Kolmogorov Complexity and Set theoretical Representations of Integers, I

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Abstract

We reconsider some classical natural semantics of integers (namely iterators of functions, cardinals of sets, index of equivalence relations) in the perspective of Kolmogorov complexity. To each such semantics one can attach a simple representation of integers that we suitably effectivize in order to develop an associated Kolmogorov theory. Such effectivizations are particular instances of a general notion of "self-enumerated system" that we introduce in this paper. Our main result asserts that, with such effectivizations, Kolmogorov theory allows to quantitatively distinguish the underlying semantics. We characterize the families obtained by such effectivizations and prove that the associated Kolmogorov complexities constitute a hierarchy which coincides with that of Kolmogorov complexities defined via jump oracles and/or

infinite computations (cf. [5]). This contrasts with the well-known fact that usual Kolmogorov complexity does not depend (up to a constant) on the chosen arithmetic representation of integers, let it be in any base $n \geq 2$ or in unary. Also, in a conceptual point of view, our result can be seen as a mean to measure the degree of abstraction of these diverse semantics.

1 Introduction

Notation 1.1. Equality, inequality and strict inequality up to a constant between total functions $D \to \mathbb{N}$, where D is any set, are denoted as follows:

$$\begin{array}{rcl} f \leq_{\mathrm{ct}} g & \Leftrightarrow & \exists c \in \mathbb{N}\ \forall x \in D\ f(x) \leq g(x) + c \\ f =_{\mathrm{ct}} g & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge\ g \leq_{\mathrm{ct}} f \\ & \Leftrightarrow & \exists c \in \mathbb{N}\ \forall x \in D\ |f(x) - g(x)| \leq c \\ f <_{\mathrm{ct}} g & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge\ \neg(g \leq_{\mathrm{ct}} f) \\ & \Leftrightarrow & f \leq_{\mathrm{ct}} g \ \wedge\ \forall c \in \mathbb{N}\ \exists x \in D\ g(x) > f(x) + c \end{array}$$

As we shall consider $\mathbb{N}$-valued partial functions with domain $\mathbb{N}$, $\mathbb{Z}$, $\mathbf{2}^*$, $\mathbb{N}^2$,..., the following definition is convenient.

Definition 1.2. A basic set $\mathbb{X}$ is any non empty finite product of sets among $\mathbb{N}, \mathbb{Z}$ or the set $\mathbf{2}^*$ of finite binary words or the set Σ^* of finite words in some finite or countable alphabet Σ.

Let's also introduce some notations for partial recursive functions.

Notation 1.3. Let $\mathbb{X}, \mathbb{Y}$ be basic sets. We denote $PR^{\mathbb{X}\to\mathbb{Y}}$ (resp. $PR^{A,\mathbb{X}\to\mathbb{Y}}$) the family of partial recursive (resp.partial A-recursive) functions $\mathbb{X} \to \mathbb{Y}$. In case $\mathbb{X} = \mathbb{Y} = \mathbb{N}$,we simply write PR and PR^A.

1.1 Kolmogorov complexity and representations of $\mathbb{N}$, $\mathbb{Z}$

Kolmogorov complexity $K : \mathbb{N} \to \mathbb{N}$ maps an integer n onto the length of any shortest binary program $\mathtt{p} \in \mathbf{2}^*$ which outputs n. The invariance theorem asserts that, up to an additive constant, K does not depend on the program semantics $\mathtt{p} \mapsto n$, provided it is a universal partial recursive function.
As a straightforward corollary of the invariance theorem, K does not depend (again up to a constant) on the representation of integers, i.e. whether the program output n is really in $\mathbb{N}$ or is a word in some alphabet $\{1\}$ or $\{0, ..., k-1\}$, for some $k \geq 2$, which gives the unary or base k representation of n. A result which is easily extended to all partial recursive representations of integers, cf. Thm.7.8.

In this paper, we show that this is no more the case when (suitably effectivized) classical set theoretical representations are considered. We particularly consider representations of integers via

- Church iterators (Church [3], 1933),
- cardinal equivalence classes (Russell [16] §IX, 1908, cf. [22] p.178),
- index equivalence classes.

Following the usual way to define $\mathbb{Z}$ from $\mathbb{N}$, we also consider representations of a relative integer $z \in \mathbb{Z}$ as pairs of representations of non negative integers x, y satisfying $z = x - y$. In the particular case of Church iterators, restricting to injective functions and considering negative iterations, leads to another direct way of representing relative integers.

Programs are at the core of Kolmogorov theory. They do not work on abstract entities but require formal representations of objects. Thus, we have to define effectivizations of the above abstract set theoretical notions in order to allow their elements to be computed by programs. To do so, we use computable functions and functionals and recursively enumerable sets.

Effectivized representations of integers constitute particular instances of *self-enumerated representation systems* (cf. Def.2.1). This is a notion of family $\mathcal{F}$ of partial functions from $\mathbf{2}^*$ to some fixed set D for which an invariance theorem can be proved using straightforward adaptation of original Kolmogorov's proof. Which leads to a notion of Kolmogorov complexity $K^D_{\mathcal{F}} : D \to \mathbb{N}$, cf. Def.2.16. The ones considered in this paper are

$$K^{\mathbb{N}}_{Church}\ ,\ K^{\mathbb{Z}}_{Church}\ ,\ K^{\mathbb{Z}}_{\Delta Church}\ ,\ K^{\mathbb{N}}_{card}\ ,\ K^{\mathbb{Z}}_{\Delta card}\ ,\ K^{\mathbb{N}}_{index}\ ,\ K^{\mathbb{Z}}_{\Delta index}$$

associated to the systems obtained by effectivization of the Church, cardinal and index representations of $\mathbb{N}$ and the passage to $\mathbb{Z}$ representations as outlined above.

The main result of this paper states that the above Kolmogorov complexities coincide (up to an additive constant) with those obtained via oracles and infinite computations as introduced in [1], 2001, and our paper [5], 2004.

Theorem 1.4 (Main result)**.**

$$\begin{array}{rclcrcl}
K^{\mathbb{N}}_{Church} & =_{\rm ct} & K^{\mathbb{Z}}_{Church}\restriction\mathbb{N} & =_{\rm ct} & K^{\mathbb{Z}}_{\Delta Church}\restriction\mathbb{N} & =_{\rm ct} & K \\
K^{\mathbb{N}}_{card} & =_{\rm ct} & K_{\max} & & K^{\mathbb{Z}}_{\Delta card}\restriction\mathbb{N} & =_{\rm ct} & K^{\emptyset'} \\
K^{\mathbb{N}}_{index} & =_{\rm ct} & K^{\emptyset'}_{\max} & & K^{\mathbb{Z}}_{\Delta index}\restriction\mathbb{N} & =_{\rm ct} & K^{\emptyset''}
\end{array}$$

Thm.1.4 gathers the contents of Thms.8.5, 8.6, 9.5, 9.7, 10.24, 10.25 and §10.10.
A preliminary "light" version of this result was presented in [4], 2002.

The strict ordering result $K >_{\rm ct} K_{\max} >_{\rm ct} K^{\emptyset'}$ (cf. Notations 1.1) proved in [1, 5] and its obvious relativization (cf. Prop.6.11) yield the following hierarchy theorem.

Theorem 1.5.

$$\log >_{\mathrm{ct}} \begin{array}{c} K^{\mathbb{N}}_{\mathit{Church}} \\ =_{\mathrm{ct}} \\ K^{\mathbb{Z}}_{\mathit{Church}} \restriction \mathbb{N} \\ =_{\mathrm{ct}} \\ K^{\mathbb{Z}}_{\Delta \mathit{Church}} \restriction \mathbb{N} \end{array} >_{\mathrm{ct}} K^{\mathbb{N}}_{\mathit{card}} >_{\mathrm{ct}} K^{\mathbb{Z}}_{\Delta \mathit{card}} \restriction \mathbb{N} >_{\mathrm{ct}} K^{\mathbb{N}}_{\mathit{index}} >_{\mathrm{ct}} K^{\mathbb{Z}}_{\Delta \mathit{index}} \restriction \mathbb{N}$$

This hierarchy result for set theoretical representations somewhat reflects their *degrees of abstraction*.

Though Church representation via iteration functionals can be considered as somewhat complex, we see that, surprisingly, the associated Kolmogorov complexities collapse to the simplest possible one.

Also, it turns out that, for cardinal and index representations, the passage from $\mathbb{N}$ to $\mathbb{Z}$, i.e. from $K^{\mathbb{N}}_{\mathit{card}}$ to $K^{\mathbb{Z}}_{\Delta \mathit{card}}$ and from $K^{\mathbb{N}}_{\mathit{index}}$ to $K^{\mathbb{Z}}_{\Delta \mathit{index}}$ does add complexity. However, for Church iterators, the passage to $\mathbb{Z}$ does not modify Kolmogorov complexity, let it be via the Δ operation (for $K^{\mathbb{Z}}_{\Delta \mathit{Church}}$) or restricting iterators to injective functions (for $K^{\mathbb{Z}}_{\mathit{Church}}$).

The results about the $\Delta \mathit{card}$ and $\Delta \mathit{index}$ classes are corollaries of those about the *card* and *index* classes and of the following result (Thm.6.12) which gives a simple normal form to functions computable relative to a jump oracle, and is interesting on its own.

Theorem 1.6. *Let $A \subseteq \mathbb{N}$. A function $G : \mathbf{2}^* \to \mathbb{Z}$ is partial A'-recursive if and only if there exist total A-recursive functions $f, g : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ such that, for all* $\mathtt{p}$*,*

$$G(\mathtt{p}) = \max\{f(\mathtt{p}, t) : t \in \mathbb{N}\} - \max\{g(\mathtt{p}, t) : t \in \mathbb{N}\}$$

(in particular, $G(\mathtt{p})$ is defined if and only if both max*'s are finite).*

1.2 Kolmogorov complexities and families of functions

The equalities in Thm.1.4 are, in fact, corollaries of equalities between families of functions $\mathbf{2}^* \to \mathbb{N}$ (namely, the associated self-enumerated representation systems, cf. §2.2) which are interesting on their own. For instance (cf. Thms.8.5, 8.6, 9.5, 9.7, 10.24, 10.25 and §10.10),

Theorem 1.7. *Denote $X \to Y$ the class of* partial *functions from X to Y.*
1. *A function $f : \mathbf{2}^* \to \mathbb{N}$ is the restriction to a Π^0_2 set of a partial recursive function if and only if it is of the form $f = \mathit{Church} \circ \Phi$ where*
- $\Phi : \mathbf{2}^* \to (\mathbb{N} \to \mathbb{N})^{(\mathbb{N} \to \mathbb{N})}$ is a computable functional,
- $\mathit{Church} : (\mathbb{N} \to \mathbb{N})^{(\mathbb{N} \to \mathbb{N})} \to \mathbb{N}$ is the functional such that

$$\mathit{Church}(\Psi) = \begin{cases} n & \text{if } \Psi \text{ is the iterator } f \mapsto f^{(n)} \\ \text{undefined} & \text{otherwise} \end{cases}$$

2. *A function $f : \mathbf{2}^* \to \mathbb{N}$ is the* max *of a total recursive (resp. total $\emptyset'$-recursive) sequence of functions (cf. Def.6.1) if and only if it is of the form*

$$\mathtt{p} \mapsto card(W^{\mathbb{N}}_{\varphi(\mathtt{p})}) \qquad (resp.\ \mathtt{p} \mapsto index(W^{\mathbb{N}^2}_{\varphi(\mathtt{p})}),\ up\ to\ 1)$$

for some total recursive $\varphi : \mathbf{2}^* \to \mathbf{2}^*$, where
- $W^{\mathbb{N}}_{\mathtt{q}}$ (resp. $W^{\mathbb{N}^2}_{\mathtt{q}}$) is the r.e. subset of $\mathbb{N}$ (resp. $\mathbb{N}^2$) with code $\mathtt{q}$,
- $card : P(\mathbb{N}) \to \mathbb{N}$ is the cardinal function (defined on the sole finite sets),
- $index : P(\mathbb{N}^2) \to \mathbb{N}$ is defined on equivalence relations with finitely many classes and gives the index (i.e. the number of equivalence classes).

3. *A function $f : \mathbf{2}^* \to \mathbb{N}$ is partial $\emptyset'$-recursive (resp. $\emptyset''$-recursive) if and only if it is of the form*

$$\mathtt{p} \mapsto card(W^{\mathbb{N}}_{\varphi_1(\mathtt{p})}) - card(W^{\mathbb{N}}_{\varphi_2(\mathtt{p})}) \ \ (resp.\ \mathtt{p} \mapsto index(W^{\mathbb{N}^2}_{\varphi_1(\mathtt{p})}) - index(W^{\mathbb{N}^2}_{\varphi_2(\mathtt{p})}))$$

for some total recursive $\varphi_1, \varphi_2 : \mathbf{2}^* \to \mathbf{2}^*$.

1.3 Road map of the paper

§2 introduces the notion of self-enumerated representation system with its associated Kolmogorov complexity.
§3 introduce simple operations on self-enumerated systems.
§4 sets up some connections between self-enumerated representation systems for $\mathbb{N}$ and $\mathbb{Z}$.
§5 considers a self-enumerated representation system for the set of recursively enumerable subsets of $\mathbb{N}$.
§6 recalls material from Becher & Chaitin & Daicz, 2001 [1] and our paper [5], 2004, about some extensions of Kolmogorov complexity involving infinite computations. This is to make the paper self-contained.
§7 introduces abstract representations and their effectivizations.
§8, 9, 10 develop the set-theoretical representations mentioned in §1.1 and prove all the mentioned theorems and some more results related to the associated self-enumerated systems, in particular the syntactical complexity of universal functions for such systems.

2 An abstract setting for Kolmogorov complexity: self-enumerated representation systems

2.1 Classical Kolmogorov complexity

Classical Kolmogorov complexity of elements of a basic set $\mathbb{X}$ is defined as follows (cf. Kolmogorov, 1965 [7]):

1. To every $\varphi : \mathbf{2}^* \to \mathbb{X}$ is associated $K^{\mathbb{X}}_{\varphi} : \mathbb{X} \to \mathbb{N}$ such that
$$K^{\mathbb{X}}_{\varphi}(\mathtt{x}) = \min\{|\mathtt{p}| : \varphi(\mathtt{p}) = \mathtt{x}\}$$

i.e. $K^{\mathbb{X}}_{\varphi}(\mathtt{x})$ is the shortest length of a "program" $\mathtt{p} \in \mathbf{2}^*$ which is mapped onto $\mathtt{x}$ by φ.

2. Kolmogorov Invariance Theorem asserts that, letting φ vary in $PR^{\mathbf{2}^* \to \mathbb{X}}$ (cf. Notation 1.3), there is a least $K^{\mathbb{X}}_{\varphi}$, up to an additive constant:
$$\exists \varphi \in PR^{\mathbf{2}^* \to \mathbb{X}} \ \ \forall \psi \in PR^{\mathbf{2}^* \to \mathbb{X}} \quad K^{\mathbb{X}}_{\varphi} \leq_{\mathrm{ct}} K^{\mathbb{X}}_{\psi}$$
Kolmogorov complexity $K_{\mathbb{X}} : \mathbb{N} \to \mathbb{N}$ is such a least $K^{\mathbb{X}}_{\varphi}$, so that it is defined up to an additive constant.

Let $A \subseteq \mathbb{N}$. The above construction relativizes to oracle A : replace $PR^{\mathbf{2}^* \to \mathbb{X}}$ by $PR^{A, \mathbf{2}^* \to \mathbb{X}}$ to get the oracular Kolmogorov complexity $K^{A}_{\mathbb{X}}$.

2.2 Self-enumerated representation systems

We introduce an abstract setting for the definition of Kolmogorov complexity: *self-enumerated representation systems.* As a variety of Kolmogorov complexities is considered, this allows to unify the multiple variations of the invariance theorem, the proofs of which repeat, mutatis mutandis, the same classical proof due to Kolmogorov (cf. Li & Vitanyi's textbook [9] p.97). This abstract setting also leads to a study of operations on self-enumerated systems, some of which are presented in §4,5 and some more are developed in the continuation of this paper.
Some intuition for the next definition is given in Note 2.2 and Rk.2.4.

Definition 2.1 (Self-enumerated representation systems)**.**
1. A self-enumerated representation system (in short "self-enumerated system") is a pair $(D, \mathcal{F})$ where D is a set — the domain of the system — and $\mathcal{F}$ is a family of partial functions $\mathbf{2}^* \to D$ satisfying the following conditions:

i. $D = \bigcup_{F \in \mathcal{F}} Range(F)$, i.e. every element of D appears in the range of some function $F \in \mathcal{F}$.

ii. If $\varphi : \mathbf{2}^* \to \mathbf{2}^*$ is a recursive total function and $F \in \mathcal{F}$ then $F \circ \varphi \in \mathcal{F}$.

iii. There exists $U \in \mathcal{F}$ (called a universal function for $\mathcal{F}$) and a total recursive function $comp_U : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ such that
$$\forall F \in \mathcal{F} \ \ \exists \mathtt{e} \in \mathbf{2}^* \ \ \forall \mathtt{p} \in \mathbf{2}^* \ \ F(\mathtt{p}) = U(comp_U(\mathtt{e}, \mathtt{p}))$$
In other words, letting $U_{\mathtt{e}}(\mathtt{p}) = U(comp_U(\mathtt{e}, \mathtt{p}))$, the sequence of functions $(U_{\mathtt{e}})_{\mathtt{e} \in \mathbb{N}}$ is an enumeration of $\mathcal{F}$.

2. (Full systems) In case condition ii holds for all *partial recursive functions* φ, the system $(D, \mathcal{F})$ is called a self-enumerated representation *full system.*

3. (Good universal functions) A universal function U for $\mathcal{F}$ is good if its associated *comp* function satisfies the condition

$$\forall \mathtt{e}\ \exists c_{\mathtt{e}}\ \forall \mathtt{p}\ |comp_U(\mathtt{e},\mathtt{p})|) \leq |\mathtt{p}| + c_{\mathtt{e}}$$

i.e. for all $\mathtt{e}$, we have $(\mathtt{p} \mapsto |comp_U(\mathtt{e},\mathtt{p})|) \leq_{\mathrm{ct}} |\mathtt{p}|$ (cf. Notation 1.1).

Note 2.2 (Intuition)**.**
1. The set $\mathbf{2}^*$ is seen as a family of programs to get elements of D. The choice of binary programs is a fairness condition in view of the definition of Kolmogorov complexity (cf. Def.2.16) based on the length of programs: larger the alphabet, shorter the programs.

2. Each $F \in \mathcal{F}$ is seen as a programming language with programs in $\mathbf{2}^*$. Special restrictions: no input, outputs are elements of D.

3. Denomination *comp* stands for "compiler" since it maps a program $\mathtt{p}$ from "language" F (with code $\mathtt{p}$) to its U-compiled form $comp_U(\mathtt{e},\mathtt{p})$ in the "language" U.

4. "Compilation" with a good universal function does not increase the length of programs but for some additive constant which depends only on the language, namely on the sole code e.

Example 2.3. If $\mathbb{X}$ is a basic set then $(\mathbb{X}, PR^{\mathbf{2}^*\to\mathbb{X}})$ is obviously a self-enumerated representation system.

Remark 2.4. In view of the enumerability condition *iii* and since there is no recursive enumeration of total recursive functions, one would a priori rather require condition *ii* to be true for all partial recursive functions $\varphi : \mathbf{2}^* \to \mathbf{2}^*$, i.e. consider the sole full systems.
However, there are interesting self-enumerated representation systems which are not full systems. The simplest one is Max_{Rec}, cf. Prop.6.2. Other examples we shall deal with involve higher order domains consisting of infinite objects, for instance the domain $RE(\mathbb{N})$ of all recursively enumerable subsets of $\mathbb{N}$, cf. §5.2. *The partial character of computability is already inherent to the objects in the domain or to the particular notion of computability and an enumeration theorem does hold for a family $\mathcal{F}$ of total functions.*

From conditions i and iii of Def.2.1, we immediately see that

Proposition 2.5. *Let $(D, \mathcal{F})$ be a self-enumerated system. Then D and $\mathcal{F}$ are countable and any universal function for $\mathcal{F}$ is surjective.*

Another consequence of condition iii of Def.2.1 is as follows.

Proposition 2.6. *Let $(\mathbb{N}, \mathcal{F})$ be a self-enumerated system. Then all universal functions for $\mathcal{F}$ are many-one equivalent.*

2.3 Good universal functions always exist

Let's recall a classical way to code pairs of words.

Definition 2.7 (Coding pairs of words)**.**
Let $\mu : \mathbf{2}^* \to \mathbf{2}^*$ be the morphism (relative to the monoid structure of concatenation product on words) such that $\mu(0) = 00$ and $\mu(1) = 01$.
The function $c : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ such that $c(\mathtt{e}, \mathtt{p}) = \mu(\mathtt{e})1\mathtt{p}$ is a recursive injection which satisfies equation

$$|c(\mathtt{e}, \mathtt{p})| = |\mathtt{p}| + 2|\mathtt{e}| + 1 \tag{1}$$

Denoting λ the empty word, we define $\pi_1, \pi_2 : \mathbf{2}^* \to \mathbf{2}^*$ as follows:

$$\pi_1(c(\mathtt{e}, \mathtt{p})) = \mathtt{e} \ , \ \pi_2(c(\mathtt{e}, \mathtt{p})) = \mathtt{p} \ , \ \pi_1(w) = \pi_2(w) = \lambda \text{ if } w \notin \mathit{Range}(c)$$

Remark 2.8. If we redefine c as $c(\mathtt{e}, \mathtt{p}) = \mu(\mathit{Bin}(|\mathtt{e}|))1\mathtt{e}\mathtt{p}$ where $\mathit{Bin}(k)$ is the binary representation of the integer $k \in \mathbb{N}$ then equation (1) can be sharpened to

$$|c(\mathtt{e}, \mathtt{p})| = |\mathtt{p}| + |\mathtt{e}| + 2\lfloor \log(|e|) \rfloor + 3$$

For an optimal sharpening with a coding of pairs involving the function

$$\log(x) + \log\log(x) + \log\log\log(x) + ...$$

see Li & Vitanyi's book [9], Example 1.11.13, p.79.

Proposition 2.9 (Existence of good universal functions)**.**
Every self-enumerated system contains a good universal function with c as associated comp function.

Proof. The usual proof works. Let U and comp_U be as in Def.2.1 and set

$$U_{\mathit{opt}} = U \circ \mathit{comp}_U \circ (\pi_1, \pi_2)$$

Then $\mathit{comp}_U \circ (\pi_1, \pi_2) : \mathbf{2}^* \to \mathbf{2}^*$ is total recursive and condition ii of Def.2.1 insures that $U_{\mathit{opt}} \in \mathcal{F}$. Now, we have

$$U_{\mathit{opt}}(c(\mathtt{e}, \mathtt{p})) = U(\mathit{comp}_U((\pi_1, \pi_2)(c(\mathtt{e}, \mathtt{p})))) = U(\mathit{comp}_U(\mathtt{e}, \mathtt{p}))$$

so that U_{opt} is universal with c as associated *comp* function. □

2.4 Relativization of self-enumerated representation systems

Def.2.1 can be obviously relativized to any oracle A. However, contrary to what can be a priori expected, this is no generalization but particularization. The main reason is Prop.2.9: there always exists a universal function with c as associated *comp* function.

Definition 2.10. Let $A \subseteq \mathbb{N}$. A self-enumerated representation A-system is a pair $(D, \mathcal{F})$ where $\mathcal{F}$ is a family of partial functions $\mathbf{2}^* \to D$ satisfying condition i of Def.2.1 and the following variants of conditions ii and iii :

ii^A. If $\varphi : \mathbf{2}^* \to \mathbf{2}^*$ is an A-recursive total function and $F \in \mathcal{F}$ then $F \circ \varphi \in \mathcal{F}$.

iii^A. There exists $U \in \mathcal{F}$ and a total A-recursive function $comp_U : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ such that

$$\forall F \in \mathcal{F}\ \exists \mathtt{e} \in \mathbf{2}^*\ \forall \mathtt{p} \in \mathbf{2}^*\ F(\mathtt{p}) = U(comp_U(\mathtt{e}, \mathtt{p}))$$

Example 2.11. If $\mathbb{X}$ is a basic set then $(\mathbb{X}, PR^{A,\mathbf{2}^* \to \mathbb{X}})$ is obviously a self-enumerated representation A-system.

Proposition 2.12. *Every self-enumerated representation A-system contains a universal function with c as associated comp function.*
In particular, every such system is also a self-enumerated representation system. Thus, $(\mathbb{X}, PR^{A,\mathbf{2}^* \to \mathbb{X}})$ is a self-enumerated representation system.

Proof. We repeat the same easy argument used for Prop.2.9. Let U and $comp_U$ be as in condition iii^A of Def.2.10 and set $U_{opt} = U \circ comp_U \circ (\pi_1, \pi_2)$. Then $comp_U \circ (\pi_1, \pi_2) : \mathbf{2}^* \to \mathbf{2}^*$ is total A-recursive and condition ii^A insures that $U_{opt} \in \mathcal{F}$ and we have

$$U_{opt}(c(\mathtt{e}, \mathtt{p})) = U(comp_U((\pi_1, \pi_2)(c(\mathtt{e}, \mathtt{p})))) = U(comp_U(\mathtt{e}, \mathtt{p}))$$

so that U_{opt} is universal with c as associated *comp* function. □

2.5 The Invariance Theorem

Definition 2.13. Let $F : \mathbf{2}^* \to D$ be any partial function. The Kolmogorov complexity $K_F^D : D \to \mathbb{N} \cup \{+\infty\}$ associated to F is the function defined as follows:

$$K_F^D(x) = \min\{|\mathtt{p}| : F(\mathtt{p}) = x\}$$

(Convention: $\min \emptyset = +\infty$)

Remark 2.14.
1. $K_F^D(x)$ is finite if and only if $x \in Range(F)$. Hence K_F^D has values in $\mathbb{N}$ (rather than $\mathbb{N} \cup \{+\infty\}$) if and only if F is surjective.

2. If $F : \mathbf{2}^* \to D$ is a restriction of $G : \mathbf{2}^* \to D$ then $K_G^D \leq K_F^D$.

Thanks to Prop. 2.9, the usual Invariance Theorem can be extended to any self-enumerated representation system, which allows to define Kolmogorov complexity for such a system.

Theorem 2.15 (Invariance Theorem, Kolmogorov, 1965 [7])**.**
Let $(D, \mathcal{F})$ be a self-enumerated representation system.

1. *When F varies in the family $\mathcal{F}$, there is a least K_F^D, up to an additive constant (cf. Notation 1.1):*

$$\exists F \in \mathcal{F}\ \ \forall G \in \mathcal{F}\quad K_F^D \leq_{\text{ct}} K_G^D$$

Such F's are said to optimal in $\mathcal{F}$.

2. *Every good universal function for $\mathcal{F}$ is optimal.*

Proof. It suffices to prove 2. The usual proof works. Consider a good universal enumeration U of $\mathcal{F}$. Let $F \in \mathcal{F}$ and let $\mathtt{e}$ be such that

$$U(comp_U(\mathtt{e}, \mathtt{p})) = F(\mathtt{p}) \text{ for all } p \in \mathbf{2}^*$$

First, since U is surjective (Prop.2.5), all values of K_U^D are finite. Thus, $K_U^D(x) < K_F^D(x)$ for $x \notin Range(F)$ (since then $K_F^D(x) = +\infty$).
For every $x \in Range(F)$, let $\mathtt{p}_x$ be a smallest program such that $F(\mathtt{p}_x) = x$, i.e. $K_F^D(x) = |\mathtt{p}_x|$. Then,

$$x = F(\mathtt{p}_x) = U(comp_U(\mathtt{e}, \mathtt{p}_x))$$

and since U is good,

$$K_U^D(x) \leq |comp_U(e, \mathtt{p}_x)| \leq |\mathtt{p}_x| + c_{\mathtt{e}} = K_F^D(x) + c_{\mathtt{e}}$$

and therefore $K_U^D \leq_{\mathrm{ct}} K_F^D$. □

As usual, Theorem 2.15 allows for an intrinsic definition of the Kolmogorov complexity associated to the self-enumerated system $(D, \mathcal{F})$.

Definition 2.16 (Kolmogorov complexity of a self-enumerated representation system)**.**
Let $(D, \mathcal{F})$ be a self-enumerated representation system.
The Kolmogorov complexity $K_{\mathcal{F}}^D : D \to \mathbb{N}$ is the function K_U^D where U is some fixed *good universal enumeration* in $\mathcal{F}$.
Up to an additive constant, this definition is independent of the particular choice of U.

The following straightforward result, based on Examples 2.3 and 2.11, insures that Def.2.16 is compatible with the usual Kolmogorov complexity and its relativizations.

Proposition 2.17. *Let $A \subseteq \mathbb{N}$ be an oracle and let $D = \mathbb{X}$ be a basic set (cf. Def.1.2). The Kolmogorov complexities $K^{\mathbb{X}}_{PR^{\mathbf{2}^* \to \mathbb{X}}}$ and $K^{\mathbb{X}}_{PR^{A,\mathbf{2}^* \to \mathbb{X}}}$ defined above are exactly the usual Kolmogorov complexity $K_{\mathbb{X}} : \mathbb{X} \to \mathbb{N}$ and its relativization $K^A_{\mathbb{X}}$ (cf. §2.1).*

3 Some operations on self-enumerated systems

3.1 The composition lemma

The following easy fact is a convenient tool to effectivize representations (cf. §7.3, 7.4). We shall also use it in §4 to go from systems with domain $\mathbb{N}$ to ones with domain $\mathbb{Z}$.

Lemma 3.1 (The composition lemma)**.**
Let $(D, \mathcal{F})$ be a self-enumerated representation system and $\varphi : D \to E$ be a surjective partial function. Set $\varphi \circ \mathcal{F} = \{\varphi \circ F : F \in \mathcal{F}\}$.

1. *$(E, \varphi \circ \mathcal{F})$ is also a self-enumerated representation system. Moreover, if U is universal or good universal for $\mathcal{F}$ then so is $\varphi \circ U$ for $\varphi \circ \mathcal{F}$.*

2. *For every $x \in E$,*

$$K^E_{\varphi \circ \mathcal{F}}(x) =_{\mathrm{ct}} \min\{K^D_{\mathcal{F}}(y) : \varphi(y) = x\}$$

In particular, $K^E_{\varphi \circ \mathcal{F}} \circ \varphi \ \leq_{\mathrm{ct}} \ K^D_{\mathcal{F}}$ and if $\varphi : D \to E$ is a total bijection from D to E then $K^E_{\varphi \circ \mathcal{F}} \circ \varphi \ =_{\mathrm{ct}} \ K^D_{\mathcal{F}}$.

Proof. Point 1 is straightforward. As for point 2, let $U : \mathbf{2}^* \to D$ be some universal function for $\mathcal{F}$ and observe that, for $x \in E$,

$$\begin{aligned} K^E_{\varphi \circ \mathcal{F}}(x) &= \min\{|\mathtt{p}| : \mathtt{p} \text{ such that } \varphi(U(\mathtt{p})) = x\} \\ &= \min\{\min\{|\mathtt{p}| : \mathtt{p} \text{ s.t. } U(\mathtt{p}) = y\} : y \text{ s.t. } \varphi(y) = x\} \\ &= \min\{K^D_{\mathcal{F}}(y) : y \text{ s.t. } \varphi(y) = x\} \end{aligned}$$

In particular, taking $x = \varphi(z)$, we get $K^E_{\varphi \circ \mathcal{F}}(\varphi(z)) \leq_{\mathrm{ct}} \ K^D_{\mathcal{F}}(z)$.
Finally, observe that if φ is bijective then z is the unique y such that $\varphi(y) = x$, so that the above min reduces to $K^D_{\mathcal{F}}(z)$. □

3.2 Product of self-enumerated representation systems

We shall need a notion of product of self-enumerated representation systems.

Theorem 3.2. *Let $(D_1, \mathcal{F}_1)$ and $(D_2, \mathcal{F}_2)$ be self-enumerated representation systems*
We identify a pair $(F_1, F_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ with the function $\mathbf{2}^* \to D_1 \times D_2$ which maps $\mathtt{p}$ to $(F_1(\mathtt{p}), F_2(\mathtt{p}))$.
Then $(D_1 \times D_2, \mathcal{F}_1 \times \mathcal{F}_2)$ is also a self-enumerated representation system.
If $(D_1, \mathcal{F}_1)$ and $(D_2, \mathcal{F}_2)$ are full systems then so is $(D_1 \times D_2, \mathcal{F}_1 \times \mathcal{F}_2)$.
If U_1, U_2 are universal for $\mathcal{F}_1, \mathcal{F}_2$ then

$$U_{1,2} = (U_1 \circ \pi_1, U_2 \circ \pi_2)$$

is universal for $\mathcal{F}_1 \times \mathcal{F}_2$.

Proof. Condition ii in Def.2.1 is obvious.

Condition i. Let $(d_1, d_2) \in D_1 \times D_2$. Applying condition i to $(D_1, \mathcal{F}_1)$ and to $(D_2, \mathcal{F}_2)$, we get $F_1 \in \mathcal{F}_1$, $F_2 \in \mathcal{F}_2$ and $\mathtt{p}_1, \mathtt{p}_2 \in \mathbf{2}^*$ such that $d_1 = F_1(\mathtt{p}_1)$ and $d_2 = F_2(\mathtt{p}_2)$. Therefore $(d_1, d_2) = (F_1 \circ \pi_1, F_2 \circ \pi_2)(c(\mathtt{p}_1, \mathtt{p}_2))$. Observe finally that $(F_1 \circ \pi_1, F_2 \circ \pi_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ (condition ii for $(D_1, \mathcal{F}_1), (D_2, \mathcal{F}_2)$).

Condition iii. Let $comp_1, comp_2 : \mathbf{2}^* \to \mathbf{2}^*$ be the *comp* functions associated to the universal functions U_1, U_2 and set

$$comp_{1,2}(\mathtt{e},\mathtt{p}) = c(comp_1(\pi_1(\mathtt{e}),\mathtt{p}), comp_2(\pi_2(\mathtt{e}),\mathtt{p}))$$

For every $(F_1, F_2) \in \mathcal{F}_1 \times \mathcal{F}_2$ there exist $\mathtt{a},\mathtt{b} \in \mathbf{2}^*$ such that $F_1(\mathtt{p}) = U_1(comp_1(\mathtt{a},\mathtt{p}))$ and $F_2(\mathtt{p}) = U_2(comp_2(\mathtt{b},\mathtt{p}))$. Therefore

$$\begin{aligned}
(F_1,F_2)(\mathtt{p}) &= (U_1(comp_1(\mathtt{a},\mathtt{p})), U_2(comp_2(\mathtt{b},\mathtt{p}))) \\
&= (U_1 \circ \pi_1, U_2 \circ \pi_2)(c(comp_1(\mathtt{a},\mathtt{p}), comp_2(\mathtt{b},\mathtt{p}))) \\
&= U_{1,2}(comp_{1,2}(c(\mathtt{a},\mathtt{b}),\mathtt{p}))
\end{aligned}$$

which proves that $U_{1,2}$ is universal for the product system $\mathcal{F}_1 \times \mathcal{F}_2$. ☐

Remark 3.3. Observe that, even if U_1, U_2 are good, the above universal function $U_{1,2}$ is not good since

$$|comp_{1,2}(\mathtt{e},\mathtt{p})| = 2|comp_1(\pi_1(\mathtt{e}),\mathtt{p})| + |comp_2(\pi_2(\mathtt{e}),\mathtt{p})| + 1$$

which is $\geq 3|\mathtt{p}|$ in general.
To get a good function $\widetilde{U_{1,2}}$, argue as in the proof of Prop.2.9:

$$\begin{aligned}
\widetilde{U_{1,2}}(\mathtt{p}) &= U_{1,2} \circ comp_{1,2} \circ (\pi_1,\pi_2)(\mathtt{p}) \\
&= U_{1,2}(comp_{1,2}(\pi_1(\mathtt{p}),\pi_2(\mathtt{p}))) \\
&= U_{1,2}(c(comp_1(\pi_1\pi_1(\mathtt{p}),\pi_2(\mathtt{p})), comp_2(\pi_2\pi_1(\mathtt{p}),\pi_2(\mathtt{p})))) \\
&= (U_1 \circ \pi_1, U_2 \circ \pi_2) \\
&\quad (c(comp_1(\pi_1\pi_1(\mathtt{p}),\pi_2(\mathtt{p})), comp_2(\pi_2\pi_1(\mathtt{p}),\pi_2(\mathtt{p})))) \\
&= (U_1(comp_1(\pi_1\pi_1(\mathtt{p}),\pi_2(\mathtt{p}))), U_2(comp_2(\pi_2\pi_1(\mathtt{p}),\pi_2(\mathtt{p}))))
\end{aligned}$$

4 From domain $\mathbb{N}$ to domain $\mathbb{Z}$

4.1 The Δ operation

Relative integers are classically introduced as equivalence classes of pairs of natural integers of which they are the differences. This give a simple way to go from a self-enumerated representation system with domain $\mathbb{N}$ to some with domain $\mathbb{Z}$.

Definition 4.1 (The Δ operation)**.**
Let $\mathit{diff} : \mathbb{N}^2 \to \mathbb{Z}$ be the function $(m,n) \mapsto m - n$.
If $(\mathbb{N},\mathcal{F})$ is a self-enumerated representation system with domain $\mathbb{N}$, using notations from Lemma 3.1 and Thm.3.2, we let $(\mathbb{Z}, \Delta\mathcal{F})$ be the system

$$(\mathbb{Z}, \mathit{diff} \circ (\mathcal{F} \times \mathcal{F}))$$

As a direct corollary of Lemma 3.1 and Thm.3.2, we have

Proposition 4.2. *If $(\mathbb{N},\mathcal{F})$ is a self-enumerated representation system (resp. full system) with domain $\mathbb{N}$ then so is $(\mathbb{Z}, \Delta\mathcal{F})$.*

4.2 $\mathbb{Z}$ systems and $\mathbb{N}$ systems

The following propositions collect some easy facts about self-enumerated systems with domain $\mathbb{Z}$ and their associated Kolmogorov complexities.

Proposition 4.3. *Let $(\mathbb{Z}, \mathcal{G})$ be a self-enumerated system.*

1. *Let $\mathcal{F} = \{G \restriction G^{-1}(\mathbb{N}) : G \in \mathcal{G}\}$. Then $(\mathbb{N}, \mathcal{F})$ is also a self-enumerated system and $K^{\mathbb{N}}_{\mathcal{F}} = K^{\mathbb{Z}}_{\mathcal{G}} \restriction \mathbb{N}$.*

2. *Denote opp : $\mathbb{Z} \to \mathbb{Z}$ the function $n \mapsto -n$. If $\mathcal{G} \circ opp = \mathcal{G}$ then $K^{\mathbb{Z}}_{\mathcal{G}} =_{\mathrm{ct}} K^{\mathbb{Z}}_{\mathcal{G}} \circ opp$.*

Proof. 1. Conditions i-ii of Def.2.1 are obvious. As for iii, observe that if $U \in \mathcal{G}$ is universal for $\mathcal{G}$ then $U \restriction U^{-1}(\mathbb{N})$ is in $\mathcal{F}$ and is universal for $\mathcal{F}$ with the same associated *comp* function. Now, $K_{U \restriction U^{-1}(\mathbb{N})} = K_U \restriction \mathbb{N}$. Whence $K^{\mathbb{N}}_{\mathcal{F}} = K^{\mathbb{Z}}_{\mathcal{G}} \restriction \mathbb{N}$.

2. Observe that if $\varphi, F \in \mathcal{G}$ and $K_\varphi \leq_{\mathrm{ct}} K_F$ then $K_{\varphi \circ opp} \leq_{\mathrm{ct}} K_{F \circ opp}$. Since $\mathcal{G} \circ opp = \mathcal{G}$, we see that if φ is optimal then so is $\varphi \circ opp$. Whence $K_\varphi =_{\mathrm{ct}} K_{\varphi \circ opp}$, and therefore $K^{\mathbb{Z}}_{\mathcal{G}} =_{\mathrm{ct}} K^{\mathbb{Z}}_{\mathcal{G}} \circ opp$. □

Proposition 4.4. *Let $A \subseteq \mathbb{N}$.*

1. $PR^{A,\mathbf{2}^* \to \mathbb{N}} = PR^{A,\mathbf{2}^* \to \mathbb{Z}} \cap (\mathbb{N} \to \mathbb{N}) = \{G \restriction G^{-1}(\mathbb{N}) : G \in PR^{A,\mathbf{2}^* \to \mathbb{Z}}\}$.
In particular, $K^{A,\mathbb{Z}} \restriction \mathbb{N} =_{\mathrm{ct}} K^{A,\mathbb{N}}$.

2. $PR^{A,\mathbf{2}^* \to \mathbb{Z}} = PR^{A,\mathbf{2}^* \to \mathbb{Z}} \circ opp = \Delta PR^{A,\mathbf{2}^* \to \mathbb{N}}$.
In particular, $K^{A,\mathbb{Z}} =_{\mathrm{ct}} K^{A,\mathbb{Z}} \circ opp$.

5 Self-enumerated representation systems for r.e. sets

We now come to examples of self-enumerated systems of a somewhat different kind, which will be used in the effectivization of set theoretical representations of integers.

5.1 Acceptable enumerations

Let's recall the notion of acceptable enumeration of partial recursive functions (cf. Rogers [15] Ex. 2.10 p.41, or Odifrreddi [12], p.215)

Definition 5.1. Let $\mathbb{X}, \mathbb{Y}$ be some basic sets and $A \subseteq \mathbb{N}$.

1. An enumeration $(\phi^A_{\mathsf{e}})_{\mathsf{e} \in \mathbf{2}^*}$ of partial A-recursive functions $\mathbb{X} \to \mathbb{Y}$ is *acceptable* if

- **i.** it is partial A-recursive as a function $\mathbf{2}^* \times \mathbb{X} \to \mathbb{Y}$

ii. and it satisfies the parametrization (also called s-m-n) property: for every basic set $\mathbb{Z}$, there exists a total A-recursive function $s^{\mathbb{Z}}_{\mathbb{X}} : \mathbf{2}^* \times \mathbb{Z} \to \mathbf{2}^*$ such that, for all $\mathtt{e} \in \mathbf{2}^*$, $\mathtt{z} \in \mathbb{Z}$, $\mathtt{x} \in \mathbb{X}$,

$$\phi^A_{\mathtt{e}}(\langle \mathtt{z}, \mathtt{x} \rangle) = \phi^A_{s^{\mathbb{Z}}_{\mathbb{X}}(\mathtt{e},\mathtt{z})}(\mathtt{x})$$

where $\langle \mathtt{z}, \mathtt{x} \rangle$ is the image of the pair $(\mathtt{z}, \mathtt{x})$ by some fixed total recursive bijection $\mathbb{Z} \times \mathbb{X} \to \mathbb{X}$.

3. An enumeration $(W^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ of A-recursively enumerable subsets of $\mathbb{X}$ is *acceptable* if, for all $\mathtt{e} \in \mathbf{2}^*$, $W^A_{\mathtt{e}} = \mathit{domain}(\phi^A_{\mathtt{e}})$ for some acceptable enumeration $(\phi^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ of partial A-recursive functions.

We shall need Rogers' theorem (cf. Odifreddi [12] p.219).

Theorem 5.2 (Rogers' theorem). *If $(\phi^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ and $(\psi^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ are two acceptable enumerations of partial A-recursive functions $\mathbb{X} \to \mathbb{Y}$, then there exists some A-recursive bijection $\theta : \mathbf{2}^* \to \mathbf{2}^*$ such that $\psi^A_{\mathtt{e}} = \phi^A_{\theta(\mathtt{e})}$ for all $\mathtt{e} \in \mathbf{2}^*$.*

Corollary 5.3. *Let $(W'^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ and $(W''^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ be two acceptable enumerations of A-r.e. subsets of $\mathbb{X}$. Then there exists an A-recursive bijection $\theta : \mathbf{2}^* \to \mathbf{2}^*$ such that $W''^A_{\mathtt{e}} = W'^A_{\theta(\mathtt{e})}$ for all $\mathtt{e} \in \mathbf{2}^*$.*

Proof. Apply Roger's theorem to acceptable enumerations $(\phi^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$, $(\psi^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ of partial A-recursive functions such that $W'^A_{\mathtt{e}} = \mathit{domain}(\phi^A_{\mathtt{e}})$ and $W''^A_{\mathtt{e}} = \mathit{domain}(\psi^A_{\mathtt{e}})$. □

5.2 Self-enumerated representation systems for r.e. sets

Cor.5.3 allows to get a natural intrinsic notion of "partial A-computable" map $\mathbf{2}^* \to RE^A(\mathbb{X})$.

Proposition 5.4. *Let $RE^A(\mathbb{X})$ be the family of A-recursively enumerable subsets of $\mathbb{X}$ and let $(W'^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ and $(W''^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ be two acceptable enumerations of A-r.e. subsets of $\mathbb{X}$. Let $G : \mathbf{2}^* \to RE^A(\mathbb{X})$.*

1. *The following conditions are equivalent:*

i. There exists a total A-recursive function $f : \mathbf{2}^* \to \mathbf{2}^*$ such that $G(\mathtt{p}) = W'^A_{f(\mathtt{p})}$ for all $\mathtt{p} \in \mathbf{2}^*$

ii. There exists a total A-recursive function $f : \mathbf{2}^* \to \mathbf{2}^*$ such that $G(\mathtt{p}) = W''^A_{f(\mathtt{p})}$ for all $\mathtt{p} \in \mathbf{2}^*$

2. *The following conditions are equivalent:*

i. There exists a partial A-recursive function $f : \mathbf{2}^* \to \mathbf{2}^*$ such that, for all $\mathtt{p} \in \mathbf{2}^*$, $G(\mathtt{p}) = \begin{cases} W'^A_{f(\mathtt{p})} & \text{if } f(\mathtt{p}) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$

ii. There exists a partial A-recursive function $f : \mathbf{2}^* \to \mathbf{2}^*$ such that, for all $\mathtt{p} \in \mathbf{2}^*$, $G(\mathtt{p}) = \begin{cases} W''^A_{f(\mathtt{p})} & \textit{if } f(\mathtt{p}) \textit{ is defined} \\ \textit{undefined} & \textit{otherwise} \end{cases}$

Proof. Applying Cor.5.3, we get $W''^A_{f(\mathtt{p})} = W'^A_{\theta(f(\mathtt{p}))}$ and $W'^A_{f(\mathtt{p})} = W'^A_{\theta^{-1}(f(\mathtt{p}))}$. To conclude, observe that $\theta \circ f$ and $\theta^{-1} \circ f$ are both total (point 1) or partial (point 2) A-recursive as is f. $\square$

We can now come to the notion of self-enumerated systems for r.e. sets.

Definition 5.5 (Self-enumerated systems for r.e. sets)**.**
Let $RE^A(\mathbb{X})$ be the class of A-r.e. subsets of the basic set $\mathbb{X}$.
Let $(W^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ be some fixed acceptable enumeration of A-r.e. subsets of $\mathbb{X}$. Cor.5.3 insures that the families defined hereafter do not depend on the chosen acceptable enumeration.

1. We let $\mathcal{F}^{RE^A(\mathbb{X})}$ be the family of all *total* functions $\mathbf{2}^* \to RE^A(\mathbb{X})$ of the form $\mathtt{p} \mapsto W^A_{f(\mathtt{p})}$ where $f : \mathbf{2}^* \to \mathbf{2}^*$ varies over *total* A-recursive functions.

2. We let $\mathcal{PF}^{RE^A(\mathbb{X})}$ be the family of all *partial* functions $\mathbf{2}^* \to RE^A(\mathbb{X})$ of the form

$$\mathtt{p} \mapsto \begin{cases} W^A_{f(\mathtt{p})} & \text{if } f(\mathtt{p}) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

where $f : \mathbf{2}^* \to \mathbf{2}^*$ varies over *partial* A-recursive functions.

The following proposition shows that, in the definition of $\mathcal{F}^{RE^A(\mathbb{X})}$, one can either relax the total "A-recursive" condition on f to "partial A-recursive" with a special convention (different from that considered in the definition of $\mathcal{PF}^{RE^A(\mathbb{X})}$) or restrict it to some particular A-recursive sequence of total functions.

Proposition 5.6. *For any acceptable enumeration $(W^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ of A-r.e. subsets of $\mathbb{X}$ there exists a total A-recursive function $\sigma : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ such that, for any total function $G : \mathbf{2}^* \to RE^A(\mathbb{X})$, the following conditions are equivalent:*

a. G is of the form $\mathtt{p} \mapsto W^A_{\sigma(\mathtt{e},\mathtt{p})}$ *for some* $\mathtt{e} \in \mathbf{2}^*$

b. $G \in \mathcal{F}^{RE^A(\mathbb{X})}$

c. For all $\mathtt{p}$, $G(\mathtt{p}) = \begin{cases} W^A_{g(\mathtt{p})} & \textit{if } g(\mathtt{p}) \textit{ is defined} \\ \emptyset & \textit{otherwise} \end{cases}$.

Proof. Since $a \Rightarrow b \Rightarrow c$ is trivial whatever be the total recursive function σ, it remains to define σ such that $c \Rightarrow a$ holds.
Let $(\phi^A_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ be an acceptable enumeration of partial A-recursive functions $\mathbb{X} \to \mathbb{N}$ such that $W^A_{\mathtt{e}} = domain(\phi^A_{\mathtt{e}})$.

Let $(\psi^A_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ be an enumeration of partial A-recursive functions $\mathbf{2}^* \to \mathbf{2}^*$ and let $\mathtt{a}$ be such that $\phi^A_{\psi^A_{\mathtt{e}}(\mathtt{p})}(\mathtt{x}) = \phi^A_{\mathtt{a}}(\langle\langle\mathtt{e},\mathtt{p}\rangle,\mathtt{x}\rangle)$ for all $\mathtt{e},\mathtt{p} \in \mathbf{2}^*$, $\mathtt{x} \in \mathbb{X}$. The parameter theorem insures that there exists a total A-recursive function $s : \mathbf{2}^* \times (\mathbf{2}^* \times \mathbf{2}^*) \to \mathbf{2}^*$ such that

$$\phi^A_{\psi^A_{\mathtt{e}}(\mathtt{p})}(\mathtt{x}) = \phi^A_{\mathtt{a}}(\langle\langle\mathtt{e},\mathtt{p}\rangle,\mathtt{x}\rangle) = \phi^A_{s(\mathtt{a},\mathtt{e},\mathtt{p})}(\mathtt{x}) = \phi^A_{\sigma(\mathtt{e},\mathtt{p})}(\mathtt{x})$$

where $\sigma(\mathtt{e},\mathtt{p}) = s(\mathtt{a},\mathtt{e},\mathtt{p})$. Whence the equality

$$W^A_{\psi^A_{\mathtt{e}}(\mathtt{p})} = W^A_{\sigma(\mathtt{e},\mathtt{p})}$$

which is also valid when $\psi^A_{\mathtt{e}}(\mathtt{p})$ is undefined, in the sense that both sets are empty.

Let G, g be as in c. Since $g : \mathbf{2}^* \to \mathbf{2}^*$ is A-recursive, there exists $\mathtt{e}$ such that $g(\mathtt{p}) = \psi^A_{\mathtt{e}}(\mathtt{p})$ for any $\mathtt{p} \in \mathbf{2}^*$. Thus,

$$W^A_{g(\mathtt{p})} = W^A_{\psi^A_{\mathtt{e}}(\mathtt{p})} = W^A_{\sigma(\mathtt{e},\mathtt{p})}$$

an equality valid also if $g(\mathtt{p})$ is undefined, in the sense that all sets are empty. This proves $c \Rightarrow a$. □

Theorem 5.7. *$(RE^A(\mathbb{X}), \mathcal{F}^{RE^A(\mathbb{X})})$ and $(RE^A(\mathbb{X}), \mathcal{PF}^{RE^A(\mathbb{X})})$ are self-enumerated representation systems.*

Proof. Conditions i, ii^A of Def.2.1, 2.10 are obvious for both systems.
If U satisfies iii^A for $PR^{A,\mathbf{2}^*\to\mathbb{X}}$ then

$$\mathtt{p} \mapsto \begin{cases} W^A_{U(\mathtt{p})} & \text{if } U(\mathtt{p}) \text{ is defined} \\ \text{undefined} & \text{otherwise} \end{cases}$$

satisfies iii^A for $\mathcal{PF}^{RE^A(\mathbb{X})}$ with the same associated *comp* function.
Prop.5.6 proves that the function $\mathtt{p} \mapsto W^A_{\mathtt{e}}$ satisfies condition iii^A with σ as *comp* function. Thus, $(RE^A(\mathbb{X}), \mathcal{F}^{RE^A(\mathbb{X})})$ and $(RE^A(\mathbb{X}), \mathcal{PF}^{RE^A(\mathbb{X})})$ are self-enumerated A-systems. We conclude using Prop.2.12. □

Remark 5.8. It is possible to improve Prop.5.6 so as to get σ total recursive (rather than A-recursive) in condition a. This will hold for particular acceptable enumerations of A-r.e. sets, with the same total recursive σ whatever be A. We sketch how this can be obtained (for more details about this type of argument, cf. our paper [6] §2.3, 2.4.).
Using partial computable functionals $\mathbb{X} \times P(\mathbb{N}) \to \mathbb{N}$, we can view partial A-recursive functions as functions obtained by freezing the second order argument in such functionals. We can also also consider A-r.e. subsets of $\mathbb{X}$ as obtained from domains of such functionals by freezing the second order argument.

When freezing the second order argument to $A \subseteq \mathbb{N}$, acceptable enumerations of partial computable functionals give acceptable enumerations of partial A-recursive functions.
In this way, consider an acceptable enumeration $(\Phi_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ of partial computable functionals $\mathbb{X}\times P(\mathbb{N}) \to \mathbb{N}$ and let $\mathcal{W}^A_{\mathtt{e}} = \{\mathtt{x} : (\mathtt{x}, A) \in domain(\Phi_{\mathtt{e}})\}$. Arguing as in the proof of Prop.5.6 (with an acceptable enumeration $(\Psi_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ of partial computable functionals $\mathbf{2}^* \times P(\mathbb{N}) \to \mathbf{2}^*$) we get

$$\Phi_{\Psi_{\mathtt{e}}(\mathtt{p},A)}(\mathtt{x}, A) = \Phi_{\mathtt{a}}(\langle(\mathtt{e}, \mathtt{p}), \mathtt{x}\rangle, A) = \Phi_{s(\mathtt{a},\mathtt{e},\mathtt{p})}(\mathtt{x}, A) = \Phi_{\sigma(\mathtt{e},\mathtt{p})}(\mathtt{x}, A)$$

where s is the total recursive function involved in the parameter property for the acceptable enumeration $(\Phi_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ and $\sigma(\mathtt{e}, \mathtt{p}) = s(\mathtt{a}, \mathtt{e}, \mathtt{p})$.
Now, let $G \in \mathcal{F}^{RE^A(\mathbb{X})}$ and let $g : \mathbf{2}^* \to \mathbf{2}^*$ be total A-recursive such that $G(\mathtt{p}) = \mathcal{W}^A_{g(\mathtt{p})}$. Let $\mathtt{e} \in \mathbf{2}^*$ be such that $g = \Psi(\mathtt{e}, A)$. Then

$$\Phi_{g(\mathtt{p})}(\mathtt{x}, A) = \Phi_{\Psi_{\mathtt{e}}(\mathtt{p},A)}(\mathtt{x}, A) = \Phi_{\sigma(\mathtt{e},\mathtt{p})}(\mathtt{x}, A) \quad \text{and} \quad G(\mathtt{p}) = \mathcal{W}^A_{g(\mathtt{p})} = \mathcal{W}^A_{\sigma(\mathtt{e},\mathtt{p})}$$

6 Infinite computations

Chaitin, 1976 [2], and Solovay, 1977 [20], considered infinite computations producing infinite objects (namely recursively enumerable sets) so as to define Kolmogorov complexity of such infinite objects.
Following the idea of possibly infinite computations leading to finite output (i.e. remove the sole halting condition), Becher & Chaitin & Daicz, 2001 [1] introduced a variant K^∞ of Kolmogorov complexity.
In our paper [5], 2004, we introduced two variants $K_{\max}, K_{\min}$ of Kolmogorov complexity and proved that $K^\infty = K_{\max}$. These variants are based on two self-enumerated representation systems, namely the classes of max and min of partial recursive sequences of partial recursive functions.

6.1 Self-enumerated systems of max of partial recursive functions

Notation 6.1. Let $A \subseteq \mathbb{N}$.
1. Let $\mathbb{X}$ be a basic set. Extending Notation 1.3, we denote $Rec^{A,\mathbf{2}^*\to\mathbb{X}}$ the family of total functions $\mathbf{2}^* \to \mathbb{X}$ which are recursive in A.

2. Let $\mathbb{X}$ be $\mathbb{N}$ or $\mathbb{Z}$. If $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{X}$, we denote $\max f$ the function $(\max f)(\mathtt{p}) = \max\{f(\mathtt{p}, t) : t \in \mathbb{N}\}$ (with the convention that $\max X$ is undefined if X is empty or infinite).
We define the families of functions

$$\begin{array}{rcl} Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A} &=& \{\max f : f \in PR^{A,\mathbf{2}^*\times\mathbb{N}\to\mathbb{X}}\} \\ Max^{\mathbf{2}^*\to\mathbb{X}}_{Rec^A} &=& \{\max f : f \in Rec^{A,\mathbf{2}^*\times\mathbb{N}\to\mathbb{X}}\} \end{array}$$

In case A is $\emptyset$, we simply write $Max^{\mathbf{2}^*\to\mathbb{X}}_{PR}$ and $Max^{\mathbf{2}^*\to\mathbb{X}}_{Rec}$.

Proposition 6.2. *Let $A \subseteq \mathbb{N}$. Then*

$$(\mathbb{N}, Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}) \quad , \quad (\mathbb{Z}, Max^{\mathbf{2}^*\to\mathbb{Z}}_{PR^A}) \quad , \quad (\mathbb{N}, Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$$

are self-enumerated representation systems.

Proof. First consider the no oracle case (i.e. $A = \emptyset$). Conditions i-ii in Def.2.1 are trivial. The classical enumeration theorem easily extends to $Max^{\mathbf{2}^*\to\mathbb{X}}_{PR}$ (cf. [5], Thm.4.1), proving condition iii for $(\mathbb{X}, Max^{\mathbf{2}^*\to\mathbb{X}}_{PR})$ where $\mathbb{X}$ is $\mathbb{N}$ or $\mathbb{Z}$.
It remains to show condition iii for $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$. We use the following straightforward fact (cf. [5], Thm.3.6):
Fact 6.3. *If $f \in PR^{\mathbf{2}^*\times\mathbb{N}\to\mathbb{N}}$ and*

$$g(\mathtt{p}, t) = \max(\{0\} \cup \{f(\mathtt{p}, i) : i \leq t \ \wedge \ f(\mathtt{p}, i) \textit{converges in at most } t \textit{ steps}\})$$

then $g \in Rec^{\mathbf{2}^*\times\mathbb{N}\to\mathbb{N}}$ and $\max g$ is an extension of $\max f$ with value 0 on $domain(\max g) \setminus domain(\max f)$ (which is the set of n's such that $f(n, t)$ is defined for no t).

Let $U \in Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ be good universal for $Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ and let V be an extension of U in $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$ given by the above fact. If $F \in Rec^{\mathbf{2}^*\to\mathbb{N}}$ then it is in $PR^{\mathbf{2}^*\to\mathbb{N}}$ and there exists $\mathtt{e}$ such that $F(\mathtt{p}) = U(comp_U(\mathtt{e}, \mathtt{p}))$ for all $\mathtt{p} \in \mathbf{2}^*$. Since V extends U and F is total, we also have $F(\mathtt{p}) = V(comp_U(\mathtt{e}, \mathtt{p}))$. Thus, V is good universal for $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$ with the same associated *comp* function.

Relativization to oracle A proves conditions ii^A, iii^A, (cf. Def.2.10) for $(\mathbb{X}, Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A})$ and $(\mathbb{N}, Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec})$. We conclude using Prop.2.12. ☐

Remark 6.4.
1. Fact 6.3 implies that $Max^{\mathbf{2}^*\to\mathbb{X}}_{PR}$ and $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec}$ contain the same total functions. However, considering partial functions, the inclusion $Max^{\mathbf{2}^*\to\mathbb{X}}_{Rec} \subset Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ is strict (cf. [5] Thm.3.6, point 1).

2. Let $\mathbb{X}$ be $\mathbb{N}$ or $\mathbb{Z}$ and let $Min^{\mathbf{2}^*\to\mathbb{X}}_{PR^A}, Min^{\mathbf{2}^*\to\mathbb{X}}_{Rec^A}$ be defined with min instead of max as in Point 2 of the above definition (with the same convention that $\min \emptyset$ is undefined). Then $(\mathbb{X}, Min^{\mathbf{2}^*\to\mathbb{X}}_{PR^A})$ is also a self-enumerated representation system.
We shall not use any min based system in this paper because they have no simple set theoretical counterparts.

3. None of the systems $(\mathbb{Z}, Max^{\mathbf{2}^*\to\mathbb{Z}}_{Rec^A})$, $(\mathbb{N}, Min^{\mathbf{2}^*\to\mathbb{N}}_{Rec})$ and $(\mathbb{Z}, Min^{\mathbf{2}^*\to\mathbb{Z}}_{Rec})$ is self-enumerated (cf. [5], Thm.4.3).

6.2 Kolmogorov complexities $K_{\max}, K_{\max}^{\emptyset'}, \ldots$

We apply Def.2.16 to the self-enumerated representation systems considered in §6.1.

Definition 6.5 (Kolmogorov complexities)**.** Let $\mathbb{X}$ be $\mathbb{N}$ or $\mathbb{Z}$. We denote $K_{\max}^{A,\mathbb{X}} : \mathbb{X} \to \mathbb{N}$ the Kolmogorov complexity of the self-enumerated representation system $(\mathbb{X}, Max_{PR^A}^{\mathbf{2}^*\to\mathbb{X}})$.

In case $\mathbb{X} = \mathbb{N}$, we omit the superscript $\mathbb{N}$.

In case $\mathbb{X} = \mathbb{N}$ and A is $\emptyset$ we simply write $K_{\max}$.

Using Remark 2.14, point 2, and Fact 6.3, it is not hard to prove the following result (cf. [5], Prop.6.3).

Proposition 6.6. *Let $A \subseteq \mathbb{N}$. Then $K_{\max}^{A}$ is also the Kolmogorov complexity of the self-enumerated system $(\mathbb{N}, Max_{Rec^A}^{\mathbf{2}^*\to\mathbb{N}})$. I.e.*

$$K^{\mathbb{N}}_{Max_{Rec^A}^{\mathbf{2}^*\to\mathbb{N}}} = K^{\mathbb{N}}_{Max_{PR^A}^{\mathbf{2}^*\to\mathbb{N}}}$$

Remark 6.7. The above proposition has no analog with $\mathbb{Z}$ since $Max_{Rec^A}^{\mathbf{2}^*\to\mathbb{Z}}$ is not self-enumerated (cf. Remark 6.4, point 3).

6.3 $Max_{Rec}^{\mathbf{2}^*\to\mathbb{N}}$ and $Max_{PR}^{\mathbf{2}^*\to\mathbb{N}}$ and infinite computations

The following simple result gives a machine characterization of functions in $Max_{Rec^A}^{\mathbf{2}^*\to\mathbb{N}}$ (resp. $Max_{PR^A}^{\mathbf{2}^*\to\mathbb{N}}$) which will be used in the proof of Thm.9.5.

Definition 6.8. Let $\mathcal{M}$ be an oracle Turing machine such that

1. the alphabet of the input tape is $\{0, 1\}$, plus an end-marker to delimitate the input,
2. the output tape is write-only and has unary alphabet $\{1\}$,
3. there is no halting state (resp. but there are some distinguished states).

The partial function $F^A : \mathbf{2}^* \to \mathbb{N}$ computed by $\mathcal{M}$ with oracle A through infinite computation (resp. with distinguished states) is defined as follows: $F^A(\mathtt{p})$ is defined with value n if and only if the infinite computation (i.e. which lasts forever) of $\mathcal{M}$ on input $\mathtt{p}$ outputs exactly n letters 1 (resp. and at some step the current state is a distinguished one).

Proposition 6.9. *Let $A \subseteq \mathbb{N}$ be an oracle. A function $F : \mathbf{2}^* \to \mathbb{N}$ is in $Max_{Rec^A}^{\mathbf{2}^*\to\mathbb{N}}$ (resp. $Max_{PR^A}^{\mathbf{2}^*\to\mathbb{N}}$) if and only if there exists an oracle Turing machine $\mathcal{M}$ which, with oracle A, computes F through infinite computation (resp. with distinguished states) in the sense of Def.6.8.*

Proof. $\Leftarrow$. The function associated to an oracle Turing machine through infinite computation (resp. with distinguished states) is clearly $\max f$ where $f(\mathtt{p}, t)$ is the current output at step t (resp. and is undefined while the machine has not been in some distinguished state).

$\Rightarrow$. Suppose $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ is total (resp. partial) A-recursive and set

$$X(\mathtt{p}, t) = \{f(\mathtt{p}, t') : t' < t \ \wedge \ f(\mathtt{p}, t') \text{ converges in } \leq t \text{ steps}\})$$

Observe that $X(\mathtt{p}, 0) = \emptyset$, so that the following is indeed an A-recursive definition:

$$\widetilde{f}(\mathtt{p}, t) \quad = \quad \begin{cases} 0 \text{ (resp. undefined)} & \text{if } X(\mathtt{p}, t) = \emptyset \\ \widetilde{f}(\mathtt{p}, t-1) + 1 & \text{if } X(\mathtt{p}, t) \neq \emptyset \ \wedge \ \widetilde{f}(\mathtt{p}, t-1) < \max X(\mathtt{p}, t) \\ \widetilde{f}(\mathtt{p}, t-1) & \text{otherwise} \end{cases}$$

Then $\max \widetilde{f} = \max f$. Also, the unary representation of $\widetilde{f}(\mathtt{p}, t)$ can be simply interpreted as the current output at step t of the infinite computation (resp. with distinguished states) of an oracle Turing machine with input $\mathtt{p}$. So that $\max \widetilde{f}$ is the function associated to that machine. $\square$

6.4 $Max_{PR}^{\mathbf{2}^* \to \mathbb{N}}$ and the jump

The following proposition is easy.

Proposition 6.10. *Let $A \subseteq \mathbb{N}$ and let $\mathbb{X}$ be $\mathbb{N}$ or $\mathbb{Z}$. Then*

$$Max_{PR^A}^{\mathbf{2}^* \to \mathbb{X}} \subset PR^{A', \mathbf{2}^* \to \mathbb{X}}$$

Proof. 1. Let $f : \mathbf{2}^* \to \mathbb{X}$ be partial A-recursive. A partial A'-recursive definition of $(\max f)(\mathtt{p})$ is as follows:

i. First, check whether there exists t such that $f(\mathtt{p}, t)$ is defined. If the check is negative then $(\max f)(\mathtt{p})$ is undefined.

ii. If check i is positive then start successive steps of the following process.
- At step t, check whether $f(\mathtt{p}, t)$ is defined,
- if defined, compute its value,
- and check whether there exists $u > t$ such that $f(\mathtt{p}, u)$ is greater than the maximum value computed up to that step.

iii. If at some step the last check in ii is negative then halt and output the maximum value computed up to now.

Clearly, oracle A' allows for the checks in i and ii. Also, the above process halts if and only if $f(\mathtt{p}, t)$ is defined for some t and $\{f(\mathtt{p}, t) : t \in \mathbb{N}\}$ is bounded, i.e. if and only if $(\max f)(\mathtt{p})$ is defined. In that case it outputs exactly $(\max f)(\mathtt{p})$.

2. To see that the inclusion is strict, observe that the graph of any function in $Max^{\mathbf{2}^*\to\mathbb{X}}_{PR^A}$ is $\Sigma^{0,A}_1 \wedge \Pi^{0,A}_1$ since

$$y = (\max f)(\mathtt{p}) \;\Leftrightarrow\; ((\exists t\ f(\mathtt{p},t) = y) \;\wedge\; \neg(\exists u\ \exists z > y\ f(\mathtt{p},u) = z))$$

Whereas the graph of functions in $PR^{A',\mathbf{2}^*\to\mathbb{X}}$ can be $\Sigma^{0,A'}_1$ and not $\Delta^{0,A'}_1$, i.e. $\Sigma^{0,A}_2$ and not $\Delta^{0,A}_2$. □

In the vein of Prop.6.10, let's mention the following result, cf. [1] (where the proof is for K^∞, cf. start of §6 above) and [5] Prop.7.2-3 & Cor.7.7.

Proposition 6.11. *Let $A \subseteq \mathbb{N}$.*

1. *K^A and $K^A_{\max}$ are recursive in A'.*

2. *$K^A >_{\mathrm{ct}} K^A_{\max} >_{\mathrm{ct}} K^{A'}$.*

6.5 The Δ operation on $Max^{\mathbf{2}^*\to\mathbb{N}}_{PR}$ and the jump

The following variant of Prop.6.10 is a normal form for partial A'-recursive $\mathbb{Z}$-valued functions. We shall use it in §8-9.

Theorem 6.12. *Let $A \subseteq \mathbb{N}$. Then*

$$PR^{A',\mathbf{2}^*\to\mathbb{Z}} = \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}) = \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$$

Thus, every partial A'-recursive function is the difference of two functions in Max_{Rec^A} (cf. Notation 6.1).

Before entering the proof of Thm.6.12, let's recall two well-known facts about oracular computation and approximation of the jump.

Lemma 6.13. *Let $(B_t)_{t\in\mathbb{N}}$ be a sequence of subsets of $\mathbb{N}$ which converges pointwise to $B \subseteq \mathbb{N}$, i.e.*

$$\forall n\ \ \exists t_n\ \ \forall t \geq t_n \quad B_t \cap \{0,1,...,n\} = B \cap \{0,1,...,n\}$$

Let $\mathbb{X}, \mathbb{Y}$ be basic sets and let $\psi : \mathbb{X} \to \mathbb{Y}$ be a partial B-recursive function computed by some oracle Turing machine $\mathcal{M}$ with oracle B. Let $\mathtt{x} \in \mathbb{X}$. Then, $\psi(\mathtt{x})$ is defined if and only if there exists $t_\mathtt{x}$ such that

i. the computation of $\mathcal{M}$ on input $\mathtt{x}$ with oracle $B_{t_\mathtt{x}}$ halts in at most $t_\mathtt{x}$ steps,

ii. for all $t \geq t_\mathtt{x}$ the computation of $\mathcal{M}$ on input $\mathtt{x}$ with oracle B_t is step by step exactly the same as that with oracle $B_{t_\mathtt{x}}$ (in particular, it asks the same questions to the oracle, gets the same answers and halts at the same computation step $\leq t_\mathtt{x}$).

Lemma 6.14. *Let $A \subseteq \mathbb{N}$ and let $A' \subseteq \mathbb{N}$ be the jump of A. There exists a total A-recursive sequence $(Approx(A',t))_{t\in\mathbb{N}}$ of subsets of $\mathbb{N}$ which is monotone increasing with respect to set inclusion and which has union A'. In particular, this sequence converges pointwise to A'.*

We can now prove Thm.6.12.

Proof of Thm.6.12.
Using Prop.6.10 and Prop.4.4, we get

$$\Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}) \subseteq \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}) \subseteq \Delta(PR^{A',\mathbf{2}^*\to\mathbb{N}}) = PR^{A',\mathbf{2}^*\to\mathbb{Z}}$$

Since $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}$ is closed by sums, we have $\Delta(\Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}) = \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$. Thus, to get the wanted equality, it suffices to prove inclusion

$$PR^{A',\mathbf{2}^*\to\mathbb{N}} \subseteq \Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$$

Let $\mathcal{M}$ be an oracle Turing machine with inputs in $\mathbf{2}^*$, which, with oracle A', computes the partial A'-recursive function $\varphi^{A'} : \mathbf{2}^* \to \mathbb{N}$.
To prove that $\varphi^{A'}$ is in $\Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$, we define total A-recursive functions $f, g : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ which are (non strictly) monotone increasing and such that $\varphi^{A'} = \max f - \max g$.

The idea to get f, g is as follows. We consider A-recursive approximations of oracle A' (as given by Lemma 6.14) and use them as fake oracles. Function f is obtained by letting $\mathcal{M}$ run with the fake oracles and restart its computation each time some better approximation of A' shows the previous fake oracle has given an incorrect answer. Function g collects all the outputs of the computations which have been recognized as incorrect in the computing process for f.

We now formally define f, g.
First, since we do not care about computation time and space, we can suppose without loss of generality, that, at any step t, $\mathcal{M}$ asks to the oracle about the integer t and writes down the oracle answer on the t-th cell of some dedicated tape.
Consider $t+1$ steps of the computation of $\mathcal{M}$ on input $\mathtt{p}$ with oracle $Approx(A',t)$ (cf. Lemma 6.14). We denote $\mathcal{C}_{\mathtt{p},t+1}$ this limited computation. We say that $\mathcal{C}_{\mathtt{p},t+1}$ halts if $\mathcal{M}$ (with that fake oracle) halts in at most $t+1$ steps.
We denote $output(\mathcal{C}_{\mathtt{p},t})$ the current value (which is in $\mathbb{Z}$) of the output tape after step t. The A-recursive definition of f, g is as follows.

i. $f(\mathtt{p}, 0) = g(\mathtt{p}, t) = 0$

ii. Suppose $Approx(A', t+1) \cap \{0, ..., t\} = Approx(A', t) \cap \{0, ..., t\}$. Then, up to the halting step of $\mathcal{C}_{\mathtt{p},t}$ or up to step t in case $\mathcal{C}_{\mathtt{p},t}$ does not halt, both computations $\mathcal{C}_{\mathtt{p},t}, \mathcal{C}_{\mathtt{p},t+1}$ are stepwise identical.

(a) If $\mathcal{C}_{\mathtt{p},t}$ halts then so does $\mathcal{C}_{\mathtt{p},t+1}$ at the same step. And both computations have the same output.
In that case, we set $f(\mathtt{p},t+1)=f(\mathtt{p},t)$, $g(\mathtt{p},t+1)=g(\mathtt{p},t)$.

(b) If $\mathcal{C}_{\mathtt{p},t}$ does not halt then let $\delta_{t+1}=output(\mathcal{C}_{\mathtt{p},t+1})-output(\mathcal{C}_{\mathtt{p},t})$, and set

$$\begin{array}{rcl} f(\mathtt{p},t+1) & = & f(\mathtt{p},t)+1+\max(0,\delta_{t+1}) \\ g(\mathtt{p},t+1) & = & g(\mathtt{p},t)+1+\max(0,-\delta_{t+1}) \end{array}$$

i.e. we add $|\delta_{t+1}|$ to f or g according to the sign of δ_{t+1}.

iii. Suppose $Approx(A',t+1)\cap\{0,...,t\}\neq Approx(A',t)\cap\{0,...,t\}$. Since these approximations are monotone increasing, we necessarily have $Approx(A',t)\cap\{0,...,t\}\neq A'\cap\{0,...,t+1\}$.
Thus, the fake oracle in $\mathcal{C}_{\mathtt{p},t}$ has given answers which are not compatible with A'. In that case, we set

$$\begin{array}{rcl} f(\mathtt{p},t+1) & = & f(\mathtt{p},t)+g(\mathtt{p},t)+1+\max(0,output(\mathcal{C}_{\mathtt{p},t+1})) \\ g(\mathtt{p},t+1) & = & f(\mathtt{p},t)+g(\mathtt{p},t)+1+\max(0,-output(\mathcal{C}_{\mathtt{p},t+1})) \end{array}$$

i.e. we uprise f,g to a common value (namely $f(\mathtt{p},t)+g(\mathtt{p},t)$) and then add $|output(\mathcal{C}_{\mathtt{p},t+1})|$ to f or g according to the sign of $output(\mathcal{C}_{\mathtt{p},t+1})$.

From the above inductive definition, we see that, for each $t>0$,

$$f(\mathtt{p},t)-g(\mathtt{p},t)=output(\mathcal{C}_{\mathtt{p},t})$$

Suppose $\varphi^{A'}(\mathtt{p})$ is defined.
Applying Lemmas 6.13, 6.14, we see that there exist $s_{\mathtt{p}}\leq t_{\mathtt{p}}$ such that
- $\mathcal{M}$, on input p, with oracle A', halts in $s_{\mathtt{p}}$ steps,
- $Approx(A',t_{\mathtt{p}})\cap\{0,...,t_{\mathtt{p}}\}=A'\cap\{0,...,t_{\mathtt{p}}\}$.
Thus, for all $t\geq t_{\mathtt{p}}$, $f_{\mathtt{p},t}=f_{\mathtt{p},t_{\mathtt{p}}}$ and $g_{\mathtt{p},t}=g_{\mathtt{p},t_{\mathtt{p}}}$ and $f_{\mathtt{p},t}-g_{\mathtt{p},t}=\varphi^{A'}(\mathtt{p})$.

Suppose $\varphi^{A'}(\mathtt{p})$ is not defined.
Observe that, each time the "fake" computation $\mathcal{C}_{\mathtt{p},t}$ with oracle $Approx(A',t)$ does not halt or appears not to be the "right" one with oracle A' (because $Approx(A',t+1)\cap\{0,...,t\}$ differs from $Approx(A',t)\cap\{0,...,t\}$), we strictly increase both f,g (this is why we put $+1$ in the equations of iib and iii).
Applying Lemmas 6.13, 6.14, we see that, if $\varphi^{A'}(\mathtt{p})$ is not defined then $\mathcal{C}_{\mathtt{p},t}$ does not halt for infinitely many t's, so that $f(\mathtt{p},t)$ and $g(\mathtt{p},t)$ increase infinitely often. Therefore, $(\max f)(\mathtt{p})$ and $(\max g)(\mathtt{p})$ are both undefined, and so is their difference.

This proves that $\varphi^{A'}=\max f-\max g$. Since the sequence $(Approx(A',t))_{t\in\mathbb{N}}$ is A-recursive, so are f,g. Thus, $\max f,\max g$ are in $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}$ and their difference $\varphi^{A'}$ is in $\Delta(Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A})$. □

7 Abstract representations and effectivizations

7.1 Some arithmetical representations of $\mathbb{N}$

As pointed in §1.1, abstract entities such as numbers can be represented in many different ways. In fact, each representation illuminates some particular role and/or property, i.e. some possible semantics chosen in order to efficiently access special operations or stress special properties of integers.

Usual arithmetical representations of $\mathbb{N}$ using words on a digit alphabet can be looked at as a (total) surjective (non necessarily injective) function $R : C \to \mathbb{N}$ where C is some simple free algebra or a quotient of some free algebra.
Such representations are the "degree zero" of abstraction for representations and, as expected, their associated Kolmogorov complexities all coincide (cf. Thm.7.8 below).

Example 7.1 (Base k representations)**.**
1. Integers in unary representation correspond to elements of the free algebra built up from one generator and one unary function, namely 0 and the successor function $x \mapsto x + 1$. The associated function $R : 1^* \to \mathbb{N}$ is simply the length function.

2. The various base k (with $k \geq 2$) representations of integers also involve term algebras, not necessarily free. They differ by the set $A \subset \mathbb{N}$ of digits they use but all are based on the usual interpretation $R : A^* \to \mathbb{N}$ such that $R(a_n \ldots a_1 a_0) = \sum_{i=0,\ldots,n} a_i k^i$. Which, written à la Hörner,

$$k(k(\ldots k(ka_n + a_{n-1}) + a_{n-2}) \ldots) + a_1) + a_0$$

is a composition of applications $S_{a_0} \circ S_{a_1} \circ \ldots \circ S_{a_n}(0)$ where $S_a : x \mapsto kx + a$. If a representation uses digits $a \in A$ then it corresponds to the algebra generated by 0 and the S_a's where $a \in A$.

i. The k-adic representation uses digits $1, 2, \ldots, k$ and corresponds to a free algebra built up from one generator and k unary functions.

ii. The usual k-ary representation uses digits $0, 1, \ldots, k - 1$ and corresponds to the quotient of a free algebra built up from one generator and k unary functions, namely 0 and the S_a's where $a = 0, 2, \ldots, k-1$, by the relation $S_0(0) = 0$.

iii. Avizienis base k representation uses digits $-k+1, \ldots, -1, 0, 1, \ldots, k-1$ (it is a much redundant representation used in computers to perform additions without carry propagation) and corresponds to the quotient of the free algebra built up from one generator and $2k - 1$ unary functions, namely 0 and the S_a's where $a = -k+1, \ldots, -1, 0, 1, \ldots, k-$

1, by the relations $\forall x\ (S_{-k+i} \circ S_{j+1}(x) = S_i \circ S_j(x))$ where $-k < j < k-1$ and $0 < i < k$.

Somewhat exotic representations of integers can also be associated to deep results in number theory.

Example 7.2.
1. $R : \mathbb{N}^4 \to \mathbb{N}$ such that $R(x, y, z, t) = x^2 + y^2 + z^2 + t^2$ is a representation based on Lagrange's four squares theorem.

2. $R : (Prime \cup \{0\})^7 \to \mathbb{N}$ such that $R(x_1, \ldots, x_i) = x_1 + \ldots + x_i$ is a representation based on Schnirelman's theorem (1931) in its last improved version obtained by Ramaré, 1995 [13], which insures that every even number is the sum of at most 6 prime numbers (hence every number is the sum of at most 7 primes).

Such representations appear in the study of the expressional power of some weak arithmetics. For instance, the representation as sums of 7 primes allows for a very simple proof of the definability of multiplication with addition and the divisibility predicate (a result valid in fact with successor and divisibility, (Julia Robinson, 1948 [14])).

7.2 Abstract representations

Foundational questions, going back to Russell, [16] 1908, and Church, [3] 1933, lead to quite different representations of $\mathbb{N}$: set theoretical representations involving abstract sets and functionals much more complex than the integers they represent.

We shall consider the following simple and general notion.

Definition 7.3 (Abstract representations)**.**
A representation of an infinite set E is a pair (C, R) where C is some (necessarily infinite) set and $R : C \to E$ is a *surjective partial* function.

Remark 7.4.
1. Though R really operates on the sole subset $domain(R)$, the underlying set C is quite significant in the effectivization process which is necessary to get a self-enumerated systen and then an associated Kolmogorov complexity.

2. We shall consider representations with arbitrarily complex domains in the Post hierarchy (cf. Prop.8.4, 9.3, 10.23, and coming papers). In fact, the sole cases in this paper where R is a total function are the usual recursive representations.

3. Representations can also involve a proper class C (cf. Rk. 8.3). However, we shall stick to the case C is a set.

7.3 Effectivizing representations: why?

Turning to a computer science (or recursion theoretic) point of view, there are some objections to the consideration of abstract sets, functions and functionals as we did in §1.1 and 7.2:

- We cannot apprehend abstract sets, functions and functionals but solely programs to compute them (if they are computable in some sense).
- Moreover, programs dealing with sets, functions and functionals have to go through some intensional representation of these objects in order to be able to compute with such objects.

To get effectiveness, we turn from set theory to computability theory. We shall do that in a somewhat abstract way using self-enumerated representation systems (cf. Def.2.1).
We shall consider higher order representations and shall "effectivize" abstract sets, functions and functionals via recursively enumerable sets, partial recursive functions or max of total or partial recursive functions, and partial computable functionals.

7.4 Effectivizations of representations and associated Kolmogorov complexities

A formal representation of an integer n is a finite object (in general a word) which describes some characteristic property of n or of some abstract object which characterizes n. To effectivize a representation $R : C \to E$, we shall process as follows:

1. Restrict the set C to a subfamily D of elements which, in some sense, are computable or partial computable. Of course, we want the restriction of R to D to be still surjective.
2. Consider a self-enumerated representation system for D.

This leads to the following definition.

Definition 7.5.
1. A set D is adapted to the representation $R : C \to E$ if $D \subseteq C$ and the partial function $R \restriction D : D \to E$ is still surjective.

2. [**Effectivization**] An effectivization of the representation $R : C \to E$ of the set E is any self-enumerated representation system $(D, \mathcal{F})$ for a domain D adapted to the representation $R : C \to E$.

Using the Composition Lemma 3.1, we immediately get

Proposition 7.6. *Let* $R : C \to E$ *be a representation of* E *and* $(D, \mathcal{F})$ *be some effectivization of* R. *Then* $(E, (R \upharpoonright D) \circ \mathcal{F})$ *is a self-enumerated representation system and the associated Kolmogorov complexity* $K^E_{(R \upharpoonright D)\circ\mathcal{F}}$ *(cf. Def.2.16) satisfies*

$$K^E_{(R \upharpoonright D)\circ\mathcal{F}}(x) = \min\{K^D_{\mathcal{F}}(y) : R(y) = x\} \quad \textit{for all } x \in E$$

Remark 7.7. Whereas abstract representations are quite natural and conceptually simple, the functions $(R \upharpoonright D) \circ F$, for $F \in \mathcal{F}$, in the self-enumerated representation families of their effectivized versions may be quite complex. In the examples we shall consider, their domains involve levels 2 or 3 of the arithmetical hierarchy. In particular, such representations are not Turing reducible one to the other.

7.5 Partial recursive representations

We already mentioned in §7.1 that all usual arithmetic representations lead to the same Kolmogorov complexity (up to an additive constant). The following result extends this assertion to all partial recursive representations.

Theorem 7.8. *We keep the notations of Notations 1.3 and Def.2.16.*
Let $A \subseteq \mathbb{N}$ *be an oracle. If* C, E *are basic sets and* $R : C \to E$ *is partial recursive (resp. partial* A*-recursive) then*

$$\begin{array}{lllll} R \circ PR^{\mathbf{2}^* \to C} & = & PR^{\mathbf{2}^* \to E} & \textit{(resp.} \quad R \circ PR^{A,\mathbf{2}^* \to C} & = \; PR^{A,\mathbf{2}^* \to E} \;) \\ K^E_{R \circ PR^{\mathbf{2}^* \to C}} & = & K_E & \textit{(resp.} \quad K^E_{R \circ PR^{A,\mathbf{2}^* \to C}} & = \; K^A_E \;) \end{array}$$

Thus, all Kolmogorov complexities associated to partial recursive (resp. partial A*-recursive) representations of* E *coincide with the usual (resp.* A*-oracular) Kolmogorov complexity on* E.

Proof. It suffices to prove that

$$R \circ PR^{A,\mathbf{2}^* \to C} = PR^{A,\mathbf{2}^* \to E}$$

Inclusion $R \circ PR^{A,\mathbf{2}^* \to C} \subseteq PR^{A,\mathbf{2}^* \to E}$ is trivial. For the other inclusion, we use the fact that $R : C \to E$ is surjective partial A-recursive.
First, define a partial A-recursive $S : E \to C$ such that, for $x \in E$, $S(\mathtt{x})$ is the element $\mathtt{y} \in C$ satisfying $R(\mathtt{y}) = \mathtt{x}$ which appears first in an A-recursive enumeration of the graph of R. Clearly, S is a right inverse of R, i.e. $R \circ S = Id_E$ where Id_E is the identity on E.
Using the trivial inclusion $S \circ PR^{A,\mathbf{2}^* \to E} \subseteq PR^{A,\mathbf{2}^* \to C}$ we get

$$PR^{A,\mathbf{2}^* \to E} = R \circ S \circ PR^{A,\mathbf{2}^* \to E} \subseteq R \circ PR^{A,\mathbf{2}^* \to C}$$

□

8 Cardinal representations of $\mathbb{N}$

8.1 Basic cardinal representation and its effectivizations

Among the conceptual representations of integers, the most basic one goes back to Russell, [16] 1908 (cf. [22] p.178), and considers non negative integers as equivalence classes of sets relative to cardinal comparison.

Definition 8.1 (Cardinal representation of $\mathbb{N}$)**.** Let $card(Y)$ denote the cardinal of Y, i.e. the number of its elements.
The cardinal representation of $\mathbb{N}$ relative to an infinite set X is the partial function

$$card_X : P(X) \to \mathbb{N}$$

with domain $P^{<\omega}(X)$, such that

$$card_X(Y) = \begin{cases} card(Y) & \text{if } Y \text{ is finite} \\ \text{undefined} & \text{otherwise} \end{cases}$$

Definition 8.2 (Effectivizations of the cardinal representation of $\mathbb{N}$)**.** We effectivize the cardinal representation by replacing $P(X)$ by $RE(\mathbb{X})$ or $RE^A(\mathbb{X})$ where $\mathbb{X}$ is some basic set and $A \subseteq \mathbb{N}$ is some oracle.
Two kinds of self-enumerated representation systems can be naturally associated to these domains (cf. §5.2 and the Composition Lemma 3.1):

$$\begin{array}{ccc} (RE(\mathbb{X}), card \circ \mathcal{F}^{RE(\mathbb{X})}) & \text{or} & (RE^A(\mathbb{X}), card \circ \mathcal{F}^{RE^A(\mathbb{X})}) \\ (RE(\mathbb{X}), card \circ \mathcal{PF}^{RE(\mathbb{X})}) & \text{or} & (RE^A(\mathbb{X}), card \circ \mathcal{PF}^{RE^A(\mathbb{X})}) \end{array}$$

Remark 8.3.
1. Historically, the cardinal representation of $\mathbb{N}$ considered the whole class of sets rather than some $P(X)$. However, the above effectivization makes such an extension unsignificant for our study.

2. One can also consider the total representation obtained by restriction to the set $P_{<\omega}(X)$ of all finite subsets of X. But this amounts to a partial recursive representation and is relevant to §7.5.

8.2 Syntactical complexity of cardinal representations

The following proposition gives the syntactical complexity of the above effectivizations of the cardinal representations.

Proposition 8.4 (Syntactical complexity)**.** *The family*

$$\{domain(\varphi) : \varphi \in card \circ \mathcal{F}^{RE^A(\mathbb{X})}\}$$

is exactly the family of $\Sigma^{0,A}_2$ subsets of $\mathbf{2}^*$. Idem with $card \circ \mathcal{PF}^{RE^A(\mathbb{X})}$.

In particular, any universal function for $card \circ \mathcal{F}^{RE^A(\mathbb{X})}$ or for $card \circ \mathcal{PF}^{RE^A(\mathbb{X})}$ is $\Sigma^{0,A}_2$-complete.

Proof. Let $(W^A_{\mathtt{e}})_{\mathtt{e}\in\mathbf{2}^*}$ be an acceptable enumeration of $RE^A(\mathbb{X})$.
1. If $g : \mathbf{2}^* \to \mathbf{2}^*$ is partial A-recursive then

$$domain(\mathtt{p} \mapsto card(W^A_{g(\mathtt{p})}) = \{\mathtt{p} : W^A_{g(\mathtt{p})} \text{ is finite}\}$$

is clearly $\Sigma^{0,A}_2$.

2. Let $X \subseteq \mathbf{2}^*$ be a $\Sigma^{0,A}_2$ set of the form $X = \{\mathtt{p} : \exists u\ \forall v\ R(\mathtt{p}, u, v)\}$ where $R \subseteq \mathbf{2}^* \times \mathbb{N}^2$ is A-recursive. Set

$$\sigma_{\mathtt{p}} \quad = \quad \begin{cases} \{u' : u' < u\} & \text{if } u \text{ is least such that } \forall v\ R(\mathtt{p}, u, v) \\ \mathbb{N} & \text{if there is no } u \text{ such that } \forall v\ R(\mathtt{p}, u, v) \end{cases}$$

It is easy to check that $\sigma_{\mathtt{p}} \subseteq \mathbb{N}$ is an A-r.e. set which can be defined by the following enumeration process described in Pascal-like instructions:

```
{Initialization}  u := 0;  v := 0;
{Loop}            DO FOREVER   BEGIN
                               WHILE R(p,u,v) DO v := v+1;
                               output u in σp;
                               u := u+1;  v := 0;
                               END;
```

Clearly, $card(\sigma_{\mathtt{p}})$ is finite if and only if $\mathtt{p} \in X$.

Now, the set $\{(\mathtt{p}, n) : n \in \sigma_{\mathtt{p}}\}$ is also A-r.e., hence of the form $W^{\mathbf{2}^*\times\mathbb{N}}_{\mathtt{a}}$ for some $\mathtt{a}$. The parameter property yields a total A-recursive function $g : \mathbf{2}^* \to \mathbf{2}^*$ such that $\sigma_{\mathtt{p}} = W_{g(\mathtt{a},\mathtt{p})}$. Finally, the function $\mathtt{p} \mapsto card(W_{g(\mathtt{a},\mathtt{p})})$ is in $card \circ \mathcal{F}^{RE^A(\mathbb{X})}$ and has domain X. $\square$

8.3 Characterization of the *card* self-enumerated systems

Theorem 8.5. *For any basic set $\mathbb{X}$ and any oracle $A \subseteq \mathbb{N}$,*

1i. $\quad card \circ \mathcal{F}^{RE^A(\mathbb{X})} \quad = \quad Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}$

ii. $\quad card \circ \mathcal{PF}^{RE^A(\mathbb{X})} \quad = \quad Max^{\mathbf{2}^*\to\mathbb{N}}_{PR^A}$

2. $\quad K^{\mathbb{N}}_{card\circ\mathcal{F}^{RE^A(\mathbb{X})}} \quad =_{\mathrm{ct}} \quad K^{\mathbb{N}}_{card\circ\mathcal{PF}^{RE^A(\mathbb{X})}} \quad =_{\mathrm{ct}} \quad K^A_{\max}$

We shall simply write $K^{\mathbb{N},A}_{card}$ in place of $K^{\mathbb{N}}_{card\circ\mathcal{F}^{RE^A(\mathbb{N})}}$.
When $A = \emptyset$ we simply write $K^{\mathbb{N}}_{card}$.

Proof. Point 2 is a direct corollary of Point 1 and Prop.6.6. Let's prove point 1.

1i. *Inclusion $\subseteq$.*
Let $g : \mathbf{2}^* \to \mathbf{2}^*$ be total A-recursive. We define a total A-recursive function $u : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ such that

$$(*) \quad \{u(\mathtt{p}, t) : t \in \mathbb{N}\} = \begin{cases} \{0, ..., n\} & \text{if } W^A_{g(\mathtt{p})} \text{ contains exactly } n \text{ points} \\ \mathbb{N} & \text{if } W^A_{g(\mathtt{p})} \text{ is infinite} \end{cases}$$

The definition is as follows. First, set $u(\mathtt{p},0)=0$ for all $\mathtt{p}$. Consider an A-recursive enumeration of $W^A_{g(\mathtt{p})}$. If at step t, some new point is enumerated then set $u(\mathtt{p},t+1)=u(\mathtt{p},t)+1$, else set $u(\mathtt{p},t+1)=u(\mathtt{p},t)$.

From $(*)$ we get $\mathit{card}(W_{\mathtt{p}})=(\max f)(\mathtt{p})$, so that $\mathtt{p}\mapsto \mathit{card}(W^A_{g(\mathtt{p})})$ is in $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}$.

1ii. *Inclusion* $\subseteq$.
Now $g:\mathbf{2}^*\to\mathbf{2}^*$ is partial A-recursive and we define $u:\mathbf{2}^*\times\mathbb{N}\to\mathbb{N}$ as a partial A-recursive function such that

$$\{u(\mathtt{p},t):t\in\mathbb{N}\}=\begin{cases}\emptyset & \text{if } g(\mathtt{p}) \text{ is undefined}\\ \{0,...,n\} & \text{if } W^A_{g(\mathtt{p})} \text{ contains exactly } n \text{ points}\\ \mathbb{N} & \text{if } W^A_{g(\mathtt{p})} \text{ is infinite}\end{cases}$$

The definition of u is as above except that, for any t, we require that $u(\mathtt{p},t)$ is defined if and only if $g(\mathtt{p})$ is.

1i. *Inclusion* $\supseteq$.
Any function in $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^A}$ is of the form $\max f:\mathbf{2}^*\to\mathbb{N}$ where $f:\mathbf{2}^*\times\mathbb{N}\to\mathbb{N}$ is total A-recursive.
The idea to prove that $\max f$ is in $\mathit{card}\circ\mathcal{F}^{RE^A(\mathbb{X})}$ is quite simple. For every $\mathtt{p}$, we define an A-r.e. subset of $\mathbb{X}$ which collects some new elements each time $f(\mathtt{p},t)$ gets greater than $\max\{f(\mathtt{p},t'):t'<t\}$.
Formally, let $\psi:\mathbf{2}^*\times\mathbb{N}\to\mathbb{N}$ be the partial A-recursive function such that

$$\psi(\mathtt{p},t)=\begin{cases}0 & \text{if } \exists u\ f(\mathtt{p},u)>t\\ \text{undefined} & \text{otherwise}\end{cases}$$

Clearly,

$$\mathit{domain}(\psi_{\mathtt{p}})=\begin{cases}\{t:0\le t<(\max f)(\mathtt{p})\} & \text{if } (\max f)(\mathtt{p}) \text{ is defined}\\ \mathbb{N} & \text{otherwise}\end{cases}$$

We define $\varphi:\mathbf{2}^*\times\mathbb{X}\to\mathbb{N}$ such that $\varphi(\mathtt{p},\mathtt{x})=\psi(\mathtt{p},\theta(\mathtt{x}))$ where $\theta:\mathbb{X}\to\mathbb{N}$ is some fixed total recursive bijection. Let's denote $\psi_{\mathtt{p}}$ and $\varphi_{\mathtt{p}}$ the functions $t\mapsto\psi(\mathtt{p},t)$ and $\mathtt{x}\mapsto\varphi(\mathtt{p},\mathtt{x})$. Let $\mathtt{e}$ be such that $W^A_{\mathtt{e}}=\{\langle\mathtt{p},\mathtt{x}\rangle:(\mathtt{p},\mathtt{x})\in \mathit{domain}(\varphi)\}$ (where $\langle\,,\rangle$ is a bijection $\mathbf{2}^*\times\mathbb{X}\to\mathbb{X}$). The parameter property yields an A-recursive function $s:\mathbf{2}^*\times\mathbf{2}^*\to\mathbf{2}^*$ such that $W^A_{s(\mathtt{e},\mathtt{p})}=\mathit{domain}(\varphi_{\mathtt{p}})$ for all $\mathtt{p}$. Thus, letting $g:\mathbf{2}^*\to\mathbf{2}^*$ be the A-recursive function such that $g(\mathtt{p})=s(\mathtt{e},\mathtt{p})$, we have

$$\mathit{card}(W^A_{g(\mathtt{p})})=\mathit{card}(\mathit{domain}(\varphi_{\mathtt{p}}))=\mathit{card}(\mathit{domain}(\psi_{\mathtt{p}}))=(\max f)(\mathtt{p})$$

Which proves that $\max f$ is in $\mathit{card}\circ\mathcal{F}^{RE^A(\mathbb{X})}$.

1ii. *Inclusion* $\supseteq$.

We argue as in the above proof of **i.** $\supseteq$. However, $f : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ is now partial A-recursive and there are two reasons for which $(\max f)(\mathtt{p})$ may be undefined: first, if $t \mapsto f(\mathtt{p}, t)$ is unbounded, second if it has empty domain. Keeping ψ and φ as defined as above, we now have,

$$domain(\psi_{\mathtt{p}}) = \begin{cases} \{v : 0 \leq v < (\max f)(\mathtt{p})\} & \text{if } (\max f)(\mathtt{p}) \text{ is defined} \\ \mathbb{N} & \text{if } range(t \mapsto f(\mathtt{p}, t)) \text{ is infinite} \\ \emptyset & \text{if } f(\mathtt{p}, t) \text{ is defined for no } t \end{cases}$$

We let $\mathtt{e}, s, g$ be as above and define $h : \mathbf{2}^* \to \mathbf{2}^*$ such that

$$h(\mathtt{p}) = \begin{cases} g(\mathtt{p}) & \text{if } f(\mathtt{p}, t) \text{ is defined for some } t \\ \text{undefined} & \text{otherwise} \end{cases}$$

Observe that
- if $t \mapsto f(\mathtt{p}, t)$ has empty domain then $h(\mathtt{p})$ is undefined,
- if $t \mapsto f(\mathtt{p}, t)$ is unbounded then $card(W^A_{h(\mathtt{p})}) = card(W^A_{g(\mathtt{p})})$ is infinite,
- otherwise $card(W^A_{h(\mathtt{p})}) = card(W^A_{g(\mathtt{p})}) = (\max f)(\mathtt{p})$.

Which proves that $\max f$ is in $card \circ \mathcal{PF}^{RE^A(\mathbb{X})}$. □

8.4 Characterization of the $\Delta card$ representation system

We now look at the self-delimited system with domain $\mathbb{Z}$ obtained from $card \circ \mathcal{F}^{RE^A(\mathbb{X})}$ by the operation Δintroduced in §4.1.

Theorem 8.6. *Let $A \subseteq \mathbb{N}$ and let A' be the jump of A. Let $\mathbb{X}$ be a basic set. Then*

$$\Delta(card \circ \mathcal{F}^{RE^A(\mathbb{X})}) = \Delta(card \circ \mathcal{PF}^{RE^A(\mathbb{X})}) = PR^{A', \mathbf{2}^* \to \mathbb{Z}}$$

Hence $K^{\mathbb{Z}}_{\Delta(card \circ \mathcal{F}^{RE^A(\mathbb{X})})} =_{\mathrm{ct}} K^{A', \mathbb{Z}}$.

We shall simply write $K^{\mathbb{N}, A}_{\Delta card}$ *in place of* $K^{\mathbb{Z}}_{\Delta(card \circ \mathcal{F}^{RE^A(\mathbb{N})})} \upharpoonright \mathbb{N}$.
When $A = \emptyset$ we simply write $K^{\mathbb{Z}}_{\Delta card}$.

Proof. The equalities about the self-enumerated systems is a direct corollary of Thm.8.5 and Thm.6.12. The equalities about Kolmogorov complexities are trivial corollaries of those about self-enumerated systems. □

9 Index representations of $\mathbb{N}$

9.1 Basic index representation and its effectivizations

A variant of the cardinal representation considers indexes of equivalence relations. More precisely, it views an integer as an equivalence class of equivalence relations relative to index comparison.

Definition 9.1 (Index representation)**.**
The index representation of $\mathbb{N}$ relative to an infinite set X is the partial function

$$index^{\mathbb{N}}_{P(X^2)} : P(X^2) \to \mathbb{N}$$

with domain the family of equivalence relations on subsets of X which have finite index, such that

$$index^{\mathbb{N}}_{P(X^2)}(R) \quad = \quad \begin{cases} index(R) & \text{if } R \text{ is an equivalence relation} \\ & \text{with finite index} \\ \text{undefined} & \text{otherwise} \end{cases}$$

(where $index(R)$ denotes the number of equivalence classes of R).

9.2 Syntactical complexity of index representations

Definition 9.2 (Effectivization of the index representation of $\mathbb{N}$)**.** We effectivize the index representation by replacing $P(X^2)$ by $RE(\mathbb{X}^2)$ or $RE^A(\mathbb{X}^2)$ where $\mathbb{X}$ is some basic set and $A \subseteq \mathbb{N}$ is some oracle.
Two kinds of self-enumerated representation systems can be naturally associated (cf. §5.2 and the Composition Lemma 3.1):

$$\begin{array}{ccc} (RE(\mathbb{X}^2), index \circ \mathcal{F}^{RE(\mathbb{X}^2)}) & \text{or} & (RE^A(\mathbb{X}^2), index \circ \mathcal{F}^{RE^A(\mathbb{X}^2)}) \\ (RE(\mathbb{X}^2), index \circ \mathcal{PF}^{RE(\mathbb{X}^2)}) & \text{or} & (RE^A(\mathbb{X}^2), index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}) \end{array}$$

The following proposition gives the syntactical complexity of the above effectivizations of the index representations.

Proposition 9.3 (Syntactical complexity)**.** *The family*

$$\{domain(\varphi) : \varphi \in index \circ \mathcal{F}^{RE^A(\mathbb{X})}\}$$

is exactly the family of $\Sigma^{0,A}_3$ subsets of $\mathbf{2}^*$.
Idem with $index \circ \mathcal{PF}^{RE^A(\mathbb{X})}$.

In particular, any universal function for $index \circ \mathcal{F}^{RE^A(\mathbb{X})}$ or for $index \circ \mathcal{PF}^{RE^A(\mathbb{X})}$ is $\Sigma^{0,A}_3$-complete.

Proof. We trivially reduce to the case $\mathbb{X} = \mathbb{N}$ and only consider the case $A = \emptyset$, relativization being straightforward.

1. Let $(W^{\mathbb{N}^2}_{\mathtt{e}})_{\mathtt{e} \in \mathbf{2}^*}$ be an acceptable enumeration of $RE(\mathbb{N}^2)$ and $g : \mathbf{2}^* \to \mathbf{2}^*$ be a partial recursive function and $\psi : \mathbf{2}^* \to \mathbb{N}$ be such that $\psi(\mathtt{p}) = index(W^{\mathbb{N}^2}_{g(\mathtt{p})})$.
To see that $domain(\psi)$ is Σ^0_3, observe that $\mathtt{p} \in domain(\psi)$ if and only if

 i. $g(\mathtt{p})$ is defined. Which is a Σ^0_1 condition.

ii. $W^{\mathbb{N}^2}_{g(\mathtt{p})}$ is an equivalence relation on its domain, i.e.

$$\forall \mathtt{x}\ \forall \mathtt{y}\ ((\mathtt{x},\mathtt{y}) \in W^{\mathbb{N}^2}_{g(\mathtt{p})} \ \Rightarrow\ ((\mathtt{x},\mathtt{x}) \in W^{\mathbb{N}^2}_{g(\mathtt{p})} \ \wedge\ (\mathtt{y},\mathtt{x}) \in W^{\mathbb{N}^2}_{g(\mathtt{p})}))$$
$$\wedge\ \forall \mathtt{x}\ \forall \mathtt{y}\ \forall \mathtt{z}\ (((\mathtt{x},\mathtt{y}) \in W^{\mathbb{N}^2}_{g(\mathtt{p})} \ \wedge\ (\mathtt{y},\mathtt{z}) \in W^{\mathbb{N}^2}_{g(\mathtt{p})}) \ \Rightarrow\ (\mathtt{x},\mathtt{z}) \in W^{\mathbb{N}^2}_{g(\mathtt{p})})$$

Which is a Π^0_2 formula (since $(\mathtt{u},\mathtt{v}) \in W^{\mathbb{N}^2}_{g(\mathtt{p})}$ is Σ^0_1).

iii. $W^{\mathbb{N}^2}_{g(\mathtt{p})}$ has finitely many classes, i.e. $\exists n\ \forall k\ \exists m \leq n\ (k,m) \in W^{\mathbb{N}^2}_{g(\mathtt{p})}$. Which is a Σ^0_3 formula.

2. Let $X \subseteq \mathbf{2}^*$ be Σ^0_3. We construct a total recursive function $g : \mathbf{2}^* \to \mathbf{2}^*$ such that $X = \{\mathtt{p} : \mathit{index}(W^{\mathbb{N}^2}_{g(\mathtt{p})})$ is finite$\}$.

A. Suppose $X = \{\mathtt{p} : \exists u\ \forall v\ \exists w\ R(\mathtt{p},u,v,w)\}$ where $R \subseteq \mathbf{2}^* \times \mathbb{N}^3$ is recursive. Let $\theta : \mathbf{2}^* \times \mathbb{N}^2 \to \mathbb{N}$ be the total recursive function such that

$$\theta(\mathtt{p},u,t) \quad = \quad \text{largest } v \leq t \text{ such that } \forall v' \leq v\ \exists w \leq t\ R(\mathtt{p},u,v',w)\}$$

Observe that θ is monotone increasing with respect to t. Also,

($*$) if $\mathtt{p} \notin X$ then, for all u, $\max_{t\in\mathbb{N}} \theta(\mathtt{p},u,t)$ is finite,

($**$) if $\mathtt{p} \in X$ and u is least such that $\forall v\ \exists w\ R(\mathtt{p},u,v,w)$ then

$$\begin{cases} \max_{t\in\mathbb{N}} \theta(\mathtt{p},u,t) & = \ +\infty \\ \max_{t\in\mathbb{N}} \theta(\mathtt{p},u',t) & \text{is finite} \quad \text{for all } u' < u \end{cases}$$

Following this observation, given $\mathtt{p} \in \mathbf{2}^*$, we define a monotone increasing sequence of equivalence relations $\rho^t_{\mathtt{p}}$ on finite initial intervals of $\mathbb{N}$ such that $\rho^t_{\mathtt{p}}$ has $t+1$ equivalence classes

$$I^t_{\mathtt{p},0}\ ,\ I^t_{\mathtt{p},1}\ ,\ \ldots\ ,\ I^t_{\mathtt{p},t}$$

which are successive finite intervals

$$[0,n^t_{\mathtt{p},0}]\ ,\ [n^t_{\mathtt{p},0}+1,n^t_{\mathtt{p},1}]\ ,\ [n^t_{\mathtt{p},1}+1,n^t_{\mathtt{p},2}]\ ,\ \ldots\ ,\ [n^t_{\mathtt{p},t-1}+1,n^t_{\mathtt{p},t}]$$

where $n^t_{\mathtt{p},1} < n^t_{\mathtt{p},2} < \ldots < n^t_{\mathtt{p},t-1} < n^t_{\mathtt{p},t}$.
The intuition is as follows:

i. the class $I^t_{\mathtt{p},u}$ is related to $\theta(\mathtt{p},u,t)$, i.e. to the best we can say at step t about the truth value of $\forall v\ \exists w\ R(\mathtt{p},u,v,w)$.

ii. if and when $\theta(\mathtt{p},u,t)$ increases, i.e. $\theta(\mathtt{p},u,t+1) > \theta(\mathtt{p},u,t)$ for some u, then we increase the class $I^t_{\mathtt{p},u}$ for the least such u.

Of course, an equivalence class which grows and remains an interval either is the rightmost one or has to aggregate some of its neighbor class(es). Whence the following inductive definition of the $\rho^t_{\mathtt{p}}$'s and $n^t_{\mathtt{p},u}$'s, $u \leq t$:

i. *(Base case).* $\rho^0_{\mathtt{p}}$ is the equivalence relation with one class $\{0\}$, i.e. $n^0_{\mathtt{p}}, 0 = 0$.

ii. *(Inductive case. Subcase 1).* Suppose $\theta(\mathtt{p}, u, t+1) = \theta(\mathtt{p}, u, t)$ for all $u \leq t$. Then $\rho^{t+1}_{\mathtt{p}}$ is obtained from $\rho^t_{\mathtt{p}}$ by adding a new singleton class on the right:

  (a) For all $u \leq t$ we let $n^{t+1}_{\mathtt{p},u} = n^t_{\mathtt{p},u}$, hence $I^{t+1}_{\mathtt{p},u} = I^t_{\mathtt{p},u}$.

  (b) $n^{t+1}_{\mathtt{p},t+1} = n^t_{\mathtt{p},t} + 1$, hence $I^{t+1}_{\mathtt{p},t+1} = \{n^t_{\mathtt{p},t} + 1\}$.

ii. *(Inductive case. Subcase 2).* Suppose $\theta(\mathtt{p}, u, t+1) > \theta(\mathtt{p}, u, t)$ for some $u \leq t$. Let u be least such. Then,

  (a) for $u' < u$, classes $I^t_{\mathtt{p},u'}$ are left unchanged: $n^{t+1}_{\mathtt{p},u'} = n^t_{\mathtt{p},u'}$ and $I^{t+1}_{\mathtt{p},u'} = I^t_{\mathtt{p},u'}$,

  (b) class $I^{t+1}_{\mathtt{p},u}$ aggregates all classes $I^t_{\mathtt{p},u''}$ for $u \leq u'' \leq t$,

  (c) $t + 1 - u$ singleton classes are added: $I^{t+1}_{\mathtt{p},u+i} = \{n^t_{\mathtt{p},t} + i\}$ where $i = 1, ..., t + 1 - u$. I.e.

$$\begin{array}{rcll} n^{t+1}_{\mathtt{p},u'} & = & n^t_{\mathtt{p},u} & \text{for all } u' \leq u \\ n^{t+1}_{\mathtt{p},u+i} & = & n^t_{\mathtt{p},t} + i & \text{for all } s \in \{i, ..., t+1-u\} \end{array}$$

B. Let $\rho_{\mathtt{p}} = \bigcup_{t \in \mathbb{N}} \rho_{\mathtt{p},t}$.

$\boxed{\text{Case } \mathtt{p} \in X}$. Let u be least such that $\forall v\ \exists w\ R(\mathtt{p}, u, v, w)$. For $u' < u$, let

$$\begin{array}{rcl} V_{u'} & = & \max\{v : \forall v' \leq v\ \exists w\ R(\mathtt{p}, u', v', w)\} \\ t & = & \min\{t' : \forall u' < u\ (V_{u'} \leq t' \ \wedge\ \forall v' \leq V_{u'}\ \exists w \leq t'\ R(\mathtt{p}, u', v', w)\} \end{array}$$

Then

- $\forall u' < u\ \forall v\ (\forall v' \leq v\ \exists w\ R(\mathtt{p}, u', v', w) \ \Rightarrow$
  $$(v \leq t\ \wedge\ \forall v' \leq v\ \exists w' \leq t\ R(\mathtt{p}, u', v', w')))$$
- $n^{t'}_{\mathtt{p},u'}, = n^t_{\mathtt{p},u'}$ and $I^{t'}_{\mathtt{p},u'} = I^t_{\mathtt{p},u'}$ for all $u' < u$ and $t' \geq t$.
- $n^{t'}_{\mathtt{p},u}$ tends to $+\infty$ with t' and $I^{t'}_{\mathtt{p},u} = [n^{t'}_{\mathtt{p},u-1} + 1, n^{t'}_{\mathtt{p},u}]$ tends to the cofinite interval $[n^t_{\mathtt{p},u-1} + 1, +\infty[$.
- for $u'' > u$, classes $I^{t'}_{\mathtt{p},u''}$ are intervals the left endpoints of which tend to $+\infty$ with t', hence they vanish at infinity.

Thus, $\rho_{\mathtt{p}}$, which is the limit of the $\rho^t_{\mathtt{p}}$'s, has $u + 1$ classes, hence has finite index.

$\boxed{\text{Case } \mathtt{p} \notin X}$. For every $u \in \mathbb{N}$, the class $I^t_{\mathtt{p},u}$ stabilizes as t tends to $+\infty$. Thus, $\rho_{\mathtt{p}}$ has infinite index.

C. Clearly, the sequence $(\rho^t_{\mathtt{p}})_{\mathtt{p}\in\mathbf{2}^*,t\in\mathbb{N}}$ is recursive. Thus,

$$\rho = \{(\mathtt{p}, m, n) : \exists t\ (m, n) \in \rho^t_{\mathtt{p}}\}$$

is r.e. Let $\mathtt{a} \in \mathbf{2}^*$ be such that $\rho = W^{\mathbf{2}^*\times\mathbb{N}^2}_{\mathtt{a}}$. Applying the parametrization property, let $s : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ be a total recursive function such that

$$\rho_{\mathtt{p}} = \{(m, n) \in \mathbb{N}^2 : (\mathtt{p}, m, n) \in W^{\mathbf{2}^*\times\mathbb{N}^2}_{\mathtt{a}}\} = W^{\mathbb{N}^2}_{s(\mathtt{a},\mathtt{p})}$$

Let $g : \mathbf{2}^* \to \mathbf{2}^*$ be total recursive such that $g(\mathtt{p}) = s(\mathtt{a}, \mathtt{p})$. Using point B, we see that $\mathtt{p} \in X$ if and only if $\mathit{index}(W^{\mathbb{N}^2}_{g(\mathtt{p})})$ is finite. □

9.3 Characterization of the *index* self-enumerated systems

We now come to the characterization of the index self-enumerated families. It turns out that these families are almost equal to $Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^{A'}}$, almost meaning here "up to 1".

Notation 9.4. If $\mathcal{G}$ is a family of functions $\mathbf{2}^* \to \mathbb{N}$, we let

$$\mathcal{G} + 1 = \{f + 1 : f \in \mathcal{G}\}$$

Theorem 9.5.
1. *For any basic set $\mathbb{X}$ and any oracle $A \subseteq \mathbb{N}$, the following* strict *inclusions hold:*

$$Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^{A'}} + 1 \subset \mathit{index} \circ \mathcal{F}^{RE^A(\mathbb{X}^2)} \subset \mathit{index} \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)} \subset Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^{A'}}$$

2. $K^{\mathbb{N}}_{\mathit{index}\circ\mathcal{F}^{RE^A(\mathbb{X}^2)}} =_{\mathrm{ct}} K^{\mathbb{N}}_{\mathit{index}\circ\mathcal{PF}^{RE^A(\mathbb{X}^2)}} =_{\mathrm{ct}} K^{A'}_{\max}$.

We shall simply write $K^{\mathbb{N},A}_{\mathit{index}}$ in place of $K^{\mathbb{N}}_{\mathit{index}\circ\mathcal{F}^{RE^A(\mathbb{N})}}$.
When $A = \emptyset$ we simply write $K^{\mathbb{N}}_{\mathit{index}}$.

Proof. Observe that if $\mathcal{F}$ is a self-enumerated system with domain D and with U as a good universal function, then $\mathcal{F} + 1$ is also a self-enumerated system with $U + 1$ as a good universal function. In particular $K^D_{\mathcal{F}} = K^D_{\mathcal{F}+1}$.
Point 2 is a direct corollary of Point 1 and Prop.6.6 and the previous observation.

Let's prove point 1.
The central inclusion $\mathit{index} \circ \mathcal{F}^{RE^A(\mathbb{X}^2)} \subset \mathit{index} \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}$ is trivial.

A. *Non strict inclusion* $\mathit{index} \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)} \subseteq Max^{\mathbf{2}^*\to\mathbb{N}}_{Rec^{A'}}$.
Let $G \in \mathit{index} \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}$ and let $g : \mathbf{2}^* \to \mathbf{2}^*$ be partial A-recursive such that

$$G(\mathtt{p}) = \begin{cases} \mathit{index}(W^{A,\mathbb{X}^2}_{g(\mathtt{p})}) & \text{if } g(\mathtt{p}) \text{ is defined and } W^{A,\mathbb{X}^2}_{g(\mathtt{p})} \text{ is an} \\ & \text{equivalence relation with finite index} \\ \text{undefined} & \text{otherwise} \end{cases}$$

We define a total A'-recursive function $u : \mathbf{2}^* \times \mathbb{N} \to \mathbb{N}$ such that

$$(*) \quad \{u(\mathtt{p}, t) : t \in \mathbb{N}\} = \begin{cases} \{0, ..., n\} & \text{if } G(\mathtt{p}) \text{ is defined and } G(\mathtt{p}) = n \\ \mathbb{N} & \text{if } G(\mathtt{p}) \text{ is undefined} \end{cases}$$

The definition is as follows. Since g is partial A-recursive and we look for an A'-recursive definition of $u(\mathtt{p}, t)$, we can use oracle A' to check if $g(\mathtt{p})$ is defined.
If $g(\mathtt{p})$ is undefined then we let $u(\mathtt{p}, t) = t$ for all t. Which insures $(*)$.
Suppose now that $g(\mathtt{p})$ is defined. First, set $u(\mathtt{p}, 0) = 0$.
Consider an A-recursive enumeration of $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$. Let R_t be the set of pairs enumerated at steps $< t$ and D_t be the set of $\mathtt{x} \in \mathbb{X}$ which appear in pairs in R_t (so that R_0 and D_0 are empty). Since at most one new pair is enumerated at each step, the set R_t contains at most t pairs and D_t contains at most $2\,t$ points.
At step $t + 1$, use oracle A' to check the following properties:

α_t. For every $\mathtt{x} \in D_{t+1}$ the pair $(\mathtt{x}, \mathtt{x})$ is in $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$.

β_t. For every pair $(\mathtt{x}, \mathtt{y}) \in R_{t+1}$ the pair $(\mathtt{y}, \mathtt{x})$ is in $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$.

γ_t. For every pairs $(\mathtt{x}, \mathtt{y}), (\mathtt{y}, \mathtt{z}) \in R_{t+1}$ the pair $(\mathtt{x}, \mathtt{z})$ is in $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$.

δ_t. For every $\mathtt{x} \in D_{t+1}$ there exists $\mathtt{y} \in D_t$ such that the pair $(\mathtt{x}, \mathtt{y})$ is in $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$.

Since R_{t+1}, D_{t+1} are finite, all these properties α_t-δ_t are *finite* boolean combinations of $\Sigma^{0,A}_1$ statements. Hence oracle A' can decide them all.

Observe that if $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$ is an equivalence relation then answers to α_t-γ_t are positive for all t. And if $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$ is not an equivalence relation then, for some $\pi \in \{\alpha, \beta, \gamma\}$, answers to π_t are negative for all t large enough .
Also, if $W^{A,\mathbb{X}^2}_{g(\mathtt{p})}$ is an equivalence relation then a new equivalence class is revealed each time δ_t is false. And every equivalence class is so revealed.

Thus, in case $g(\mathtt{p})$ is defined, we insure $(*)$ by letting

$$u(\mathtt{p}, t + 1) = \begin{cases} u(\mathtt{p}, t) & \text{if all answers to } \alpha_t\text{-}\delta_t \text{ are positive} \\ u(\mathtt{p}, t) + 1 & \text{otherwise} \end{cases}$$

From $(*)$, we get $G = \max u$. Since u is total A'-recursive, this proves that G is in $Max^{\mathbf{2}^* \to \mathbb{X}}_{Rec^{A'}}$.

B. *Non strict inclusion* $Max^{\mathbf{2}^* \to \mathbb{N}}_{Rec^{A'}} + 1 \subseteq index \circ \mathcal{F}^{RE^A(\mathbb{X}^2)}$.
We reduce to the case $\mathbb{X} = \mathbb{N}$.
Let $F \in Max^{\mathbf{2}^* \to \mathbb{N}}_{PR^{A'}}$. Using Prop.6.9, let $\mathcal{M}$ be an oracle Turing machine

which on input $\mathtt{p}$ and oracle A' computes $F(\mathtt{p})$ through an infinite computation.
The idea to prove that F is in $index \circ \mathcal{F}^{RE^A(\mathbb{N}^2)}$ is as follows. We consider A-recursive approximations of oracle A' and use them as fake oracles. For each $\mathtt{p}$ we build an A-r.e. equivalence relation $\rho_{\mathtt{p}} \subseteq \mathbb{N}^2$ with domain $\mathbb{N}$ which consists of one big class containing 0 and some singleton classes. Each time the computation with the fake oracle outputs a new digit 1, we put some new singleton class in $\rho_{\mathtt{p}}$. When, with a better approximation of A', we see that the fake oracle has given an incorrect answer, all singleton classes which were put in $\rho_{\mathtt{p}}$ because of the oracle incorrect answer are annihilated: they are aggregated to the class of 0. Since we are going to consider $index(\rho_{\mathtt{p}})$, this process will lead to the correct value $F(\mathtt{p})+1$.

Formally, we consider an A-recursive monotone increasing sequence $(Approx(A',t))_{t\in\mathbb{N}}$ such that $A' = \bigcup_{t\in\mathbb{N}} Approx(A',t)$ (cf. Lemma 6.14). Though all oracles $Approx(A',t)$ are false approximations of oracle A', they are nevertheless “less and less false” as t increases.

Without loss of generality, we can suppose that at each computation step of $\mathcal{M}$ there is a question to the oracle (possibly the same one many times).

Let $\mathcal{C}_{\mathtt{p},t}$ be the computation of $\mathcal{M}$ on input $\mathtt{p}$ with oracle $Approx(A',t)$, reduced to the sole t first steps.
Increasing parts of oracle $Approx(A',t)$ are questioned during $\mathcal{C}_{\mathtt{p},t}$. Let $\Omega_{\mathtt{p},t} : \{1,...,t\} \to P_{fin}(\mathbb{N})$ (where $P_{fin}(\mathbb{N})$ is the set of finite subsets of $\mathbb{N}$) be such that $\Omega_{\mathtt{p},t}(t')$ is the set of k such that the oracle has been questioned about k during the t' first steps, $1 \leq t' \leq t$. Clearly, $\Omega_{\mathtt{p},t}$ is (non strictly) monotone increasing with respect to set inclusion.
Let $1^{n_{\mathtt{p},t}}$ be the output of $\mathcal{C}_{\mathtt{p},t}$ (recall that $\mathcal{M}$ outputs a finite or infinite sequence of digits 1's).
The successive digits of this output are written down at increasing times (all $\leq t$). Let $OT_{\mathtt{p},t} : \{0,...,n_{\mathtt{p},t}\} \to \{0,...,t\}$ be such that $OT_{\mathtt{p},t}(n)$ is the least step at which the current output is 1^n (OT stands for output time). Clearly, $OT_{\mathtt{p},t}(0) = 0$.

We construct A-recursive sequences $(\rho_{\mathtt{p},t})_{\mathtt{p}\in\mathbf{2}^*,t\in\mathbb{N}}$ and $(w_{\mathtt{p},t})_{\mathtt{p}\in\mathbf{2}^*,t\in\mathbb{N}}$ (where w stands for witness) such that

- i_t. $\rho_{\mathtt{p},t}$ is an equivalence relation on $\{0,...,2^t-1\}$ with index equal to $1+n_{\mathtt{p},t}$ (there is nothing essential with 2^t, it is merely a large enough bound convenient for the construction),
- ii_t. all equivalence classes of $\rho_{\mathtt{p},t}$ are singleton sets except possibly the equivalence class of 0.
- iii_t. if $t>0$ then $\rho_{\mathtt{p},t}$ contains $\rho_{\mathtt{p},t-1}$.

iv_t. $w_{\mathtt{p},t}$ is a bijection between $\{1, ..., n_{\mathtt{p},t}\}$ and the set of point $s \in \{1, ..., 2^t - 1\}$ such that $\{s\}$ is a singleton class of $\rho_{\mathtt{p},t}$ (in case $n_{\mathtt{p},t} = 0$ then $w_{\mathtt{p},t}$ is the empty map).

First, $w_{\mathtt{p},0}$ is the empty map and $\rho_{\mathtt{p},0} = \{(0,0)\}$, i.e. the trivial equivalence relation on $\{0\}$.

The inductive construction of the $\rho_{\mathtt{p},t}$'s uses the above conditions i_t-iv_t as an induction hypothesis.

Case $Approx(A', t+1) \cap \Omega_{\mathtt{p},t}(t) = Approx(A', t) \cap \Omega_{\mathtt{p},t}(t)$.
Then the computation $\mathcal{C}_{\mathtt{p},t}$ is totally compatible with $\mathcal{C}_{\mathtt{p},t+1}$. Now, that last computation may possibly output one more digit 1, i.e. $n_{\mathtt{p},t+1} = n_{\mathtt{p},t}$ or $n_{\mathtt{p},t+1} = n_{\mathtt{p},t} + 1$. Hence the two following subcases.

Subcase $n_{\mathtt{p},t+1} = n_{\mathtt{p},t}$. Then $\rho_{\mathtt{p},t+1}$ is obtained from $\rho_{\mathtt{p},t}$ by putting $2^t, 2^t + 1, ..., 2^{t+1} - 1$ as new points in the class of 0. In particular, $\rho_{\mathtt{p},t+1}$ and $\rho_{\mathtt{p},t}$ have the same index. We also set $w_{\mathtt{p},t+1} = w_{\mathtt{p},t}$.

Subcase $n_{\mathtt{p},t+1} = n_{\mathtt{p},t} + 1$. Then $\rho_{\mathtt{p},t+1}$ is obtained from $\rho_{\mathtt{p},t}$ as follows:

- Add a new singleton class $\{2^t\}$.
- Put $2^t + 1, ..., 2^{t+1} - 1$ as new points in the class of 0.

We also set $w_{\mathtt{p},t+1} = w_{\mathtt{p},t} \cup \{(n_{\mathtt{p},t+1}, 2^t)\}$.

In both subcases, conditions i_{t+1}-iv_{t+1} are clearly satisfied.

Case $Approx(A', t+1) \cap \Omega_{\mathtt{p},t}(t) \neq Approx(A', t) \cap \Omega_{\mathtt{p},t}(t)$.
Let $\tau \leq t$ be least such that $Approx(A', t+1) \cap \Omega_{\mathtt{p},t}(\tau) \neq Approx(A', t) \cap \Omega_{\mathtt{p},t}(\tau)$. Though the computation $\mathcal{C}_{\mathtt{p},t}$ is not entirely compatible with $\mathcal{C}_{\mathtt{p},t+1}$, it is compatible up to step $\tau - 1$.
Let $n \leq n_{\mathtt{p},t}$ be greatest such that $OT_{\mathtt{p},t}(n) < \tau$. Then the n first digits output by $\mathcal{C}_{\mathtt{p},t}$ are also output by $\mathcal{C}_{\mathtt{p},t+1}$ at the same computation steps. In particular, $n_{\mathtt{p},t+1} \geq n$.
Then $\rho_{\mathtt{p},t+1}, w_{\mathtt{p},t+1}$ are obtained from $\rho_{\mathtt{p},t}, w_{\mathtt{p},t}$ as follows:

- Put all $w_{\mathtt{p},t}(m)$, where $n < m \leq n_{\mathtt{p},t}$, as new points in the class of 0. This annihilates the singleton classes of $\rho_{\mathtt{p},t}$ corresponding (via $w_{\mathtt{p},t}(m)$) to the part of the output which was created by answers of oracle $Approx(A', t)$ which are known to be false at step $t+1$.
- Add a new singleton class $\{2^t - 1 + i\}$ for each $i > 0$ such that $n + i \leq n_{\mathtt{p},t+1}$. Together with the singleton classes of $\rho_{\mathtt{p},t}$ which have not been aggragated by the above point, this allows to get exactly $n_{\mathtt{p},t+1}$ singleton classes in $\rho_{\mathtt{p},t+1}$
  Accordingly, set
  $$w_{\mathtt{p},t+1} = (w_{\mathtt{p},t} \upharpoonright \{1, ..., n\}) \ \cup \ \{(n+i, 2^t - 1 + i) : 0 < i \leq n_{\mathtt{p},t+1} - n\}$$

- Put the $2^t - 1 + j$'s, where $j \geq \max(1, n_{\mathtt{p},t+1} - n)$, as new points in the class of 0.

Again, conditions i_{t+1}-iv_{t+1} are clearly satisfied.

Let $\rho_{\mathtt{p}} = \bigcup_{t \in \mathbb{N}} \rho_{\mathtt{p},t}$. Condition iii_t insures that $\rho_{\mathtt{p}}$ is also an equivalence relation. Condition ii_t goes through the limit when $t \to +\infty$, so that all classes of $\rho_{\mathtt{p}}$ are singleton sets except the class of 0.

The computation we are really interesting in is that which gives $F(\mathtt{p})$, i.e. the infinite computation of $\mathcal{M}$ on input $\mathtt{p}$ with oracle A'. Let denote it $\mathcal{C}_{\mathtt{p}}$. When t increases, the common part of $\mathcal{C}_{\mathtt{p}}$ with computation $\mathcal{C}_{\mathtt{p},t}$ gets larger and larger (though not monotonously).

We now prove the equality

$$(\dagger) \quad index(\rho_{\mathtt{p}}) = \begin{cases} 1 + F(\mathtt{p}) & \text{if } F(\mathtt{p}) \text{ is defined} \\ +\infty & \text{otherwise} \end{cases}$$

Case $F(\mathtt{p})$ is defined and $F(\mathtt{p}) = z$.
Let τ be the computation time at which $\mathcal{C}_{\mathtt{p}}$ has output z. Let $\Omega_{\mathtt{p}}$ be the set of k such that oracle A' has been questioned about during the first τ steps of $\mathcal{C}_{\mathtt{p}}$. For t large enough, say $t \geq t_z$, we have $Approx(A', t) \cap \Omega_{\mathtt{p}} = A' \cap \Omega_{\mathtt{p}}$. In particular, the τ first steps of $\mathcal{C}_{\mathtt{p},t}$ and $\mathcal{C}_{\mathtt{p}}$ will be exactly the same and both computations output z. The same with the τ first steps of $\mathcal{C}_{\mathtt{p},t}$ and $\mathcal{C}_{\mathtt{p},t+1}$.
Thus, $w_{\mathtt{p},t+1} \restriction \{1, ..., z\} = w_{\mathtt{p},t} \restriction \{1, ..., z\}$.
Let $w_{\mathtt{p}} = w_{\mathtt{p},t+1} \restriction \{1, ..., z\}$. Then all singleton sets $\{w_{\mathtt{p}}(i)\}$, where $1 \leq i \leq z$, are equivalence classes for the $\rho_{\mathtt{p},t}$'s, hence for $\rho_{\mathtt{p}}$.

Now, if $n_{\mathtt{p},t} > z$ then oracle $Approx(A', t)$ has been questioned on $\Omega_{\mathtt{p},t}(n_{\mathtt{p},t})$ and differs from A' on that set. Let $u > t$ be first such that $Approx(A', u)$ agrees with A' on $\Omega_{\mathtt{p},t}(z+1)$. Then the singleton class $\{w_{\mathtt{p},t}(z+1)\}$ of $\rho_{\mathtt{p},t}$ is aggregated at step u to the class of 0 in $\rho_{\mathtt{p},t+1}$, hence also in $\rho_{\mathtt{p}}$.

Thus, the $\{w_{\mathtt{p}}(i)\}$'s, where $1 \leq i \leq z$, are the sole singleton equivalence classes of $\rho_{\mathtt{p}}$. And the class of 0 contains all other points in $\mathbb{N}$.
In particular, $index(\rho_{\mathtt{p}}) = 1 + F(\mathtt{p})$.

Case $F(\mathtt{p})$ is undefined because the output of $\mathcal{M}$ on input $\mathtt{p}$ with oracle A' is infinite.
As in the above case, we see that there are more and more singleton set classes of $\rho_{\mathtt{p},t}$ which are never annihilated. Thus, the index of $\rho_{\mathtt{p}}$ is infinite.

This proves $(\dagger)$.

Observing that all the construction of the $\rho_{\mathtt{p},t}$'s is A-recursive, we see that

$$\rho = \bigcup_{\mathtt{p} \in \mathbf{2}^*} \rho_{\mathtt{p}}$$

is A-r.e. Thus, $\rho = W_{\mathtt{a}}^{A,\mathbf{2}^*\times\mathbb{N}^2}$ for some $\mathtt{a}$. The parameter property gives a total A-recursive function $s : \mathbf{2}^* \times \mathbf{2}^* \to \mathbf{2}^*$ such that

$$\rho_{\mathtt{p}} = W_{s(\mathtt{a},\mathtt{p})}^{A,\mathbb{N}^2}$$

Thus, $p \mapsto index(\rho_{\mathtt{p}})$ is indeed in $index \circ \mathcal{F}^{RE^A(\mathbb{X}^2)}$. Thanks to (†), the same is true of $1 + F$.

C. *Inclusion* $Max_{Rec^{A'}}^{\mathbf{2}^*\to\mathbb{N}} + 1 \subseteq index \circ \mathcal{F}^{RE^A(\mathbb{X}^2)}$ *is strict.*
The constant 0 function is an obvious counterexample to equality.

D. *Inclusion* $index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)} \subseteq Max_{Rec^{A'}}^{\mathbf{2}^*\to\mathbb{N}}$ *is strict.*
We exhibit a function κ_X in $PR^{A'} \setminus index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)} \neq \emptyset$.
Let $X \subset \mathbf{2}^*$ be A'-recursive, i.e. $\Delta_2^{0,A}$, but not a boolean combination of $\Sigma_1^{0,A}$ sets. Let $\kappa_X : \mathbf{2}^* \to \mathbb{N}$ be the $\{0,1\}$-valued characteristic function of X. Then κ_X is A'-recursive (hence in $Max_{Rec^{A'}}^{\mathbf{2}^*\to\mathbb{N}}$) and $\kappa_X^{-1}(0) = X$ is a $\Delta_2^{0,A}$ set which is not a boolean combination of $\Sigma_1^{0,A}$ sets.

Now, suppose G is in $index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}$ and $G = index(W_{g(\mathtt{p})}^{A,\mathbb{X}^2})$ where $g : \mathbf{2}^* \to \mathbf{2}^*$ is in PR^A. Then

$$\begin{aligned} G(\mathtt{p}) = 0 \quad &\Leftrightarrow \quad (g(\mathtt{p}) \text{ is defined } \wedge\ W_{g(\mathtt{p})}^{A,\mathbb{X}^2} = \emptyset) \\ &\Leftrightarrow \quad (g(\mathtt{p}) \text{ is defined} \\ &\qquad\quad \wedge\ \forall t\ \forall \mathtt{e}\ (g(\mathtt{p}) \text{ converges to } \mathtt{e} \text{ in } t \text{ steps } \Rightarrow\ W_{\mathtt{e}}^{A,\mathbb{X}^2} = \emptyset) \end{aligned}$$

so that $G^{-1}(0)$ is $\Sigma_1^{0,A} \wedge \Pi_1^{0,A}$.

This shows that no $G \in index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}$ can be equal to the above κ_X. Therefore, the considered inclusion cannot be an equality. □

Let's finally observe a simple fact contrasting inclusions in Thm.9.5.

Proposition 9.6. $1 + PR^{A',\mathbf{2}^*\to\mathbf{2}^*}$ *(a fortiori* $1 + Max_{PR^{A'}}^{\mathbf{2}^*\to\mathbb{N}}$*) is not included in* $index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}$.

Proof. The proof is analog to that of point D in the proof of Thm.9.5.
1. We show that $G^{-1}(1)$ is $\Pi_2^{0,A}$ for every $G \in index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}$.
Suppose $G = index(W_{g(\mathtt{p})}^{A,\mathbb{X}^2})$ where $g : \mathbf{2}^* \to \mathbf{2}^*$ is partial A-recursive.
Let's denote $W_{\mathtt{e},t}^{A,\mathbb{X}^2}$ the finite part of $W_{\mathtt{e}}^{A,\mathbb{X}^2}$ obtained after t steps of its enumeration. Let's also denote $CV_g(\mathtt{p}, \mathtt{e}, t)$ the A-recursive relation stating

that $g(\mathtt{p})$ converges to $\mathtt{e}$ in $\leq t$ steps. Then

$$
\begin{aligned}
G(\mathtt{p}) = 1 \quad &\Leftrightarrow \quad (g(\mathtt{p}) \text{ is defined } \wedge\ W^{A,\mathbb{X}^2}_{g(\mathtt{p})} \neq \emptyset \\
&\qquad \wedge\ W^{A,\mathbb{X}^2}_{g(\mathtt{p})} \text{ is an equivalence relation with index 1)} \\
&\Leftrightarrow \quad (g(\mathtt{p}) \text{ is defined } \wedge\ W^{A,\mathbb{X}^2}_{g(\mathtt{p})} \neq \emptyset \\
&\qquad \wedge\ \forall t\ \forall \mathtt{e}\ (CV_g(\mathtt{p},\mathtt{e},t) \ \Rightarrow \\
&\qquad\qquad W^{A,\mathbb{X}^2}_{\mathtt{e}} \text{ is an equivalence relation with index 1)}
\end{aligned}
$$

The first two conjuncts are clearly $\Sigma^{0,A}_1$. As for the last one, observe that $W^{A,\mathbb{X}^2}_{\mathtt{e}}$ is an equivalence relation if and only if

$$
\begin{aligned}
&\forall \mathtt{x},\mathtt{y} \in \mathbb{X}\ ((\mathtt{x},\mathtt{y}) \in W^{A,\mathbb{X}^2}_{\mathtt{e}} \ \Rightarrow\ (\mathtt{x},\mathtt{x}) \in W^{A,\mathbb{X}^2}_{\mathtt{e}} \ \wedge\ (\mathtt{y},\mathtt{x}) \in W^{A,\mathbb{X}^2}_{\mathtt{e}}) \\
\wedge\ &\forall \mathtt{x},\mathtt{y},\mathtt{z} \in \mathbb{X}\ ((\mathtt{x},\mathtt{y}) \in W^{A,\mathbb{X}^2}_{\mathtt{e}} \ \wedge\ (\mathtt{y},\mathtt{z}) \in W^{A,\mathbb{X}^2}_{\mathtt{e}}) \ \Rightarrow\ (\mathtt{x},\mathtt{z}) \in W^{A,\mathbb{X}^2}_{\mathtt{e}})
\end{aligned}
$$

Which is $\Pi^{0,A}_2$ since $W^{A,\mathbb{X}^2}_{\mathtt{e}}$ is $\Sigma^{0,A}_1$.
Also, if $W^{A,\mathbb{X}^2}_{\mathtt{e}}$ is a non empty equivalence relation then it has index 1 if and only if

$$
\forall \mathtt{x},\mathtt{y},\mathtt{x}',\mathtt{y}' \in \mathbb{X}\ ((\mathtt{x},\mathtt{x}') \in W^{A,\mathbb{X}^2}_{\mathtt{e}} \ \wedge\ (\mathtt{y},\mathtt{y}') \in W^{A,\mathbb{X}^2}_{\mathtt{e}},) \ \Rightarrow\ (\mathtt{x},\mathtt{y}) \in W^{A,\mathbb{X}^2}_{\mathtt{e}})
$$

Which is again $\Pi^{0,A}_2$.

This proves that $G^{-1}(1)$ is indeed $\Pi^{0,A}_2$.

2. Now, let $X \subset \mathbb{X}$ be $\Sigma^{0,A'}_1$ and not A'-recursive. Thus, X is $\Sigma^{0,A}_2$ and not $\Pi^{0,A}_2$. Let $\pi_X : \mathbf{2}^* \to \mathbb{N}$ be such that

$$
\pi_X(\mathtt{p}) = \begin{cases} 1 & \text{if } \mathtt{p} \in X \\ \text{undefined} & \text{otherwise} \end{cases}
$$

Then $\pi_X \in 1 + PR^{A',\mathbf{2}^* \to \mathbb{N}}$.

Since $\pi_X^{-1}(1) = X$ is not $\Pi^{0,A}_2$, π_X cannot be in $index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}$. $\square$

9.4 Characterization of the $\Delta index$ self-enumerated systems

Theorem 9.7.
Let $A \subseteq \mathbb{N}$ and let A'' be the second jump of A. Let $\mathbb{X}$ be a basic set.

1. $\Delta(index \circ \mathcal{F}^{RE^A(\mathbb{X})})) = \Delta(index \circ \mathcal{PF}^{RE^A(\mathbb{X})})) = PR^{A'',\mathbf{2}^* \to \mathbb{Z}}$

2. $K^{\mathbb{Z}}_{\Delta(index \circ \mathcal{F}^{RE^A(\mathbb{X})})} =_{\mathrm{ct}} K^{A'',\mathbb{Z}}$.

We shall simply write $K^{\mathbb{N},A}_{\Delta index}$ in place of $K^{\mathbb{Z}}_{\Delta(index \circ \mathcal{F}^{RE^A(\mathbb{N})})} \upharpoonright \mathbb{N}$.
When $A = \emptyset$ we simply write $K^{\mathbb{Z}}_{\Delta index}$.

Proof. Point 2 is a direct corollary of Point 1. Let's prove point 1. Using Thm.9.5, and applying the Δ operator, we get

$$\Delta(Max_{Rec^{A'}}^{\mathbf{2}^*\to\mathbb{N}}+1) \subseteq \Delta(index \circ \mathcal{F}^{RE^A(\mathbb{X}^2)})$$
$$\subseteq \Delta(index \circ \mathcal{PF}^{RE^A(\mathbb{X}^2)}) \subseteq \Delta(Max_{Rec^{A'}}^{\mathbf{2}^*\to\mathbb{N}})$$

But, for any family $\mathcal{G}$ of functions $\mathbf{2}^* \to \mathbb{N}$, we trivially have $\Delta(\mathcal{G}+1) = \Delta(\mathcal{G})$. This proves that the above inclusions are, in fact, equalities. We conclude with Thm.6.12. □

10 Functional representations of $\mathbb{N}$

Notation 10.1 (Functions sets)**.** We denote

- Y^X the set of total functions from X into Y.
- $X \to Y$ the set of partial functions from X into Y.
- $X \overset{1-1}{\to} X$ the set of injective partial functions from X into X.
- Id_X the identity function over X.

10.1 Basic Church representation of $\mathbb{N}$

First, let's introduce some simple notations related to function iteration.

Definition 10.2 (Iteration)**.**
1) If $f : X \to X$ is a partial function, we inductively define for $n \in \mathbb{N}$ the n-th iterate $f^{(n)} : X \to X$ of f as the partial function such that:

$$f^{(0)} = Id_X \ , \ f^{(n+1)} = f^{(n)} \circ f$$

2) $It_X^{(n)} : (X \to X) \to (X \to X)$ is the total functional $f \mapsto f^{(n)}$.
$It_X^{\mathbb{N}} : \mathbb{N} \to (X \to X)^{(X\to X)}$ is the total functional $n \mapsto It_X^n$.

The following Proposition is easy.

Proposition 10.3. *The total functional $It_X^{\mathbb{N}} : \mathbb{N} \to (X \to X)^{(X\to X)}$ is injective (hence admits a left inverse) if and only if X is an infinite set.*

We can now come to the functional representation of integers introduced by Church, 1933 [3].

Definition 10.4 (Church representation of $\mathbb{N}$)**.**
If X is an infinite set, the Church representation of $\mathbb{N}$ relative to X is the function

$$Church_X^{\mathbb{N}} : (X \to X)^{(X\to X)} \to \mathbb{N}$$

which is the unique left inverse of $It_X^{\mathbb{N}}$ with domain $Range(It_X^{\mathbb{N}}) = \{It_X^n : n \in \mathbb{N}\}$, i.e.

$$\begin{array}{rcl} Church_X^{\mathbb{N}} \circ It_X^{\mathbb{N}} & = & Id_{\mathbb{N}} \\ Church_X^{\mathbb{N}}(F) & = & \left\{ \begin{array}{ll} n & \text{if } F = It_X^n \\ \text{undefined} & \text{if } \forall n \in \mathbb{N}\ F \neq It_X^n \end{array} \right. \end{array}$$

For future use in Def.10.17, let's introduce the following variant of $Church_X^{\mathbb{N}}$.

Definition 10.5. We denote $church_X^{\mathbb{N},A} : (PR^{A,\mathbb{X}\to\mathbb{X}})^{PR^{A,\mathbb{X}\to\mathbb{N}}}$ the functional which is the unique left inverse of the restriction of $It_X^{\mathbb{N}}$ to $(PR^{A,\mathbb{X}\to\mathbb{X}})^{PR^{A,\mathbb{X}\to\mathbb{X}}}$, i.e.

$$church_X^{\mathbb{N},A}(F) \quad = \quad \begin{cases} n & \text{if } F = It_X^n \restriction (PR^{A,\mathbb{X}\to\mathbb{X}})^{PR^{A,\mathbb{X}\to\mathbb{X}}} \\ \text{undefined} & \text{if } \forall n \in \mathbb{N}\ F \neq It_X^n \restriction (PR^{A,\mathbb{X}\to\mathbb{X}})^{PR^{A,\mathbb{X}\to\mathbb{X}}} \end{cases}$$

10.2 Computable and effectively continuous functionals

We recall the two classical notions of partial computability for functionals, cf. Odifreddi's book [12] p.178, 188, 197.

Definition 10.6 (Kleene partial computable functionals)**.**
1. Let $\mathbb{X}, \mathbb{Y}, \mathbb{S}, \mathbb{T}$ be some basic space and fix some suitable representations of their elements by words. An $(\mathbb{X} \to \mathbb{Y})$-oracle Turing machine with inputs and outputs respectively in $\mathbb{S}, \mathbb{T}$ is a Turing machine $\mathcal{M}$ which has a special oracle tape and is allowed at certain states to ask an oracle $f \in (\mathbb{X} \to \mathbb{X})$ what are the successive digits of the value of $f(\mathtt{q})$ where $\mathtt{q}$ is the element of $\mathbb{X}$ currently written on the oracle tape.
The functional $\Phi_{\mathcal{M}} : ((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$ associated to $\mathcal{M}$ maps the pair $(f, \mathtt{s})$ on the output (when defined) computed by $\mathcal{M}$ when f is given as the partial function oracle and $\mathtt{s}$ as the input.
If on input $\mathtt{x}$ and oracle f the computation asks the oracle its value on an element on which f is undefined then $\mathcal{M}$ gets stuck, so that $\Phi_{\mathcal{M}}(f, \mathtt{x})$ is undefined.

2. A functional $\Phi : ((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$ is partial computable (also called partial recursive) if $\Phi = \Phi_{\mathcal{M}}$ for some $\mathcal{M}$.
A functional obtained via curryfications from such a functional is also called partial computable.

We denote PC^τ the family of partial computable functionals with type τ.
If $A \subseteq \mathbb{N}$, we denote A-PC^τ the analog family with the extra oracle A.

Definition 10.7 (Uspenskii (effectively) continuous functionals)**.** Denote $Fin(\mathbb{X} \to \mathbb{Y})$ the class of partial functions $\mathbb{X} \to \mathbb{Y}$ with finite domains. Observe that, for $\alpha, \beta \in Fin(\mathbb{X} \to \mathbb{Y})$ are compatible if and only if $\alpha \cup \beta \in Fin(\mathbb{X} \to \mathbb{Y})$.

1. Let's say that the relation $R \subseteq Fin(\mathbb{X} \to \mathbb{Y}) \times \mathbb{S} \times \mathbb{T}$ is functional if

$$\alpha \cup \beta \in Fin(\mathbb{X} \to \mathbb{Y}) \ \wedge\ (\alpha, \mathtt{s}, \mathtt{t}) \in R \ \wedge\ (\beta, \mathtt{s}, \mathtt{t}') \in R \ \Rightarrow\ \mathtt{t} = \mathtt{t}'$$

To such a functional relation R can be associated a functional

$$\Phi_R : ((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$$

such that, for every $f, \mathtt{s}, \mathtt{t}$,

$$(\dagger) \qquad \Phi(f, \mathtt{s}) = \mathtt{t} \;\Leftrightarrow\; \exists u \subseteq f\ R(u, \mathtt{s}, \mathtt{t})$$

2. (Uspenskii [21], Nerode [11]) A functional $\Phi : ((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$ is **continuous** if it is of the form Φ_R for some functional relation R.

Φ is **effectively continuous (resp. (A-effectively continuous)** if R is r.e. (resp. A-r.e.). Effectively continuous functionals are also called recursive operators (cf. Rogers [15], Odifreddi [12]).
A functional obtained via curryfications from such a functional is also called effectively continuous.
We denote $\mathit{EffCont}^{\tau}$ the family of effectively continuous functionals with type τ.
If $A \subseteq \mathbb{N}$, we denote A-$\mathit{EffCont}^{\tau}$ the analog family with the extra oracle A.

Effective continuity is more general than partial computability (cf. [12] p.188).

Theorem 10.8. *Let $A \subseteq \mathbb{N}$.*
1. (Uspenskii [21], Nerode [11]) *Partial A-computable functionals are A-effectively continuous.*

2. (Sasso [17, 18]) *There are A-effectively continuous functionals which are not partial A-computable.*

However, restricted to total functions, both notions coincide.

Proposition 10.9. *A functional $\Phi : (\mathbb{Y}^{\mathbb{X}}) \times \mathbb{S} \to \mathbb{T}$ is the restriction of a partial A-computable functional $((\mathbb{X} \to \mathbb{Y}) \times \mathbb{S}) \to \mathbb{T}$ if and only if it is the restriction of an A-effectively continuous functional.*

10.3 Effectiveness of the *Apply* functional

The following result will be used in §10.7-10.5.

Proposition 10.10. *Let $\phi : \mathbf{2}^* \to PR^{A, \mathbb{X} \to \mathbb{X}}$ be partial A-recursive (as a function $\mathbf{2}^* \times \mathbb{X} \to \mathbb{X}$) and $\Phi : \mathbf{2}^* \to A\text{-}\mathit{EffCont}^{\mathbf{2}^* \to ((\mathbb{X} \to \mathbb{X}) \to (\mathbb{X} \to \mathbb{X}))}$ be effectively continuous. There exists a partial A-recursive function $g : \mathbf{2}^* \times \mathbf{2}^* \times \mathbb{X}$ such that, for all $\mathtt{e}, \mathtt{p} \in \mathbf{2}^*$ and $\mathtt{x} \in \mathbb{X}$,*

$$(*) \qquad g(\mathtt{p}, \mathtt{e}, \mathtt{x}) = (\Phi(\mathtt{e})(\phi(\mathtt{p})))(\mathtt{x})$$

Proof. Let $R \subseteq \mathbf{2}^* \times Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ be an A-r.e. set such that, for all $\mathtt{e}$, $R^{(\mathtt{e})} = \{(\alpha, \mathtt{x}, \mathtt{y}) : (\mathtt{e}, \alpha, \mathtt{x}, \mathtt{y}) \in R\}$ is functional and $\Phi(\mathtt{e}) = \Phi_{R^{(\mathtt{e})}}$. We define $g(\mathtt{p}, \mathtt{e}, \mathtt{x})$ as follows:

i. A-effectively enumerate $R^{(\mathtt{e})}$ and the graph of $\phi(\mathtt{p})$ up to the moment we get $(\alpha, \mathtt{x}, \mathtt{y}) \in R^{(\mathtt{e})}$ and a finite part γ of $\phi(\mathtt{p})$ such that $\alpha \subseteq \gamma$.

ii. If and when i halts then output $\mathtt{y}$.

It is clear that g is partial A-recursive and satisfies $(*)$. $\square$

10.4 Functionals over $PR^{\mathbb{X}\to\mathbb{Y}}$ and computability

Using indexes, one can also consider computability for functionals operating on the sole partial recursive or A-recursive functions.

Definition 10.11. Let $A \subseteq \mathbb{N}$ and let $(\varphi_{\mathsf{e}}^{\mathbb{X}\to\mathbb{Y},A})_{\mathsf{e}\in\mathbf{2}^*}$ denote some acceptable enumeration of $PR^{A,\mathbb{X}\to\mathbb{Y}}$ (cf. Def.5.1).

1. A functional $\Phi : PR^{A,\mathbb{X}\to\mathbb{Y}} \times \mathbb{S} \to \mathbb{T}$ is an A-effective functional on partial A-recursive functions if there exists some partial A-recursive function $f : \mathbf{2}^* \to \mathbf{2}^*$ such that, for all $\mathsf{s} \in \mathbb{S}, \mathsf{e} \in \mathbf{2}^*$,

$$\Phi(\varphi_{\mathsf{e}}^{\mathbb{X}\to\mathbb{Y},A}) = f(\mathsf{e})$$

We denote A-$\mathit{Eff}^{PR^{A,\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}\to\mathbb{T}}$ the family of such functionals.

2. We denote A-$\mathit{Eff}^{PR^{A,\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}_1\to PR^{A,\mathbb{S}_2\to\mathbb{T}}}$ the family of functionals obtained by curryfication of the above class with $\mathbb{S} = \mathbb{S}_1 \times \mathbb{S}_2$.
An easy application of the parameter property shows that these functionals are exactly those for which there exists some partial A-recursive function $g : \mathbf{2}^* \times \mathbb{S}_1 \to \mathbf{2}^*$ such that, for all $\mathsf{s}_1 \in \mathbb{S}_1, \mathsf{e} \in \mathbf{2}^*$,

$$\Phi(\varphi_{\mathsf{e}}^{\mathbb{X}\to\mathbb{Y},A}, \mathsf{s}_1) = \varphi_{g(\mathsf{e},\mathsf{s}_1)}^{\mathbb{S}_2\to\mathbb{T},A}$$

Note 10.12.
1. Thanks to Rogers' theorem (cf. Thm.5.2), the above definition does not depend on the chosen acceptable enumerations.

2. The above functions f, g should have the following properties:

$$\begin{array}{rcl}
\varphi_{\mathsf{e}}^{\mathbb{X}\to\mathbb{Y},A} = \varphi_{\mathsf{e}'}^{\mathbb{X}\to\mathbb{Y},A} & \Rightarrow & f(\mathsf{e},\mathsf{s}) = f(\mathsf{e}',\mathsf{s}) \\
\varphi_{\mathsf{e}}^{\mathbb{X}\to\mathbb{Y},A} = \varphi_{\mathsf{e}'}^{\mathbb{X}\to\mathbb{Y},A} & \Rightarrow & \varphi_{g(\mathsf{e},\mathsf{s}_1)}^{\mathbb{S}_2\to\mathbb{T},A} = \varphi_{g(\mathsf{e}',\mathsf{s}_1)}^{\mathbb{S}_2\to\mathbb{T},A}
\end{array}$$

As shown by the following remarkable result, such functionals essentially reduce to those of Def.10.7 (cf. Odifreddi's book [12] p.206–208).

Theorem 10.13 (Uspenskii [21], Myhill & Shepherdson [10])**.**
Let $A \subseteq \mathbb{N}$. The A-effective functionals $PR^{A,\mathbb{X}\to\mathbb{Y}} \to PR^{A,\mathbb{S}\to\mathbb{T}}$ are exactly the restrictions to $PR^{A,\mathbb{X}\to\mathbb{Y}}$ of A-effectively continuous functionals $(\mathbb{X} \to \mathbb{Y}) \to (\mathbb{S} \to \mathbb{T})$.

10.5 Effectivizations of Church representation of $\mathbb{N}$

Observe the following trivial fact (which uses notations from Def.10.6,10.7).

Proposition 10.14. *Let $A \subseteq \mathbb{N}$ and τ be any 2d order type.*
Functionals in A-$PC^{\mathbf{2}^*\to\tau}$ (resp. A-$\mathit{EffCont}^{\mathbf{2}^*\to\tau}$) are total maps $\mathbf{2}^* \to A$-PC^{τ} (resp. $\mathbf{2}^* \to A$-$\mathit{EffCont}^{\tau}$).

Theorem 10.15. *Let τ be any 2d order type. The systems*

$$(A\text{-}PC^{\tau}, A\text{-}PC^{\mathbf{2}^*\to\tau}) \quad , \quad (A\text{-}\mathit{EffCont}^{\tau}, A\text{-}\mathit{EffCont}^{\mathbf{2}^*\to\tau})$$

are self-enumerated representation A-systems.

Proof. Points i-ii of Def.2.1 are trivial. As for point iii, we use the classical enumeration theorem for partial computable (resp. effectively continuous) functionals: consider a function $V \in A\text{-}PC^{\mathbf{2}^*\to(\mathbf{2}^*\to\tau)}$ which enumerates $A\text{-}PC^{\mathbf{2}^*\to\tau}$ and set $U(c(\mathtt{e},\mathtt{p})) = V(\mathtt{e})(\mathtt{p})$. Idem with A-*EffCont*. □

As an easy corollary of Thms.10.15 and 10.13, we get the following result.

Theorem 10.16. *Let $A \subseteq \mathbb{N}$. Let $A\text{-}\mathit{Eff}^{\mathbf{2}^*\to(PR^{A,\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}\to\mathbb{T})}$ be obtained by curryfication from $A\text{-}\mathit{Eff}^{(PR^{A,\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}\times\mathbf{2}^*)\to\mathbb{T}}$. The systems*

$$\begin{array}{lcl}(A\text{-}\mathit{Eff}^{PR^{A,\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}\to\mathbb{T}} & , & A\text{-}\mathit{Eff}^{\mathbf{2}^*\to(PR^{A,\mathbb{X}\to\mathbb{Y}}\times\mathbb{S}\to\mathbb{T})}) \\ (A\text{-}\mathit{Eff}^{PR^{A,\mathbb{X}\to\mathbb{Y}}\to PR^{A,\mathbb{X}\to\mathbb{Y}}} & , & A\text{-}\mathit{Eff}^{\mathbf{2}^*\to(PR^{A,\mathbb{X}\to\mathbb{Y}}\to PR^{A,\mathbb{S}\to\mathbb{T}})})\end{array}$$

are self-enumerated representation A-systems.

Definition 10.17 (Effectivizations of Church representation of $\mathbb{N}$)**.** We effectivize the Church representation by replacing $(X \to X) \to (X \to X)$ by one of the following classes:

$$A\text{-}PC^{(\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X})} \;,\; A\text{-}\mathit{EffCont}^{(\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X})} \;,\; A\text{-}\mathit{Eff}^{PR^{A,\mathbb{X}\to\mathbb{Y}}\to PR^{A,\mathbb{X}\to\mathbb{Y}}}$$

where $\mathbb{X}$ is some basic set. and $A \subseteq \mathbb{N}$ is some oracle. Using Def.10.5, this leads to three self-enumerated systems with domain $\mathbb{N}$:

$$\begin{array}{lcl}\mathcal{F}_1 & = & (\mathbb{N} \;,\; \mathit{Church}^{\mathbb{N}}_{\mathbb{X}} \circ A\text{-}PC^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}) \\ \mathcal{F}_2 & = & (\mathbb{N} \;,\; \mathit{Church}^{\mathbb{N}}_{\mathbb{X}} \circ A\text{-}\mathit{EffCont}^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}) \\ \mathcal{F}_3 & = & (\mathbb{N} \;,\; \mathit{church}^{\mathbb{N},A}_{\mathbb{X}} \circ A\text{-}\mathit{Eff}^{\mathbf{2}^*\to(PR^{A,\mathbb{X}\to\mathbb{Y}}\to PR^{A,\mathbb{X}\to\mathbb{Y}})})\end{array}$$

The following result greatly simplifies the landscape.

Theorem 10.18. *The three systems $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$ of Def.10.17 coincide.*

Before proving the theorem (cf. the end of this subsection), we state some convenient tools in the next three propositions, the first of which will also be used in §10.7.

Proposition 10.19. *Suppose $R \subset Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ is functional (cf. Def.10.7). The following conditions are equivalent*

i. $\Phi_R = It^{(n)}_{\mathbb{X}}$

ii. $\Phi_R \restriction Fin(\mathbb{X} \to \mathbb{X}) = It^{(n)}_{\mathbb{X}} \restriction Fin(\mathbb{X} \to \mathbb{X})$

iii. $\forall \alpha \in Fin(\mathbb{X} \to \mathbb{X})\ \forall \mathtt{x}\ (\alpha^{(n)}(\mathtt{x})$ *is defined* $\Rightarrow$

$$(\alpha \restriction \{\alpha^{(i)}(\mathtt{x}) : 0 \leq i < n\}, \mathtt{x}, \alpha^{(n)}(\mathtt{x})) \in R)$$

and

$\forall \alpha \in Fin(\mathbb{X} \to \mathbb{X})\ \forall \mathtt{x}\ \forall \mathtt{y}$

$$((\alpha, \mathtt{x}, \mathtt{y}) \in R \ \Rightarrow\ (\alpha^{(n)}(\mathtt{x}) \textit{ is defined} \ \wedge\ \mathtt{y} = \alpha^{(n)}(\mathtt{x})))$$

Proof. $iii \Rightarrow i$ and $i \Rightarrow ii$ are trivial.
$ii \Rightarrow iii$. Assume ii. Suppose $(\alpha, \mathtt{x}, \mathtt{y}) \in R$ then $\Phi_R(\alpha)(\mathtt{x}) = \mathtt{y}$. Since $\alpha \in Fin(\mathbb{X} \to \mathbb{X})$, ii insures that $\alpha^{(n)}(\mathtt{x})$ is defined and $\alpha^{(n)}(\mathtt{x}) = \mathtt{y}$. This proves the second part of iii.
Suppose $\alpha^{(n)}(\mathtt{x})$ is defined and let $\alpha^{(n)}(\mathtt{x}) = \mathtt{y}$. Then

$$\begin{aligned} \Phi_R(\alpha \restriction \{\alpha^{(i)}(\mathtt{x}) : 0 \leq i < n\})(\mathtt{x}) &= It^{(n)}_{\mathbb{X}}(\alpha \restriction \{\alpha^{(i)}(\mathtt{x}) : 0 \leq i < n\})(\mathtt{x}) \\ &= It^{(n)}_{\mathbb{X}}(\alpha)(\mathtt{x}) \\ &= \mathtt{y} \end{aligned}$$

So that there exists a restriction β of $\alpha \restriction \{\alpha^{(i)}(\mathtt{x}) : 0 \leq i < n\}$ such that $(\beta, \mathtt{x}, \mathtt{y}) \in R$. Thus, $\Phi_R(\beta)(\mathtt{x}) = \mathtt{y}$. Applying ii, this yields that $\beta^{(n)}(\mathtt{x})$ is defined and $\beta^{(n)}(\mathtt{x}) = \mathtt{y}$. Since β is a restriction of $\alpha \restriction \{\alpha^{(i)}(\mathtt{x}) : 0 \leq i < n\}$, this insures that $\beta = \alpha \restriction \{\alpha^{(i)}(\mathtt{x}) : 0 \leq i < n\}$. This proves the first part of iii. □

Proposition 10.20. *Let $n \in \mathbb{N}$. If $\Phi_R(f)$ is a restriction of $f^{(n)}$ for every $f : \mathbb{X} \to \mathbb{X}$ then either $\Phi_R = It^{(n)}_{\mathbb{X}}$ or Φ_R is not an iterator.*

Proof. We reduce to the case $\mathbb{X} = \mathbb{N}$. Let $Succ : \mathbb{N} \to \mathbb{N}$ be the successor function. Since $\Phi_R(Succ)$ is a restriction of $Succ^{(n)}$, either $\Phi_R(Succ)(0)$ is undefined or $\Phi_R(Succ)(0) = n$. In both cases it is different from $Succ^{(p)}(0)$ for any $p \neq n$. Which proves that $\Phi_R \neq It^{(p)}_{\mathbb{N}}$ for every $p \neq n$. Hence the proposition. □

Proposition 10.21.
1. *Let $(W_\mathsf{e})_{\mathsf{e} \in \mathbf{2}^*}$ be an acceptable enumeration of r.e. subsets of $Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$. There exists a total recursive function $\xi : \mathbf{2}^* \to \mathbf{2}^*$ such that, for all e,*

a. $W_{\xi(\mathsf{e})} \subseteq W_\mathsf{e}$ *and* $W_{\xi(\mathsf{e})}$ *is functional (cf. Def.10.7, point 1),*

b. $W_{\xi(\mathsf{e})} = W_\mathsf{e}$ *whenever* W_e *is functional.*

2. *There exists a partial recursive function $\lambda : \mathbf{2}^* \to \mathbb{N}$ such that if R_e is functional and Φ_{R_e} is an iterator then $\lambda(\mathsf{e})$ is defined and $\Phi_{R_\mathsf{e}} = It^{(\lambda(\mathsf{e}))}_{\mathbb{X}}$. (However, $\lambda(\mathsf{e})$ may be defined even if R_e is not functional or Φ_{R_e} is not an iterator).*

3. *There exists a total recursive function $\theta : \mathbf{2}^* \to \mathbf{2}^*$ such that, for all $\mathsf{e} \in \mathbf{2}^*$,*

a. *if* $\Phi_{R_{\mathtt{e}}}$ *is an iterator then the* $(\mathbb{X} \to \mathbb{X})$*-oracle Turing machine* $\mathcal{M}_{\theta(\mathtt{e})}$ *with code* $\theta(\mathtt{e})$ *(cf. Def.10.6) computes the functional* $\Phi_{R_{\mathtt{e}}}$*,*

b. *if* $\Phi_{R_{\mathtt{e}}}$ *is not an iterator then neither is the functional computed by the* $(\mathbb{X} \to \mathbb{X})$*-oracle Turing machine* $\mathcal{M}_{\theta(\mathtt{e})}$ *with code* $\theta(\mathtt{e})$*.*

In other words, $Church(\Phi_{R_{\mathtt{e}}}) = Church(\Phi_{\mathcal{M}_{\theta(\mathtt{e})}})$

4. *The above points relativize to any oracle* $A \subseteq \mathbb{N}$*.*

Proof. 1. This is the classical fact underlying the enumeration theorem for effectively continuous functionals. To get $W_{\xi(\mathtt{e})}$, enumerate $W_{\mathtt{e}}$ and retain a triple if and only if, together with the already retained ones, it does not contradict functionality (cf. Odifreddi's book [12] p.197).

2. We reduce to the case $\mathbb{X} = \mathbb{N}$. Let $\alpha_n : \mathbb{N} \to \mathbb{N}$ be such that

$$domain(\alpha_n) = \{0, ..., n\} \quad , \quad \alpha_n(i) = i + 1 \text{ for } i = 0, ..., n$$

Suppose R is functional and $\Phi_R = It_{\mathbb{N}}^{(n)}$. Prop.10.19 insures $(\alpha_n, 0, n) \in R$. Also, for $m \neq n$, since α_m and α_n are compatible and R is functional, R cannot contain $(\alpha_m, 0, m)$. Thus, if $\Phi_R = It_{\mathbb{N}}^{(n)}$ then n is the unique integer such that R contains $(\alpha_n, 0, n)$.

This leads to the following definition of the wanted partial recursive function $\lambda : \mathbf{2}^* \to \mathbb{N}$:

- enumerate $R\mathtt{e}$,
- if and when some triple $(\alpha_n, 0, n)$ appears, halt and output $\lambda(\mathtt{e}) = n$.

3. Given a code $\mathtt{e}$ of a functional relation $R_{\mathtt{e}}$, we let θ be the total recursive function which gives a code for the oracle Turing machine $\mathcal{M}$ which acts as follows:

i. First, it computes $\lambda(\mathtt{e})$.

ii. If $\lambda(\mathtt{e})$ is defined then, on input $\mathtt{x}$ and oracle f, $\mathcal{M}$ tries to compute $It_{\mathbb{X}}^{(\lambda(\mathtt{e}))}(f)(\mathtt{x})$ in the obvious way: ask the oracle the values of $f^{(i)}(\mathtt{x})$ for $i \leq \lambda(\mathtt{e})$.

iii. Finally, in case i and ii halt, $\mathcal{M}$ enumerates $R\mathtt{e}$ and halts and accepts (with the output computed at phase ii) if and only if $(f \upharpoonright \{f^{(i)}(\mathtt{x}) : i \leq \lambda(\mathtt{e})\}, \mathtt{x}, f^{(\lambda(\mathtt{e}))}(\mathtt{x}))$ appears in $R_{\mathtt{e}}$. I.e. if and only if $f^{(\lambda(\mathtt{e}))}(\mathtt{x}) = \Phi_R(f)(\mathtt{x})$

Clearly, the functional $\Phi_{\mathcal{M}}$ computed by $\mathcal{M}$ is such that $\Phi_{\mathcal{M}}(f)$ is equal to or is a restriction of $It_{\mathbb{X}}^{(\lambda(\mathtt{e}))}(f)$.
If $\Phi_{R_{\mathtt{e}}}$ is an iterator then point 2 insures that $\Phi_{R_{\mathtt{e}}} = It_{\mathbb{X}}^{(\lambda(\mathtt{e}))}$ and Prop.10.19 insures that phase iii is no problem, so that $\mathcal{M}$ computes exactly $\Phi_{R_{\mathtt{e}}}$.

Suppose $\Phi_{R_{\mathtt{e}}}$ is not an iterator.
If $\lambda(\mathtt{e})$ is undefined then $\mathcal{M}$ computes the constant functional with value the nowhere defined function. Thus, $\mathcal{M}$ does not compute an iterator.
If $\lambda(\mathtt{e})$ is defined then, on input $\mathtt{x}$, $\mathcal{M}$ computes $f^{(\lambda(\mathtt{e}))}(\mathtt{x})$ and halt and accepts if and only $f^{(\lambda(\mathtt{e}))}(\mathtt{x}) = \Phi_R(f)(\mathtt{x})$. Since Φ_R is not an iterator, there exists f and $\mathtt{x}$ such that $f^{(\lambda(\mathtt{e}))}(\mathtt{x})$ is defined and $\Phi_R(f)(\mathtt{x}) \neq f^{(\lambda(\mathtt{e}))}(\mathtt{x})$. Hence $\Phi_{\mathcal{M}}(f)$ is a strict restriction of $It_{\mathbb{X}}^{(\lambda(\mathtt{e}))}(f)$, so that $\Phi_{\mathcal{M}} \neq It_{\mathbb{X}}^{(\lambda(\mathtt{e}))}$. Finally, Prop.10.20 insures that $\Phi_{R_{\mathtt{e}}}$ cannot be an iterator. $\square$

Proof of Theorem 10.18.
1. Since $Fin(\mathbb{X} \to \mathbb{X}) \subset PR^{A,\mathbb{X}\to\mathbb{Y}}$, condition ii of Prop.10.19 and Thm.10.13 prove equality $\mathcal{F}_2 = \mathcal{F}_3$.

2. Inclusion $\mathcal{F}_1 \subseteq \mathcal{F}_2$ is a corollary of Thm.10.8, point 1. Let's prove the converse inclusion. Suppose $\Phi : (\mathbf{2}^* \times (\mathbb{X} \to \mathbb{X})) \to (\mathbb{X} \to \mathbb{X})$ is effectively continuous and let $R \subseteq \mathbf{2}^* \times Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ be a functional r.e. set such that $\Phi = \Phi_R$. Using the parameter property, let $h : \mathbf{2}^* \to \mathbf{2}^*$ be a total recursive function such that $h(\mathtt{e})$ is an r.e. code for $R^{(\mathtt{e})} = \{(\alpha, \mathtt{x}, \mathtt{y}) : (\mathtt{e}, \alpha, \mathtt{x}, \mathtt{y}) \in R\}$. Prop.10.21, point 3, gives a total recursive $\theta : \mathbf{2}^* \to \mathbf{2}^*$ such that $Church(\Phi_{R^{(\mathtt{e})}}) = Church(\Phi_{\mathcal{M}_{\theta(\mathtt{e})}})$. Thus, $\mathtt{e} \mapsto Church(\Phi_{R^{(\mathtt{e})}})$ is partial computable with a $(\mathbb{X} \to \mathbb{X})$-oracle Turing machine having inputs in $\mathbf{2}^* \times \mathbb{X}$. $\square$

10.6 Some examples of effectively continuous functionals

For future use in sections §10.7-10.8, let's get the following examples of effectively continuous functionals.

Proposition 10.22. *If $\varphi : \mathbf{2}^* \to \mathbb{N}$ is partial A-recursive and $S \subseteq \mathbf{2}^*$ is $\Pi_2^{0,A}$ then there exists an A-effectively continuous functional*

$$\Phi : \mathbf{2}^* \to (\mathbb{X} \to \mathbb{X})^{\mathbb{X}\to\mathbb{X}}$$

such that, for all $\mathtt{p}$,

$$\begin{array}{lll} (*) & \mathtt{p} \in S \cap domain(\varphi) \;\Rightarrow\; \Phi(\mathtt{p}) = It_{\mathbb{X}}^{(\varphi(\mathtt{p}))} \\ (**) & \mathtt{p} \notin S \cap domain(\varphi) \;\Rightarrow\; \Phi(\mathtt{p}) \textit{ is not an iterator} \end{array}$$

Proof. We consider the sole case $A = \emptyset$, relativization being straightforward. Let $S = \{\mathtt{e} : \forall u\ \exists v\ (\mathtt{e}, u, v) \in \sigma\}$ where σ is a recursive subset of $\mathbf{2}^* \times \mathbb{N} \times \mathbb{N}$. We construct a total recursive function $g : \mathbf{2}^* \to \mathbf{2}^*$ such that, for all $\mathtt{p}$, $W_{g(\mathtt{p})}$ is functional and

$$\begin{array}{lll} \mathtt{p} \in S \cap domain(\varphi) & \Rightarrow & \Phi_{W_{g(\mathtt{p})}} = It_{\mathbb{X}}^{(\varphi(\mathtt{p}))} \\ \mathtt{p} \notin S \cap domain(\varphi) & \Rightarrow & \Phi_{W_{g(\mathtt{p})}} \text{ is not an iterator} \end{array}$$

Let

$$\begin{aligned}\mathcal{S}_n &= \{(\alpha, \mathtt{x}, \mathtt{y}) : \alpha \in Fin(\mathbb{X} \to \mathbb{X}) \ \wedge \ \alpha^{(n)}(\mathtt{x}) \text{ is defined } \wedge \ \mathtt{y} = \alpha^{(n)} \\ &\qquad \wedge \ domain(\alpha) = \{\alpha^{(i)} : i \leq n\}\}\end{aligned}$$

Let $\gamma : \mathbb{N}^2 \to \bigcup_{n \in \mathbb{N}} \mathcal{S}_n$ be a total recursive function such that, for all n, $u \mapsto \gamma(n, u)$ is a bijection $\mathbb{N} \to \mathcal{S}_n$. Set

$$\rho_{\mathsf{e}} = \{\gamma(\varphi(\mathsf{e}), u) : \varphi(\mathsf{e}) \text{ is defined } \wedge \ \exists v \ (\mathsf{e}, u, v) \in \sigma\}$$

Clearly, ρ_{e} is functional. Also, the construction of the ρ_{e}'s is effective and the parametrization property yields a total recursive function $g : \mathbf{2}^* \to \mathbf{2}^*$ such that $\rho_{\mathsf{e}} = W_{g(\mathsf{e})}$.
If $\varphi(\mathsf{e})$ is not defined then $\rho_{\mathsf{e}} = \emptyset$ so that $\Phi_{\rho_{\mathsf{e}}}$ is the constant functional which maps any function to the nowhere defined function. In particular, $\Phi_{\rho_{\mathsf{e}}}$ is not an iterator.
Suppose $\varphi(\mathsf{e})$ is defined. Condition *iii* of Prop.10.19 and the definition of ρ_{e} show that

$$\begin{aligned}\Phi_{\rho_{\mathsf{e}}} \text{ is an iterator } &\Leftrightarrow \ \Phi_{\rho_{\mathsf{e}}} = It_{\mathbb{X}}^{(\varphi(n))} \\ &\Leftrightarrow \ \rho_{\mathsf{e}} \supseteq range(u \mapsto \gamma(\varphi(n), u)) \\ &\Leftrightarrow \ \forall u \ \exists v \ (\mathsf{e}, u, v) \in \sigma \\ &\Leftrightarrow \ \mathsf{e} \in S\end{aligned}$$

Since $\rho_{\mathsf{e}} = W_{g(\mathsf{e})}$, the functional $\Phi : \mathsf{e} \mapsto \Phi_{\rho_{\mathsf{e}}}$ is effectively continuous. Clearly, it satisfies $(*)$ and $(**)$. $\square$

10.7 Syntactical complexity of Church representation

Proposition 10.23 (Syntactical complexity)**.** *The family*

$$\{domain(\varphi) : \varphi \in Church_X^{\mathbb{N}} \circ A\text{-}EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}\}$$

is exactly the family of $\Pi_2^{0,A}$ *subsets of* $\mathbf{2}^*$*.*

Thus, any universal function for $Church_X^{\mathbb{N}} \circ A\text{-}EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}$ *has* $\Pi_2^{0,A}$*-complete domain.*

Proof. To simplify notations, we only consider the case $A = \emptyset$. Relativization being straightforward.
1. Prop.10.22 insures that every Π_2^0 set is the domain of $Church_{\mathbb{X}}^{\mathbb{N}} \circ \Phi$ for some effectively continuous functional Φ.

2. Conversely, we prove that every function in $Church_X^{\mathbb{N}} \circ EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}$ has Π_2^0 domain.
Suppose $\Phi : (\mathbf{2}^* \times (\mathbb{X} \to \mathbb{X})) \to (\mathbb{X} \to \mathbb{X})$ is effectively continuous and let

$R \subseteq \mathbf{2}^* \times Fin(\mathbb{X} \to \mathbb{X}) \times \mathbb{X} \times \mathbb{X}$ be a functional r.e. set such that $\Phi = \Phi_R$. For $\mathtt{e} \in \mathbf{2}^*$, let $R^{\mathtt{e}} = \{(\alpha, \mathtt{x}, \mathtt{y}) : (\mathtt{e}, \alpha, \mathtt{x}, \mathtt{y}) \in R\}$. Then

$$domain(Church_X^{\mathbb{N}} \circ \Phi) = \{\mathtt{e} : \Phi_{R^{\mathtt{e}}} \text{ is an iterator}\}$$

Now, an r.e. code for the functional relation $R^{\mathtt{e}}$ is given by a total recursive function $h : \mathbf{2}^* \to \mathbf{2}^*$. Applying Prop.10.20, point 2, the partial recursive function $\lambda \circ h$ is such that if $\Phi_{R^{\mathtt{e}}}$ is an iterator then $\Phi_{R^{\mathtt{e}}} = It_{\mathbb{X}}^{(\lambda(h(\mathtt{e})))}$.
Thus, $\Phi_{R^{\mathtt{e}}}$ is an iterator if and only if

a. $\lambda(h(\mathtt{e}))$ is convergent,

b. condition *iii* of Prop.10.19 with $n = \lambda(h(\mathtt{e}))$ holds.

Condition a is Σ_1^0 and condition b is Π_2^0. Thus, $domain(Church_X^{\mathbb{N}} \circ \Phi)$ is Π_2^0. □

10.8 Characterization of the *Church* representation system

Theorem 10.24. *Let's denote $PR^{A,\mathbf{2}^* \to \mathbb{N}} \upharpoonright \Pi_2^{0,A}$ the family of restrictions to $\Pi_2^{0,A}$ subsets of partial A-recursive functions $\mathbf{2}^* \to \mathbb{N}$.*
Let $\mathbb{X}$ be some basic set and $A \subseteq \mathbb{N}$ be some oracle.

1. $Church \circ A\text{-}EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))} = PR^{A,\mathbf{2}^* \to \mathbb{N}} \upharpoonright \Pi_2^{0,A}$

2. $K^{\mathbb{N}}_{Church \circ A\text{-}EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}} =_{\mathrm{ct}} K^A$

We shall simply write $K^{\mathbb{N},A}_{Church}$, or $K^{\mathbb{N}}_{Church}$ when $A = \emptyset$.

Proof. 1A. First, we prove that, for any A-effectively continuous functional $\Phi : \mathbf{2}^* \to (\mathbb{X} \to \mathbb{X})^{\mathbb{X}\to\mathbb{X}}$, the function $Church \circ \Phi : \mathbf{2}^* \to \mathbb{N}$ has a partial A-recursive extension. We reduce to the case $\mathbb{X} = \mathbb{N}$.
Let $Succ : \mathbb{N} \to \mathbb{N}$ be the successor function. Observe that, for all $n \in \mathbb{N}$,

$$(It_{\mathbb{N}}^{(n)}(Succ))(0) = n$$

Thus, if $Church(\Phi(\mathtt{e})$ is defined then $Church(\Phi(\mathtt{e})) = (\Phi(\mathtt{e})(Succ))(0)$. Applying Prop.10.10, we see that $\mathtt{e} \mapsto (\Phi(\mathtt{e})(Succ))(0)$ is a partial A-recursive extension of $Church \circ \Phi : \mathbf{2}^* \to \mathbb{N}$.

1B. Prop.10.23 insures that $Church \circ \Phi : \mathbf{2}^* \to \mathbb{N}$ has $\Pi_2^{0,A}$ domain. Together with point 1A, this insures that $Church \circ \Phi : \mathbf{2}^* \to \mathbb{N}$ is the restriction of a partial A-recursive function to a $\Pi_2^{0,A}$ set. This proves the inclusion

$$Church \circ A\text{-}EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))} \subseteq PR^{A,\mathbf{2}^* \to \mathbb{N}} \upharpoonright \Pi_2^{0,A}$$

1C. The converse inclusion is Prop.10.22.

2. Inclusion $PR^{A,\mathbf{2}^* \to \mathbb{N}} \subseteq Church \circ A\text{-}EffCont^{\mathbf{2}^* \to ((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}$ yields the

inequality $K^{\mathbb{N}}_{Church\circ A\text{-}EffCont^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}} \leq_{\mathrm{ct}} K^A$.

Consider a function $\phi \in Church \circ A\text{-}EffCont^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}$. Let $\widehat{\phi}$ be a partial A-recursive extension of ϕ. Then $K_\phi \geq K_{\widehat{\phi}}$. This proves inequality $K^{\mathbb{N}}_{Church\circ A\text{-}EffCont^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}} \geq_{\mathrm{ct}} K^A$. $\square$

10.9 Characterization of the $\Delta Church$ self-enumerated systems

Theorem 10.25. *Let $\mathbb{X}$ be some basic set and $A \subseteq \mathbb{N}$ be some oracle.*

1. $\Delta(Church \circ A\text{-}EffCont^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))}) = PR^{A,\mathbf{2}^*\to\mathbb{Z}} \restriction \Pi^{0,A}_2$

2. $K^{\mathbb{Z}}_{\Delta(Church\circ A\text{-}EffCont^{\mathbf{2}^*\to((\mathbb{X}\to\mathbb{X})\to(\mathbb{X}\to\mathbb{X}))})} =_{\mathrm{ct}} K^A_{\mathbb{Z}}$

We shall simply write $K^{\mathbb{Z},A}_{\Delta Church}$, or $K^{\mathbb{Z}}_{\Delta Church}$ when $A = \emptyset$.

Proof. 1. Observe that $\Delta(PR^{A,\mathbf{2}^*\to\mathbb{N}} \restriction \Pi^{0,A}_2) = PR^{A,\mathbf{2}^*\to\mathbb{Z}} \restriction \Pi^{0,A}_2$ and apply Thm.10.24.

2. Argue as in point 2 of the proof of Thm.10.24. $\square$

10.10 Functional representations of $\mathbb{Z}$

Specific to Church representation, there is another approach for an extension to $\mathbb{Z}$: *positive and negative iterations* of injective functions over some infinite set X. Formally, I.e., letting $X \overset{1-1}{\to} X$ denote the family of injective functions, consider the $\mathbb{Z}$-iterator functional

$$It^{\mathbb{Z}}_X : \mathbb{Z} \to (X \overset{1-1}{\to} X)^{X \overset{1-1}{\to} X}$$

such that, for $n \in \mathbb{N}$, $It^{\mathbb{Z}}_X(n)(f) = f^{(n)}$ and $It^{\mathbb{Z}}_X(-n)(f) = It^{\mathbb{Z}}_X(n)(f^{-1})$. Effectivization can be done as in §10.5. Thm.10.18, Prop.10.23 and Thm.10.24 go through the $\mathbb{Z}$ context.

11 Conclusion

We have characterized Kolmogorov complexities associated to some set theoretical representations of $\mathbb{N}$ in terms of the Kolmogorov complexities associated to oracular and/or infinite computations (Thm.1.4). As a corollary, we got a hierarchy result (Thm.1.5).

These results can be improved in two directions.
First, one can consider higher order (higher than type 2) effectivizations of set theoretical representations of $\mathbb{N}$. This is the contents of a forthcoming continuation of this paper.
Second, using the results of our paper [6], the hierarchy result Thm.1.5 can

be improved with finer orderings than $<_{\rm ct}$. These orderings $\ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ are such that $f \ll_{\mathcal{F}}^{\mathcal{C},\mathcal{D}}$ g if and only if

1. $f \leq_{\rm ct} g$
2. For every infinite set $X \in \mathcal{C}$ and every total monotone increasing function $\phi \in \mathcal{F}$ there exists an infinite set $Y \in \mathcal{D}$ such that
$$Y \subseteq \{z \in X : f(z) < \phi(g(x))\}$$
3. The above property is effective: relative to standard enumerations of $\mathcal{C}, \mathcal{D}, \mathcal{F}$, a code for Y can be recursively computed from codes for X and ϕ.

Thm.1.5 can be restated in the following improved form.

Theorem 11.1. *Denote Min_{PR} (resp. Min_{PR^A}) the family of functions $\mathbb{N} \to \mathbb{N}$ which are infima of partial recursive (resp.partial A-recursive) sequences of functions $\mathbb{N} \to \mathbb{N}$ (cf. Rk.6.4). Then*

$$\log \ \gg_{\rm PR}^{\Sigma_1^0,\Sigma_1^0} \ \begin{array}{c} K^{\mathbb{N}}_{\mathit{Church}} \\ =_{\rm ct} \\ K^{\mathbb{Z}}_{\mathit{Church}} \restriction \mathbb{N} \\ =_{\rm ct} \\ K^{\mathbb{Z}}_{\Delta\mathit{Church}} \restriction \mathbb{N} \end{array} \ \gg_{\rm Min_{PR}}^{\Sigma_1^0\cup\Pi_1^0,\Delta_2^0} \ \mathrm{K}^{\mathbb{N}}_{\mathit{card}} \ \gg_{\rm PR^{\emptyset'}}^{\Sigma_2^0,\Sigma_2^0} \ \mathrm{K}^{\mathbb{Z}}_{\Delta\mathit{card}} \restriction \mathbb{N}$$
$$\gg_{\rm Min_{PR^{\emptyset'}}}^{\Sigma_2^0\cup\Pi_2^0,\Delta_3^0} \ \mathrm{K}^{\mathbb{N}}_{\mathit{index}} \ \gg_{\rm PR^{\emptyset''}}^{\Sigma_3^0,\Sigma_3^0} \ \mathrm{K}^{\mathbb{Z}}_{\Delta\mathit{index}} \restriction \mathbb{N}$$

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