I$ A_i is a set. Moreover $A = \prod_{i \in I} A_i / U$ denotes the ultraproduct of A_i 's modulo an ultrafilter U.

Every ordinal is the set of smaller ordinals and natural numbers are identified with finite ordinals. Throughout ω denotes the smallest infinite ordinal and cf denotes the cofinality operation.

In order to simplify notation, sometimes we will identify ${}^k(\Pi_{i\in I}A_i)$ with $\Pi_{i\in I}{}^kA_i$ by the natural way, that is, k-tuples of sequences are identified with single sequences whose terms are k-tuples.

If X is a topological space and $A \subseteq X$ then cl(A) denotes the closure of A. Suppose $k \in \omega$, $\langle A_i : i \in I \rangle$ is a sequence of sets, U is an ultrafilter on I and $R_i \subseteq {}^k A$ is a given relation for every $i \in I$. Then the *ultraproduct* relation $\prod_{i \in I} R_i / U$ is defined as follows.

$$\Pi_{i \in I} R_i / U = \{ s / U \in ({}^k \Pi_{i \in I} A_i / U) : \{ i \in I : s(i) \in R_i \} \in U \}.$$

As we mentioned, we assume that the reader is familiar with the notions of "choice function", "ultratopology", "a point is close to a relation", "T(a,R)", etc. These notions were introduced in [2] and a short (but fairly complete) survey can be found at the beginning of [3].

2 Homeomorphisms between different ultraproducts

In [3] Problem 5.2 asks whether is it possible to choose ultrafilters U_1 , U_2 and sequences of natural numbers $s = \langle n_i, i \in I \rangle$ and $z = \langle m_j, j \in J \rangle$ so that

- (i) $n_i \neq m_j$ for all $i \in I, j \in J$ and
- (ii) for every $k \in \omega$ there are k-dimensional ultratopologies C_k in $\Pi_{i \in I} n_i / U_1$ and D_k in $\Pi_{j \in J} m_j / U_2$ such that C_k and D_k are homeomorphic?

We will give an affirmative answer. In fact, we prove the following theorem from which the above question can be easily answered.

Theorem 2.1 Suppose $\langle n_i, i \in I \rangle$ is an infinite sequence of natural numbers and U is a good ultrafilter on I such that $A = \prod_{i \in I} n_i / U$ is infinite. Then for every $k \in \omega$ there is a k-dimensional choice function on A such that the family of closed sets in the induced ultratopology consists of the finite subsets of kA and kA .

Proof. Let c be any k-dimensional choice function on kA . By modifying c, we will construct another choiche function $\hat{}$ which induces the required ultratopology. Let E be the set of all triples $\langle s, i, m \rangle$ where $s \in \Pi_{l \in I} \mathcal{P}({}^k n_l), i \in I$ such that $\Pi_{l \in I} s(l)/U$ is infinite and $m \in {}^k n_i - s(i)$. It is easy to see that $|\Pi_{i \in I} \mathcal{P}({}^k n_i)| \leq |I^I \omega| = 2^{|I|}$. Therefore $|E| \leq 2^{|I|} \times |I| \times \aleph_0 = 2^{|I|}$. Let $\{\langle s_\alpha, i_\alpha, m_\alpha \rangle : \alpha < 2^{|I|}\}$ be an enumeration of E.

By transfinite recursion we construct an injective function $f: E \to {}^k A$ such that for every $\langle s, i, m \rangle \in E$ one has $f(\langle s, i, m \rangle) \in \Pi_{l \in I} s(l) / U$. Suppose f_{β} has already been defined on $\{\langle s_{\alpha}, i_{\alpha}, m_{\alpha} \rangle : \alpha < \beta\}$ for all $\beta < \gamma \leq 2^{|I|}$ such that

$$\beta_1 < \beta_2 < \gamma \Rightarrow f_{\beta_1} \subseteq f_{\beta_2}$$
 and $|f_{\beta}| \leq |\beta|$.

If γ is a limit ordinal, then let $f_{\gamma} = \bigcup_{\beta < \gamma} f_{\beta}$. Now suppose γ is a successor ordinal, say $\gamma = \delta + 1$. Since U is a good ultrafilter, by Theorem VI, 2.13 of [4] it follows, that the cardinality of $B = \prod_{l \in I} s(l)/U$ is $2^{|I|}$. Therefore there is an element $b \in B$ which is not in the range of f_{δ} . Let f_{γ} be $f_{\delta} \cup \{\langle\langle s_{\delta}, i_{\delta}, m_{\delta} \rangle, b\rangle\}$. Clearly, $f = f_{2^{|I|}}$ is the required function.

Now we construct a k-dimensional choice function $\hat{}$ as follows. If $a \notin rng(f)$ then let $\hat{a} = c(a)$. Otherwise there is a unique $e = \langle s, i, m \rangle \in E$ such that f(e) = a. Let

$$\hat{a}(j) = \begin{cases} c(a)(j) & \text{if } i \neq j, \\ m & \text{otherwise.} \end{cases}$$

In this way we really defined a k-dimensional choice function on A. We claim that the closed sets of the induced ultratopology are exactly the finite subsets of kA and kA .

By Theorem 2.5 of [3] every ultratoplogy is T_1 therefore every finite subset of kA is closed. Let F be an infinite closed subset of kA and suppose, seeking a contradiction, that there is an element

$$(*) b \in {}^k A - F.$$

By Corollary 2.2 of [3] F is a decomposable relation, say $F = \prod_{l \in I} s(l)/U$. Therefore, there is a $J \in U$ such that for every $i \in J$ one has $\hat{b}(i) \notin s(i)$. Hence, for every $i \in J$ $e_i = \langle s, i, \hat{b}(i) \rangle \in E$. By construction, for every $i \in J$ one has $f(e_i) \in F$ and $f(e_i) \cap (i) = \hat{b}(i)$. This means that

$$\{i \in I : \exists a \in F : \hat{a}(i) = \hat{b}(i)\} \supseteq J \in U.$$

That is, $T(F, b) \in U$ (where T is understood according to the new choice function $\hat{}$). Since we assumed that F is closed, this implies $b \in F$ which contradicts to (*).

Corollary 2.2 There are ultrafilters U_1 , U_2 (respectively, over I and J) and sequences of natural numbers $s = \langle n_i, i \in I \rangle$ and $z = \langle m_j, j \in J \rangle$ so that

- (i) $n_i \neq m_j$ for all $i \in I, j \in J$ and
- (ii) for every $k \in \omega$ there are k-dimensional ultratopologies C_k in $\Pi_{i \in I} n_i / U_1$ and D_k in $\Pi_{j \in J} m_j / U_2$ such that C_k and D_k are homeomorphic.

Proof. Let U_1, U_2 be good ultrafilters and let s and z be arbitrary sequences of natural numbers satisfying the requirements of the corollary such that |I| = |J| and both $A = \prod_{i \in I} s_i/U_1$ and $B = \prod_{j \in J} z_j/U_2$ are infinite. Let $k \in \omega$ be arbitrary. By Theorem 2.1 there are ultratopologies C_k , D_k , respectively on A and B such that

the closed sets of C_k are exactly the finite subsets of kA and kA and the closed sets of \mathcal{D}_k are exactly the finite subsets of kB and kB .

By Theorem VI, 2.13 of [4] $|A| = |B| = 2^{|I|}$. Let $f: A \to B$ be any bijection. Then $f_k: {}^kA \to {}^kB$, $f(\langle a_0, ..., a_{k-1} \rangle = \langle f(a_0), ..., f(a_{k-1}) \rangle$ is clearly a bijection from kA onto kB mapping finite subsets of kA to finite subsets of kB . Thus, f is the required homeomorphism.

3 cardinalities of dense sets

Problem 5.3 (A) of [3] asks whether is it possible to choose a sequence of finite sets s and an ultrafilter U so that there is an utratopology on $A = \prod_{i \in I} n_i / U$ in which every dense set has cardinality |A|. In this section we will show that this is impossible if A is infinite. We start by a simple observation: every k-dimensional ultratopology is homeomorphic with an appropriate 1-dimensional ultratopology.

Theorem 3.1 Suppose C_k is a k-dimensional ultratopology on A. Then there is a 1-dimensional ultratopology D which is homeomorphic to C.

Proof. The idea is to identify k-tuples of sequences by sequences of k-tuples. By a slight abuse of notation, we will use this identification freely. Let $A = \prod_{i \in I} A_i/U$ (here the A_i 's are not necessarily finite) and suppose $\hat{}$ is a k-dimensional choice function inducing C_k . Let $\mathcal{B} = \prod_{i \in I} {}^k A_i/U$. We define a 1-dimensional choice function c in B as follows. If $s = \langle s_i : i \in I \rangle/U \in B$ then for each $j \in k$ let $s^j = \langle s_i(j) : i \in I \rangle/U$. Define $c(s) = \langle s^0, ..., s^{k-1} \rangle^{\hat{}}$ and $\varphi : {}^k A \to B$, $\varphi(\langle s^0/U, ..., s^{k-1}/U \rangle) = \langle \langle s^0(i), ..., s^{k-1}(i) \rangle : i \in I \rangle/U$. Then clearly, c is a 1-dimensional choice function which induces an ultratopology \mathcal{D} on B. Then for any $a \in {}^k A$ and $i \in I$ one has $\hat{a}(i) = c(\varphi(a))(i)$. Now it is straightforward to check that φ is a homeomorphism between \mathcal{C} and \mathcal{D} .

Let \mathcal{C} be an ultratopology on an ultraproduct $A = \prod_{i \in I} n_i / U$ of finite sets. Suppose \mathcal{C} can be induced by a choice function $\hat{\ }$. Let

$$G = \{\langle i, m \rangle : i \in I, m \in n_i \text{ and } (\exists a \in A)(\hat{a}(i) = m)\}$$

and for each $\langle i, m \rangle \in G$ let $a_{i,m} \in A$ be such that $\hat{a}_{i,m}(i) = m$. Clearly, if I is infinite, then $|G| \leq |I|$. We claim that there is a dense subset R of A such that $|R| \leq |G|$. In fact, R can be chosen to be $R = \{a_{i,m} : \langle i, m \rangle \in G\}$. To see this, suppose $a \in A$. Then for every $i \in I$ one has $\langle i, \hat{a}(i) \rangle \in G$ and therefore $T(R, a) = I \in U$. Hence cl(R) = A, as desired.

Now we are able to provide a negative answer for Problem 5.3 (A) of [3].

Theorem 3.2 Suppose C is a 1-dimensional ultratopology on an infinite ultraproduct $A = \prod_{i \in I} n_i / U$ where each n_i is a finite set. Then d(C) < |A|.

Proof. Suppose, seeking a contradiction, that \mathcal{C} is an ultratopology on A such that the cardinality of every dense set in \mathcal{C} is equal with $|A| = \kappa$. Using the notation just introduced in the remark before the theorem, R is a dense subset of A and therefore |R| = |A|. Let $<^A$ be a well ordering of A (having order type κ). By transfinite recursion we define a sequence $\langle b_i \in A : i < \kappa \rangle$ as follows. Assume $j < \kappa$ and b_l has already been defined for every l < j. Let $W_j = \{b_l : l < j\}$ and let $V_j = \{a \in A : T(W_j, a) \in U\}$. Since $j < \kappa$, $V_j \subseteq cl(W_j) \neq A$. If j is an odd ordinal, then let b_j be the $<^A$ -first element of $R - W_j$. Otherwise let b_j be the $<^A$ -first element in $A - V_j$. Clearly, the following conditions are satisfied:

- (i) for every $\langle i, m \rangle \in G$ there is a $j < \kappa$ such that $\hat{b}_j(i) = m$, in fact, $R \subseteq \{b_l : l < \kappa\} = W_{\kappa}$.
- (ii) for every $a \in A$ there is a $j < \kappa$ such that $a \in V_j$ (the smallest such j will be denoted by j_a),
- (iii) for every $j < \kappa$ there is an ordinal j < s(j) such that $s(j) < \kappa$ and $b_{s(j)} \notin cl(W_j)$. (This is true because otherwise by (i) one would have $cl(W_j) \supseteq cl(R) = A$ which is impossible since $|W_j| < \kappa$.)

Now let $H = \{i \in I : n_i = \{\hat{b}_j(i) : j < \kappa\}\}$. We show that $H \in U$.

Again, seeking a contradiction, assume $I - H \in U$. For every $i \in I - H$ let $e_i \in n_i - \{\hat{b}_j(i) : j < \kappa\}$ be arbitrary, let $e = \langle e_i : i \in I - H \rangle / U$ and let $O = \{i \in I : \hat{e}(i) = e_i\}$. Clearly, $O \in U$. In addition, if $i \in O \cap (I - H)$ then by (i) there is a $j < \kappa$ such that $\hat{b}_j(i) = \hat{e}(i) = e_i$ contradicting to the selection of e_i 's.

For every $i \in H$ we introduce a binary relation \prec^i as follows. If $n, m \in n_i$ then $n \prec^i m$ means that there is a $j \in \kappa$ such that $\hat{b}_j(i) = n$ but for every $l \leq j$ $\hat{b}_l(i) \neq m$. Clearly, for each $i \in H$ the relation \prec^i is irreflexive, transitive, and trichotome. Since n_i is finite, for every $i \in H$ there is an \prec^i -maximal element $y_i \in n_i$. Let $y = \langle y_i : i \in H \rangle / U$ and let $K = \{i \in I : \hat{y}(i) = y_i\}$. Now by (ii) and (iii) $y \in V_{jy}$ and $b_{s(jy)} \notin cl(W_{jy}) = cl(V_{jy})$. Thus, for every $i \in K \cap T(W_{jy}, y)$ one has

$$(*) \quad \hat{y}(i) = y_i \in \{\hat{b}_l(i) : l < j_y\}.$$

 $cl(V_{j_y})$ is closed, therefore there is $L \in U$ such that for all $i \in L$ one has $\hat{b}_{s(j_y)}(i) \notin \{\hat{b}_l(i) : l < j_y\}$ and thus $\hat{b}_l(i) \prec^i \hat{b}_{s(j_y)}(i)$ for every $i \in L \cap H$ and for every $l < j_y$. Particularly, (*) implies that if $i \in L \cap H \cap K \cap T(W_{j_y}, y)$ then $\hat{y}(i) = y_i \prec^i \hat{b}_{s(j_y)}(i)$ which is impossible since by construction, y_i is the \prec^i -maximal element in n_i . This contradiction completes the proof.

Using Theorem 3.1 the above results can be generalized to higher dimensional ultratopologies as well.

Corollary 3.3 Let $k \in \omega$ be arbitrary and suppose C is a k-dimensional ultratopology on an infinite ultraproduct $A = \prod_{i \in I} n_i / U$ where each n_i is a finite set. Then d(C) < |A|.

Proof. Assume, seeking a contradiction, that C is a k-dimensional ultratopology on A whose density is |A|. Then, by Theorem 3.1 there is a 1-dimensional ultratopology with the above property, contradicting to Theorem 3.2

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