ON LONG INCREASING CHAINS MODULO FLAT IDEALS

SAHARON SHELAH

ABSTRACT. We prove that, e.g., in $(\omega_3)(\omega_3)$ there is no sequence of length ω_4 increasing modulo the ideal of countable sets.

This note is concerned with the depth of the partial order of the functions in ${}^{\kappa}\gamma$ modulo the ideal of the form $\mathcal{I} = [\kappa]^{<\mu}$. Let us recall the following definitions.

Definition 1. For a partial order (P, \sqsubset) we define

- Depth $(P, \Box) = \sup\{|\mathcal{F}| : \mathcal{F} \subseteq P \text{ is well-ordered by } \Box\}$ [the depth]
- cf $(P, \sqsubset) = \min\{|\mathcal{F}| : \mathcal{F} \subseteq P \text{ is } \sqsubset \text{-cofinal which mean that for every } p \in P \text{ there is } q \in \mathcal{F} \text{ such that } p \sqsubseteq p \} \text{ [the cofinality].}$

Our result (Theorem 4) states that under suitable assumptions the depth of the partial order $(\kappa_{\gamma}, <_{[\kappa]^{<\mu}})$ is at most $|\gamma|$. In particular, letting $\mu = \aleph_1$, $\kappa = |\gamma| = \aleph_3$ we obtain that in $(\omega_3)(\omega_3)$ there is no sequence of length ω_4 increasing modulo the ideal of countable sets.

Let $\kappa = \operatorname{cf}(\kappa) > \aleph_0$. If $\mu = \kappa$, then $\operatorname{Depth}({}^{\kappa}\kappa, <_{J_{\kappa}^{\operatorname{bd}}})$ can be (forced to be) large. But for $\mu > \operatorname{Depth}({}^{\kappa}\mu, <_{J_{\operatorname{bd}}})$ this implies pcf results (see [Sh 410], [Sh 589]).

However, e.g., for the ideal $\mathcal{I} = [\omega_3]^{\leq \aleph_0}$ it is harder to get long increasing sequence, as above for "high μ ", this leads to pcf results e.g. if we assume that $\bar{\lambda} = \langle \lambda_i : i < \omega_3 \rangle \in {}^{\omega_3}\mathrm{Reg}$, and in $(\prod \bar{\lambda}, <_{\mathcal{I}})$ there is an increasing sequence moduo \mathcal{I} of length say $> 2^{\aleph_3} + \sup\{\lambda_i : i < \omega_3\}$ are much stronger than known consistency results. Even for $I = [\omega_1]^{\leq \aleph_0}$ we do not know, for $I = [\beth_{\omega}]^{\leq \aleph_0}$ we know ([Sh 460]), so even $[\aleph_{\omega}]^{\leq \aleph_0}$ would be interesting good news.

We hope sometime to prove, e.g.,

Conjecture 2. For every $\mu > \theta$, in $(\theta^{+3})\mu$ there is no increasing sequence of length μ^+ modulo $[\theta^{+3}]^{\leq \theta}$.

Problem 3. Is it consistent that ${}^{\theta}\theta$ contains $<_{\mathcal{I}}$ -increasing sequence of length θ^+ when $\theta = \kappa^+$ and $\mathcal{I} = [\theta]^{<\kappa}$?

Notation: Our notation is rather standard and compatible with that of classical textbooks (like Jech [J]).

(1) Ordinal numbers will be denoted be the lower case initial letters of the Greek alphabet $\alpha, \beta, \gamma, \delta, \ldots$ (with possible subscripts). Cardinal numbers will be called $\kappa, \lambda, \mu, \theta$.

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(2) For a set X and a cardinal θ , $[X]^{\theta}$ (or $[X]^{<\theta}$, respectively) stands for the family of subsets of X of size θ ($<\theta$, respectively).

Theorem 4. Assume $\mu^+ < \kappa \le \theta$ and $\mathcal{J} = [\kappa]^{<\mu}$ and $\operatorname{cf}([\theta]^{\mu}, \subseteq) \le \theta$. Let $\gamma < \theta^+$. Then $\operatorname{Depth}(^{\kappa}\gamma, <_{\mathcal{J}}) \le \theta$, i.e., there is no $<_{\mathcal{J}}$ -increasing sequence $\langle f_{\alpha} : \alpha < \theta^+ \rangle$ of functions from $^{\kappa}\gamma$ modulo \mathcal{J} .

Proof. Assume towards contradiction that there is a $<_{\mathcal{J}}$ -increasing sequence $\langle f_{\zeta} : \zeta < \theta^{+} \rangle \subseteq {}^{\kappa} \gamma$.

Let $S \subseteq [\gamma]^{\mu}$ be cofinal of cardinality $\leq \theta$ (exists as $|\gamma| \leq \theta$ and $\mathrm{cf}([\theta]^{\mu}, \subseteq) \leq \theta$). For every $s \in S$ and $\beta < \kappa$, $\zeta < \theta^+$ we let

- $I(\beta) := [\beta, \beta + \mu),$
- $f_{\zeta}^s \in {}^{\kappa}(\gamma+1)$ be defined by $f_{\zeta}^s(i) = \min(s \cup \{\gamma\} \setminus f_{\zeta}(i)),$
- $f_{\zeta}^{s,\beta} \in I(\beta)(\gamma+1)$ be defined as $f_{\zeta}^{s} \upharpoonright I(\beta)$.

Now, for each $s \in \mathcal{S}$ we have

- (*)₁ (a) for every $\zeta < \theta^+$, $f_{\zeta}^{s,\beta} : I(\beta) \longrightarrow s \cup \{\gamma\}$,
 - (b) if $\zeta < \xi < \theta^+$, then $f_{\zeta}^{s,\beta} \le f_{\xi}^{s,\beta} \mod [I(\beta)]^{<\mu}$.

For $s \in \mathcal{S}$ we define

$$(*)_2 \ B_s = \{\beta < \kappa : (\forall \zeta < \theta^+)(\exists \xi > \zeta) \neg (f_{\zeta}^{s,\beta} = f_{\xi}^{s,\beta} \mod [I(\beta)]^{<\mu})\}.$$

Plainly, we may choose a sequence $\langle C^s_{\beta} : \beta < \kappa, \ s \in \mathcal{S} \rangle$ such that

- $(*)_3$ (a) C^s_{β} is a club of θ^+ ,
 - (b) if $\beta \in B_s$ and $\xi, \zeta \in C^s_\beta$ are such that $\zeta < \xi$, then $\neg (f^{s,\beta}_\zeta = f^{s,\beta}_\xi \mod [I(\beta)]^{<\mu})$,
 - (c) if $\beta \in \kappa \setminus B_s$, then $f_{\zeta}^{s,\beta} = f_{\xi}^{s,\beta} \mod [I(\beta)]^{<\mu}$ whenever $\min(C_{\beta}^s) \leq \zeta \leq \xi < \theta^+$.

Then, as $|S| \leq \theta$ and $\kappa \leq \theta$, we have

$$(*)_4$$
 the set $C := \bigcap \{C^s_\beta : s \in \mathcal{S} \text{ and } \beta < \kappa\}$ is a club of θ^+ .

Choose a sequence $\langle \alpha_{\varepsilon} : \varepsilon < \mu^+ \rangle \subseteq C$ increasing with ε . Then, for all $\varepsilon < \zeta < \mu^+$,

$$(*)_5 \ u_{\varepsilon,\zeta} := \{i < \kappa : f_{\alpha_{\varepsilon}}(i) \ge f_{\alpha_{\zeta}}(i)\} \in \mathcal{J}.$$

We have assumed that $\mu^+ < \kappa$, so we can find $\delta < \kappa$ such that

- $(*)_6$ (a) $I(\delta) = [\delta, \delta + \mu)$ is disjoint from $\bigcup \{u_{\varepsilon,\zeta} : \varepsilon < \zeta < \mu^+\}$, and hence
 - (b) the sequence $\langle f_{\alpha_{\varepsilon}}(i) : \varepsilon < \mu^{+} \rangle$ is increasing for each $i \in I(\delta)$.

As $|I(\delta)| = \mu$ and $S \subseteq [\gamma]^{\leq \mu}$ is cofinal (for the partial order \subseteq), we can find $s \in S$ such that

$$(*)_7 \{f_{\alpha_0}(i), f_{\alpha_1}(i) : i \in I(\delta)\} \subseteq s.$$

It follows from $(*)_6 + (*)_7$ that for every $i \in I(\delta)$

$$(*)_8 f_{\alpha_0}^s(i) = f_{\alpha_0}(i) < f_{\alpha_1}(i) = f_{\alpha_1}^s(i).$$

As $\alpha_0 < \alpha_1$ are from C and $I(\delta) \notin \mathcal{J}$, recalling $(*)_2 + (*)_3 + (*)_4$, clearly $(*)_9 \ \delta \in B_s$.

Therefore, as $\alpha_{\varepsilon} \in C \subseteq C_{\delta}^{s}$ for $\varepsilon < \mu^{+}$ and α_{ε} is increasing with ε , we have

- $(*)_{10}$ for every $\varepsilon < \mu^+$ there is $i_{\varepsilon} \in I(\delta)$ such that
 - $(\alpha) \ f_{\alpha_{\varepsilon}}^{s}(i_{\varepsilon}) < f_{\alpha_{\varepsilon+1}}^{s}(i_{\varepsilon}),$

and hence there is $j_{\varepsilon} \in s$ such that

$$(\beta) f_{\alpha_{\varepsilon}}^{s}(i_{\varepsilon}) \leq j_{\varepsilon} < f_{\alpha_{\varepsilon+1}}^{s}(i_{\varepsilon})$$

and therefore

$$(\gamma) \ f_{\alpha_{\varepsilon}}(i_{\varepsilon}) \le j_{\varepsilon} < f_{\alpha_{\varepsilon+1}}(i_{\varepsilon}).$$

But $|I(\delta)| + |s| = \mu < \mu^+$, so for some pair $(j_*, i_*) \in s \times I(\delta)$ we may choose $\varepsilon_1 < \varepsilon_2 < \mu^+$ such that

$$(*)_{11}$$
 $j_{\varepsilon_1} = j_{\varepsilon_2} = j_*$ and $i_{\varepsilon_1} = i_{\varepsilon_2} = i_*$.

But the sequence $\langle f_{\alpha_{\varepsilon}}(i_*) : \varepsilon < \theta^+ \rangle$ is increasing by $(*)_6(b)$ (see the choice of δ), so

$$f_{\alpha_{\varepsilon_1}}(i_*) < f_{\alpha_{\varepsilon_1+1}}(i_*) \le f_{\alpha_{\varepsilon_2}}(i_*) < f_{\alpha_{\varepsilon_2+1}}(i_*).$$

It follows from $(*)_{10}(\gamma) + (*)_{11}$ that the ordinal j_* belongs to $[f_{\alpha_{\varepsilon_1}}(i_*), f_{\alpha_{\varepsilon_1+1}}(i_*))$ and to $[f_{\alpha_{\varepsilon_2}}(i_*), f_{\alpha_{\varepsilon_2+1}}(i_*))$, which are disjoint intervals, a contradiction.

Similarly,

Theorem 5. Assume that

- (a) \mathcal{J} is an ideal on κ ,
- (b) $I_{\beta} \in [\kappa]^{\mu}$, $I_{\beta} \notin \mathcal{J}$ for $\beta < \kappa$,
- (c) $\theta = |\gamma| + \kappa \text{ and } \operatorname{cf}([\theta]^{\mu}, \subseteq) < \lambda,$
- (d) if $u_{\varepsilon} \in \mathcal{J}$ for $\varepsilon < \mu^{+}$, then for some $\beta < \kappa$ the set I_{β} is disjoint from $\bigcup_{\varepsilon < \mu^{+}} u_{\varepsilon}$.

Then there is no $<_{\mathcal{J}}$ -increasing sequence of functions from κ to γ of length λ .

Proof. Without loss of generality λ is the successor of $\operatorname{cf}([\theta]^{\mu}, \subseteq)$ hence is regular. The proof is similar to the proof of Theorem 4.

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 $E ext{-}mail\ address: shelah@math.huji.ac.il}$

THE HEBREW UNIVERSITY OF JERUSALEM, EINSTEIN INSTITUTE OF MATHEMATICS, EDMOND J. SAFRA CAMPUS, GIVAT RAM, JERUSALEM 91904, ISRAEL, AND DEPARTMENT OF MATHEMATICS, HILL CENTER-BUSCH CAMPUS, RUTGERS, THE STATE UNIVERSITY OF NEW JERSEY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019 USA