Zariski-type Topology for Implication Algebras

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In this work we provide a new topological representation for implication algebras in such a way that its one-point compactification is the topological space given in [1]. Some applications are given thereof.

1 Introduction and Preliminaries

Among the algebraic structures associated with logical systems, implication structures are particularly frequent. Generally, they consist of a partially ordered set where the order is characterized by a binary implication operation $\rightarrow$. If the ordered set is a join-semilattice whose principal filters are Boolean algebras, we obtain implication algebras [2, 3], which are also known as Tarski algebras [7] - the variety of $\{\rightarrow\}$-subreducts of Boolean algebras.

In this work we continue our study of implication algebras. In [1] we represent an implication algebra as a union of a unique family of filters of a suitable Boolean algebra $\mathbf{Bo}(\boldsymbol{A})$, and we use the Stone space of $\mathbf{Bo}(\boldsymbol{A})$ to obtain a topological representation for $\boldsymbol{A}$. Now we define a Zariski type topology on the set $Spec(\boldsymbol{A})$ of maximal implicative filters of $\boldsymbol{A}$ in such a way that the Stone space of $\mathbf{Bo}(\boldsymbol{A})$ is homeomorphic to the one-point compactification of the topological space $Spec(\boldsymbol{A})$. This is an intrinsic construction in the sense that it does not depend on the embedding of $\boldsymbol{A}$ into $\mathbf{Bo}(\boldsymbol{A})$.

To start, let us recall the definition of implication algebras.

An *implication algebra* is an algebra $\langle A, \rightarrow\rangle$ of type $\langle 2\rangle$ that satisfies the equations:

(I1) $(x \rightarrow y) \rightarrow x = x$,

(I2) $(x \rightarrow y) \rightarrow y = (y \rightarrow x) \rightarrow x$,

(I3) $x \rightarrow (y \rightarrow z) = y \rightarrow (x \rightarrow z)$.

Just to put this class of algebras in a wider context, let us say that an implication algebra is a BCK-algebra that satisfies the equation $(x \rightarrow y) \rightarrow x = x$.

In any implication algebra $\boldsymbol{A}$ the term $x \rightarrow x$ is constant, which we represent by 1. The relation $x \leq y$ if and only if $x \rightarrow y = 1$ is a partial order, called *the natural order of* A, with 1 as its greatest element. Relative to this partial order, $\boldsymbol{A}$ is a join-semilattice and the join of two elements a and b is given by $a \vee b = (a \rightarrow b) \rightarrow b$. Besides, for each a in A, $[a) = \{x \in A : a \leq x\}$ is a Boolean algebra in which, for $b, c \geq a$, $b \wedge c = (b \rightarrow (c \rightarrow a)) \rightarrow a$ gives the meet and $b \rightarrow a$ is the complement of b in $[a)$. In fact, following [2, Theorems 6 and 7], implication algebras are precisely join-semilattices with greatest element such that for each element a, $[a)$ with the inherited order is a Boolean algebra.

If $\boldsymbol{A}$ is an implication algebra, there is a Boolean algebra $\boldsymbol{B}$ such that $\boldsymbol{A}$ is an implication subalgebra of $\boldsymbol{B}$ (see [2, Theorem 17]). Let $B(\boldsymbol{A})$ be the Boolean subalgebra of $\boldsymbol{B}$ generated by $\boldsymbol{A}$, and $F(\boldsymbol{A})$ the filter generated

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by A in $B(\boldsymbol{A})$. A is increasing in $B(\boldsymbol{A})$ [1] (a new shorter proof is given in Lemma 2.2), and consequently, A is a union of filters of $B(\boldsymbol{A})$.

A subset C of a Boolean algebra $\boldsymbol{B}$ satisfies the *finite meet property* (fmp for short), provided that 0 cannot be obtained with finite meets of elements of C, that is, the lattice filter generated by C in $\boldsymbol{B}$ is proper. The fmp is the analogue of the *finite intersection property* for set boolean algebras.

Consider the following Boolean algebra, called the *Boolean closure* of $\boldsymbol{A}$:

$$\mathbf{Bo}(\boldsymbol{A}) = \begin{cases} B(\boldsymbol{A}) & \text{if } F(\boldsymbol{A}) \neq B(\boldsymbol{A}) \\ B(\boldsymbol{A}) \times \{0,1\} & \text{if } F(\boldsymbol{A}) = B(\boldsymbol{A}) \end{cases}$$

Theorem 1.1 [1] *Let $\boldsymbol{A}$ be an implication algebra. Then*

(1) *A is an increasing subset of $\mathbf{Bo}(\boldsymbol{A})$ and $\boldsymbol{A}$ satisfies the* fmp.

(2) *If $h\colon \boldsymbol{A} \to \boldsymbol{B}$ is an $\{\to\}$-homomorphism from the implication algebra $\boldsymbol{A}$ into a Boolean algebra $\boldsymbol{B}$, such that $h[\boldsymbol{A}]$ has the* fmp *in $\boldsymbol{B}$, then there is a Boolean homomorphism $\widehat{h}\colon \mathbf{Bo}(\boldsymbol{A}) \to \boldsymbol{B}$ such that $\widehat{h}\restriction\boldsymbol{A} = h$, i.e., the diagram*

$$\begin{array}{ll} \boldsymbol{A} \subseteq & \mathbf{Bo}(\boldsymbol{A}) \\ h \searrow & \downarrow \widehat{h} \\ & \boldsymbol{B} \end{array}$$

commutes.

Moreover, the proper filter $F(\boldsymbol{A})$ generated by A in $\mathbf{Bo}(\boldsymbol{A})$ is an ultrafilter.

Two implication algebras may have the same Boolean closure, but they can be distinguished by means of the filters contained in them. Indeed, if $\mathcal{M}(\boldsymbol{A})$ is the family of all maximal elements in the set of all filters of $\mathbf{Bo}(\boldsymbol{A})$ contained in the implication algebra $\boldsymbol{A}$, ordered by inclusion, then:

(a) $A = \bigcup_{F \in \mathcal{M}(\boldsymbol{A})} F$,

(b) $\mathcal{M}(\boldsymbol{A})$ is an antichain, relative to inclusion,

(c) if M is a filter of $\mathbf{Bo}(\boldsymbol{A})$ contained in A, then $M \subseteq F$ for some $F \in \mathcal{M}(\boldsymbol{A})$.

Moreover, these properties characterize $\mathcal{M}(\boldsymbol{A})$, in the sense that if $\boldsymbol{A}$ is an implication algebra and $\mathcal{G}$ is an antichain of filters of $\mathbf{Bo}(\boldsymbol{A})$ contained in A satisfying (a), (b) and (c), then $\mathcal{G} = \mathcal{M}(\boldsymbol{A})$. Notice that the case $\mathcal{M}(\boldsymbol{A}) = \{A\}$ is not excluded.

We denote by $St(\boldsymbol{B})$ the Stone space of a Boolean algebra $\boldsymbol{B}$ [4].

By an *implication space* we mean a 4-tuple $\langle X, \tau, u, \mathcal{C}\rangle$ such that

(i) $\langle X, \tau\rangle$ is a Boolean space,

(ii) u is a fixed element of X,

(iii) $\mathcal{C}$ is an antichain, with respect to inclusion, of closed sets of X such that $\bigcap \mathcal{C} = \{u\}$,

(iv) if C is a closed subset of X such that for every clopen N of X, $C \subseteq N$ implies $D \subseteq N$ for some $D \in \mathcal{C}$, then there exists $D' \in \mathcal{C}$ such that $D' \subseteq C$.

If $\langle X_1, \tau_1, u_1, \mathcal{C}_1\rangle$ and $\langle X_2, \tau_2, u_2, \mathcal{C}_2\rangle$ are implication spaces, we say that a map $f : X_1 \longrightarrow X_2$ is *i-continuous* provided that f is continuous, $f(u_1) = u_2$ and for all $C \in \mathcal{C}_2$, there is $D \in \mathcal{C}_1$ such that $D \subseteq f^{-1}[C]$.

In [1] it is proved that there exists a dual equivalence between the category of implication algebras and homomorphisms and the category of implication spaces and i-continuous functions.

2 Compactification of $Spec(\boldsymbol{A})$

In this section we define a topology on the set $Spec(\boldsymbol{A})$ of *maximal* implicative filters of an implication algebra $\boldsymbol{A}$ in such a way that the one-point compactification of $Spec(\boldsymbol{A})$ will be homeomorphic to the Stone space of $\mathbf{Bo}(\boldsymbol{A})$.

We call a subset F of an implication algebra $\boldsymbol{A}$ an *implicative filter* if

(a) $1 \in F$,

(b) for all $x, y \in A$ such that $x, x \to y \in F$, we have that $y \in F$.

In particular, every implicative filter is upwardly closed. A *prime* implicative filter is a proper implicative filter such that $x \vee y \in F$ implies $x \in F$ or $y \in F$. Observe that in this variety, maximal implicative filters and prime filters coincide. The filter generated by a subset X of A is $Fg(X) = \{b \in A :$ there exists $x_1, \ldots, x_n \in X$ such that $x_1 \to (x_2 \to \ldots (x_n \to b) \ldots) = 1\}$. Let us write $x \overset{0}{\to} y = y$, and $x \overset{k+1}{\to} y = x \to (x \overset{k}{\to} y)$ for $k < \omega$. If $F \subseteq A$ is an implicative filter and $a \in A$, $Fg(F \cup \{a\}) = \{b \in A :$ there exists an $n < \omega$ such that $a \overset{n}{\to} b \in F\}$.

Lemma 2.1 *Let M be a proper implicative filter of an implication algebra $\boldsymbol{A}$. Then M is maximal if and only if for every $a \notin M$, $a \to b \in M$ for every $b \in A$.*

P r o o f. Let M be a maximal implicative filter of $\boldsymbol{A}$ and assume $a \notin M$ and $b \in A$. Then

$$A = Fg(M \cup \{a\}) = \{x \in A : a \overset{n}{\to} x \in M \text{ for some } n < \omega\}.$$

This implies that $a \overset{n}{\to} b \in M$ for some $n \in \mathbb{N}$. Now, since the identity $x \overset{2}{\to} y \approx x \to y$ holds in any implication algebra, we get that $a \to b \in M$.

Conversely, suppose M is a proper implicative filter of $\boldsymbol{A}$ such that $a \to b \in M$ whenever $a \notin M$. Let F be an implicative filter of $\boldsymbol{A}$ such that $M \subsetneqq F$. Let $a \in F \setminus M$ and $b \in A$. By hypothesis, $a \to b \in M$, so $a \to b \in F$. Since $a \in F$, we get that $b \in F$. This shows that $F = A$ and so M is a maximal implicative filter. $\square$

Lemma 2.2 [1, Lemma 1.1] *Let $\boldsymbol{B}$ be a Boolean algebra, $\boldsymbol{A}$ an implication subalgebra of $\boldsymbol{B}$ and $B(\boldsymbol{A})$ the Boolean subalgebra of $\boldsymbol{B}$ generated by A. Then A is increasing in $B(\boldsymbol{A})$.*

P r o o f. Let $a \in A$, $b \in B(\boldsymbol{A})$ such that $a \leq b$. Let us see that $b \in A$. From $b \in B(\boldsymbol{A})$, there exist a_{ki}, $c_{ki} \in A$ such that

$$b = \bigwedge_{k=1}^{r} \left((\bigvee_{i \in I_k} \neg a_{ki}) \vee (\bigvee_{i \in J_k} c_{ki}) \right),$$

where $r \geq 1$ and for every $k = 1, \ldots r$, I_k and J_k are finite subsets of $\mathbb{N}$ with $I_k \cup J_k \neq \emptyset$.

Let $a_k = (\bigvee_{i \in I_k} \neg a_{ki}) \vee (\bigvee_{i \in J_k} c_{ki})$, $k = 1, \ldots, r$. As $a \leq b$, $a \leq a_k$ for every $k = 1, \ldots, r$, so, in order to prove that $b \in A$ it is enough to prove that $a_k \in A$ for every k.

For every k such that $J_k \neq \emptyset$, we have that $\bigvee_{i \in J_k} c_{ki} \in A$. So, $a_k = (\bigvee_{i \in I_k} \neg a_{ki}) \vee (\bigvee_{i \in J_k} c_{ki}) = \bigvee_{i \in I_k} (a_{ki} \to \bigvee_{i \in J_k} c_{ki}) \in A$. If k is such that $J_k = \emptyset$, then $a \leq \bigvee_{i \in I_k} \neg a_{ki}$, and consequently, $a_k = \bigvee_{i \in I_k} \neg a_{ki} = \bigvee_{i \in I_k} \neg a_{ki} \vee a = \bigvee_{i \in I_k} (a_{ki} \to a) \in A$. $\square$

Observe that as a consequence of the previous lemma, the collection of filters of $B(\boldsymbol{A})$ contained in $\boldsymbol{A}$ is just the family of lattice filters of $\boldsymbol{A}$.

If A is an increasing subset of a Boolean algebra $\boldsymbol{B}$, it is clear that $\boldsymbol{A}$ and the filter $F(\boldsymbol{A})$ generated by A in $\boldsymbol{B}$ are implication subalgebras of $\boldsymbol{B}$.

Lemma 2.3 *Let A be an increasing subset of a Boolean algebra $\boldsymbol{B}$. If M is a maximal implicative filter of $\boldsymbol{A}$, then the (implicative) filter $F(M)$ generated by M in $F(\boldsymbol{A})$ is a maximal implicative filter of $F(\boldsymbol{A})$ and $F(M) \cap A = M$.*

Proof. Since M is an increasing subset of $F(\boldsymbol{A})$, it is easy to see that $F(M) \cap A = M$. This, in turn, implies that $F(M)$ is a proper implicative filter of $F(\boldsymbol{A})$.

In order to prove that $F(M)$ is maximal, let $x \in F(\boldsymbol{A}) \backslash F(M)$ and let us prove that $x \to y \in F(M)$ for every $y \in F(\boldsymbol{A})$.

Since $x \in F(\boldsymbol{A})$, $x = \bigwedge_{i=1}^{n} x_i$, $x_i \in A$, and since $x \notin F(M)$, there exists $i_0 = 1, \ldots, n$ such that $x_{i_0} \notin M$. Let $y \in F(\boldsymbol{A})$, $y = \bigwedge_{j=1}^{m} y_j$, $y_j \in A$. Then

$$x \to y = (\bigwedge_{i=1}^{n} x_i) \to (\bigwedge_{j=1}^{m} y_j) = \neg(\bigwedge_{i=1}^{n} x_i) \vee (\bigwedge_{j=1}^{m} y_j) = (\bigvee_{i=1}^{n} \neg x_i) \vee (\bigwedge_{j=1}^{m} y_j) =$$

$$= \bigwedge_{j=1}^{m} [(\bigvee_{i=1}^{n} \neg x_i) \vee y_j] = \bigwedge_{j=1}^{m} [(\bigvee_{i \neq i_0} \neg x_i) \vee (x_{i_0} \to y_j)].$$

Since $x_{i_0} \notin M$ and M is maximal in $\boldsymbol{A}$, $x_{i_0} \to y_j \in M$ for every $j = 1, \ldots, m$. As M is increasing, then $(\bigvee_{i \neq i_0} \neg x_i) \vee (x_{i_0} \to y_j) \in M$ for every j. Hence $\bigwedge_{j=1}^{m} [(\bigvee_{i \neq i_0} \neg x_i) \vee (x_{i_0} \to y_j)] \in F(M)$. That is, $x \to y \in F(M)$, for every $y \in F(\boldsymbol{A})$. □

Lemma 2.4 *If $M \in Spec(\boldsymbol{A})$, then $U = F(M) \cup (\neg F(\boldsymbol{A}) \setminus \neg F(M)) \in St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$.*

Proof. We verify that U is an ultrafilter of $\mathbf{Bo}(\boldsymbol{A})$.

(1) $x \wedge y \in U$ whenever $x, y \in U$. Indeed, if $x, y \in F(M)$, $x \wedge y \in F(M)$. Suppose $x, y \in \neg F(\boldsymbol{A}) \setminus \neg F(M)$. Since $\neg F(\boldsymbol{A})$ is an ideal of $\mathbf{Bo}(\boldsymbol{A})$, $x \wedge y \in \neg F(\boldsymbol{A})$. Now assume $x \wedge y \in \neg F(M)$, then $x \wedge y = \neg z$ for some $z \in F(M)$. Then $\neg x \vee \neg y = z \in F(M)$. As $F(M)$ is prime in $F(\boldsymbol{A})$, then $\neg x \in F(M)$ or $\neg y \in F(M)$, that is, $x \in \neg F(M)$ or $y \in \neg F(M)$, a contradiction, since $x, y \notin \neg F(M)$. Hence, $x \wedge y \in \neg F(\boldsymbol{A}) \setminus \neg F(M)$. Suppose now that $x \in F(M)$ and $y \in \neg F(\boldsymbol{A}) \setminus \neg F(M)$. If $x \wedge y \in F(\boldsymbol{A})$, $y \in F(\boldsymbol{A})$, a contradiction. So $x \wedge y \in \neg F(\boldsymbol{A})$. As above we have that $x \wedge y \notin \neg F(M)$. Hence, $x \wedge y \in \neg F(\boldsymbol{A}) \setminus \neg F(M)$.

(2) For $x \in \mathbf{Bo}(\boldsymbol{A})$, $x \in U$ or $\neg x \in U$, but not both. Indeed, suppose that $x \notin U$. Since $F(\boldsymbol{A})$ is an ultrafilter of $\mathbf{Bo}(\boldsymbol{A})$, it follows that $x \in F(\boldsymbol{A})$ or $\neg x \in F(\boldsymbol{A})$. If $x \in F(\boldsymbol{A})$, then $\neg x \in \neg F(\boldsymbol{A})$. Besides, since $x \notin U$, $x \notin F(M)$ and so $\neg x \notin \neg F(M)$. Therefore, $\neg x \in \neg F(\boldsymbol{A}) \setminus \neg F(M) \subseteq U$. If $\neg x \in F(\boldsymbol{A})$, since $x \notin \neg F(\boldsymbol{A}) \setminus \neg F(M)$, $x \in \neg F(M)$. Consequently $\neg x \in F(M) \subseteq U$. Finally, it is easy to see that $U \cap \neg U = \emptyset$, so $x \in U$ or $\neg x \in U$, but not both.

(3) U is increasing in $\mathbf{Bo}(\boldsymbol{A})$. Indeed, suppose $x \leq y$, where $x \in U$ and $y \in \mathbf{Bo}(\boldsymbol{A})$. Assume $y \notin U$, then $\neg y \in U$. Now, if $x \in F(M)$, we get that $y \in U$ since $F(M)$ is increasing in $\mathbf{Bo}(\boldsymbol{A})$. If $x \in \neg F(\boldsymbol{A}) \setminus \neg F(M)$, we consider two possibilities for $\neg y$. If $\neg y \in F(M)$, since $\neg y \leq \neg x$, it follows that $\neg x \in F(M)$, contradiction. If $\neg y \in \neg F(\boldsymbol{A}) \setminus \neg F(M)$, then $y \in F(\boldsymbol{A}) \setminus F(M)$. By the previous lemma, $F(M)$ is maximal in $F(\boldsymbol{A})$ and so we must have $y \to \neg x \in F(M)$. But $y \to \neg x = \neg y \vee \neg x = \neg x$, since $\neg y \leq \neg x$. Hence, $\neg x \in F(M)$, contradiction.

By the above conditions and the fact that $U \neq F(\boldsymbol{A})$, we conclude that $U \in St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$. □

The previous lemmas lead us to the following crucial relationship between $St(\mathbf{Bo}(\boldsymbol{A}))$ and $Spec(\boldsymbol{A})$.

Theorem 2.5 *There exists a bijection $\varphi : Spec(\boldsymbol{A}) \longrightarrow St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$.*

Proof. Define $\varphi : Spec(\boldsymbol{A}) \longrightarrow St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$ by $\varphi(M) = F(M) \cup (\neg F(\boldsymbol{A}) \setminus \neg F(M))$, for $M \in Spec(\boldsymbol{A})$. By Lemma 2.4, φ is a well defined mapping.

Let us define the inverse map of φ. In order to do this, observe that if U is an ultrafilter of $\mathbf{Bo}(\boldsymbol{A})$, $U \neq F(\boldsymbol{A})$, then $U \cap A$ is a maximal implicative filter of A. Indeed, it is clear that $U \cap A \neq A$, and for $x \in A$, $y \in A \setminus U$, $y \to x \in U$ since U is maximal, so $y \to x \in A \cap U$ for every $x \in A$. This allows us to define a map $\psi : St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\} \longrightarrow Spec(\boldsymbol{A})$ such that $\psi(U) = U \cap A$ for every $U \in St(\mathbf{Bo}(\boldsymbol{A}))$.

We now show that ψ is one-to-one. Let $U_1, U_2 \in St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$, $U_1 \neq U_2$, and let $x \in \mathbf{Bo}(\boldsymbol{A})$ such that $x \in U_1$ and $x \notin U_2$. There are two possibilities: $x \in F(\boldsymbol{A})$ or $\neg x \in F(\boldsymbol{A})$. If $x \in F(\boldsymbol{A})$, then $x = \bigwedge_{i=1}^{n} x_i$, $x_i \in A$. Since $x \in U_1$, $x_i \in U_1$ for every $i = 1, \ldots, n$, and since $x \notin U_2$, there exists $i_0 \in \{1, \ldots, n\}$ such that

$x_{i_0} \notin U_2$. So $x_{i_0} \in U_1 \cap A$ and $x_{i_0} \notin U_2 \cap A$. In case $\neg x \in F(\boldsymbol{A})$, we can argue as before taking into account that $\neg x \notin U_1$ and $\neg x \in U_2$. Consequently, ψ is one-to-one.

Finally, given $M \in Spec(\boldsymbol{A})$, let $U = \varphi(M) \in St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$. By Lemma 2.3 we have that $\psi(U) = M$. This shows that ψ is onto and completes the proof. □

Now we are going to define a Zariski type topology τ on $Spec(\boldsymbol{A})$. For each $a \in A$, let $N_a = \{M \in Spec(\boldsymbol{A}) : a \in M\}$ and let $\mathcal{B} = \{Spec(\boldsymbol{A}) \setminus N_b : b \in A\}$. Let τ be the topology generated by $\mathcal{B}$. Observe that $\mathcal{B}$ is, in fact, a basis for τ since $\mathcal{B}$ is closed by finite intersections. Indeed, let $b_1, \ldots, b_n \in A$ and $b = b_1 \vee \ldots \vee b_n$. Since maximal implicative filters are prime, it follows that

$$N_b = \bigcup_{i=1}^{n} N_{b_i},$$

hence

$$Spec(\boldsymbol{A}) \setminus N_b = \bigcap_{i=1}^{n} (Spec(\boldsymbol{A}) \setminus N_{b_i}).$$

The one-point compactification of a topological space X is the set $X^* = X \cup \{\infty\}$ with the topology whose members are the open subsets of X and all subsets U of X^* such that $X^* \setminus U$ is a closed compact subset of X.

A set U is open in X^* if and only if (a) $U \cap X$ is open in X and (b) whenever $\infty \in U$, $X \setminus U$ is compact.

It is known (see for example [6]) that the one-point compactification X^* of a topological space X is compact and X is a subspace. The space X^* is Hausdorff if and only if X is locally compact and Hausdorff.

Theorem 2.6 *φ is a homeomorphism between the spaces $\langle Spec(\boldsymbol{A}), \tau\rangle$ and $St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$ with the relative topology.*

P r o o f. We already know that φ is a bijection. It remains to show that φ and $\varphi^{-1} = \psi$ are continuous.

Let $X = St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$ with the relative topology and consider an open subset G of X. Then $G = G' \cap X$ for some open subset G' in $St(\mathbf{Bo}(\boldsymbol{A}))$. Since $St(\mathbf{Bo}(\boldsymbol{A}))$ is Hausdorff, $\{F(\boldsymbol{A})\}$ is closed in $St(\mathbf{Bo}(\boldsymbol{A}))$, so $G = G' \setminus \{F(\boldsymbol{A})\}$ is open in $St(\mathbf{Bo}(\boldsymbol{A}))$. Therefore, there exists some subset $Y \subseteq \mathbf{Bo}(\boldsymbol{A})$ such that

$$G = \bigcup_{b \in Y} G_b$$

where $G_b = \{U \in St(\mathbf{Bo}(\boldsymbol{A})) : b \in U\}$. Since

$$\varphi^{-1}(G) = \bigcup_{b \in Y} \varphi^{-1}(G_b)$$

it suffices to show that $\varphi^{-1}(G_b)$ is open in $Spec(\boldsymbol{A})$ for every $b \in Y$.

Since $F(\boldsymbol{A}) \notin G$, then $F(\boldsymbol{A}) \notin G_b$ for any $b \in Y$. As $F(\boldsymbol{A})$ is an ultrafilter of $\mathbf{Bo}(\boldsymbol{A})$, it follows that $b \notin F(\boldsymbol{A})$, so $\neg b \in F(\boldsymbol{A})$ and we have that $\neg b = \bigwedge_{i=1}^{n} a_i$ for some $a_1, \ldots, a_n \in A$. Then $b = \bigvee_{i=1}^{n} \neg a_i$. It follows immediately that $G_b = \bigcup_{i=1}^{n} G_{\neg a_i}$.

We claim that $\varphi^{-1}(G_{\neg a_i}) = Spec(\boldsymbol{A}) \setminus N_{a_i}$, which completes the proof of the continuity of φ. Indeed, if $M \in \varphi^{-1}(G_{\neg a_i}) = \psi(G_{\neg a_i})$, there exists some $U \in G_{\neg a_i}$ such that $M = U \cap A$. Since $\neg a_i \in U$, $a_i \notin U$, so $a_i \notin M$. Hence $M \in Spec(\boldsymbol{A}) \setminus N_{a_i}$. The converse is similar.

It remains to show that ψ is continuous. It is enough to prove that $\psi^{-1}(Spec(\boldsymbol{A}) \setminus N_a)$ is open in X for every $a \in A$. Indeed,

$$\begin{array}{rcl} \psi^{-1}(Spec(\boldsymbol{A}) \setminus N_a) & = & \{U \in X : a \notin U \cap A\} \\ & = & \{U \in X : a \notin U\} \\ & = & \{U \in X : \neg a \in U\} \\ & = & G_{\neg a} \cap X \end{array}$$

which is open in X. □

Remark 2.7 Let Y be a Hausdorff compact space and consider $Y \setminus \{a\}$, $a \in Y$, with the relative topology. Then Y is the one-point compactification of $Y \setminus \{a\}$. Indeed, it is easy to see that the open sets of Y and those of $(Y \setminus \{a\})^*$ coincide. In particular, $St(\mathbf{Bo}(\boldsymbol{A}))$ is the one-point compactification of $St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$.

Corollary 2.8 *$St(\mathbf{Bo}(\boldsymbol{A}))$ is homeomorphic to the one-point compactification of $\langle Spec(\boldsymbol{A}), \tau\rangle$.*

Corollary 2.9 *$\langle Spec(\boldsymbol{A}), \tau\rangle$ is Hausdorff and locally compact.*

Corollary 2.10 *$\langle Spec(\boldsymbol{A}), \tau\rangle$ has a basis of clopen compact subsets.*

Proof. In the proof of Theorem 2.6 we showed that for any $a \in A$, $Spec(\boldsymbol{A}) \setminus N_a = \psi(G_{\neg a})$. Now, since $G_{\neg a}$ is compact, $Spec(\boldsymbol{A}) \setminus N_a$ must also be compact in $Spec(\boldsymbol{A})$. Finally, since $Spec(\boldsymbol{A})$ is Hausdorff, it follows that $Spec(\boldsymbol{A}) \setminus N_a$ is closed. Therefore, $\mathcal{B} = \{Spec(\boldsymbol{A}) \setminus N_a : a \in A\}$ is a basis of clopen compact sets for $Spec(\boldsymbol{A})$. $\square$

Remark 2.11 We could have shown directly that $Spec(\boldsymbol{A}) \setminus N_a$ is closed for every $a \in A$. Indeed, using Lemma 2.1, it is immediately verified that

$$Spec(\boldsymbol{A}) \setminus N_a = \bigcap_{b \in A} N_{a \to b}.$$

Since the sets N_b, $b \in A$, are also open, this shows that the topology on $Spec(\boldsymbol{A})$ is analogous to the Priestly topology on the prime filters of a bounded distributive lattice (see [5]).

The compactness of $Spec(\boldsymbol{A}) \setminus N_a$ for every $a \in A$ also follows directly from the definition of the topology on $Spec(\boldsymbol{A})$. Suppose $Spec(\boldsymbol{A}) \setminus N_a \subseteq \bigcup_{i \in I}(Spec(\boldsymbol{A}) \setminus N_{a_i})$. Then $\bigcap_{i \in I} N_{a_i} \subseteq N_a$. Now, note that the intersection of the maximal implicative filters in N_a is $F(\{a\})$. Similarly, the intersection of the maximal implicative filters of $\bigcap_{i \in I} N_{a_i}$ is $F(\{a_i : i \in I\})$. It follows that $F(\{a\}) \subseteq F(\{a_i : i \in I\})$, so there must be some finite subset $J \subseteq I$ such that $a \in F(\{a_i : i \in J\})$. We have then that $\bigcap_{i \in J} N_{a_i} \subseteq N_a$ whence $Spec(\boldsymbol{A}) \setminus N_a \subseteq \bigcup_{i \in J}(Spec(\boldsymbol{A}) \setminus N_{a_i})$. This shows that $Spec(\boldsymbol{A}) \setminus N_a$ is compact.

Definition 2.12 We say that a topological space X is a **locally Stone space** if X^* is Stone, i.e., the one-point compactification of X has a base of clopens.

Observe that if X is a locally Stone space, then X is Hausdorff and locally compact.

Consequently, $\langle Spec(\boldsymbol{A}), \tau\rangle$ is a locally Stone space.

Proposition 2.13 *A topological space X is locally Stone if and only if it is Hausdorff and has a basis of clopen compact sets.*

Proof. Let X be a locally Stone space. Then X is Hausdorff. Now, since X^* is a Stone space, X^* has a basis of clopen sets, which are compact because of the compactness of the space. Let $\mathcal{B}^*$ be such a basis and consider $\mathcal{B} = \{N \in \mathcal{B}^* : N \subseteq X\}$. It is clear that the elements of $\mathcal{B}$ are clopen compact sets in X. It remains to show that $\mathcal{B}$ is a basis for X. Indeed, let G be open in X, then G is open in X^*, so G is a union of element in $\mathcal{B}^*$. However, since $\infty \notin G$, every element in this union is in fact in $\mathcal{B}$.

Conversely, let X be a Hausdorff topological space with a basis $\mathcal{B}$ of clopen compact subsets. It is clear then that X is locally compact. In order to show that X is a locally Stone space, we only need to show that X^* has a basis of clopen sets. This basis will be noted $\mathcal{B}^*$ and is defined thus

$$\mathcal{B}^* = \mathcal{B} \cup \left\{ X^* \setminus \bigcup_{i=1}^{n} N_i : n \in \mathbb{N}, N_i \in \mathcal{B} \right\}.$$

The elements of $\mathcal{B}$ are trivially clopen in X^*. A set $H = X^* \setminus \bigcup_{i=1}^n N_i$, $N_i \in \mathcal{B}$, is open because $\infty \in H$ and $X^* \setminus H = \bigcup_{i=1}^n N_i$ is compact (and closed because X is Hausdorff). Moreover, H is closed because $\bigcup_{i=1}^n N_i$ is open in X^*. Finally, we must prove that $\mathcal{B}^*$ is a basis for X^*. To do that, consider an arbitrary open set G in X^*. If $\infty \notin G$, then G is open in X, so G is a union of elements in $\mathcal{B}$ and hence in $\mathcal{B}^*$. On the other hand, if $\infty \in G$, then $X^* \setminus G$ is compact in X. Then, there must be $N_1, \ldots, N_n \in \mathcal{B}$ such that $X^* \setminus G \subseteq \bigcup_{i=1}^n N_i$.

Hence, $X^* \setminus \bigcup_{i=1}^{n} N_i \subseteq G$. Besides, since X^* is Hausdorff $G \setminus \{\infty\}$ is open in X^* and so it is also open in X. So $G \setminus \{\infty\} = \bigcup_{i\in I} N'_i$, $N'_i \in \mathcal{B}$. This shows that

$$G = \bigcup_{i\in I} N'_i \cup \left(X^* \setminus \bigcup_{i=1}^{n} N_i \right).$$

This completes the proof. □

Definition 2.14 We say that a triple $\langle X, \tau, \mathcal{C}\rangle$ is a **Zariski implication space** (Z-space) if

(i1) $\langle X, \tau\rangle$ is a locally Stone space,

(i2) $\mathcal{C}$ is a nonempty family of closed subsets of X such that $\mathcal{C}$ is an antichain and $\bigcap \mathcal{C} = \emptyset$,

(i3) if C is a closed subset of X such that for every clopen N of X whose complement is compact, $C \subseteq N$ implies $D \subseteq N$ for some $D \in \mathcal{C}$, then there exists $D' \in \mathcal{C}$ such that $D' \subseteq C$.

Let $\langle X, \tau, \mathcal{C}\rangle$ be a Z-space and let $\langle X^*, \tau^*, \infty, \mathcal{C}^*\rangle$ be such that $\langle X^*, \tau^*\rangle$ is the one-point compactification of $\langle X, \tau\rangle$ (recall that $\langle X^*, \tau^*\rangle$ is a Stone space) and $\mathcal{C}^* = \{C \cup \{\infty\} : C \in \mathcal{C}\}$. Observe that if C is a closed subset of X then $X \setminus C$ is an open subset of X. Thus $X \setminus C$ is an open set in X^* and therefore $X^* \setminus (X \setminus C) = C \cup \{\infty\}$ is a closed set of X^*.

Lemma 2.15 *If $\langle X, \tau, \mathcal{C}\rangle$ is a Z-space, then $\langle X^*, \tau^*, \infty, \mathcal{C}^*\rangle$ is an i-space.*

Proof. We already know that $\langle X^*, \tau^*\rangle$ is a Stone space and it is clear that $\mathcal{C}^*$ is an antichain of closed sets in X^* such that $\bigcap \mathcal{C}^* = \{\infty\}$. Now, let C be a closed set in X^* such that for every clopen set N in X^*, $C \subseteq N$ implies $D \subseteq N$ for some $D \in \mathcal{C}^*$.

First we show that $\infty \in C$. Indeed, if $\infty \notin C$, C is compact in X. Since X has a basis of clopen compact sets, $C \subseteq \bigcup_{i=1}^{n} N_i$ where each N_i is a clopen compact subset of X. Then $\bigcup N_i$ is clopen in X^*, $C \subseteq \bigcup N_i$, but $\bigcup N_i$ does not contain any $D \in \mathcal{C}^*$, because $\infty \notin \bigcup N_i$. This contradicts our hypothesis on C. Hence ∞ must lie in C.

Now, as C is closed in X^*, $C \cap X = C \setminus \{\infty\}$ is closed in X. Now suppose N' is a clopen of X such that $X \setminus N'$ is compact and $C \setminus \{\infty\} \subseteq N'$. Then $N = N' \cup \{\infty\}$ is a clopen in X^* such that $C \subseteq N$. By hypothesis, there exists some $D \in \mathcal{C}^*$ such that $D \subseteq N$, so $D \setminus \{\infty\} \in \mathcal{C}$ and $D \setminus \{\infty\} \subseteq N'$. Using now condition $(i3)$ in the definition of Z-space, we get that there must be some $D' \in \mathcal{C}$ such that $D' \subseteq C \setminus \{\infty\}$. Then $D' \cup \{\infty\} \in \mathcal{C}^*$ and $D' \cup \{\infty\} \subseteq C$.

This completes the proof that $\langle X^*, \tau^*, \infty, \mathcal{C}^*\rangle$ is an implication space. □

Let $(X_1, \tau_1, \mathcal{C}_1)$ and $(X_2, \tau_2, \mathcal{C}_2)$ be two $Z-$spaces. We say that a *partial map* $f : X_1 \longrightarrow X_2$ is Z-*continuous* if the following conditions hold:

(1) f is continuous, i.e., for every open G in X_2, $f^{-1}[G]$ is open in X_1.

(2) for every compact K in X_2, $f^{-1}[K]$ is compact in X_1,

(3) for all $C \in \mathcal{C}_2$, there is $D \in \mathcal{C}_1$ such that $D \subseteq f^{-1}[C]$.

Given a Z-continuous partial map $f : X_1 \longrightarrow X_2$, let $Dom(f) = \{x \in X_1 : f(x) \text{ exists}\}$. We associate with f a function $f^* : X_1^* \longrightarrow X_2^*$ given by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in Dom(f) \\ \infty_2 & \text{if } x \notin Dom(f) \end{cases}$$

Lemma 2.16 *Given a Z-continuous partial map $f : X_1 \longrightarrow X_2$, f^* is an i-continuous map from X_1^* into X_2^*.*

Proof. Let G be an open subset of X_2^*. If $\infty_2 \notin G$, then G is an open subset of X_2 and $(f^*)^{-1}[G] = f^{-1}[G]$ which is open in X_1 and also in X_1^*. If $\infty_2 \in G$, then $X_2^* \setminus G$ is compact in X_2 and so $f^{-1}[X_2^* \setminus G]$ is compact in X_1. But $f^{-1}[X_2^* \setminus G] = (f^*)^{-1}[X_2^* \setminus G] = X_1^* \setminus (f^*)^{-1}[G]$. This shows that $(f^*)^{-1}[G]$ is open in X_1^*. Therefore f^* is continuous.

It is trivially verified that $f^*(\infty_1) = \infty_2$ and that for each $D_2 \in \mathcal{C}_2^*$, there exists $D_1 \in \mathcal{C}_1^*$ such that $D_1 \subseteq (f^*)^{-1}[D_2]$. Hence f^* is an i-continuous map. □

Let $\mathfrak{Z}$ be the category of $Z-$spaces with Z-continuous partial maps, and let $\mathfrak{X}$ denote the category of implication spaces with i-continuous maps. Let $\star : \mathfrak{Z} \longrightarrow \mathfrak{X}$ be such that $\star(\langle X, \tau, \mathcal{C}\rangle) = \langle X^*, \tau^*, \infty, \mathcal{C}^*\rangle$ and if $f : \langle X_1, \tau_1, \mathcal{C}_1\rangle \longrightarrow \langle X_2, \tau_2, \mathcal{C}_2\rangle$ is a Z-continuous partial map, then $\star(f) = f^*$. The previous lemmas directly imply the following theorem.

Theorem 2.17 $\star : \mathfrak{Z} \longrightarrow \mathfrak{X}$ *is a covariant functor.*

Now we are going to define an inverse for $\star$. Given an implication space $\langle X, \tau, u, \mathcal{C}\rangle$, let $\circ(\langle X, \tau, u, \mathcal{C}\rangle) = \langle X^\circ, \tau^\circ, \mathcal{C}^\circ\rangle$ where $X^\circ = X \setminus \{u\}$, τ° is the relative topology on X° and $\mathcal{C}^\circ = \{C \setminus \{u\} : C \in \mathcal{C}\}$.

Lemma 2.18 *For every implication space* $\langle X, \tau, u, \mathcal{C}\rangle$*,* $\langle X^\circ, \tau^\circ, \mathcal{C}^\circ\rangle$ *is a* Z*-space.*

Proof. Straightforward. □

It remains to define the correspondence between morphisms.

Let $\langle X_1, \tau_1, u_1, \mathcal{C}_1\rangle$ and $\langle X_2, \tau_2, u_2, \mathcal{C}_2\rangle$ be two implication spaces. Given an i-continuous map $f : X_1 \longrightarrow X_2$, we define $f^\circ : X_1^\circ \longrightarrow X_2^\circ$ such that $f^\circ = f\restriction_S$, where $S = \{x \in X_1 : f(x) \neq u_2\} = X_1 \setminus f^{-1}(u_2)$. Observe that f° is a partial map since $f(x)$ is not defined for those $x \in X_1^\circ$ such that $f(x) = u_2$.

Lemma 2.19 *If* $f : X_1 \longrightarrow X_2$ *is an* i*-continuous map between implication spaces, then* $f^\circ : X_1^\circ \longrightarrow X_2^\circ$ *is a* Z*-continuous partial map between* Z*-spaces.*

Proof. Let G be an open subset of X_2°. Then G is open in X_2, so $f^{-1}[G]$ is open in X_1 and consequently $f^{-1}[G] = f^{-1}[G] \cap X_1^\circ = (f^\circ)^{-1}[G]$ is open in X_1°. This shows that f° is continuous.

Now let K be a compact set in X_2°. Then $X_2 \setminus K$ is open in X_2, so $f^{-1}[X_2 \setminus K] = X_1 \setminus f^{-1}[K]$ is open in X_1 and contains u_1. Hence $f^{-1}[K] = (f^\circ)^{-1}[K]$ is compact in X_1°.

This completes the proof that f° is a Z-continuous partial map, since condition (3) is trivial. □

We summarize the last two lemmas in the following theorem.

Theorem 2.20 $\circ : \mathfrak{X} \longrightarrow \mathfrak{Z}$ *is a covariant functor.*

Our objective now is to show that the functors $\star$ and $\circ$ define a category equivalence between the categories $\mathfrak{X}$ and $\mathfrak{Z}$.

Given a Z-space $\langle X, \tau, \mathcal{C}\rangle$, we have that $\circ\star(\langle X, \tau, \mathcal{C}\rangle) = \langle X^{*\circ}, \tau^{*\circ}, \mathcal{C}^{*\circ}\rangle$. It is immediate that $X^{*\circ} = X$ and $\mathcal{C}^{*\circ} = \mathcal{C}$. Using the definition of one-point compactification and the fact that $\langle X, \tau\rangle$ is a Hausdorff space, it is easily shown that $\tau^{*\circ} = \tau$. So, in fact, upon applying the functors $\circ$ and $\star$ we get the original Z-space back.

Conversely, suppose $\langle X, \tau, \infty, \mathcal{C}\rangle$ is an implication space, where we used ∞ for the distinguished element instead of u for the sake of simplicity in the following argument. Then, $\star\circ(\langle X, \tau, \infty, \mathcal{C}\rangle) = \langle X^{\circ*}, \tau^{\circ*}, \infty, \mathcal{C}^{\circ*}\rangle$. It is easily seen that $X^{\circ*} = X$ and $\mathcal{C}^{\circ*} = \mathcal{C}$. Moreover, by Remark 2.7, we also have that $\tau^{\circ*} = \tau$. Therefore, after applying $\star\circ$ we obtain the original implication space we started with.

Since $\circ\star = id_{\mathfrak{Z}}$ and $\star\circ = id_{\mathfrak{X}}$, we have the following equivalence theorem.

Theorem 2.21 *The functors* $\star$ *and* $\circ$ *define an equivalence between the categories* $\mathfrak{Z}$ *and* $\mathfrak{X}$*.*

Let $\mathfrak{I}$ be the category of implication algebras and homomorphisms. Let $\mathbb{I}$ be the functor that establishes a duality between the categories $\mathfrak{X}$ and $\mathfrak{I}$ [1]. As a consequence of the previous theorem we have that

$$\eta = \mathbb{I}\star : \mathfrak{Z} \longrightarrow \mathfrak{I}$$

is a contravariant functor between the categories $\mathfrak{Z}$ and $\mathfrak{I}$. Observe that for any Z-space $\langle X, \tau, \mathcal{C}\rangle$, we have that

$$\mathbb{I}\star(\langle X, \tau, \mathcal{C}\rangle) = \mathbb{I}(\langle X^*, \tau^*, \infty, \mathcal{C}^*\rangle) = \langle\{N \in Clop(X^*) : C \subseteq N, \text{ for some } C \in \mathcal{C}^*\}, \rightarrow\rangle,$$

where $N_1 \to N_2 = N_1^c \cup N_2$, and it is easily seen (as we will see in the following section) that

$$\langle \{N \in Clop(X^*) : C \subseteq N, \text{ for some } C \in \mathcal{C}^*\}, \to \rangle \cong$$

$$\cong \langle \{N \in Clop(X) : X \setminus N \text{ is compact, and } C \subseteq N \text{ for some } C \in \mathcal{C}\}, \to \rangle.$$

The following corollary is immediate.

Corollary 2.22 *The functor η defines a duality between the categories $\mathfrak{Z}$ and $\mathfrak{I}$.*

As an application, we give a topological representation for generalized Boolean algebras. Recall that a generalized Boolean algebra is an implication algebra $\boldsymbol{A}$ such that the infimum is defined for every pair of elements of A, and it is a meet-semilattice with the implication as residuum. In this case we have that $F(\boldsymbol{A}) = A$, and so the corresponding implication space is $\mathbb{X}(\boldsymbol{A}) = (\mathbf{Bo}(\boldsymbol{A}), \tau, \{F(\boldsymbol{A})\})$. Hence, the associated Z-space is $\langle Spec(\boldsymbol{A}), \tau', \{\emptyset\}\rangle$, where $\langle Spec(\boldsymbol{A}), \tau'\rangle$ is a locally Stone space. Conversely, if $\langle X, \tau, \{\emptyset\}\rangle$ is a Z-space, then the corresponding implication algebra is a generalized Boolean algebra. This shows that generalized Boolean algebras correspond to Z-spaces where $\mathcal{C} = \{\emptyset\}$.

Let $g\mathfrak{Z}$ be the full subcategory of $\mathfrak{Z}$ whose objects are those Z-spaces for which $\mathcal{C} = \{\emptyset\}$. Besides, let $g\mathfrak{B}$ be the full subcategory of $\mathfrak{I}$ consisting of generalized Boolean algebras. Thus, the restriction $g\eta$ of the functor η to $g\mathfrak{Z}$ gives a duality between the categories $g\mathfrak{Z}$ and $g\mathfrak{B}$.

Observe that in the category $g\mathfrak{X}$ we can drop the symbol $\{\emptyset\}$ and consider its objects simply as locally Stone spaces. Moreover, in the definition of the morphisms in $g\mathfrak{Z}$ we can drop condition (3) since it is trivially implied by the fact that $\mathcal{C} = \{\emptyset\}$.

In the following section we will describe explicitly the duality between $\mathfrak{Z}$ and $\mathfrak{I}$ in order to avoid passing through $\mathfrak{X}$.

3 Duality between $\mathfrak{I}$ and $\mathfrak{Z}$

In what follows, we describe in detail the duality between the categories $\mathfrak{I}$ and $\mathfrak{Z}$ developed in the previous section. Specifically, we make explicit the correspondence between implication algebras and Z-spaces as well as the correspondence between implicative homomorphisms and Z-continuous partial maps. In addition, we will characterize monomorphisms and epimorphisms in both categories as well as give a dual counterpart of surjective homomorphisms in $\mathfrak{Z}$.

3.1 Description of the duality

We now give a direct description of the duality between $\mathfrak{I}$ and $\mathfrak{Z}$.

We have the functor $\eta : \mathfrak{Z} \longrightarrow \mathfrak{I}$ such that

$$\eta(\langle X, \tau, \mathcal{C}\rangle) = \mathbb{I}(\langle X^*, \tau^*, \infty, \mathcal{C}^*\rangle) = \langle A, \to \rangle,$$

where $A = \{N \in Clop(X^*) : C \subseteq N \text{ for some } C \in \mathcal{C}^*\}$ and $N_1 \to N_2 = N_1^c \cup N_2$ for every $N_1, N_2 \in A$.

Since $\infty \in C$ for every $C \in \mathcal{C}^*$, it follows that $\infty \in N$ for every $N \in A$. Besides, it is easy to see that $N \in Clop(X^*)$ such that $\infty \in N$ if and only if $N' = N \setminus \{\infty\}$ is clopen in X and $X \setminus N'$ is compact. If we identify the clopen sets in X^* and the clopen sets in X whose complement is compact, we may consider

$$A = \{N \in Clop(X) : X \setminus N \text{ is compact and } C \subseteq N \text{ for some } C \in \mathcal{C}\}.$$

In this way we obtain the implication algebra associated with the Z-space $\langle X, \tau, \mathcal{C}\rangle$ without referring to the implication space $\langle X^*, \tau^*, \infty, \mathcal{C}^*\rangle$.

Now consider a Z-continuous partial map $f : \langle X_1, \tau_1, \mathcal{C}_1\rangle \longrightarrow \langle X_2, \tau_2, \mathcal{C}_2\rangle$. Applying $\star$ we get $f^* : \langle X_1^*, \tau_1^*, \infty_1, \mathcal{C}_1^*\rangle \longrightarrow \langle X_2^*, \tau_2^*, \infty_2, \mathcal{C}_2^*\rangle$ given by

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in Dom(f) \\ \infty_2 & \text{if } x \notin Dom(f) \end{cases}$$

where $Dom(f) = \{x \in X_1 : f(x) \text{ exists}\}$. Let $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ be the corresponding implication algebras. Then $\mathbb{I}(f^*) : \boldsymbol{A}_2 \longrightarrow \boldsymbol{A}_1$ is given by $\mathbb{I}(f^*)(N) = (f^*)^{-1}(N)$ for every $N \in A_2$. If we consider $N' = N \setminus \{\infty_2\}$ and identify N' with N, we get that

$$\eta(f)(N') = f^{-1}(N') \cup (X \setminus Dom(f)).$$

Let $\boldsymbol{A}$ be an implication algebra and $\mathbb{X}(\boldsymbol{A}) = \langle X, \tau, \infty, \mathcal{C}\rangle$ its associated implication space. Then $\circ\mathbb{X}(\boldsymbol{A}) = \langle X^\circ, \tau^\circ, \mathcal{C}^\circ\rangle$, where $X^\circ = St(\mathbf{Bo}(\boldsymbol{A})) \setminus \{F(\boldsymbol{A})\}$. If we identify M with $\varphi(M)$, we have $X^\circ = Spec(\boldsymbol{A})$ and $C \in \mathcal{C}^\circ$ iff $C = \{M \in Spec(\boldsymbol{A}) : F \subseteq M\}$ where $F \in \mathcal{M}(\boldsymbol{A})$.

Now if $h : \boldsymbol{A}_1 \longrightarrow \boldsymbol{A}_2$ is a homomorphism between two implication algebras, then $\mathbb{X}(h) : \mathbb{X}(\boldsymbol{A}_2) \longrightarrow \mathbb{X}(\boldsymbol{A}_1)$ is given by $\mathbb{X}(h)(U) = \widehat{h}^{-1}(U)$ where $U \in St(\mathbb{X}(\boldsymbol{A}_1))$ and $\widehat{h} : \mathbf{Bo}(\boldsymbol{A}_1) \longrightarrow \mathbf{Bo}(\boldsymbol{A}_2)$ is the boolean homomorphism given in Theorem 1.1. It follows that $\circ\mathbb{X}(h) : \circ\mathbb{X}(\boldsymbol{A}_2) \longrightarrow \circ\mathbb{X}(\boldsymbol{A}_1)$ is given by

$$\circ\mathbb{X}(h)(M) = \begin{cases} \widehat{h}^{-1}(F(M) \cup \neg F(\boldsymbol{A}_2) \setminus \neg F(M)) \cap A_1 & \text{if } \widehat{h}^{-1}(F(M) \cup \neg F(\boldsymbol{A}_2) \setminus \neg F(M)) \neq F(\boldsymbol{A}_1) \\ \text{not defined} & \text{otherwise} \end{cases}$$

where $M \in Spec(\boldsymbol{A}_2)$.

It is easy to show that this may be shortened to

$$\circ\mathbb{X}(M) = \begin{cases} h^{-1}(M) & \text{if } h^{-1}(M) \neq A_1 \\ \text{not defined} & \text{otherwise} \end{cases}$$

3.2 Special morphisms

Since $\mathfrak{I}$ is an equational category, monomorphisms in $\mathfrak{I}$ are simply injective homomorphisms. However, epimorphisms do not coincide with surjective homomorphisms. For example, consider the four-element boolean implication algebra with universe $B = \{0, a, a', 1\}$ and its subuniverse $A = \{a, a', 1\}$. Then the inclusion map $i : \boldsymbol{A} \longrightarrow \boldsymbol{B}$ may be easily shown to be an epimorphism which is not onto.

We now turn to the task of characterizing monomorphisms and epimorphisms in the category $\mathfrak{Z}$. We also find the dual counterparts of surjective homomorphisms.

Proposition 3.1 *A $\mathfrak{Z}$-continuous partial map $f : \langle X_1, \tau_1, \mathcal{C}_1\rangle \longrightarrow \langle X_2, \tau_2, \mathcal{C}_2\rangle$ is monic if and only if f is an injective map.*

P r o o f. Suppose f is monic and let us see that f is a map, rather than a partial map. Assume there were some $x \notin Dom(f)$. Consider the Z-space $Z = \{a, b\}$ with $\mathcal{C}_Z = \{\emptyset\}$ and two partial maps $g, h : Z \longrightarrow X_1$ given by $g(a) = x$ and $h(b) = x$. It is immediate to see that g, h are Z-continuous partial maps and that $f \circ g = f \circ h$. Since f is monic, we get $g = h$, a contradiction. This shows that $Dom(f) = X_1$. The injectivity of f is immediate.

The converse is trivial. □

Corollary 3.2 *Let $h : \boldsymbol{A}_1 \longrightarrow \boldsymbol{A}_2$ be a homomorphism between two implication algebras. Then h is an epimorphism in the category $\mathfrak{I}$ if and only if the following conditions hold:*

$(e1)$ $h^{-1}(M) \in Spec(\boldsymbol{A}_1)$ *for every* $M \in Spec(\boldsymbol{A}_2)$.

$(e2)$ *If* $M_1, M_2 \in Spec(\boldsymbol{A}_2)$ *and* $M_1 \neq M_2$, *then* $h^{-1}(M_1) \neq h^{-1}(M_2)$.

The following two propositions are immediate.

Proposition 3.3 *A $\mathfrak{Z}$-continuous partial map $f : \langle X_1, \tau_1, \mathcal{C}_1\rangle \longrightarrow \langle X_2, \tau_2, \mathcal{C}_2\rangle$ is epic if and only if $f^{-1}(N_1) \neq f^{-1}(N_2)$ whenever $N_1, N_2 \in \eta(X_1)$, $N_1 \neq N_2$.*

Proposition 3.4 *Let $f : \langle X_1, \tau_1, \mathcal{C}_1\rangle \longrightarrow \langle X_2, \tau_2, \mathcal{C}_2\rangle$ be a $\mathfrak{Z}$-continuous partial map. Then $\eta(f)$ is a surjective homomorphism if and only if given any $N_1 \in \eta(X_1)$, there exists $N_2 \in \eta(X_2)$ such that $N_1 = f^{-1}(N_2) \cup (X_1 \setminus Dom(f))$.*

4 Congruences and Products

Theorem 4.1 *Let $\boldsymbol{A}$ be an implication algebra and $(X, \tau, \mathcal{C})$ its corresponding Z-space. Then, there is a one-one correspondence between the implicative filters in $\boldsymbol{A}$ and the closed subsets of X.*

Proof. Let F be an implicative filter in $\boldsymbol{A}$ and consider $C_F = \{M \in Spec(\boldsymbol{A}) : F \subseteq M\}$. Since $C_F = \bigcap_{a \in F} N_a$, it is clear that C_F is closed in X. Note also that $\bigcap C_F = F$. We will show that the mapping $F \longmapsto C_F$ is a one-one correspondence between the implicative filters in $\boldsymbol{A}$ and the closed subsets of X. Indeed, suppose F_1 and F_2 are two implicative filters in $\boldsymbol{A}$ such that $C_{F_1} = C_{F_2}$. Then $F_1 = \bigcap C_{F_1} = \bigcap C_{F_2} = F_2$. Besides, if C is any closed set in X, then there exists a family $\{a_i\}_{i \in I}$ of element of A such that $C = \bigcap_{i \in I} N_{a_i}$. It now follows immediately that $C = \{M \in Spec(\boldsymbol{A}) : Fg(\{a_i\}_{i \in I}) \subseteq M\}$. □

Corollary 4.2 *The congruence lattice of a finite implication algebra is boolean.*

Proof. Let $\boldsymbol{A}$ be a finite implication algebra and $(X, \tau, \mathcal{C})$ its corresponding Z-space. Since X is finite and Hausdorff, τ must be the discrete topology. Hence every subset of X is closed and the congruence lattice of $\boldsymbol{A}$ is isomorphic to the power set of X. □

Theorem 4.3 *Let $\boldsymbol{A}_1, \boldsymbol{A}_2$ be two implication algebras and $(X_1, \tau_1, \mathcal{C}_1)$, $(X_2, \tau_2, \mathcal{C}_2)$ its corresponding Z-spaces. Assume that $X_1 \cap X_2 = \emptyset$. Then the corresponding Z-space for $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ is the space $(X, \tau, \mathcal{C})$ where $X = X_1 \cup X_2$, $\tau = \{U_1 \cup U_2 : U_i \in \tau_i, i = 1, 2\}$ and $\mathcal{C} = \{C_1 \cup C_2 : C_i \in \mathcal{C}_i, i = 1, 2\}$.*

Proof. First observe that $(X, \tau, \mathcal{C})$ is a Z-space. Indeed, it is easy to see that (X, τ) is a Hausdorff topological space. Besides, if we let $\mathcal{B}_i$ be a basis of clopen compact sets for X_i, $i = 1, 2$, it may be shown that $\mathcal{B} = \{N_1 \cup N_2 : N_i \in \mathcal{B}_i, i = 1, 2\}$ is a basis of clopen compact sets for X. It is also clear that $\mathcal{C}$ is a family of closed sets in X and we have that

$$\bigcap \mathcal{C} = \bigcap_{U \in \mathcal{C}_1} \bigcap_{V \in \mathcal{C}_2} (U \cup V) = \bigcap_{U \in \mathcal{C}_1} \left(U \cup \bigcap_{V \in \mathcal{C}_2} V \right) = \bigcap_{U \in \mathcal{C}_1} U = \emptyset.$$

Finally, let C be a closed set in X such that for every clopen N whose complement is compact, $C \subseteq N$ implies $D \subseteq N$ for some $D \in \mathcal{C}$. We know that $C = C_1 \cup C_2$ with each C_i closed in X_i, $i = 1, 2$. Assume $C_1 \subseteq N_1$ for some clopen N_1 in X_1 whose complement is compact. Then $N_1 \cup X_2$ is trivially clopen in X and $X \setminus (N_1 \cup X_2) = X_1 \setminus N_1$ is still compact in X. Since $C \subseteq N_1 \cup X_2$, there must exist $D \in \mathcal{C}$ such that $D \subseteq N_1 \cup X_2$. But $D = D_1 \cup D_2$, with $D_i \subseteq X_i$, $i = 1, 2$, so $D_1 \subseteq N_1$. Since X_1 is a Z-space, we get that there must exist $D_1' \in \mathcal{C}_1$ such that $D_1' \subseteq C_1$. Analogously, there exists $D_2' \in \mathcal{C}_2$ such that $D_2' \subseteq C_2$, so $D' = D_1' \cup D_2' \in \mathcal{C}$ and $D' \subseteq C$. This completes the proof that $(X, \tau, \mathcal{C})$ is a Z-space.

It now remains to show that $(X, \tau, \mathcal{C})$ is indeed the corresponding Z-space for $\boldsymbol{A}_1 \times \boldsymbol{A}_2$. We first claim that the maximal implicative filters in $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ are those of the form $M_1 \times A_2$ with $M_1 \in Spec(\boldsymbol{A}_1)$ and $A_1 \times M_2$ with $M_2 \in Spec(\boldsymbol{A}_2)$. For brevity we put $\overline{M}_1 = M_1 \times A_2$ and $\overline{M}_2 = A_1 \times M_2$. By Lemma 2.1, it is clear that $\overline{M}_i$ is a maximal implicative filter in $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ for each $M_i \in Spec(\boldsymbol{A}_i)$, $i = 1, 2$. Conversely, let $M \in Spec(\boldsymbol{A}_1 \times \boldsymbol{A}_2)$. It is easy to show that $M = F_1 \times F_2$ for some implicative filters F_i in $\boldsymbol{A}_i$, $i = 1, 2$. Suppose $F_1 \neq A_1$ and let $x \in A_1 \setminus F_1$, thus $(x, 1) \notin M$. By Lemma 2.1, for each $x' \in A_1$ and $y \in A_2$, $(x, 1) \to (x', y) = (x \to x', y) \in M$, so $x \to x' \in F_1$ and $y \in F_2$. This shows that $F_1 \in Spec(\boldsymbol{A}_1)$ and $F_2 = A_2$. Likewise, if we suppose that $F_2 \neq A_2$ we obtain that $F_1 = A_1$ and $F_2 \in Spec(\boldsymbol{A}_2)$. This completes the proof of our claim.

It is now clear that the elements in $X = X_1 \cup X_2$ may be identified with those in $Spec(\boldsymbol{A}_1 \times \boldsymbol{A}_2)$ via $M_i \longmapsto \overline{M}_i$, $M_i \in Spec(\boldsymbol{A}_i)$, $i = 1, 2$. In addition, for every $(a_1, a_2) \in A_1 \times A_2$ we have that

$$Spec(\boldsymbol{A}_1 \times \boldsymbol{A}_2) \setminus N_{(a_1, a_2)} = \{\overline{M} : M \in Spec(\boldsymbol{A}_1) \setminus N_{a_1}\} \cup \{\overline{M} : M \in Spec(\boldsymbol{A}_2) \setminus N_{a_2}\}.$$

This shows that the basis $\mathcal{B}$ defined above is the correct basis for the Z-space of $\boldsymbol{A}_1 \times \boldsymbol{A}_2$. Finally, it is easy to notice that the lattice filters in $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ are precisely those filters of the form $F_1 \times F_2$ where F_i is a lattice filter in $\boldsymbol{A}_i$, $i = 1, 2$. So the set of maximal implicative filters in $\boldsymbol{A}_1 \times \boldsymbol{A}_2$ that contain $F_1 \times F_2$ is the set $\{\overline{M} : M \in Spec(\boldsymbol{A}_1), F_1 \subseteq M\} \cup \{\overline{M} : M \in Spec(\boldsymbol{A}_2), F_2 \subseteq M\}$. This implies that the choice of $\mathcal{C}$ is also correct. □

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