

MORE RESULTS ON REGULAR ULTRAFILTERS IN ZFC

PAOLO LIPPARINI

ABSTRACT. We prove, in ZFC alone, some new results on regularity and decomposability of ultrafilters; among them:

(a) If $m \geq 1$ and the ultrafilter D is $(\beth_m(\lambda^{+n}), \beth_m(\lambda^{+n}))$ -regular then D is κ -decomposable for some κ with $\lambda \leq \kappa \leq 2^\lambda$ (Theorem 4.3(a')).

(b) If λ is a strong limit cardinal and D is $(\beth_m(\lambda^{+n}), \beth_m(\lambda^{+n}))$ -regular then either D is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular or there are arbitrarily large $\kappa < \lambda$ for which D is κ -decomposable (Theorem 4.3(b)).

(c) Suppose that λ is singular, $\lambda < \kappa$, $\text{cf } \kappa \neq \text{cf } \lambda$ and D is (λ^+, κ) -regular. Then:

(i) D is either $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, or (λ', κ) -regular for some $\lambda' < \lambda$ (Theorem 2.2).

(ii) If κ is regular then D is either (λ, κ) -regular, or (ω, κ') -regular for every $\kappa' < \kappa$ (Corollary 6.4).

(iii) If either (1) λ is a strong limit cardinal and $\lambda^{<\lambda} < 2^\kappa$, or (2) $\lambda^{<\lambda} < \kappa$, then D is either λ -decomposable, or (λ', κ) -regular for some $\lambda' < \lambda$ (Theorem 6.5).

(d) If λ is singular, D is $(\mu, \text{cf } \lambda)$ -regular and there are arbitrarily large $\nu < \lambda$ for which D is ν -decomposable then D is κ -decomposable for some κ with $\lambda \leq \kappa \leq \lambda^{<\mu}$ (Theorem 5.1; actually, our result is stronger and involves a covering number).

(e) $D \times D'$ is (λ, μ) -regular if and only if there is a ν such that D is (ν, μ) -regular and D' is (λ, ν') -regular for all $\nu' < \nu$ (Proposition 7.1).

We also list some problems, and furnish applications to topological spaces and to extended logics (Corollaries 4.6 and 4.8).

1. INTRODUCTION

The notion of a (λ, μ) -regular ultrafilter has proven particularly useful in Model Theory, Set Theory and even General Topology, see e.g.

1991 *Mathematics Subject Classification.* 03E20, 03E55; 03E65, 03E75, 03E35, 03C95, 54D20.

Key words and phrases. (λ, μ) -regular, κ -decomposable, λ -descendingly incomplete ultrafilters; products, sums of ultrafilters; cofinalities, cardinalities of ultrapowers.

[BS, CK, BF, KSV, Lp1, Lp11]; [CN, KM, Ka, Fo, DN]; [KV, Ca, Ko, GF, Sa, Lp2, Lp8].

In this paper we are concerned with theorems of the form “every (λ, μ) -regular ultrafilter is (λ', μ') -regular”: many results are known in this direction, but most of them rely on assumptions not decided by ZFC (Zermelo-Fraenkel Set Theory with Choice): see [Do, DD] and [FMS, Fo], [Wo, p. 427-431] for recent advances.

However, some theorems hold without special assumptions: in this paper we prove some new results of this kind; moreover we furnish some simplified proofs or slight improvements of known results. We (try to) collect all known results valid in ZFC alone (that is, that do not refer to large cardinals, inner models or other special assumptions of Set Theory). We also state some problems, and deal with the closely related notion of κ -decomposability.

Apart from the results we prove, and their applications, we hope to convince the reader that the study of (λ, μ) -regular ultrafilters in ZFC has some interest in itself, and that many theorems are still to be discovered. Even those interested solely in independence results might find some delight in trying to measure the exact consistency strength of the failure of some natural (but false) generalizations of the results provable in ZFC. See, e.g., Problems 2.8, 5.6, Remarks 2.4 5.5 and the comments after Theorems 2.13, 2.15, Question 3.1, Problems 5.2, 6.8, Proposition 6.7 and Definition 6.9.

Let us recall the basic notions (see [CK, Lp1, CN], or [KM] for other unexplained notions).

$S_\mu(X)$ denotes the set of all subsets of X of cardinality $< \mu$, and $S(X)$ denotes the set of all subsets of X .

We shall give the definition of (μ, λ) -regularity in several equivalent forms.

An ultrafilter D is (μ, λ) -regular if and only if

(FORM I) There is a family of λ members of D such that the intersection of any μ members of the family is empty.

The above notion is due to [Kei], who gave it in the following equivalent form: an ultrafilter D over I is (μ, λ) -regular if and only if

(FORM II) There is a function $f : I \rightarrow S_\mu(\lambda)$ such that, for every $\alpha \in \lambda$, $\{i \in I \mid \alpha \in f(i)\} \in D$.

The two forma are indeed equivalent. If f is a function as given by Form II, then define, for $\alpha \in \lambda$, $X_\alpha = \{i \in I \mid \alpha \in f(i)\}$. $(X_\alpha)_{\alpha \in \lambda}$ is then a family witnessing (μ, λ) -regularity as given by Form I (cf. [Kei, Lemma 1.2]).

Conversely, if $(X_\alpha)_{\alpha \in \lambda}$ is a family witnessing (μ, λ) -regularity as given by Form I, define $f : I \rightarrow S_\mu(\lambda)$ by $f(i) = \{\alpha \in \lambda \mid i \in X_\alpha\}$. Then f witnesses (μ, λ) -regularity as given by Form II.

There is a useful (apparently weaker but actually equivalent) version of Form II.

(FORM II') There is a function $f : I \rightarrow S_\mu(\lambda)$ such that $|\{\alpha \in \lambda \mid \{i \in I \mid \alpha \in f(i)\} \in D\}| = \lambda$.

Form II trivially implies Form II'; conversely if Form II' holds, let $X = \{\alpha \in \lambda \mid \{i \in I \mid \alpha \in f(i)\} \in D\}$, and let $f'(i) = f(i) \cap X$. Since $|X| = \lambda$, then $\langle S_\mu(X), \subseteq \rangle$ and $\langle S_\mu(\lambda), \subseteq \rangle$ are isomorphic; thus, f' composed with an isomorphism witnesses the (μ, λ) -regularity of D as given by Form II.

It is interesting to translate the above equivalent conditions in terms of the ultrapower of $S_\mu(\lambda)$ taken modulo D . It is immediate (from Forms II, II') to see that an ultrafilter D is (μ, λ) -regular if and only if

(FORM III) In the ultrapower $\prod_D \langle S_\mu(\lambda), \subseteq, \{\alpha\} \rangle_{\alpha \in \lambda}$ there is an element x such that $d(\{\alpha\}) \subseteq x$, for every $\alpha \in \lambda$.

Equivalently,

(FORM III') In the ultrapower $\prod_D \langle S_\mu(\lambda), \subseteq, \{\alpha\} \rangle_{\alpha \in \lambda}$ there is an element x such that $|\{\alpha \in \lambda \mid d(\{\alpha\}) \subseteq x\}| = \lambda$.

Here and in what follows d denotes the *natural embedding* [CK].

The definitions given according to Forms III, III' are particularly useful for two reasons: first, we can work in a model of the form $\langle S_\mu(\lambda), \subseteq, \dots \rangle$ and freely use Loš Theorem. Second, and more important, these reformulations allow us to translate arguments concerning (μ, λ) -regularity of ultrafilters into results about models of the above kind. This aspect will play no role in the present paper, but we hope that most results presented here can be generalized to this extended setting (Problem 8.4). The whole matter is described in detail in [Lp1, Section 0], and applications are given in [Lp1, Lp3, Lp4] (in some of those references the order of λ and μ is exchanged). See also [Lp5, Theorem 2].

In the above definitions we assume that μ and λ are infinite cardinals. The notion of an (α, μ) -regular ultrafilter can be defined even for α an ordinal [BK], and some results are indeed theorems in ZFC [Ta], but we shall not deal with this generalized notion here. See also [Lp5, Corollary 5].

We now briefly discuss two notions closely related to (μ, λ) -regularity: λ -descending incompleteness and λ -decomposability. It turns out that λ -descending incompleteness is equivalent to $(\text{cf } \lambda, \text{cf } \lambda)$ -regularity, so

that it is nothing but a reformulation of (λ, λ) -regularity (for λ regular). However, it is useful since it can be defined in terms of the ultrapower of a linear order (rather than of the partial order $S_\lambda(\lambda)$).

As far as λ -decomposability is concerned, it is equivalent to (λ, λ) -regularity for λ regular, but it is a stronger notion for λ singular. Essentially, D is λ -decomposable if and only if some quotient of D is uniform over λ . The main point in applications of decomposability is that, for essentially all purposes, it is enough to consider uniform ultrafilters, and any uniform ultrafilter *must* be uniform on some set: from the cardinality of such a set we can get information about regularity properties of the ultrafilter; see Remark 1.5(b) below and, e.g., the proofs of Proposition 3.3, Theorem 5.1, and Corollary 5.3.

Now for the definitions: D is λ -*descendingly incomplete* if and only if there is a decreasing sequence $(X_\alpha)_{\alpha \in \lambda}$ of sets in D with empty intersection. In terms of ultrapowers, D is λ -descendingly incomplete if and only if in $\prod_D \langle \lambda, < \rangle$ there is an element x such that $d(\alpha) < x$, for every $\alpha \in \lambda$.

An ultrafilter D over I is said to be *uniform* if and only if $|X| = |I|$ for every $X \in D$. It is enough to consider uniform ultrafilters because, were D not uniform, it could be replaced by $D|X$, with X a set in D of minimal cardinality. An ultrafilter D over I is *principal* if and only if there is $i \in I$ such that, for every $X \subseteq I$, $X \in D$ if and only if $i \in X$. Principal ultrafilters are the trivial ones: if D is principal and uniform, then $|I| = 1$.

If D is over I , D is λ -*decomposable* if and only if there is a partition of I into λ classes, the union of $< \lambda$ classes of which never belongs to D . A partition as above will be called a λ -*decomposition* (of D). If Π is a partition of I , we say that Π *has κ classes modulo D* if and only if κ is the least cardinal for which there is $X \in D$ such that Π restricted to X has κ classes. Notice that, if this is the case, then Π induces a κ -decomposition of D : just consider $\Pi' = \Pi|_X \cup \{I \setminus X\}$. It is easy to see that D is λ -decomposable if and only if there is a function $f : I \rightarrow \lambda$ such that whenever $X \subseteq \lambda$ and $|X| < \lambda$ then $f^{-1}(X) \notin D$. Such an f will be called a λ -*decomposition*, too. Notice that every ultrafilter is 1-decomposable, and no ultrafilter is m -decomposable for $1 < m < \omega$.

If D is over I and D' is over I' , then $D' \leq D$ in the *Rudin Keisler* (pre-)order means that there is a surjection $f : I \rightarrow I'$ such that $X \in D'$ if and only if $f^{-1}(X) \in D$. In the above situation, some authors say that D' is a *quotient* or a *projection* of D .

We now recall some facts about the above notions; most of these facts are trivial or easy, but it is hard to find all of them collected in a single place (most of them can be found in [DJK, Section 4]).

Properties 1.1. Assume that λ , μ and κ are infinite cardinals.

- (i) (μ, λ) -regularity is preserved by making μ larger or λ smaller.
- (ii) If D' is (μ, λ) -regular and $D' \leq D$ then D is (μ, λ) -regular.
- (iii) D is λ -decomposable if and only if there is a D' which is uniform on λ and $\leq D$. In particular, every ultrafilter uniform over λ is λ -decomposable.
- (iv) D is λ -descendingly incomplete if and only if it is $\text{cf } \lambda$ -descendingly incomplete.
- (v) Every $(\text{cf } \lambda, \text{cf } \lambda)$ -regular ultrafilter is (λ, λ) -regular.
- (vi) Every ultrafilter uniform on λ is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular and (λ, λ) -regular.
- (vii) Every λ -decomposable ultrafilter is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular and (λ, λ) -regular.
- (viii) If λ is regular, then every (λ, λ) -regular ultrafilter is λ -decomposable.
- (ix) If λ is regular, then an ultrafilter is λ -decomposable if and only if it is λ -descendingly incomplete.
- (x) If D' is λ -decomposable and $D' \leq D$ then D is λ -decomposable.
- (xi) In particular, if λ is regular, then D is (λ, λ) -regular if and only if D is λ -descendingly incomplete, if and only if D is λ -decomposable, if and only if there is a D' which is uniform over λ and $\leq D$.
- (xii) If D is (μ, λ) -regular, κ is regular, and $\mu \leq \kappa \leq \lambda$ then D is κ -decomposable.
- (xiii) If $\mu > \lambda$ then every ultrafilter is (μ, λ) -regular.

Proof. A proof of (iv) can be found, e.g., in [CN, p. 198–199]; (v) and (vi) come from [Kei, Lemma 1.3(iv)(iii)]; (vii) is immediate from (iii), (vi) and (ii). See e.g. [KM, p. 179] for a proof of (viii). (xi) follows from (iii), (vii), (viii) and (ix). (xii) follows from (i) and (viii). All other statements are trivial. \square

As a consequence of 1.1, most results on regularity of ultrafilters have many equivalent reformulations. For example:

Consequence 1.2. If κ is a regular cardinal, and μ, λ are cardinals, then the following are equivalent:

- (a) Every ultrafilter uniform on κ is (μ, λ) -regular.
- (b) Every κ -decomposable ultrafilter is (μ, λ) -regular.
- (c) Every (κ, κ) -regular ultrafilter is (μ, λ) -regular.

Proof. The equivalence of (b) and (c) is immediate from 1.1(xi), since κ is assumed to be regular. (c) \Rightarrow (a) follows from 1.1(vi). Finally, if D is κ -decomposable, then by 1.1(iii) there is $D' \leq D$, D' uniform on κ ; if (a) holds then D' is (μ, λ) -regular, and D , too, is (μ, λ) -regular by 1.1(ii). Thus, (a) \Rightarrow (b). \square

Many results on regularity of ultrafilters have the form described in 1.2, and are usually stated as in clause (a). However, we believe that clause (c) is the most convenient way to state the results. Formally, (c) is more natural in the sense that involves just one notion, regularity, rather than two notions, regularity and uniformity (or decomposability). For example, a classical result (see Theorem 2.1(b)) states, when expressed as in clause (a), that

(*) every uniform ultrafilter over κ^+ is (κ, κ) -regular.

If we state this result as in clause (c), that is

(**) every (κ^+, κ^+) -regular ultrafilter is (κ, κ) -regular,

we immediately get that every $(\kappa^{++}, \kappa^{++})$ -regular ultrafilter is (κ, κ) -regular, a corollary which is not that obvious, if we keep the theorem in the form (*).

Moreover, in many applications, (c) is what is really used (see Corollaries 4.6 and 4.8, as far as this paper is concerned).

The main advantage of clause (c), however, is that it naturally lends itself to generalizations. We can take a known result in the form given by (c): every (κ, κ) -regular ultrafilter is (μ, λ) -regular, and try to see whether it generalizes to: every (κ, κ') -regular ultrafilter is (μ, λ') -regular, for appropriate $\kappa' \geq \kappa$ and $\lambda' \geq \lambda$.

It turns out that usually such a generalized statement holds, and many examples are provided in the present paper: the statements of Theorems 2.2, 2.13(ii), 6.5(a), and Proposition 7.3 have all been devised by applying the above described pattern (there are more possibilities: see Conjecture 2.16, and Problem 2.20(b)).

The following result has a very easy proof (for example, it is the easy part of [BK, Theorem 1.3]), but it has many interesting consequences.

Proposition 1.3. *If λ is regular and the ultrafilter D is (λ, κ) -regular, then $\text{cf}(\prod_D \langle \lambda, < \rangle) > \kappa$.*

The following cardinality result has many consequences, too. It is just a particular case of [Kei, Theorem 2.1].

Proposition 1.4. *If the ultrafilter D is (μ, λ) -regular then $|\prod_D 2^{<\mu}| \geq 2^\lambda$.*

Remarks 1.5. (a) It is also interesting to note that if λ is regular then the following are equivalent: (i) D is (λ, λ) -regular; (ii) $\text{cf}(\prod_D \langle \lambda, < \rangle) > \lambda$; (iii) $\text{cf}(\prod_D \langle \lambda, < \rangle) \neq \lambda$.

(i) \Rightarrow (ii) is an instance of Proposition 1.3; (ii) \Rightarrow (iii) is trivial, and (iii) implies that D is λ -descendingly incomplete, hence (λ, λ) -regular by 1.1(xi).

Actually, the above remark is the particular case $\lambda = \kappa$ of Theorem 2.13(iii).

(a') If D is over I , λ is regular, and D is (λ, λ) -regular, then $|I| \geq \lambda$. Indeed, by 1.1(viii), D is λ -decomposable, and by 1.1(iii) there is $D' \leq D$ uniform on λ , thus $|I| \geq \lambda$.

Moreover, if D is over I , $\mu < \lambda$, and D is (μ, λ) -regular, then $|I| \geq \lambda$. Indeed, by 1.1(i), D is (λ', λ') -regular for all λ' with $\mu \leq \lambda' \leq \lambda$. By the preceding paragraph, we get $\lambda' \leq |I|$ for all regular cardinals λ' with $\mu \leq \lambda' \leq \lambda$, and this implies $\lambda \leq |I|$.

Notice, however, that if μ is singular then every ultrafilter uniform over $\text{cf}\mu$ is (μ, μ) -regular, by 1.1(vi)(v) (take $\lambda = \text{cf}\mu$).

(b) A subset X of $S_\mu(\lambda)$ is *cofinal* if and only if for every $y \in S_\mu(\lambda)$ there is $x \in X$ such that $y \subseteq x$; the minimal cardinality of such an X is the *cofinality* of $S_\mu(\lambda)$, and is denoted by $\text{cf}S_\mu(\lambda)$. Notice that in Forms II, II' of the definition of (μ, λ) -regularity we can equivalently ask that $f : I \rightarrow X$, where X is a cofinal subset of $S_\mu(\lambda)$. Similarly, in Forms III, III' it is enough to refer to the ultrapower of a cofinal subset X of $S_\mu(\lambda)$, assuming, without loss of generality, that $\{\alpha\} \in X$ for every $\alpha \in \lambda$.

Whence if D is (μ, λ) -regular then there is a (μ, λ) -regular quotient of D which is uniform over some $\kappa \leq \text{cf}S_\mu(\lambda)$; moreover, if either $\mu = \lambda$ is regular, or $\mu < \lambda$ then we have $\kappa \geq \lambda$ by (a'). (See also Proposition 6.7(ii).)

In particular, if $\mu < \lambda$, D is (μ, λ) -regular and $\text{cf}S_\mu(\lambda) = \lambda$ then D is λ -decomposable; this applies, for example, when $\lambda^{<\mu} = \lambda$, since $|S_\mu(\lambda)| = \lambda^{<\mu}$; in particular, every (ω, λ) -regular ultrafilter is λ -decomposable. Notice also that if μ is regular then $\text{cf}S_\mu(\mu^{+n}) = \mu^{+n}$, for every natural number n .

An ultrafilter D is λ -complete if and only if the intersection of any family of $< \lambda$ members of D belongs to D . It is quite easy to show that if $\lambda > \omega$, then D is λ -complete if and only if for no infinite $\lambda' < \lambda$ D is λ' -decomposable, if and only if for no infinite $\lambda' < \lambda$ D is (λ', λ') -regular.

A cardinal $\lambda > \omega$ is *measurable* if and only if there exists a λ -complete ultrafilter uniform over λ . By the preceding remark and 1.1(iii)(x), if an ultrafilter D is κ -decomposable for some infinite cardinal κ , the first such κ is either ω or a measurable cardinal. Moreover, if an ultrafilter D is (κ, κ) -regular for some infinite cardinal κ , the first such κ is either ω or a measurable cardinal; this is proved as follows: because of (κ, κ) -regularity D is not principal, hence uniform over some infinite cardinal, hence κ' -decomposable for some infinite κ' (by 1.1(iii)); the first such

κ' is either ω or a measurable cardinal, by above, and D is (κ', κ') -regular by 1.1(vii); then D is κ' -complete, and D is not (κ, κ) -regular for $\kappa < \kappa'$, by the remark after the definition of λ -completeness, thus κ' is also the first κ for which D is (κ, κ) -regular.

The cardinal $\lambda > \omega$ is κ -compact if and only if there is a λ -complete (λ, κ) -regular ultrafilter. λ is *strongly compact* if and only if it is κ -compact for all κ . It is well known that the above definitions are equivalent to the more usual ones (see e.g. [Ket1, Theorems 5.9 and 5.10] or [KM, Section 15]).

We shall sometimes use the following known fact (see e.g. [KM, p. 190]): if $\lambda \leq \kappa$ are regular, and λ is κ -compact then $\kappa^{<\lambda} = \lambda$. It follows from the above identity and trivial cardinality arithmetic that if λ is regular, κ is any cardinal, cf $\kappa \geq \lambda$ and λ is κ' -compact for all $\kappa' < \kappa$, then $\kappa^{<\lambda} = \lambda$ still holds. Moreover, if κ is singular, λ is regular, λ is κ^+ -compact and cf $\kappa < \lambda$ then $\kappa^+ \leq \kappa^{\text{cf } \kappa} \leq \kappa^{<\lambda} \leq (\kappa^+)^{<\lambda} = \kappa^+$, hence $\kappa^{<\lambda} = \kappa^+$. In particular, by 1.1(ii) and Remark 1.5(b), if λ is regular, then λ is κ -compact if and only if there is a λ -complete (λ, κ) -regular ultrafilter over κ (a fact first stated as Theorem 5.10 in [Ket1]).

We have promised to consider only results in ZFC and, needless to say, measurable and strongly compact cardinals are large cardinals; but we shall use them only in order to get counterexamples, that is, in order to show that certain statements are not theorems of ZFC (as usual, whenever we mention any such large cardinal, we implicitly assume its consistency). In recent developments of set theory the notion of *supercompactness* has played a very central role (see e.g. [Ka]): supercompactness is a stronger property than strong compactness and, in certain respects, it is better behaved; however, in our counterexamples we need only the weaker notion of strong compactness. The relationship between strong compactness and supercompactness has been analyzed in several recent papers by A. Apter and others; see e.g. [Ap], and further references there.

Clearly, if there is a measurable cardinal, there are largely irregular ultrafilters. In most cases, from an irregular ultrafilter, it is possible to construct a model of set theory with a large cardinal.

Theorem 1.6. [Do, Theorem 4.5] *If there is no inner model with a measurable cardinal then, for every cardinal κ , every ultrafilter uniform over κ is (ω, κ') -regular for every $\kappa' < \kappa$.*

It is conceivable that the conclusion in Theorem 1.6 can be improved to (ω, κ) -regular (the maximum of regularity attainable), but, to the best of our knowledge, a proof has not been found yet (however, [Do] contains some more results towards this direction).

If λ is a limit cardinal and D is an ultrafilter, we say that there are arbitrarily large $\nu < \lambda$ such that D is ν -decomposable if and only if for every $\nu' < \lambda$ there is ν such that $\nu' < \nu < \lambda$ and D is ν -decomposable.

Occasionally, we shall use the following principle.

Definition 1.7. If λ is a limit cardinal, $U'(\lambda)$ means that for every ultrafilter D , if there are arbitrarily large $\nu < \lambda$ such that D is ν -decomposable, then D is (λ, λ) -regular.

Slightly weaker principles have been used in [Lp1, Lp4]. See Definitions 6.9 and the subsequent discussion for the consistency strength of these principles.

$\lambda^{+\alpha}$ denotes the α^{th} successor of λ : that is, if $\lambda = \omega_\beta$ then $\lambda^{+\alpha}$ is $\omega_{\beta+\alpha}$. $\lambda^{<\mu}$ is $\sup\{\lambda^{\mu'} \mid \mu' < \mu\}$.

2. FROM SUCCESSORS TO PREDECESSORS.

As the main result of this section, we will prove a generalization (with a new proof) of the following known result.

Theorem 2.1. (a) *If the ultrafilter D is (λ^+, λ^+) -regular, then D is either $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, or (λ', λ^+) -regular for some regular $\lambda' \leq \lambda$.*

(b) *In particular, every (λ^+, λ^+) -regular ultrafilter is (λ, λ) -regular.*

(b) follows from (a) because of 1.1(v) and 1.1(i). Notice that (a) is stronger than (b) only in the case when λ is singular. If λ is regular, then $\text{cf}\lambda = \lambda$, hence $(\text{cf}\lambda, \text{cf}\lambda)$ -regularity is the same as (λ, λ) -regularity, so that (b) implies (a), and, actually, (b) implies that the first alternative holds in the conclusion of (a).

As we remarked in 1.2, and the comment below, Theorem 2.1 is usually stated in some equivalent form.

Under instances of GCH, [Ch] proved 2.1(b) for λ regular, and a slightly weaker form of 2.1(a). Without assuming GCH, Theorem 2.1 is proved in [ČČ, Theorem 1] and [KP, Theorem 2.1]. In the particular case when λ is regular, case (b) can be obtained also as a consequence of either [BK, Corollary 1.8] or [Jo], using 1.1(i) and the characterization of (λ, λ) -regularity given in Remark 1.5(a). See also [CN, Theorem 8.35, Corollary 8.36, and p. 203]. Now we present our generalization of Theorem 2.1.

Theorem 2.2. *If $\mu \geq \lambda^+$, $\text{cf}\mu \neq \text{cf}\lambda$ and D is a (λ^+, μ) -regular ultrafilter, then D is either $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, or (λ', μ) -regular for some regular $\lambda' \leq \lambda$.*

Proof. For every $z \subseteq \mu$ with $|z| \leq \lambda$ let $\phi(z, -) : z \rightarrow \lambda$ be an injection; and for every $\beta < \lambda$ let $F(z, \beta) = \{\alpha \in z \mid \phi(z, \alpha) < \beta\}$. Thus, for every $z \subseteq \mu$ and $\beta < \lambda$, $|F(z, \beta)| \leq |\beta| < \lambda$.

Let us work in $\mathbf{B} = \prod_D \mathbf{A}$, where \mathbf{A} is an appropriate expansion of $S_{\lambda^+}(\mu)$. Suppose that D is (λ^+, μ) -regular, and let $x \in B$ witness it (as given by Form III).

Case (a): there is $\alpha < \mu$ such that $\phi_{\mathbf{B}}(x, d(\alpha)) >_{\mathbf{B}} d(\beta)$ for all $\beta < \lambda$. Then $\phi_{\mathbf{B}}(x, d(\alpha))$ witnesses the λ -descending incompleteness of D , hence D is $(\text{cf}\lambda, \text{cf}\lambda)$ -regular because of 1.1(iv) and of 1.1(xi) (applied with $\text{cf}\lambda$ in place of λ).

Case (b): otherwise. For every $\alpha < \mu$ choose some $\beta_\alpha < \lambda$ such that $\phi_{\mathbf{B}}(x, d(\alpha)) \leq_{\mathbf{B}} d(\beta_\alpha)$.

If we show that there is $\beta < \lambda$ such that $|\{\alpha < \mu \mid \phi_{\mathbf{B}}(x, d(\alpha)) \leq_{\mathbf{B}} d(\beta)\}| = \mu$ then $|\{\alpha < \mu \mid d(\{\alpha\}) \subseteq_{\mathbf{B}} F_{\mathbf{B}}(x, d(\beta + 1))\}| = \mu$, and this implies that D is $(|\beta|^+, \mu)$ -regular (Form III'), since $F_{\mathbf{B}}(x, d(\beta + 1))$ belongs to $\prod_D S_{|\beta|^+}(\mu)$.

In order to show the existence of a β as above, consider an increasing sequence of ordinals $(\varepsilon_\delta)_{\delta \in \text{cf}\lambda}$ cofinal in λ , and let $X_\delta = \{\alpha < \mu \mid \beta_\alpha < \varepsilon_\delta\}$. It is enough to show that $|X_\delta| = \mu$ for some $\delta \in \text{cf}\lambda$, since in this case we can take $\beta = \varepsilon_\delta$.

Since we are in Case (b), $\bigcup_{\delta \in \text{cf}\lambda} X_\delta = \mu$. If $\text{cf}\mu > \text{cf}\lambda$ then it is trivial that $|X_\delta| = \mu$ for some $\delta \in \text{cf}\lambda$. Otherwise by hypothesis $\text{cf}\mu < \text{cf}\lambda$. For every $\nu < \mu$ there is X_{δ_ν} of cardinality $\geq \nu$ (otherwise there is $\nu < \mu$ such that $\mu \leq \nu \cdot \text{cf}\lambda < \mu$, absurd). We can consider a sequence of $\text{cf}\mu$ -many ν 's converging to μ ; then the δ_ν 's are bounded by some $\delta \in \text{cf}\lambda$, since $\text{cf}\mu < \text{cf}\lambda$, and then $|X_\delta| = \mu$, since $\delta' < \delta$ implies $X_{\delta'} \subseteq X_\delta$. \square

Theorem 2.1(a) is the particular case $\mu = \lambda^+$ of Theorem 2.2.

Theorem 2.2 strengthens the classical result Theorem 2.1 only in the case when λ is singular: if $\mu \geq \lambda^+$ and D is (λ^+, μ) -regular, then D is trivially (λ^+, λ^+) -regular, by 1.1(i), hence (λ, λ) -regular, by Theorem 2.1(b); if λ is regular, $(\text{cf}\lambda, \text{cf}\lambda)$ -regularity is the same as (λ, λ) -regularity, and we get the first alternative in the conclusion of Theorem 2.2. Anyway, the proof we have given has the advantage of a greater simplicity (at least, in our opinion).

Remark 2.3. The assumption $\mu \geq \lambda^+$ is not needed in Theorem 2.2, but this is the only interesting case. Since every ultrafilter is (λ, μ) -regular for $\mu < \lambda$, the theorem is trivially true for $\mu < \lambda$.

For $\mu = \lambda$ Theorem 2.2 would be false: any ultrafilter is (λ^+, λ) -regular, but any principal ultrafilter is neither $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, nor

(λ', λ) -regular for $\lambda' \leq \lambda$ (of course, the case $\mu = \lambda$ is prevented by the hypothesis $\text{cf } \mu \neq \text{cf } \lambda$).

Remark 2.4. The dichotomy in the conclusion of Theorem 2.2 cannot be avoided. On one side, if κ is $\kappa^{+\omega+1+\alpha}$ -compact then there is a $(\kappa^{+\omega+1}, \kappa^{+\omega+1+\alpha})$ -regular (in fact, $(\kappa, \kappa^{+\omega+1+\alpha})$ -regular) ultrafilter D which is κ -complete, and hence not $(\omega, \omega) = (\text{cf } \kappa^{+\omega}, \text{cf } \kappa^{+\omega})$ -regular.

On the other side, [BM] shows that if it is consistent to have a κ^+ -compact cardinal κ then it is consistent to have an $(\omega_{\omega+1}, \omega_{\omega+1})$ -regular ultrafilter D which for no $n > 0$ is (ω_n, ω_n) -regular (see also [AH]). Hence, by 1.1(i), for no $\lambda' < \omega_\omega$ D is $(\lambda', \omega_{\omega+1})$ -regular.

We do not know the exact consistency strengths (for each $\alpha > 0$) of a $(\lambda^+, \lambda^{+\alpha})$ -regular ultrafilter which is not $(\lambda, \lambda^{+\alpha})$ -regular (cf. also Remark 5.5).

We do not know whether the hypothesis $\text{cf } \mu \neq \text{cf } \lambda$ in Theorem 2.2 can be omitted (when λ is regular the hypothesis is unnecessary, since we always get (λ, λ) -regularity from (λ^+, λ^+) -regularity, hence from (λ^+, μ) -regularity). Another case in which $\text{cf } \mu \neq \text{cf } \lambda$ is not necessary is presented in Proposition 8.2.

We do not know whether the proof of Theorem 2.2 can be extended in order to show:

Conjectures 2.5. If $\mu \geq \lambda^{+n}$ and D is a (λ^{+n}, μ) -regular ultrafilter, then D is either $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, or (λ', μ) -regular for some regular $\lambda' \leq \lambda$.

If λ is regular Conjecture 2.5 is true: by 1.1(i) we get $(\lambda^{+n}, \lambda^{+n})$ -regularity, hence (λ, λ) -regularity, by iterating Theorem 2.1(b).

In case $n = 0$ Conjecture 2.5 has an affirmative answer, too, by the next proposition. Then Theorem 2.15 implies that Conjecture 2.5 is true also in case when μ is singular and $\text{cf } \mu < \text{cf } \lambda$.

The next proposition is an immediate consequence of [Lp1, Theorem 0.20(iv)].

Proposition 2.6. *If λ is singular then every (λ, μ) -regular ultrafilter is either $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, or (λ', μ) -regular for some $\lambda' < \lambda$.*

If $\mu \geq \lambda$ and λ is regular then every (λ, μ) -regular ultrafilter is $(\lambda, \lambda) = (\text{cf } \lambda, \text{cf } \lambda)$ -regular, so that the hypothesis λ singular is not necessary in 2.6, if $\mu \geq \lambda$.

The following proposition is a variation (and relies heavily) on [Do, Theorem 4.5] (stated here as Theorem 1.6).

Proposition 2.7. *If there is no inner model with a measurable cardinal, then, for every cardinal λ , every (λ, λ) -regular ultrafilter is both $(\text{cf } \lambda, \text{cf } \lambda)$ -regular and (ω, ν) -regular, for every $\nu < \text{cf } \lambda$.*

Proof. If λ is regular, the statement in 2.7 is equivalent to the statement in Theorem 1.6, by 1.2.

If λ is singular, and D is (λ, λ) -regular then, by Proposition 2.6, D is either $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, or (λ', λ) -regular for some $\lambda' < \lambda$.

In the first case, D is $\text{cf}\lambda$ -decomposable by 1.1(viii), and the conclusion follows from Theorem 1.6 (with $\text{cf}\lambda$ in place of κ).

In the second case, there is a regular $\mu \geq \lambda'$, with $\text{cf}\lambda < \mu < \lambda$ and such that D is (μ, λ) -regular (by 1.1(i)), hence D is μ -decomposable by 1.1(xii). By applying Theorem 1.6 with μ in place of κ , we get that D is $(\omega, \text{cf}\lambda)$ -regular (more than requested, by 1.1(i)). \square

Essentially, the proof of Proposition 2.7 is implicit in the proof of [Lp1, Proposition 4.2].

Problem 2.8. Find the exact consistency strength of a (λ, λ) -regular not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular ultrafilter.

Remark 2.4 gives an example of a (λ, λ) -regular not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular ultrafilter.

The following theorem, reformulated here using 1.1(xi), is proved in [Lp11, Corollary 7].

Theorem 2.9. *If λ is a singular cardinal and the ultrafilter D is not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, then the following conditions are equivalent:*

- (a) *There is $\lambda' < \lambda$ such that D is (κ, κ) -regular for all regular cardinals κ with $\lambda' < \kappa < \lambda$.*
- (b) *D is (λ^+, λ^+) -regular.*
- (c) *There is $\lambda' < \lambda$ such that D is (λ', λ^+) -regular.*
- (d) *D is (λ, λ) -regular.*
- (e) *There is $\lambda' < \lambda$ such that D is (λ', λ) -regular.*
- (f) *There is $\lambda' < \lambda$ such that D is (λ'', λ'') -regular for every λ'' with $\lambda' < \lambda'' < \lambda$.*

By Proposition 2.7, if there is a (λ, λ) -regular not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular ultrafilter then there is an inner model with a measurable cardinal. Using Theorem 2.9, a much stronger result can be proved.

The principle \square_μ has been introduced by R. Jensen in his study of the fine structure of L , and is now “ubiquitous in set theory” [SZ]. For our purposes here, the exact definition of \square_μ is not relevant: we only need to know the following classical result.

Theorem 2.10. *If \square_μ holds then every (μ^+, μ^+) -regular ultrafilter is (κ, κ) -regular, for every $\kappa \leq \mu$.*

A proof for the case κ regular can be found, e. g., in [KM, p. 219-221], using 1.1(xi)(ii), of course. The case κ singular then follows by 1.1(v).

In fact, a stronger result holds: [Do, Theorem 1.4] implies that if $\square_{\mu^+}^-$ (a principle weaker than \square_μ) holds, then every (μ^+, μ^+) -regular ultrafilter is (ω, μ) -regular, hence κ -decomposable for all κ with $\omega \leq \kappa \leq \mu$, by 1.1(i) and Remark 1.5(b).

Proposition 2.11. *Suppose that λ is singular, and D is a (λ, λ) -regular not $(\text{cf } \lambda, \text{cf } \lambda)$ -regular ultrafilter. Then \square_λ fails.*

Proof. By Theorem 2.9(d) \Rightarrow (b), D is (λ^+, λ^+) -regular. Suppose by contradiction that \square_λ holds. Then Theorem 2.10 implies that D is $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, contradiction. \square

As far as we know, the exact consistency strength of the failure of \square_λ for some singular λ has not been evaluated yet. However, [SZ] announces that in many cases we get the consistency of many Woodin cardinals. See [SZ, St] and references there for more details. See Proposition 8.6 for a strengthening of Proposition 2.11.

Maybe the following is also true.

Conjectures 2.12. If $\lambda < \mu$ and D is (λ^+, μ) -regular then D is either λ -decomposable, or (λ', μ) -regular for some regular $\lambda' \leq \lambda$.

The significant case in Conjecture 2.12 is when λ is singular. When λ is regular, we get (λ, λ) -regularity from 1.1(i) and Theorem 2.1(b), hence λ -decomposability from 1.1(viii), so that the conjecture always holds when λ is regular.

For λ singular, Conjecture 2.12 is true under some cardinality and cofinality assumptions: see Theorem 6.5.

The following are proved in [Lp5, Theorem A and Corollary 1] (we left (ii) as an open problem in the first version of the present paper).

Theorem 2.13. *(i) If D is a (λ^+, κ) -regular ultrafilter then either:*

- (a) D is (λ, κ) -regular, or*
- (b) the cofinality of the linear order $\prod_D \langle \lambda, < \rangle$ is $\text{cf } \kappa$, and D is (λ, κ') -regular for all $\kappa' < \kappa$.*
- (ii) In particular, by 1.1(i), every (λ^+, κ^+) -regular ultrafilter is (λ, κ) -regular.*

(iii) If $\lambda \leq \kappa$, λ is regular and D is (λ^+, κ) -regular then the following are equivalent:

- (a) D is (λ, κ) -regular; (b) the cofinality of $\prod_D \langle \lambda, < \rangle$ is $> \kappa$; (c) the cofinality of $\prod_D \langle \lambda, < \rangle$ is $\neq \text{cf } \kappa$.*

Shortly after [Lp5] was published, we realized that Theorem 2.13 can be proved also by a slight extension of the techniques in [Pr2]. The equivalence of (a) and (b) in Theorem 2.13(iii) is due to [BK].

[FMS] shows that, modulo the consistency of (something less than) a huge cardinal, for every regular λ it is consistent to have a (λ^+, λ^+) -regular not (λ, λ^+) -regular ultrafilter (see also [Ka2] [Hu], [Wo, p. 427-431]).

Even more irregular ultrafilters have been constructed by M. Foreman [Fo]: modulo some large cardinal consistency assumption, it is consistent to have a uniform ultrafilter D over ω_2 such that $|\prod_D \omega| = \omega_1$ ([Fo, Corollary 3.2]); as remarked by Foreman, the above identity implies $|\prod_D \omega_1| = \omega_2$, and the same argument gives $|\prod_D \omega_2| = \omega_3$. Notice also that if D is uniform over ω_2 and $|\prod_D \omega| = \omega_1$ then D is not (ω, ω_1) -regular, because of [Kei, Theorem 2.1] (see Proposition 1.4 here); moreover, D is not (ω_1, ω_2) -regular, by Theorem 2.13(ii), or by easy cardinality considerations. On the other hand, D is (ω_2, ω_2) -regular by 1.1(vi), (ω_1, ω_1) -regular and (ω, ω) -regular by Theorem 2.1(b). Since by Proposition 1.4 $\omega_1 = |\prod_D \omega| \geq 2^\omega \geq \omega_1$, we get $2^\omega = \omega_1$, hence $2^{\omega_1} = \omega_2$, by Theorem 2.18(ii), hence also $2^{\omega_2} = \omega_3$, again by 2.18.

It should be mentioned that the construction of an irregular ultrafilter is only one among many other important applications of results in [Fo]. According to [Fo], the problem of whether similar results can be obtained for ω_n ($2 < n < \omega$) in place of ω_2 *looks like only a “technical problem”, but perhaps not*. Thus we do not know whether for $n > 2$ it is consistent to have some ultrafilter uniform over ω_n not (ω, ω_1) -regular.

An infinitary generalization of Foreman’s result, if possible, probably would be more than a “technical problem”!

Problem 2.14. Is it consistent to have an ultrafilter D which is (ω_n, ω_n) -regular, for every $n < \omega$, but which for no $n < \omega$ is (ω_n, ω_{n+1}) -regular? (see the remark after Proposition 7.4)

Can we get $|\prod_D \omega_n| = \omega_{n+1}$ for all $n \in \omega$?

Be that as it may, the situation changes for singular cardinals.

Theorem 2.15. (i) [Ka1, Corollary 2.4] *If λ is singular then every (λ^+, λ^+) -regular ultrafilter is (λ, λ^+) -regular.*

(ii) [Lp5, Corollary B] *Suppose that κ is a singular cardinal, $\kappa > \lambda$ and either λ is regular, or $\text{cf} \kappa < \text{cf} \lambda$. Then every (λ^+, κ) -regular ultrafilter is (λ, κ) -regular.*

Notice that the conclusion in Theorem 2.15(i) cannot be improved to “there is $\lambda' < \lambda$ such that D is (λ', λ^+) -regular”, because of the result from [BM] mentioned in Remark 2.4.

Similarly, the conclusion in Theorem 2.15(ii) cannot be improved to “there is $\lambda' < \lambda$ such that D is (λ', κ) -regular”: if there are ω_1 strongly compact cardinals, say λ_α ($\alpha \in \omega_1$), λ_α increasing, let $\lambda = \sup_{\alpha \in \omega_1} \lambda_\alpha$ and let κ be any cardinal $> \lambda$ (in particular, we can have $\text{cf } \kappa = \omega$). For each $\alpha \in \omega_1$ there is an ultrafilter D_α which is (λ_α, κ) -regular and not (ν, ν) -regular for all $\nu < \lambda_\alpha$. Let D be uniform over ω_1 , and consider the sum $D' = \sum_D D_\alpha$ (see Section 7 for the definition). By Proposition 7.4(c) (with $\mu = \nu = \kappa$, using 1.1(i)(xiii)) D' is (λ, κ) -regular; by 7.4(d) D is not (λ', λ') -regular, for every regular cardinal λ' with $\omega_1 < \lambda' < \lambda$, hence not (λ', κ) -regular for every $\lambda' < \lambda$, by 1.1(i).

We expect that just one strongly compact cardinal is sufficient in order to obtain a counterexample as above: if λ is $\lambda^{+\alpha}$ -compact then it is probably possible to make λ singular (by some variation on Prikry forcing [Pr1]) in such a way that in the resulting model there is a $(\lambda, \lambda^{+\alpha})$ -regular ultrafilter which for no $\lambda' < \lambda$ is $(\lambda', \lambda^{+\alpha})$ -regular.

Notice also that 2.1(b) and 2.15(i) imply that if λ is singular and $n > 0$ then every $(\lambda^{+n}, \lambda^{+n})$ -regular ultrafilter is (λ, λ^+) -regular.

In Theorem 2.15(ii) the hypothesis “ κ is a singular cardinal” cannot be weakened to “ κ is a limit cardinal”: see Remark 5.5.

Conjectures 2.16. If λ is singular and $\lambda < \mu$ then every (λ^+, μ) -regular ultrafilter is (λ, μ) -regular (maybe some assumption on $\text{cf } \mu$ is necessary, say $\text{cf } \mu \neq \text{cf } \lambda$, or some similar condition).

A positive answer to Conjecture 2.16 would encompass (the case when λ is singular of) Theorem 2.2, in view of Proposition 2.6. Thus, the dichotomy in the conclusion of Theorem 2.2 might depend only on the dichotomy in the conclusion of 2.6. Also, an affirmative solution to Conjecture 2.16 with no assumption on $\text{cf } \mu$ would make the assumption $\text{cf } \mu \neq \text{cf } \lambda$ in Theorem 2.2 unnecessary. Theorem 2.15(ii) shows that if Conjecture 2.16 is true whenever $\text{cf } \lambda \leq \text{cf } \mu$ then it is true for every λ and μ . See Theorem 2.15, Corollary 6.4 and Propositions 6.7 and 8.2 for partial answers to Conjecture 2.16.

Problem 2.17. The proof of Theorem 2.15(i) uses notions of a large cardinal nature, as least functions, weak normality, and the like (see Section 6). Is there a simpler proof which makes use of elementary arguments only?

Also the following results can be proved:

Theorem 2.18. (a) ([BK, Corollary 2.2], [Ket, Theorem 1.11]) *If $2^\kappa = \kappa^+$ and $2^{\kappa^+} > \kappa^{++}$ then every (κ^+, κ^+) -regular ultrafilter is (κ, κ^+) -regular.*

(b) *If $2^{\kappa^{+n}} = \kappa^{+n+1}$ and $2^{\kappa^{+n+1}} > \kappa^{+n+2}$ then every $(\kappa^{+n+1}, \kappa^{+n+1})$ -regular ultrafilter is (κ, κ^+) -regular.*

(c) [JP, Theorem 7.2.1] *If $\omega_{\omega_1} > 2^\omega > \omega_1$ and $2^{\omega_1} > 2^\omega$ then every (ω_1, ω_1) -regular ultrafilter is (ω, ω_1) -regular.*

Proof of (b). Suppose that D is $(\kappa^{+n+1}, \kappa^{+n+1})$ -regular. By (a) (with κ^{+n} in place of κ) D is $(\kappa^{+n}, \kappa^{+n+1})$ -regular. Now apply Theorem 2.13(ii) n times. \square

For sake of completeness we shall mention also the following result, though it deals with (moderately) large cardinals.

Theorem 2.19. [ČČ]; see also [Pr, Theorem 3.1] *If κ is inaccessible and not ω -Mahlo, or is weakly inaccessible and not ω -weakly-Mahlo then every (κ, κ) -regular ultrafilter is either (λ, κ) -regular for some $\lambda < \kappa$, or (λ, λ) -regular for all $\lambda < \kappa$.*

Problems 2.20. (a) Does Theorem 2.19 apply to more inaccessible cardinals (for example, to κ 's which are not $\omega + 1$ -Mahlo)? It probably applies to more cardinals (see [Ha]), but [Shr] imposes limitations.

(b) We do not know whether a two cardinals version of 2.19 holds, that is whether (for κ as in the statement of 2.19, and appropriate κ') it is true that every (κ, κ') -regular ultrafilter is either (λ, κ') -regular for some $\lambda < \kappa$, or (λ, λ) -regular for all $\lambda < \kappa$.

3. DOWN FROM EXPONENTS (PART I)

One could ask whether a version of Theorem 2.1(b) holds when successors are replaced by exponents; namely whether it can be proved that every $(2^\lambda, 2^\lambda)$ -regular ultrafilter is (λ, λ) -regular. In this form, the problem has a negative answer: if we start with a model satisfying GCH and with a measurable cardinal μ , and we add μ Cohen reals, then in the resulting model $2^\omega = 2^{\omega_1} = \mu$, and there is a μ -decomposable ultrafilter which is not λ -decomposable, for every λ with $\omega < \lambda < \mu$ (see e.g. [Ket, p. 62]; see [Shr] for related results). Thus, by 1.1(xi), there is a $(2^{\omega_1}, 2^{\omega_1})$ -regular not (ω_1, ω_1) -regular ultrafilter.

On the other side, if there is no inner model with a measurable cardinal, then by Proposition 2.7 every $(2^\lambda, 2^\lambda)$ -regular ultrafilter is (ω, ν) -regular for all $\nu < \text{cf} 2^\lambda$, hence (λ, λ) -regular, by 1.1(i) and since $\lambda < \text{cf} 2^\lambda$.

By the above remarks, the existence of a measurable cardinal is equiconsistent with the existence, for some λ , of a $(2^\lambda, 2^\lambda)$ -regular not (λ, λ) -regular ultrafilter.

In a previous version of the present paper we refined the above problem to:

Question 3.1. Suppose that $2^\kappa = \lambda$, and that κ is the first cardinal such that $2^\kappa = \lambda$. Is it true that every (λ, λ) -regular ultrafilter is (κ, κ) -regular?

Also 3.1 has a negative answer: start with a model of GCH in which μ is $\mu^{+\omega_1+1}$ -compact, and add $\mu^{+\omega_1}$ Cohen reals. Then in the resulting model $2^\omega = \mu^{+\omega_1}$ and there is a $(\mu, \mu^{+\omega_1+1})$ -regular ultrafilter which is not ν -decomposable, for every ν with $\omega < \nu < \mu$. If we put $\kappa = \omega_1$ and $\lambda = \mu^{+\omega_1+1}$, we have that κ is the first cardinal such that $2^\kappa = \lambda$, and there exists a (λ, λ) -regular not (κ, κ) -regular ultrafilter.

The above example also shows that 3.1 can be false even if we strengthen the hypothesis of (λ, λ) -regularity to λ -decomposability.

However, we do not know whether it is possible to get a counterexample to 3.1 starting with something less than a $\mu^{+\omega_1+1}$ -compact cardinal μ . Does a measurable suffice?

In spite of the above counterexamples, we have some positive results, and we can actually show that some amount of decomposability (and hence regularity) can be brought down from exponents.

Let $\beth_n(\lambda)$ denote the n^{th} iteration of the power set of λ ; that is, $\beth_0(\lambda) = \lambda$, and $\beth_{n+1}(\lambda) = 2^{\beth_n(\lambda)}$.

Theorem 3.2. (a) If D is $(2^{2^\lambda}, 2^{2^\lambda})$ -regular (or just $|\prod_D 2^{2^\lambda}| > 2^{2^\lambda}$) then D is κ -decomposable for some κ with $\lambda \leq \kappa \leq 2^\lambda$.

(d') More generally, if $n \geq 1$ and D is $(\beth_n(\lambda), \beth_n(\lambda))$ -regular (or just if $|\prod_D \beth_n(\lambda)| > \beth_n(\lambda)$) then D is κ -decomposable for some κ with $\lambda \leq \kappa \leq 2^\lambda$.

(b) Suppose that λ is a strong limit cardinal, and that D is $(\beth_n(\lambda), \beth_n(\lambda))$ -regular for some $n \geq 0$ (or just that $|\prod_D \beth_n(\lambda)| > \beth_n(\lambda)$). Then either D is $(\text{cf}\lambda, \text{cf}\lambda)$ -regular or there are arbitrarily large $\kappa < \lambda$ for which D is κ -decomposable. If in addition $U'(\lambda)$ holds (recall Definition 1.7) then D is (λ, λ) -regular.

In order to prove Theorem 3.2 we need the following proposition, a slight improvement on [AJ, Theorem 1], and which has independent interest.

Proposition 3.3. Suppose that $|\prod_D \mu| > \mu$, and let ν be the smallest cardinal such that $\mu^\nu > \mu$. Then either:

- (i) D is κ -decomposable for some κ with $\nu \leq \kappa \leq \mu$; or
- (ii) for every $\nu' < \nu$ there is κ such that $\nu' \leq \kappa < \nu$ and D is κ -decomposable; in addition, D is κ -decomposable for some κ with $\mu < \kappa \leq 2^\nu$; and moreover $|\prod_D \mu| = \sup_{\nu' < \nu} |\prod_D \nu'|$.

In particular, if ν is a successor cardinal, then D is κ -decomposable for some κ with $\nu^- \leq \kappa \leq \mu$ (ν^- denotes the predecessor of ν).

Proof of 3.3. Let D be over I . Recall from the introduction that if Π is a partition of I , Π has κ classes modulo D if and only if κ is the least cardinal for which there is $X \in D$ such that Π restricted to X has κ classes. If this is the case, then Π induces a κ -decomposition of D .

Any representative $f : I \rightarrow \mu$ of an element $f_D \in \prod_D \mu$ induces the partition $\Pi_f = \{(i, j) | f(i) = f(j)\}$, which has at most μ classes. If for some $f : I \rightarrow \mu$ Π_f has κ classes modulo D , and $\nu \leq \kappa \leq \mu$, then (i) holds, so that we can suppose that

(*) for every $f : I \rightarrow \mu$, Π_f has $< \nu$ classes modulo D (whence every $f_D \in \prod_D \mu$ has a representative f such that Π_f has $< \nu$ classes).

We now find an ordinal ρ and construct a chain of partitions Π_α ($\alpha \leq \rho$) of I according to the following rules:

- (a) Π_0 is the trivial partition;
- (b) if α is limit, Π_α is the common refinement of the Π_β 's, for $\beta < \alpha$;
- (c) if $\alpha = \beta + 1$, there are two cases:
 - (c₁) every element of $\prod_D \mu$ can be represented as f_D , for some f such that $f(i) = f(j)$ whenever i and j belong to the same Π_β class. In this case, take $\rho = \beta$, and the construction ends.

(c₂) Otherwise: take an element f_D which cannot be represented in that way, and choose by (*) a representative f in such a way that Π_f has $< \nu$ classes; then define Π_α to be the common refinement of Π_β and Π_f . Thus, Π_α properly refines Π_β .

Notice that

(**) if $\alpha < \nu$ then, by (c₂), Π_α (if defined) has at most $\nu^{|\alpha|} \leq \mu^{<\nu} = \mu$ classes; and that Π_ν (if defined) has at most 2^ν classes.

Π_ρ has at least ν classes modulo D , since if it has only $\nu' < \nu$ classes modulo D then by (c₁) $|\prod_D \mu| \leq \mu^{\nu'} = \mu$, a contradiction. Whence if $\rho < \nu$ then Π_ρ induces a κ -decomposition of D for some κ with $\nu \leq \kappa \leq \mu$, and we are in case (i).

Hence, we can suppose $\rho \geq \nu$.

We now show:

Claim. If $\nu' \leq \rho$ then $\Pi_{\nu'}$ has at least ν' classes modulo D .

Proof of the Claim. Fix $\nu' \leq \rho$, and suppose that $\Pi_{\nu'}$ has κ classes modulo D , witnessed by $X \in D$. For $\alpha \leq \nu'$, let Π_α^* be Π_α restricted

to X ; for every $\alpha < \nu'$ $\Pi_{\alpha+1}^*$ is a proper refinement of Π_α^* , by (c₂) and since $f_D = g_D$, if $f(i) = g(i)$ for every $i \in X$.

We shall define by induction a sequence $(C_\alpha)_{\alpha \leq \nu'}$ of subsets of X in such a way that, for $\alpha \leq \nu'$, $|C_\alpha| = |\alpha|$, and different elements of C_α belong to different Π_α^* classes. Let $C_0 = \emptyset$; $C_\alpha = \bigcup_{\beta < \alpha} C_\beta$ for α limit; and, if $\alpha = \beta + 1$, let $C_\alpha = C_\beta \cup \{p\}$, where $p \in X$ is such that no element of C_β is in the same Π_α^* class of p (such a p exists since Π_α^* properly refines Π_β^*).

Thus, $|C_{\nu'}| = |\nu'|$, and different elements of $C_{\nu'}$ belong to different $\Pi_{\nu'}^*$ classes; hence $\Pi_{\nu'}^*$ has at least ν' classes, that is $\nu' \leq \kappa$, so that the claim is proved. \square

Proof of 3.3 (continued). Now consider Π_ν : the claim shows that Π_ν has at least ν classes modulo D ; on the other side, Π_ν has at most 2^ν classes, so that D is κ -decomposable for some κ with $\nu \leq \kappa \leq 2^\nu$ (whence $\mu < \kappa \leq 2^\nu$ if we are not in case (i)).

Moreover, for every $\nu' < \nu$ the claim shows that $\Pi_{\nu'}$ has at least ν' classes modulo D ; by (**) $\Pi_{\nu'}$ has at most μ classes, so that $\Pi_{\nu'}$ induces a κ -decomposition of D for some κ with $\nu' \leq \kappa \leq \mu$ (whence $\nu' \leq \kappa < \nu$ if we are not in case (i)).

The above proof is essentially taken from [AJ]. The only difference is that [AJ] applies the arguments in the proof of the claim only to the case $\nu' = \nu$. Considering the general case $\nu' \leq \nu$ provides a strengthening without which we could not prove Theorem 3.2. Notice that statements in [AJ] talk about descending incompleteness; however, proofs actually deal with decomposability.

It remains to prove the last identity in (ii). This is easy: we mentioned that, if (i) fails, we can suppose that for every f_D Π_f has $< \nu$ classes modulo D . Then $\prod_D \mu = \bigcup \{\prod_D x \mid x \in S_\nu(\mu)\}$; but $|\prod_D \mu| > \mu$ and $|S_\nu(\mu)| = \mu^{<\nu} = \mu$, so that $|\prod_D \mu| = \sup_{\nu' < \nu} |\prod_D \nu'|$, since $|x| = \nu'$ implies $|\prod_D x| = |\prod_D \nu'|$. \square

The proof of Proposition 3.3 should be compared with the proof of [Si, Lemma 2].

Proof of 3.2. It is well known (at least for μ regular) that if D is (μ, μ) -regular then $|\prod_D \mu| > \mu$ (if μ is singular, use eventually different functions from $S_\mu(\mu)$ to μ : see [Lp1, Theorem 0.25]).

Moreover, standard arguments (e.g. [AJ, Lemma 4]) show that, for every μ ,

(*) $|\prod_D 2^\mu| > 2^\mu$ implies $|\prod_D \mu| > \mu$.

Whence, in case (a), $|\prod_D 2^{2^\lambda}| > 2^{2^\lambda}$, by hypothesis and the first remark, hence $|\prod_D 2^\lambda| > 2^\lambda$ by (*).

Arguing in a similar way and iterating (*) $n-1$ times we get $|\prod_D 2^\lambda| > 2^\lambda$ also from the hypothesis of (a').

Now take $\mu = 2^\lambda$ in Proposition 3.3: by standard cardinal arithmetic, the least ν such that $(2^\lambda)^\nu > 2^\lambda$ is $\geq \lambda^+$, so that the conclusion of (a) and (a') follows from Proposition 3.3.

As for (b), arguing as before, we get $|\prod_D \lambda| > \lambda$ in each case.

If λ is regular, then λ is the least ν such that $\lambda^\nu > \lambda$, since λ is supposed to be a strong limit cardinal. Hence the conclusion follows from Proposition 3.3 and 1.1(vii).

If λ is singular and D is not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, then $|\prod_D \lambda| > \lambda$ and an easy argument ([AJ, Lemma 2] or [Lp1, Lemma 2.1]) show that $|\prod_D \lambda'| > \lambda$, for some $\lambda' < \lambda$. If $\lambda' \leq \mu < \lambda$ then $|\prod_D 2^\mu| \geq |\prod_D \lambda'| > \lambda > 2^\mu$, since λ is a strong limit cardinal. By (a') applied with $n = 1$ and μ in place of λ , D is κ -decomposable for some κ with $\mu \leq \kappa \leq 2^\mu < \lambda$. By taking arbitrarily large μ 's $< \lambda$ we get arbitrarily large κ 's $< \lambda$ for which D is κ -decomposable.

As for the last statement, if D is $(\text{cf}\lambda, \text{cf}\lambda)$ -regular then D is (λ, λ) -regular by 1.1(v). If D is not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, we have just shown that there are arbitrarily large κ 's $< \lambda$ for which D is κ -decomposable, hence D is (λ, λ) -regular by applying the definition of $U'(\lambda)$ (Definition 1.7). \square

Remarks 3.4. (a) The hypothesis that λ is a strong limit cardinal in Theorem 3.2(b) is necessary. This is particularly evident in the case $n = 0$: if $2^\omega > \lambda$, $\text{cf}\lambda > \omega$, and D is uniform over ω , then, by Proposition 1.4, $|\prod_D \lambda| \geq |\prod_D \omega| = 2^\omega > \lambda$, but $\kappa = \omega$ is the only infinite κ for which D is κ -decomposable.

(b) The above counterexample involves the weaker assumption $|\prod_D \lambda| > \lambda$; indeed, if we assume (λ, λ) -regularity then case $n = 0$ of 3.2(b) is always true for every limit cardinal λ , because of Proposition 2.6 and 1.1(i)(viii).

(c) However, case $n = 1$ of 3.2(b) may fail even when D is assumed to be $(2^\lambda, 2^\lambda)$ -regular, if λ is not a strong limit cardinal. This goes exactly as in the example at the beginning of this section: start with a model of GCH in which μ is measurable and add μ Cohen reals. Take $\lambda < \mu$ singular with $\omega \neq \text{cf}\lambda$. In the resulting model $2^\omega = 2^\lambda = \mu$, and there is an ultrafilter D which is κ -decomposable exactly for $\kappa = \omega$ and $\kappa = \mu$; thus D is not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, by 1.1(viii), but D is $(2^\lambda, 2^\lambda)$ -regular (actually, 2^λ -decomposable). Hence the conclusion of 3.2(b) fails.

(d) We were led to the formulation of Theorem 3.2 by easy results of the following kind: *every $(2^\omega, 2^\omega)$ -regular ultrafilter is (ω, ω) -regular*. This fact can be obtained as a consequence of Theorem 3.2(b), taking

$\lambda = \omega$ and $n = 1$, and since no ultrafilter is m -decomposable, if $1 < m < \omega$. However, the following is a simpler proof: as we mentioned after the definition of measurability, the first cardinal μ for which an ultrafilter is (μ, μ) -regular is either ω or a measurable cardinal, and it is well-known that measurable cardinals are strongly inaccessible, thus $2^\omega < \text{the first measurable cardinal}$.

Problems 3.5. (a) Does (i) follow from the hypotheses of Proposition 3.3? Notice that this would improve the conclusion of Theorem 3.2(a)(a') to $\lambda < \kappa \leq 2^\lambda$, and would render the hypothesis $U'(\lambda)$ unnecessary in 3.2(b), in the case when λ is regular.

[AJ, p. 832] asked something slightly weaker, namely whether, under the hypotheses of 3.3, D is κ -descendingly incomplete for some κ with $\nu \leq \kappa \leq \mu$.

(a') Can we show, at least, that if (i) in 3.3 fails then D is 2^ν -decomposable?

(b) A more general problem: find pairs of cardinals $\mu \leq \mu'$ and $\kappa < \kappa'$ such that D is (κ, κ') -regular whenever $|\prod_D \mu| > \mu'$. Notice that, under particular assumptions on cardinal arithmetic, 1.1(xi), Proposition 3.3 and Theorems 2.1(b) and 2.15(i) actually furnish examples of such pairs; more examples can be found in combination with Theorems 2.18 3.2, 4.3.

We notice also the following corollary of Proposition 3.3 which deals with exponentiation with a larger base.

Corollary 3.6. *If $|\prod_D \mu^\lambda| > \mu^\lambda$ then D is κ -decomposable for some κ with $\lambda \leq \kappa \leq \mu^\lambda$.*

Proof. If ν is the first cardinal such that $(\mu^\lambda)^\nu > \mu^\lambda$, then $\nu \geq \lambda^+$. Then apply the last statement in Proposition 3.3. \square

4. DOWN FROM EXPONENTS (PART II)

Proposition 4.1. *For every ultrafilter D , and every cardinals κ, ν , $|\prod_D \nu^\kappa| \leq |\prod_D \nu^{<\kappa}|^{\text{cf}(\prod_D \langle \kappa, < \rangle)}$.*

Proof. Consider a model \mathbf{A} of the form $\langle A, U, <, V, W, F, G, H, J \rangle$, where U, V, W, J are unary predicates, $\langle U, < \rangle = \langle \kappa, < \rangle$, $|J| = \nu$ V is (in a one to one correspondence with) ${}^\kappa \nu$, the set of all functions from κ to ν , and W is (in a one to one correspondence with) $\bigcup_{\beta < \kappa} {}^\beta \nu$, the set of all functions from β to ν , for all $\beta < \kappa$.

F is a function from W to κ , and we require that if $W(w)$ then w is (corresponds to) a function from $F(w)$ to ν . G and H are functions

which represent the functions in V, W ; namely if $V(v)$ and v is (corresponds to) $f : \kappa \rightarrow \nu$ then $G(v, \alpha) = f(\alpha)$, for all $\alpha < \kappa$. Similarly, if $W(w)$ and w is (corresponds to) $g : F(w) \rightarrow \nu$ then $H(w, \alpha) = g(\alpha)$, for all $\alpha < F(w)$.

What matters is that (in \mathbf{A}) $|V| = \nu^\kappa$ and $|W| = \nu^{<\kappa}$; we shall get the desired inequality by taking the ultrapower of \mathbf{A} modulo D , and then computing the cardinality of the unary predicates V, W there.

Let \mathbf{B} be any model elementarily equivalent to \mathbf{A} . The following formula holds in \mathbf{A} , hence in \mathbf{B} :

$$\forall v \forall u (V(v) \wedge U(u) \Rightarrow \exists w (W(w) \wedge F(w) = u \wedge \forall x (x < u \Rightarrow G(v, x) = H(w, x))))),$$

in words, for every function f in V and for every element u in U there is a function g in W such that g is f restricted to the domain $[0, u) = \{x | U(x) \wedge x < u\}$. Thus, all “initial segments” of functions in V can be found in W ; in particular, working in \mathbf{B} , if λ is the cofinality of $\langle U, < \rangle$ and u_γ ($\gamma < \lambda$) is a cofinal sequence, then any f in V is determined by the functions $f|_{[0, u_\gamma)}$ ($\gamma < \lambda$).

For each $\gamma < \lambda$ there are at most $|W|$ (computed in \mathbf{B}) functions with domain $[0, u_\gamma)$, whence in \mathbf{B} $|V| \leq |W|^\lambda$ (notice that in \mathbf{A} different elements in V correspond to different functions, and this can be expressed by a first order sentence, using G).

Now let $\mathbf{B} = \prod_D \mathbf{A}$. By Łoś Theorem, \mathbf{B} is elementarily equivalent to \mathbf{A} , and the above argument gives the result, recalling that in \mathbf{A} $\langle U, < \rangle = \langle \kappa, < \rangle$. \square

Corollary 4.2. *Suppose that κ and λ are infinite cardinals, ν is a cardinal, $|\prod_D \nu^\kappa| > \nu^\lambda$, and $|\prod_D \nu^{<\kappa}| \leq \nu^\lambda$.*

Then $\nu^\lambda < |\prod_D \nu^\kappa| \leq |\prod_D \nu^{<\kappa}|^{\text{cf}(\prod_D \langle \kappa, < \rangle)} \leq \nu^{\lambda \cdot \text{cf}(\prod_D \langle \kappa, < \rangle)}$, hence $\text{cf}(\prod_D \langle \kappa, < \rangle) > \lambda$.

If in addition D is (κ^+, λ) -regular, then D is (κ, λ) -regular (by Theorem 2.13(i)).

The following theorem improves Theorem 3.2.

Theorem 4.3. (a) *If $|\prod_D \nu^{\kappa^{+n}}| > \nu^{\kappa^{+n}}$ (in particular, if D is $(\nu^{\kappa^{+n}}, \nu^{\kappa^{+n}})$ -regular) then D is μ -decomposable for some μ with $\kappa \leq \mu \leq \nu^\kappa$. If in addition $\nu^\kappa = \kappa^{+p}$ for some $p < \omega$ then D is (κ, κ) -regular.*

(a') *If $m \geq 1$ and $|\prod_D \beth_m(\kappa^{+n})| > \beth_m(\kappa^{+n})$ (in particular, if D is $(\beth_m(\kappa^{+n}), \beth_m(\kappa^{+n}))$ -regular) then D is μ -decomposable for some μ with $\kappa \leq \mu \leq 2^\kappa$. If in addition $2^\kappa = \kappa^{+p}$ for some $p < \omega$ then D is (κ, κ) -regular.*

(b) Suppose that κ is a strong limit cardinal and that $|\prod_D \beth_m(\kappa^{+n})| > \beth_m(\kappa^{+n})$ (in particular, this holds when D is $(\beth_m(\kappa^{+n}), \beth_m(\kappa^{+n}))$ -regular). Then either D is $(\text{cf}\kappa, \text{cf}\kappa)$ -regular or there are arbitrarily large $\mu < \kappa$ for which D is μ -decomposable.

Proof. (a) Let q be the smallest integer such that $|\prod_D \nu^{\kappa^{+q}}| > \nu^{\kappa^{+q}}$.

If $q = 0$ the conclusion follows from Corollary 3.6.

Otherwise, $|\prod_D \nu^{\kappa^{+q-1}}| = \nu^{\kappa^{+q-1}}$ and $\nu^{\kappa^{+q}} < |\prod_D \nu^{\kappa^{+q}}|$, hence $\text{cf}(\prod_D \langle \kappa^{+q}, < \rangle) > \kappa^{+q}$ by Corollary 4.2, by taking $\kappa = \lambda$ there to be κ^{+q} .

Hence D is $(\kappa^{+q}, \kappa^{+q})$ -regular by Remark 1.5(a), and (κ^+, κ^+) -regular by an iteration of Theorem 2.1(b), since $q > 0$. Hence D is κ^+ -decomposable by 1.1(viii), and the conclusion holds with $\mu = \kappa^+$.

If $\nu^\kappa = \kappa^{+p}$ for some $p < \omega$, then the μ given by the preceding statement satisfies $\kappa \leq \mu \leq \nu^\kappa = \kappa^{+p}$, hence $\mu = \kappa^{+p'}$ for some $p' \leq p$. Since D is μ -decomposable, D is $(\kappa^{+p'}, \kappa^{+p'})$ -regular by 1.1(vii). If $p' = 0$ this is what we want; otherwise it is enough to use Theorem 2.1(b) a sufficient number of times.

A remark: the reader might observe that in the course of the proof we have obtained (κ^+, κ^+) -regularity, which implies (κ, κ) -regularity, so that the hypothesis $\nu^\kappa = \kappa^{+p}$ might appear to be unnecessary. However, we get (κ^+, κ^+) -regularity only in the case $q > 0$, while we do not necessarily have it in the case $q = 0$, which uses Corollary 3.6. In fact, the hypothesis $\nu^\kappa = \kappa^{+p}$ is necessary: as in the example at the beginning of Section 3, take μ measurable, and add μ Cohen reals. Then $2^\lambda = \mu$, for every $\lambda < \mu$: take λ regular with $\mu > \lambda > \omega$. As in the beginning of Section 3, we have an ultrafilter D μ -decomposable but not λ -decomposable; D is not (λ, λ) -regular, and is (μ, μ) -regular (by 1.1(vii)), hence $|\prod_D \mu| > \mu$ (by the first lines in the proof of 3.2). $\mu = 2^{\lambda^{+n}}$, so that $|\prod_D 2^{\lambda^{+n}}| > 2^{\lambda^{+n}}$, but D is not (λ, λ) -regular, hence the conclusion in the third statement of 4.3(a) fails.

(a') From the hypothesis we get $|\prod_D 2^{\kappa^{+n}}| > 2^{\kappa^{+n}}$ by iterating the fact (already mentioned in the proof of 3.2) that $|\prod_D 2^{2^\lambda}| > 2^{2^\lambda}$ implies $|\prod_D 2^\lambda| > 2^\lambda$. The conclusion then follows from (a), with $\nu = 2$ and $n = 0$.

(b) follows from Theorem 3.2(b) (case $n = 0$) and iterating the fact that both $|\prod_D \lambda^+| > \lambda^+$ and $|\prod_D 2^\lambda| > 2^\lambda$ imply $|\prod_D \lambda| > \lambda$. \square

Theorems 3.2 and 4.3 can be improved in many ways. For example, Theorem 4.3 holds when in place of $\beth_m(\kappa^{+n})$ we consider any iteration (in any order) of any finite number of the \beth and of the successor functions (with at least one occurrence of \beth in (a')). This is because both $|\prod_D \lambda^+| > \lambda^+$ and $|\prod_D 2^\lambda| > 2^\lambda$ imply $|\prod_D \lambda| > \lambda$.

More generally, one can use the formula $|\prod_D \lambda^\mu| \leq |\prod_D \lambda|^{|\prod_D \mu|}$, so that, for example, $(\lambda^\mu, \lambda^\mu)$ -regularity implies $|\prod_D \lambda^\mu| > \lambda^\mu$, hence either $|\prod_D \lambda| > \lambda$ or $|\prod_D \mu| > \mu$: then we can apply the statements of 3.2 or 4.3 (or the methods of proof). Other results can be obtained from Propositions 3.3, 4.1 and theorems about cardinalities of ultrapowers (e.g. [Kei], or [Lp1, Theorem 0.25]). One can also mix in the results of Section 2. We leave details to the reader, since statements become quite involved, and since we do not know whether Proposition 3.3 is the best possible result.

Notice also that if Problem 3.5(a) has an affirmative answer, then we can improve the conclusion in Theorem 4.3(a') to “for some μ with $\kappa < \mu \leq 2^\kappa$ ”.

Problems 4.4. (a) [Lp7] Suppose that $2^\kappa < \mu < 2^{\kappa^+}$ and D is μ -decomposable. Is D necessarily λ -decomposable for some λ with $\kappa \leq \lambda \leq 2^\kappa$? (maybe the assumption μ regular is necessary)

(b) [Lp6] Is it consistent to have an ω_1 -complete ultrafilter uniform on some μ with $\mu^\omega > \mu$?

Theorems 3.2 and 4.3 have the following consequences for topological spaces and for extensions of first-order logic (see [Ca, Lp2]; [Ma, Lp1]).

Recall that a topological space is $[\kappa, \mu]$ -compact if and only if every open cover by μ many sets has a subcover by $< \kappa$ many sets. A family \mathcal{F} of topological spaces is *productively* $[\kappa, \mu]$ -compact if and only if every product of members of \mathcal{F} is $[\kappa, \mu]$ -compact.

In [Lp2, Theorem 3] we have shown, improving results from [Ca]:

Theorem 4.5. *Let λ, μ be infinite cardinals, and $(\kappa_i)_{i \in I}$ be a family of infinite cardinals. Then the following are equivalent:*

(i) *Every productively $[\lambda, \mu]$ -compact topological space is $[\kappa_i, \kappa_i]$ -compact for some $i \in I$.*

(ii) *Every productively $[\lambda, \mu]$ -compact family of topological spaces is productively $[\kappa_i, \kappa_i]$ -compact for some $i \in I$.*

(iii) *Every (λ, μ) -regular ultrafilter is (κ_i, κ_i) -regular for some $i \in I$.*

Corollary 4.6. *Any productively $[2^{2^\kappa}, 2^{2^\kappa}]$ -compact family of topological spaces is productively $[\mu, \mu]$ -compact for some μ with $\kappa \leq \mu \leq 2^\kappa$.*

More generally, if $m \geq 1$ then any productively $[\beth_m(\kappa^{+n}), \beth_m(\kappa^{+n})]$ -compact family of topological spaces is productively $[\mu, \mu]$ -compact for some μ with $\kappa \leq \mu \leq 2^\kappa$.

If κ is a strong limit cardinal, and $U'(\kappa)$ holds, then any productively $[2^\kappa, 2^\kappa]$ -compact (even, every productively $[\beth_m(\kappa^{+n}), \beth_m(\kappa^{+n})]$ -compact) family of topological spaces is productively $[\kappa, \kappa]$ -compact.

Proof. By 1.1(vii), Theorems 3.2, 4.3 and 4.5(iii) \Rightarrow (ii) (actually, this implication in Theorem 4.5 is a corollary of results in [Ca]). \square

In what follows, by a *logic*, we mean a *regular logic* in the sense of [Eb]. Typical examples of regular logics are extensions of first-order logic obtained by adding new quantifiers (e.g., cardinality quantifiers, asserting “there are at least ω_α x ’s such that ...”), or by allowing infinitary conjunctions and disjunctions, and possibly simultaneous quantification over infinitely many variables (infinitary logics).

Roughly, a logic is regular if and only if it shares all the good properties of the above typical examples. The reader is invited to look at [BF] for more information about logics.

A logic L is $[\lambda, \mu]$ -compact if and only if for every pair of sets Γ and Σ of sentences of L , with $|\Sigma| \leq \lambda$, the following holds: if $\Gamma \cup \Sigma'$ has a model for every $\Sigma' \subseteq \Sigma$ with $|\Sigma'| < \mu$, then $\Gamma \cup \Sigma$ has a model.

There is an older (and weaker) notion, called (λ, μ) -compactness, which corresponds to the above definition in the particular case when $\Gamma = \emptyset$. In a series of papers, J. Makowsky and S. Shelah showed that the new stronger $[\lambda, \mu]$ -compactness is much better behaved (see [Ma] for a survey). In particular, Makowsky and Shelah defined what it means for an ultrafilter to be *related* to a logic, and showed that a logic L is $[\lambda, \mu]$ -compact if and only if there is a (μ, λ) -regular ultrafilter related to L . An immediate consequence is:

Corollary 4.7. *Let λ, μ be cardinals, and $(\kappa_i)_{i \in I}$ be a family of cardinals. If*

- (i) for every (μ, λ) -regular ultrafilter D there is $i \in I$ such that D is (κ_i, κ_i) -regular,*
- then*
- (ii) for every $[\lambda, \mu]$ -compact logic L there is $i \in I$ such that L is $[\kappa_i, \kappa_i]$ -compact.*

The analysis of connections between regularity of ultrafilters and compactness of logics can be further elaborated: see [Lp1, Section 0], and some references there. Indeed, we recently proved that conditions (i) and (ii) in Corollary 4.7 are actually equivalent. The case when all the κ_i ’s are assumed to be regular cardinals is easier and had been proved in [Lp1, Theorem 4.1] (in an equivalent form, by Proposition 7.6). The general case when the κ_i ’s are allowed to be singular cardinals is proved in [Lp11, Theorem 10], is connected with Problem 2.8, and has led to the formulation of the principle $U'(\lambda)$ of Definition 1.7. See [Lp1, Section 4]; see also the end of Section 6 in the present paper.

Anyway, from Corollary 4.7, 1.1(vii) and Theorems 3.2 and 4.3, we get:

Corollary 4.8. *Every $[2^{2^\kappa}, 2^{2^\kappa}]$ -compact logic is $[\mu, \mu]$ -compact for some μ with $\kappa \leq \mu \leq 2^\kappa$.*

More generally, if $m \geq 1$ then every $[\beth_m(\kappa^{+n}), \beth_m(\kappa^{+n})]$ -compact logic is $[\mu, \mu]$ -compact for some μ with $\kappa \leq \mu \leq 2^\kappa$.

If κ is a strong limit cardinal, and $U'(\kappa)$ holds, then every $[2^\kappa, 2^\kappa]$ -compact logic (even, every $[\beth_m(\kappa^{+n}), \beth_m(\kappa^{+n})]$ -compact logic) is $[\kappa, \kappa]$ -compact.

Of course, we could use Theorem 4.3(b) in order to improve the last part of Corollary 4.6 to: *if κ is a strong limit cardinal then any productively $[\beth_m(\kappa^{+n}), \beth_m(\kappa^{+n})]$ -compact family \mathcal{F} of topological spaces either is productively $[\text{cf}\kappa, \text{cf}\kappa]$ -compact, or there are arbitrarily large $\mu < \kappa$ for which \mathcal{F} is productively $[\mu, \mu]$ -compact.*

A similar remark holds for logics and Corollary 4.8.

Remark 4.9. The remarks at the beginning of Section 3, together with [Lp3, Theorem 2.5(iv) \Leftrightarrow (vi)], show that it is possible to have a $(2^\lambda, 2^\lambda)$ -compact not (λ, λ) -compact logic, thus solving negatively [Lp4, Problem III]. Actually, what we get is a $[2^{\omega_1}, 2^{\omega_1}]$ -compact not (ω_1, ω_1) -compact logic.

5. ABOVE LIMITS.

If λ is a limit cardinal and an ultrafilter enjoys some form of regularity for arbitrarily large cardinals below λ then in some cases regularity can be lifted up to λ . In this section we provide some examples and counterexamples. Again, many problems seem still open.

For $\lambda \geq \lambda' \geq \mu$, let $\text{COV}(\lambda, \lambda', \mu)$ denote the minimal cardinality of a family of subsets of λ , each of cardinality $< \lambda'$, such that every subset of λ of cardinality $< \mu$ is contained in at least one set of the family. In particular, $\text{COV}(\lambda, \mu, \mu)$ is the cofinality of (the partial order) $S_\mu(\lambda)$. Notice that $\text{COV}(\lambda, \lambda', \mu) \leq |S_\mu(\lambda)| = \lambda^{<\mu}$.

Theorem 5.1. *Suppose that λ is a limit cardinal, and that there are arbitrarily large $\nu < \lambda$ for which D is ν -decomposable. Then:*

(a) *D is κ -decomposable for some κ with $\lambda \leq \kappa \leq \lambda^{\text{cf}\lambda}$.*

(b) *In addition, suppose that λ is singular and D is $(\mu, \text{cf}\lambda)$ -regular. Then D is κ -decomposable for some κ with $\lambda \leq \kappa \leq \lambda^{<\mu}$. More generally, for every λ' with $\mu \leq \lambda' < \lambda$ there exists κ such that D is κ -decomposable and $\lambda \leq \kappa \leq \text{COV}(\lambda, \lambda', \mu)$.*

Proof. Let $(\nu_\alpha)_{\alpha \in \text{cf } \lambda}$ be a sequence cofinal in λ , such that D is ν_α -decomposable, for $\alpha \in \text{cf } \lambda$; and for $\alpha \in \text{cf } \lambda$ let Π_α be a ν_α -decomposition.

In order to prove (a) it is enough to consider the common refinement of all the Π_α 's (compare with the proofs of [Pr, Proposition 2] and [AJ, Theorem 2]).

The last statement in (b) implies the statement that precedes it, since, trivially, $\text{COV}(\lambda, \lambda', \mu) \leq \lambda^{<\mu}$. Anyway, we shall give a proof for the first statement, too, because it is rather simpler. First, notice that if $\mu > \text{cf } \lambda$ then (b) follow from (a), hence without loss of generality we can suppose $\mu \leq \text{cf } \lambda$.

Let the Π_α 's be as above, and for $X \subseteq \text{cf } \lambda$ with $|X| < \mu$ let Π_X be the common refinement of $\{\Pi_\alpha \mid \alpha \in X\}$. Π_X has $\leq \lambda^{<\mu}$ classes. Let D be over I , and let $f : I \rightarrow S_\mu(\text{cf } \lambda)$ witness the $(\mu, \text{cf } \lambda)$ -regularity of D , as given by Form II. Define Π on I by $i \sim j$ if and only if $f(i) = f(j)$ and $(i, j) \in \Pi_{f(i)}$. Π has $\leq \lambda^{<\mu} \cdot (\text{cf } \lambda)^{<\mu} = \lambda^{<\mu}$ classes.

On the other side, for every α , Π has more than ν_α classes (modulo D): for every $\alpha \in \text{cf } \lambda$, $I_\alpha = \{i \in I \mid \alpha \in f(i)\} \in D$, and $\Pi|_{I_\alpha}$ refines $\Pi_\alpha|_{I_\alpha}$ (which has ν_α classes modulo D). Thus, Π has κ classes modulo D for some κ with $\lambda \leq \kappa \leq \lambda^{<\kappa}$, and this proves the κ -decomposability of D .

In order to prove the last statement in (b), let D be over I , let $f : I \rightarrow S_\mu(\text{cf } \lambda)$ witness the $(\mu, \text{cf } \lambda)$ -regularity of D (Form II), and for $\alpha \in \text{cf } \lambda$ let $f_\alpha : I \rightarrow \nu_\alpha$ be a ν_α -decomposition.

Let H be a family of subsets of λ as given by $\text{COV}(\lambda, \lambda', \mu)$: thus, for every $i \in I$ there is $h(i) \in H$ such that $\{f_\alpha(i) \mid \alpha \in f(i)\} \subseteq h(i)$, and $|h(i)| < \lambda'$. Choose f , H , the f_α 's and the $h(i)$'s in such a way that the cardinality of $K = \{h(i) \mid i \in I\}$ is minimal; this choice makes the function $h : I \rightarrow K$ a $|K|$ -decomposition of D : if $h^{-1}(K') \in D$ for some $K' \subseteq K$ with $|K'| < |K|$ then we could change the values of $f(i)$, $f_\alpha(i)$ and of $h(i)$ for $i \notin h^{-1}(K')$, thus making K' work in place of K , contradicting the minimal cardinality of K (the argument is identical with the one in the proof of [Lp1, Lemma 4.7]).

Since $|K| \leq |H| = \text{COV}(\lambda, \lambda', \mu)$, it remains to show that $|K| \geq \lambda$. By $(\mu, \text{cf } \lambda)$ -regularity, for every $\alpha \in \text{cf } \lambda$, $I_\alpha = \{i \in I \mid \alpha \in f(i)\} \in D$, whence $\nu_\alpha = |f_\alpha(I_\alpha)|$, since f_α is a ν_α -decomposition. Since $h(i)$ has been chosen in such a way that $f_\alpha(i) \in h(i)$ when $\alpha \in f(i)$, that is, when $i \in I_\alpha$, we have that $f_\alpha(I_\alpha) = \{f_\alpha(i) \mid i \in I_\alpha\} \subseteq \bigcup_{i \in I_\alpha} h(i)$, hence $|f_\alpha(I_\alpha)| \leq \lambda' \cdot |h(I_\alpha)|$, since $|h(i)| < \lambda'$, for all i .

Putting the above inequalities together, we get: $\nu_\alpha = |f_\alpha(I_\alpha)| \leq \lambda' \cdot |h(I_\alpha)| \leq \lambda' \cdot |h(I)|$, for all $\alpha \in \text{cf } \lambda$. Hence $\nu_\alpha \leq |h(I)| = |K|$, for

all α 's such that $\nu_\alpha > \lambda'$. Since $\lambda' < \lambda$, and the ν_α 's are cofinal in λ , we get $|K| \geq \sup_{\alpha \in \text{cf}\lambda} \nu_\alpha = \lambda$. \square

Notice that Theorem 5.1(b) is a common generalization of [DJK, Lemma 4.9], [Pr, Proposition 1] and [Lp1, Lemma 4.7, Remark 4.8].

We do not know whether Theorem 5.1 can be improved; actually, we do not even know whether:

Problem 5.2. Is the following true? If λ is a limit cardinal, and there are arbitrarily large $\nu < \lambda$ for which D is ν -decomposable, then D is either λ -decomposable or λ^+ -decomposable.

We believe that Problem 5.2 has a negative answer, in general (and we believe that the case when λ is regular is much easier to falsify). Be that as it may, 5.2 is the best we can expect: if κ is $\kappa^{+\omega+1}$ -compact there is a $(\kappa, \kappa^{+\omega+1})$ -regular ultrafilter which is κ -complete, and hence not (ω, ω) -regular and not $\kappa^{+\omega}$ -decomposable, by 1.1(vii), but κ^{+n} -decomposable for all $n < \omega$, and $\kappa^{+\omega+1}$ -decomposable, by 1.1(xii). On the other hand, for every λ there is a (ω, λ) -regular ultrafilter uniform over λ (e. g. [CK]), hence not λ^+ -decomposable by 1.1(iii), but κ -decomposable for all $\kappa \leq \lambda$ by Remark 1.5(b).

Of course, an affirmative solution of 5.2 would show that $U'(\lambda)$ (see Definition 1.7) holds for every limit cardinal λ . Indeed, 5.2 and $U'(\lambda)$ have the same hypothesis, and the conclusion in $U'(\lambda)$ follows from the conclusion in 5.2, since, by 1.1(vii), every λ -decomposable ultrafilter is (λ, λ) -regular, and, by 1.1(vii) and Theorem 2.1(b), every λ^+ -decomposable ultrafilter is (λ, λ) -regular.

Anyway, Theorem 5.1 shows that Problem 5.2 has an affirmative answer for many singular cardinals.

Corollary 5.3. *Suppose that λ is singular, and either*

- (i) $\text{COV}(\lambda, \lambda, (\text{cf}\lambda)^+) = \lambda^{+n}$, for some $n < \omega$; or*
- (i') $\lambda^{\text{cf}\lambda} = \lambda^{+n}$, for some $n < \omega$; or*
- (ii) $\text{cf}\lambda = \omega$ and there is no measurable cardinal; or*
- (iii) $\text{cf}\lambda = \omega$ and $\text{COV}(\lambda, \lambda, \omega_1) < \lambda^{+\mu_0}$, where μ_0 is the smallest measurable cardinal; or*
- (iv) $\lambda = \omega_{\alpha+\delta}$, $\text{cf}\delta = \omega$, $\delta < \omega_\alpha$, and $\delta < \mu_0$, where μ_0 is the smallest measurable cardinal.*

If D is an ultrafilter, and there are arbitrarily large $\nu < \lambda$ for which D is ν -decomposable, then D is either λ -decomposable or λ^+ -decomposable.

Proof. (i) It is not difficult to show that there is a $\lambda' < \lambda$ such that $\text{COV}(\lambda, \lambda, (\text{cf}\lambda)^+) = \text{COV}(\lambda, \lambda', (\text{cf}\lambda)^+)$: see [Sh, Analytical Guide,

6.2, or II, Observation 5.3(10)]; the notation for $\text{COV}(\lambda, \mu, \nu)$ is $\text{cov}(\lambda, \mu, \nu, 2)$ in [Sh]: see [Sh, II, Definition 5.1].

By Theorem 5.1(b) and 1.1(xiii), D is κ -decomposable for some κ with $\lambda \leq \kappa \leq \text{COV}(\lambda, \lambda', (\text{cf } \lambda)^+) = \text{COV}(\lambda, \lambda, (\text{cf } \lambda)^+) = \lambda^{+n}$. If D is λ -decomposable we are OK, otherwise, D is λ^{+m} -decomposable for some $m > 0$. Since successors are regular cardinals, then 1.1(xi) and an iteration of Theorem 2.1(b) imply that D is λ^+ -decomposable. Essentially, (i) had been already noticed in [Lp1, Remark 4.8] (under the assumption $\text{COV}(\lambda, \lambda', (\text{cf } \lambda)^+) = \lambda^{+n}$).

(i') follows from (i), since $\text{COV}(\lambda, \lambda, (\text{cf } \lambda)^+) \leq \lambda^{\text{cf } \lambda}$.

In order to prove (ii) and (iii), let $\text{cf } \lambda = \omega$. If D is (ω, ω) -regular, then D is λ -decomposable because of Theorem 5.1(b), since $\lambda^{<\omega} = \lambda$.

Otherwise, D is ω_1 -complete (so (ii) cannot hold), hence μ_0 -complete. Let κ be the smallest cardinal $\geq \lambda$ such that D is κ -decomposable. By Theorem 5.1, at least one such κ exists, and, by the last statement in 5.1(b) and 1.1(xiii), $\kappa \leq \text{COV}(\lambda, \lambda', \omega_1) < \lambda^{+\mu_0}$ for some $\lambda' < \lambda$ (as at the beginning of the proof of (i), we can suppose that $\text{COV}(\lambda, \lambda, \omega_1) = \text{COV}(\lambda, \lambda', \omega_1)$, for some $\lambda' < \lambda$).

By Theorem 2.1(b), 1.1(xi) and the minimality of κ , κ cannot be the successor of a regular cardinal.

If $\kappa = \lambda$ the result is proved. If $\kappa > \lambda$ then κ cannot be limit: all limit cardinals between λ and $\lambda^{+\mu_0}$ are singular cardinals of cofinality $< \mu_0$; if κ is singular then D is not $(\text{cf } \kappa, \text{cf } \kappa)$ -regular, since $\text{cf } \kappa < \mu_0$ and D is μ_0 -complete. By 1.1(vii) this implies that D is not κ -decomposable.

The remaining case is when κ is a successor of a singular cardinal, say $\kappa = \nu^+$. As above, D is not $(\text{cf } \nu, \text{cf } \nu)$ -regular, and, by Theorem 2.1 and 1.1(vii), D is (κ', κ) -regular for some $\kappa' < \kappa$; by 1.1(xii), this contradicts the minimality of κ , unless $\kappa = \lambda^+$.

(iv) Since $\lambda = \omega_{\alpha+\delta}$ and $\delta < \omega_\alpha$, then, by [Sh, IX, Claim 3.7(1) and Theorem 2.2], $\text{COV}(\lambda, \lambda, (\text{cf } \lambda)^+) = \text{pp}(\lambda) \leq \omega_{\alpha+|\delta|+4}$.

Since $\text{cf } \lambda = \omega$, we get from above $\text{COV}(\lambda, \lambda, \omega_1) \leq \omega_{\alpha+|\delta|+4} = \lambda^{+|\delta|+4}$, since $\lambda = \omega_{\alpha+\delta}$ and $\delta + |\delta|^{+4} = |\delta|^{+4}$. Since $\delta < \mu_0$, and μ_0 is a measurable cardinal, hence inaccessible, $|\delta|^{+4} < \mu_0$ and the hypotheses of (iii) apply. \square

In (iv) above we have used Shelah's tight bounds on $\text{COV}(\lambda, \lambda, (\text{cf } \lambda)^+)$ in order to show that Problem 5.2 has an affirmative answer for a large class of singular cardinals of countable cofinality. Can the argument be extended in order to cover *all* singular cardinals of countable cofinality? Can the argument be adapted in some way to singular cardinals of uncountable cofinality?

Having dealt with decomposability, we now turn to the problem of lifting up regularity.

Proposition 5.4. *Suppose that μ is singular, $\lambda < \mu$ and D is (λ, μ') -regular for all $\mu' < \mu$ (equivalently, by 1.1(i), for a sequence of μ' 's cofinal in μ). Then D is (λ, μ) -regular, provided at least one of the following holds:*

- (a) λ is regular; or
- (b) D is $(\text{cf}\lambda, \text{cf}\mu)$ -regular (in particular, this holds when $\text{cf}\lambda > \text{cf}\mu$, by 1.1(xiii)); or
- (c) the cofinality of $\prod_D \langle \text{cf}\lambda, < \rangle$ is $\neq \text{cf}\mu$; or
- (d) $\text{cf}\lambda \neq \text{cf}\mu$ and D is not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular.

Proof. The easy proof of (a) is given in [Do, Lemma 1.1], slightly generalizing [BK, Proposition 1.1].

We now prove (c): by 1.1(i) D is (λ^+, μ') -regular for all $\mu' < \mu$, hence we can apply (a), with λ^+ in place of λ , obtaining (λ^+, μ) -regularity. Now, (λ, μ) -regularity follows from Theorem 2.13(i), since the cofinality of $\prod_D \langle \text{cf}\lambda, < \rangle$ equals the cofinality of $\prod_D \langle \lambda, < \rangle$.

(b) is a consequence of (c) because of Proposition 1.3. Actually, the arguments in the proof of [Do, Lemma 1.1] give a proof of (b).

Notice that (b) is trivial when $\text{cf}\lambda > \text{cf}\mu$: let D be over I , let μ_α ($\alpha \in \text{cf}\mu$) be cofinal in μ , and for $\alpha \in \text{cf}\mu$ let $f_\alpha : I \rightarrow S_\lambda(\mu_\alpha)$ witness the (λ, μ_α) -regularity (Form II) of D . Then f defined by $f(i) = \bigcup_{\alpha \in \text{cf}\mu} f_\alpha(i)$ witnesses the (λ, μ) -regularity of D .

As for (d), we have, as in the proof of (c), that D is (λ^+, μ) -regular, and the conclusion follows from Theorem 2.2. Alternatively, use (c) and 1.5(a) with $\text{cf}\lambda$ in place of λ . \square

We do not know whether Proposition 5.4 can be proved without any of the assumptions (a)-(d).

Of course, a positive answer to Conjecture 2.16 would make the assumptions unnecessary. This is proved as follows: because of 5.4(a), the only case to be discussed is λ singular, and, as in the proof of 5.4(c), we get (λ^+, μ) -regularity from the hypotheses of 5.4; then Conjecture 2.16, if true, would imply (λ, μ) -regularity.

See also the remark after Proposition 8.2.

Remark 5.5. On the contrary, Proposition 5.4 does not generalize to the case when μ is a regular limit cardinal. As pointed out in [Ket1, p. 67], if κ is huge then there exist an inaccessible cardinal $\mu > \kappa$ and a κ -complete ultrafilter which is (κ^+, μ) -regular, not (κ, μ) -regular, but (κ, μ') -regular for all $\mu' < \mu$ (for the last statement use Theorem 2.13(i), or [BK, Theorem 1.3]). The example shows that, even for fairly large

α , it is possible to have a $(\kappa, \lambda^{+\alpha})$ -regular and (λ, μ) -regular ultrafilter which is not (κ, μ) -regular.

As in [Ket1, Theorem 1.10], we can collapse κ to a *regular* cardinal, still having a (κ^+, μ) -regular not (κ, μ) -regular ultrafilter.

Problem 5.6. We do not know whether, starting from the above example, it is possible to make κ *singular* by forcing, still having a (κ, μ') -regular (for all $\mu' < \mu$), (κ^+, μ) -regular not (κ, μ) -regular ultrafilter in the extension. Of course, this would falsify Conjecture 2.16.

Meanwhile, we have proved that Problem 5.2 has an affirmative answer in the particular case when λ is a singular cardinal, and provided that D is (ν, ν) -regular for *all* sufficiently large regular cardinals $\nu < \lambda$. The proof of the following two results can be found in [Lp11], along with some generalizations and connections with Shelah's pcf theory.

Theorem 5.7. *Suppose that λ is a singular cardinal, $\lambda' < \lambda$, and the ultrafilter D is ν -decomposable for all regular cardinals ν with $\lambda' < \nu < \lambda$. Then D is either λ -decomposable or λ^+ -decomposable (hence (λ, λ) -regular, by 1.1(vii)(xi) and Theorem 2.1(b)).*

Corollary 5.8. *If λ is an infinite cardinal, then an ultrafilter is (λ, λ) -regular if and only if it is either cf λ -decomposable or λ^+ -decomposable.*

Corollary 5.8 is stated in [Lp11] under the assumption that λ is a singular cardinal. The case when λ is regular (that is $\text{cf } \lambda = \lambda$) is immediate from 1.1(xi) and Theorem 2.1(b).

6. LEAST FUNCTIONS AND APPLICATIONS.

The notion of a least function arises directly from the theory of ω_1 -complete ultrafilters, hence from large cardinals. However, there are consequences holding in ZFC; in particular, we shall find partial answers to many conjectures, by a slight extension of important and well-known results of A. Kanamori and J. Ketonen. Needless to say, our arguments lead to new open problems (we expect that, at this point, the reader has become accustomed to this phenomenon).

The next definition gives us the possibility of extending some results from [Ka1], and to provide partial answers to Conjectures 2.16 and 2.12.

Definition 6.1. If D is an ultrafilter over some set I , and κ is a cardinal, we say that a function $f : I \rightarrow \kappa$ is a κ -least function for D if and only if $\{i \in I \mid \alpha < f(i)\} \in D$, for every $\alpha \in \kappa$, yet for every $g : I \rightarrow \kappa$, $\{i \in I \mid g(i) < f(i)\} \in D$ implies that there is $\alpha \in \kappa$ such that $\{i \in I \mid g(i) < \alpha\} \in D$. In other words, in $\prod_D \langle \kappa, < \rangle$, f_D is the

least element larger than all the $d(\alpha)$'s ($\alpha \in \kappa$) (as usual, f_D denotes the class of f modulo D).

When D is over κ we get the usual notion of a *least function*. We need the above definition since we will be dealing with arbitrary (λ, κ) -regular ultrafilters, not necessarily over κ (however, see Proposition 6.7(ii)).

Notice that if D has a κ -least function then D is κ -descendingly incomplete, hence if κ is regular then D is (κ, κ) -regular by 1.1(xi). The case when κ is singular shall be discussed in Remark 8.5(a).

The following theorem generalizes [Ka1, Corollary 2.6] (see also [Ket2, Section 1]), and is proved in a very similar way.

Theorem 6.2. *If κ is regular, and the ultrafilter D is (κ, κ) -regular and has no κ -least function, then D is (ω, κ') -regular for every $\kappa' < \kappa$.*

Proof. The proof goes exactly as in the proofs of [Ka1, Theorem 2.5 and Corollary 2.6]; let D be over I : just replace everywhere “ $f : \kappa \rightarrow \kappa$ ” with “ $f : I \rightarrow \kappa$ ”, and “ $\xi < \kappa$ ” with “ $i \in I$ ”. \square

The following is proved in [Ket1, Theorem 1.4] under the assumption of the existence of a least function, but the proof uses only a κ -least function.

Theorem 6.3. *Suppose that κ is regular, and that the ultrafilter D over I has a κ -least function f . Then D is (λ, κ) -regular if and only if $\{i \in I \mid \text{cf } f(i) < \lambda\} \in D$.*

From Theorems 6.2 and 6.3 we get the following partial answer to Conjecture 2.16.

Corollary 6.4. *If $\kappa > \lambda$, κ is regular, λ is singular and the ultrafilter D is (λ^+, κ) -regular, then D is either (λ, κ) -regular, or (ω, κ') -regular for every $\kappa' < \kappa$.*

Proof. Since $\kappa > \lambda$, D is (κ, κ) -regular, by 1.1(i).

If D has no κ -least function we are done by Theorem 6.2.

Otherwise, let D be over I , and let f be a κ -least function. By Theorem 6.3, $\{i \in I \mid \text{cf } f(i) < \lambda^+\} \in D$, hence $\{i \in I \mid \text{cf } f(i) < \lambda\} \in D$, since cofinalities are regular cardinals. Now Theorem 6.3 implies that D is (λ, κ) -regular. \square

See Theorem 2.15(ii) for a similar result in the case when κ is singular.

Corollary 6.4 puts some restrictions on possible solutions of Problem 5.6.

Corollary 6.4 furnishes another proof of Theorem 2.13(ii) in the particular case when λ is singular: take κ^+ in place of κ in 6.4, and observe that, by 1.1(i), both (λ, κ^+) -regularity and (ω, κ) -regularity imply (λ, κ) -regularity.

The case λ singular of Theorem 2.2, too, can be obtained as a consequence of Corollary 6.4 (and of Proposition 2.6): if μ in 2.2 is regular, take $\kappa = \mu$ in 6.4; then if μ is singular, one can use an argument similar to the one used at the end of the proof of Theorem 6.5 below.

Corollary 6.4 shows that at least the following weak form of Conjectures 2.16 and 2.12 holds: *if μ is regular, λ is singular, $\lambda < \mu$, and D is (λ^+, μ) -regular then D is either (λ, μ) -regular or λ -decomposable* (since (ω, λ) -regularity implies λ -decomposability, by Remark 1.5(b)).

Actually, we can use Theorems 6.2 and 6.3 in order to show that Conjecture 2.12 holds under some assumptions on cardinal arithmetic and cofinalities (recall that Conjecture 2.12 is always true when μ is regular, as remarked after its statement). Recall [Sh] that $\kappa \in \text{pcf}\mathbf{a}$ means that \mathbf{a} is a set of (distinct) regular cardinals, and there is an ultrafilter D' over \mathbf{a} such that $\kappa = \text{cf} \prod_{D'} \mathbf{a}$.

Theorem 6.5. *Suppose that $\kappa > \lambda$, λ is singular, $\text{cf} \lambda \neq \text{cf} \kappa$, the ultrafilter D is (λ^+, κ) -regular, and either*

(a) λ is a strong limit cardinal and $\text{cf} S_\lambda(\lambda) < 2^\kappa$ (in particular, this holds if $\lambda^{<\lambda} < 2^\kappa$), or

(b) $\lambda^{<\lambda} < \kappa$, or

(c) κ is regular and for no $\mathbf{a} \subseteq \lambda$ with $|\mathbf{a}| < \lambda$ and $\sup \mathbf{a} = \lambda$ it happens that $\kappa \in \text{pcf}\mathbf{a}$.

Then D is either λ -decomposable, or (λ', κ) -regular for some $\lambda' < \lambda$.

Proof. First we prove the result in the case when κ is regular. We shall suppose that D is not λ -decomposable, and get (λ', κ) -regularity for some $\lambda' < \lambda$.

Were D (ω, κ') -regular for all $\kappa' < \kappa$ then in particular D would be (ω, λ) -regular (by 1.1(i), and since $\lambda < \kappa$) and trivially λ -decomposable (because of Remark 1.5(b)). So, by 1.1(i) and Theorem 6.2, we can suppose that D has a κ -least function f , and that D is (λ, κ) -regular, by Corollary 6.4.

(a) By Proposition 1.4, $|\prod_D 2^{<\lambda}| \geq 2^\kappa$, but $2^{<\lambda} = \lambda$ since λ is a strong limit cardinal, so we get $|\prod_D \lambda| \geq 2^\kappa$.

Let X be cofinal in $S_\lambda(\lambda)$ with $|X| = \text{cf} S_\lambda(\lambda)$. Since D is supposed to be not λ -decomposable, $\prod_D \lambda = \bigcup_{x \in X} \prod_D x$. Thus, there is $x \in X$ such that $|\prod_D x| \geq \lambda$, since $|X| = \text{cf} S_\lambda(\lambda) < 2^\kappa$, $2^\kappa > \kappa > \lambda$ and $|\prod_D \lambda| \geq 2^\kappa$.

Arguing as in the last part of the proof of Theorem 3.2, we get that there are arbitrarily large ν 's $< \lambda$ for which D is ν -decomposable; were D $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, then D would be λ -decomposable because of Theorem 5.1(b) with $\mu = \text{cf}\lambda$ ($\lambda^{<\text{cf}\lambda} = \lambda$ since λ is a strong limit cardinal).

Hence D is not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, but then Theorem 2.2 implies that D is (λ', κ) -regular for some $\lambda' < \lambda$.

(b) Suppose that D is over I and f is a κ -least function. By Theorem 6.3, $\{i \in I \mid \text{cf}f(i) < \lambda\} \in D$. If for some $\lambda' < \lambda$ $\{i \in I \mid \text{cf}f(i) < \lambda'\} \in D$ then D is (λ', κ) -regular again by Theorem 6.3.

Otherwise, let $g : I \rightarrow \lambda$ be defined by $g(i) = \text{cf}f(i)$ if $\text{cf}f(i) < \lambda$ and $g(i) = 0$ if $\text{cf}f(i) > \lambda$. Let $D' = g(D)$ over $I' = g(I)$. Since f is a least function, $\text{cf}(\prod_D \langle f(i), < \rangle) = \kappa$, and, trivially, $\text{cf}(\prod_D \langle g(i), < \rangle) = \kappa$. In general, it is not necessarily the case that also $\text{cf}(\prod_{D'} \langle g(i), < \rangle) = \kappa$, but we shall prove that this follows from the assumption that D is not λ -decomposable.

Indeed, let $h_D \in \prod_D g(i)$; since $g(i) = \text{cf}f(i) < \lambda$, for every i in a set in D , then we can choose a representative h of h_D which is a function from I to λ . If h is not a λ -decomposition, there is $X_h \subseteq \lambda$, $|X_h| < \lambda$ such that $\{i \in I \mid h(i) \in X_h\} \in D$. We have that $\{i \in I \mid |X_h| < g(i)\} \in D$, since otherwise $\{i \in I \mid \text{cf}f(i) \leq |X_h|\} \in D$, contrary to the first paragraph, since $|X_h| < \lambda$.

If $i' \in I'$ and $|X_h| < i'$, define $h'(i') = \sup\{h(j) \mid j \in I \text{ is such that } h(j) \in X_h \text{ and } g(j) = i'\}$. Notice that $h'(i') < i'$, since $|X_h| < i'$, and i' is a regular cardinal. If $|X_h| \geq i'$, then the definition of $h'(i')$ is arbitrary and irrelevant, since $\{i \in I \mid |X_h| \geq g(i)\} \notin D$ hence $\{i' \in I' \mid |X_h| \geq i'\} \notin D'$.

Trivially, $\prod_{D'} \langle g(i), < \rangle$ (can be identified with) a substructure of $\prod_D \langle g(i), < \rangle$; since the above argument shows that for every $h_D \in \prod_D g(i)$ there is $h'_D \in \prod_{D'} g(i)$ such that $h_D \leq h'_D \pmod{D}$, we have that the two ultraproducts above have the same cofinality, hence $\text{cf}(\prod_{D'} \langle g(i), < \rangle) = \kappa$.

Moreover, if g is not a λ -decomposition, we can assume without loss of generality that $|I'| = |g(I)| < \lambda$, but then $\kappa = \text{cf}\kappa \leq |\prod_{D'} g(i)| \leq \lambda^{|g(I)|}$ contradicts the hypothesis $\lambda^{<\lambda} < \kappa$.

(c) The arguments used in the proof of (b) give also (c), just letting $\mathbf{a} = g(I)$: the proof shows that if the conclusion fails then $|\mathbf{a}| < \lambda$, $\sup \mathbf{a} = \lambda$, and $\kappa = \text{cf} \prod_{D'} \mathbf{a}$, contradicting the hypothesis of (c).

Now let κ be singular.

In case (a), if $\text{cf}\kappa < \text{cf}\lambda$ then we get from Theorem 2.15(ii) that D is (λ, κ) -regular, and we can proceed as in the case κ regular, since in

the proof of (a) the existence of a least function has been used only in order to get (λ, κ) -regularity.

The following argument, however, covers all cases when κ is singular.

Having already proved the theorem in the case κ regular, we prove the case κ singular by applying the case κ regular to a set of sufficiently large regular cardinals $\kappa' < \kappa$.

First notice that, in case (a), if $\lambda < \kappa'$, then $\text{cf } S_\lambda(\lambda) \leq \lambda^{<\lambda} \leq 2^\lambda \leq 2^{\kappa'}$; moreover, since $\text{cf } S_\lambda(\lambda) < 2^\kappa$, there is no κ'' with $\lambda < \kappa'' < \kappa$ such that $2^{\kappa'} = \text{cf } S_\lambda(\lambda)$ for all κ' with $\kappa'' < \kappa' < \kappa$. Indeed, κ singular, together with classical cardinal arithmetic (see e.g. [KM, p. 257]), would imply $2^\kappa = \text{cf } S_\lambda(\lambda)$, contradicting the hypothesis. Hence, $2^{\kappa'} > \text{cf } S_\lambda(\lambda)$, for large enough $\kappa' < \kappa$.

By what we have proved, the theorem is true for every regular and sufficiently large $\kappa' < \kappa$ (in (b) it is enough to take $\kappa' > \lambda^{<\lambda}$, and this is possible since κ is singular $> \lambda^{<\lambda}$). Hence, if D is not λ -decomposable, then for every sufficiently large regular $\kappa' < \kappa$ there is $\lambda_{\kappa'} < \lambda$ such that D is $(\lambda_{\kappa'}, \kappa')$ -regular.

Choose a set C of sufficiently large cardinals, $C \subseteq \kappa$, C cofinal in κ , with $|C| = \text{cf } \kappa$, and consider $C' = \{\lambda_{\kappa'} | \kappa' \in C\}$. If $\text{cf } \kappa < \text{cf } \lambda$ let $\lambda' = \sup C'$ and notice that $\text{cf } \kappa < \text{cf } \lambda$ implies that $\lambda' < \lambda$. If $\text{cf } \kappa > \text{cf } \lambda$, then there is $\lambda' < \lambda$ such that $|\{\kappa' \in C | \lambda_{\kappa'} < \lambda'\}| \geq \text{cf } \kappa$, since if $|\{\kappa' \in C | \lambda_{\kappa'} < \lambda'\}| < \text{cf } \kappa$ for all $\lambda' < \lambda$, then $|C| < \text{cf } \kappa$, contradicting the assumption that C is cofinal in κ .

In both cases, if λ' is as above, then by 1.1(i), D is (λ', κ') -regular for $\text{cf } \kappa$ -many $\kappa' \in C$, hence for a set of κ' unbounded in κ and, again by 1.1(i), for all $\kappa' < \kappa$. Without loss of generality, we can choose λ' regular, so that Proposition 5.4(a) implies that D is (λ', κ) -regular. \square

The main argument in the proof of 6.5(a), in the case when κ is regular, is taken from [Pr, Theorems 11 and 12], where a slightly less general result is obtained for the particular case $\kappa = \lambda^+$. The arguments for cases (b) and (c), as well as for the case κ singular, seem to be new. See [Lp10, Lp11] for other applications of Shelah's pcf-theory to decomposability and regularity of ultrafilters.

The results in [Ka1], as well as the generalization given in Theorem 6.2, suggest the following problems. The problems are quite natural, but can hardly be found in the literature; see, anyway, [Ket2, p.231], [Sh1, Remark 6.9].

Problem 6.6. Are the following theorems of ZFC?

(A) If κ is regular, and the ultrafilter D is uniform over κ and has no least function, then D is (ω, κ) -regular.

(B) If κ is regular, and the ultrafilter D is (κ, κ) -regular and has no κ -least function, then D is (ω, κ) -regular.

Clearly, 6.6(B) implies 6.6(A), by 1.1(vi). We state below two other consequences of 6.6(B).

Proposition 6.7. *Suppose that 6.6(B) holds. Then:*

(i) *If μ is regular and λ is singular then every (λ^+, μ) -regular ultrafilter is (λ, μ) -regular (that is, Conjecture 2.16 holds for μ regular).*

(ii) *If $\lambda \leq \kappa$, κ is regular and D is (λ, κ) -regular then there exists a (λ, κ) -regular $D' \leq D$ (in the Rudin-Keisler order), D' uniform over κ .*

Proof. (i) is proved as Corollary 6.4, using 6.6(B) in place of Theorem 6.2.

(ii) If D over I is (ω, κ) -regular, then there is a function $f : I \rightarrow S_\omega(\kappa)$ witnessing it. But $|S_\omega(\kappa)| = \kappa^{<\omega} = \kappa$ so that $D' = f(D)$ is uniform on κ and (ω, κ) -regular.

If, on the contrary, D is not (ω, κ) -regular, then by 6.6(B) there is a κ -least function $f : I \rightarrow \kappa$. So $D' = f(D)$ is uniform on κ and the identity function id is a least function for D' (such an ultrafilter is usually called *weakly normal*). By Theorem 6.3, $\{i \in I \mid \text{cf } f(i) < \lambda\} \in D$, hence $\{\alpha \in \kappa \mid \text{cf } id(\alpha) < \lambda\} \in D'$, so, by Theorem 6.3 again, the least function id proves the (λ, κ) -regularity of D' . \square

6.7(ii) is a quite natural requirement: it imports that, for most purposes, it is enough to consider only (λ, κ) -regular ultrafilters over κ .

Notice that the assumption κ regular is necessary in 6.7(ii): every ultrafilter uniform over ω is $(\omega_\omega, \omega_\omega)$ -regular, by 1.1(vi)(v). A more significant example, in which $\kappa > \lambda$, is the following: if λ is a strongly compact cardinal, then there is an ω_1 -complete $(\lambda, \lambda^{+\omega})$ -regular ultrafilter, but no such ultrafilter can be uniform over $\lambda^{+\omega}$, otherwise it would be (ω, ω) -regular by 1.1(vi), and this contradicts ω_1 -completeness.

However, in the above example, the ultrafilter can be chosen to be uniform over $\lambda^{+\omega+1}$, so that, as far as we know, the following might be a theorem of ZFC:

6.7(ii)* If $\lambda < \kappa$ and D is (λ, κ) -regular then there exists a (λ, κ) -regular $D' \leq D$ such that D' is uniform either over κ^+ or over κ .

Notice also that 6.7(ii) is true when $\kappa = \lambda^{+n}$ and λ is regular, since we can always get a $D' \leq D$ over $\text{cf } S_\lambda(\kappa)$, by 1.5(b), and it is easy to show that $\text{cf } S_\lambda(\lambda^{+n}) = \lambda^{+n}$, if λ is regular.

Of course, in case 6.6(B) turned out to be unprovable in ZFC, we might ask the problem whether 6.7(ii) or 6.7(ii)* are theorems of ZFC.

We suspect that 6.7(ii)* and 6.7(ii) are equivalent.

Is it true that 6.7(ii) implies that 6.6(A) and 6.6(B) are equivalent?
We now turn to another kind of problem.

Problem 6.8. For which sets K of infinite cardinals is there an ultrafilter which is κ -decomposable exactly for those $\kappa \in K$?

In other words, if D is an ultrafilter, let $K_D = \{\kappa \geq \omega \mid D \text{ is } \kappa\text{-decomposable}\}$. Which are the possible values K_D can take?

Many constraints are known on K_D : first, every κ -decomposable ultrafilter is $\text{cf}\kappa$ -decomposable, by 1.1(vii)(viii); and the least κ for which an ultrafilter is κ -decomposable is always a measurable cardinal or ω , as mentioned shortly after 1.5. Moreover, if κ is regular, then every κ^+ -decomposable ultrafilter is κ -decomposable, by [ČČ], [KP], or Theorem 2.1 here, and 1.1(xi). By the same results, if κ is singular, then every κ^+ -decomposable ultrafilter is either $\text{cf}\kappa$ -decomposable or κ' -decomposable for all sufficiently large regular $\kappa' < \kappa$ (use 1.1(xi)(xii)).

Further constraints on K_D are given by Theorems 2.9, 2.10 2.19, 3.2, 4.3, 5.1, 5.7 6.5, Propositions 3.3, 6.10, 8.1, 8.7 and Corollary 5.3. See also Propositions 7.1, 7.6, and Problems 3.5, 4.4, 5.2, 7.2.

We now consider in detail the cases when $|K_D| \leq 2$.

If $|K_D| = 1$, say $K_D = \{\mu\}$, then μ is either ω , or a measurable cardinal.

There are many open problems already for the case $|K_D| = 2$. The case when $K_D = \{\omega, \kappa\}$ has originally been studied by J. Silver [Si]; in this case, D is usually called *indecomposable*, and only the following possibilities can occur, because of 1.1 and Theorem 2.1: either κ has cofinality ω , or κ is weakly inaccessible, or κ is the successor of a cardinal of cofinality ω .

If μ is a measurable cardinal and D is a μ -complete uniform ultrafilter over μ , then D is μ -decomposable, but not λ -decomposable for all $\lambda < \mu$, and, trivially, not λ -decomposable for all $\lambda > \mu$. If D' is uniform over ω , then D' is ω -decomposable, by 1.1(iii). Thus, by Proposition 7.1, $K_{D \times D'} = \{\omega, \mu\}$. If μ is measurable, and μ is made singular by Prikry forcing [Pr1], then in the resulting model there is an ultrafilter which is κ -decomposable exactly for $\kappa = \omega$ and $\kappa = \mu$.

We can get a similar example without using forcing, but starting with ω measurable cardinals. Let D be uniform over ω , and let $(\mu_n)_{n \in \omega}$ be a strictly increasing sequence of measurable cardinals, and take D_n to be a μ_n -complete uniform ultrafilter over μ_n . Set $\mu = \sup\{\mu_n \mid n \in \omega\}$ and $D^* = \sum_D D_n$ (see Section 7 for the definition). It is easy to show that $K_{D^*} = \{\omega, \mu\}$.

The example can be modified in order to get: $K_{D^{**}} = \{\omega, \mu_0, \mu_1, \mu_2, \dots, \mu_n, \dots, \mu\}$: just take $D'_n = D_0 \times D_1 \times \dots \times D_{n-1} \times D_n$, and $D^{**} = \sum_D D'_n$.

In the model constructed in [BM] there is an ultrafilter D such that $K_D = \{\omega, \omega_\omega, \omega_{\omega+1}\}$ (ω_ω -decomposability follows from GCH and Theorem 6.5(a) with $\kappa = \lambda^+$). By taking a projection along some ω_ω -decomposition, we get an ultrafilter D' such that $K_{D'} = \{\omega, \omega_\omega\}$. See also the result by H. Woodin stated in [Ma, Theorem 1.5.6(iv)]. The constructions given in [BM, AH] might shed further light on the problem of the possible values K_D can take.

[Shr] has constructed an ultrafilter for which $K_D = \{\omega, \kappa\}$, where κ is inaccessible and not weakly compact. However, it is not exactly known ([Shr, p. 1007]) for which inaccessible cardinals κ we can have $K_D = \{\omega, \kappa\}$; by Theorem 2.19 and 1.1(xi)(xii), κ must be ω -Mahlo, but it is not known whether we can have, say, κ not $\omega + 1$ -Mahlo.

As far as the remaining possibilities are concerned in the case $|K_D| = 2$, we do not know whether we can have:

- (a) $K_D = \{\omega, 2^{\omega_1}\}$ and $2^\omega < 2^{\omega_1}$, or
- (b) $K_D = \{\omega, \omega_{\omega+1}\}$, or, more generally,
- (c) $K_D = \{\omega, \kappa^+\}$ with $\text{cf} \kappa = \omega$, and $\kappa \neq \omega$.

An affirmative answer to Conjecture 2.12 would prevent (b) and (c), by 1.1(xi)(xii). If Problem 3.5(a) has an affirmative answer then (a) cannot hold: this is proved as follows. If D is 2^{ω_1} -decomposable, then D is $(2^{\omega_1}, 2^{\omega_1})$ -regular by 1.1(vii); now, take $\kappa = \omega$ and $m = n = 1$ in the statement of Theorem 4.3(a'), and recall that if Problem 3.5(a) has an affirmative answer then the conclusion of 4.3(a') can be improved to $\kappa < \mu \leq 2^\kappa$, so that, in the present case, D is μ -decomposable for some μ with $\omega < \mu \leq 2^\omega < 2^{\omega_1}$.

Apparently, the case when $|K_D| = 2$ and $\inf K_D > \omega$ has never been studied. There are trivial cases, e.g., if D is μ -complete and uniform over μ , and D' is μ' -complete and uniform over μ' , then $K_{D \times D'} = \{\mu, \mu'\}$. As another example, if κ is κ^+ -compact, then there is an ultrafilter D which is κ -complete, (κ, κ^+) -regular, and uniform over κ^+ , hence $K_D = \{\kappa, \kappa^+\}$, by 1.1(xii).

In case K_D is infinite Shelah's **pcf** theory [Sh] deeply influences the possible values of K_D ([Lp10, Lp11]).

The possibility that K_D is an interval can always occur: if D is uniform over λ and (ω, λ) -regular, then $K_D = [\omega, \lambda]$, by 1.1(i) and Remark 1.5(b). If there is no inner model with a measurable cardinal, then by Donder's Theorem 1.6, if D is uniform over λ then K_D is always equal to $[\omega, \lambda]$, since D is λ' -decomposable for all $\lambda' < \lambda$ by 1.1(i) and Remark 1.5(b); moreover, D is λ -decomposable by 1.1(iii).

If κ is λ -compact and λ is regular then there is a κ -complete (κ, λ) -regular ultrafilter uniform over λ , hence $K_D = \{\mu \mid \kappa \leq \mu \leq \lambda \text{ and } \text{cf } \mu \geq \kappa\}$, by 1.1(xii) in the case μ regular; then apply Theorem 5.1(b) in order

to get the case μ singular (since $\mu^{<\kappa} = \mu$, by the result we mentioned shortly after the definition of strong compactness). If $\kappa \leq \mu \leq \lambda$ and cf $\mu < \kappa$ then $\mu \notin K_D$ by 1.1(vii).

Hence, a general solution of Problem 6.8 appears to be quite difficult.

Problem 6.8 appears to be connected also with some variations on the principle we have denoted by $U'(\lambda)$ (see Definition 1.7).

Definition 6.9. Let λ be a limit cardinal.

(i) [Lp1, Definition 4.4] $U^*(\lambda)$ means that for every ultrafilter D , if there are arbitrarily large regular $\kappa < \lambda$ such that D is κ -decomposable, then D is (λ, λ) -regular.

(ii) $U(\lambda)$ means that for every ultrafilter D , if there is $\lambda' < \lambda$ such that D is κ -decomposable for all regular κ with $\lambda' < \kappa < \lambda$, then D is (λ, λ) -regular.

In [Lp4, p. 132] $U(\lambda)$ has been stated in the following equivalent form: “for every ultrafilter D , if there is $\lambda' < \lambda$ such that D is (κ, κ) -regular for all κ with $\lambda' < \kappa < \lambda$, then D is (λ, λ) -regular”. The equivalence follows from 1.1(xi) and Theorem 2.1(b).

Theorem 5.7 shows that if λ is a singular cardinal, then $U(\lambda)$ holds.

Clearly, for every limit cardinal λ , $U'(\lambda)$ implies $U^*(\lambda)$, which implies $U(\lambda)$. [Lp1, Lp4] contain applications of the above principles to abstract logics. See also [Lp11] for improved results, whose proofs implicitly use $U(\lambda)$.

As far as we know, it is possible that $U^*(\lambda)$ is a theorem of ZFC, for every singular cardinal λ (our guess is that it is not a theorem). For sure, the failure of $U^*(\lambda)$ for λ singular has a quite large consistency strength; see [Lp1, Proposition 4.5], Corollary 5.3 and the next proposition. We originally thought that the above principles were quite similar in strength, but now we know that it is much harder to make $U(\lambda)$ fail. In fact, if $U(\lambda)$ fails then λ is weakly inaccessible, by Theorem 5.7.

The following Proposition gives a condition equivalent to the failure of $U^*(\lambda)$.

Proposition 6.10. *Let λ be a singular cardinal. Then $U^*(\lambda)$ fails if and only if there is an ultrafilter D such that:*

- (a) *D is not cf λ -decomposable;*
- (b) *there are arbitrarily large regular $\kappa < \lambda$ for which D is κ -decomposable;*
- (c) *there are arbitrarily large regular $\kappa < \lambda$ for which D is not κ -decomposable.*

Proof. (i) By definition, if $U^*(\lambda)$ fails there is D which satisfies (b) and is not (λ, λ) -regular.

D satisfies (a) since every cf λ -decomposable ultrafilter is (λ, λ) -regular, by 1.1(xi)(v).

Were (c) false, there would be $\lambda' < \lambda$ such that D is κ -decomposable for all regular κ with $\lambda' < \kappa < \lambda$, but then Theorem 5.7 would imply that D is (λ, λ) -regular, a contradiction.

Conversely, let D satisfy (a), (b), and (c). Because of (b), if D is not (λ, λ) -regular then $U^*(\lambda)$ fails. By Proposition 2.6 and 1.1(xi), every (λ, λ) -regular ultrafilter is either cf λ -decomposable or (λ', λ) -regular for some $\lambda' < \lambda$. But this is impossible: the first possibility cannot occur because of (a), and the second possibility cannot occur because of (c), since every (λ', λ) -regular ultrafilter is κ -decomposable for all regular κ with $\lambda' \leq \kappa \leq \lambda$, by 1.1(xii). \square

Proposition 6.10 still holds if we replace everywhere $U^*(\lambda)$ by $U'(\lambda)$ and we delete the word “regular” in condition (b).

Notice that the example mentioned shortly before Definition 6.9 can be used in order to provide a singular cardinal λ and a (λ, λ) -regular not cf λ -decomposable ultrafilter D such that there are arbitrarily large regular $\kappa < \lambda$ for which D is κ -decomposable and there are arbitrarily large singular $\kappa < \lambda$ for which D is not κ -decomposable. Just take λ singular with $\omega < \text{cf } \lambda < \kappa$.

We expect to be able to find more applications of Kanamori and Ketonen’s results, as well as of their generalizations Theorems 6.2 and 6.3.

7. REGULARITY OF PRODUCTS.

Given certain regular ultrafilters, we sometimes can “sum” their regularities by taking products. This is because the regularity properties of $D \times D'$ are determined by the regularity properties of D and of D' (Proposition 7.1).

In this section we shall present some examples, and, more generally, we shall consider D -sums; some similar results appeared in [Ket1, Section 5] under much stronger assumptions, such as ω_1 -completeness.

The product $D \times D'$ of two ultrafilters D and D' (over I , I' , respectively) is the ultrafilter over $I \times I'$ defined by: $X \in D \times D'$ if and only if $\{i \in I \mid \{i' \in I' \mid (i, i') \in X\} \in D'\} \in D$.

The following proposition is useful and has a simple proof, but might be new.

Proposition 7.1. *For D , D' ultrafilters, the following are equivalent:*

- (a) $D \times D'$ is (λ, μ) -regular;
- (b) there is a cardinal ν such that D is (ν, μ) -regular and D' is (λ, ν') -regular for all $\nu' < \nu$.

Thus, in particular, $D \times D'$ is (λ, λ) -regular if and only if either D is (λ, λ) -regular or D' is (λ, λ) -regular.

Proof. We believe that the proof can be best viewed in model-theoretical terms, using Form III of the definition of regularity (at least, we discovered the result by reasoning in model-theoretical terms). An alternative proof of 7.1 using Form I can be obtained from the proof of 7.4 below (which is a result stronger than 7.1).

It is well known that, for every model \mathbf{A} , $\prod_{D \times D'} \mathbf{A} \cong \prod_D \prod_{D'} \mathbf{A}$ (throughout, we shall use the same names for corresponding elements of $\prod_{D \times D'} \mathbf{A}$ and of $\prod_D \prod_{D'} \mathbf{A}$).

Now, suppose that there exists a ν as in statement (b) of the Proposition. Then the following is true: whenever $X \subseteq \mu$ and $|X| < \nu$ then there is $x_X \in \prod_{D'} \langle S_\lambda(\mu), \subseteq, \{\alpha\} \rangle_{\alpha < \mu}$ such that $d(\{\alpha\}) \subseteq x_X$ for every $\alpha \in X$ (by Form III, since if $|X| = \nu' < \nu$ then D' is (λ, ν') -regular, and $S_\lambda(X) \subseteq S_\lambda(\mu)$ is isomorphic to $S_\lambda(\nu')$). Further, we can have $X = \{\alpha \in \mu \mid d(\alpha) \subseteq x_X\}$: just replace x_X by $x'_X(i') = X \cap x_X(i')$.

In other words, $\prod_{D'} \langle S_\lambda(\mu), \subseteq, \{\alpha\} \rangle_{\alpha < \mu}$ contains a “copy” of $\langle S_\nu(\mu), \subseteq, \{\alpha\} \rangle_{\alpha < \mu}$, whence, by the (ν, μ) -regularity of D , in $\prod_D \prod_{D'} \langle S_\lambda(\mu), \subseteq, \{\alpha\} \rangle_{\alpha < \mu} \cong \prod_{D \times D'} \langle S_\lambda(\mu), \subseteq, \{\alpha\} \rangle_{\alpha < \mu}$ there is an element y such that $d(\{\alpha\}) \subseteq y$ for every $\alpha \in \mu$, so that $D \times D'$ is (λ, μ) -regular.

If one needs the actual definition of a (λ, μ) -regularizing function, this goes as follows: for every $X \subseteq \mu$ with $|X| < \nu$ let x_X be as above. Thus, x_X is (the equivalence class modulo D of) a function $x_X : I' \rightarrow S_\lambda(X) \subseteq S_\lambda(\mu)$ such that if $\alpha \in X$ then $\{i' \in I' \mid \alpha \in x_X(i')\} \in D'$. Now, let $g : I \rightarrow S_\nu(\mu)$ witness the (ν, μ) -regularity of D , as given by Form II. Thus, for every $\alpha \in \mu$, $\{i \in I \mid \alpha \in g(i)\} \in D$. Then $f : I \rightarrow \prod_{D'} S_\lambda(\mu)$ defined by $f(i) = x_{g(i)}$ witnesses the (λ, μ) -regularity of $D \times D'$, since, for every $\alpha \in \mu$, $\{(i, i') \mid \alpha \in f(i)(i')\} = \{(i, i') \mid \alpha \in x_{g(i)}(i')\} \in D \times D'$, since $\{i \in I \mid \{i' \in I' \mid \alpha \in x_{g(i)}(i')\} \in D'\} \supseteq \{i \in I \mid \alpha \in g(i)\} \in D$.

Having proved that (b) \Rightarrow (a), let us prove (a) \Rightarrow (b). Suppose that $D \times D'$ is (λ, μ) -regular, so that, by Form III, in $\prod_{D \times D'} \langle S_\lambda(\mu), \subseteq, \{\alpha\} \rangle_{\alpha < \mu}$ there is an element x such that $d(\{\alpha\}) \subseteq x$ for every $\alpha \in \mu$. Thus, x is in $\prod_D \prod_{D'} S_\lambda(\mu)$, and this means that there is a function $f : I \rightarrow \prod_{D'} S_\lambda(\mu)$ such that for every $\alpha \in \mu$ $\{i \in I \mid \prod_{D'} S_\lambda(\mu) \models d(\{\alpha\}) \subseteq f(i)\} \in D$.

Define $g : I \rightarrow S(\mu)$ by $g(i) = \{\alpha \in \mu \mid \prod_{D'} S_\lambda(\mu) \models d(\{\alpha\}) \subseteq f(i)\}$. Since $\alpha \in g(i)$ if and only if $\prod_{D'} S_\lambda(\mu) \models d(\{\alpha\}) \subseteq f(i)$, then for every $\alpha \in \mu$ $\{i \in I \mid \alpha \in g(i)\} \in D$.

Let $\nu = \sup_{i \in I} (|g(i)|^+)$; thus, $g : I \rightarrow S_\nu(\mu)$ makes D (ν, μ) -regular, according to Form II. By the definition of ν , for every $\nu' < \nu$ there is $i \in I$ such that $|\{\alpha \in \mu \mid \alpha \in g(i)\}| \geq \nu'$.

Given any $\nu' < \nu$, fix some i as above. Then $|\{\alpha \in \mu \mid \prod_{D'} S_\lambda(\mu) \models d(\{\alpha\}) \subseteq f(i)\}| \geq \nu'$. Choose $X \subseteq \{\alpha \in \mu \mid \prod_{D'} S_\lambda(\mu) \models d(\{\alpha\}) \subseteq f(i)\}$ with $|X| = \nu'$, and, for $i' \in I'$, define $f'(i') = X \cap f(i)(i')$. Since $S_\lambda(\nu')$ is isomorphic to $S_\lambda(X)$, $f' : I' \rightarrow S_\lambda(X)$ witnesses the (λ, ν') -regularity of D' , as given by Form II, since if $\alpha \in X$ then $\prod_{D'} S_\lambda(\mu) \models d(\{\alpha\}) \subseteq f(i)$, that is, $\{i' \in I' \mid d(\{\alpha\}) \subseteq f(i)(i')\} \in D'$, thus $\{i' \in I' \mid d(\{\alpha\}) \subseteq f'(i')\} \in D'$.

As for the last statement in the Proposition, the if-part follows from 1.1(ii) and the fact that both $D \leq D \times D'$ and $D' \leq D \times D'$.

On the other side, if $D \times D'$ is (λ, λ) -regular, then, by the equivalence of (a) and (b), there is a cardinal ν such that D is (ν, λ) -regular and D' is (λ, ν') -regular for all $\nu' < \nu$. Thus, by 1.1(i), if $\nu \leq \lambda$ then D is (λ, λ) -regular, and if $\nu > \lambda$ then D' is (λ, λ) -regular \square

Thus, for example, if D is (ν^+, μ) -regular and D' is (λ, ν) -regular, and neither D nor D' is (κ, κ) -regular, then $D \times D'$ is (λ, μ) -regular and not (κ, κ) -regular (see also Proposition 7.6).

As another example, if D is not (λ^+, λ^+) -regular and if D' is not (λ, λ^+) -regular, then $D \times D'$ is not (λ, λ^+) -regular.

On the contrary, if D is (λ^+, λ^+) -regular, and D' is (λ, λ) -regular then $D \times D'$ is (λ, λ^+) -regular (this improves [Ket1, Theorem 5.8]); in particular, Theorem 2.1(b) implies that if D is (λ^+, λ^+) -regular then $D \times D$ is (λ, λ^+) -regular. More generally, if D is $(\lambda^{+n}, \lambda^{+n})$ -regular then $D \times D \times \dots \times D$ ($n+1$ factors) is (λ, λ^{+n}) -regular. Moreover, if D is $(\lambda^{+n+1}, \lambda^{+2n+1})$ -regular then $D \times D$ is $(\lambda, \lambda^{+2n+1})$ -regular, since if D is $(\lambda^{+n+1}, \lambda^{+2n+1})$ -regular then D is (λ, λ^{+n}) -regular by iterating Theorem 2.13(ii) $n+1$ times; then apply Proposition 7.1 with $D = D'$ and $\nu = \lambda^{+n+1}$.

Notice that Proposition 7.1, together with 1.1(xi), implies that, if κ is regular, then $D \times D'$ is κ -decomposable if and only if either D or D' is κ -decomposable. This is not necessarily true when κ is singular: let D be uniform over ω , and suppose that κ is $\kappa^{+\omega}$ -compact; thus, there is an ω_1 -complete $(\kappa, \kappa^{+\omega})$ -regular ultrafilter D' , which is not $\kappa^{+\omega}$ -decomposable, by 1.1(vii), but which is κ^{+n} -decomposable for all $n < \omega$, by 1.1(xii). Now, $D \times D'$ is (ω, ω) -regular and κ^{+n} -decomposable for all $n < \omega$, by 1.1(vi)(xi) and the last statement in Theorem 7.1; then Theorem 5.1(b) implies that $D \times D'$ is $\kappa^{+\omega}$ -decomposable, since $(\lambda^{+\omega})^{<\omega} = \lambda^{+\omega}$. However, neither D nor D' is $\kappa^{+\omega}$ -decomposable.

We do not know whether we have a counterexample as above in which $D = D'$.

Problem 7.2. Can there be a D not κ -decomposable such that $D \times D$ is κ -decomposable?

As an application of Proposition 7.1 we can get a generalization (with a simpler proof) of a result by Ketonen. [Ket2, Theorem 1.1] is the particular case $\lambda = \kappa$ of the next proposition.

Proposition 7.3. *If κ is regular, $\lambda \geq \kappa$, and the ultrafilter D is (κ, λ) -regular and has no κ -least function then $D \times D$ is (ω, λ) -regular.*

Proof. Immediate from 1.1(i), Theorem 6.2 and Proposition 7.1. \square

There is a version of Proposition 7.1 for sums of ultrafilters.

If D is an ultrafilter over I , and for every $i \in I$ D_i is an ultrafilter over some set I_i , the D -sum $\sum_D D_i$ of the D_i 's modulo D is the ultrafilter over $\{(i, j) | i \in I, j \in I_i\}$ defined by $X \in \sum_D D_i$ if and only if $\{i \in I | \{j \in I_i | (i, j) \in X\} \in D_i\} \in D$ (cf. e.g. [Ket1, Definition 0.4]).

In the particular case when all the D_i 's are equal (to, say, D') we get the product $D \times D'$, thus the following proposition generalizes Proposition 7.1.

Proposition 7.4. *(a) $\sum_D D_i$ is (λ, μ) -regular if and only if there is a function $g : I \rightarrow S(\mu)$ such that $\{i \in I | \alpha \in g(i)\} \in D$ for every $\alpha \in \mu$ and such that for every $i \in I$ D_i is $(\lambda, |g(i)|)$ -regular (equivalently, we can just ask that $\{i \in I | D_i \text{ is } (\lambda, |g(i)|)\text{-regular}\} \in D$).*

(a') $\sum_D D_i$ is (λ, μ) -regular if and only if there is a family $(X_\alpha)_{\alpha \in \mu}$ of elements in D such that for every $i \in I$ D_i is $(\lambda, |\{\alpha \in \mu | i \in X_\alpha\}|)$ -regular (equivalently, we can just ask that $\{i \in I | D_i \text{ is } (\lambda, |\{\alpha \in \mu | i \in X_\alpha\}|)\text{-regular}\} \in D$).

(b) If $\sum_D D_i$ is (λ, μ) -regular then for every cardinal ν either D is (ν, μ) -regular, or $\{i \in I | D_i \text{ is } (\lambda, \nu)\text{-regular}\} \in D$.

(c) If D is (ν^+, μ) -regular and $\{i \in I | D_i \text{ is } (\lambda, \nu)\text{-regular}\} \in D$, then $\sum_D D_i$ is (λ, μ) -regular.

(d) $\sum_D D_i$ is (λ, λ) -regular if and only if either D is (λ, λ) -regular or $\{i \in I | D_i \text{ is } (\lambda, \lambda)\text{-regular}\} \in D$.

Proof. (a) can be proved in a way similar to the proof of Proposition 7.1, noticing that, for every model \mathbf{A} , if $E = \sum_D D_i$ then $\prod_E \mathbf{A} \cong \prod_D \prod_{D_i} \mathbf{A}$. A direct proof of (a') is given below. However, the two proofs are interchangeable, since it is immediate to see that (a) and (a') are equivalent. Indeed, if g is a function as given by (a), then define, for each $\alpha \in \mu$, $X_\alpha = \{i \in I | \alpha \in g(i)\}$: the X_α 's then satisfy (a'). Conversely, given X_α 's as in (a'), let $g(i) = \{\alpha \in \mu | i \in X_\alpha\}$: then g satisfies (a) (indeed, this is nothing but the usual proof for the equivalence of Forms I and II in the definition of (λ, μ) -regularity).

Notice that both in (a) and in (a') the condition inside the parenthesis is equivalent to the condition outside, since if $X \notin D$ then $\sum_D D_i$ does not depend on the D_i 's ($i \in X$).

Now, let us prove (a'). Suppose that $\sum_D D_i$ is (λ, μ) -regular, that is, by Form I, there exist sets $(Z_\alpha)_{\alpha \in \mu}$ in $\sum_D D_i$ such that the intersection of any λ of them is empty. For every $\alpha \in \mu$, let $X_\alpha = \{i \in I \mid \{j \in I_i \mid (i, j) \in Z_\alpha\} \in D_i\}$. Thus, for every $\alpha \in \mu$, $X_\alpha \in D$, since $Z_\alpha \in \sum_D D_i$.

If $i \in X_\alpha$, let $Y_{\alpha i} = \{j \in I_i \mid (i, j) \in Z_\alpha\}$. Thus, $Y_{\alpha i} \in D_i$. We claim that, for each $i \in I$, the family $\{Y_{\alpha i} \mid \alpha \in \mu \text{ is such that } i \in X_\alpha\}$ witnesses the $(\lambda, |\{\alpha \in \mu \mid i \in X_\alpha\}|)$ -regularity of D_i . If not, for some i there is $B \subseteq \{\alpha \in \mu \mid i \in X_\alpha\}$ with $|B| = \lambda$ such that $\bigcap_{\alpha \in B} Y_{\alpha i} \neq \emptyset$. If $j \in \bigcap_{\alpha \in B} Y_{\alpha i}$, then $(i, j) \in \bigcap_{\alpha \in B} Z_\alpha$, absurd, since $(Z_\alpha)_{\alpha \in \mu}$ was supposed to be a (λ, μ) -regularizing family for $\sum_D D_i$.

Conversely, suppose that there are $(X_\alpha)_{\alpha \in \mu}$ as given in (a'), and for every $i \in I$ let $\{Y_{\alpha i} \mid \alpha \in \mu \text{ is such that } i \in X_\alpha\}$ witness the $(\lambda, |\{\alpha \in \mu \mid i \in X_\alpha\}|)$ -regularity of D_i . For $\alpha \in \mu$, let $Z_\alpha = \{(i, j) \mid i \in X_\alpha \text{ and } j \in Y_{\alpha i}\}$. Thus, $Z_\alpha \in \sum_D D_i$.

We claim that the Z_α 's witness the (λ, μ) -regularity of $\sum_D D_i$. Indeed, if by contradiction $\bigcap_{\alpha \in B} Z_\alpha \neq \emptyset$ for some B with $|B| = \lambda$, say $(i, j) \in \bigcap_{\alpha \in B} Z_\alpha$, then $j \in \bigcap_{\alpha \in B} Y_{\alpha i}$, and this contradicts the assumption that $\{Y_{\alpha i} \mid \alpha \in \mu \text{ is such that } i \in X_\alpha\}$ witnesses the $(\lambda, |\{\alpha \in \mu \mid i \in X_\alpha\}|)$ -regularity of D_i .

Hence, we have proved (a) and (a').

(b) Since $\sum_D D_i$ is (λ, μ) -regular, there is a function $g : I \rightarrow S(\mu)$ as given by (a). If $\{i \in I \mid |g(i)| < \nu\} \in D$ then D is (ν, μ) -regular by Form II since we can change the values of g for a set not in D hence, without loss of generality, we can suppose that $g : I \rightarrow S_\nu(\mu)$.

Otherwise, $\{i \in I \mid |g(i)| \geq \nu\} \in D$, hence $\{i \in I \mid D_i \text{ is } (\lambda, \nu)\text{-regular}\} \in D$, by (a) and 1.1(i).

(c) If D is (ν^+, μ) -regular, then, by Form II, there is $g : I \rightarrow S_{\nu^+}(\mu)$ witnessing it. Thus, $|g(i)| \leq \nu$, for every $i \in I$, hence, by 1.1(i) and (a), $\sum_D D_i$ is (λ, μ) -regular.

(d) The only-if part is immediate from (b) with $\mu = \nu = \lambda$.

Conversely, if $\{i \in I \mid D_i \text{ is } (\lambda, \lambda)\text{-regular}\} \in D$ then (c) with $\mu = \nu = \lambda$ implies that $\sum_D D_i$ is (λ, λ) -regular, by 1.1(xiii). On the other side, if D is (λ, λ) -regular, then $\sum_D D_i$ is (λ, λ) -regular, by 1.1(ii), since trivially $D \leq \sum_D D_i$.

Notice that in (b) and (d) we cannot replace " $\{i \in I \mid D_i \text{ is } (\lambda, \nu)\text{-regular}\} \in D$ " by "for every $i \in I$ D_i is (λ, ν) -regular". This is because if $X \notin D$ then $\sum_D D_i$ does not really depend on the ultrafilters D_i ($i \in X$), hence the D_i 's ($i \in X$) can be chosen arbitrarily. \square

For example, suppose that, in the same model of Set Theory, for every $n < \omega$ there is an ultrafilter D_n which is (ω_n, ω_n) -regular, and

which for no $m < n$ is (ω_m, ω_{m+1}) -regular. Then, if we take D uniform over ω , the sum $\sum_D D_n$ is (ω_n, ω_n) -regular, for every $n < \omega$, but for no $n < \omega$ $\sum_D D_n$ is (ω_n, ω_{n+1}) -regular (compare with Problem 2.14). Notice that, without loss of generality, each D_n can be chosen uniform over ω_n , by 1.1(viii)(iii)(ii); in this case, $\sum_D D_n$ is uniform over ω_ω .

When D has a least function, the condition in Proposition 7.4 can be simplified.

Corollary 7.5. *Suppose that μ is regular, the ultrafilter D is (μ, μ) -regular and has a μ -least function f . Then $\sum_D D_i$ is (λ, μ) -regular if and only if $\{i \in I \mid D_i \text{ is } (\lambda, \text{cf } f(i))\text{-regular}\} \in D$.*

Proof. If $\sum_D D_i$ is (λ, μ) -regular then there is a function $g : I \rightarrow S(\mu)$ as given by Proposition 7.4(a), such that for every $i \in I$ D_i is $(\lambda, |g(i)|)$ -regular.

For $i \in I$, let $f'(i) = \sup(g(i) \cap f(i))$. Thus, $f'(i) \leq f(i)$ for every $i \in I$. Since for every $\alpha < \mu$ both $\{i \in I \mid \alpha < f(i)\}$ and $\{i \in I \mid \alpha \in g(i)\}$ belong to D , we have that $\{i \in I \mid \alpha < f'(i)\} \in D$. Since f is a μ -least function, $\{i \in I \mid f(i) = f'(i)\} \in D$, hence $\{i \in I \mid \text{cf } f(i) \leq |g(i)|\} \in D$, and we are done by 1.1(i).

For the converse, suppose that $\{i \in I \mid D_i \text{ is } (\lambda, \text{cf } f(i))\text{-regular}\} \in D$. The arguments in the proof of [Ket1, Theorem 1.3] show that there is a $g : I \rightarrow S(\mu)$ such that $\{i \in I \mid |g(i)| = \text{cf } f(i)\} \in D$ and for every $\alpha \in \mu$ $\{i \in I \mid \alpha \in g(i)\} \in D$. Then the conclusion is immediate from Proposition 7.4(a). \square

Corollary 7.5 improves [Ket1, Theorem 5.6].

Notice that if D has a μ -least function, it is not necessarily the case that $D \times D$ has a μ -least function (see Remark 8.5(b))

It is natural to ask whether Proposition 7.4 can be improved (and simplified) to: “ $\sum_D D_i$ is (λ, μ) -regular if and only if there is a cardinal ν such that D is (ν, μ) -regular and for every $\nu' < \nu$ $\{i \in I \mid D_i \text{ is } (\lambda, \nu')\text{-regular}\} \in D$ ”. The next example shows that the above statement is false.

Let D be $(\omega_\omega, \omega_{\omega+1})$ -regular, but not (ω_n, ω_n) -regular for $n > 0$ (as we mentioned, [BM] produced such an ultrafilter). By Theorem 6.2 and 1.1(i), D has an $\omega_{\omega+1}$ -least function f and, by Theorem 6.3 and 1.1(i), for every $n < \omega$ $\{i \in I \mid \text{cf } f(i) > \omega_n\} \in D$. For every $i \in I$, if $\text{cf } f(i) = \omega_n$ and $n > 0$, let D_i be an (ω, ω_{n-1}) -regular ultrafilter over ω_{n-1} ; notice that $\{i \in I \mid \text{cf } f(i) = \omega\} \notin D$, so that if $\text{cf } f(i) = \omega$ then D_i can be chosen arbitrarily. Corollary 7.5 implies that $\sum_D D_i$ is not $(\omega, \omega_{\omega+1})$ -regular, but for every $n < \omega$ $\{i \in I \mid D_i \text{ is } (\omega, \omega_n)\text{-regular}\} \in D$, since $\{i \in I \mid \text{cf } f(i) > \omega_{n+1}\} \in D$.

Notice that, in the above example, $\sum_D D_i$ is $(\omega_\omega, \omega_{\omega+1})$ -regular, (ω, ω_n) -regular for all $n < \omega$, hence (ω, ω_ω) -regular by Proposition 5.4, but not $(\omega_n, \omega_{\omega+1})$ -regular for $n < \omega$, again by Corollary 7.5.

The following is a generalization of [Ket1, Theorem 5.9].

Proposition 7.6. *For every cardinals λ, μ, χ , and for every set \mathcal{K} of cardinals, the following are equivalent:*

- (a) *There is a χ -complete (λ, μ) -regular ultrafilter which for no $\kappa \in \mathcal{K}$ is (κ, κ) -regular.*
- (b) *For every ν with $\lambda \leq \nu \leq \mu$ there is a χ -complete (ν, ν) -regular ultrafilter which for no $\kappa \in \mathcal{K}$ is (κ, κ) -regular.*
- (c) *For every ν with $\lambda \leq \nu \leq \mu$ there are an $n \in \omega$ and a χ -complete (ν^{+n}, ν^{+n}) -regular ultrafilter which for no $\kappa \in \mathcal{K}$ is (κ, κ) -regular.*

Proof. (a) \Rightarrow (b) is trivial, by 1.1(i).

First, we prove the converse in the case $\chi = \omega$. Let D_ν ($\lambda \leq \nu \leq \mu$) be ultrafilters as given by (b). Construct inductively, for each ν with $\lambda \leq \nu \leq \mu$, a chain of models \mathbf{A}_ν as follows.

\mathbf{A}_λ is $\prod_{D_\lambda} \langle S_\lambda(\mu), S_\kappa(\kappa), \subseteq, \{\alpha\} \rangle_{\kappa \in \mathcal{K}, \alpha \in \mu \cup \sup \mathcal{K}}$;

\mathbf{A}_{ν^+} is $\prod_{D_{\nu^+}} \mathbf{A}_\nu$; and

\mathbf{A}_ν is $\prod_{D_\nu} (\lim_{\nu' < \nu} \mathbf{A}_{\nu'})$, if ν is limit, where $\lim_{\nu' < \nu} \mathbf{A}_{\nu'}$ denotes the direct limit of the $\mathbf{A}_{\nu'}$'s with respect to the natural embeddings.

Iterating the arguments in the proof of Proposition 7.1 it can be shown by induction on ν ($\lambda \leq \nu \leq \mu$) that whenever $X \subseteq \mu$ and $|X| \leq \nu$ then there is $x_X \in \mathbf{A}_\nu$ such that $d(\{\alpha\}) \subseteq x_X$ for every $\alpha \in X$. In other words, for every ν , \mathbf{A}_ν contains a copy of $S_{\nu^+}(\mu)$.

The base $\nu = \lambda$ of the induction is just Form III of the definition of (λ, λ) -regularity, since if $|X| = \lambda$ then $S_\lambda(X)$ is isomorphic to $S_\lambda(\lambda)$.

The successor step is dealt exactly as in the proof of 7.1: by the inductive hypothesis, \mathbf{A}_ν contains a copy of $S_{\nu^+}(\mu)$, hence, by the (ν^+, ν^+) -regularity of D_{ν^+} , $\mathbf{A}_{\nu^+} = \prod_{D_{\nu^+}} \mathbf{A}_\nu$ contains a copy of $S_{\nu^{++}}(\mu)$.

The case ν limit is similar: $\lim_{\nu' < \nu} \mathbf{A}_{\nu'}$ contains a copy of $S_{\nu'}(\mu)$ for each $\nu' < \nu$, hence a copy of $\bigcup_{\nu' < \nu} S_{\nu'}(\mu) = S_\nu(\mu)$, thus $\prod_{D_\nu} (\lim_{\nu' < \nu} \mathbf{A}_{\nu'})$ contains a copy of $S_{\nu^+}(\mu)$, by the (ν, ν) -regularity of D_ν .

Thus, in the final model \mathbf{A}_μ there is an element x such that $d(\{\alpha\}) \subseteq x$ for every $\alpha \in \mu$; now, \mathbf{A}_μ is a *complete extension* (see [CK, Section 6.4]) of $\langle S_\lambda(\mu), \subseteq \rangle$, and by [CK, Theorem 6.4.4] there is an ultrafilter D such that $\prod_D \langle S_\lambda(\mu), \subseteq \rangle$ is embeddable in \mathbf{A}_μ and x is in the range of the embedding. Thus, D is (λ, μ) -regular (Form III).

Let $\kappa \in \mathcal{K}$. Since no D_ν is (κ, κ) -regular, we have that D is not (κ, κ) -regular, again by using the arguments in the proof of Proposition 7.1. Indeed, at no stage of the construction of the \mathbf{A}_ν 's there can appear an

element witnessing (κ, κ) -regularity. A fortiori, no such element can be in $\prod_D S_\kappa(\kappa)$. (This is the reason why we have included the $S_\kappa(\kappa)$'s in our models).

Having proved (b) \Rightarrow (a) in the case $\chi = \omega$, let now χ be arbitrary. Since, as we mentioned in the introduction, an ultrafilter D is χ -complete if and only if for no $\chi' < \chi$ D is (χ', χ') -regular, then the result for χ -complete ultrafilters follows from the case $\chi = \omega$, by appropriately extending the set \mathcal{K} .

Thus, (b) \Rightarrow (a) is proved.

(c) is equivalent to (b) by Theorem 2.1(b). \square

Less direct proofs of Proposition 7.6 can be obtained from the proof of [Lp2, Theorem 3] (stated here as Theorem 4.5) or from the proof of [Lp11, Theorem 10] (cf. also [Lp9, Theorem 7] ; notice that there the order of λ and μ is exchanged, in the definition of regularity).

If I is a finite set, we can generalize Proposition 7.6 to the effect that there is a χ -complete ultrafilter which is (λ_i, μ_i) -regular for every $i \in I$ and not (κ, κ) -regular for $\kappa \in \mathcal{K}$ if and only if for every $i \in I$ and for every ν with $\lambda_i \leq \nu \leq \mu_i$ there is a χ -complete ultrafilter which is (ν, ν) -regular and not (κ, κ) -regular for $\kappa \in \mathcal{K}$.

This is because for each $i \in I$ Proposition 7.6 gives a (λ_i, μ_i) -regular not (κ, κ) -regular ultrafilter D_i , and then, letting $I = \{i_1, \dots, i_n\}$, $D_{i_1} \times D_{i_2} \times \dots \times D_{i_n}$ is the desired ultrafilter, by Proposition 7.1.

The above statement is not true when I is infinite, already in the case $\lambda_i = \mu_i$. Suppose that GCH holds, and that $(\mu_i)_{i \in \omega}$ is a strictly increasing sequence of measurable cardinals, and let $\kappa = \sup \mu_i$. Then for every $i < \omega$ there is a (μ_i, μ_i) -regular not (κ, κ) -regular ultrafilter; but GCH, 1.1(vii)(xi) and Theorems 5.1(a) and 2.1(b) imply that any ultrafilter which is (μ_i, μ_i) -regular for every $i < \omega$ is (κ, κ) -regular.

8. FURTHER REMARKS.

In this section we add a few disparate remarks (of course, we state some problems, too, in order to keep on with the tradition).

Proposition 8.1. *Suppose that λ, μ, κ are regular cardinals, and that there is a sequence $(f_\alpha)_{\alpha \in \kappa}$ of functions from λ to μ which is increasing modulo eventual dominance (that is, for every $\alpha < \beta < \kappa$ there is $\gamma < \lambda$ such that $f_\alpha(\delta) < f_\beta(\delta)$, for every $\delta > \gamma$), and suppose that there is no function from λ to μ which eventually dominates all the f_α 's. Then every (κ, κ) -regular ultrafilter is either (λ, λ) -regular or (μ, μ) -regular.*

Proof. Consider a model \mathbf{A} with unary predicates U, V, W representing κ, λ, μ respectively, with a binary predicate $<$ representing the well

orders of κ, λ, μ , and a ternary relation R such that for $\alpha \in \kappa$ $R(\alpha, -, -)$ represents the diagram of f_α . Thus, **A** satisfies

$$\forall xy(U(x) \wedge U(y) \wedge x < y \Rightarrow \exists z(V(z) \wedge \forall z' > z(V(z') \Rightarrow \forall w, w'(R(x, z', w) \wedge R(y, z', w') \Rightarrow w < w'))))$$

Let D be an ultrafilter, consider the ultrapower of the above model, and recall that, by 1.1(xi), if κ is regular, then κ -descending incompleteness is equivalent to (κ, κ) -regularity. If D is κ -descendingly incomplete, then in $\prod_D U$ there is an element x greater than all $d(\alpha)$'s ($\alpha \in \kappa$). Now, $R(x, -, -)$ is the diagram of a function g from $\prod_D V$ to $\prod_D W$, and, by the above-displayed formula and Łoś Theorem, g eventually dominates all the functions with diagram given by $R(d(\alpha), -, -)$ ($\alpha \in \kappa$).

If D is not (λ, λ) -regular, that is, not λ -descendingly incomplete, then the z whose existence is asserted by (*) is bounded by some $d(\gamma)$ with $\gamma < \lambda$, hence, without loss of generality, we can assume $z = d(\gamma)$. In particular, for every $\alpha \in \kappa$ there is $\gamma_\alpha \in \lambda$ such that, from γ_α on, g dominates the function with diagram given by $R(d(\alpha), -, -)$.

If D is not (μ, μ) -regular, that is, not μ -descendingly incomplete, define, for $\gamma \in \lambda$, $g'(\gamma) = \inf\{\delta \in \mu \mid g(d(\gamma)) \leq d(\delta)\}$. Thus, $g' : \lambda \rightarrow \mu$ and $g|_\lambda \leq g'$ pointwise. Translating (in **A**) the fact that, from γ_α on, g dominates the function with diagram given by $R(d(\alpha), -, -)$, we have that, from γ_α on, g' dominates f_α , thus g' dominates all the f_α 's, contradicting our assumption. \square

The above proposition might be relevant to the problems discussed in Section 3. See [BK], [KM, p. 180] and [Lp1, Theorem 0.25] for connections between regularity of ultrafilters and the existence of families of eventually different functions. Probably, the argument in Proposition 8.1 can be elaborated further.

In some cases, we can prove Theorem 2.2 without the hypothesis $\text{cf}\mu \neq \text{cf}\lambda$. The simplest case is when $\text{cf}\lambda = \text{cf}\mu = \omega$, λ has the form $\nu^{+\omega}$ for some ν , while $\mu = \sup_{n \in \omega} \mu_n$, where the μ_n 's can be chosen to be limit cardinals. This is a consequence of the next proposition (it is case $\alpha = 2$). In order to prove the general form, we need a definition.

If $\text{cf}\mu = \omega$, define as follows the relation “the *order* of μ is $\geq \alpha$ ”, for $\alpha \neq 0$ an ordinal.

The order of every μ of cofinality ω is ≥ 1 ;

If $\alpha > 1$, the order of μ is $\geq \alpha$ if and only if for every $\beta < \alpha$ μ is a limit of some sequence of cardinals, each of cofinality ω and of order $\geq \beta$.

Of course, we could define the order of μ to be the least α such that the order of μ is $\geq \alpha$, but we shall not actually need this.

Proposition 8.2. *Suppose that λ and μ are infinite cardinals, and*
 (a) *$\alpha > 0$ and $\alpha \leq$ the first weakly inaccessible cardinal (or there is none); and*
 (b) *μ has cofinality ω and order $\geq \alpha$; and*
 (c) *$\lambda < \mu$, and $\lambda = \nu^{+\gamma}$, for some $\gamma < \omega^\alpha$ (ordinal exponentiation), and some regular cardinal ν .*

If D is a (λ^+, μ) -regular ultrafilter, then D is (λ, μ) -regular. Moreover, D is either (ω, ω) -regular, or (ν, μ) -regular.

Proof. If $\text{cf } \lambda > \omega$ then D is (λ, μ) -regular by Theorem 2.15(ii).

If $\text{cf } \lambda = \omega$ and D is (ω, ω) -regular then D is (λ, μ) -regular by Theorem 2.13(iii) and 1.5(a), since $\text{cf } \prod_D \lambda = \text{cf } \prod_D \text{cf } \lambda$.

We shall prove by induction on α that the proposition is true for every ultrafilter which is not (ω, ω) -regular.

The case $\alpha = 1$ is a particular case of Theorem 2.15(ii), since in this case $\lambda = \nu^{+n}$ for some $n < \omega$, and ν is regular.

Suppose the statement of the proposition true for all $\beta < \alpha$, and let $\lambda, \mu, \nu, \gamma$ and D be given. Let δ be the smallest ordinal such that D is $(\nu^{+\delta}, \mu)$ -regular. Notice that $\delta \leq \gamma + 1$, since D is $(\nu^{+\gamma+1}, \mu)$ -regular, thus $\nu^{+\delta} < \mu$. Notice also that $\gamma + 1 < \omega^\alpha$, since $\alpha > 0$, hence ω^α is limit.

We want to show that $\delta = 0$. If not, by Theorem 2.15(ii), either δ is limit, or $\delta = \varepsilon + 1$ and $\text{cf } \varepsilon = \omega$.

Since D is not (ω, ω) -regular, D is ω_1 -complete, hence μ^* -complete, where μ^* is the first measurable cardinal, hence for all $\kappa < \mu^*$ D is not (κ, κ) -regular. We now show that δ is not limit. If δ is limit, then, since $\delta \leq \gamma + 1 < \omega^\alpha$, we necessarily have $\text{cf } \delta < \sup\{\alpha, \omega_1\} \leq$ the first weakly inaccessible cardinal, hence δ cannot be limit by Proposition 2.6, since then $\nu^{+\delta}$ would be a singular cardinal, δ being smaller than the first weakly inaccessible cardinal, and since D is not $(\text{cf } \nu^{+\delta}, \text{cf } \nu^{+\delta})$ -regular, $\text{cf } \nu^{+\delta}$ being smaller than the first measurable cardinal.

So, let $\delta = \varepsilon + 1$ and $\text{cf } \varepsilon = \omega$. Since $\varepsilon < \delta \leq \gamma + 1 < \omega^\alpha$, then, by expressing ε in normal form, we get that ε has the form $\varepsilon' + \omega^\beta$, for some $\beta < \alpha$, $\beta > 0$. By hypothesis, μ has order $\geq \alpha$, so that $\mu = \sup_{n \in \omega} \mu_n$ for certain μ_n 's of cofinality ω and order $\geq \beta$. Without loss of generality, we can assume that $\mu_n > \nu^{+\varepsilon}$, for every $n \in \omega$, since $\varepsilon \leq \gamma$, $\sup_{n \in \omega} \mu_n = \mu > \lambda = \nu^{+\gamma}$, and μ is limit.

Let us fix n . By the definition of δ , D is $(\nu^{+\delta}, \mu)$ -regular, that is, $(\nu^{+\varepsilon+1}, \mu)$ -regular, hence, by 1.1(i), $(\nu^{+\varepsilon+1}, \mu_n^+)$ -regular, and $(\nu^{+\varepsilon}, \mu_n)$ -regular by Theorem 2.13(ii). By Proposition 2.6, and since D is not (ω, ω) -regular, there is $\eta < \varepsilon = \varepsilon' + \omega^\beta$ such that D is $(\nu^{+\eta}, \mu_n)$ -regular; hence either $\eta \leq \varepsilon'$ or $\eta = \varepsilon' + \eta'$ with $\eta' < \omega^\beta$. In both cases, D is

$(\nu^{+\varepsilon'+1}, \mu_n)$ -regular: if $\eta \leq \varepsilon'$, by 1.1(i); if $\eta = \varepsilon' + \eta'$ with $\eta' < \omega^\beta$ by the inductive hypothesis with β in place of α , μ_n in place of μ , η or $\eta - 1$ in place of γ , $\nu^{+\varepsilon'+1}$ in place of ν , and ν^{+n} in place of λ), and since $\nu^{+\varepsilon'+1}$ is a regular cardinal.

Since ε' does not depend on n , then D is $(\nu^{+\varepsilon'+1}, \mu)$ -regular by Proposition 5.4(a), but this contradicts the choice of δ , so $\delta = 0$ and D is (ν, μ) -regular. \square

Proposition 8.2 can be used in order to improve Proposition 5.4 a bit. Anyone of the assumptions (a)-(d) in the hypotheses of Proposition 5.4 can be replaced by the conjunction of the assumptions (a)-(c) stated in the hypothesis of Proposition 8.2. We do not know whether the hypothesis $\text{cf}\lambda \neq \text{cf}\kappa$ in Theorem 6.5 (cases (a)(b)), too, can be replaced by the conjunction of the assumptions (a)-(c) of Proposition 8.2.

We do not know whether a result similar to Proposition 8.2 can be proved for cardinals λ, μ of cofinality $> \omega$. The argument in the proof of 8.2 breaks since if D is not $(\text{cf}\lambda, \text{cf}\lambda)$ -regular, nonetheless D might be (λ', λ') -regular for some $\lambda' < \text{cf}\lambda$. In other words, we can prove a result analogous to 8.2 only if we assume $(\text{cf}\lambda)^+$ -completeness. The situation is similar to the proof of Corollary 5.3.

We turn to another kind of problem.

Problem 8.3. Are the regularity properties of an ultrafilter D determined by the function $\lambda \rightarrow \text{cf}(\prod_D \langle \lambda, < \rangle)$?

More precisely, is it always true that if D and D' are ultrafilters and $\text{cf}(\prod_D \langle \lambda, < \rangle) = \text{cf}(\prod_{D'} \langle \lambda, < \rangle)$ for every cardinal λ then for every pair of cardinals ν, μ D is (μ, ν) -regular if and only if D' is (μ, ν) -regular?

Theorem 2.13 suggests that the answer to Problem 8.3 might be affirmative.

Problem 8.4. Which results of the present paper generalize to the notion $(\lambda, \mu) \Rightarrow (\lambda', \mu')$ of [Lp1, Definition 0.12] or [L13, p. 139]. See also [Lp3, Definition 1.2] and [L12].

We suspect that most results generalize; the problem is whether we need the parameter κ of [Lp1, 0.21(c)] and, if this is necessary, to determine the smallest possible value of this parameter. Most results in Section 5 do not generalize, unless in the definition of $(\lambda, \mu) \Rightarrow (\lambda', \mu')$ one considers only *simple* extensions (that is, extensions generated by a single element: in order to make sense, one has to deal only with models having Skolem functions).

Remarks 8.5. (a) The notions of a least function and of a κ -least function (Definition 6.1) are interesting only in the case κ regular. Indeed, an ultrafilter D has a κ -least function if and only if it has a cf κ -least function, as we are going to show.

Let κ be singular, and let $C = (\beta_\alpha)_{\alpha \in \text{cf } \kappa}$ be a sequence of order type $\text{cf } \kappa$ closed and unbounded in κ .

If D is over I , and $g : I \rightarrow \kappa$ is a κ -least function, let $h : I \rightarrow \kappa$ be defined by $h(i) = \sup\{\beta_\alpha \in C \mid \beta_\alpha \leq g(i)\}$: for every i , $h(i)$ belongs to C , since C is closed. Moreover, $h(i) \leq g(i)$. Notice that, for every $\alpha \in \text{cf } \kappa$, $\beta_\alpha \leq h(i)$ if and only if $\beta_\alpha \leq g(i)$. It cannot be the case that h is bounded mod D by some $\beta < \kappa$: were this the case, take $\beta_\alpha \in C$ such that $\beta_\alpha \geq \beta$; then $\{i \in I \mid \beta < h(i)\} \supseteq \{i \in I \mid \beta_\alpha < h(i)\} \in D$. Since g is a κ -least function, then $\{i \in I \mid h(i) = g(i)\} \in D$. This implies that the function h' defined by $h'(i) = \text{the } \alpha \in \kappa \text{ such that } \beta_\alpha = h(i)$ is a cf κ -least function.

Conversely, suppose that $h : I \rightarrow \text{cf } \kappa$ is a cf κ -least function. Without loss of generality, we can suppose that $\{i \in I \mid h(i) \text{ is limit}\} \in D$. Define $g : I \rightarrow \kappa$ by $g(i) = \beta_{h(i)}$. We want to show that g is a κ -least function. Indeed, g is clearly unbounded in $\kappa \pmod{D}$. If $f : I \rightarrow \kappa$, $f \leq g \pmod{D}$ and f is unbounded in $\kappa \pmod{D}$, let $f' : I \rightarrow \text{cf } \kappa$ be defined by $f'(i) = \text{the least } \alpha \in \text{cf } \kappa \text{ such that } f(i) \leq \beta_\alpha$. f' is unbounded in $\text{cf } \kappa \pmod{D}$, since were f' bounded \pmod{D} by $\alpha \in \text{cf } \kappa$, then f would be bounded \pmod{D} in κ by β_α , absurd. Since h is a cf κ -least function, then $h \leq f' \pmod{D}$; thus $g(i) = \beta_{h(i)} \leq \beta_{f'(i)}$ for i in a set in D . Thus, for a set in D , $f(i) \leq g(i) \leq \beta_{f'(i)}$. Since the image of g is contained in C , by the definition of f' , we have $g(i) = \beta_{f'(i)}$. Again by the definition of f' , and since $g(i)$ is a limit point of C , we get $f(i) = g(i)$. Thus, $f = g \pmod{D}$, hence g is a κ -least function.

(b) It may happen that D has a least function but $D \times D$ has no least function. [FMS] constructed an ultrafilter D uniform over ω_1 , hence (ω_1, ω_1) -regular, which is not (ω, ω_1) -regular, hence D has an ω_1 -least function (equivalently, a least function, since D is uniform over ω_1) by [BK, Section 2], [Ka1, Theorem 2.3]. However $D \times D$ is (ω, ω_1) -regular by Theorem 7.1, and cannot have a least function by [Ket1, Theorem 2.4] (see Theorem 6.3).

(c) The above remarks lead to the following definition. If $\langle X, \leq \rangle$ is a linear order, let us say that an ultrafilter D over I has an $\langle X, \leq \rangle$ -least function (or, simply, an X -least function, if the order on X is understood) if and only if there exists a function $f : I \rightarrow X$ such that $\{i \in I \mid x < f(i)\} \in D$, for every $x \in X$, yet for every $g : I \rightarrow X$, $\{i \in I \mid g(i) < f(i)\} \in D$ implies that there is $x \in X$ such that $\{i \in$

$I|g(i) < x\} \in D$. In other words, in $\prod_D \langle X, \leq \rangle$, f_D is the least element larger than all the $d(x)$'s ($x \in X$).

Thus, a κ -least function as defined in 6.1 is the same as a $\langle \kappa, \leq \rangle$ -least function in the above sense.

Of course, there are orders X for which the existence of an X -least function is forbidden, for example, take X to be any order in which every element has an immediate predecessor (by elementarity, this is true in $\prod_D X$, too, hence no least function is possible).

Notice that if D is the ultrafilter given in (b), then D has an ω_1 -least function but not an X -least function, where $X = \{x \in \prod_D \omega_1 \mid \text{for some } \alpha < \omega_1, x < d(\alpha)\}$.

However, the arguments in (a) give the following. Suppose that X is a linear order, and $C \subseteq X$ is a well order cofinal in X of type κ (κ a regular cardinal) and such that

(*) whenever $H \subseteq C$ is nonempty, $\sup H$ computed in X exists and is the same as computed in C .

Then an ultrafilter D has an X -least function if and only if it has a κ -least function.

(d) The remarks in (b) and (c) above are connected as follows. We get a condition under which $D \times D'$ has a κ -least function.

Suppose that D, D' are ultrafilters, and let $X = \{x \in \prod_{D'} \kappa \mid \text{for some } \alpha < \kappa, x < d(\alpha)\}$. Thus, X has cofinality κ . If there exists a well order C cofinal in X and satisfying (*) above, then D has a κ -least function if and only if $D \times D'$ has a κ -least function: just apply (c) with X as above, recalling that $\prod_{D \times D'} \kappa \cong \prod_D \prod_{D'} \kappa$.

In Proposition 2.11 we showed that if D is a (λ, λ) -regular non $(\text{cf } \lambda, \text{cf } \lambda)$ -regular ultrafilter then \square_λ fails. We can show that \square_μ fails for many more cardinals.

Proposition 8.6. *Suppose that λ is singular, and D is a (λ, λ) -regular not $(\text{cf } \lambda, \text{cf } \lambda)$ -regular ultrafilter. Then there is $\mu < \lambda$ such that for every κ with $\mu \leq \kappa \leq \lambda$ D is (κ^+, κ^+) -regular and \square_κ fails. Moreover, either*

(a) μ is singular, or

(b) $\lambda = \mu^{+\omega}$, μ is a regular limit ω -weakly-Mahlo cardinal, and D is (μ, λ) -regular.

Proof. We shall show that there exists some $\mu < \lambda$ such that $\mu > \text{cf } \lambda$ and D is (κ^+, κ^+) -regular for every κ with $\mu \leq \kappa \leq \lambda$. This necessarily implies that \square_κ fails. Suppose by contradiction that \square_κ holds for some κ with $\mu \leq \kappa \leq \lambda$. Then Theorem 2.10 implies that D is (κ', κ') -regular for every $\kappa' \leq \mu$, but this contradicts the assumption that D is not $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, since $\mu > \text{cf } \lambda$.

Now we show how to find μ as above. By Proposition 2.6, D is (λ', λ) -regular for some $\lambda' < \lambda$. By Corollary 5.8 and 1.1(xi), D is (λ^+, λ^+) -regular.

Suppose that λ has not the form $\nu^{+\omega}$ for some ν . Then there is a singular cardinal μ with $\lambda' < \mu < \lambda$ and $\mu > \text{cf } \lambda$, hence by 1.1(i) D is (κ^+, κ^+) -regular for every κ with $\mu \leq \kappa < \lambda$, and we fall in case (a).

Hence we can suppose that λ has the form $\nu^{+\omega}$ for some ν ; hence $\text{cf } \lambda = \omega$; moreover, without loss of generality, we can suppose that ν is a limit cardinal. D is (ν^+, λ) -regular by [Lp5, Corollary B], stated here as Theorem 2.15(ii), hence, by 1.1(i), D is (κ^+, κ^+) -regular for every κ with $\nu \leq \kappa < \lambda$.

If ν is singular, just take $\mu = \nu$, and we are in case (a). Hence, suppose that ν is a regular cardinal. Again by [Lp5, Corollary B], D is (ν, λ) -regular. Since $\text{cf } \lambda = \omega$ and D is not $(\text{cf } \lambda, \text{cf } \lambda)$ -regular, then necessarily $\nu > \omega$. If ν is ω -weakly-Mahlo, then (b) holds with $\mu = \nu$. By 1.1(i), D is (ν, ν) -regular, and if ν is not ω -weakly-Mahlo, then the result from [ČČ] stated here as Theorem 2.19 implies that D is either (μ, ν) -regular for some $\mu < \nu$, or (μ, μ) -regular for all $\mu < \nu$. By 1.1(i), and since ν is a limit cardinal of uncountable cofinality, we can choose μ to be a singular cardinal in such a way that $\mu > \text{cf } \lambda = \omega$. Hence we are in case (a). \square

Parts of the above proof of Proposition 8.6 essentially appeared, in a somewhat hidden form, in the course of the proof of [Lp1, Proposition 0.22].

The next proposition appeared in a previous version of this paper, where it has been used in order to show that if μ is a singular cardinal of cofinality ω , $\lambda^\omega < \mu$ and the ultrafilter D is (λ, μ) -regular, then D is either μ -decomposable or μ^+ -decomposable, a result now subsumed by Theorem 5.7, via 1.1(i).

However, the next proposition appears to be of independent interest; the main idea of its proof is probably due to R. Solovay (see [Ket1, p. 74]).

Proposition 8.7. *If D is (λ, μ) -regular and κ -complete then D is $((\lambda^{<\kappa})^+, \mu^{<\kappa})$ -regular. Actually, if $\nu^{<\kappa} < \lambda^{<\kappa}$ for every $\nu < \lambda$ then D is $(\lambda^{<\kappa}, \mu^{<\kappa})$ -regular.*

Proof. Let X_α ($\alpha \in \mu$) witness the (λ, μ) -regularity of D (Form I). For $y \subseteq \mu$ with $|y| < \kappa$ let $Z_y = \bigcap_{\alpha \in y} X_\alpha$. Each Z_y is in D , since D is κ -complete. If $x \subseteq \mu$ with $|x| < \lambda$ then $|\{Z_y \mid y \subseteq x, |y| < \kappa\}| \leq |x|^{<\kappa}$. Thus, if $Y \subseteq S_\kappa(\mu)$ and $|Y| > \nu^{<\kappa}$ for every $\nu < \lambda$, then $|\bigcup Y| \geq \lambda$; hence $\bigcap_{y \in Y} Z_y = \bigcap_{\alpha \in \bigcup Y} X_\alpha = \emptyset$. Since $|S_\kappa(\mu)| = \mu^{<\kappa}$ the Z_y 's

witness the $((\lambda^{<\kappa})^+, \mu^{<\kappa})$ -regularity of D , respectively, the $(\lambda^{<\kappa}, \mu^{<\kappa})$ -regularity of D if the assumption in the second statement holds. \square

The following interesting corollary is hardly mentioned in the literature.

Corollary 8.8. *If κ is μ -compact, μ is singular and $\text{cf } \mu < \kappa$ then κ is μ^+ -compact.*

Proof. By μ -compactness there is a κ -complete (κ, μ) -regular ultrafilter D . By Proposition 8.7 with $(\text{cf } \mu)^+$ in place of κ , D is $(\kappa^{\text{cf } \mu}, \mu^{\text{cf } \mu})$ -regular, since κ is strongly inaccessible. Since $\mu^{\text{cf } \mu} > \mu$, then D is (κ, μ^+) -regular by 1.1(i), hence κ is μ^+ -compact. \square

Some of the results presented in this paper have been obtained in 1995, while the author was visiting the University of Cagliari. A preliminary version of this paper has been circulating since 1996. That version contained essentially all the results proved in Sections 2, 3 and 5 here. The introduction, too, had been written in 1996 (at that time, one had the feeling that independence results were taken in much more consideration than ZFC results; now things are rapidly changing).

Slightly less general versions of the results in Section 4 have been announced in the abstract [Lp7].

We have announced further results about regularity of ultrafilters in the abstracts [Lp6] and [Lp7]; however we have found a gap in a proof, so that some statements in [Lp6, Lp7] have to be considered as problems, or conjectures, so far.

This work has been performed under the auspices of GNSAGA (CNR).

Problem 8.9. Which results about regularity of ultrafilters (in particular, which results of the present paper) hold assuming just the Prime Ideal Theorem, rather than the Axiom of Choice?

We wish to express our warmest gratitude to an anonymous referee for a careful reading of the manuscript, for a great deal of suggestions that helped improve the exposition and for detecting some inaccuracies. Last but not least, we appreciate encouragement in our efforts to “keep this neglected area of set theory alive”.

REFERENCES

- [AJ] A. Adler and M. Jorgensen, Descendingly incomplete ultrafilters and the cardinality of ultrapowers, *Can. J. Math.* **24**, 830–834 (1972).
- [Ap] A. W. Apter, An Easton theorem for level by level equivalence, *Math. Log.* **51**, 247–253 (2005).
- [AH] A. W. Apter and J. M. Henle, On box, weak box and strong compactness, *Bull. London Math. Soc.* **24**, 513–518 (1992).

- [BF] J. Barwise and S. Feferman (eds.), *Model-theoretic logics*, Berlin (1985).
- [BS] J. L. Bell and A. B. Slomson, *Models and ultraproducts: an introduction*, Amsterdam (1969).
- [BM] S. Ben-David and M. Magidor, The weak \square^* is really weaker than the full \square , *J. Symbolic Logic* **51**, 1029–1033 (1986).
- [BK] M. Benda and J. Ketonen, Regularity of ultrafilters, *Israel J. Math.* **17**, 231–240 (1974).
- [Ca] X. Caicedo, The Abstract Compactness Theorem revisited, in: *Logic in Florence*, edited by A. Cantini, E. Casari, P. Minari (1999), 131–141.
- [Ch] C. C. Chang, Descendingly incomplete ultrafilters, *Trans. Amer. Math. Soc.* **126**, 108–111 (1967).
- [CK] C. C. Chang and J. Keisler, *Model Theory*, Amsterdam (1977).
- [CN] W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Berlin (1974).
- [ČČ] G. V. Čudnovskii and D. V. Čudnovskii, Regular and descending incomplete ultrafilters (English translation), *Soviet Math. Dokl.* **12**, 901–905 (1971).
- [DD] O. Deiser and H. D. Donder, Canonical functions, non-regular ultrafilters and Ulam’s problem on ω_1 , *J. Symbolic Logic* **68**, 713–739 (2003).
- [DN] M. Di Nasso, Hyperordinals and nonstandard α -models, in: *Logic and algebra* (Pontignano, 1994), 457–475, New York (1996).
- [Do] H. D. Donder, Regularity of ultrafilters and the core model, *Israel J. Math.* **63**, 289–322 (1988).
- [DJK] H. D. Donder, R. B. Jensen and B. J. Koppelberg, Some applications of the core model, in: *Set Theory and Model Theory*, edited by R. B. Jensen and A. Prestel, 55–97, Berlin (1981).
- [Eb] H.-D. Ebbinghaus, *Extended logics: the general framework*, Chapter II in [BF].
- [Fo] M. Foreman, An \aleph_1 -dense ideal on \aleph_2 , *Israel J. Math.* **108**, 253–290 (1998).
- [FMS] M. Foreman, M. Magidor and S. Shelah, Martin’s Maximum, saturated ideals and non-regular ultrafilters. Part II, *Annals of Mathematics* **127**, 521–545, (1988).
- [GF] S. Garcia Ferreira, On two generalizations of pseudocompactness, in *Proceedings of the 14th Summer Conference on General Topology and its Applications* (Brookville, NY, 1999). *Topology Proc.* **24** (1999), Summer, 149–172 (2001).
- [Ha] A. Hajnal, Ulam-matrices for Inaccessible Cardinals, *Bulletin de l’Academie Polonaise des Sciences Series de Sciences Math. Astr. Phys.* **XVII** (11), 683–688 (1969).
- [Hu] M. Huberich, Non-regular ultrafilters, *Israel J. Math.* **87**, 275–288 (1994).
- [JP] T. Jech and K. Prikry, Ideals over uncountable sets: application of almost disjoint functions and generic ultrapowers, *Mem. Amer. Math. Soc.* **18**, no. 214 (1979).
- [Jo] M. Jorgensen, Regular ultrafilters and long ultrapowers, *Canad. Math. Bull.* **18**, 41–43 (1975).
- [Ka] A. Kanamori, *The Higher Infinite*, Berlin (1994).
- [Ka1] A. Kanamori, Weakly normal filters and irregular ultrafilters, *Trans. Amer. Math. Soc.* **220**, 393–399 (1974).
- [Ka2] A. Kanamori, Finest partitions for ultrafilters, *J. Symbolic Logic* **51**, 327–332 (1986).

- [KM] A. Kanamori and M. Magidor, The evolution of large cardinal axioms in Set Theory, in: Higher Set Theory, edited by G. H. Müller and D. S. Scott, 99–275, Berlin (1978).
- [KSV] J. Kennedy, S. Shelah and J. Väänänen, Regular ultrafilters and finite square principles, *J. Symb. Logic* **73**, 817–823 (2008).
- [Kei] J. Keisler, On cardinalities of ultraproducts, *Bull. Amer. Math. Soc.* **70**, 644–647 (1964).
- [Ket] J. Ketonen, Nonregular ultrafilters and large cardinals, *Trans. Amer. Math. Soc.* **224**, 61–73 (1976).
- [Ket1] J. Ketonen, Strong compactness and other cardinal sins, *Ann. Math. Logic* **5**, 47–76 (1972).
- [Ket2] J. Ketonen, Some combinatorial properties of ultrafilters, *Fundamenta Mathematicae* **107**, 225–235 (1980).
- [Ko] P. Komjáth, A second category set with only first category functions, *Proc. Amer. Math. Soc.* **112**, 1129–1136 (1991).
- [KP] K. Kunen and K. L. Prikry, On descendingly incomplete ultrafilters, *J. Symbolic Logic* **36**, 650–652 (1971).
- [KV] K. Kunen and J. E. Vaughan (editors), *Handbook of Set Theoretical Topology*, Amsterdam (1984).
- [Lp1] P. Lipparini, Ultrafilter translations, I: (λ, λ) -compactness of logics with a cardinality quantifier, *Arch. Math. Logic* **35**, 63–87 (1996).
- [Lp2] P. Lipparini, Productive $[\lambda, \mu]$ -compactness and regular ultrafilters, *Topology Proceedings* **21**, 161–171 (1996).
- [Lp3] P. Lipparini, The compactness spectrum of abstract logics, large cardinals and combinatorial principles, *Boll. Unione Matematica Italiana ser. VII* **4-B**, 875–903 (1990).
- [Lp4] P. Lipparini, Compactness of cardinality logics and constructibility, in: *Proceedings of the 10th Easter Conference on Model Theory*, Seminarberichte of Humboldt Universität No 93-1, 130–133, Berlin (1993).
- [Lp5] P. Lipparini, Every (λ^+, κ^+) -regular ultrafilter is (λ, κ) -regular, *Proc. Amer. Math. Soc.* **128**, 605–609 (1999).
- [Lp6] P. Lipparini, Decomposable ultrafilters and cardinal arithmetic (abstract), *Bull. Symbolic Logic* **5**, 281–282 (1999).
- [Lp7] P. Lipparini, Still more on regular ultrafilters (abstract), *Bull. Symbolic Logic* **6**, 394 (2000).
- [Lp8] P. Lipparini, Regular ultrafilters and $[\lambda, \lambda]$ -compact products of topological spaces (abstract), *Bull. Symbolic Logic* **5**, 121 (1999).
- [Lp9] P. Lipparini, Consequences of compactness properties for abstract logics, *Atti Accademia Nazionale dei Lincei* **80**, 501–503 (1986).
- [Lp10] P. Lipparini, A connection between decomposability of ultrafilters and possible cofinalities, <http://arXiv.org/math.LO/0604191>; II. <http://arXiv.org/math.LO/0605022> (2006); III, in preparation.
- [Lp11] P. Lipparini, Decomposable ultrafilters and possible cofinalities, *Notre Dame J. Form. Log.* **49**, 307–312 (2008).
- [L12] P. Lipparini, Combinatorial and model-theoretical principles related to regularity of ultrafilters and compactness of topological spaces. I, <http://arxiv.org/abs/0803.3498>; II. <http://arxiv.org/abs/0804.1445>; III. <http://arxiv.org/abs/0804.3737>; IV <http://arxiv.org/abs/0805.1548> (2008).

- [L13] P. Lipparini, About some generalizations of (λ, μ) -compactness, Proceedings of the 5th Easter conference on model theory (Wendisch Rietz, 1985), Seminarber., Humboldt-Univ. Berlin, Sect. Math. **93**, 139–141 (1987). Available also at the author's web page <http://www.mat.uniroma2.it/~lipparin/art/easter85.pdf>.
- [Ma] J. A. Makowsky, Compactness, embeddings and definability, Chapter 17 in [BF].
- [Pr] K. Prikry, On descendingly complete ultrafilters, in: Cambridge Summer School in Mathematical Logic, edited by A. R. D. Mathias and H. Rogers, 459–488, Berlin (1973).
- [Pr1] K. Prikry, Changing measurable into accessible cardinal, Dissertationes Mathematicae (Rozprawy Matematyczne) LXVIII (1970).
- [Pr2] K. Prikry, On the regularity of ultrafilters, in: Surveys in set theory, edited by A. R. D. Mathias, 162–166, Cambridge, 1983.
- [Sa] M. Sakai, Non- (ω, ω_1) -regular ultrafilter and perfect κ -normality of product spaces, Topology and its Applications **45**, 165–172 (1992).
- [SZ] E. Schimmerling and M. Zeman, Square in Core Models, Bull. Symbolic Logic **7**, 305–314 (2001).
- [Shr] M. Sheard, Indecomposable ultrafilters over small large cardinals, J. Symbolic Logic **48**, 1000–1007 (1983).
- [Sh] S. Shelah, Cardinal Arithmetic, Oxford (1994).
- [Sh1] S. Shelah, Applications of pcf theory, J. Symbolic Logic **65**, 1624–1674 (2000).
- [Si] J. H. Silver, Indecomposable ultrafilters and $0^\#$, in: Proceedings of the Tarski Symposium, Proc. Sympos. Pure Math. XXV, Univ. Calif., Berkeley, Calif., 357–363 (1971).
- [St] J. R. Steel, PFA implies $\text{AD}^{L(\mathbb{R})}$, J. Symbolic Logic **70**, 1255–1296 (2005).
- [Ta] A. D. Taylor, Regularity properties of ideals and ultrafilters, Ann. Math. Logic **16**, 33–55 (1979).
- [Wo] W. H. Woodin, The axiom of determinacy, forcing axioms, and the nonstationary ideal, Berlin (1999).

DIPARTIMENTO DI MATEMATICA, VIALE DELLA CICERCA SCIENTIFICA, II UNIVERSITÀ DI ROMA (TOR VERGATA), I-00133 ROME ITALY