

# STABLE THEORIES AND REPRESENTATION OVER SETS

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ABSTRACT. In this paper we explore the representation property over sets. This property generalizes constructibility, however is weak enough to enable us to prove that the class of theories  $T$  whose models are representable is exactly the class of stable theories. Stronger results are given for  $\omega$ -stable.

## 1. PRELIMINARIES

**Convention 1.** We use  $\mathfrak{k}$  to denote an arbitrary class of structures (of a given language, closed under isomorphism). The class of structures of the language  $\{=\}$  is denoted  $\mathfrak{k}^{\text{eq}}$ .

- (1)  $\mathfrak{C}$  is a “monster” model for  $T$ . i.e. a sufficiently saturated one.
- (2) for a sequence of sets  $\langle A_\beta : \beta < \alpha \rangle$  let  $A_{<\alpha} := \bigcup_{\beta < \alpha} A_\beta$ ,  $A_{\leq \alpha} := A_{<\alpha+1}$ .
- (3)  $\text{tp}(a, A) := \text{tp}(a, A, \mathfrak{C})$ .

**Definition 1.1.** Let  $\mathfrak{k}$  be a class of structures of a given vocabulary  $\tau$ .

- $\text{Ex}_{\mu, \kappa}^1(\mathfrak{k})$  denotes the minimal class of structures  $\mathfrak{k}' \supseteq \mathfrak{k}$  with the property that for each structure  $I \in \mathfrak{k}$  there exists an enrichment  $I^+ \in \mathfrak{k}'$  by a partition  $\langle P_\alpha^{I^+} : \alpha < \kappa \rangle$ , partial unary functions  $\langle F_\beta^{I^+} : \beta < \mu \rangle$  such that  $F_\beta(P_\alpha) \subseteq P_{<\alpha}$  and  $P_\alpha, F_\beta \notin \tau_I$  hold for every  $\alpha < \kappa$ ,  $\beta < \mu$
- For a given model  $I \in \mathfrak{k}$ , we define the free algebra  $\mathcal{M} = \mathcal{M}_{\mu, \kappa}(I)$  as the model having the language  $\tau^+ := \tau_I \cup \{F_{\alpha, \beta}, \}_{\alpha < \mu, \beta < \kappa}$  where each  $F_{\alpha, \beta}$  is a  $\beta$ -place function.  $\|\mathcal{M}_{\mu, \kappa}(I)\|$  consists of all the terms constructed in the usual (well-founded inductive) way from elements of  $I$  using the functions  $F_{\alpha, \beta}$ . The functions and relations of  $\tau_I$  are interpreted in  $\mathcal{M}$  as partial functions and restricted relations on  $I \subseteq \mathcal{M}$ .
- Let  $\theta_{\mu, \kappa} := |\mathcal{M}_{\mu, \kappa}(\kappa)|$  (The power of the set of  $\mu$ -terms with  $\kappa$  constants).
- $\text{Ex}_{\mu, \kappa}^2(\mathfrak{k})$  denotes the class of models of the form  $\mathcal{M}_{\mu, \kappa}(I)$  for every  $I \in \mathfrak{k}$ .

**Definition 1.2.** Let  $M \models T$ ,  $I$  a structure.  $f : M \rightarrow I$  is a  $(\Gamma, \Delta)$ -representation of  $M$  in- $I$  iff  $\text{Rang}(f)$  is closed under functions in  $I$  (both partial and full), and for

$\bar{a}, \bar{b} \in {}^{<\omega}M$  the following holds:

$$\text{tp}_\Gamma(f(\bar{a}), \emptyset, I) = \text{tp}_\Gamma(f(\bar{b}), \emptyset, I) \Rightarrow \text{tp}_\Delta(\bar{a}, \emptyset, M) = \text{tp}_\Delta(\bar{b}, \emptyset, M)$$

- We say that  $M$  is  $(\mathfrak{k}, \Gamma, \Delta)$ -representable if  $I \in \mathfrak{k}$  and there exists a  $(\Gamma, \Delta)$ -representation  $f : M \rightarrow I$ .
- we say that the theory  $T$  is  $(\mathfrak{k}, \Gamma, \Delta)$ -representable if every  $M \models T$  is  $(\mathfrak{k}, \Gamma, \Delta)$ -representable.
- We omit  $\Delta, \Gamma$  from the notation if  $\Gamma = \text{qf}_{\mathcal{L}[\tau_I]}$ ,  $\Delta = \mathcal{L}[\tau_M]$ .

**Observation 1.1.** let  $M \models T$

- $M$  is  $\mathfrak{k}$ -representable implies  $M$  is  $\text{Ex}_{\mu, \kappa}^i(\mathfrak{k})$ -representable ( $i = 1, 2$ ).
- For  $i = 1, 2$ :  $M$  is  $\text{Ex}_{\mu_2, \kappa_2}^i(\text{Ex}_{\mu_1, \kappa_1}^i(\mathfrak{k}))$ -representable iff  $M$  is  $\text{Ex}_{\mu_1 + \mu_2, \kappa_1 + \kappa_2}^i(\mathfrak{k})$ -representable.
- $M$  is  $\text{Ex}_{\mu_2, \kappa_2}^2(\text{Ex}_{\mu_1, \kappa_1}^1(\mathfrak{k}))$ -representable implies  $M$  is  $\text{Ex}_{\mu_1, \kappa_1}^1(\text{Ex}_{\mu_2, \kappa_2}^2(\mathfrak{k}))$ -representable.

**Fact 1.** A map  $f : M \rightarrow I^+$  is a representation of  $M$  in  $I^+ \in \text{Ex}_{\mu, \kappa}^1(\mathfrak{k}^{\text{eq}})$ , if  $\text{tp}(\bar{a}, \emptyset, M) = \text{tp}(\bar{b}, \emptyset, M)$  holds for every  $U, \tilde{h}, \bar{a}, \bar{b}$  fulfilling the following condition:

$U \subseteq |I^+|$  is such that  $\text{cl}_{\{F_\beta^{I^+}\}} U = U$ ,  $\tilde{h}$  is a partial automorphism of  $I^+$  whose domain contains  $U$ , and  $\bar{a}, \bar{b} \in {}^m M$  are sequences such that  $\tilde{h}(f(\bar{a})) = f(\bar{b})$  and  $f(\bar{a}) \subseteq U$ .

## 2. STABLE THEORIES

**Discussion 2.1.** In this section we prove the equivalence  $\text{stable} = \text{Ex}_{\mu_1, \kappa_1}^1(\text{Ex}_{\mu_2, \kappa_2}^2(\mathfrak{k}^{\text{eq}}))$ -representable

**Theorem 2.1.** Let  $T$  be  $\text{Ex}_{\mu_1, \kappa_1}^1(\text{Ex}_{\mu_2, \kappa_2}^2(\mathfrak{k}^{\text{eq}}))$ -representable. If  $\bar{b} = \langle \bar{b}_\alpha : \alpha < \lambda \rangle \subseteq \mathfrak{C}$ , is such that  $\text{lg } \bar{b}_\alpha < \mu = \mu_1 + \kappa_2$ ,  $\lambda > \kappa_1 + \mu_1 + \kappa_2 +$ , and  $\lambda > \chi^{<\mu}$  for every  $\chi < \lambda$  then there exists an  $S \subseteq \lambda$  of cardinality  $\lambda$  such that  $\langle \bar{b}_\alpha : \alpha \in S \rangle$  is an indiscernible set.

*Proof.* Let  $M \models T$  contain  $\bar{a}$ ,  $f : M \rightarrow \mathcal{M}^{++} := (\mathcal{M}_{\mu_2, \kappa_2}(I), P_\alpha, F_\beta)_{\alpha < \kappa_1, \beta < \mu_1}$  a representation. Let  $\bar{a}_\alpha = f(\bar{b}_\alpha)$ .

assume w.l.o.g:

- Every  $\bar{a}_\alpha$  is closed under subterms in  $\mathcal{M}_{\mu_2, \kappa_2}(I)$ .
- Every  $\bar{a}_\alpha$  is closed under the partial functions  $F_\beta$ .
- $\text{lg } \bar{a}_\alpha = \xi$  ( for all  $\alpha < \lambda$  ).

$\lambda = \text{cf } \lambda > (\theta_{\mu_1, \kappa_1})^\xi$  and therefore there exists  $\bar{\sigma}(\bar{x})$ ,  $\text{lg } \bar{x} < \kappa_2$ ,  $S_0 \in [\lambda]^\lambda$  such that for all  $\alpha \in S_0$  there exists  $\bar{t}_\alpha \subseteq {}^{<\kappa_2}I$  such that  $\bar{a}_\alpha = \bar{\sigma}(\bar{t}_\alpha)$ .

similarly,  $(\kappa_2)^\xi < \lambda$  and there exists an  $S_1 \in [S_0]^\lambda$  on which the map

$$\alpha \mapsto \{(i, \beta) \in \xi \times \kappa_2 : a_\alpha^i \in P_\beta\}$$

is constant and equal to a binary relation  $R_1$ .

Also,  $\xi^{\mu_1+\xi} < \lambda$  implies that there exists an  $S_2 \in [S_1]^\lambda$  on which the map

$$\alpha \mapsto \{(\beta, \zeta_0, \zeta_1) : \zeta_0, \zeta_1 < \xi, \beta < \mu_1, F_\beta(a_\alpha^{\zeta_0}) = a_\alpha^{\zeta_1}\}$$

is constant and equal to the relation  $R_2$ .

From lemma 3.3 it follows that there exist  $S_3 \in [S_2]^\lambda$ ,  $U \subseteq \xi$ ,  $E \subseteq \xi \times \xi$  such that:

- $\bar{a}_\alpha \upharpoonright U = \bar{a}_\beta \upharpoonright U$  for all  $\alpha, \beta \in S_3$
- $E$  an equivalence such that  $a_\alpha^i = a_\alpha^j \leftrightarrow (i, j) \in E$  for all  $\alpha \in S_3$ .
- $a_\alpha^i = a_\beta^j \rightarrow i, j \in U$  for all  $\alpha \neq \beta \in S_3$ .

We show that for every  $\bar{u}, \bar{v} \subseteq S_3$  without repetitions and of length  $\ell < \omega$ , there exists a partial automorphism  $h$  of  $\mathcal{M}^{++}$  such that  $h(\bar{a}_{\bar{v}}) = \bar{a}_{\bar{u}}$ .

Indeed, define  $h(a_{v_k}^j) = a_{u_k}^j$  for all  $j < \xi, k < \ell$ .  $E$  and  $U$  show that  $a_{v_{k_0}}^{j_0} = a_{v_{k_1}}^{j_1} \rightarrow a_{u_{k_0}}^{j_0} = a_{u_{k_1}}^{j_1}$ . Hence,  $h$  is well-defined.

Let the term  $\bar{\sigma}(\bar{t})$  be in  $\text{Dom}(h)$ . Since  $\bar{a}_{\bar{v}}$  is closed under subterms it follows that  $h(\bar{\sigma}(\bar{t})) = \bar{\sigma}(h(\bar{t}))$ .

$h$  respects  $P_\alpha$ :

$$a_{u_\kappa}^j \in P_\alpha \leftrightarrow (j, \alpha) \in R_1 \leftrightarrow a_{v_k}^j \in P_\alpha$$

$h$  commutes with  $F_\alpha$ : for all  $a_{v_{k_0}}^{j_0}, a_{v_{k_1}}^{j_1} \in \text{Dom}(h)$ , since  $\bar{a}_{v_{k_0}}$  is closed under it follows that there exists  $j < \xi$  such that  $F_\alpha(a_{v_{k_0}}^{j_0}) = a_{v_{k_0}}^j$ . Therefore  $(j, j_0) \in E$  and by the definition it follows

$$F_\alpha(h(a_{v_{k_0}}^{j_0})) = F_\alpha(a_{v_{k_1}}^{j_0}) = a_{v_{k_1}}^j = h(a_{v_{k_0}}^j) = h(F_\alpha(a_{v_{k_0}}^{j_0}))$$

□

**Theorem 2.2.** *Quote theorem [2, II.2.13]*

**Theorem 2.3.**  *$T$  is  $\text{Ex}_{\mu_1, \kappa_1}^1(\text{Ex}_{\mu_2, \kappa_2}^2(\mathfrak{E}^{\text{eq}}))$ -representable implies  $T$  stable.*

*Proof.* Let  $T$  be unstable. from theorem 2.2 and compactness, there exist  $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}_T$ ,  $M \models T$  and a sequence  $\langle \bar{a}_i : i < \lambda \rangle$ ,  $\lambda = (\kappa^\mu)^+ + \beth_2(\mu)^+$  such that  $\models \varphi(\bar{a}_i, \bar{a}_j)^{\text{if}(i < j)}$  for all  $i, j < \lambda$ . Assume towards contradiction that  $f : M \rightarrow I^+$  is a representation of  $M$  in  $I^+ \in \text{Ex}_{\mu_1, \kappa_1}^1(\text{Ex}_{\mu_2, \kappa_2}^2(\mathfrak{E}^{\text{eq}}))$ . Then theorem 2.1 implies in particular the existence of  $i, j < \lambda$ , a partial automorphism  $g$  of  $I^+$  with domain and range closed under functions, such that:

$$g(f(\bar{a}_i \widehat{\ } \bar{a}_j)) = f(\bar{a}_j \widehat{\ } \bar{a}_i)$$

from the definition of a representation we get

$$\text{tp}(\bar{a}_i \hat{\ } \bar{a}_j, \emptyset, M) = \text{tp}(\bar{a}_j \hat{\ } \bar{a}_i, \emptyset, M)$$

a contradiction to the definition of  $\varphi$ .  $\square$

**Discussion 2.2.** *We now turn to the proof of the other direction of equivalence. This will require more facts on stable theories and strongly independent sets, defined below.*

**Definition 2.1.** *A set  $\mathbb{I} \subseteq \mathfrak{C}$  will be called strongly independent over  $A$  if the following holds:*

$\otimes$  *for all  $a \in \mathbb{I}$ ,  $\text{tp}(a, A \cup \mathbb{I} \setminus \{a\}, M)$  is the unique  $p \in \mathbf{S}(A \cup \mathbb{I} \setminus \{a\})$  such that  $p \supseteq \text{tp}(a, A \cup \mathbb{I} \setminus \{a\})$  and  $p$  does not fork over  $A$ .*

**Definition 2.2.** *We call the a sequence  $\langle \mathbb{I}_\alpha : \alpha < \mu \rangle$  of subsets of  $M$  a strongly-independent decomposition (in short: *s.i.d*) of length  $\mu$  of  $M$  if for all  $\alpha < \mu$ , it holds that  $\mathbb{I}_\alpha$  is strongly independent over  $\mathbb{I}_{<\alpha}$ , and that  $|M| = \mathbb{I}_{<\mu}$ . [2, II.2.13]*

**Convention 2.** *We assume from this point onwards that  $T$  is stable.*

**Claim 2.4.** *let  $a_1, a_2 \in \mathfrak{C}$ ,  $A \supseteq B_1, B_2$  such that  $\text{tp}(a_i, A \cup \{a_{3-i}\})$  is non-forking over  $B_i$ , and  $\text{tp}(a_i, A)$  is the unique nonforking extension in  $\mathbf{S}(A)$  of  $\text{tp}(a_i, B_i)$ . Then  $(*)_1 \Leftrightarrow (*)_2$  where:*

$(*)_i$   *$\text{tp}(a_i, B_i)$  has a unique nonforking extension whose domain is  $A \cup \{a_{3-i}\}$ .*

*Proof.* it is sufficient to prove  $\neg(*)_2 \Rightarrow \neg(*)_1$ , since the converse follows by symmetry.

Assume that  $\text{tp}(a_2, B_2)$  has two distinct nonforking extensions  $p_1, p_2 \in \mathbf{S}(A \cup \{a_1\})$ .

Then, there exists  $\varphi \in p_1$ ,  $\neg\varphi \in p_2$ ,  $\varphi = \varphi(x, a_1, \bar{c})$ . Let  $b_1, b_2$  realize  $p_1, p_2$ , respectively.

$\text{tp}(b_i, A) = p_i \upharpoonright A$  is a nonforking extension of  $p$  implies  $p_1 \upharpoonright A = p_2 \upharpoonright A$ . Thus, for  $i < 2$  There exist elementary maps  $F_i$  in  $\mathfrak{C}$  so that  $F_i \upharpoonright A = \text{id}_A$ ,  $F_i(b_i) = a_2$ .

Let  $q_i \in \mathbf{S}(A \cup \{b_i\})$  be a nonforking extension of  $\text{tp}(a_1, B_1)$ .

Then  $F_i(q_i) \in \mathbf{S}(A \cup \{a_2\})$  is a nonforking extension of  $\text{tp}(a_1, B_1)$  ( $F_i \upharpoonright A = \text{id}_A$ , and elementary maps preserve nonforking).

Now note that  $\models \varphi(b_1, a_1, \bar{c}) \wedge \neg\varphi(b_2, a_1, \bar{c})$ , hence  $\varphi(a_2, x, \bar{c}) \in F_1(q_1)$  and  $\neg\varphi(a_2, x, \bar{c}) \in F_2(q_2)$ . Therefore,  $F_i(q_i)$  are distinct extensions, as required.  $\square$

**Definition 2.3.** *An ordered partition  $\langle \mathbb{J}_\alpha : \alpha < \mu' \rangle$  is called an order-preserving refinement of the ordered partition  $\langle \mathbb{I}_\alpha : \alpha < \mu' \rangle$  if it is a refinement as a partition and  $\alpha' < \beta'$  for all  $\alpha < \beta < \mu$ ,  $\alpha', \beta' < \mu'$  such that  $\mathbb{I}_\alpha \supseteq \mathbb{J}_{\alpha'}$ ,  $\mathbb{I}_\beta \supseteq \mathbb{J}_{\beta'}$ .*

**Claim 2.5.** *If  $\langle \mathbb{I}_\alpha : \alpha < \mu \rangle$  is an s.i.d of  $M$ , then every order-preserving refinement of it is an s.i.d. of  $M$ .*

*Proof.* Let  $\alpha' < \mu'$ . we show that  $\mathbb{J}_{\alpha'} \subseteq \mathbb{I}_\alpha$  is strongly independent over  $\mathbb{J}_{<\alpha'}$ . Then let  $a \in \mathbb{J}_{\alpha'}$ .

$\text{tp}(a, \mathbb{I}_{\leq\alpha} \setminus \{a\})$  is nonforking over  $\mathbb{I}_{<\alpha}$  and hence, the reduct  $\text{tp}(a, \mathbb{J}_{\leq\alpha'})$  is nonforking over  $\mathbb{I}_{<\alpha}$ , nor does it fork over the larger  $\mathbb{J}_{<\alpha'}$ . On the other hand, if  $\text{tp}(a, \mathbb{J}_{<\alpha'}) \subseteq q \in \mathbf{S}(\mathbb{J}_{\leq\alpha'} \setminus \{a\})$  is nonforking over  $\mathbb{J}_{<\alpha}$ , it has an extension  $q \subseteq q' \in \mathbf{S}(\mathbb{I}_{\leq\alpha} \setminus \{a\})$  which is nonforking over  $\mathbb{J}_{<\alpha}$ .  $a \in \mathbb{I}_\alpha$  implies that  $\text{tp}(a, \mathbb{J}_{\leq\alpha'} \setminus \{a\}) \subseteq \text{tp}(a, \mathbb{I}_{\leq\alpha} \setminus \{a\})$  is nonforking over  $\mathbb{I}_{<\alpha}$  and since  $\mathbb{I}_\alpha$  is strongly independent over  $\mathbb{I}_{<\alpha}$  we get  $q' = \text{tp}(a, \mathbb{I}_{\leq\alpha} \setminus \{a\})$ , and in particular,

$$q' \upharpoonright (\mathbb{J}_{\leq\alpha'} \setminus \{a\}) = \text{tp}(a, \mathbb{J}_{\leq\alpha'} \setminus \{a\})$$

therefore  $\mathbb{J}_{\alpha'}$  is as required, nonforking over  $\mathbb{J}_{<\alpha'}$ .  $\square$

**Theorem 2.6.** *Let  $p, q \in \mathbf{S}(B)$  be distinct, nonforking over  $A \subseteq B$ . Then there exists an  $E \in \text{FE}(A)$  such that:*

$$p(x) \cup q(y) \vdash \neg E(x, y)$$

(cf [2, III;2.9(2)])

**Claim 2.7.** *Let  $A \subset B$  be such that if  $\varphi$  is a formula over  $B$  which is almost over  $A$ , then there exists a formula over  $A$  which is equivalent to  $\varphi$  modulo  $T$ . If  $p, q \in \mathbf{S}(B)$  are distinct nonforking over  $A$ , There exists a  $\varphi_*(x, \bar{c})$  such that  $p \vdash \varphi_*$ ,  $q \vdash \neg \varphi_*$ .*

*Proof.* By 2.6, there exists an  $E \in \text{FE}(A)$  such that  $p(x) \cup q(y) \vdash \neg E(x, y)$ . Let  $\{b_i : i < n(E)\} \subseteq \mathfrak{C}$  represent the equivalence classes of  $E$ . Define  $w := \{i < n(E) : p(x) \cup \{E(x, b_i)\} \text{ is consistent}\}$ , and let  $\varphi(x) := \bigvee_{i \in w} E(x, b_i)$ . Then

- w.l.o.g for all  $i \in w$ ,  $b_i \in \mathfrak{C}$  realizes  $p$ .
- $p(x) \vdash \varphi(x)$  ( if  $a$  realizes  $p$  there exists a  $b_i$  such that  $\models aEb_i$  since the  $b_i$  are representatives of the equivalence classes of  $E$ . on the other hand,  $i$  must belong to  $w$ , which implies that  $\varphi(a)$  holds ) .
- Similarly,  $q(x) \vdash \neg \varphi(x)$  since if  $a$  realizes  $q$  then  $p(x) \cup q(y) \vdash \neg E(x, y)$  therefore  $\neg E(b_i, a)$  for all  $i \in w$ , therefore  $\models \neg \varphi(a)$ .
- $\varphi(x)$  is preserved by members of  $\text{Aut}(\mathfrak{C}, B)$ : Let  $f \in \text{Aut}(\mathfrak{C}, B)$ . Then  $f$  preserves  $E$  (and its equivalence classes in  $\mathfrak{C}$ ) and  $p(\text{Dom}(p) = B)$  implying:
  - $p(x) \cup \{E(x, b_i)\} \Leftrightarrow p(x) \cup \{E(x, f(b_i))\}$  holds for all  $i < n(E)$ .
  - $\neg E(f(b_i), f(b_j))$  for all  $i, j < n(E)$ ,  $i \neq j$ .

- $f$  acts as a permutation on  $\mathfrak{C}/E$ , and when reduced also on  $\{b_i/E : i \in w\}$ , therefore:

$$f(\varphi(\mathfrak{C})) = f\left(\bigcup_{i \in w} b_i/E\right) = \bigcup_{i \in w} f(b_i)/E = \varphi(\mathfrak{C})$$

implying  $\models \varphi(x) \equiv f(\varphi(x))$ . Lemma Sh:c,III.2.3] implies that  $\varphi(x)$  has an equivalent formula  $\varphi_*$  over  $B$ , as needed.  $\square$

**Claim 2.8.** *For all  $p \in \mathbf{S}^m(B)$  there exists  $A \subseteq B$ ,  $|A| < \kappa(T)$  such that  $p$  does not fork over  $A$ . Also,  $\kappa(T) \leq |T|^+$ .*

(cf. [2, III;3.2, 3.3])

**Claim 2.9.** *The number of formulas almost over  $A$  is (up to logical equivalence) at most  $|A| + |T|$*

(cf. [2, III;2.2(2)])

**Lemma 2.10.** *if  $M \models T$  then there exists an s.i.d of length  $\mu = |T|^+$*

*Proof.* We construct inductively a sequence  $\langle \mathbb{I}_\alpha : \alpha < \mu \rangle$  such that  $\mathbb{I}_\alpha$  is strongly independent over  $\mathbb{I}_{<\alpha}$  and is moreover maximal with respect to this property ( for all  $\mathbb{I} \supseteq \mathbb{I}_\alpha$  is not strongly independent over  $\mathbb{I}_{<\alpha}$ ), for all  $\alpha$ .

Assume towards contradiction that  $a \in M \setminus \mathbb{I}_{<\mu}$ . By the definition of  $\kappa(T)$  and 2.8 we get a set

$$B \subseteq \mathbb{I}_{<\mu}, |B| < \kappa(T) \leq |T|^+$$

such that  $p(x) := \text{tp}(a, \mathbb{I}_{<\mu})$  is nonforking over  $B$ , and there exists an  $\alpha_0(*) < \mu$  such that  $\mathbb{I}_{<\alpha_0(*)} \supseteq B$ .

Let

$$\Gamma := \{ \varphi(x; \bar{c}) : \varphi(x, \bar{c}) \text{ is almost over } B, \varphi(x; \bar{y}) \in \mathcal{L}, \bar{c} \in {}^{\text{lg}} \bar{y} \mathbb{I}_{<\mu} \}$$

By claim 2.9, there exists a set  $\Gamma_* \subseteq \Gamma$ ,  $|\Gamma_*| \leq |B| + |T| < \text{cf}(|T|^+)$  of representatives (by logical equivalence) of the formulas almost over  $B$ . Hence, there exists  $\alpha_1(*) < \mu$  such that  $\bar{b} \subseteq \mathbb{I}_{<\alpha_1(*)}$  for all  $\varphi(x, \bar{b}) \in \Gamma_*$ . Let  $\alpha(*) = \max_{i < 2} \{\alpha_i(*)\}$ .

We now show that  $p \upharpoonright \mathbb{I}_{\leq \alpha(*)}$  is the only extension in  $\mathbf{S}(\mathbb{I}_{\leq \alpha(*)})$  of  $p \upharpoonright \mathbb{I}_{<\alpha(*)}$  which is nonforking over  $\mathbb{I}_{<\alpha(*)}$ :

Indeed,  $p$  is nonforking over  $B$ . Let  $q \in \mathbf{S}(\mathbb{I}_{\leq \alpha(*)})$  a nonforking extension of  $p \upharpoonright \mathbb{I}_{<\alpha(*)}$ .

By transitivity of non-forking,  $q$  is nonforking over  $B$ . Assume towards contradiction that  $q \neq p$ . Then by 2.6 there exists an  $E \in FE(B)$  such that  $q(x) \cup p(y) \vdash \neg E(x, y)$ , and particularly  $q(x) \vdash \neg E(x, a)$ .

The formula  $E(x, a)$  is almost over  $B$ , therefore by the choice of  $\alpha_1(*)$ , there exists a  $\varphi(x, \bar{b})$  logically equivalent to  $E(x, a)$  in  $T$ , with  $\bar{b} \subseteq \mathbb{I}_{<\alpha(*)}$ .

Now, since  $E(a, a)$ , it also holds that  $\models \varphi(a, \bar{b})$ , and  $\bar{b} \subseteq \mathbb{I}_{<\alpha(*)}$  implies  $\varphi(x, \bar{b}) \in \text{tp}(a, \mathbb{I}_{<\alpha(*)}) = q \upharpoonright \mathbb{I}_{<\alpha(*)}$ , a contradiction.

In particular,  $\text{tp}(a, \mathbb{I}_{\leq\alpha(*)})$  is the only nonforking extension of  $\text{tp}(a, \mathbb{I}_{<\alpha(*)})$  in  $\mathbf{S}(\mathbb{I}_{\leq\alpha(*)} \setminus \{b\})$ . By the choice of  $\mathbb{I}_{\alpha(*)}$  it follows for all  $b \in \mathbb{I}_{\alpha(*)}$  that  $\text{tp}(b, \mathbb{I}_{\leq\alpha(*)} \setminus \{b\})$  is the only nonforking extension of  $\text{tp}(b, \mathbb{I}_{<\alpha(*)})$  in  $\mathbf{S}(\mathbb{I}_{\leq\alpha(*)} \setminus \{b\})$ .

From claim 2.4 it follows that  $\text{tp}(b, \mathbb{I}_{\leq\alpha(*)} \setminus \{b\} \cup \{a\})$  is the only nonforking extension of  $\text{tp}(b, \mathbb{I}_{<\alpha(*)})$  in  $\mathbf{S}(\mathbb{I}_{\leq\alpha(*)} \setminus \{b\} \cup \{a\})$ .

So,  $\circledast$  holds for  $\mathbb{I}_{\alpha(*)} \cup \{a\}$  (with respect to  $\mathbb{I}_{<\alpha(*)}$ ) contradicting the maximality of  $\mathbb{I}_{\alpha(*)}$ .  $\square$

**Claim 2.11.** *Forking is preserved under elementary maps [2, III.1.5]*

**Theorem 2.12.** *definability for types cf. [2, II;2.2]*

### 2.1. Representing stable theories.

**Theorem 2.13.** *If  $M \models T$ , then  $M$  is  $\text{Ex}_{|T|^+, |T|}^1(\mathfrak{e}^{\text{eq}})$ -representable.*

*Proof.* By 2.10 we get a strongly independent decomposition of  $M$ :  $\langle \mathbb{I}_\alpha : \alpha < |T|^+ \rangle$ .

By Claim 2.5 we assume w.l.o.g  $|\mathbb{I}_1| = |\mathbb{I}_0| = 1$ .

Define the structure  $I^+ \in \text{Ex}_{|T|^+, |T|}^1(\mathfrak{e}^{\text{eq}})$  as follows:

- (1)  $|I^+| = |M|$ .
- (2) for all  $\alpha < |T|^+$ ,  $P_\alpha^{I^+} = \mathbb{I}_\alpha$ .
- (3) for all  $\varphi(x, \bar{y}) \in \mathcal{L}_M$  define  $n$  one-place partial functions (let  $n = \text{lg } \bar{z}$ )  $\{F_{\varphi(x, \bar{y}), j}^{I^+}(x) : j < n\}$  as follows:
  - (a)  $\text{Dom} F_{\varphi(x, \bar{y}), j}^{I^+} = |M| \setminus (\mathbb{I}_0 \cup \mathbb{I}_1)$ .
  - (b) By Theorem 2.12 we get for every  $\varphi(x, \bar{y}) \in \mathcal{L}_M$  another formula  $\psi_\varphi(\bar{y}, \bar{z}) \in \mathcal{L}_M$ , such that for all  $2 \leq \alpha < \mu$ ,  $a \in \mathbb{I}_\alpha$  there exists  $\bar{c}_a \in {}^{\text{lg } \bar{z}} \mathbb{I}_{<\alpha}$  such that for all  $\bar{b} \in \mathbb{I}_\alpha$ ,  $\models \varphi[a, \bar{b}] \Leftrightarrow \models \psi_\varphi[\bar{b}, \bar{c}_a]$  holds.
  - (c) For all  $2 \leq \alpha < \mu$  and  $a \in \mathbb{I}_\alpha$ , let  $F_{\varphi(x, \bar{y}), j}^{I^+}(a) := (\bar{c}_a)_j$
- (4) Add  $|T|$  partial functions  $\langle (F_i^*)^{I^+} : i < |T| \rangle$  as follows:
  - (a)  $\text{Dom} F_i^* = |M| \setminus \mathbb{I}_{<2}$
  - (b) Fix  $\alpha > 1$ , then there exists  $|B| \leq |T|$  such that for every  $\varphi(\bar{x}, \bar{c})$  over  $\mathbb{I}_{<\alpha}$  which is almost over  $B$  there exists a  $\theta(\bar{x}, \bar{d})$  over  $B$  such that  $\models \forall \bar{x} (\theta(\bar{x}, \bar{d}) \leftrightarrow \varphi(\bar{x}, \bar{c}))$ :
    - (i) Let  $|B_0| < \kappa(T) \leq |T|^+$ ,  $B_0 \subseteq \mathbb{I}_{<\alpha}$  such that  $\text{tp}(a, \mathbb{I}_{<\alpha})$  does not fork over  $B_0$ .
    - (ii) Assume  $B_n$  is defined and let

$$B_{n+1} := B_n \cup \{\bar{c} : \varphi(\bar{x}, \bar{c}) \in S'\}$$

where  $S'$  is a complete set of representatives of  $S$ , relative to logical equivalence in  $T$

$$S := \{\varphi(\bar{x}, \bar{c}) \in \mathcal{L}_T : \bar{c} \subseteq \mathbb{I}_{<\alpha}, \varphi \text{ is almost over } B_n\}$$

by 2.9 we can assume w.l.o.g  $|S'| \leq |T| + |B_n| = |T|$ .

(iii) Then the set  $B = \bigcup_{n < \omega} B_n$  is as required.

(c) Let  $\langle b_i : i < |T| \rangle$  enumerate  $B$  (possibly with repetitions). We define  $F_i(a) = b_i$ .

(5) Let  $f : M \rightarrow I^+$  be defined as  $f(a) = a$  for all  $a \in |M|$ .

Let  $h$  be a partial automorphism of  $I^+$  whose domain and range are closed under partial functions of  $I^+$ .

We show that  $\text{tp}(h(\bar{a}), \emptyset, M) = \text{tp}(\bar{a}, \emptyset, M)$  holds for all  $\bar{a} \subseteq \text{Dom}(h)$ :

- It is sufficient to show for all  $\alpha < |T|^+$ ,  $n < \omega$ ,  $\bar{a} \in \mathbb{I}_\alpha \cap \text{Dom}(h)$  without repetitions ( $n := \text{lg } \bar{a}$ ) the following holds:

$$h(\text{tp}(\bar{a}, \mathbb{I}_{<\alpha} \cap \text{Dom}(h))) = \text{tp}(h(\bar{a}), \mathbb{I}_{<\alpha} \cap \text{Rang}(h)) \quad \boxtimes_{\alpha, n}$$

we prove this by induction on the lexicographic well-order  $|T|^+ \times \omega$

- For  $\boxtimes_{\alpha, n}$  holds for  $\alpha < 2$  since  $\mathbb{I}_\alpha$  is a singleton.
- Let  $\alpha \geq 2$ , and assume  $\boxtimes_{\beta, n}$  for all  $n < \omega$  and  $\beta < \alpha$ .
- $\boxtimes_{\alpha, 1}$  holds, since let  $a \in \mathbb{I}_\alpha$ ,  $\varphi(x, \bar{c})$  a formula over  $\text{Dom}(h) \cap \mathbb{I}_{<\alpha}$  such that  $\varphi[a, \bar{c}]$  holds. Then by the definitions of the  $F$ 's above  $\psi_\varphi[\bar{c}, F_{\varphi, 0}(a) \dots F_{\varphi, \text{lg } \bar{y}-1}(a)]$  holds. Since the latter is a formula over  $\text{Dom}(h) \cap \mathbb{I}_{<\alpha}$  and by the induction hypothesis it follows that  $\psi_\varphi[h(\bar{c}), h(F_{\varphi, 0}(a)) \dots h(F_{\varphi, \text{lg } \bar{y}-1}(a))]$  holds. Also by the induction hypothesis,  $h$  commutes with the functions of  $I^+$  over the domain  $\mathbb{I}_{<\alpha} \cap \text{Dom}(h)$ . Hence,  $\psi_\varphi[h(\bar{c}), F_{\varphi, 0}(h(a)) \dots F_{\varphi, \text{lg } \bar{y}-1}(h(a))]$  holds. The definition of  $F_{\varphi, j}(x)$  implies that  $M \models \varphi[h(a), h(\bar{c})]$ .
- For  $n > 1$  We continue by induction, but first we prove the following:

**Claim 2.14.** *Let  $A \subseteq I^+$  be closed under functions of  $I^+$ . Then  $A \cap \mathbb{I}_\alpha$  is strongly independent over  $A \cap \mathbb{I}_{<\alpha}$ .*

proof Let  $A_\alpha = \mathbb{I}_\alpha \cap A$ ,  $a \in A_\alpha$ ,  $B := \{F_i^*(a) : i < |T|\}$ . Then,

- (1)  $B \subseteq A_{<\alpha}$ .
- (2) By the choice of the  $F_i^*$ 's it holds that  $\text{tp}(a, \mathbb{I}_{<\alpha})$  is nonforking over  $B$
- (3) By 2 and by transitivity of nonforking,  $\text{tp}(a, \mathbb{I}_{<\alpha} \setminus \{a\})$  is nonforking over  $B$ .  $\text{tp}(a, \mathbb{I}_{<\alpha} \setminus \{a\})$  is a nonforking extension of  $\text{tp}(a, \mathbb{I}_{<\alpha})$ .
- (4) For any formula over  $\mathbb{I}_{<\alpha}$  which is almost over  $B$  there exists an equivalent formula (in  $T$ ) over  $B$  (by the choice of the  $F_i^*$ )

The first two properties imply that  $\text{tp}(a, A_{\leq\alpha} \setminus \{a\}) \subseteq \text{tp}(a, \mathbb{I}_{<\alpha} \setminus \{a\})$  is nonforking over  $A_{<\alpha}$ .

We turn to proving the uniqueness. Let  $q_0 \in \mathbf{S}(A_{\leq\alpha} \setminus \{a\})$  be a nonforking extension of  $\text{tp}(a, A_{<\alpha})$ .

- $q_0$  has a nonforking extension  $q \in \mathbf{S}(\mathbb{I}_{\leq\alpha} \setminus \{a\})$ .
- $q$  is nonforking over  $A_{\leq\alpha} \setminus \{a\}$  and by transitivity nonforking over  $A_{<\alpha}$  and therefore nonforking over  $A_{<\alpha} \subseteq \mathbb{I}_{<\alpha}$ .
- $q \upharpoonright \mathbb{I}_{<\alpha} = \text{tp}(a, \mathbb{I}_{<\alpha})$  - since otherwise, a formula  $\varphi(x)$  over  $\mathbb{I}_{<\alpha}$  exists such that  $q(x) \vdash \varphi(x)$ ,  $\text{tp}(a, \mathbb{I}_{<\alpha}) \vdash \neg\varphi(x)$ . By 4 above ( as  $B$  was chosen ) and claim 2.7  $\varphi(x)$  is equivalent to a formula over  $B$ . Hence,  $q \upharpoonright B \neq \text{tp}(a, B)$  contradicting the choice of  $q$ .
- So,  $q$  is a nonforking extension of  $q \upharpoonright \mathbb{I}_{<\alpha}$ , unique by the strong independence of  $\mathbb{I}_\alpha$  over  $\mathbb{I}_{<\alpha}$ , and therefore equal to  $\text{tp}(a, \mathbb{I}_{\leq\alpha} \setminus \{a\})$ .
- The above arguments imply the required conclusion -  $q_0 = q \upharpoonright (A_{\leq\alpha} \setminus \{a\}) = \text{tp}(a, A_{\leq\alpha} \setminus \{a\})$

□

- We continue the main proof, letting  $D_\gamma := \text{Dom}(h) \cap \mathbb{I}_\gamma$ ,  $R_\gamma := \text{Rang}(h) \cap \mathbb{I}_\gamma$  ( for all  $\gamma < |T|^+$ ,  $h''(D_\gamma) = R_\gamma$ ).

Let  $\bar{a} \in^n (D_\alpha)$  and  $b \in D_\alpha \setminus \bar{a}$ .

- $h \upharpoonright (D_{<\alpha} \cup \bar{a})$  is elementary by the induction hypothesis.
- $\text{tp}(b, D_{\leq\alpha} \setminus \{b\})$  does not fork over  $D_{<\alpha}$  (by the last claim, and since  $\text{Dom}(h)$  is closed under functions), therefore  $\text{tp}(b, D_{<\alpha} \cup \bar{a})$  also does not fork over  $D_{<\alpha}$ .

The above, with claim 2.11 imply that  $q := h(\text{tp}(b, D_{<\alpha} \cup \bar{a}))$  does not fork over  $h(\text{Dom}(h) \cap \mathbb{I}_{<\alpha}) = \text{Rang}(h) \cap \mathbb{I}_{<\alpha}$ .

- $\boxtimes_{\beta,1}$  holds for all  $\beta < \alpha$ , and in particular  $q \in \mathbf{S}(R_{<\alpha} \cup \bar{a})$  is a nonforking extension of  $\text{tp}(h(b), R_{<\alpha})$ . Also,  $q$  has a nonforking extension  $q' \in \mathbf{S}(R_{\leq\alpha} \setminus h(b))$  which does not fork over  $R_{<\alpha}$  by transitivity.
- On the other hand, since  $\text{Rang}(h)$  is closed under functions and by the last claim, it follows that  $R_\alpha$  is strongly independent over  $R_{<\alpha}$ . hence,  $q' = \text{tp}(h(b), R_{\leq\alpha} \setminus \{h(b)\})$ . After reduction to  $R_{<\alpha} \cup h(\bar{a})$  we get

$$\text{tp}(h(a), R_{<\alpha} \cup h(\bar{a})) = h(\text{tp}(b, D_{<\alpha} \cup \bar{a}))$$

implying the inductive step from  $(\alpha, n)$  to  $(\alpha, n+1)$

$$\text{tp}(h(b \frown \bar{a}), R_{<\alpha}) = h(\text{tp}(b \frown \bar{a}, D_{<\alpha}))$$

□

## 2.2. Representation for $\omega$ -stable theories.

**Convention 3.** For the remainder of the section  $T$  is  $\omega$ -stable

**Claim 2.15.** Let  $p \in \mathbf{S}(A)$ . Then, there exists a finite  $B \subseteq A$  such that  $p$  is a nonforking extension of  $p \upharpoonright B$ . (See: [2])

**Claim 2.16.** For every  $p \in \mathbf{S}(A)$  there exists a finite  $B \subseteq A$  such that  $p$  is the unique nonforking extension of  $p \upharpoonright B$  in  $\mathbf{S}(A)$ .

**Claim 2.17.** Let  $M \models T$ .  $M$  has a strongly independent decomposition  $\langle \mathbb{I}_n : n < \omega \rangle$ , so that

- (1)  $\mathbb{I}_0$  is an indiscernible set over  $\emptyset$  (possibly finite), and
- (2) For every  $a \in \mathbb{I}_n$ ,  $n < \omega$  there exists a finite  $B_a \subseteq \mathbb{I}_{<n}$  so that  $\text{tp}(a, \mathbb{I}_{\leq n} \setminus \{a\})$  is the unique nonforking extension of  $\text{tp}(a, B_a)$  in  $\mathbf{S}(\mathbb{I}_{\leq n} \setminus \{a\})$ .

*Proof.* The first condition is fulfilled by a singleton, so it is possible to find a  $\mathbb{I}_0 \subseteq |M|$  as above. For  $n > 0$ , Construct a sequence  $\langle \mathbb{I}_n : n < \omega \rangle$  such that  $\mathbb{I}_n \subseteq |M|$  is maximal with respect to the second condition (possibly empty) for every  $n < \omega$ . Assume towards contradiction that there exists  $a \in M \setminus \mathbb{I}_{<\omega}$ . By 2.16 it follows that there exists a finite  $B_a \subseteq \mathbb{I}_{<\omega}$  such that  $\text{tp}(a, \mathbb{I}_{<\omega})$  is the unique nonforking extension of  $\text{tp}(a, B_a)$  in  $\mathbf{S}(\mathbb{I}_{<\omega})$ . Clearly, this implies that  $\mathbb{I}_n \neq \emptyset$  for all  $n < \omega$ . Therefore, there exists  $0 < n_* < \omega$  such that  $B_a \subseteq \mathbb{I}_{<n_*}$ . In particular it follows that  $\text{tp}(a, \mathbb{I}_{\leq n_*})$  is the unique nonforking extension of  $\text{tp}(a, B_a)$  in  $\mathbf{S}(\mathbb{I}_{\leq n_*})$  (otherwise, by transitivity of nonforking we would have two nonforking extensions in  $\mathbf{S}(\mathbb{I}_{<\omega})$ ).

The construction above implies that there exists a finite  $B_b \subseteq \mathbb{I}_{<n_*}$  such that  $\text{tp}(b, \mathbb{I}_{\leq n_*} \setminus \{b\})$  is the unique nonforking extension of  $\text{tp}(b, B_b)$  in  $\mathbf{S}(\mathbb{I}_{\leq n_*} \setminus \{b\})$ . Claim 2.4 implies that for every  $b \in \mathbb{I}_{n_*}$ ,  $\text{tp}(b, \mathbb{I}_{\leq n_*} \setminus \{b\} \cup \{a\})$  is the unique nonforking extension of  $\text{tp}(b, B_b)$  in  $\mathbf{S}(\mathbb{I}_{\leq n_*} \setminus \{b\} \cup \{a\})$ . Thus,  $\mathbb{I}_{n_*} \cup \{a\}$  fulfills the second condition, contradicting the maximality of  $\mathbb{I}_{n_*}$ . □

**Theorem 2.18.** Let  $M \models T$ , then  $M$  is  $\text{Ex}_{\omega, \omega}^2(\text{feq})$ -representable.

*Proof.* Let  $\langle \mathbb{I}_n : n < \omega \rangle$  as in 2.17,  $I = |\mathbb{I}_0|$ . Since  $T$  is  $\omega$ -stable,  $\mathbf{S}^m(\emptyset)$  is countable for all  $m < \omega$ . For convenience we replace the enumeration of the functions of  $\mathcal{M}(I)$  to  $\{F_p : p \in \mathbf{S}^{<\omega}(\emptyset)\}$ , and for every  $m+1$ -type  $F_p$  is an  $m$ -ary function. Define by induction an increasing series of functions  $f_i : \mathbb{I}_{\leq i} \rightarrow \mathcal{M}(I)$  as follows: Let  $f_0$  be a bijective map from  $\mathbb{I}_0$  onto  $I$ . Let  $f_{n+1}$  be defined from  $f_n$  as follows:

- $f_{n+1} \upharpoonright \mathbb{I}_{\leq n} = f_n$
- For all  $a \in \mathbb{I}_{n+1}$ , let  $\bar{c}_a \in {}^\ell(\mathbb{I}_{\leq n})$  enumerate  $B_a$  from Claim 2.16,  $p = \text{tp}(a \frown \bar{c}_a, \emptyset, M) \in \mathbf{S}^{\ell+1}(\emptyset)$ . Define  $f_{n+1}(a) = F_p(f_n(\bar{c}_a))$ . Now let  $f = \bigcup_{n < \omega} f_n$ . Now we will show that  $f$  is a  $\text{Ex}_{\omega, \omega}^2(\text{feq})$ -representation. Let  $h$  be a partial automorphism of  $\mathcal{M}(I)$  with domain and range closed under

subterms. Let  $\bar{a}, \bar{b} \in M$  such that  $h(f(\bar{a})) = f(\bar{b})$ , and  $n$  so that  $\bar{a}, \bar{b} \in \mathbb{I}_{<n}$ . Assume w.l.o.g  $m < \omega$ ,  $i < \lg \bar{a} - 1$ :  $a_{\lg \bar{a}-1}, b_{\lg \bar{b}-1} \in \mathbb{I}_{\leq m} \rightarrow a_i, b_i \in \mathbb{I}_{\leq m}$ . We prove  $\text{tp}_{\text{qf}}(\bar{a}, \emptyset) = \text{tp}_{\text{qf}}(\bar{b}, \emptyset)$  by induction on  $\langle n, |\bar{a} \cap \mathbb{I}_n| \rangle \in \omega \times \omega$ .

**case  $n = 0$ ::** the claim holds since  $\mathbb{I}_0$  is an indiscernible set.

**case  $n = m + 1$ ::**

**case  $|\bar{a} \cap \mathbb{I}_n| = 0$ ::**  $\bar{a} \subseteq \mathbb{I}_{\leq m}$ , hence, the claim holds by the induction hypothesis.

**case  $|\bar{a} \cap \mathbb{I}_n| > 0$ ::** Let  $k = \lg \bar{a} - 1$ . By the definition,  $f(a_k) = F_p(\bar{c}_{a_k})$  where  $\bar{c}_{a_k} \subseteq \mathbb{I}_{<n}$ .  $h$  commutes with  $F_p$ , implying  $f(b_k) = h(f(a_k)) = h(F_p(f(\bar{c}_{a_k}))) = F_p(h(f(\bar{c}_{a_k})))$ . Therefore,  $h(f(\bar{c}_{a_k})) = f(\bar{c}_{b_k})$ . Now, since  $|\bar{c}_{a_k} \hat{\ } \bar{a} \upharpoonright k \cap \mathbb{I}_n| = |\bar{a} \cap \mathbb{I}_n| - 1$ , and by the induction hypothesis, the map  $F : \bar{c}_{a_k} \hat{\ } \bar{a} \upharpoonright k \mapsto \bar{c}_{b_k} \hat{\ } \bar{b} \upharpoonright k$  is elementary. Consider the type  $q = F(\text{tp}(a_k, \bar{a} \upharpoonright k \cup \bar{c}_{a_k}))$ . Note  $\text{tp}(a_k \hat{\ } \bar{c}_{a_k}) = p = \text{tp}(b_k \hat{\ } \bar{c}_{b_k})$ , so  $F(\text{tp}(a_k, \bar{c}_{a_k})) = \text{tp}(b_k, \bar{c}_{b_k})$ . Then,  $q$  is a nonforking extension of  $\text{tp}(b_k, \bar{c}_{b_k})$ . Moreover,  $F$  being elementary and  $\text{tp}(a_k, \bar{a} \upharpoonright k \cup \bar{c}_{a_k})$  is a nonforking extension of  $\text{tp}(a_k, \bar{c}_{a_k})$  imply that  $q$  is a nonforking extension of  $\text{tp}(b_k, \bar{c}_{b_k})$ . Now let  $q \subseteq q', \text{tp}(b_k, \bar{b} \upharpoonright k \cup \bar{c}_{b_k}) \subseteq q'', q', q'' \in \mathbf{S}(\mathbb{I}_{\leq n} \setminus \{b_k\})$  be nonforking extensions. By monotonicity of nonforking extensions,  $q', q''$  are nonforking extensions of  $\text{tp}(b_k, \bar{c}_{b_k})$ . The definition of  $\bar{c}_{b_k}$  implies  $q' = q''$ . Thus,  $q = \text{tp}(b_k, \bar{b} \upharpoonright k \cup \bar{c}_{b_k})$ , therefore  $\text{tp}(\bar{a} \hat{\ } \bar{c}_{a_k}) = \text{tp}(\bar{b}, \bar{c}_{b_k})$ .  $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$  follows.  $\square$

### 3. APPENDIX - COMBINATORIAL CLAIMS.

**Theorem 3.1.** (Fodor) *Let  $\lambda$  a regular cardinal, and  $f : \lambda \rightarrow \lambda$  such that  $f(\alpha) < \alpha$  for all  $0 < \alpha < \lambda$ . (such  $f$  is called regressive) Then there exists an ordinal  $\beta < \lambda$  such that the set  $\{\alpha < \lambda : f(\alpha) = \beta\}$  is stationary in  $\lambda$ .*

**Corollary 3.2.** *Let  $f : \lambda \rightarrow \mu$ ,  $\lambda > \mu$  ( $\lambda$  regular). There exists an  $\alpha < \mu$  such that  $f^{-1}(\{\alpha\}) \subseteq \lambda$  is stationary*

**Theorem 3.3.** ( $\Delta$ -system Lemma) *Let  $\lambda$  regular,  $|W| = \lambda$  a set,  $|S_t| < \mu$  ( $t \in W$ ) such that  $\chi^{<\mu} < \lambda$  for all  $\chi < \lambda$ . then:*

- (1) *There exist  $W' \subseteq W$ ,  $|W'| = \lambda$  and  $S$  such that  $s \neq t$  implies  $S_t \cap S_s = S$  for all  $s, t \in W'$ .*
- (2) *Moreover, if  $\langle z_t^\alpha : \alpha < \alpha(t) \rangle$  lists  $S_t$ , also:*
  - (a) *There exists  $\alpha_0$  such that  $\alpha(t) = \alpha_0$  for all  $t \in W'$ .*
  - (b) *There exists  $U \subseteq \alpha_0$  such that for all  $s, t \in W'$  implies  $S_t \upharpoonright U = S_s \upharpoonright U$ ,  $U = \{\alpha < \alpha_0 : z_t^\alpha = z_s^\alpha\}$ .*

- (c) *There exists an equivalence  $E$  on  $\alpha_0$  such that  $z_t^\alpha = z_t^\beta \leftrightarrow (\alpha, \beta) \in E$ , for all  $t \in W'$ .*

*Proof.* Proofs for the first part can be found in [1].

The map  $t \rightarrow \alpha(t)$  is regressive ( $\alpha(t) < \mu < \lambda$ ), so by Fodor's theorem there exists  $W_0 \subseteq W$  such that 2a holds. By the first part there exists  $S \subseteq \{z_t^\alpha : \alpha < \alpha_0, t \in W_0\}$ ,  $W_1 \subseteq W_0$  such that  $S = \bar{z}_t \cap \bar{z}_s$  for all  $t \neq s$ . Define the map  $W_1 \ni t \rightarrow U_t$  where  $U_t = \{\alpha < \alpha_0 : z_t^\alpha \in S\}$ . The range has power at most  $2^{|\alpha_0|} \leq 2^{<\mu} < \lambda$  implying that the map is regressive, and the existence of  $W_2 \subseteq W_1$ ,  $U$  such that  $t \in W_2 \rightarrow U_t = U$ . The range of the map  $t \rightarrow S_t \upharpoonright U$  is  ${}^U S$  and it has power  $\leq |\alpha_0|^{|\alpha_0|} < \lambda$ . By another use of Fodor's theorem there exists  $W_3 \subseteq W_2$  such that (b) holds. The range of the map  $t \rightarrow E_t$  where  $E_t = \{(\alpha, \beta) : z_t^\alpha = z_t^\beta, \alpha, \beta < \alpha_0\}$  has power at most  $|\alpha_0|^{|\alpha_0|}$ . And by another application of Fodor's theorem there are  $E$  and  $W' \subseteq W_3$  as required.  $\square$

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