# TWO CARDINALS MODELS WITH GAP ONE REVISITED

Saharon Shelah

Institute of Mathematics
The Hebrew University
Jerusalem, Israel

Rutgers University
Mathematics Department
New Brunswick, NJ USA

Abstract. We succeed to say something on the identities of $(\mu^+, \mu)$ when $\mu > \theta > \mathrm{cf}(\mu), \mu$ strong limit $\theta$-compact or even $\mu$ limit of compact cardinals. This hopefully will help to prove that

(a) the pair $(\mu^+, \mu)$ is compact and

(b) the consistency of "some pair $(\mu^+, \mu)$ is not compact", however, this has not been proved.

The author would like to thank the Israel Science Foundation for partial support of this research (Grant No. 242/03). Publication 824.
I would like to thank Alice Leonhardt for the beautiful typing.

## Annotated Content

§0 Introduction

[We give the basic definitions.]

§1 2-simplicity for gap one

[We prove that if $\mu = 2^{<\mu}$ then the family of identities of $(\mu^+, \mu)$ is 2-simple. So this applies to $\mu$ singular strong limit but also, e.g., to triples $(\mu^+, \mu, \kappa), \mu = 2^{<\mu} > \kappa$.]

§2 Successor of strong limit above supercompact:2-identities

[Consider a pair $(\mu^+, \mu)$ with $\mu$ strong limit singular $> \theta >$ $\mathrm{cf}(\mu), \theta$ a compact cardinal. We point out quite simply 2-identities which belong to $\mathrm{ID}_2(\mu^+, \mu)$ but not to $\mathrm{ID}_2(\aleph_1, \aleph_0)$.]

## §0 Introduction

There has been much work on $\kappa$-compactness of pairs $(\lambda, \mu)$ of cardinals, i.e., when: if $T$ is a set of first order sentences of cardinality $\leq \kappa$ and every finite subset has a $(\lambda, \mu)$ mod $M$ (i.e., $\|M\| = \lambda, |P^M| = \mu$ for a fixed unary $P$). Then $T$ has a $(\lambda, \mu)$-model.

A particularly important case is $\lambda = \mu^+$ in which case this can be represented as a problem on the $\kappa$-compactness of the logic $\mathbb{L}(\mathbf{Q}^{\text{card}}_\lambda)$, i.e., $(\mathbf{Q}^{\text{card}}_{\geq\lambda} x)\varphi$ says that there are at least $\lambda$ element $x$ satisfying $\varphi_i$. We deal here only with this case. See Furkhen [Fu65], Morley and Vaught [MoVa62], Keisler [Ke70], Mitchel [M1]; for more history see [Sh 604].

Now two cardinal theorems can be translated to partition problems: see [Sh 8], [Sh:E17], lately Shelah and Vaananan [ShVa 790].

Restricting ourselves to pairs $(\mu^+, \mu)$, the identities of $(\aleph_1, \aleph_0)$ were sorted out in [Sh 74], but we do not know of the identities of any really different pair $(\mu^+, \mu)$, i.e., one for which $(\aleph_1, \aleph_0) \nrightarrow (\mu^+, \mu)$. We know of some such pairs is suitable set theory. By Mitchel $(\aleph_2, \aleph_1)$ after suitably collapsing of a Mahlo strongly inaccessible to $\aleph_2$. The other, when there is a compact cardinal in $(\text{cf}(\mu), \mu)$ by Litman and Shelah. So it would be nice to know (taking the extreme case).

0.1 Question: Assume $\mu$ is a singular cardinal the limit of compact and even supercompact cardinals.
1) What are the identities of $(\mu^+, \mu)$?
2) Is $(\mu^+, \mu)$ $\aleph_0$-compact (equivalently $\mu$-compact)?

Note that though we already know that there are some identities of $(\mu^+, \mu)$ which are not identities of $(\aleph_1, \aleph_0)$ we have no explicit example. We give here a partial solution to 0.1(1) by finding families of such identities.

Another problem is consistency of failure of compactness.
In [Sh 604] we have dealt with the simplest case for pairs $(\lambda, \mu)$ by a reasonable criterion: including no use of large cardinals. From another perspective the simplest case is the consistency of non compactness of $\mathbb{L}(\mathbf{Q}), \mathbf{Q}$ one cardinality quantifier, and the simplest one is $\mathbf{Q} = \exists^{\geq \mu^+}$. So we are again drawn to pairs $(\mu^+, \mu)$, that is gap one instead of gap 2 as in [Sh 604], so necessarily we need to use large cardinals as if, e.g., $\neg 0^\#$ then every such pair is compact.

**0.2 Definition.** 1) A partial identity[1] $\mathbf{s}$ is a pair $(a, e) = (\text{Dom}_{\mathbf{s}}, e_{\mathbf{s}})$ where $a$ is a finite set and $e$ is an equivalence relation on a subfamily of the family of the finite subsets of $a$, having the property

[1] identification in the terminology of [Sh 8]

$$b\, e\, c \Rightarrow |b| = |c|.$$

The equivalence class of $b$ with respect to $e$ will be denoted $b/e$.
1A) We say $\mathbf{s}$ is a full identity or identity if $\mathrm{Dom}(e) = \mathscr{P}(a)$.
1B) We say that partial identities $\mathbf{s}_1 = (a_1, e_1), \mathbf{s}_2 = (a_2, e_2)$ are isomorphic if there is an isomorphism $h$ from $\mathbf{s}_1$ onto $\mathbf{s}_2$ which mean that $h$ is a one-to-one function from $a_1$ onto $a_2$ such that for every $b_1, c_1 \subseteq a_1$ we have $(b_1 e_1 c_1) \equiv h(b_1) e_2 h(b_2)$ (so $h$ maps $\mathrm{Dom}(e_1)$ onto $\mathrm{Dom}(e_2)$). We define similarly "$h$ is an embedding of $\mathbf{s}_1$ into $\mathbf{s}_2$.
2) We say that $\lambda \rightarrow (a, e)_\mu$, <u>if</u> $(a, e)$ is an identity or a partial identity and for every function $f : [\lambda]^{<\aleph_0} \rightarrow \mu$, there is a one-to-one function $h : a \rightarrow \lambda$ such that

$$b\, e\, c \Rightarrow f(h''(b)) = f(h''(c)).$$

(Instead $\mathrm{Rang}(f) \subseteq \mu$ we may just require $|\mathrm{Rang}(f)| \leq \mu$, this is equivalent).
3) We define

$$\mathrm{ID}(\lambda, \mu) =: \{(n, e) : n < \omega \ \& \ (n, e) \text{ is an identity and } \lambda \rightarrow (n, e)_\mu\}$$

and for $f : [\lambda]^{<\aleph_0} \rightarrow X$ we let

$$\begin{aligned} \mathrm{ID}(f) =: \{(n, e) :& (n, e) \text{ is an identity such that for some one-to-one function} \\ & h \text{ from } n = \{0, \ldots, n-1\} \text{ to } \lambda \text{ we have} \\ & (\forall b, c \subseteq n)(b\, e\, c \Rightarrow f(h''(b)) = f(h''(c)))\}. \end{aligned}$$

Clearly two-place functions are easier to understand; this motivates:

**0.3 Definition.** 1) A two-identity or 2-identity[2] is a pair $(a, e)$ where $a$ is a finite set and $e$ is an equivalence relation on $[a]^2$. Let $\lambda \rightarrow (a, e)_\mu$ mean $\lambda \rightarrow (a, e^+)_\mu$ where $be^+c \leftrightarrow [(bec) \vee (b = c \subseteq a)]$ for any $b, c \subseteq a$.
2) We defined

$$\mathrm{ID}_2(\lambda, \mu) =: \{(n, e) : \ (n, e) \text{ is a 2-identity and } \lambda \rightarrow (n, e)_\mu\}$$

we define $\mathrm{ID}_2(f)$ when $f : [\lambda]^2 \rightarrow X$ as

[2] it is not an identity as $e$ is an equivalence relation on too small set but it is a partial identity

$$\Big\{(n,e) : (n,e) \text{ is a two-identity such that for some } h,$$
$$\text{a one-to-one function from } \{0,\dots,n-1\} \text{ into } \lambda$$
$$\text{we have } \{\ell_1,\ell_2\}e\{k_1,k_2\} \text{ implies that } \ell_1 \neq \ell_2 \in \{0,\dots,n-1\},$$
$$k_1 \neq k_2 \in \{0,\dots,n-1\} \text{ and } f(\{h(\ell_1),h(\ell_2)\}) = f(\{h(k_1),h(k_2)\})\Big\}.$$

3) Let us define

$$\begin{aligned} \mathrm{ID}_2^{\circledast} =: \{({}^n2,e) :\ & ({}^n2,e) \text{ is a two-identity and if} \\ & \{\eta_1,\eta_2\} \neq \{\nu_1,\nu_2\} \text{ are } \subseteq {}^n2, \text{ then} \\ & \{\eta_1,\eta_2\}e\{\nu_1,\nu_2\} \Rightarrow \eta_1 \cap \eta_2 = \nu_1 \cap \nu_2\}. \end{aligned}$$

4) In parts (1) and (2) we may replace 2 by $k < \omega$ (only $k < |a_{\mathbf{s}}|$ is interesting) and by $(\leq k)$.

<u>0.4 Discussion</u>: By [Sh 49], under the assumption $\aleph_\omega < 2^{\aleph_0}$, the families $\mathrm{ID}_2(\aleph_\omega,\aleph_0)$ and $ID_2^{\circledast}$ coincide (up to an isomorphism of identities). In Gilchrist and Shelah [GcSh 491] and [GcSh 583] we considered the question of the equality between these $\mathrm{ID}_2(2^{\aleph_0},\aleph_0)$ and $\mathrm{ID}_2^{\circledast}$ under the assumption $2^{\aleph_0} = \aleph_2$. We showed that consistently the answer may be "yes" and may be "no".

Note that $(\aleph_n,\aleph_0) \nrightarrow (\aleph_\omega,\aleph_0)$ so $\mathrm{ID}(\aleph_2,\aleph_0) \neq \mathrm{ID}(\aleph_\omega,\aleph_0)$, but for identities for pairs (i.e. $\mathrm{ID}_2$) the question is meaningful.

We can look more at ordered identities

**0.5 Definition.** 1) An ord-identity or order identity is an identity $\mathbf{s}$ such that $a_s \subseteq$ Ord or just: $a$ is an ordered set.
2) $\lambda \rightarrow_{or} (\mathbf{s})_\mu$ if $\mathbf{s}$ is an ord-identity and for every $\mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu$ we have $\mathbf{s} \in \mathrm{OID}(\mathbf{c})$, see below (equivalently $\mathrm{Dom}(\mathbf{c}) = [\lambda]^{<\aleph_0}, |\mathrm{Rang}(\mathbf{c})| \leq \mu$).
3) For $\mathbf{c} : [\lambda]^{<\aleph_0} < \mu$ let $\mathrm{OID}(\mathbf{c}) = \{(a,e) : a$ is a set of ordinals and there is an order preserving function $f : a \rightarrow \lambda$ such that $b_1 e b_2 \Rightarrow \mathbf{c}(f''(b_1)) = \mathbf{c}(f''(b_2))\}$.
4) $\mathrm{OID}(\lambda,\mu) = \{(n,e) : (n,e) \in \mathrm{OID}(\mathbf{c})$ for every $\mathbf{c} : [\lambda]^{<\aleph_0} \rightarrow \mu$ we say $(n,e) \in \mathrm{OID}(\mathbf{c})\}$.
5) Similarly $\mathrm{OID}_2$, $\mathrm{OID}_k$, $\mathrm{OID}_{\leq k}$.

Of course,

**0.6 Claim.** *1)* $\mathrm{ID}(\lambda,\mu)$ *can be computed from* $\mathrm{OID}(\lambda,\mu)$.
*2) Let* $a$ *be a finite set of ordinals and* $e$ *a function. If* $(a,e)$ *is an identity,* $a$ *a set of ordinals and* $\lambda>\mu$, *then* $(a,e)\in \mathrm{ID}(\lambda,\mu)$ *iff for some permutation* $\pi$ *of* $a$ *we have* $(a,e^\pi)\in \mathrm{OID}(\lambda,\mu)$ *where* $e^\pi=\{(b,c):(\pi''(b),\pi''(c))\in e\}$.
*3) Let* $A$ *be a set of ordinals,* $(a,e)$ *an ord-identity and* $\mathbf{c}$ *a function with domain* $[A]^{<\aleph_0}$. *Then* $(a,e)\in \mathrm{ID}(\mathbf{c})$ *iff for some permutation* $\pi$ *of* $a,(a,e^\pi)\in \mathrm{OID}(\mathbf{c})$.
*4) Similarly for 2-identities and* $k$*-identities and* $(\leq k)$*-identities and partial identities.*

**0.7 Claim.** *For* $n\in[1,\omega)$ *and* $\mathbf{s}$ *an ordered partial identity then there is a first order sentence* $\psi_{\mathbf{s}}$ *such that:* $\psi_{\mathbf{s}}$ *has a* $(\mu^{+n},\mu)$*-model iff* $\mathbf{s}\notin \mathrm{OID}(\mu^{+n},\mu)$.

*Proof.* Easy as for some first order $\psi$ sentence if $M$ is a $(\mu^{+n},\mu)$-model of $\psi$ then $<^M$ is a linear order of $M$ (of cardinality $\mu^{+n}$) which is $\mu^{+n}$-like (i.e. every initial segment has cardinality). $\square_{0.7}$

We define simplicity:

**0.8 Definition.** 1) For $k\leq\aleph_0$, we say $(\lambda,\mu)$ has $k$-simple identities when $(a,e)\in \mathrm{ID}(\lambda,\mu)\Rightarrow(a,e')\in\mathrm{ID}(\lambda,\mu)$ whenever:

$(*)_k$ $a\subseteq\omega,(a,e)$ is an identity of $(\lambda,\mu)$ and $e'$ is defined by

$$be'c \text{ iff } |b|=|c| \;\&\; (\forall b'c')[b'\subseteq b \;\&\; |b'|\leq k \;\&\; c'=\mathrm{OP}_{c,b}(b')\rightarrow b'ec];$$

$$\text{recall } \mathrm{OP}_{A,B}(\alpha)=\beta \text{ iff } \alpha\in A \;\&\; \beta\in B \;\&\; \mathrm{otp}(\alpha\cap A)=\mathrm{otp}(\beta\cap B).$$

2) We define "$(\lambda,\mu)$ for $k$-simple ordered identities".

We can ask
0.9 Question: 1) Define reasonably a pair $(\lambda,\mu)$ such that consistently

- $\circledast$ $\mathrm{ID}(\lambda,\mu)$ is not recursive
- $\circledast'$ $\mathrm{ID}(\lambda,\mu)$ is not, in a reasonable way, finitely generated.

2) Similarly for $\mathrm{ID}_2(\lambda,\mu)$.
3) Restrict yourself to $(\mu^+,\mu)$.

## §1 2-simplicity for gap one

**1.1 Claim.** *1) If $\mu$ is strong limit singular <u>then</u> $\mathrm{ID}_2(\mu^+,\mu)$ is 2-simple.*
*2) If $\mu = 2^{<\mu}$ and $c_0 : [\mu^+]^{<\aleph_0} \to \mu$ <u>then</u> we can find $c^* : [\mu^+]^2 \to \mu$ such that:*

- $(\alpha)$ *if $n \in [2,\omega)$ and $\alpha_0,\dots,\alpha_{n-1} < \mu^+$ are with no repetitions and $\beta_0,\dots,\beta_{n-1} < \mu^+$ are with no repetitions and $\ell < k < n \Rightarrow c^*\{\alpha_\ell,\alpha_k\} = c^*\{\beta_\ell,\beta_k\}$ <u>then</u> $c_0\{\alpha_0,\dots,\alpha_{n-1}\} = c_0\{\beta_0,\dots,\beta_{n-1}\}$ and even $c^*\{\alpha_0,\dots,\alpha_{n-1}\} = c^*\{\beta_0,\dots,\beta_{n-1}\}$*
- $(\beta)$ *if in addition $\alpha_0 < \alpha_1 < \dots < \alpha_{n-1}$ <u>then</u> $\beta_0 < \beta_1 < \dots < \beta_{n-3} < \beta_{n-2}, \beta_{n-3} < \beta_{n-1}$.*

*1.2 Remark.* 1) We may wonder what is the gain in 1.1(2) as compared to 1.1(1), as if $\mu = 2^{<\mu}$ is regular then we know all relevant theory on $(\mu^+,\mu)$? The answer is that it clarifies identities of triples $(\mu^+,\mu,\kappa)$, e.g.

- $(a)$ $(\mu^+,\mu,\kappa), \mu$ strong limit singular $> \kappa \geq \mathrm{cf}(\mu)$
- $(b)$ $(\mu^+,\mu,\kappa), \mu = \mu^{\beth_\omega(\kappa)}$.

2) Replacing $\mu^+, 2$ by $\mu^{+k}, k+1 \geq 2$ is similar and easier.

*Proof.* 1) By part (2).
2) By $\boxdot_1 - \boxdot_5$ below the claim is easy (see details in the end).

- $\boxdot_1$ There is $c_1 : [\mu^+]^2 \to \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0,\beta_1,\beta_2 < \mu^+$ are with no repetitions and $c_1\{\beta_\ell,\beta_k\} = c_1\{\alpha_\ell,\alpha_k\}$ for $\ell < k < 3$ <u>then</u> at least two of the following holds $\beta_0 < \beta_1, \beta_0 < \beta_2, \beta_1 < \beta_2$.

[Why? Let $\eta_\alpha \in {}^\mu 2$ for $\alpha < \mu^+$ be pairwise distinct and for $\alpha \neq \beta < \mu^+$ let $\varepsilon\{\alpha,\beta\} = \mathrm{Min}\{\varepsilon : \eta_\alpha \restriction \varepsilon \neq \eta_\beta \restriction \varepsilon\}$ and define the function $c'_1$ with domain $[\mu^+]^2$ by $c'_1\{\alpha,\beta\} = \{\eta_\alpha \restriction \varepsilon\{\alpha,\beta\}, \eta_\beta \restriction \varepsilon\{\alpha,\beta\}\}$, now $|\mathrm{Rang}(c'_1)| \leq \mu$ holds because $\mu = 2^{<\mu}$. For $\alpha \neq \beta$, let $c''_1\{\alpha,\beta\}$ be 1 if $(\eta_\alpha <_{\mathrm{lex}} \eta_\beta) \equiv (\alpha < \beta)$ and 0 otherwise (the Sierpinski colouring). Lastly, define $c_1$ by $c_1, c_1\{\alpha,\beta\} = (c'_1\{\alpha,\beta\}, c''_1\{\alpha,\beta\})$, it is a function with domain $[\mu^+]^2$ and range of cardinality $\leq \mu$ and easily it is as required.]

- $\boxdot_2$ for every $c : [\mu^+]^{<\aleph_0} \to \mu$ there is $c_2 : [\mu^+]^2 \to \mu$ such that: if $n \geq 2, \alpha_0 < \alpha_1 < \dots < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \dots < \beta_{n-1} < \mu^+$ and $\ell < k < n \Rightarrow c_2\{\alpha_\ell,\alpha_k\} = c_2\{\beta_\ell,\beta_k\}$ then $c\{\alpha_0,\dots,\alpha_{n-1}\} = c\{\beta_0,\dots,\beta_{n-1}\}$.

[Why? We are given $c : [\mu^+]^{<\aleph_0} \to \mu$ and for each $\alpha < \mu^+$ let $f_\alpha$ be a one-to-one function from $\alpha$ onto the ordinal $|\alpha| \le \mu$ and we shall use those $f_\alpha$'s also later. We define an equivalence relation $E$ on $[\mu^+]^2$

$(*)$ for $\alpha_1 < \beta_1 < \mu^+$ and $\alpha_2 < \beta_2 < \mu^+$ we have $\{\alpha_1, \beta_1\} E \{\alpha_2, \beta_2\}$ <u>iff</u>

- $(a)$ $f_{\beta_1}(\alpha_1) = f_{\beta_2}(\alpha_2)$ and
- $(b)$ for any $n < \omega$ and $\gamma_0 < \ldots < \gamma_{n-1} < f_{\beta_1}(\alpha_1)$ we have

$$c\{\alpha_1, \beta_1, f^{-1}_{\beta_1}(\gamma_0), \ldots, f^{-1}_{\beta_1}(\gamma_{n-1})\} = c\{\alpha_2, \beta_2, f^{-1}_{\beta_2}(\gamma_0), \ldots, f^{-1}_{\beta_1}(\gamma_{n-1})\}$$

and similarly if we omit $\alpha_1, \alpha_2$ and/or $\beta_1, \beta_2$.

So $[\mu^+]^2/E$ has cardinality $\le {}^{\mu >}2 = \mu$ and let $c_2 : [\mu^+] \to \mu$ be such that $c_2\{\alpha_1, \beta_1\} = c_2\{\alpha_2, \beta_2\}$ iff $\{\alpha_1, \beta_1\}/E = \{\alpha_2, \beta_2\}/E$. We now check that it is as required in $\boxdot_2$. Let $n, \langle \alpha_\ell : \ell < n \rangle, \langle \beta_\ell : \ell < n \rangle$ be as in $\boxdot_2$; so $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_n\} = c_2\{\beta_\ell, \beta_n\}$, hence by $(*)(a)$ above (for $k = n-1$) we have $\ell < n-1 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\beta_{n-1}}(\beta_\ell)$, call it $\gamma_\ell$. Let $\ell(*) < n(*)$ be such that $\gamma_\ell$ is maximal. Now apply $(*)(b)$ with $\alpha_{\ell(*)}, \alpha_{n-1}, \beta_{\ell(*)}, \beta_{n-2}$ here standing for $\alpha_1, \beta_1, \alpha_2, \beta_2$ there and we get the desired result.]

$\boxdot_3$ In $\boxdot_2$, using $f_\alpha : \alpha \to \mu$ as in its proof, we have $c\{\alpha_0, \ldots, \alpha_{n-1}\} = c\{\beta_0, \ldots, \beta_{n-2}\}$ also when

$(*)$ $n \ge 2, \alpha_0 < \alpha_1 < \ldots < \alpha_{n-3} < \alpha_{n-2} < \alpha_{n-1} < \mu^+, \beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}$ and $\ell < n-2 \Rightarrow f_{\alpha_{n-1}}(\alpha_\ell) = f_{\alpha_{n-2}}(\alpha_\ell)$ and $\ell < k < n \Rightarrow c_2\{\alpha_\ell, \alpha_k\} = c_2\{\beta_\ell, \beta_k\}$.

[Why? Just the same proof.]

$\boxdot_4$ there is $c_4 : [\mu^+] \to \mu$ such that if $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0, \beta_1, \beta_2 < \mu^+$ with no repetitions, $c_4\{\beta_\ell, \beta_k\} = c_4\{\alpha_\ell, \alpha_k\}$ for $\ell < k < 3$ then $\beta_0 < \beta_1$ & $\beta_0 < \beta_2$.

[Why? For $\alpha < \beta < \mu^+$ we let $c'\{\alpha, \beta\} = \{f_\beta(\gamma) : \gamma < \alpha \ \& \ f_\beta(\gamma) < f_\beta(\beta)\}$ and let $c_4\{\alpha, \beta\} = (c'\{\alpha, \beta\}, c_1\{\alpha, \beta\}, f_\beta(\alpha))$ where $c_1$ is from $\boxdot_1$ and $\langle f_\gamma : \gamma < \mu^+ \rangle$ is from the proof of $\boxdot_2$. Clearly $|\text{Rang}(c')| \le \sum\limits_{\zeta < \mu} 2^{|\zeta|} = \mu$ hence $|\text{Rang}(c_4)| \le \mu^3 = \mu$. If $\alpha_\ell, \beta_\ell (\ell < 3)$ form a counterexample, then $c_1\{\alpha_\ell, \alpha_k\} = c_1\{\beta_\ell, \beta_k\}$ for $\ell < k < 3$ hence by $\boxdot_1$ we have four cases according to which one of the inequalities $\beta_\ell < \beta_k, \ell < k < 3$ fail. So the proof of $\boxdot_4$ splits to three cases.

<u>Case 0</u>: $\beta_0 < \beta_1 < \beta_2$.

Trivial: the desired conclusion holds.

Case 1: $\beta_1 < \beta_0$ so $\beta_1 < \beta_0 < \beta_2$.

Let $\zeta_\ell = f_{\alpha_2}(\alpha_\ell)$ for $\ell = 0, 1$ hence $\zeta_0 \neq \zeta_1$ as $f_{\alpha_2}$ is one to one and $\zeta_\ell = f_{\beta_2}(\beta_\ell)$. Now on the one hand if $\zeta_0 < \zeta_1$ then $c'\{\alpha_1, \alpha_2\} \neq c'\{\beta_1, \beta_2\}$ (as $\zeta_0 \in c'\{\alpha_1, \alpha_2\}, \zeta_0 \notin c'\{\beta_1, \beta_2\}$), contradiction. On the other hand if $\zeta_1 < \zeta_0$ then $c'\{\alpha_0, \alpha_2\} \neq c'\{\beta_0, \beta_2\}$ (as $\zeta_1 \in c'\{\beta_0, \beta_2\}, \zeta_1 \notin c'\{\alpha_0, \alpha_2\}$), a contradiction, too.

Case 2: $\beta_2 < \beta_0$.

Then at least one of $\beta_1 < \beta_0, \beta_2 < \beta_1$ hold contradicting $\boxdot_1$, (i.e., the case we are in).

Case 3: $\beta_2 < \beta_1$.

By $\boxdot_1$ we have $\beta_0 < \beta_2 < \beta_1$.

This is O.K. for $\boxdot_4$.]

$\boxdot_5$ for every $c : [\mu^+]^2 \to \mu$ there is $c_5 : [\mu^+]^2 \to \mu$ such that

- $(a)$ $c_5\{\alpha_1, \beta_1\} = c_5\{\alpha_2, \beta_2\} \Rightarrow c_2\{\alpha_1, \beta_1\} = c_2\{\alpha_2, \beta_2\}$ where $c_2$ is from $\boxdot_2$ (so also $\boxdot_3$)
- $(b)$ there are no $\alpha_0 < \alpha_1 < \alpha_2 < \mu^+$ and $\beta_0 < \beta_1 < \beta_2 < \mu^+$ such that $f_{\alpha_2}(\alpha_0) \neq f_{\alpha_1}(\alpha_0), c_5\{\alpha_0, \alpha_1\} = c_5\{\beta_0, \beta_2\}, c_5\{\alpha_0, \alpha_2\} = c_5\{\beta_0, \beta_1\}$ and $c_5\{\alpha_1, \alpha_2\} = c_5\{\beta_1, \beta_2\}$
- $(c)$ $c_5\{\alpha_1, \beta_1\} = c_5\{\alpha_2, \beta_2\} \Rightarrow c_4\{\alpha_1, \beta_1\} = c_4\{\alpha_2, \beta_2\}$ where $c_4$ is from $\boxdot_4$.

[Why? Let $\kappa = \mathrm{cf}(\mu) \leq \mu$ and $\mu = \sum\limits_{i<\kappa} \lambda_i$ be such that if $\mu$ is a limit cardinal then $\lambda_i$ is (strictly) increasing continuous and if $\mu$ is a successor cardinal then $\mu = \lambda^+$ and $\lambda_i = \lambda$ for $i < \kappa$. We can find $d : [\mu^+]^2 \to \kappa$ and $\bar{g}$ such that

$\circledast_0$
- $(i)$ for $\beta < \mu^+, i < \kappa$ the set $A_{\beta,i} =: \{\alpha < \beta : d\{\alpha, \beta\} \leq i\}$ has cardinality $\leq \lambda_i$ and
- $(ii)$ if $\alpha < \beta < \gamma < \mu^+$ then $d\{\alpha, \gamma\} \leq \max\{d\{\alpha, \beta\}, d\{\beta, \gamma\}\}$
- $(iii)$ $\bar{g}$ is a sequence $\langle g_\alpha : \alpha < \mu^+ \rangle$
- $(iv)$ $g_\alpha : \alpha \to \mu$ is one to one and $\lambda_i^+ < \mu \ \& \ i < \kappa \ \& \ \alpha < \beta \Rightarrow ((g_\beta(\alpha) < \lambda_i^+) \equiv (d\{\alpha, \beta\} \leq i))$
- $(v)$ if $\alpha < \beta, d\{\alpha, \beta\} = i$ and $\lambda_i^+ = \mu$ then $g_\beta(\alpha) < d\{\alpha, \beta\}$.

[Why we can find them? By induction on $\beta < \mu^+$ by induction on $i < \mu$ for $\alpha = f_\beta^{-1}(i)$ we choose $d\{\alpha,\beta\}$ and $g_\beta(\alpha)$ as required.]

Define the functions $c'_6$ and $c'_7$ with domain $[\mu^+]^2$ as follows: if $\alpha < \beta$ then $c'_6\{\alpha,\beta\} = \{(t,\zeta_0,\zeta_1) : \zeta_0,\zeta_1 \le g_\beta(\alpha), t < 2$ and $t = 0 \Rightarrow g_\beta^{-1}(\zeta_1) < g_\beta(\zeta_2), t = 1 \Rightarrow g_\beta(\zeta_1) > g_\beta(\zeta_2)\}$ and $c'_7\{\alpha,\beta\} = \{(t,\zeta,\xi) : \zeta \in \lambda^+_{d\{\alpha,\beta\}} \cap \mathrm{Rang}(g_\alpha)$ and $\xi \in \lambda^+_{d\{\alpha,\beta\}} \cap \mathrm{Rang}(g_\beta)$ and $[\lambda^+_{d\{\alpha,\beta\}} = \mu \Rightarrow \zeta < d\{\alpha,\beta\}\ \ \&\ \ \xi < d\{\alpha,\beta\}]$ and $g_\alpha^{-1}(\zeta) < g_\beta^{-1}(\xi)\ \ \&\ \ t = 0$ or $g_\alpha^{-1}(\zeta) = g_\beta^{-1}(\xi)\ \ \&\ \ t = 1$ or $g_\alpha^{-1}(\zeta) > g_\beta^{-1}(\xi)\ \ \&\ \ t = 2\}$.

Now for $\alpha < \beta < \mu^+$ we define $c'_5\{\alpha,\beta\} \in \Pi\{\lambda^+_j : j \le d\{\alpha,\beta\}\}$, we do this by induction on $\beta$ and for a fixed $\beta$ by induction $i = d\{\alpha,\beta\}$ and for a fixed $\beta$ and $i$ by induction on $\alpha$.

Arriving to $\alpha < \beta$ so $\zeta < \lambda^+_{d\{\alpha,\beta\}}$, for each $j \le d\{\alpha,\beta\}$, let $(c'_5\{\alpha,\beta\})(j)$ be the first ordinal $\xi < \lambda^+_j$ such that:

$\circledast_1$ if $\gamma < \beta\ \ \&\ \ d\{\gamma,\beta\} \le j\ \ \&\ \ (d\{\gamma,\beta\} = d\{\alpha,\beta\} \Rightarrow \gamma < \alpha)$ then
$$(c'_5\{\alpha,\gamma\})(j) < \xi.$$

Clearly possible. The colouring we use is $c_5$ where for $\alpha < \beta < \mu^+$ we let $c_5\{\alpha,\beta\} = (d\{\alpha,\beta\}, g_\beta(\alpha), f_\beta(\alpha), c_2\{\alpha,\beta\}, c'_5\{\alpha,\beta\}, c'_6\{\alpha,\beta\}, c'_7\{\alpha,\beta\}, c_4\{\alpha,\beta\})$, recalling $c_4$ is from $\boxdot_4$ and $c_2$ is from $\boxdot_2$. Obviously, $|\mathrm{Rang}(c_5)| \le \mu$ and clauses (a) + (c) of $\boxdot_5$ holds. So assume $\alpha_0 < \alpha_1 < \alpha_2, \beta_0 < \beta_1 < \beta_2$ form a counterexample to clause (b) of $\boxdot_5$ and we shall eventually derive a contradiction.

Clearly

$\circledast_2$ (*i*) $d\{\alpha_0,\alpha_2\} = d\{\beta_0,\beta_1\}, d\{\alpha_0,\alpha_1\} = d\{\beta_0,\beta_2\}, d\{\alpha_1,\alpha_2\} = d\{\beta_1,\beta_2\}$

(*ii*) similarly for $c', c'_0, c'_1, c_4$.

By clause (ii) above we have $d\{\alpha_0,\alpha_2\} \le \max\{d\{\alpha_0,\alpha_1\}, d\{\alpha_1,\alpha_2\}\}$, and applying clause (ii) to $\beta_0 < \beta_1 < \beta_2$ and using $\circledast_2$ we have $d\{\alpha_0,\alpha_1\} \le \max\{d\{\alpha_0,\alpha_2\}, d\{\alpha_1,\alpha_2\}$. Hence $d\{\alpha_0,\alpha_1\} = d\{\alpha_0,\alpha_2\} > d\{\alpha_1,\alpha_2\}$ or $\bigwedge\limits_{\ell=1}^{2}[d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\}]$; we deal with those two cases separately.

<u>Case 1</u>: $\varepsilon = d\{\alpha_0,\alpha_1\} = d\{\alpha_0,\alpha_2\} > d\{\alpha_1,\alpha_2\}$.
So (see the definition of $c'_5$, with $\alpha_0,\alpha_2,\alpha_1,\varepsilon$ here standing for $\alpha,\beta,\gamma,j$ there recalling that $\alpha_0 < \alpha_1 < \alpha_2$) we have $\lambda^+_\varepsilon > (c'_5\{\alpha_0,\alpha_2\})(\varepsilon) > (c'_5\{\alpha_0,\alpha_1\})(\varepsilon)$. Similarly, $\lambda^+_\varepsilon > (c'_5\{\beta_0,\beta_2\})(\varepsilon) > (c'_5\{\beta_0,\beta_1\})(\varepsilon)$. This contradicts $c'_5\{\alpha_0,\alpha_\ell\} = c'_5\{\beta_0,\beta_{3-\ell}\}$ for $\ell = 1,2$.

<u>Case 2</u>: $d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\}$ for $\ell = 1,2$.

Let $\varepsilon = d\{\alpha_1,\alpha_2\}$. Let $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$ for $\ell = 1,2$ so $\zeta_\ell = g_{\beta_{3-\ell}}(\beta_0)$ for $\ell = 1,2$. By the assumption toward contradiction, i.e., by a demand in clause (b) of $\boxdot_5$ we have $\zeta_1 \ne \zeta_2$. Clearly $\zeta_\ell < \lambda^+_{d\{\alpha_0,\alpha_\ell\}} \le \lambda^+_{d\{\alpha_1,\alpha_2\}} = \lambda^+_\varepsilon$ and $\lambda^+_\varepsilon = \mu \Rightarrow \zeta_\ell < d\{\alpha_0,\alpha_\ell\} \le d\{\alpha_1,\alpha_2\} \le \varepsilon$.

As $c'_7\{\alpha_1,\alpha_2\} = c'_7\{\beta_1,\beta_2\}$ and $g^{-1}_{\alpha_1}(\zeta_1) = g^{-1}_{\alpha_2}(\zeta_2)$ clearly $g^{-1}_{\beta_1}(\zeta_1) = g^{-1}_{\beta_2}(\zeta_2)$ and they are well defined.

For $\ell = 1,2$ as $c_5\{\alpha_0,\alpha_\ell\} = c_5\{\beta_0,\beta_{3-\ell}\}$ by the choice of $\zeta_\ell$ (that is $\zeta_\ell = g_{\alpha_\ell}(\alpha_0)$) we have $g_{\beta_\ell}(\beta_0) = \zeta_{3-\ell}$ so $g^{-1}_{\beta_\ell}(\zeta_{3-\ell}) = \beta_0$ for $\ell = 1,2$ hence $g^{-1}_{\beta_1}(\zeta_2) = g^{-1}_{\beta_2}(\zeta_1)$. As $c_5\{\alpha_1,\alpha_2\} = c_5\{\beta_1,\beta_2\}$ we have $c'_7\{\alpha_1,\alpha_2\} = c'_7\{\beta_1,\beta_2\}$ but $\zeta_1,\zeta_2 \le g_{\alpha_2}(\alpha_1)$ hence

$\circledast_3$ $(g^{-1}_{\alpha_\ell}(\zeta_1) < g^{-1}_{\alpha_\ell}(\zeta_2)) \equiv (g^{-1}_{\beta_\ell}(\zeta_1) < g^{-1}_{\beta_\ell}(\zeta_2))$ for $\ell = 1,2$.

As $\zeta_1 \ne \zeta_2$ we have $g^{-1}_{\alpha_1}(\zeta_1) \ne g^{-1}_{\alpha_1}(\zeta_2)$.

By symmetry without loss of generality $\zeta_1 > \zeta_2$ so $g^{-1}_{\beta_1}(\zeta_1) < g^{-1}_{\beta_1}(\zeta_2)$ <u>iff</u> (by equalities above) $g^{-1}_{\beta_2}(\zeta_2) < g^{-1}_{\beta_2}(\zeta_1)$ <u>iff</u> (the equivalence in $\circledast_3$) $g^{-1}_{\alpha_2}(\zeta_2) < g^{-1}_{\alpha_2}(\zeta_1)$ <u>iff</u> by the choice of $\zeta_1, g^{-1}_{\alpha_2}(\zeta_1) = \alpha_0), g^{-1}_{\alpha_2}(\zeta_2) < \alpha_0$ <u>iff</u> (as $c'_5\{\alpha_0,\alpha_2\} = c'_5\{\beta_0,\beta_1\}$ and $\zeta_2 < \zeta_1 = g_{\alpha_1}(\beta_0)), g^{-1}_{\beta_1}(\zeta_2) < \beta_0$ <u>iff</u> (as $\beta_0 = g^{-1}_{\beta_1}(\zeta_1)), g^{-1}_{\beta_1}(\zeta_2) < g^{-1}_{\beta_1}(\zeta_1)$, clear contradiction.

So we have proved $\boxdot_5$.

We can now sum up, i.e.:

*Proof of 1.1(2) from $\boxdot_1 - \boxdot_5$.* We are given $c_0 : [\mu^+]^{<\aleph_0} \to \mu$. First we apply $\boxdot_2$ for $c = c_0$ and get $c_2 : [\mu^+]^2 \to \mu$ as there.

Second, we apply $\boxdot_5$ for $c = c_2$ and get $c_5$ as there. Let us check that $c_5$ is as required on $c^*$ in 1.1(2). So assume $(*)_0 + (*)_1$ below and (as the case $n = 2$ is trivial) assume $n \ge 3$ where

$(*)_0$ $\{\alpha_0,\ldots,\alpha_{n-1}\} \in [\mu^+]^n$ and $\{\beta_0,\ldots,\beta_{n-1}\} \in [\mu^+]^n$ and

$(*)_1$ $\ell < k < n \Rightarrow c_5\{\alpha_\ell,\alpha_k\} = c_5\{\beta_\ell,\beta_k\}$.

Without loss of generality (by renaming)

$(*)_2$ $\alpha_0 < \ldots < \alpha_{n-1}$.

and it is enough to prove that $c_0\{\alpha_0,\ldots,\alpha_{n-1}\} = c_0\{\beta_0,\ldots,\beta_{n-1}\}$. By clause (a) of $\boxdot_5$ we have

$(*)_3$ $\ell < k < n \Rightarrow c_2\{\alpha_\ell,\alpha_k\} = c_2\{\beta_\ell,\beta_k\}$.

By clause (c) of $\boxdot_5$ we have

$(*)_4$ $\ell < k < n \Rightarrow c_4\{\alpha_\ell,\alpha_k\} = c_4\{\beta_\ell,\beta_k\}$.

Hence by $\boxdot_4$ we have

$(*)_5$ if $\ell < k < n$ and $\ell < n-2$ then $\beta_\ell < \beta_k$.

[Why? Apply $\boxdot_4$ to $\alpha_\ell, \alpha_{\ell+1}, \alpha_k; \beta_\ell, \beta_{\ell+1}, \beta_k$ if $\ell + 1 < k$, and apply $\boxdot_4$ to $\alpha_\ell, \alpha_{\ell+1}, \alpha_{\ell+2}; \beta_\ell, \beta_{\ell+1}, \beta_{\ell+2}$ if $\ell + 1 = k$.]
So

$(*)_6(i)$ $\beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-2} < \beta_{n-1}$ or
$(ii)$ $\beta_0 < \beta_1 < \ldots < \beta_{n-3} < \beta_{n-1} < \beta_{n-2}$.

So clause $(\beta)$ of 1.1 holds.

If $(i)$ of $(*)_6$ holds, then the choice of $c_2$, i.e., by $\boxdot_2$ and $(*)_3$ above we get $c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}$ so we are done. Otherwise we have $(ii)$ of $(*)_6$ so by clause (b) of $\boxdot_5$ we have

$(*)_7$ if $\ell < n-2$ then $f_{\alpha_{n-1}}(\alpha_\ell) = f_{\beta_{n-2}}(\beta_\ell)$.

[Why? Apply $\boxdot_5(b)$ to $\alpha_\ell, \alpha_{n-2}, \alpha_{n-1}; \beta_\ell, \beta_{n-2}, \beta_{n-1}$.]
So by $\boxdot_3$ we get $c_0\{\alpha_0, \ldots, \alpha_{n-1}\} = c_0\{\beta_0, \ldots, \beta_{n-1}\}$ finishing. $\square_{1.1}$

**1.3 Claim.** *Defining* $\mathrm{ID}(\lambda, \mu)$, *we can restrict ourselves to* $c : [\lambda]^{<\aleph_0} \to \mu$ *such that* $c \restriction [\lambda]^1$ *is constant* $\underline{if}$ $\mathrm{cf}(\lambda) > \mu$.

**1.4 Claim.** *1) Assume* $\mu = \mu^{<\mu}$ *and* $n \in [1, \omega)$. *The identities of* $\mathrm{ID}(\mu^{+n}, \mu)$ *are* $(n+1)$*-simple (and also* $\mathrm{OID}(\mu^+, \mu)$*).*

*Proof.* As in 1.1, only easier in the additional cases. $\square_{2.1}$

## §2 Successor of strong limit above supercompact: 2-identities

So we know that if $\mu$ is strong limit singular and there is a compact cardinal in $(\mathrm{cf}(\mu), \mu)$ then $\mathrm{ID}_2(\mu^+, \mu) \neq \mathrm{ID}_2(\aleph_1, \aleph_0)$. It seems desirable to find explicitly such 2-identity.

The proof of the following does much more.

**2.1 Claim.** *Assume*

(*a*) $\mathbf{s}_k = (k + \binom{k}{2}, e_{\mathbf{s}_k})$ *where the non-singleton* $e_{\mathbf{s}_k}$*-equivalence classes are the set sets here* $\binom{1}{2} = 0$*)*
$\{\{\ell_0, \ell_2\} : \ell_0 < k$ *and for some* $\ell_1 \in \{\ell_0 + 1, \dots, k-1\}$ *we have* $\ell_2 = k + \binom{\ell_1}{2} + \ell_0\}$ *and*
$\{\{\ell_1, \ell_2\} : \ell_1 < k$ *and for some* $\ell_0 < \ell_1$ *we have* $\ell_2 = k + \binom{\ell_1}{2} + \ell_0\}$

(*b*) $\mu$ *is strong limit,* $\theta$ *a compact cardinal and* $\mathrm{cf}(\mu) < \theta < \mu$.

*1)* $\mathbf{s}_k \in \mathrm{ID}_2(\mu^+, \mu)$*, moreover* $\mathbf{s}_k \in \mathrm{OID}_2(\mu^+, \mu)$.
*2)* $\mathbf{s}_k \notin \mathrm{ID}_2(\aleph_1, \aleph_0)$ *for* $k \geq 3$ *so for* $k = 3$ *we have* $\mathbf{s}_k = (6, e_{\mathbf{s}})$ *and the non-singleton equivalence classes, after permuting* $\{3, 5\}$ *are* $\{\{1,3\}, \{0,4\}, \{0,5\}\}$ *and* $\{\{1,5\}, \{2,3\}, \{2,4\}\}$.

*Proof.* Part (1) follows from subclaim 2.2(3) below and part (2) follows from 2.3 below. $\square_{2.1}$

**2.2 Claim.** *Assume*

(*a*) $\mu$ *is strong limit,*

(*b*) $\theta$ *is compact and* $\mathrm{cf}(\mu) < \theta < \mu$

(*c*) $\kappa = \mathrm{cf}(\mu), \langle \lambda_i : i < \kappa \rangle$ *is increasing with limit* $\mu$

(*d*) $c : [\mu^+]^2 \to \mu$

(*e*) $d\{\alpha, \beta\} = \mathrm{Min}\{i : c\{\alpha, \beta\} < \lambda_i\}$.

*1) We can find* $i(*), A, f$ *such that*

$(*)(i)$ $i(*) < \kappa, A \in [\mu^+]^{\mu^+}$ *and* $f : A \to \lambda_{i(*)}$

$(ii)$ *for every set* $B \subseteq A$ *of cardinality* $< \theta$ *there are* $\mu^+$ *ordinals* $\gamma \in A$ *satisfying* $(\forall \alpha \in B)[d\{\alpha, \gamma\} = f(\alpha)]$.

*2) In part (1) we also have: if $A_1 \subseteq A, |A_1| \geq \beth_n(\lambda)^+$ and $\lambda_{i(*)} \leq \lambda < \mu$, <u>then</u> for some $\langle \gamma_\ell : \ell < n \rangle \in {}^n(\lambda_{i(*)})$ and $B \in [A_1]^\lambda$ for every $\alpha_0 < \ldots < \alpha_{n-1}$ from $B$ for arbitrarily large $\beta < \lambda$ we have $\ell < n \Rightarrow c\{\alpha_\ell, \beta\} = \gamma_\ell$.*
*3) $\mathbf{s}_k \in \mathrm{ID}_2(c)$ where $\mathbf{s}_k$ is from clause (a) of 2.1.*

*Proof.* 1) Let $D$ be a uniform $\theta$-complete ultrafilter on $\mu^+$.

Define $f : \mu^+ \to \kappa$ by $f(\alpha) = i \Leftrightarrow \{\gamma < \mu^+ : d\{\alpha, \gamma\} = i\} \in D$, note that the function $f$ is well defined as $D$ is a $\theta$-complete ultrafilter on $\mu^+$ and $\theta > \kappa$. So for some $i(*)$, the set $A =: \{\alpha < \mu^+ : f(\alpha) = i(*)\}$ belongs to $D$ and check that $(*)$ holds, that is (i) + (ii) hold.
2) Define $c^* : [A]^n \to {}^n(\lambda_{i(*)})$ such that

- $\circledast$ if $\alpha_0 < \ldots < \alpha_{n-1}$ are from $A$ then for $\mu^+$ ordinals $\beta < \mu^+$ we have $\langle c\{\alpha_\ell, \beta\} : \ell < n\} \rangle = c^*\{\alpha_0, \ldots, \alpha_{n-1}\}$.

So $\mathrm{Rang}(c^*)$ has cardinality $\leq (\lambda_{i(*)})^n = \lambda_{i(*)}$ hence by the Erdös-Rado theorem there is $B \subseteq A_1$ infinite (even of any pregiven cardinality $< \lambda$) such that $c^* \restriction [B]^n$ is constant.
3) Straight: in part (2) use $n = 2, A_1 = A$ and get $B$ and $\langle \gamma_0, \gamma_1 \rangle \in {}^2(\lambda_{i(*)})$ as there and choose $\alpha_0 < \ldots < \alpha_{k-1}$ from $B$. Next choose $\alpha_\ell$ for $\ell = 0, 1, \ldots, \binom{k}{2} - 1$, choosing $\beta_\ell$ by induction on $\ell$. If $\ell = \binom{\ell_1}{2} + \ell_0$ and $\ell_0 < \ell_1 < k$ choose $\beta_\ell \in A$ satisfying $\beta_\ell > \alpha_{k-1}$ and $\beta_\ell > \beta_m$ for $m < \ell$ such that $c\{\alpha_{\ell_0}, \beta_\ell\} = \gamma_0, c\{\alpha_{\ell_1}, \beta_\ell\} = \gamma_1$.
Now let $\alpha_{k+\ell} = \beta_\ell$ for $\ell < \binom{k}{2}$, and clearly $\langle \alpha_\ell : \ell < k + \binom{k}{2} \rangle$ realize the identity $\mathbf{s}_k$.
$\square_{2.2}$

**2.3 Subclaim.** *1) If $\mathbf{s} \in \mathrm{ID}_2(\aleph_1, \aleph_0)$, <u>then</u> we can find a function $h : [\mathrm{Dom}_\mathbf{s}]^2/\mathbf{s} \to \omega$ respecting $e_\mathbf{s}$ (i.e. $\{\ell_1, \ell_2\} e_\mathbf{s} \{\ell_3, \ell_4\} \Rightarrow h\{\ell_1, \ell_2\} = h\{\ell_3, \ell_4\}$) and there is a linear order $<$ of $\mathrm{Dom}_\mathbf{s}$ satisfying*

- $\circledast$ *for any equivalence class $\mathbf{a}$ of $e$ there are $a_0, a_1$ such that*
  - *(i) $a_0, a_1$ are disjoint finite subsets of $\mathrm{Dom}_\mathbf{s}$*
  - *(ii) if $\{\ell_0, \ell_1\} \in \mathbf{a}$ and $\ell_0 < \ell_1$ then $\ell_0 \in a_0$ & $\ell_1 \in a_1$*
  - *(iii) if $\ell_0 \neq \ell_1$ are from $a_0 \cup a_1$ and $\{\ell_0, \ell_1\} \notin \mathbf{a}$ <u>then</u> $h(\{\ell_0, \ell_1\}) > h(\mathbf{a})$.*

*2) We can add in $\circledast$*

- *(iv) if $\mathbf{a}_0, \mathbf{a}_1$ are distinct $\mathbf{e}_\mathbf{s}$-equivalence classes <u>then</u> for some $m \in \{0, 1\}$ we have $[\cup \mathbf{a}_m]^2 \backslash \mathbf{a}_m$ is disjoint to $\mathbf{a}_{1-m}$*

(v) *in* $\circledast$ *above* $a_0, a_1$ *can be defined as* $\{\ell_0 : \{\ell_0, \ell_1\} \in \mathbf{a}, \ell_0 < \ell_1\}, \{\ell_1 : \{\ell_0, \ell_1\} \in \mathbf{a}, \ell_0 < \ell_1\}$ *respectively.*

*3) If* $k \geq 3, \mathbf{s}_k$ *from 2.1 clause (a)* <u>*then*</u> $\mathbf{s}_k$ *does not belong to* $\mathrm{ID}_2(\aleph_1, \aleph_0)$.

*Proof.* 1) Remember that by 0.6 we can deal with $\mathrm{OID}(\aleph_1, \aleph_0)$. By [Sh 74] we know what is $\mathrm{ID}(\aleph_1, \aleph_0)$, i.e., the family of identities in $\mathrm{OID}(\aleph_1, \aleph_0)$ is generated by two operations; one is called duplication and the other of restriction (see below) from the trivial identity (i.e. $|\mathrm{dom}_{\mathbf{s}}| = 1$) and we prove $\circledast$ by induction on $n$, the number of times we need to apply the operations.
Recall that $(a, e)$ is gotten by duplication if we can find sets $a_0, a_1, a_2$ and a function $g$ such that

$\circledast_1(a)$ $a_0 < a_1 < a_2$ (i.e. $\ell_0 \in a_0, \ell_1 \in a_1, \ell_2 \in a_2 \Rightarrow \ell_0 < \ell_1 < \ell_2$)

(b) $a = a_0 \cup a_1 \cup a_2$

(c) $g$ a one-to-one order preserving function from $a_0 \cup a_1$ onto $a_0 \cup a_1$ (so $g \restriction a_0 = \mathrm{id}_{a_0}$; let $g_1 = g, g_2 = g^{-1}$

(d) for $\ell_0 \neq \ell_1 \in (a_0 \cup a_1)$ we have $\{\ell_0, \ell_1\} e \{g(\ell_0), g(\ell_1)\}$

(e) if $\ell_1 \in a_1, \ell_2 \in a_2$ then $\{\ell_1, \ell_2\}/e$ is a singleton

(f) $\mathbf{s}_\ell = (a_0 \cup a_\ell, e \restriction [a_0 \cup a_\ell]^2)$ is from a lower level (up to isomorphism).

Recall that $(a, e)$ is gotten by restriction from $(a', e')$ if $a \subseteq a', e = e' \restriction [a]^2$.

Now we prove the existence of $h$ as required by induction on the level. If $|\mathrm{Dom}_{\mathbf{s}}| = 1$ this is trivial. If $\mathbf{s}$ is gotten by restriction it is trivial too, (as if $\mathbf{s} = (a, e), s' = (a', e'), a' \subseteq a, e' = e \restriction a'$ and $h : [a]^2/e$ is as guaranteed then we let $h'(\{\ell_0, \ell_1\}/e') = h(\{\ell_0, \ell_1\}/e)$ for $\ell_0 < \ell_1$. Easily $h'$ is as required). So assume $\mathbf{s} = (a, e)$ is gotten by duplication, so let $a_0, a_1, a_2, g_1, g_2$ be as in $\circledast_1$ and let $h_1$ be as required for $\mathbf{s}_1 = (a_0 \cup a_1, e \restriction [a_0 \cup a_1)^2)$ and similarly define $h_2$ by $h_2\{\alpha, \beta\} = h_1\{g_2(\alpha), g_2(\beta)\}$. Let $n^* =$ max $\mathrm{Rang}(h_1)$ and define $h : [a_0 \cup a_1 \cup a_2]^2 \Rightarrow \omega$ by $h \supseteq h_1, h \supseteq h_2$ and if $k \in a_1, \ell \in a_2$ then we let $h\{k, \ell\} = n^* + 1$. Now check.
2) By symmetry, without loss of generality $h(\mathbf{a}_0) < h(\mathbf{a}_1)$ and now $m = 1$ satisfies the requirement by applying $\circledast_1$ to the equivalence class $\mathbf{a} = \mathbf{a}_1$.
3) It is enough to deal with $\mathbf{s}_3$. By direct checking the criterion in part (2) fails.
$\square_{2.3}$

The following is like 2.1 with $\mu$ just limit (not necessarily a strong limit cardinal) so

**2.4 Claim.** *Assume*

(*a*) $\mathbf{s}'_n \in \mathrm{OID}_2$ *is* $(2n+n^2, e_{\mathbf{s}'_n})$ *where the non-singleton* $e_{\mathbf{s}'_n}$*-equivalence classes are*
$\{\{\ell_0, 2n+n\ell_0+\ell_1\} : \ell_0, \ell_1 < n\}$ *and*
$\{\{n+\ell_1, 2n+n\ell_0+\ell_1\} : \ell_0, \ell_1 < n\}$

(*b*) $\mu$ *is a limit cardinal,* $\mu > \theta > \mathrm{cf}(\mu)$ *and* $\theta$ *is a compact cardinal*

(*c*) $s''_n \in \mathrm{OID}_n$ *is* $(2^n + 2^{2n}, e_{\mathbf{s}''_s})$ *where the non-singleton* $e_{\mathbf{s}''_n}$*-equivalence classes are: for* $m < n, \eta \in {}^m 2, i = 0,1$ *let* $\mathbf{a}^i_\eta = \{\{\ell_i, 2^n + \binom{2^n}{\ell_0} + \ell_1\} : \ell_0, \ell_1 < 2^n$ *and for some* $\nu_0, \nu_1 \in {}^n 2$ *we have* $\eta\hat{\ }\langle 0\rangle \trianglelefteq \nu_0, \eta\hat{\ }\langle 1\rangle \trianglelefteq \nu_1$ *and* $\ell_0 = \Sigma\{\nu_0(j)2^j : j < n\}$ *and* $\ell_1 = \Sigma\{\nu_1(j)2^j : j < n\}\}$.

*1)* $\mathbf{s}'_n \in \mathrm{ID}_2(\mu^+, \mu)$*, moreover* $\mathbf{s}'_n \in \mathrm{OID}_2(\mu^+, \mu)$ *similarly for* $\mathbf{s}'_n$.
*2)* $\mathbf{s}'_n \notin \mathrm{ID}_2(\aleph_1, \aleph_0)$ *for* $n \geq 2$*, similarly for* $\mathbf{s}''_n$.

*Proof.* 1) Like the proof of 2.2 using [Sh 49] (or just [Sh 604, §5]) instead of the Erdös-Rado theorem.
2) Otherwise there is $(a, e) \in \mathrm{ID}_2(\aleph_1, \aleph_0)$ and an embedding $h$ of $\mathbf{s}'_n$ into $(a, e)$ and by 0.6 without loss of generality $h$ is order preserving and $(a, e) \in \mathrm{OID}_2(\aleph_1, \aleph_0)$. Now

$(*)_1$ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ <u>then</u> $h(\ell_0) < h(\ell)$.
[Why? Choose $\ell'_1 < n, \ell'_1 \neq \ell_1$ and $\ell' = 2n + n\ell_0 + \ell'_1$, so $\ell \neq \ell'$ and $\{\ell_0, \ell\} e_{\mathbf{s}'_n} \{\ell_0, n+\ell'\}$ hence $\{h(\ell_0), h(\ell)\}, \{h(\ell_0), h(\ell')\}$ are $e$-equivalent and $h(\ell) \neq h(\ell')$. But on $(a, e)$ we know that if $\{m_0, m_1\} e \{m_0, m_2\}$ then $m_2 < m_1 < m_0$ and $m_2 < m_0 < m_1$ are impossible (see 2.5(2) below) so we are done.]

$(*)_2$ if $\ell_0 < n, \ell_1 < n$ and $\ell = 2n + n\ell_0 + \ell_1$ then $h(\ell_1) < h(\ell)$.
[Why? Like $(*)_1$.]

Now we apply 2.3(1) + (2) above so $\mathbf{s}'_n \notin \mathrm{ID}_2(\aleph_2, \aleph_1)$. The conclusion about $\mathbf{s}''_n$ follows. $\square_{2.4}$

*2.5 Observation.* 1) If $k \geq 2, \mathbf{s} = (n, e) \in \mathrm{OID}_2(\mu^+, \mu)$ <u>then</u> we can find $\mathbf{s}' = (n', e')$ in fact $n' = 2n - 1$ such that:

(*i*) $e' \restriction [n]^2 = e$

(*ii*) $\mathbf{s}' \in \mathrm{ID}(\mu^+, \mu)$

(*iii*) for every $c : [\mu^+]^{<\aleph_0} \to \mu$ there is $c' : [\mu^+]^{<\aleph_0} \to \mu$ refining $c$ (i.e. $c'(u_1) = c'(u_2) \Rightarrow c(u_1) = c(u_2)$) such that: if $h : \{0, \ldots, 2n-2\} \to \mu^+$ is one to one and satisfies $u_1 e' u_2 \Rightarrow c'(h''(u_1)) = c'(h''(u_2))$ then $h \restriction \{0,\dots,n-1\}$ is increasing.

2) There is $c : [\mu^+]^2 \to \mu$ such that:
if $\alpha, \beta, \gamma$ are distinct and $c\{\alpha,\beta\} = c\{\alpha,\gamma\}$ then $\alpha < \beta \ \& \ \alpha < \gamma$.
3) We can replace in (1), $(\mu^+, \mu)$ by $(\lambda, \mu)$ if there is $\mathbf{s} = (n, e) \in \mathrm{ID}(\lambda, \mu)$ such that for some $c : [\lambda]^{<\aleph_0} \to \mu$ such that

$\circledast$ if $h : n \to \lambda$ induces $e_{\mathbf{s}}$ then $h(0) < h(1)$.

*Proof.* 1) Define $e' : u_1 e' u_2$ <u>iff</u> $u_1 e u_2 \vee u_1 = u_2 \vee \bigvee\limits_{\ell < n-1} (u_1 = \{\ell, n+\ell+1\} \ \& \ u_2 e \{\ell, \ell+1\}) \vee \bigvee\limits_{\ell < n} (u_2 = \{\ell, n+\ell+1\} \ \& \ u_1 e \{\ell, \ell+1\})$. Now use (2).
2) Let $f_\alpha : \alpha \to \mu$ be one to one and let $<^*$ a dense linear order on $\mu^+$ with $\{\alpha : \alpha < \mu\}$ a dense subset. Now choose $c_1 : [\mu^+]^2 \to \mu$ such that $\alpha < \beta \Rightarrow \alpha \le^* c_1\{\alpha,\beta\} <^* \beta$ and $c : [\mu^+]^2 \to \mu$ be $\alpha < \beta \Rightarrow c\{\alpha,\beta\} = \mathrm{pr}(f_\beta(\alpha), c_1\{\alpha,\beta\})$ for some pairing function pr.
3) Similar to part (1) only $|\mathrm{Dom}_{\mathbf{s}'}|$ is larger. $\square_{2.5}$

REFERENCES.

[Fu65] E. G. Furkhen. Languages with added quantifier "there exist at least $\aleph_\alpha$". In J. V. Addison, L. A. Henkin, and A. Tarski, editors, *The Theory of Models*, pages 121–131. North-Holland Publishing Company, 1965.

[GcSh 491] Martin Gilchrist and Saharon Shelah. Identities on cardinals less than $\aleph_\omega$. *Journal of Symbolic Logic*, **61**:780–787, 1996. math.LO/9505215.

[GcSh 583] Martin Gilchrist and Saharon Shelah. The Consistency of $\mathrm{ZFC} + 2^{\aleph_0} > \aleph_\omega + \mathbf{I}(\aleph_2) = \mathbf{I}(\aleph_\omega)$. *Journal of Symbolic Logic*, **62**:1151–1160, 1997. math.LO/9603219.

[Ke70] Jerome H. Keisler. Logic with the quantifier "there exist uncountably many". *Annals of Mathematical Logic*, **1**:1–93, 1970.

[M1] J. Donald Monk. *Cardinal Invariants of Boolean Algebras*. Lectures in Mathematics. ETH Zurich, Birkhauser Verlag, Basel Boston Berlin, 1990.

[MoVa62] M. D. Morley and R. L. Vaught. Homogeneous and universal models. *Mathematica Scandinavica*, **11**:37–57, 1962.

[Sh 604] Saharon Shelah. The pair $(\aleph_n, \aleph_0)$ may fail $\aleph_0$–compactness. In M. Baaz, S. Friedman, and J. Krajicek, editors, *Proceedings of LC'2001*, submitted. ASL.

[Sh:E17] Saharon Shelah. Two cardinal and power like models: compactness and large group of automorphisms. *Notices of the AMS*, **18**:425, 1968.

[Sh 8] Saharon Shelah. Two cardinal compactness. *Israel Journal of Mathematics*, **9**:193–198, 1971.

[Sh 49] Saharon Shelah. A two-cardinal theorem and a combinatorial theorem. *Proceedings of the American Mathematical Society*, **62**:134–136, 1976.

[Sh 74] Saharon Shelah. Appendix to: "Models with second-order properties. II. Trees with no undefined branches" (Annals of Mathematical Logic 14(1978), no. 1, 73–87). *Annals of Mathematical Logic*, **14**:223–226, 1978.

[ShVa 790] Saharon Shelah and Jouko Väänänen. Recursive logic frames. *Preprint*.