

On some peculiar aspects of the constructive theory of point-free spaces*

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Abstract

This paper presents several independence results concerning the topos-valid and the intuitionistic (generalised) predicative theory of locales. In particular, certain consequences of the consistency of a general form of Troelstra's uniformity principle with constructive set theory and type theory are examined.

Key words: Locales, formal spaces, constructive set theory and type theory, topos logic, independence results, uniformity principle.

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1 Introduction

It may be argued that the well-known equivalence of theorems such as Tychonoff theorem, or Stone-Čech compactification, with the axiom of choice, or other similarly non-constructive principles, is a consequence of the chosen formulation of the concept of space, rather than being an intrinsic feature of these results. By replacing the ordinary notion of topological space with that of *locale* (or frame, or complete Heyting algebra) one obtains fully general versions of these theorems that can be proved without any choice, and often with no application of the principle of excluded middle [21, 14, 13, 20].

The notion of locale is for this reason the concept of space generally adopted in choice-free intuitionistic settings, such as toposes or intuitionistic set theory (IZF) [26]. By not assuming as available impredicative principles as the existence of powersets, the concept of *formal space*, or *set-generated locale*, plays a corresponding role in even weaker systems, as constructive set theory (CZF) or constructive type theory (CTT) [31, 10, 1, 24, 11]. The main criterion of adequacy of this notion is that, considered in fully impredicative settings as ZF or IZF¹, the category FSp of formal spaces is equivalent to the ordinary category of locales.

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¹In this paper a system is defined 'fully impredicative' if it is at least as strong as full higher order arithmetic HHA (topos logic).

This paper deals with certain peculiar features of both the constructive and the intuitionistic theory of locales. In fact, we will mainly be concerned with certain independence results that follow from the consistency of CZF and CTT with a generalised form of Troelstra's *principle of uniformity* [34].

Our main results are mostly related to the following fundamental ‘structural’ aspect of the theory of locales: considered in any topos, the category of locales is complete and cocomplete, i.e., all limits (in particular products) and colimits exist in this category. By contrast, the existence of binary products of arbitrary formal spaces already seems to require the use of strongly impredicative principles, that are not available in the generalised predicative settings under consideration. In particular to remedy this deficiency, the concept of *inductively generated formal space* was introduced in [10, 1]: inductively generated formal spaces define a full subcategory \mathbf{FSp}_i of the category of formal spaces, in which limits and colimits do exist (albeit under the assumption of strong principles for the existence of inductively defined sets, such as the axiom REA in constructive set theory, see e.g. [24, 1]).

By exhibiting a particular formal space that CTT cannot prove to be inductively generated, \mathbf{FSp}_i has been shown to form a proper subcategory of \mathbf{FSp} [10] (see [15] for a similar result in CZF). Nevertheless, since \mathbf{FSp}_i contains in particular all locally compact formal spaces, and since, considered in a fully impredicative setting, this category is still equivalent to the category of locales (as every formal space is inductively generated in such a setting), the concept of inductively generated formal space has generally been regarded as providing the proper constructive analogue of the notion of locale.

In this paper we show that the restriction to the category \mathbf{FSp}_i is, however, a very severe one: we prove that CTT, CZF, as several extensions of CZF, including REA and the impredicative unbounded separation scheme, cannot prove that a non-trivial *Boolean formal space* - i.e., a formal space whose associated frame is a non-trivial complete Boolean algebra -, is inductively generated. This result provides us with an example, for every given formal space S (inductively generated or not), of a formal space that these systems cannot prove to be inductively generated, namely the least dense subspace of S . Similar facts also hold for De Morgan (or extremally disconnected) formal spaces, and for formal spaces whose associated frame is the the Dedekind–MacNeille completion of a poset.

Further independence results, concerning compactness, overtiness (openness) and existence of points, will then be shown to hold with respect to the internal language of toposes (HHA), IZF, and/or CTT and CZF. In particular, we show that CZF (+REA+...), CTT cannot prove that a non-trivial formal space is compact and De Morgan. This is in contrast with a well-known result of M. H. Stone, valid in any topos: in HHA, or IZF, the frame of ideals of a complete Boolean algebra is a compact De Morgan locale [19]. It follows in particular that, for no non-trivial compact regular formal space S , the Gleason cover of S [19, 20] can be constructed in CZF, CTT.

The paper is organized as follows: basic facts on formal spaces/locales as treated in constructive settings are recalled in Section 2. In Section 3 the version of the uniformity principle that we shall exploit is presented and its incompatibility with De Morgan law is exhibited. The independence results concerning Boolean and De Morgan formal spaces are described in Section 4; the case of spaces arising via the Dedekind–MacNeille completion of a poset, and a problem left open in [10], are discussed in Section 5.

2 Preliminaries

The reader is referred to [3, 27] for background on Aczel's constructive set theory (CZF) and constructive type theory (CTT), respectively. In the following, we shall use CZF strictly to indicate the basic formulation of Aczel's theory. An extension of CZF that is often considered, particularly in connection with constructive locale theory, is the theory CZF+REA. The regular extension axiom REA is needed to ensure that certain inductively defined classes are sets [3]. Extending CZF with the full separation scheme (Sep) and the powerset axiom yields the fully impredicative set theory IZF. Adding the law of excluded middle to either theory (CZF or IZF) gives ZF. In the following, we use HHA (for intuitionistic higher-order Heyting arithmetic) to indicate topos logic [26, 34].

General information on locales may be found in [13, 14, 20]; for basic facts concerning the theory of formal spaces in constructive predicative settings such as CZF, CTT see [1, 10, 31, 24] or [2, 11]. Here we synthetically recall the notions needed in this note.

A *formal topology*, or *formal space*, is a pair $S \equiv (S, \triangleleft)$ where S , the *base*, is a set, and \triangleleft , the *covering relation*, is a relation between elements and subsets of S satisfying:

- i. $a \in U$ implies $a \triangleleft U$,
- ii. if $a \triangleleft U$ and $U \triangleleft V$, then $a \triangleleft V$,
- iii. $a \triangleleft U$ and $a \triangleleft V$ imply $a \triangleleft U \downarrow V$,

where $U \triangleleft V \equiv (\forall u \in U) u \triangleleft V$, and $U \downarrow V \equiv \{d \in S : (\exists u \in U) (d \triangleleft \{u\}) \ \& \ (\exists v \in V) (d \triangleleft \{v\})\}$. In CZF, the covering is formally a subclass of $S \times \mathbf{Pow}(S)$, where $\mathbf{Pow}(S)$ is the *class* of subsets of S ; in addition to i – iii, a further requirement in that context is that the class $\mathcal{S}(U) \equiv \{a : a \triangleleft U\}$ be a set for all U (see [1] for more). Two subsets U, V of S are the same *formal open*, $U =_S V$, exactly when $U \triangleleft V$ & $V \triangleleft U$. Observe that one may always assume that S has a ‘top’ element, i.e., an element 1_S such that $S =_S \{1_S\}$. An implication operation is defined on formal opens by $U \rightarrow V \equiv \{a \in S : a \downarrow U \triangleleft V\}$. The *pseudocomplement* U^* of U (the largest open disjoint from U) is given by:

$$U^* \equiv U \rightarrow \emptyset \equiv \{a \in S : a \downarrow U \triangleleft \emptyset\}.$$

A *morphism* $f : S_1 \longrightarrow S_2$ of formal topologies is a mapping $f : S_1 \longrightarrow \mathbf{Pow}(S_2)$ satisfying, for all $a, b \in S_1$, $U \subseteq S_1$,

- i. $f(S_1) =_{S_2} S_2$,
- ii. $f(a) \downarrow f(b) \triangleleft f(a \downarrow b)$,
- iii. $a \triangleleft U$ implies $f(a) \triangleleft f(U)$

where, for U a subset of S_1 , $f(U) \equiv \bigcup_{a \in U} f(a)$. Two morphisms $f, g : S_1 \longrightarrow S_2$ are defined to be equal precisely when $f(a) =_{S_2} g(a)$ for all $a \in S_1$. On any formal topology S , the identity morphism is given by $id_S(a) = \{a\}$, for all a .

A (*formal*) *point* of a formal space S is a subset α of S satisfying:

- i. $(\exists a \in S) a \in \alpha$

ii. $a, b \in \alpha$ implies $(\exists c \in \alpha) c \in a \downarrow b$.

iii. $a \in \alpha$ and $a \triangleleft U$ imply $(\exists b \in U) b \in \alpha$.

In particular, the top element 1_S (when it is present) belongs to every point, and for no $a \in \alpha$, is $a \triangleleft \emptyset$.

Even classically, a formal space may well have no points and be *non-trivial*, i.e., such that $\neg(S \triangleleft \emptyset)$. In terms of logic, this is because points of a formal space are the $\mathbf{Pow}(\{\top\})$ -valued (classically two-valued) models of a geometric theory, which may be consistent without having a model [13].

A *subspace* of a formal space $S \equiv (S, \triangleleft)$ is a formal space $S' \equiv (S, \triangleleft')$, on the same base, and with \triangleleft' satisfying *i.* $\triangleleft \subseteq \triangleleft'$, and *ii.* $x \downarrow' y \triangleleft' x \downarrow y$. See [11] for a more detailed discussion. For example, a (formal) open subset $V \subseteq S$ determines the closed subspace $S^V \equiv (S, \triangleleft^V)$, with $a \triangleleft^V U \iff a \triangleleft U \cup V$ (intuitively, S^V represents the complement of the open V as a subspace).

A formal space S is *set-presented* iff there are families of sets $I(x)$, for x in S , and $C(x, i) \subseteq S$, for $x \in S, i \in I(x)$, such that

$$a \triangleleft U \iff (\exists i \in I(a)) C(a, i) \subseteq U.$$

Observe that this implies $a \triangleleft C(a, i)$ for all i . In CZF+REA, CTT, S is set-presented if and only if it is *inductively generated* in the sense of [1, 10]. In CZF, CTT, the class of set-presented formal spaces contains all locally compact spaces [1]. In a topos, or in (I)ZF, all formal spaces are trivially set-presented: one simply defines $I(x) = \{U \in \mathbf{Pow}(S) : x \triangleleft U\}$, $C(x, U) = U$. The full subcategory of set-presented formal spaces has limits and colimits in sufficiently strong versions of constructive set theory and type theory.

In CZF, a *class-frame* (or class-locale) L is a partially ordered class that has a top element, binary meets, and suprema for arbitrary *sets* of elements of L , and that is such that meets distribute over the set suprema. A class-frame is said to be *set-generated* by a subclass B if: *i.* B is a set; *ii.* the class $\{b \in B : b \leq x\}$ is a set and $x = \bigvee \{b \in B : b \leq x\}$, for all $x \in L$.

Morphisms of set-generated frames are class-functions respecting meets, the top, and arbitrary set joins.

Given a formal topology S , let the collection of *saturated subsets* of S , i.e., the class $\{U \subseteq S : \mathcal{S}(U) = U\}$, be denoted by $Sat(S)$. Endowed with the operations $U \wedge V \equiv U \cap V = U \downarrow V$ and $\bigvee_{i \in I} U_i \equiv \mathcal{S}(\bigcup_{i \in I} U_i)$, $Sat(S)$ is a set-generated frame. The implication operation previously recalled defines an implication operation on $Sat(S)$ in the usual sense, making it in a complete Heyting algebra. In particular, U^* , for $U \in Sat(S)$ is the pseudocomplement of U in the ordinary lattice-theoretic sense.

With their respective morphisms, formal topologies and set-generated class-frames form equivalent categories [1, 2]. With powersets, every set-generated class-frame has a set of elements, so it is just an ordinary frame (locale). Therefore, in fully impredicative settings such as toposes, the category FT of formal topologies is equivalent to that of frames (see also [31]). Its opposite $\mathbf{FSp} = \mathbf{FT}^{op}$, here referred to as the category of *formal spaces* (often simply spaces) and *continuous functions*, is thus equivalent in such settings to the category of locales.

3 Uniformity principles

To distinguish the behavior of formal spaces in constructive settings from that in an intuitionistic but fully impredicative context, we will exploit a generalised form of the so called uniformity principle [34, 30]. In constructive set theory this is so formulated: for every set I ,

$$(\forall x)(\exists y \in I)A(x, y) \rightarrow (\exists y \in I)(\forall x)A(x, y) \quad (\mathbf{GUP-CZF}).$$

In [4, 5] this principle has been proved to be consistent (in particular) with CZF+REA+PA+Sep, where REA is the regular extension axiom, PA is the presentation axiom, and Sep is impredicative unbounded separation (see [5] for a list of other principles compatible with GUP-CZF. Consistency of these principles with CZF is shown in [5] by the definition of a model that has independently been noted also in [25] and [33]; see also [29]). Note that GUP-CZF follows from its instance:

$$(\forall x)(\exists y \in \omega)A(x, y) \rightarrow (\exists y \in \omega)(\forall x)A(x, y) \quad (\mathbf{UP-CZF})$$

(ω is the set of natural numbers), and the principle that every set is subcountable, also valid in the model of GUP described in [5, 25, 33]. It will be convenient to note explicitly the following consequence of GUP-CZF: for every set I ,

$$(\forall p \in \mathbf{Pow}(\{\top\}))(\exists i \in I)A(p, i) \rightarrow (\exists i \in I)(\forall p \in \mathbf{Pow}(\{\top\}))A(p, i) \quad (\mathbf{GUP'-CZF})$$

where $\mathbf{Pow}(\{\top\})$ is the powerclass of the one-element set (the antecedent of GUP'-CZF yields $(\forall x)(\exists y \in I)((\exists z)((\forall w)(w \in z \leftrightarrow w \in x \ \& \ w \in \{\top\}) \ \& \ A(z, y)))$; one can then apply GUP-CZF). The type-theoretic formulation of this principle, first exploited in [10], is recalled in the Appendix.

We write **EM**, **DML** for the principle of excluded middle and De Morgan law, respectively. Recall that De Morgan law $\neg(P \wedge Q) \rightarrow \neg P \vee \neg Q$ for all propositions P, Q , is equivalent to

$$\neg\neg P \vee \neg P$$

for all P . By the identification of subsets of the one-element set with restricted formulas (those in which all quantifiers are bounded) [3], in CZF this principle for restricted formulas can be formulated as

$$(\forall p \in \mathbf{Pow}(\{\top\}))p^{**} \cup p^* = \{\top\} \quad ([\mathbf{R}]\mathbf{DML})$$

where $p^* \equiv \{x \in \{\top\} : x \notin p\}$. Note that, considered in IZF, [R]DML expresses De Morgan Law for arbitrary formulas.

The generalised uniformity principle conflicts in CZF with [R]DML, and with DML in CTT. We prove the first fact: assume that $p^{**} \cup p^* = \{\top\}$ for all $p \in \mathbf{Pow}(\{\top\})$. Define a relation $F \subseteq \mathbf{Pow}(\{\top\}) \times \{0, 1\}$ by letting

$$(x, y) \in F \iff (\top \in x^{**} \ \& \ y = 1) \vee (\top \in x^* \ \& \ y = 0).$$

By the assumption, $(\top \in x^{**}) \vee (\top \in x^*)$ for all $x \in \mathbf{Pow}(\{\top\})$, so that trivially

$$(\forall x \in \mathbf{Pow}(\{\top\}))(\exists y \in \{0, 1\})(x, y) \in F.$$

By GUP-CZF, this gives

$$(\exists y \in \{0, 1\})(\forall x \in \text{Pow}(\{\top\}))(x, y) \in F,$$

which yields a contradiction (consider $x = \{\top\}$, $x = \emptyset$, for $y = 0$, $y = 1$, respectively).

Of course, this implies that GUP is inconsistent with the principle of excluded middle EM in CTT (see also [28]), in CZF with excluded middle for restricted formulas, or equivalently, with

$$(\forall p \in \text{Pow}(\{\top\}))p \cup p^* = \{\top\} \quad (\mathbf{[R]EM})$$

(again, considered in IZF, $\mathbf{[R]EM}$ is equivalent to the full law of excluded middle).

In the following, we shall write CZF^* or CTT^* for a generic fixed extension of CZF or CTT, respectively, that is compatible with the generalised uniformity principle. For simplicity, we call any extension of this kind a ‘constructive setting’ (this terminology is quite improper, given that CZF^* may be taken to be given by CZF plus the impredicative unbounded separation scheme Sep). With ‘intuitionistic setting’ we indicate any of CTT^* , CZF^* , IZF, HHA.

We shall make free use of the fact that all the settings that we consider may consistently be extended with the negation of the (restricted) De Morgan law. The assertion that a space of a certain type cannot be proved to have a certain property in a certain setting will invariably be proved by showing that in the setting extended with some compatible non-classical principle (as GUP, or $\neg\mathbf{[R]DML}$), the assumption that the space has the property is contradictory.

For definiteness, in what follows we always argue in the setting of constructive set theory. The proof for a different system for which a given result is claimed, is obtained by the expected modifications of the given argument.

4 Boolean and De Morgan locales/formal spaces

By exploiting the generalised uniformity principle it is shown in [10] that there is a formal space that CTT cannot prove to be set-presented; in [16] it is shown by other means that the system CZF cannot prove the so-called ‘double-negation’ formal space $\text{Pow}(\{\top\})_{\neg\neg}$ (see Section 5) to be set-presented². We shall see in Section 5 that $\text{Pow}(\{\top\})_{\neg\neg}$ and the space considered in [10] are in fact isomorphic, and that their associated frame is a complete Boolean algebra. The same argument given in [16] can then be used to show that any formal space whose associated frame is a non-trivial (complete) Boolean algebra cannot be proved to be set-presented over the basic set of axioms of CZF³.

In this section, using the generalised uniformity principle we give a simple proof that no system CZF^* , CTT^* can prove a non-trivial Boolean formal space to be set-presented. In particular, thus, this holds for $\text{CZF}^* = \text{CZF} + \text{REA} + \text{PA} + \text{Sep}$. A similar result is also shown to hold for De Morgan (or extremally disconnected) formal spaces.

Further independence results concerning overtiness, compactness and existence of points are also obtained. Aside from Theorem 4.10, these make no use of the consistency of GUP

² As noted in [16], R. Grayson [17] had obtained a corresponding result for certain formulations of intuitionistic set theory without the powerset axiom.

³ This observation is essentially due to S. Vickers; in fact also the corresponding of this result was known to Grayson [17] in connection with the set theories he considered (cf. footnote 2).

with the given setting and hold true, *mutatis mutandis*, also with respect to topos logic (HHA), or IZF. All results in which the generalised uniformity principle is involved, which thus only concern the constructive settings, will be marked with GUP.

Call a formal space S such that $Sat(S)$ is a Boolean frame a *Boolean formal space*. From now on, let for simplicity S have a top basic element 1_S . If S is set-presented, also the enlargement of its base with a top element 1_S can be proved to be set-presented (this is proved in type theory using type-theoretic choice [11], in constructive set theory exploiting the Subset Collection scheme). Thus, a Boolean formal space is one such that $1_S =_S U \cup U^*$ for all $U \in \mathbf{Pow}(S)$.

A formal space S is *De Morgan* if it satisfies $1_S \triangleleft U^{**} \cup U^*$ for all $U \in \mathbf{Pow}(S)$. Classically, a topological space is extremally disconnected iff its frame of open subsets is De Morgan [20]. Obviously, S Boolean implies S De Morgan.

For $p \in \mathbf{Pow}(\{\top\})$, we shall suggestively write P to stand for $\top \in p$, while ‘ $\forall p$ ’ will always stand for ‘for all p in $\mathbf{Pow}(\{\top\})$ ’. We set

$$U_P = \{x \in S : x = 1_S \ \& \ P\} \equiv \{x \in S : x = 1_S \ \& \ \top \in p\}.$$

Note that, in any formal space, $\{1_S\}^* = S^* =_S \emptyset$, and $\emptyset^* = S =_S \{1_S\}$.

Lemma 4.1 *Let S be any formal space.*

- i. *If $(\forall p)(\exists x)x \in U_P \cup U_P^* \ \& \ \neg(x \triangleleft \emptyset)$, then $(\forall p)P \vee \neg P$, i.e., $(\forall p \in \mathbf{Pow}(\{\top\}))p \cup p^* = \{\top\}$.*
- ii. *If $(\forall p)(\exists x)x \in U_P^{**} \cup U_P^* \ \& \ \neg(x \triangleleft \emptyset)$, then $(\forall p)\neg\neg P \vee \neg P$, i.e., $(\forall p \in \mathbf{Pow}(\{\top\}))p^{**} \cup p^* = \{\top\}$.*

Proof. From $x \in U_P$ one gets P . From $x \in U_P^*$ and $\neg(x \triangleleft \emptyset)$ one obtains $\neg P$ as follows: assuming P , one has $U_P = \{x \in S : x = 1_S\}$, so that $U_P^* \triangleleft \emptyset$; together with $x \in U_P^*$ and $\neg(x \triangleleft \emptyset)$, this yields a contradiction, so that $\neg P$. Finally, assuming $\neg P$ gives $U_P = \emptyset$, and thus also $U_P^{**} =_S \emptyset$; by $x \in U_P^{**}$, $\neg(x \triangleleft \emptyset)$ one derives $\neg\neg P$. The reader may then easily fill in the details. \square

In [14] one finds an ‘arrow-theoretic’ proof that no Boolean frame may have points, unless classical logic is accepted. Here is another formulation of that proof, and the corresponding fact for De Morgan locales.

Proposition 4.2 *No Boolean formal space can have a point unless $[R]EM$ is accepted in CZF^* (EM in CTT^* , HHA , IZF). No De Morgan formal space can have a point unless $[R]DML$ is accepted in CZF^* (DML in CTT^* , HHA , IZF).*

Proof. Assume Boolean formal space S has a point.

$$S \text{ Boolean implies } (\forall p)1_S \triangleleft U_P \cup U_P^*;$$

$$S \text{ has a point } \alpha \text{ implies that } (\forall p)(\exists a)a \in U_P \cup U_P^* \ \& \ a \in \alpha.$$

By $a \in \alpha$ one has $\neg(a \triangleleft \emptyset)$, and then one concludes using Lemma 4.1. The proof for the De Morgan case is similar. \square

As the settings in consideration (CZF*, CTT*, HHA, IZF) can be extended consistently by $\neg[\mathbf{R}]\mathbf{DML}$ ($\neg\mathbf{DML}$), in these settings De Morgan locales/formal spaces cannot be proved to have points. This implies that no such formal space is a topological space, i.e., no non-trivial De Morgan (in particular Boolean) frame can be obtained as the frame of opens of a non-empty (inhabited) topological space. Classically, of course, the lattice of open subsets of any discrete (non-empty) space is a Boolean frame with points.

Despite this result, at least in HHA/IZF, there are Boolean locales that are *proper*, i.e., such that, for all $U \in \mathbf{Pow}(S)$, $S \triangleleft U$ implies $\exists a \in U$ (see [14]). Properness is a stronger formulation of non-triviality.

A formal space (S, \triangleleft) is *open* (or *overt*, or has a positivity predicate [22, 31, 10]) iff there is a predicate $\mathbf{Pos}(x)$, for x in S , satisfying

- i. $\mathbf{Pos}(a)$ and $a \triangleleft U$ imply $(\exists b \in S)b \in U$ & $\mathbf{Pos}(b)$ (*monotonicity*);
- ii. $a \triangleleft U$ implies $a \triangleleft U^+ \equiv \{b \in U : \mathbf{Pos}(b)\}$ (*positivity*).

(Note that classically, all formal spaces are open, with $\mathbf{Pos}(a) \equiv \neg(a \triangleleft \emptyset)$). Then, although a Boolean locale can be proper, it cannot be open.

Proposition 4.3 *No non-trivial De Morgan formal space S (in particular, no non-trivial Boolean formal space) can be proved to be open in the intuitionistic settings considered.*

Proof. Assume S is open. Then, by positivity, $1_S \triangleleft \{1_S\}^+$. Assume $1_S \in \{1_S\}^+$, so that $\mathbf{Pos}(1_S)$ holds. Then,

$$S \text{ De Morgan implies } (\forall p)1_S \triangleleft U_P^{**} \cup U_P^*;$$

by $\mathbf{Pos}(1_S)$ and monotonicity of \mathbf{Pos} , for all p there is $a \in U_P^{**} \cup U_P^*$ with $\mathbf{Pos}(a)$.

Thus, since $\mathbf{Pos}(a)$ implies $\neg(a \triangleleft \emptyset)$ (by monotonicity), by Lemma 4.1 one obtains $(\forall p)\neg\neg P \vee \neg P$. As the settings under consideration can be extended with the negation of (restricted) De Morgan law, one has that $1_S \in \{1_S\}^+$ leads to a contradiction in the extended setting, so that $\{1_S\}^+ = \emptyset$. But this cannot be, as, by positivity $1_S \triangleleft \{1_S\}^+$, and we assumed the space to be non-trivial. \square

This proof shows that Boolean formal spaces can be proper only because the elements one extracts from each cover of the whole space are not required to be different from the empty open, let alone positive.

Theorem 4.4 (GUP) *Let S be*

- i. *a non-trivial Boolean formal space, or*
- ii. *a non-trivial De Morgan formal space such that $(\forall x \in S)x \triangleleft \emptyset \vee \neg(x \triangleleft \emptyset)$;*

then S cannot be proved to be set-presented (inductively generated) in CZF or in CTT*.*

Proof. *i.* Assume S has a set-presentation in CZF^* . Then

S Boolean implies: $(\forall p)1_S \triangleleft U_P \cup U_P^*$;

S set-presented implies: $(\forall p)(\exists i \in I(1_S))C(1_S, i) \subseteq U_P \cup U_P^*$;

assuming GUP, this implies: $(\exists i \in I(1_S))(\forall p)C(1_S, i) \subseteq U_P \cup U_P^*$.

In particular, taking $p = \{\top\}$ so that $P \equiv \top \in \{\top\}$ is true, this gives

$$(*) \quad \forall x \in C(1_S, i)(x = 1_S \vee x \triangleleft \emptyset).$$

Then assume $x = 1_S \in C(1_S, i)$. Since $C(1_S, i) \subseteq U_P \cup U_P^*$, and since S is non-trivial, by Lemma 4.1, one gets (in $\text{CZF}^* + \text{GUP}$) $(\forall p)P \vee \neg P$. We saw that GUP is incompatible with [R]EM. Therefore, by $(*)$, one gets $\forall x \in C(1_S, i)x \triangleleft \emptyset$; but $1_S \triangleleft C(1_S, i)$, and we assumed S to be non-trivial, whence S is not set-presented in $\text{CZF}^* + \text{GUP}$. This shows that S cannot be proved to be set-presented in CZF^* .

ii. Assume S is set-presented in CZF^* . Note first that if S satisfies $(\forall x \in S)x \triangleleft \emptyset \vee \neg(x \triangleleft \emptyset)$, but has no top element 1_S , also the isomorphic formal space with a top [11] will satisfy the given decidability condition, as, for the top element, we have by hypothesis that $\neg(1_S \triangleleft \emptyset)$. Then, in $\text{CZF}^* + \text{GUP}$, one finds $i \in I(1_S)$ such that $(\forall p)C(1_S, i) \subseteq U_P^{**} \cup U_P^*$. Let $x \in C(1_S, i)$, and assume $\neg(x \triangleleft \emptyset)$. By Lemma 4.1, one obtains $(\forall p)\neg\neg P \vee \neg P$. As this contradicts GUP, one has $\neg\neg(x \triangleleft \emptyset)$. By the decidability of $x \triangleleft \emptyset$, it follows that $x \triangleleft \emptyset$ for all $x \in C(1_S, i)$. But S is non-trivial, so that it (is not set-presented in $\text{CZF}^* + \text{GUP}$ and thus) cannot be proved to be set-presented in CZF^* . \square

An example of a De Morgan non-Boolean formal space satisfying the condition in Theorem 4.4 is presented in the next section.

This theorem shows that the formal spaces that cannot be inductively generated consist not just of few pathological cases. In particular, one has:

Corollary 4.5 *Given any non-trivial formal space S , the formal space $S^{**} \equiv (S, \triangleleft^{**})$, where $a \triangleleft^{**} U \iff \{a\}^{**} \triangleleft U^{**}$, can not be proved set-presented in CTT^* , CZF^* .*

The space S^{**} is indeed the Boolean formal subspace corresponding to the $**$ -nucleus on the frame defined by S , i.e., the space associated with the frame of ‘regular’ elements of $\text{Sat}(S)$ (see e.g. [14, 20] for a discussion of the $**$ -nucleus on a locale L).⁴ S^{**} is non-trivial as soon as S is non-trivial.

These subspaces/nuclei are, also classically, a peculiarity of locale theory (as opposed to point-set topology), since, given *any* locale L , the $**$ -nucleus on L yields the least dense sublocale of L . (This need not exist in a topological space; consider e.g. the real line: the rationals and the irrationals define dense disjoint subspaces).

Remark 4.6 Although not set-presentable, Boolean formal spaces are constructively useful: an example of the use of a Boolean formal space to obtain a concrete (constructive and predicative) description of ideal non-effective objects can be found in [7, Theorem 6.1].

⁴In the literature on locales this nucleus is also known as the Booleanization of S . There are many Boolean sublocales of a given locale L , but each of them can be seen as defined by a $**$ -nucleus over a closed sublocale of L [14].

The following proposition shows that being Boolean also conflicts with being compact.

Proposition 4.7 *No non-trivial Boolean formal space S can be compact unless $[R]EM$ in CZF^* , EM in CTT^* , HHA , IZF , is accepted.*

Proof. Assume S is compact.

S Boolean implies $(\forall p)1_S \triangleleft U_P \cup U_P^*$;

S compact implies that, for all p , there is a finite u_0 such that $1_S \triangleleft u_0 \subseteq U_P \cup U_P^*$.

It is a standard fact that $u_0 \subseteq V \cup W$, with u_0 finite, implies intuitionistically $u_0 = v_0 \cup w_0$, with $v_0 \subseteq V$, $w_0 \subseteq W$ both finite [6]. Thus we have finite $v_0 \subseteq U_P, w_0 \subseteq U_P^*$, with $1_S \triangleleft v_0 \cup w_0$; moreover, ‘finite’ implies ‘either empty or inhabited’. By cases: $v_0, w_0 = \emptyset$ cannot be, by non-triviality. Then one of the following alternatives holds:

1. v_0, w_0 inhabited, or
2. v_0 inhabited and $w_0 = \emptyset$, or
3. $v_0 = \emptyset$ and w_0 inhabited.

The first and second case directly give $P \vee \neg P$. For the last, assuming P one gets $w_0 \triangleleft \emptyset$, that together with $v_0 = \emptyset$, gives $1_S \triangleleft \emptyset$, so that, by non-triviality of S , $\neg P$, and then again $P \vee \neg P$. Therefore, if S is compact, the law $(\forall p)P \vee \neg P$ holds. \square

In CZF^* , or CTT^* , more generally, no non-trivial Boolean space S can be proved to be locally compact, since locally compact formal spaces are set-presented [1], and by Theorem 4.4 no non-trivial Boolean space S can be proved to be set-presented in these settings (‘more generally’: any Boolean S is regular, and a compact regular locale is locally compact, e.g. [20]). Classically (e.g. in ZF), every finite discrete space has a compact Boolean frame of opens.

So far the generalised uniformity principle has only been used to show that a constructive system cannot prove that formal spaces of a certain type can be set-presented. We conclude this section with two other important consequences of the consistency of this principle with the constructive settings we are considering.

First let us note that, contrary to what one may expect, in HHA or IZF , *De Morgan locales can be compact*: the classical result (due to M. Stone) that the (compact) frame $Idl(B)$ of ideals over a Boolean algebra B is De Morgan if and only if B is complete (e.g. [19]) is topos-valid. The following is one half of this result, formulated for formal spaces. Recall that in e.g. IZF , frames and set-generated class-frames come to the same thing, so that $Sat(S)$ is carried by a set for every space S .

Proposition 4.8 (Stone) *In any of the intuitionistic settings we are considering, let S be a Boolean formal space. In the context of CZF^* , or CTT^* , assume further that $Sat(S)$ is (carried by) a set. Then the formal space $S_\beta \equiv (Sat(S), \triangleleft_\beta)$, with $U \triangleleft_\beta \{U_i\}_{i \in I} \iff U \triangleleft U_{i_1} \cup \dots \cup U_{i_n}$ for $\{i_1, \dots, i_n\}$ a (possibly empty) finite subset of I , is a compact De Morgan formal space.*

Proof. The proof that S_β is a formal space is left to the reader. One has to prove

$$S \triangleleft_\beta (\{U_i\}_{i \in I})^{*\beta} \cup (\{U_i\}_{i \in I})^{*\beta * \beta}$$

for any given set $\{U_i\}_{i \in I}$ of elements of $Sat(S)$. Routine calculations show that

$$(\{U_i\}_{i \in I})^{*\beta} =_\beta \{(\bigcup_{i \in I} U_i)^{*S}\} \quad (1)$$

($*^S$ is pseudo-complementation in S), so that

$$((\{U_i\}_{i \in I})^{*\beta})^{*\beta} =_\beta \{(\bigcup_{i \in I} U_i)^{*S * S}\}. \quad (2)$$

Since S is Boolean, $(\bigcup_{i \in I} U_i)^{*S * S} =_S \bigcup_{i \in I} U_i$. As

$$S \triangleleft (\bigcup_{i \in I} U_i)^{*S} \cup (\bigcup_{i \in I} U_i) =_S (\bigcup_{i \in I} U_i)^{*S} \cup (\bigcup_{i \in I} U_i)^{*S * S},$$

one has $S \triangleleft_\beta \{(\bigcup_{i \in I} U_i)^{*S}, (\bigcup_{i \in I} U_i)^{*S * S}\}$ by definition of \triangleleft_β (note that pseudo-complements are saturated). Therefore, by 1,2 above, $S \triangleleft_\beta (\{U_i\}_{i \in I})^{*\beta} \cup (\{U_i\}_{i \in I})^{*\beta * \beta}$, as wished. \square

Despite this fact, one has:

Proposition 4.9 *No non-trivial De Morgan formal space S such that $(\forall x \in S) \neg\neg(x \triangleleft \emptyset) \vee \neg(x \triangleleft \emptyset)$ is compact unless $[R]DML$ holds in CZF^* (DML holds in CTT^* , HHA , IZF).*

Proof. For all $p \in \text{Pow}(\{\top\})$, one finds a finite v_0 with $v_0 \subseteq U_P^{**} \cup U_P^*$, $1_S \triangleleft v_0$. One has, in particular, $(\forall x \in v_0) \neg\neg(x \triangleleft \emptyset) \vee \neg(x \triangleleft \emptyset)$. By a general intuitionistic principle (see [22, Lemma 2.4]), this gives $(\forall x \in v_0) (\neg\neg(x \triangleleft \emptyset)) \vee (\exists x \in v_0) (\neg(x \triangleleft \emptyset))$. Since v_0 is finite, we get $\neg\neg(\forall x \in v_0) (x \triangleleft \emptyset) \vee (\exists x \in v_0) (\neg(x \triangleleft \emptyset))$. It cannot be that $x \triangleleft \emptyset$ for all $x \in v_0$. Thus, there is $x \in v_0$ with $\neg(x \triangleleft \emptyset)$, so that one concludes by Lemma 4.1. \square

As a consequence, in a topos that does not satisfy De Morgan law, no frame of the form $Idl(B)$, with B complete Boolean algebra, can have a base satisfying the decidability condition in the above proposition. Note also that such frames are examples of De Morgan frames that are never intuitionistically Boolean, given that no Boolean frame can be proved compact.

Using the generalised uniformity principle, the above proposition may be strengthened.

Theorem 4.10 (GUP) *No non-trivial De Morgan formal space S can be proved to be compact in CZF^* , CTT^* .*

Proof. Using GUP, one has that a finite subset $u_0 = \{x_1, \dots, x_n\}$ of S exists such that $(\forall p) 1_S \triangleleft u_0 \subseteq U_P^* \cup U_P^{**}$ (u_0 is non-empty, as S is non-trivial). Assume $\neg(x_1 \triangleleft \emptyset) \vee \dots \vee \neg(x_n \triangleleft \emptyset)$. By Lemma 4.1, one has that $(\forall p) \neg\neg P \vee \neg P$ holds. We saw that this principle is incompatible with GUP, so that $\neg(\neg(x_1 \triangleleft \emptyset) \vee \dots \vee \neg(x_n \triangleleft \emptyset))$. This gives $\neg\neg(x_1 \triangleleft \emptyset) \& \dots \& \neg\neg(x_n \triangleleft \emptyset)$, that is $\neg\neg(x_1 \triangleleft \emptyset \& \dots \& x_n \triangleleft \emptyset)$. On the other hand, from

$1_S \triangleleft u_0$ one gets $\neg(x_1 \triangleleft \emptyset \ \& \dots \& \ x_n \triangleleft \emptyset)$, so that S (is not compact in $\text{CZF}^* + \text{GUP}$, and hence) cannot be proved to be compact in CZF^* . \square

Recall that the *Gleason cover* of a compact regular formal space S is a minimal surjection $\gamma S \rightarrow S$, with γS a compact, regular, De Morgan formal space [19, 20]. It then follows from Theorem 4.10 that, in contrast with what happens in a topos, *for no non-trivial compact regular formal space S the Gleason cover of S can be constructed in CZF^* , CTT^* .*

By Theorem 4.10 it also follows that no non-trivial frame can be assumed to be carried by a set in a constructive setting (see [12] for a more direct proof).

Corollary 4.11 (GUP) *Every non-trivial frame $\text{Sat}(S)$, for S Boolean, is carried by a proper class in CZF^* , CTT^* . Thus, no non-trivial frame $\text{Sat}(S)$ may be proved to have a set of elements in these contexts.*

Proof. If the collection of elements of $\text{Sat}(S)$ could be proved to be constructively a set, by Proposition 4.8 the formal space S_β would be compact and De Morgan, contradicting Theorem 4.10. Now assume a frame $\text{Sat}(S)$ has a set of elements; then all frames $\text{Sat}(S')$, for S' a subspace of S , are carried by a set, too, so that also $\text{Sat}(S^{**})$ should be. \square

Remark 4.12 The property of Boolean formal spaces that has been exploited in the proofs of Propositions 4.2, 4.7, and Theorem 4.4, is that the whole space S is covered by $U_P \cup U_P^*$, for all p in $\text{Pow}(\{\top\})$ (this is also true for Proposition 4.3, if one proves the result just for the Boolean case). It is easy to check that a morphism $f : S \rightarrow S'$, with S Boolean, preserves pseudocomplements. It follows that whenever such a morphism exists, one also has $(\forall p) 1_{S'} \triangleleft V_P \cup V_P^*$, with $V_P \equiv \{x \in S' : x = 1_{S'} \ \& \ P\}$. Then, Propositions 4.2, 4.3, 4.7, and Theorem 4.4 hold true more generally if one replaces the Boolean space S with any non-trivial codomain of a morphism with Boolean domain. Similar considerations also hold in connection with the results concerning De Morgan spaces in Propositions 4.2, 4.3, 4.9, and Theorems 4.4, 4.10, when $f : S \rightarrow S'$ is any morphism that preserves pseudocomplements (in particular, when f defines an *open* continuous functions of locales/formal spaces [22]).

In contrast with the Boolean case, one cannot hope to prove that every subspace of a De Morgan formal space is De Morgan: classically, an extremally disconnected space may have Hausdorff subspaces that are not extremally disconnected. In [18], the following law is considered: for all propositions P, Q

$$(P \rightarrow Q) \vee (Q \rightarrow P).$$

This principle is stronger than De Morgan's (take Q to be $\neg P$), and is inherited by the internal logic of sheaf subtoposes [18]. Call *strongly De Morgan* a formal space such that the associated frame models this formula. A strongly De Morgan formal space is De Morgan. It is easy to prove that the class of strongly De Morgan formal spaces is closed for subspaces.

5 Dedekind–MacNeille completions

Given a set S , and any (class-)relation $R(a, U)$, for $a \in S$ and $U \in \text{Pow}(S)$, one may define R to be set-presented precisely as for coverings. Let $\Phi(P)$ be an instance of a law in one

variable P that is incompatible with GUP, e.g. $\Phi(P) \equiv P \vee \neg P$ (in CZF^* , $p \cup p^* = \{\top\}$, for $p \in \text{Pow}(\{\top\})$).

Proposition 5.1 (GUP) *Let S be a set, and let $R \subseteq S \times \text{Pow}(S)$ be such that, for some a in S , $\neg R(a, \emptyset)$, and $(\forall p)R(a, U_{\Phi(P)})$, where $U_{\Phi(P)} \equiv \{x \in S : x = a \ \& \ \Phi(P)\}$. Then R cannot be proved to be set-presented in CZF^* , CTT^* .*

Proof. Assume R is set-presented by $C(x, i)$ with $x \in S$ and $i \in I(x)$. By GUP, there is $i \in I(a)$ such that $(\forall p)C(a, i) \subseteq U_{\Phi(P)}$ and $R(a, C(a, i))$. Assume $x \in C(a, i)$. Then $(\forall p)\Phi(P)$. By hypothesis this contradicts GUP. Therefore, $C(a, i) = \emptyset$, and $R(a, \emptyset)$, against what we have assumed. \square

Recall that the Dedekind–MacNeille completion of a partial order makes it possible to embed a given partially ordered set in a complete lattice preserving meets and joins that exist (see e.g. [32, 34]). Given a partially ordered set (S, \leq) , one may define a relation $R_c(x, U)$ by letting

$$R_c(x, U) \iff (\forall y)[(\forall u \in U)u \leq y] \rightarrow x \leq y \iff x \in \bigcap_{U \subseteq \downarrow y} \downarrow y.$$

To have that R_c is a covering relation, the *Dedekind–MacNeille covering*, the partial order has to satisfy some further conditions. In particular, if S is a Heyting algebra this is always the case. The frame $\text{Sat}(S, R_c)$ of saturated subsets of the formal topology (S, R_c) is then the complete lattice in which the Heyting algebra S is embedded via $e : S \rightarrow \text{Sat}(S, R_c)$, $e(a) = S(\{a\})$. Recall that, as over any frame, an implication operation making $\text{Sat}(S, R_c)$ a complete Heyting algebra can be defined by letting $U \rightarrow V \equiv \{a \in S : a \downarrow U \triangleleft V\}$. The Heyting algebra structure of $\text{Sat}(S, R_c)$ then extends that of S (see e.g. [34], vol. II).

T. Coquand has suggested⁵ that no Dedekind–MacNeille covering can be constructively proved to be set-presented (see also [9]). We prove here that this holds for every relation $R_c(x, U)$ on a given poset (S, \leq) , but with a further hypothesis.

Proposition 5.2 (GUP) *Let (S, \leq) be a partial order having at least one element a that is not the least of S , and that is ‘stable’, in the sense that $\neg\neg(a \leq x)$ implies $a \leq x$, for all x . Then the relation $R_c(x, U)$ cannot be proved to be set-presented in CZF^* , CTT^* .*

Proof. By Proposition 5.1, it suffices to show that $\neg R_c(a, \emptyset)$ and $(\forall p)R_c(a, U_{\Phi(P)})$, with $U_{\Phi(P)} \equiv \{a : \Phi(P)\}$, and $\Phi(P) \equiv P \vee \neg P$. If $R_c(a, \emptyset)$, then $a \in \bigcap_{\emptyset \subseteq \downarrow y} \downarrow y$. This gives $(\forall y \in S)a \leq y$, against the hypothesis. For the second, let $U_{\Phi(P)} \subseteq \downarrow y$, and assume $\neg(a \leq y)$ and $\Phi(P)$. Then $U_{\Phi(P)} = \{a\} \subseteq \downarrow y$, so that $a \leq y$, against what we have assumed. This gives $\neg\Phi(P)$. As $\neg\neg\Phi(P)$ is intuitionistically provable, we get $\neg\neg(a \leq y)$, whence $a \leq y$. We conclude that $a \in \bigcap_{U_{\Phi(P)} \subseteq \downarrow y} \downarrow y$, for all p , i.e., $(\forall p)R_c(a, U_{\Phi(P)})$. \square

Note that this proof is, in essence, a simplification and a generalization of the proof for the special case considered in [10].

⁵On the occasion of the presentation of the material in the preceding sections at the workshop “Trends in constructive mathematics”, Chiemsee (Germany) June 19–23, 2006.

Corollary 5.3 (GUP) *If (S, \leq) is any poset with at least two elements and a decidable order relation, then $R_c(x, U)$ cannot be proved to be set-presented in CZF^* , CTT^* .*

These results may be used to produce examples of non-De Morgan formal spaces that cannot be constructively set-presented.

Corollary 5.4 (GUP) *Let $H \equiv (S, \wedge, \vee, \rightarrow, 0, 1)$ be a Heyting algebra satisfying the hypothesis in Proposition 5.2 (w.r.t. the partial order associated with H). Assume in H De Morgan law is false, i.e., there is $b \in S$ such that $1 \neq b^* \vee b^{**}$. Then the relation $R_c(x, U)$, defining the Dedekind–MacNeille cover on H , defines a non-De Morgan formal space that cannot be proved to be set-presented in CZF^* , CTT^* .*

Proof. As already recalled, the (set-generated) frame $Sat(S, R_c)$ associated with the Dedekind–MacNeille cover defined over an Heyting algebra H is a complete Heyting algebra in which the Heyting algebra operations extend the corresponding operations on H . As $b^* = b \rightarrow 0$ one can conclude. \square

The set $T = \{0, \frac{1}{2}, 1\}$ endowed with the natural order is a non-Boolean Heyting algebra. The Dedekind–MacNeille cover over this poset defines a De Morgan non-Boolean formal space (T, \triangleleft_{DM}) . That T is non-Boolean again follows by the fact that the complete Heyting algebra $Sat(T)$ is such that the Heyting algebra operations are extensions of the corresponding operations of T . To prove that (T, \triangleleft_{DM}) is De Morgan, i.e., that for every $U \in \mathbf{Pow}(T)$, $1 \in \bigcap K(U)$, with $K(U) \equiv \{\downarrow y : U^* \cup U^{**} \subseteq \downarrow y\}$, first one notes that $U^* \cup U^{**} \subseteq \downarrow 1$; unwinding the definitions, one then proves that assuming $1 \notin U^* \cup U^{**}$ leads to a contradiction, so that $\neg 1 \notin U^* \cup U^{**}$ (sketch: $1 \notin U^* \cup U^{**}$ implies $\neg(1 \in U^*) \ \& \ \neg(1 \in U^{**})$). The second conjunct yields $\neg(\frac{1}{2} \notin U^* \ \& \ 1 \notin U^*)$, that in turn gives $(\frac{1}{2} \notin U \ \& \ 1 \notin U)$; on the other, by the first conjunct one gets $\neg(\frac{1}{2} \notin U \ \& \ 1 \notin U)$, so that a contradiction is reached). Therefore, $\downarrow \frac{1}{2} \notin K(U)$ and $\downarrow 0 \notin K(U)$, whence $1 \in \bigcap K(U)$. By Proposition 5.2 (or Theorem 4.4) one has that (T, \triangleleft_{DM}) is not constructively set-presentable.

Remark 5.5 If in the hypotheses of Proposition 5.1, R is a covering \triangleleft on a set S , and if $a \equiv 1_S =_S S$, then for every morphism $f : S \rightarrow S'$, by $1_s \triangleleft U_{\Phi(P)}$ one gets $1_{S'} \triangleleft U'_{\Phi(P)}$, with $U'_{\Phi(P)} \equiv \{x \in S' : x = 1_{S'} \ \& \ \Phi(P)\}$. Therefore, if S' is non-trivial, by Proposition 5.1 it is not set-presentable. As an immediate corollary one has in particular that no formal space defined by the Dedekind–MacNeille cover on a poset (S, \leq) with the properties in Proposition 5.2, and such that a is also the greatest element of (S, \leq) , may have points (as points are in a bijective correspondence with morphisms from the given topology to the set-presentable topology $\mathbf{Pow}(\{\top\})$), and that every non-trivial formal subspace of S is not set-presentable (if $S' \equiv (S, \triangleleft')$ is a subspace of S , letting $e(x) = \{x\}$ for all $x \in S$ defines a morphism $e : S \rightarrow S'$).

In fact, the next result can be obtained without any reference to the uniformity principle (see also [9]).

Proposition 5.6 *Let S be a Dedekind–MacNeille topology defined on a poset with the property in Proposition 5.2, and having a as greatest element. If S has a point, or is compact, then $[R]EM$ in CZF^* (EM in CTT^* , HHA , IZF) holds. Furthermore, S cannot be proved to be open in the intuitionistic settings considered.*

Proof. Follows immediately by the fact that it holds $\neg R_c(a, \emptyset)$ and $R_c(a, U_{\Phi(P)})$ for all p , with $\Phi(P) \equiv (p \cup p^* = \{\top\})$ (cf. proof of Proposition 5.2). \square

We conclude this section with a discussion of Open Problem 4.5 of [10]. Taking $S = \{0, 1\}$ with the natural order, and $a \triangleleft_{DM} U \equiv R_c(a, U)$ we obtain the formal space that is in [10] shown not to be set-presentable in CTT. Let us denote this space by S_{DM} .

We already pointed out that a uniform method for the definition of products of arbitrary formal spaces is generally regarded as being beyond constructive means. Open Problem 4.5 of [10] asked whether at least the particular product of S_{DM} with itself is predicatively definable. The answer in this (indeed very special) case is yes. We show this with a slight detour. The ‘double negation’ formal topology $\mathbf{Pow}(\{\top\})_{\neg\neg}$, is defined by

$$S = \{\top\}, \top \triangleleft U \iff \neg\neg(\top \in U).$$

In [15, 16] it is shown that the system CZF cannot prove $\mathbf{Pow}(\{\top\})_{\neg\neg}$ to be set-presentable. Note that $\mathbf{Pow}(\{\top\})_{\neg\neg}$ is isomorphic with the Boolean formal space $\mathbf{Pow}(\{\top\})^{**}$ of regular elements of $\mathbf{Pow}(\{\top\})$ (cf. section 4).

It is then easy to see that $\mathbf{Pow}(\{\top\})_{\neg\neg}$ and S_{DM} are the ‘same’ formal space.

Lemma 5.7 $S_{DM} \cong \mathbf{Pow}(\{\top\})_{\neg\neg}$.

Proof. It is an exercise in intuitionistic logic to prove that $0 =_{S_{DM}} \emptyset$, $\neg\neg(1 \in U) \iff 1 \triangleleft_{DM} U$, and $\neg\neg(\top \in U)$ implies $1 \triangleleft_{DM} \{1 : \top \in U\}$. It follows that the homomorphisms $f : S_{DM} \rightarrow \mathbf{Pow}(\{\top\})_{\neg\neg}$ and $g : \mathbf{Pow}(\{\top\})_{\neg\neg} \rightarrow S_{DM}$, given by $f(0) = \emptyset$, $f(1) = \{\top\}$, and $g(\top) = \{1\}$, yield the required isomorphism. \square

Now the product $\mathbf{Pow}(\{\top\})_{\neg\neg} \times \mathbf{Pow}(\{\top\})_{\neg\neg}$ in FSp is simply $\mathbf{Pow}(\{\top\})_{\neg\neg}$ itself: $\mathbf{Pow}(\{\top\})_{\neg\neg}$ is ‘almost’ a terminal object in FSp, if a morphism with $\mathbf{Pow}(\{\top\})_{\neg\neg}$ as domain exists, then it is unique.

Conclusion

The generalised uniformity principle has been systematically exploited in this paper to obtain non-derivability results for the main formal systems for constructive mathematics, in particular with the aim of distinguishing topos-valid from intuitionistic generalised predicative mathematics. It has already been pointed out that it is improper to define ‘constructive’ an extension of CZF or CTT that is consistent with the generalised uniformity principle, as e.g. CZF plus the highly impredicative unbounded separation principle is one such extension. On the other hand, it may be reasonable to define ‘non-constructive’ a result that cannot be derived within some extension of CZF or CTT that is compatible with GUP. This note has thus shown that some standard topos-valid results are in fact non-constructive in this sense. A further important result, valid in any topos, that turns out to be a non-constructive theorem in the present sense is described in [12].

Appendix: The generalised uniformity principle in type theory

In the type-theoretic context the generalised uniformity principle reads informally as follows: given any set I , and any mapping R into the type of propositions $PROP$ taking a proposition and an element of I as arguments, if a mapping F is given from $PROP$ to I such that $R(P, F(P))$ holds for all P , then one may find an element $\bar{i} \in I$ such that $R(P, \bar{i})$ holds for all P . We denote this version of the principle by **GUP-CTT**. Observe that, due to the propositions-as-sets identification, the type $PROP$ of propositions may be replaced in GUP-CTT by the type SET of sets. In [10] this principle is formulated implicitly and it is claimed that GUP-CTT can be ‘added’ consistently to type theory. Models, due to T. Coquand, of type theory validating this form of the uniformity principle have then been discussed in [28].

Formally, GUP-CTT can be expressed in type theory (more specifically, in the logical framework [27, 8]) by the addition of two constants UP1, UP2 as follows (cf. [28]):

$$\begin{aligned} UP1 : (I : SET) \rightarrow R : (PROP \rightarrow El(I) \rightarrow PROP) \rightarrow F : (PROP \rightarrow El(I)) \rightarrow \\ G : (P : PROP \rightarrow El(R(P, F(P)))) \rightarrow El(I); \\ UP2 : (I : SET) \rightarrow R : (PROP \rightarrow El(I) \rightarrow PROP) \rightarrow F : (PROP \rightarrow El(I)) \rightarrow \\ G : (P : PROP) \rightarrow El(R(P, F(P))) \rightarrow ((P : PROP) \rightarrow El(R(P, UP1(IRFG))))). \end{aligned}$$

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