Completely nonmeasurable unions

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ABSTRACT. Assume that there is no quasi-measurable cardinal smaller than 2^{ω} . (κ is quasi measurable if there exists κ -additive ideal \mathscr{I} of subsets of κ such that the Boolean algebra $P(\kappa)/\mathscr{I}$ satisfies c.c.c.) We show that for a c.c.c. σ -ideal I with a Borel base of subsets of an uncountable Polish space, if \mathscr{A} is a point-finite family of subsets from I then there is an uncountable collection of pairwise disjoint subfamilies of \mathscr{A} whose union is completely non-measurable i.e. its intersection with every non-small Borel set does not belong to the σ -field generated by Borel sets and the ideal I. This result is a generalization of Four Poles Theorem (see [1]) and results from [2] and [4].

1. Notation and motivation

In this paper X will denote an uncountable Polish space. Borel will denote all Borel subsets of X. A family $\mathbb{I} \subseteq P(X)$ will be a σ -ideal of subsets of X with Borel base containing singletons. Let us recall that \mathbb{I} has Borel base means that $(\forall I \in \mathbb{I})(\exists J \in \mathbb{I} \cap \text{Borel})(I \subseteq J)$. We have the following cardinal coefficients

$$\begin{array}{ll} \operatorname{add}(\mathbb{I}) = \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I}, \ \bigcup \mathscr{C} \notin \mathbb{I}\},\\ \operatorname{cov}(\mathbb{I}) = \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I}, \ \bigcup \mathscr{C} = X\},\\ \operatorname{cov}_h(\mathbb{I}) = \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I}, \ (\exists B \in \operatorname{Borel} \setminus \mathbb{I})(\bigcup \mathscr{C} \supseteq B)\},\\ \operatorname{cof}(\mathbb{I}) = \min\{|\mathscr{C}| : \ \mathscr{C} \subseteq \mathbb{I}, \ (\forall I \in \mathbb{I})(\exists C \in \mathscr{C})(I \subseteq C)\}. \end{array}$$

Similarly for a family $\mathscr{A} \subseteq P(X)$ we can define

 $\begin{array}{ll} \operatorname{add}(\mathscr{A}) = \min\{|\mathscr{C}|: \ \mathscr{C} \subseteq \mathscr{A}, \ \bigcup \mathscr{C} \notin \mathbb{I}\}, \\ \operatorname{cov}_h^{\mathbb{I}}(\mathscr{A}) = \min\{|\mathscr{C}|: \ \mathscr{C} \subseteq \mathscr{A}, \ (\exists B \in \operatorname{Borel} \setminus \mathbb{I})(\bigcup \mathscr{C} \supseteq B)\}. \end{array}$

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We start our consideration with the following theorem from [1]. It is known in literature as Four Poles Theorem.

Theorem 1.1 (Brzuchowski, Cichoń, Grzegorek, Ryll-Nardzewski). Let $\mathscr{A} \subseteq \mathbb{I}$ be a point finite cover of X i.e. $(\forall x \in X) | \{A \in \mathscr{A} : x \in A\} | < \omega$. Then there exists a subfamily \mathscr{A}' such that $\bigcup \mathscr{A}'$ is not \mathbb{I} -measurable i.e. does not belong to the σ -field generating by Borel and \mathbb{I} .

There is a hypothesis stated by J. Cichoń saying that we can improve the conclusion of the above theorem to get $\bigcup \mathscr{A}'$ completely I-nonmeasurable.

Definition 1.1. We say that C is completely \mathbb{I} -nonmeasurable in D iff

$$(\forall B \in \text{Borel} \setminus \mathbb{I}) (B \cap D \notin \mathbb{I} \longrightarrow (B \cap C \notin \mathbb{I} \land B \cap (D \setminus C) \notin \mathbb{I})).$$

Recall that \mathbb{I} is c.c.c. if every family $\mathscr{A} \subseteq \text{Borel} \setminus \mathbb{I}$ such that

 $(\forall A, A' \in \mathscr{A})(A = A' \lor A \cap A' \in \mathbb{I})$

is at most countable.

If \mathbb{I} is c.c.c then we can define $[D]_{\mathbb{I}}$ to be a minimal (modulo \mathbb{I}) Borel set B containing D i.e. $D \setminus B \in \mathbb{I}$ and if $D \subseteq C$ and C is Borel then $B \setminus C \in \mathbb{I}$.

Assume that $\mathscr{A} \subseteq \mathbb{I}$. Let \mathscr{I} be an ideal on $P(\mathscr{A})$ associated with \mathbb{I} in the following way

$$(\forall \mathscr{X} \in P(\mathscr{A}))(\mathscr{X} \in \mathscr{I} \longleftrightarrow \bigcup \mathscr{X} \in \mathbb{I}).$$

Then $W \subseteq P(\mathscr{A})$ is an antichain in $P(\mathscr{A})/\mathscr{I}$ iff $(\forall a, b \in W)(a \neq b \longrightarrow a \cap b \in \mathscr{I})$. We say that $P(\mathscr{A})/\mathscr{I}$ is c.c.c. iff every antichain on $P(\mathscr{A})/\mathscr{I}$ is at most countable.

We say that the cardinal number κ is quasi-measurable if there exists κ -additive ideal \mathscr{I} of subsets of κ such that the Boolean algebra $P(\kappa)/\mathscr{I}$ satisfies c.c.c. Cardinal κ is weakly inaccessible if κ is regular cardinal and for every cardinal $\lambda < \kappa$ we have that $\lambda^+ < \kappa$. Recall that every quasi-measurable cardinal is weakly inaccessible (see [3]), so it is a large cardinal.

Let us recall a result from [4].

Theorem 1.2 (Zeberski). Assume that there is no quasi-measurable cardinal not greater than 2^{ω} . Assume that \mathbb{I} satisfies c.c.c. Let $\mathscr{A} \subseteq \mathbb{I}$ be a point-finite family such that $\bigcup \mathscr{A} \notin \mathbb{I}$. Then there exists a subfamily $\mathscr{A}' \subseteq \mathscr{A}$ such that $\bigcup \mathscr{A}'$ is completely \mathbb{I} -nonmeasurable in $\bigcup \mathscr{A}$.

2. Results

Let us recall three technical lemmas from [4] (Theorem 3.3, Lemma 3.4, Lemma 3.5).

Lemma 2.1 (Zeberski). Assume that \mathbb{I} satisfies c.c.c. Let $\{A_{\xi} : \xi \in \omega_1\}$ be any family of subsets of X. Then we can find a family $\{I_{\alpha}\}_{\alpha \in \omega_1}$ of pairwise disjoint countable subsets of ω_1 such that for $\alpha < \beta < \omega_1$ we have that $[\bigcup_{\xi \in I_{\alpha}} A_{\xi}]_{\mathbb{I}} = [\bigcup_{\xi \in I_{\beta}} A_{\xi}]_{\mathbb{I}}$.

Next lemma is a reformulation of a result obtained in [4].

Lemma 2.2 (Zeberski). Assume that \mathbb{I} satisfies c.c.c. Let $\mathscr{A} \subseteq \mathbb{I}$ be a point-finite family such that $\bigcup \mathscr{A} \notin \mathbb{I}$ and the algebra $P(\mathscr{A})/\mathscr{I}$ is not c.c.c. Then there exists a family $\{\mathscr{A}_{\alpha}\}_{\alpha \in \omega_1}$ satisfying the following conditions

- (1) $(\forall \alpha < \omega_1)(\mathscr{A}_{\alpha} \subseteq \mathscr{A} \land \bigcup \mathscr{A}_{\alpha} \notin \mathbb{I}),$
- (2) $(\forall \alpha < \beta < \omega_1) (\mathscr{A}_{\alpha} \cap \mathscr{A}_{\beta} = \emptyset),$
- (3) $(\forall \alpha, \beta < \omega_1)([\bigcup \mathscr{A}_{\alpha}]_{\mathbb{I}} = [\bigcup \mathscr{A}_{\beta}]_{\mathbb{I}}).$

Lemma 2.3 (Żeberski). Assume that I satisfies c.c.c. Let $\mathscr{A} \subseteq P(X)$ be any point-finite family. Then there exists a subfamily $\mathscr{A}' \subseteq \mathscr{A}$ such that $|\mathscr{A} \setminus \mathscr{A}'| \leq \omega$ and

$$(\forall B \in \text{Borel} \setminus \mathbb{I})(\forall A \in \mathscr{A}')(B \cap \bigcup \mathscr{A} \notin \mathbb{I} \to \neg(B \cap \bigcup \mathscr{A} \subseteq B \cap A)).$$

In paper [2] it is shown that if $\operatorname{cov}_h(\mathbb{I}) = \operatorname{cof}(\mathbb{I})$ and $\mathscr{A} \subseteq \mathbb{I}$ is a cover of X such that $\bigcup \{A \in \mathscr{A} : x \in A\} \in \mathbb{I}$ for every $x \in X$, then there is a family $\mathscr{A}' \subseteq \mathscr{A}$ such that $\bigcup \mathscr{A}'$ is completely \mathbb{I} nonmeasurable. This result can be generalized. Namely, we have the following theorem.

Theorem 2.1. Let $\mathscr{A} \subseteq \mathbb{I}$ be a family satisfying the following conditions:

$$\begin{array}{ll} (1) \ (\forall B \in \operatorname{Borel} \setminus \mathbb{I}) | \{ \mathscr{A}(x) : \bigcup \mathscr{A}(x) \cap B \neq \emptyset, x \in X \} | = 2^{\omega}, \\ (2) \ \operatorname{cov}_{h}^{\mathbb{I}}(\{\bigcup \mathscr{A}(x) : \ x \in X \}) = 2^{\omega}, \end{array}$$

where $\mathscr{A}(x) = \{A \in \mathscr{A} : x \in A\}$. Then there exists continuum many pairwise disjoint subfamilies $\{\mathscr{A}_{\alpha} : \alpha \in 2^{\omega}\}$ of a family \mathscr{A} such that for every $\alpha \in 2^{\omega}$ a set $\bigcup \mathscr{A}_{\alpha}$ is completely \mathbb{I} -nonmeasurable.

PROOF. Let us enumerate the set of all Borel I positive sets Borel \setminus I = { B_{α} : $\alpha < 2^{\omega}$ }. By transfinite induction we will construct a sequence

$$((A_{\xi,\eta}, d_{\xi}) \in \mathscr{A} \times B_{\xi} : \xi, \eta < 2^{\omega})$$

with the following conditions:

(1) $(\forall \xi, \eta < 2^{\omega})(A_{\xi,\eta} \cap B_{\xi} \neq \emptyset),$

(2) $\bigcup_{\xi,\eta<2^{\omega}} A_{\xi,\eta} \cap \{d_{\xi} : \xi < 2^{\omega}\} = \emptyset,$ (3) $(\forall \xi, \xi' < 2^{\omega})(\forall \eta, \eta' < 2^{\omega})(\eta \neq \eta' \longrightarrow A_{\xi,\eta} \neq A_{\xi',\eta'}).$

Let us fix $\alpha < 2^{\omega}$ and assume that we have defined the sequence

$$((A_{\xi,\eta}, d_{\xi}) \in \mathscr{A} \times B_{\xi} : \xi, \eta < \alpha)$$

with the following conditions:

 $(1) \quad (\forall \xi, \eta < \alpha) (A_{\xi,\eta} \cap B_{\xi} \neq \emptyset),$ $(2) \quad \bigcup_{\xi,\eta < \alpha} A_{\xi,\eta} \cap \{d_{\xi} : \xi < \alpha\} = \emptyset,$ $(3) \quad (\forall \xi, \xi' < \alpha) (\forall \eta, \eta' < \alpha) (\eta \neq \eta' \longrightarrow A_{\xi,\eta} \neq A_{\xi',\eta'}).$

For every $\xi < \alpha$ let us consider the set $\mathscr{A}(d_{\xi}) = \{A \in \mathscr{A} : d_{\xi} \in A\}$. By assumption (2) the family $\bigcup_{\xi < \alpha} \mathscr{A}(d_{\xi})$ does not cover any I-possitive Borel set. So, assumptions (1) and (2) guaranties that we can choose the set $\{A_{\alpha,\eta} \in \mathscr{A} : \eta < \alpha\}$ such that

- (1) $(\forall \xi, \xi' \leq \alpha) (\forall \eta, \eta' < \alpha) (\eta \neq \eta' \longrightarrow A_{\xi,\eta} \neq A_{\xi',\eta'}),$
- (2) $(\forall \eta < \alpha)(A_{\alpha,\eta} \cap B_{\alpha} \neq \emptyset),$
- (3) $(\forall \xi, \eta < \alpha)(d_{\xi} \notin A_{\alpha,\eta}).$

The same argument gives us the set $\{A_{\xi,\alpha} \in \mathscr{A} : \xi \leq \alpha\}$ with the following properties:

- (1) $(\forall \xi, \xi' \leq \alpha) (\forall \eta < \alpha) (A_{\xi,\eta} \neq A_{\xi',\alpha}),$
- (2) $(\forall \xi \leq \alpha) (A_{\xi,\alpha} \cap B_{\xi} \neq \emptyset \land A_{\xi,\alpha} \cap \{d_{\xi'} : \xi' < \alpha\} = \emptyset).$

Once again by assumption (2) we can find $d_{\alpha} \in B_{\alpha}$ such that $(\bigcup_{\xi,\eta \leq \alpha} A_{\xi,\eta}) \cap \{d_{\alpha}\} = \emptyset$. It finishes the α -step of our construction.

Now, let us put $\mathscr{A}_{\eta} = \{A_{\xi,\eta} \in \mathscr{A} : \xi < 2^{\omega}\}$ for any $\eta < 2^{\omega}$. The family $\{\mathscr{A}_{\eta} : \eta < 2^{\omega}\}$ fulfills the assertion of our Theorem. \Box

Proposition 2.1. If $\operatorname{cov}_h(\mathbb{I}) = 2^{\omega}$ is a regular cardinal and $\mathscr{A} \subseteq \mathbb{I}$ is a cover of X such that each point is covered by less than continuum many members of \mathscr{A} then there exists continuum many pairwise disjoint subfamilies $\{\mathscr{A}_{\alpha} : \alpha \in 2^{\omega}\}$ of a family \mathscr{A} such that for every $\alpha \in 2^{\omega}$ a set $\bigcup \mathscr{A}_{\alpha}$ is completely \mathbb{I} -nonmeasurable.

Proposition 2.2. If $\operatorname{cov}_h(\mathbb{I}) = 2^{\omega}$ and $\mathscr{A} \subseteq \mathbb{I}$ is a point-finite family such that $\bigcup \mathscr{A} \notin \mathbb{I}$ then there exists continuum many pairwise disjoint subfamilies $\{\mathscr{A}_{\alpha} : \alpha \in 2^{\omega}\}$ of a family \mathscr{A} such that for every $\alpha \in 2^{\omega}$ a set $\bigcup \mathscr{A}_{\alpha}$ is completely \mathbb{I} -nonmeasurable in $\bigcup \mathscr{A}$.

PROOF. First, we use Theorem 2.1 to obtain continuum many pairwise disjoint subfamilies \mathscr{A}_{α} for $\alpha < 2^{\omega}$ such that $\bigcup \mathscr{A}_{\alpha}$ is completely I-nonmeasurable in $[\bigcup \mathscr{A}]_{\mathbb{I}}$. Then by point-finiteness of family \mathscr{A} the family $\{\bigcup \mathscr{A}_{\alpha}\}_{\alpha<2^{\omega}}$ is also point-finite. Using Lemma 2.3 we can find a countable set $C \in [2^{\omega}]^{\omega}$ such that each member of the family $\{\bigcup \mathscr{A}_{\alpha} : \mathcal{A}_{\alpha} : \mathcal{$ $\alpha \in 2^{\omega} \setminus C$ does not contain any \mathbb{I} -possitive Borel set with respect to $\bigcup \mathscr{A}$. So, the family $\{\mathscr{A}_{\alpha} : \alpha \in 2^{\omega} \setminus C\}$ satisfies required conditions. \Box

Recall that the σ -ideal I has Steinhaus property if for any two Ipositive Borel sets $A, B \in \text{Borel} \setminus I$ the complex sum $A + B = \{a + b : a \in A, b \in B\}$ contains nonempty open set. Let us remark that if the ideal I has Steinhaus property then $\text{cov}_h(I) = \text{cov}(I)$.

Theorem 2.2. Assume that 2^{ω} is the smallest quasi-measurable cardinal. Let $\mathscr{A} \subseteq \mathbb{I}$ be a point-finite family such that $\bigcup \mathscr{A} \notin \mathbb{I}$. Then $P(\mathscr{A})/\mathscr{I}$ is not c.c.c.

PROOF. Assume that $\mathscr{A} \subseteq \mathbb{I}$ satisfies the following conditions

- (1) $\bigcup \mathscr{A} \notin \mathbb{I}$,
- (2) $P(\mathscr{A})/\mathscr{I}$ is c.c.c.

Since 2^{ω} is the minimal quasi-measurable cardinal, $|\mathscr{A}| = 2^{\omega}$. Moreover $\operatorname{add}(\mathscr{A}) = 2^{\omega}$. By point-finiteness of the family \mathscr{A} we get that $\operatorname{add}(\{\bigcup \mathscr{A}(x): x \in X\}) = 2^{\omega}$, where $\mathscr{A}(x) = \{A \in \mathscr{A}: x \in A\}$. So the family \mathscr{A} fulfils the assumptions of Theorem 2.1 (for $X = [\bigcup \mathscr{A}]_{\mathbb{I}}$). By Theorem 2.1 there exists $\{\mathscr{C}_{\alpha}: \alpha < 2^{\omega}\}$ such that

- (1) $\mathscr{C}_{\alpha} \subseteq \mathscr{A}$ for any $\alpha < 2^{\omega}$,
- (2) $\forall \alpha < 2^{\omega} \quad \bigcup \mathscr{C}_{\alpha}$ is completely \mathbb{I} -nonmeasurable in $[\bigcup \mathscr{A}]_{\mathbb{I}}$,
- (3) $\forall \alpha, \beta < 2^{\omega} \ \alpha \neq \beta \longrightarrow \mathscr{C}_{\alpha} \cap \mathscr{C}_{\beta} = \emptyset.$

In particular, a family $\{\mathscr{C}_{\alpha}: \alpha < 2^{\omega}\}$ forms an antichain in $P(\mathscr{A})/\mathscr{I}$, what gives a contradiction.

Theorem 2.3. Assume there is no quasi-measurable cardinal smaller than 2^{ω} . Assume that the ideal \mathbb{I} is c.c.c. Let $\mathscr{A} \subseteq \mathbb{I}$ be a family satisfying the following conditions:

- (1) $\bigcup \mathscr{A} \notin \mathbb{I},$
- (2) $(\forall x \in X) | \{A \in \mathscr{A} : x \in A\} | < \omega.$

Then there exists pairwise disjoint subfamilies $\{\mathscr{A}_{\xi} : \xi \in \omega_1\}$ of a family \mathscr{A} such that each of the union $\bigcup \mathscr{A}_{\xi}$ is completely \mathbb{I} -nonmeasurable in $\bigcup \mathscr{A}$.

PROOF. By transfinite induction we construct a family $\{B_{\alpha}\}$ of pairwise disjoint Borel sets and a family $\{\{\mathscr{A}_{\xi}^{\alpha}\}_{\xi\in\omega_1}\}$ of subfamilies of \mathscr{A} satisfying the following conditions

- (1) $B_{\alpha} \cap \bigcup \mathscr{A} \notin \mathbb{I}$,
- (2) $(\forall \xi < \zeta < \omega_1)(\mathscr{A}^{\alpha}_{\xi} \cap \mathscr{A}^{\alpha}_{\zeta} = \emptyset),$
- (3) $(\forall \xi < \omega_1)([\bigcup \mathscr{A}_{\xi}^{\alpha} \setminus \bigcup_{\beta < \alpha} B_{\beta}]_{\mathbb{I}} = B_{\alpha}).$

At α -step we consider the family $\mathscr{A}^{\alpha} = \{A \setminus \bigcup_{\xi < \alpha} B_{\xi} : A \in \mathscr{A} \setminus \bigcup_{\xi < \alpha} \mathscr{A}_{\xi}\}$. If $\bigcup \mathscr{A}^{\alpha} \in \mathbb{I}$ then we finish our construction. If $\bigcup \mathscr{A}^{\alpha} \notin \mathbb{I}$

I then by Theorem 2.2 the algebra $P(\mathscr{A}^{\alpha})/\mathscr{I}$ is not c.c.c. We use Lemma 2.2 to obtain a required family $\{\mathscr{A}_{\xi}^{\alpha}\}_{\xi\in\omega_1}$. We put $B_{\alpha} = [\bigcup \mathscr{A}_0^{\alpha} \setminus$ $\bigcup_{\zeta < \alpha} B_{\zeta}]_{\mathbb{I}}.$

Since I satisfies c.c.c. the construction have to end up at some step $\gamma < \omega_1$.

Now put $\mathscr{A}'_{\xi} = \bigcup_{\alpha < \gamma} \mathscr{A}^{\alpha}_{\xi}$. By construction for each $\xi < \omega_1$ we have $[\bigcup \mathscr{A}'_{\xi}]_{\mathbb{I}} = \bigcup_{\alpha < \gamma} B_{\alpha} = [\bigcup \mathscr{A}]_{\mathbb{I}}$. The family $\{\bigcup \mathscr{A}'_{\xi} : \xi \in \omega_1\}$ is point finite because for every $x \in X$

$$\left|\left\{\bigcup\mathscr{A}'_{\xi}: x \in \bigcup\mathscr{A}'_{\xi}\right\}\right| \le |\{A \in \mathscr{A}: x \in A\}| < \omega.$$

Now using Lemma 2.3 we can find a countable set $C \in [\omega_1]^{\omega}$ such that each member of the family $\{\bigcup \mathscr{A}'_{\xi} : \xi \in \omega_1 \setminus C\}$ does not contain any I-positive set of the form $B \cap \bigcup \mathscr{A}$, where B is Borel. So, the family $\{\mathscr{A}'_{\xi}: \xi \in \omega_1 \setminus C\}$ satisfies required conditions.

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